# The Role of Conceptual Structure in Mathematical Explanation

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**Abstract**

People's reasoning about physical and social explanations is well understood (Keil, 2008). However, less is known about how people reason about mathematical explanations (Johnson et. al., 2017). Experiment 1 replicates the central result of Johnson et. al (2017), that people impose order on simple arithmetic explanations, as well as sets the limits of that preference. Experiment 2 extends the results of a second factor, the character of the relationship between the operations related by the explanation.

**Keywords:** explanations, mathematical reasoning

## Introduction

Take a calculator and form a six-digit 'calculator number' by taking the three digits on any row, column, or main diagonal forward and then back (e.g., 789987, formed by moving across the top row from left to right and back). It is simple to check that every calculator number is divisible by 37. We might be led to ask why every calculator number is divisible by 37, and in doing so we are requesting an explanation for a mathematical fact. Typically, this is done via proof, though a proof is not necessarily explanatory: We can show that all calculator numbers are divisible by 37 by exhaustively checking all cases, but such a proof could hardly be regarded as explanatory (see Lange, 2014 for a full discussion of this example).

In contrast to physical explanations, which seek to explain one event by reference to how it is caused by another, mathematical explanations seek to demonstrate that one mathematical fact is implied by another via proof (Mancosu, 2001; Steiner, 1978). However, physical and mathematical explanations may share an important common feature: both may depend on an asymmetry between what is being explained and what is doing the explaining.

A commonly noted feature of physical explanations is that they are asymmetric: though the length of a building's shadow and the height of the sun on the horizon are related in regular and lawful ways, such that either one can be deduced from the other, explanations that relate the two abide by the fact that it is the height of the sun that explains the length of the shadow, and not the other way around (Bromberger, 1966). Plausibly, this is due to two asymmetrical relations that frequently appear in physical explanations: causal and reductive.

### Ordering in Physical Explanations

Causes proceed their effects. Thus, it is not a surprise that when seeking to explain a particular event (e.g., why Matt is thirsty), people look for temporally prior information that would help identify potential causal mechanisms (e.g., that it is exceptionally hot today; Ahn, Kalish, Medin, & Gelman, 1995). In this way, the temporal order of cause and effect provides one basis for asymmetric explanations in science.

A second source of asymmetry appears in reductive explanations: accounting for a phenomena in terms of more basic components. Indeed, finding reductive explanations is often a major goal of modern science. The reductive relations between different levels of a system impose ordering constraints on possible explanations: It would be inappropriate, for instance, to explain the chemical properties of dopamine by reference to its effects on neurons, but explanations that run the other way (explaining how neurons communicate via dopamine) would be acceptable.

### Ordering in Mathematical Explanations

Whereas in causal domains inverting the order of an explanation typically results in a falsehood, many mathematical explanations may be inverted and still be true (e.g., "$1+2=3$ *because* $3-2=1$", and "$3-2=1$ *because* $1+2=3$"). Despite this, we might expect to find order imposed on explanations in mathematics. People preferentially write equations in a particular order to reflect the structure of the system being represented (Mochon & Sloman, 2004), sometimes leading them to generate false or uninterpretable equations (Landy, Brookes, & Smout, 2014). Indeed, two critical features of modern mathematical practice, axiomatic treatment of domains and the reduction of mathematical structures to set or category theory, may reflect the imposition of order on explanations in mathematics.

First, many advanced mathematical texts begin by stating the axioms of the theory under discussion, and then showing how the major theorems of the theory are implied by the axioms. Second, mathematical objects are frequently reduced to more basic components, analogous to reduction in science. Reduction is often seen as a major accomplishment, even if it

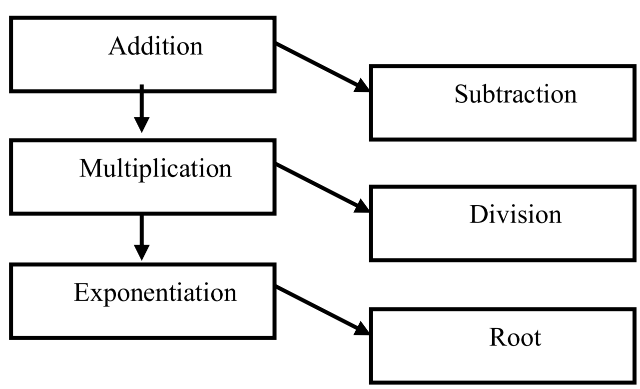


Figure 1 — Adapted from Johnson et al. (2017). Arrows between boxes indicate the direction of grounding relations. Diagonal relations are inverses; vertical relations involve repeated operations.

doesn't directly lead to new theorems. For instance, Nicolas Bourbaki, the pseudonym of a group of early 20th century French mathematicians, sought to ground all of mathematics in the theory of sets (Bourbaki, 1968). The goal of this project was not necessarily to obtain new results, but to provide a characterization of known results in more basic or simpler terms (though new results may be obtained).

