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MODELLING THE ABSOLUTE RETURNS OF DIFFERENT STOCK INDICES:
EXPLORING THE FORECASTABILITY OF AN ALTERNATIVE MEASURE OF RISK

BY

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Modelling the Absolute Returns of Different Stock Indices: Exploring the Forecastability of an Alternative Measure of Risk

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Abstract

Conventional measures of the risk of a financial asset make use of the *unobserved* (conditional) variance or standard deviation of its return. In this paper, we treat the *observed* absolute return as a measure of risk and explore its forecastability. Two simple models are considered. One is a new *AR-like* model which is applied to the absolute return. The other is an *ARCH-like* model called Asymmetric Power ARCH. The forecastability is evaluated with the average log-likelihood of absolute return, instead of that of return itself. While the absolute return is interpreted as “volatility”, some quantities of its entire distribution, such as the 95-th quantiles, can be interpreted as “volatility of volatility”. We apply both models to three stock indices, namely Hang Seng Index, Nikkei 225 Index and Standard and Poors 500 Index. The new model by and large outperforms the ARCH-like model in both in-sample goodness of fit and post-sample forecastability. It performs exceptionally well in the post-sample period after the outbreak of the Asian financial crisis.

KEY WORDS absolute return; asymmetric least squares; log-likelihood; return; risk; volatility.

Preliminary. Comments Welcome

1 INTRODUCTION

Conventional measures of the risk¹ of a financial asset make use of the *unobserved* (conditional) variance or the standard deviation of its return. This can be due to the overwhelming mean-variance analysis in the study of investment. Further, the standard deviation enters directly in the celebrated Black-Scholes model, used by both practitioners and academics.

In this paper, we treat the *observed* absolute return as a measure of risk and explore its forecastability. Modelling the absolute return can be traced back to Taylor (1986). See Sections 3.6 and 3.7 of the book. Point forecastability of absolute return, in terms of high autocorrelations of different lags, is well documented in Taylor (1986) (Chapter 2), Kariya, Tsukuda and Maru (1990), Cao and Tsay (1992), Ding, Engle and Granger (1993), and Granger and Ding (1995). For different stock prices, the autocorrelations of absolute returns are higher than those of squared returns, which are unsurprisingly higher than those of returns. This stylized fact is a property of the so-called *Taylor effect*.²

Simple arguments (see the beginning of the next section) show that the absolute return and the conditional standard deviation have an intricate relation to each other, in a class of scale models. This is also the case for squared return and conditional variance, though estimating models of the squared returns may be too sensitive to extreme values. The observability of absolute return permits an *autoregression-like* (AR-like) model, as suggested in Sin and Granger (1999). Along the lines in that paper, we apply this model not only to the mean/median but also to a number of expectiles/quantiles. This model is readily estimated with asymmetric least squares (ALS), first suggested in Aigner, Amemiya and Poirier (1976) in the context of estimating production frontiers. ALS is an alternative to the usual asymmetric least absolute deviation (ALAD) first suggested in

¹ Throughout the paper, we use *risk* and *volatility* interchangeably.

² Taylor effect states that the absolute return has the highest autocorrelations among all of its power transformation. See Section 4 in Granger and Ding (1995).

Koenker and Bassett (1978) in the context of robust estimation. Further details can be found in the next section.

For comparison, we also consider a conventional *autoregressive conditional heteroskedasticity-like (ARCH-like)* model called Asymmetric Power ARCH (A-PARCH), first suggested in Ding et al. (1993). More precisely, A-PARCH is a wide class of models for the power transformations of the conditional standard deviation. As with the well-known related models such as GARCH and EGARCH, latent variables (namely, the lagged standard deviations) are involved and the standard deviation responses asymmetrically to the positive and negative “shocks”. The former is for parsimony while the latter captures the leverage effect suggested in the finance literature. As claimed in Appendix A of that paper, this model is general enough to unify seven ARCH-like models in the literature. See also a recent survey of ARCH models in Bollerslev, Engle and Nelson (1994).

In this paper, the forecastability of absolute return is evaluated with the average log-likelihood. For the AR-like model, the log-likelihoods are computed with a whole bunch of expectiles/quantiles (of the absolute returns). For the ARCH-like model, we follow a somewhat congenital approach and first estimate the conditional standard deviation (of the return). The log-likelihoods are then computed with the assumption on the normalized (normalized by the conditional standard deviation) absolute return. Contrast to criteria such as mean squared errors or mean absolute errors (of point forecasts), the log-likelihoods consider the entire density instead of focusing on the mean or the median of the distribution. Interpreting the absolute return as a measure of risk or volatility, information in its entire density potentially addresses the issue of “volatility of volatility”, which is raised by practitioners in financial markets.