Deductive relations in math and causal relations in science may be thought of as analogous. Ordering stems from the manner that one set of propositions or events is deemed prior to the other: conceptually for mathematical propositions, and temporally for events. Similarly, reductive relations in mathematics and physical science are analogous in the sense that the truth of the reducing proposition (or the existence of the reducing phenomena) guarantees the truth of the reduced proposition (or the existence of the reduced phenomena). Here, ordering stems from how one domain is considered more basic or simpler than the other.

While deductive and reductive relations are most evident in advanced mathematics, they might also be found in more elementary mathematics. For instance, operations in integer arithmetic can be conceptualized as having ordered relationships among themselves: addition is more basic than subtraction and multiplication, and multiplication is more basic than division and exponentiation. In this way, relations among mathematical operations may come to be structured as they are in Figure 1.

This structure corresponds to the standard way that arithmetic is taught in American schools: addition is introduced as aggregating groups of objects before subtraction is taught as the taking away of objects from a group. Later, multiplication is taught as aggregating many groups of even size, and division as dividing a whole into equal parts. Explanations that ground facts concerning one operation (subtraction) in a more basic operation (addition) can be thought of as following this conceptual order, while explanations that do the opposite (e.g., explaining addition in terms of subtraction) violate this order.

Johnson, Johnson, Koven, and Keil (2017) presented participants with a variety of simple arithmetic explanations and asked them to rate the quality of the explanations on a ten-point scale. Johnson et al. found that participants rated explanations that followed the conceptual structure of mathematics in Figure 1 as better than those that proceeded in the opposite direction. Johnson et al. interpreted this as indicating that participants have an ordered conceptual structure of arithmetic, and use these ordered features to structure their explanations.

While Johnson et al. (2017) investigated the role that conceptual order plays in mathematical explanations, order is not the only structural factor that may be relevant when evaluating mathematical explanations. In particular, the manner in which one operator explains another may be important. Subtraction is the inverse of addition, in the sense that subtracting X from Y 'undoes' the effect of adding X to Y (similarly with multiplication and division). In contrast, the effect of multiplying Y by X is the same as repeatedly adding Y to itself X times (similarly with multiplication and exponentiation). We can think of addition and subtraction as different sides of the same processes, a conceptualization that is borne out by their treatment in the standard axioms for the real numbers (Rudin, 1976). In contrast, addition and multiplication enjoy no such relationship in real analysis, and instead are treated as fundamentally distinct operations by those axioms. Multiplying 187.3 by $\pi$ can't be explained as adding 187.3 copies of $\pi$. It is possible that students who are familiar with the real number system are sensitive to the different kinds of relationships being captured by these explanations. If so, they may treat the diagonal relations in Figure 1 as more explanatory than the vertical ones.

The present research has two goals: first, to replicate the main results of Johnson et al. (2017) in a novel paradigm, thus strengthening the original result and validating the new paradigm. While Johnson et al. asked participants to numerically evaluate explanations, the present paradigm asks participants to compare pairs of explanations and select the better one. The second objective is to use the new paradigm to explore the influence of a particular factor, the character of the relationship between the operators expressed by an explanation, on participants' judgments of that explanation's value. Do people believe the diagonal relations in Figure 1 (e.g., the relation of addition to subtraction) are more explanatory than the vertical ones (e.g., the relation of addition to multiplication)?

## Experiment 1

### Method

**Procedure** Participants were shown pairs of explanations and asked to indicate which explanation they preferred, and whether they preferred that explanation *Strongly, Moderately,* or *Weakly*. Each trial asked participants to compare a pair of explanations that used the same operations, one of which was forward, and the other backward (e.g., "$3 \times 3 \times 3 = 27\ because\ 3^3 = 27$" vs. "$2^4 = 16\ because\ 2 \times 2 \times 2 \times 2 = 16$). The explanations in the pairs contained different numbers, and presentation order was randomized across participants.

Participants were told, "Please note that for many of the questions, both explanations are true, and you must decide which explanation you prefer based on other criteria." Participants allowed to use a calculator to check any of the math. On average, participants took approximately 4 minutes to complete the task.

**Materials** Four lists containing 16 test questions and 4 check questions were created. Lists differed only in the specific numbers used in the various explanations. All questions consisted of two explanations presented simultaneously. All explanations were of the form "X because Y," where X and Y were arithmetic identities that could be transformed into each other by a simple manipulation, (e.g. "$7 - 2 = 5\ because\ 7 = 5 + 2$"). Explanations were generated using four pairs of operators: addition and subtraction, addition and multiplication, multiplication and division, and multiplication and exponentiation.

For each test item, the *forward* explanation used a more basic operator to explain a more advanced one (e.g., using addition to explain subtraction, as in the previous example), and the *backward* explanation used a more advanced operator to explain a more basic one (e.g., using exponentiation to explain multiplication, as in "$3 \times 3 \times 3 = 27\ because\ 3^3 = 27$").

In addition, explanations varied in terms of the relationship between the operations in the explanation. *Inverse* explanations explained an operation in terms of its inverse (e.g., addition and subtraction), while *repeated* explanations explained an operation in terms of the repeated application of another operation, (e.g., addition and multiplication), or the opposite.

Check questions were of the same form as test questions, except one of the explanation pairs either contained an arithmetic error (e.g., "$13 - 4 = 8\ because$ ...") or explained an arithmetic fact using an irrelevant arithmetic fact (e.g., "$13 - 6 = 7\ because\ 2 \times 2 \times 5 = 20$").

**Participants** 29 undergraduate students were recruited from the Northwestern Introduction to Psychology pool. Participants were granted course credit for their participation. 7 participants were excluded because they failed the check questions.

## Results

Participant responses were coded so that preference for forward explanations were positive and preference for backwards explanations were negative. Strong preferences were coded as ±2.5, moderate preferences as ±1.5, and weak preferences as ±.5.

Mean preference roughly corresponded to a *Weak Preference* for forward explanations (M=0.44, SD=.92). Coded responses from each participant were averaged and entered into a one sample t-test. As expected from Johnson et al., participants' average preference was significantly higher than zero, $t(21) = 3.92$, $p < .001$, indicating that they preferred explanations that conformed to the standard conceptual order of arithmetic. Participants preferred the forward explanation 60% of the time, not significantly higher than chance (One-sample binomial, p=.076, two-sided).

To examine whether the strength of this preference varied across operator pairs, a two-way (2x2) repeated measures ANOVA with operator used in the explanation base (addition vs. multiplication) and relation between the explanation base and explanation target (inverse vs. repeated) as within subject factors was performed. The analysis found no effect of explanation base, relation type, or their interaction (Fs < 1, ps > .7). As shown in Figure 2, preference for the forward direction was consistent across operator pairs.

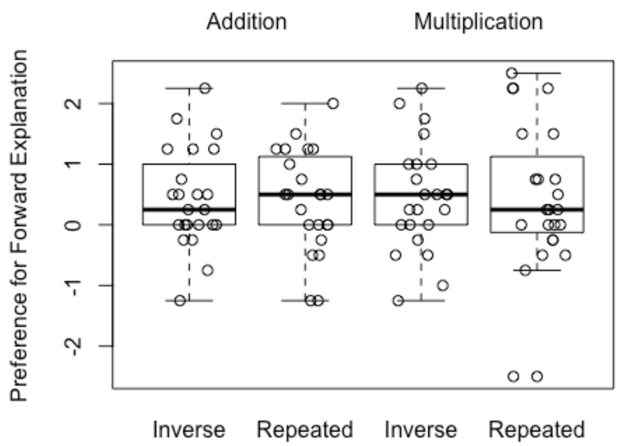


Figure 2 — The top axis labels indicate the operator used in the explanatory base, while the bottom axis labels indicate the relationship expressed by the explanation. Box edges indicate 1st and 3rd quartile of the distribution, while whiskers indicate 1.5 times the interquartile range.