We applied both models to the daily absolute return of the three stock indices: Hang Seng Index, Nikkei 225 Index and Standard and Poors 500 Index. The paper is organized as follows. We start with a *brief* discussion on the two models. Detailed

discussion can be found in Sin and Granger (1999) and Ding et al. (1993). The section after next contains the data description and the estimation results. Section 4 contains a comparison of the two models, both in-sample and post-samples. The last section concludes.

2 A BRIEF DISCUSSION ON THE METHODOLOGY

Throughout the paper, *daily* return, for trading day t , is defined as:

$$r_t \equiv \ln(p_t) - \ln(p_{t-1}),$$

where p_t is one of the three stock price indices. The absolute return is simply $|r_t|$. This contrasts to those used in Cao and Tsay (1992) in which they used the excess absolute return (the return minus the risk-free rate). Moreover, while they used *monthly* return, we used *daily* return and thus it is safe for us to assume a zero unconditional mean. To focus our attention to absolute return, we also do not subtract the conditional mean from it, which is modelled as a moving average process in Ding et al. (1993). Interpreting the absolute return itself as a measure of risk, we may not need to bother the conditional mean.

Ignoring the location (such as the conditional mean) and consider a scale model for the return:

$$r_t = \sigma_{t-1} \eta_t,$$

where σ_{t-1} is realized in period $(t-1)$ and η_t is realized in period (t) .³ In the ARCH

³ Note this contrasts with that in the stochastic volatility literature, where the first term is also realized in period (t) .

literature, it is often assumed that η_t is i.i.d., σ_{t-1} is strictly positive with probability one. and σ_{t-1} is interpreted as the conditional standard deviation. Rewrite the above equation as: $|r_t| = \sigma_{t-1}|\eta_t|$, it is easy to see that the absolute return follows another scale model in which it is simply a product of the conditional standard deviation and an i.i.d random variable $|\eta_t|$. The ARCH-like model below exploits this fact.

Modelling Expectiles/Quantiles of the Absolute Return: an AR-Like Model

This new model takes the “semi-parametric” approach in which the conditional density of the absolute return is estimated via a number of *expectiles/quantiles*. The expectiles are modelled as a linear function of the lags of the absolute return. More precisely, for $\omega \in (0,1)$, the 100 ω -th conditional (on the information set I_{t-1}) expectile of the absolute return is modelled as:

$$\tau_{t-1}(\omega) = \phi_{\omega 0} + \sum_{i=1}^p \phi_{\omega i} |r_{t-i}|.$$

This is a model very close to that first suggested in Sin and Granger (1999), in which they modelled the expectiles of growth rate or return rate rather than absolute return. Note for each ω , we can always define the “error” $u_t(\omega) \equiv |r_t| - \tau_{t-1}(\omega)$. Here, we make no attempt to impose some explicit and implausible assumptions, such as zero conditional mean, homoskedasticity, or conditional normality, on $\{u_t(\omega)\}$. This is the major difference of this AR-like model from the usual AR model.

The expectiles are estimated with asymmetric least squares (ALS).⁴ That is, the parameter $\phi_\omega \equiv [\phi_{\omega 0}, \phi_{\omega 1}, \dots, \phi_{\omega p}]^T$ is estimated by minimizing:

⁴ The computer codes, which are written in FORTRAN, are available from the second author upon request. Description of the algorithm can be found in Section 2 of Efron (1991) and Section 2 and Appendix I of Sin and Granger (1999).

$$R(\phi_\omega) \equiv \sum_{t=1}^n [\rho_\omega(|r_t| - \tau_{t-1}(\omega))],$$

where $\tau_{t-1}(\omega)$ is defined above, and

$$\begin{aligned} \rho_\omega(u) &\equiv (1-\omega)u^2 & \text{if } u \leq 0, \\ &\equiv \omega u^2 & \text{if } u > 0. \end{aligned}$$

ALS resembles ordinary least squares (OLS) with a squared loss function except that different weights are assigned to positive and negative residuals. For $\omega = 1/2$, it reduces to OLS. These estimators were first discussed and used by Aigner et al. (1976). While they estimated production frontiers under a particular setting in that paper, Newey and Powell (1987) used these estimators to test for heteroskedasticity and asymmetry.

Exploiting the computational simplicity of ALS, Efron (1991) suggested estimating a number of *quantiles* by counting the proportion of in-sample observations (of the absolute return in our case) lying below the *expectile* curves. This is based on the fact that, for each 100ω -th expectile $\{\tau_{t-1}(\omega)\}$, there is a corresponding 100α -th quantile $\{Q_{t-1}(\alpha)\}$, though ω is typically *not* equal to α . Yao and Tong (1996) gave the theoretical justification. See also Section 3 of Sin and Granger (1999). Based on the quantiles, the densities can be estimated and be compared with those derived from other models. See Section 6 in Sin and Granger (1999).