## Discussion

The hypothesis guiding Experiment 1 was that, as in Johnson et al., participants would favor explanations that followed the conceptual order of arithmetic over those that did not. The results from Experiment 1 corroborate those from Johnson et al., strengthening the hypothesis that participants prefer explanations that proceed in a particular direction, even in domains without temporal or causal ordering. However, participants did not distinguish between the inverse and repeated relations of Figure 1.

# Experiment 2

Experiment 2 was designed to retest the hypothesis that participants take into consideration the kind of relationship between the arithmetic facts in math explanations. Experiment 2 also contains a conceptual reproduction of Experiment 1, with the modification that participants were permitted to indicate that they don't have a preference. This provided a better sense of explanatory direction's importance is when participants evaluate arithmetic explanations. In addition, by having the same participants compare forward to backward explanations, and inverse to repeated explanations, we can compare the relative strength of the any preferences found.

## Method

**Procedure** The procedure was identical to that used in Experiment 1, with the modification that the response scale included a middle item. This middle item allowed participants to indicate that they had *No Preference* between the explanations being compared. Participants took approximately ten minutes to complete the study.

| | Inverse Base | Inverse Explanation | Repeated Base | Repeated Explanation |
|---|---|---|---|---|
| 1 | Addition | $9 - 3 = 6\ because\ 3 + 6 = 9$ | Addition | $3 \times 3 = 9\ because\ \overbrace{3 + 3 + 3}^{3} = 9$ |
| 2 | Addition | $9 - 3 = 6\ because\ 3 + 6 = 9$ | Multiplication | $3^3 = 27\ because\ \overbrace{3 \times 3 \times 3}^{3} = 27$ |
| 3 | Multiplication | $\frac{9}{3} = 3\ because\ 3 \times 3 = 9$ | Addition | $3 \times 3 = 9\ because\ \overbrace{3 + 3 + 3}^{3} = 9$ |
| 4 | Multiplication | $\frac{9}{3} = 3\ because\ 3 \times 3 = 9$ | Multiplication | $3^3 = 27\ because\ \overbrace{3 \times 3 \times 3}^{3} = 27$ |
| 5 | Exponentiation | $\sqrt[2]{9} = 3\ because\ 3^2 = 9$ | Addition | $3 \times 3 = 9\ because\ \overbrace{3 + 3 + 3}^{3} = 9$ |
| 6 | Exponentiation | $\sqrt[2]{9} = 3\ because\ 3^2 = 9$ | Multiplication | $3^3 = 27\ because\ \overbrace{3 \times 3 \times 3}^{3} = 27$ |

Table 1 — Six example trials from Experiment 2. Inverse explanations on the left were compared to repeated explanations on the right.

**Materials** Materials for Experiment 2 were similar to those for Experiment 1, but new explanations were generated to test the primary hypothesis at question.

***Inverse versus repeated.*** Inverse explanations explained an operator in terms of its inverse (e.g., addition and subtraction, as in “$9 - 3 = 6\ because\ 3 + 6 = 9$”), while repeated explanations explained an operator in terms of the repeated application of another operator (e.g., addition and multiplication, as in “$3 \times 3 = 9\ because\ 3 + 3 + 3 = 9$”).

As table 1 shows, test items for the main portion of the experiment compared inverse and repeated explanations. To construct test items, operator pairs were crossed as best possible: Test items compared inverse addition to repeated addition and repeated multiplication, inverse multiplication to repeated addition and repeated multiplication, and inverse exponentiation to repeated addition and repeated multiplication. Example explanation pairs are show in Table 1. Thus, there were 6 trials in this portion of the experiment. Because the standard course of mathematical education does not include treatment of operations that can be conceptualized as the outcome of repeated exponentiations, there are no repeated exponentiation explanations (though such explanations would be possible, see Goodstein 1947). All explanations for this portion of the study were forward.

***Forward versus backward.*** Test items for the reproduction of Experiment 1 paired forward and backward explanations using the same operator pairs, as in Experiment 1. In addition to the four pairs of operators used in Experiment 1, an additional pair of operators, exponentiation and root taking, were used in the generation of the explanations to test the generality and strength of any preferences found. Thus, there were 5 trials in this portion of the experiment. Test items in the reproduction of Experiment 1 used a different set of numbers than those used in the first part of the experiment.

Equations relating operators via repeated application could be very long (e.g., "$3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 24$"). Furthermore, the large number of terms in the equation could obscure the relationship being illustrated (i.e., that there are eight 3’s). To get around this, equations expressing repeated operations were displayed so that (a) the number of terms being repeated was indicated by an overhand bracket, and (b) equations with many terms were abbreviated so that only the first and last elements of the expression were shown, with the middle terms replaced with ellipsis. The instructions made clear how the participants were to interpret this notation.

***Check questions.*** As in Experiment 1, Experiment 2 included 4 check questions. All check questions in Experiment 2 contrasted a correct explanation with an explanation containing a simple arithmetic error.

**Participants** Thirty-seven participants were recruited from Amazon Mechanical Turk. Those that completed the survey were compensated $1. Average time to complete the survey was approximately 15 minutes, resulting in an effective hourly rate of $4. Seventeen participants were excluded because they failed the check questions.

## Results

Strong preferences were coded as ±3, moderate preferences as ±2, weak preferences as ±1, and no preference as 0.

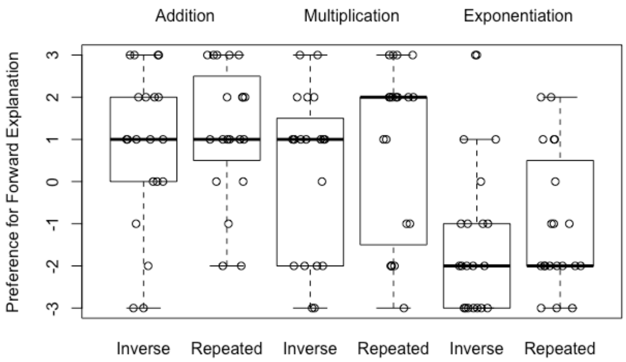


Figure 3 — Top axis labels indicate the base operator used in the Inverse explanation, while bottom axis labels indicate the base operator used in the repeated explanation. Box edges indicate 1st and 3rd quartile of the distribution, while whiskers indicate 1.5 interquartile range.

Responses were entered into a 3x2 repeated measures ANOVA, with the base used in the inverse explanation (addition, multiplication, or exponentiation) and the base used in the repeated explanation (addition or multiplication) as within-subject factors. This test revealed a significant effect of the base used in the inverse explanation, $F(2, 38) = 20.18$, $p < .0001$, but no effect of the base used in the repeated explanation, nor of their interaction (Fs <2.5). A post-hoc paired t-test found that the preference for the inverse explanation was weaker when the inverse explanation used exponentiation as a base, $t(19)=-5.6$, $p<.001$. A second post-hoc paired t-test found no differences on trials where the inverse explanation used addition compared to those that used multiplication, $t < 1.5$.

In light of these results, a second round of post-hoc analyses was carried out, this time omitting trials where the inverse explanation used exponentiation. Preference on these trials fell roughly between *No* and *Weak Preference* for Inverse explanations (M=0.76, SD=1.25). Participants selected *No Preference* on these trials only ~7% of the time. On the ~93% of the trials where the participant expressed a preference, 73% of the time that preference was for the inverse explanation, significantly different from chance ($p<.0001$, two-sided). Preference for inverse explanations was reliably above chance, $t(19)=2.73$, $p=.013$.

**Replication of Experiment 1** The second main question of Experiment 2 was whether preference for forward explanations would generalize when participants are given the option of expressing *No Preference*. In contrast to Experiment 1, participants did not reliably favor forward explanations over backward explanations, as revealed by a one sample t-test of participant means against 0, $t(19)=1.4$, $p = .174$. Indeed, the modal response was *No Preference* (40%). However, of the 60% of trials where participants expressed a preference, participants favored forward explanations at a similar rate as in Experiment 1 (61%), though this did not rise above chance ($p=.08$, two-sided). A 2x2 repeated measures ANOVA with base (addition vs multiplication) and relation type (inverse vs repeated) as within-subject factors failed to find an effect of explanatory base, relation type, or their interaction (Fs <1.0).

**Comparison of Preferences**. Embedding the reproduction of Experiment 1 in Experiment 2 allows us to compare the relative strength of preferences in the two comparisons. A paired t-test comparing participants' preference for forward over backward explanations to their preference for inverse over repeated explanations, omitting inverse exponentiation explanations, found that preference was stronger for the latter, $t(19)=-4.6$, $p<.001$.