It is noteworthy that the distribution under which expectiles equal quantiles (that is, ω equals α in the previous paragraph) is very extreme and in that case, the asymptotic theory of the expectiles breaks down. (Koenker, 1992). Fortunately, it is *not* the case in our empirical examples. On the other hand, Yao and Tong (1996) showed that there is a one-to-one mapping from expectiles to quantiles: It suffices to assume a location-scale model. It remains to see if it is preferable to estimate quantiles *directly* with asymmetric least absolute deviation (ALAD). Theoretical discussion on this issue can be found in

Portnoy and Koenker (1997) and Ellis (1998). Nevertheless, it should be reminded that with either estimation method, smoothing is required when the log-likelihood is compared with that of another model such as the following one.

Asymmetric Power ARCH Model: an ARCH-Like Model

For comparison, we consider a special case of A-PARCH developed in Ding et al. (1993) in which the *return* is modelled as:

$$r_t = \sigma_{t-1} \eta_t,$$

where $E[\eta_t | I_{t-1}] = 0$ and $\sigma_{t-1} = \alpha_0 + \alpha_1(|r_{t-1}| - \gamma_1 r_{t-1}) + \beta_1 \sigma_{t-2}$, which is assumed to be strictly positive.

First of all, the notation here deviates from that in Ding et al. (1993) as the conditional standard deviation is denoted as σ_{t-1} , which signifies that it *realizes* in period (t-1). On the other hand, while the return there follows a moving average process, here we assume zero conditional mean. Also, we focus on the case that the exponent (δ in their notation) is 1 and a simple model of A-PARCH (1,1) is considered. This A-PARCH(1,1), which models the conditional standard deviation, is a variant of the huge class of ARCH models.

Moreover, $|\eta_t|$ is assumed to be i.i.d. *exponential* with parameter λ . As a result, the absolute return is *conditionally* exponentially distributed.⁵ Given the fact that $E(|\eta_t|) = 1/\lambda$ and $\text{var}(|\eta_t|) = 1/\lambda^2$ (See, for instance, p.133 in Hogg and Craig, 1995), It is easy to verify that the conditional mean and the conditional variance of the absolute return are:

$$E(|r_t| | I_{t-1}) = \sigma_{t-1}/\lambda \text{ and } \text{var}(|r_t| | I_{t-1}) = \sigma_{t-1}^2/\lambda^2.$$

Thus, the conditional mean equals the conditional standard deviation. A conventional quasi maximum likelihood (QML) assuming normality is used to estimate the parameter in σ_{t-1} while an estimate of λ is simply the reciprocal of the (in-sample) average of $|r_t|/\hat{\sigma}_{t-1}$'s.⁶

Model Comparison with Average Log-likelihoods

For the ARCH-like model, we assume $|r_t| = \sigma_{t-1}|\eta_t|$ where $|\eta_t|$ is i.i.d. exponential with parameter λ . It is easy to verify (with p.133 and Section 4.3 of Hogg and Craig, 1995) that for *each* (in-sample or post-sample) observation t , the log-likelihood can be estimated as:

$$\log(\hat{\lambda}) - \hat{\lambda} |r_t| / \hat{\sigma}_{t-1} - \log(\hat{\sigma}_{t-1}).$$

For the AR-like model, the estimation is more involved.⁷ First recall that for each 100 ω -th expectile we estimate (in this paper, we try 101 different expectiles), there is an estimate of the 100 $\hat{\alpha}(\omega)$ -quantile, by counting the proportion of *in-sample* observations lying below the expectile curves. Denote the quantile as $\{\hat{Q}_{t-1}(\hat{\alpha}(\omega)) = \hat{\tau}_{t-1}(\omega)\}$. Therefore, for each (in-sample or post-sample) observation t , we can fit a smooth function and its derivative of $\hat{Q}_{t-1}(\hat{\alpha}(\omega))$ against $\hat{\alpha}(\omega)$ with, say, a cubic spline smoother. Note by the definition of a quantile, the inverse of the corresponding function is the distribution while the inverse of the derivative is the density. As a result, for each *pair of w and t* , we have an estimate of the density, denoted as $\hat{g}_{t-1}^\omega$. Thus for *each* (in-sample or post-sample) observation t , the log-likelihood is estimated by:

⁵ In Section 5 of Granger and Ding (1995), the *marginal* distribution of $|r_t|$ is exponential.

⁶ The computer codes for this two-step algorithm, which are written in RATS, are available from the second author upon request.

$$\log((\hat{g}_{t-1}^{\omega 1} + \hat{g}_{t-1}^{\omega 2})/2),$$

where $|r_t|$ lies between the $100\omega 1$ -th expectile (the $100\hat{\alpha}(\omega 1)$ -th quantile) and the $100\omega 2$ -th expectile (the $100\hat{\alpha}(\omega 2)$ -th quantile).

3 DATA DESCRIPTION AND ESTIMATION RESULTS

Data Description

All daily data of the three stock price indices, Hang Seng Index (HSI), Nikkei 225 Index (NIKKEI) and Standard and Poors 500 Index (S&P), are retrieved from *Datastream*. The entire sample is divided into the following three parts:

In-Sample: January 1, 1988 - December 31, 1994

Pre-Crisis Post-Sample: January 1, 1995 - April 30, 1997

Post-Crisis Post-Sample: May 1, 1997 - December 31, 1998

The entire sample start from 1988 such that the 1987 world stock crisis is not included. The post-sample is further divided by the Asian financial crisis, which is believed to start in some Asian economies as early as May of 1997. The absolute return is defined without considering the day-of-week effect, as that in the previous section. The summary statistics and the time plots, where it is expressed in percentage points, are depicted in Table 1 and Figures 1 to 3 respectively.

⁷ A detailed discussion can be found in Section 6 of Sin and Granger (1999).

Table 1: Summary Statistics of the Absolute Return (in Percentage Points)**(i) Hang Seng Index (HSI)**

	<i>In-sample</i>	<i>Post-sample: Pre-crisis</i>	<i>Post-sample: Post-crisis</i>
<i>Number of Observations</i>	1727	577	411
<i>Mean</i>	0.999	0.844	1.970
<i>Standard Deviation</i>	1.152	0.807	2.112
<i>Skewness</i>	5.555	2.315	2.999
<i>Kurtosis</i>	70.215	10.090	14.998
<i>Maximum</i>	21.745	7.313	18.824
<i>Minimum</i>	0.001	0.003	0.005

(ii) Nikkei 225 Index (NIKKEI)

	<i>In-sample</i>	<i>Post-sample: Pre-crisis</i>	<i>Post-sample: Post-crisis</i>
<i>Number of Observations</i>	1715	575	413
<i>Mean</i>	0.945	0.963	1.306
<i>Standard Deviation</i>	1.009	0.865	1.166
<i>Skewness</i>	2.983	1.910	2.796
<i>Kurtosis</i>	18.179	5.503	4.698
<i>Maximum</i>	13.236	6.267	7.962
<i>Minimum</i>	0.001	0.003	0.001

(iii) Standard & Poors 500 Index (S&P)

	<i>In-sample</i>	<i>Post-sample: Pre-crisis</i>	<i>Post-sample: Post-crisis</i>
<i>Number of Observations</i>	1756	588	422
<i>Mean</i>	0.578	0.513	0.903
<i>Standard Deviation</i>	0.553	0.468	0.875
<i>Skewness</i>	2.233	1.679	2.763
<i>Kurtosis</i>	9.823	3.943	12.269
<i>Maximum</i>	6.117	3.083	6.866
<i>Minimum</i>	0.002	0.001	0.015

Figure 1: Time Plots of HSI Absolute Return

Figure 2: Time Plots of NIKKEI Absolute Return

Figure 3: Time Plots of S&P Absolute Return

From Table 1, one can see that the number of observations for the three markets vary in all samples, due to different holidays in different markets. For each market, the in-sample mean is more or less the same as the pre-crisis (pre-crisis post-sample) mean. The post-crisis (post-crisis post-sample) mean increases tremendously, showing the effect of the financial crisis on the risk of the stock market. For HSI, the mean even doubles in that period. Interestingly, the pre-crisis standard deviations are smaller than the in-sample standard deviations. For HSI, it is reduced by one third. As with the means, the standard deviations increase tremendously in the post-crisis period. In sum, while the risk and the risk of risk increases in the post-crisis period, the risk of risk decreases in the pre-crisis period.

In all markets and all samples, as expected, the skewness is positive. It is rather stable across time and across market except for HSI, which in-sample skewness is almost double of that in the post-sample periods. This huge in-sample skewness and kurtosis (larger than 70) may due to several or two extreme values. The maximum in that period is 21%. It is unclear if these extreme values affects our empirical results though.

Moreover, while in in-samples, the sample means are very close to the sample standard deviations. It is not the case for the post-samples, especially the post-crisis post-sample of HSI.