## Discussion

While on the whole participants did not take into account whether the explanation expressed an inverse or repeated relation, participants gave lower ratings to inverse explanations that related exponentiation to root taking, decreasing the average preference for inverse explanations. After omitting those trials, participants reliably selected the inverse explanation. In conjunction with Experiment 1, these data suggest that participants take into account the conceptual relation expressed by an explanation, but that other factors, perhaps their familiarity with the concepts involved (as suggested by the dis-preference for explanations that use roots) can outweigh this preference.

The specific nature of this preference — inverse explanations over repeated explanations — suggests that something like conceptual fit between the operators being related is taken into account. Because addition and subtraction are inter-defined operations (as are multiplication and division), using one to explain the other is better than using conceptually distinct operations, such as addition and multiplication.

An alternative interpretation, suggested by Johnson et al., is that when evaluating explanations, participants construct a mental proof which derives the explanatory target from the explanatory base. Evaluation of explanation-proofs penalize proofs that require many steps. Inverse explanations have short proofs: simply move one term to the other side of the equation. In contrast, repeated application proofs require grouping many terms and then relating the grouped operation to another operation. While this account has merits, the consensus among mathematicians and philosophers is that proof is distinct from explanation, and the latter does not cleanly map onto formal features of the former: for instance, proofs by induction or enumeration are generally thought to be non-explanatory, despite being rigorous (Lange, 2009).

A secondary goal of Experiment 2 was to test whether the results of Experiment 1, that participants favored forward explanations over backwards explanations, would generalize when participants were given the option of indicating that they have no preference. In contrast to the results of Johnson et al. and Experiments 1, the results of Experiment 2 indicate that participants are on the whole ambivalent to the distinction between forward and backwards explanations: the most common response was "No Preference."

This should not be considered a failure to reproduce the results of (Johnson et al., 2017), however: the preference for forward explanations was weak in both Johnson et al. (2017) and Experiment 1, and the lower powered design of Experiment 2 plausibly rendered the effect too small to detect with small samples. In addition, when participants did express a preference, it was generally for forward explanations. However, this result does place limits on how important conceptual direction is when evaluating arithmetic explanations. Other factors, such as familiarity with the operations involved, may be more important.

## General Discussion

Across two experiments, participants took structural factors into account when evaluating explanations in arithmetic. They favored explanations that proceeded from conceptually basic operations to advanced ones over those that did the opposite, and preferred explanations that linked conceptually similar operations over those that linked dissimilar operations.

The dis-preference for backward explanations was found to be weaker than other factors, such as a dis-preference for explanations that use unfamiliar operations. Indeed, when given the choice to indicate that ordering didn't matter, participants often did so. These results provide evidence against theories of explanation that require explanations to be asymmetric and are more consistent with theories of explanation in which order emerges out of other factors.

Plausibly, the symmetry of arithmetic explanations is a consequence of a lack of an objective ordering in arithmetic. Both "$3 - 1 = 2\ because\ 1 + 2 = 3$" and "$1 + 2 = 3\ because\ 3 - 1 = 2$" are valid, in the sense that both summarize valid proofs, and participants' indication of "No Preference" may indicate that they are sensitive to this feature of arithmetic explanations. Thus, it may be that validity of the explanation, rather than the explanation's ordering *per se*, is the primary criteria by which explanations are evaluated. This is substantiated by participants' response to the check questions in Experiments 1. Participants equally dis-preferred explanations containing an error and those that explained a mathematical fact via an unrelated fact, indicating that the possibility of constructing a deductively valid proof linking the explanatory base and target is an important criterion.

More generally, these results are consistent with a view of explanations that holds that their formal properties are inherited from the domain in which they are applied. Causal explanations are asymmetric because they are generated by a domain that is best described using asymmetric causal relations, not because explanations are inherently asymmetric. In contrast, arithmetic is best described by symmetric derivation relations, and as such the ordering constraint is weaker. This would not explain the preference for forward explanations when forced to choose, however. Further experiments may be needed to tease out the respective roles of ordering and validity in causal and acausal domains.

## Acknowledgments

I am grateful to Lance Rips and Fabrizio Cariani for feedback on the early stages of the project, and to Natalie Gallagher, Sandy LaTourrette, Camdyn Parrott and Meghan Salomon-Amend for comments on an early draft, as well as other members of the Higher Order Cognition Lab.

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