In line with the findings in the ARCH literature, all the time plots in Figures 1 to 3 suggest high autocorrelation among the absolute returns. Casual observation on the figures may suggest contemporaneous correlation between HSI and S&P, in all three periods; while in the post-crisis period, all three markets seem to be correlated.

Estimation Results

To minimize the size of this section, only the estimates of the 50th expectile (that is, the mean with $\omega = 1/2$) out of 101 expectiles, as well as the estimates from the ARCH-like model, are reported. In both estimates, the autocorrelation-heteroskedasticity-insensitive standard errors are computed (see Newey and West, 1987), which are reported in the brackets.

Hang Seng Index (nobs = 1727)

$$\begin{aligned}\hat{\tau}_{t-1} = & 0.366 + 0.158|r_{t-1}| + 0.101|r_{t-2}| + 0.116|r_{t-3}| + 0.056|r_{t-4}| + 0.041|r_{t-5}| \\ & (0.081) \quad (0.054) \quad (0.031) \quad (0.037) \quad (0.057) \quad (0.025) \\ & -0.078|r_{t-6}| + 0.084|r_{t-7}| - 0.017|r_{t-8}| + 0.108|r_{t-9}| + 0.065|r_{t-10}| \\ & (0.038) \quad (0.063) \quad (0.037) \quad (0.062) \quad (0.067)\end{aligned}$$

$$\hat{\alpha} = 0.611.$$

$$\begin{aligned}\hat{\sigma}_{t-1} = & 0.543 + 0.275(|r_{t-1}| - 0.581r_{t-1}) + 0.443\hat{\sigma}_{t-2} \\ & (0.006) \quad (0.021) \quad (0.038) \quad (0.014)\end{aligned}$$

$$\hat{\lambda} = 1.441.$$

Nikkei 225 Index (nobs = 1715)

$$\begin{aligned}\hat{\tau}_{t-1} = & 0.231 + 0.089|r_{t-1}| + 0.118|r_{t-2}| + 0.078|r_{t-3}| + 0.116|r_{t-4}| + 0.109|r_{t-5}| \\ & (0.037) \quad (0.033) \quad (0.031) \quad (0.027) \quad (0.040) \quad (0.043) \\ & + 0.056|r_{t-6}| + 0.063|r_{t-7}| + 0.044|r_{t-8}| + 0.056|r_{t-9}| + 0.026|r_{t-10}| \\ & (0.034) \quad (0.036) \quad (0.029) \quad (0.029) \quad (0.029)\end{aligned}$$

$$\hat{\alpha} = 0.605.$$

$$\begin{aligned}\hat{\sigma}_{t-1} = & 0.074 + 0.190(|r_{t-1}| - 0.383r_{t-1}) + 0.795\hat{\sigma}_{t-2} \\ & (0.000) \quad (0.008) \quad (0.024) \quad (0.005)\end{aligned}$$

$$\hat{\lambda} = 1.371.$$

Standard & Poors 500 Index (nobs = 1756)

$$\begin{aligned}
 \hat{\tau}_{t-1} &= 0.318 + 0.037|r_{t-1}| + 0.016|r_{t-2}| + 0.047|r_{t-3}| + 0.076|r_{t-4}| + 0.060|r_{t-5}| \\
 &\quad (0.035) \quad (0.030) \quad (0.026) \quad (0.026) \quad (0.023) \quad (0.027) \\
 &\quad + 0.051|r_{t-6}| + 0.008|r_{t-7}| + 0.068|r_{t-8}| + 0.025|r_{t-9}| + 0.060|r_{t-10}| \\
 &\quad (0.031) \quad (0.023) \quad (0.030) \quad (0.023) \quad (0.023) \\
 \hat{\alpha} &= 0.613. \\
 \hat{\sigma}_{t-1} &= 0.420 + 0.122 (|r_{t-1}| - 0.413r_{t-1}) + 0.399 \hat{\sigma}_{t-2} \\
 &\quad (0.042) \quad (0.021) \quad (0.085) \quad (0.055) \\
 \hat{\lambda} &= 1.398.
 \end{aligned}$$

For all 101 expectiles of all three markets, an AR(10)-like model is used. That is, we use 10 lags. This is based on a preliminary estimation on one of the series. As one can see from the empirical results, some of the coefficients are not significant. Redundant parameter should not be an issue as we have a huge sample size of over 1700 observations. For HSI, some of the coefficients are negative: that of the 6-th lag is marginally significant (using 1.96 as the critical value) and that of the 8-th lag is marginally insignificant. Fortunately, none of the estimated expectiles for any observation is negative. The estimate $\hat{\alpha}$ is typically greater than 0.50, which suggests that the conditional mean is larger than the conditional median. This is consistent with the positive skewness of the conditional density.

For the ARCH-like model, all the estimates in the standard deviations are highly significant (once again, we apply the critical value 1.96). All the signs are expected. Moreover, the estimate $\hat{\lambda}$ is typically greater than 1. Recall that the conditional variance equals σ_{t-1}^2/λ^2 (see Section 2). This suggests that the conditional variance is smaller than that of the return.

In Figures 4 - 6, we depict the estimated lower, middle, and upper quantiles, which correspond, roughly, to the 25-th, 50-th and 75-th quantiles. Counting the *in-sample* observations (see the previous section), we obtain the following mappings:

Table 2

	<i>Lower Quantile</i>		<i>Middle Quantile</i>		<i>Upper Quantile</i>	
	w	$\hat{a}(w)$	w	$\hat{a}(w)$	w	$\hat{a}(w)$
<i>HSI</i>	0.061	0.252	0.301	0.496	0.741	0.748
<i>NIKKEI</i>	0.061	0.254	0.331	0.501	0.751	0.753
<i>S&P</i>	0.061	0.255	0.391	0.551	0.751	0.753

The in-sample and post-sample quantiles are computed with the coefficients of the corresponding estimated expectiles.

Figure 4: Estimated Quantiles of HSI Absolute Return

Figure 5: Estimated Quantiles of NIKKEI Absolute Return

Figure 6: Estimated Quantiles of S&P Absolute Return

It is apparent from the above figures that the quantiles of the AR-like model do not differ from a time-invariant constant. This is in line with the ARCH-like model under which the difference between two quantiles is proportional to the time-varying conditional standard deviation. In other words, the coefficients of different expectiles of the AR-like model are different. By construction of the AR-like model, all the quantiles are somewhat moving averages of the lags of the absolute returns, as one can compare the dynamics of the quantiles and the time plots in Figures 1-3. Further, the lower quantiles are in general less “volatile” than the upper quantiles. This is because in estimating the lower expectile, a larger weight is taken by the observations below the expectile curve, which are bounded below by zero and thus less variable. In contrast, the variation of different quantiles in the ARCH-like model *solely* depends on the variation of the conditional standard deviation.

4 FORECASTABILITY IN TERMS OF LOG-LIKELIHOODS

In this section, we compare the in-sample goodness of fit and the post-sample forecastability. For each in-sample *and* post-sample observation, we compute the conditional densities and the log-likelihoods of the two models with the methods described in Section 2. Figures 7-9 depict examples of the conditional densities, each of which is *arbitrarily* selected from the middle of each in-sample or post-sample. Needless to say, the smooth ones are those from the ARCH-like model. Table 3 reports the average log-likelihoods, with the estimated standard errors in brackets. Further, we perform a classical z-test of zero-mean of the difference in the log-likelihoods. To take into account the plausible heteroskedasticity and autocorrelation (of the difference in the log-likelihoods), we also compute the non-nested likelihood ratio (LR) test first proposed in Vuong (1989) and modified in Appendix IV of Sin and Granger (1999) along the lines of Wooldridge (1991).

Figure 7: Examples of Conditional Density Estimate of HSI Absolute Return

Figure 8: Examples of Conditional Density Estimate of NIKKEI Absolute Return

Figure 9: Examples of Conditional Density Estimate of S&P Absolute Return

Table 3 Average Log-Likelihoods and Tests for Equality of Means**(i) Hang Seng Index**

	<i>AR-Like</i>	<i>ARCH-Like</i>	<i>z-test</i>	<i>LR- test</i>
<i>In-Sample (nobs = 1645)</i>	-0.934 (0.023)	-0.960 (0.022)	-2.976*	-2.643*
<i>Post-Sample: Pre-Crisis (nobs = 540)</i>	-0.867 (0.039)	-0.835 (0.033)	1.886*	1.899*
<i>Post-Sample: Post-Crisis (nobs = 385)</i>	-1.652 (0.058)	-1.721 (0.069)	-2.192*	-2.130*

(ii) Nikkei 225 Index

	<i>AR-Like</i>	<i>ARCH-Like</i>	<i>z-test</i>	<i>LR- test</i>
<i>In-Sample (nobs = 1636)</i>	-0.803 (0.022)	-0.859 (0.022)	-7.423*	-6.052*
<i>Post-Sample: Pre-Crisis (nobs = 538)</i>	-0.949 (0.038)	-0.979 (0.037)	-2.090*	-1.981*
<i>Post-Sample: Post-Crisis (nobs = 382)</i>	-1.249 (0.047)	-1.292 (0.048)	-2.354*	-2.668*

(iii) Standard & Poors 500 Index

	<i>AR-Like</i>	<i>ARCH-Like</i>	<i>z-test</i>	<i>LR- test</i>
<i>In-Sample (nobs = 1665)</i>	-0.415 (0.021)	-0.454 (0.021)	-5.593*	-5.000*
<i>Post-Sample: Pre-Crisis (nobs = 556)</i>	-0.349 (0.035)	-0.352 (0.032)	-0.305	-0.299
<i>Post-Sample: Post-Crisis (nobs = 398)</i>	-0.755 (0.045)	-0.825 (0.052)	-3.859*	-3.128*

* indicates significance using the critical value ± 1.65 , which is the 5% critical value of a one-tailed z-test or a one-tailed non-nested likelihood ratio test under the usual assumptions and estimation procedure.

For comparison of log-likelihoods, about 5% of each sample are deleted. This is because we estimate the 100 ω -th expectiles with ω equal to 0.001, 0.011, ..., 0.501, 0.511, ..., 0.981, 0.991. In result, the two tails are *not* estimated and the conditional densities at the tails are not *interpolated* with the spline smoother. The average log-likelihoods of the ARCH-like model before deletion are more or less the same though.

Comparing Table 3 with Table 1 (summary statistics of the absolute return), one can see that the smaller the sample standard deviation, the larger the average log-likelihoods, across all markets and all periods, except for the pre-crisis NIKKEI. The z-test and the LR-test are essentially the same. The AR-like model outperforms the ARCH-like model except for the pre-crisis HSI. Further, only the pre-crisis S&P test statistics are insignificant. In sum, we conclude that the AR-like model by and large performs better than the ARCH-like model.

In Tables 4 and 5, we investigate the models in further detail. First, by definition, quantile minimizes the expected asymmetric least absolute deviation (ALAD). More precisely, the “true” 100α -th quantile minimizes (over q):

$$E[(1-\alpha) |u_t(\alpha)| 1(u_t(\alpha) \leq 0) + \alpha |u_t(\alpha)| 1(u_t(\alpha) > 0)],$$

where $1(A)$ denotes the indicator function for the event A ; and the “error” $u_t(\alpha)$ is defined as $|r_t| - q$.

As a result, it is natural to compare the (sample) mean ALAD (MALAD) of the quantiles implied by the models. More precisely, consider the MALAD:

$$n^{-1} \sum [(1-\hat{\alpha}) |\hat{u}_t(\hat{\alpha})| 1(\hat{u}_t(\hat{\alpha}) \leq 0) + \hat{\alpha} |\hat{u}_t(\hat{\alpha})| 1(\hat{u}_t(\hat{\alpha}) > 0)],$$

where $\hat{u}_t(\hat{\alpha}) \equiv |r_t| - \hat{\tau}_{t-1}(\hat{\alpha})$ for the AR-like model, and

$$\hat{u}_t(\hat{\alpha}) \equiv |r_t| + \ln(1-\hat{\alpha}) \hat{\sigma}_{t-1} / \hat{\lambda} \text{ for the ARCH-like model.}$$

Results are reported in Table 4. The upper panel is the MALAD of the AR-like model while that of the ARCH-like model is in the lower panel. The classical z-test for equality of means are in brackets. Comparing Table 4 with Table 3, unsurprisingly, one can see that the smaller the MALAD, the larger the average log-likelihoods, across all markets and all periods, except for the pre-crisis HSI and the pre-crisis NIKKEI. Except for the in-sample HSI and for the pre-crisis S&P, whenever there is (are) significant difference in the quantile(s), there is significant difference in the log-likelihoods.

Alternatively, we consider the means of the normalized absolute return from the ARCH-like model: $|r_t|/\sigma_{t-1}$. Results with σ_{t-1} replaced by $\hat{\sigma}_{t-1}$ are reported in Table 5. The classical z-test for equality of means are in brackets. From the table, we may conclude that

the inferior performance of the ARCH-model may, at least partially, be due to the change in the means in the post-samples.

Table 4 Mean Asymmetric Least Absolute Deviation

(i) Hang Seng Index

	$\hat{a} = 0.252$	$\hat{a} = 0.496$	$\hat{a} = 0.748$	$\hat{a} = 0.959$
<i>In-sample</i>	0.2080	0.3237	0.3199	0.1286
	0.2090 (0.8466)	0.3259 (0.8791)	0.3216 (0.3695)	0.1292 (0.0852)
<i>Pre-crisis</i>	0.1797	0.2828	0.2866	0.1075
	0.1790 (-0.4508)	0.2777 (-1.5673)	0.2747 (-2.1411)*	0.1078 (0.0416)
<i>Post crisis</i>	0.4121	0.6435	0.6636	0.2641
	0.4241 (2.4128)*	0.6697 (2.3252)*	0.6900 (1.3397)	0.2523 (-0.4483)

(ii) Nikkei 225 Index

	$\hat{a} = 0.254$	$\hat{a} = 0.501$	$\hat{a} = 0.753$	$\hat{a} = 0.949$
<i>In-sample</i>	0.1933	0.2931	0.2771	0.1203
	0.1963 (2.5062)*	0.2977 (1.7944)*	0.2806 (1.0793)	0.1255 (1.5990)
<i>Pre-crisis</i>	0.1992	0.3006	0.2868	0.1271
	0.2023 (1.7120)*	0.3003 (-0.0707)	0.2917 (0.9896)	0.1287 (0.3375)
<i>Post crisis</i>	0.2743	0.4088	0.3888	0.1747
	0.2782 (1.2104)	0.4217 (1.9262)*	0.4025 (1.8411)*	0.1665 (-1.1724)

(iii) Standard & Poor 500 Index

	$\hat{a} = 0.255$	$\hat{a} = 0.551$	$\hat{a} = 0.753$	$\hat{a} = 0.955$
<i>In-sample</i>	0.1226	0.1955	0.1823	0.0710
	0.1237 (2.6933)*	0.1989 (3.1495)*	0.1861 (2.6870)*	0.0737 (2.0211)*
<i>Pre-crisis</i>	0.1085	0.1702	0.1582	0.0620
	0.1093 (1.4828)	0.1732 (2.0439)*	0.1600 (0.9761)	0.0659 (2.6939)*
<i>Post crisis</i>	0.1854	0.2977	0.2829	0.1196
	0.1929 (6.2038)*	0.3110 (3.1851)*	0.2956 (2.1134)*	0.1240 (-0.7904)

* indicates significance using the critical value ± 1.65 , which is the 5% critical value of a one-tailed z-test under the usual assumptions and estimation procedure.

Table 5 Means of the Normalized Absolute Returns

	<i>In-Sample</i>	<i>Post-sample: Pre-crisis</i>	<i>Post-sample: Post-crisis</i>
<i>HSI</i>	0.688	0.628 (-2.024)*	1.031 (6.766)*
<i>NIKKEI</i>	0.722	0.770 (1.487)	0.872 (3.725)*
<i>SP&</i>	0.709	0.641 (-2.347)*	1.019 (6.634) *

* indicates significance using the critical value ± 1.96 , which is the 5% critical value of a two-tailed z-test under the usual assumptions and estimation procedure.

5 CONCLUSION

Conventional measures of the risk of a financial asset make use of the unobserved (conditional) variance or standard deviation of its return. In this paper, we treat the observed absolute return as a measure of risk and explore its forecastability. Two simple models are considered. One is a new AR-like model which is applied to the absolute return. The other is an ARCH-like model called Asymmetric Power ARCH. The forecastability is evaluated with the average log-likelihood of absolute return, instead of that of return itself. While the absolute return is interpreted as “volatility”, some quantities of its entire distribution, such as the 95-th quantiles, can be interpreted as “volatility of volatility”. We apply both models to three stock indices, namely Hang Seng Index, Nikkei 225 Index and Standard and Poors 500 Index. The new model by and large outperforms the ARCH-like model in both in-sample goodness of fit and post-sample forecastability. It performs exceptionally well in the post-sample period after the outbreak of the Asian financial crisis.

Throughout the paper, we assume asymptotic normality of the t-statistics of the coefficients in estimating expectiles, the z-test and the non-nested LR-test for the equality in means of log-likelihoods. This may preclude the long-memory properties of the absolute return, as discussed in Section 4 of Granger and Ding (1995). The results proved in Ho and Lin (1998) and the references therein may be found useful. On the other hand, for some expectiles, up to 10 lags are found significant. The non-parametric local polynomial

estimation, which is in fact suggested in Yan and Tong (1996), may not be applied to our case because of the curse of dimensionality.

Empirical studies on asset prices are overwhelmed by the forecastability of their *returns* and the main focus is on their standard errors or variance, which are shown to be successfully modelled by the celebrated class of ARCH models. While these model can be easily applied to studies on *absolute returns*, in this paper, we suggest a new model and there is evidence that it outperforms the former one. The new model is similar to an AR model, usually applied to the return rate (or the growth rate) instead of the absolute return. Further, we apply this AR-like model not only to the mean or median, but also to a number of expectiles or quantiles. Interpreting the absolute return as a measure of risk, we have not only a point forecast but also interval forecasts of risk, the latter of which may be interpreted as the risk of risk or volatility of volatility. Nevertheless, a usual aggregation argument first raised in Granger and Morris (1976) points to the deficiency of an AR model. In sum, our model may be improved with an ARMA-like model, which is a natural extension in our future work.

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