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Generic properties of some geometric deformations

A dissertation submitted in partial satisfaction
of the requirements for the degree

Doctor of Philosophy
in
Math

by

Ricky Lee

Committee in charge:

Professor Darren Long, Co-Chair
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June 2026

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June 2026

This thesis is dedicated to my father.

Abstract

Generic properties of some geometric deformations

by

Ricky Lee

We will show that $SL(4, \mathbb{R})$ and $SU(3, 1)$ contain strongly dense hyperbolic 3-manifold subgroups. Then we will establish the existence of Schottky groups in $PSL(2, \mathbb{C})$ where no pair of non-commuting elements have translation axes that lie on a common plane.

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2. *Self-replicating 3-manifolds* (with Ryan Blair)
Topology Appl. 318 (2022),no. 1, 108–197.
3. *Widths of links via diagram colorings* (with Puttipong Pongtanapaisan & Hanh Vo)
Geometriae Dedicata 219.5 (2025): 71.
4. *Strongly Dense Representations of Hyperbolic 3-Manifold Groups*
Proceedings of the American Mathematical Society 154.03 (2026): 1311-1323. (2026)
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Chapter 1

Introduction

This thesis studies representations varieties modeling deformations of geometric structures. We briefly review what this is.

1.0.1 Background

Let Γ be a finitely presented group and G a Lie group. Let $Hom(\Gamma, G)$ denote the set of all representations of Γ into G . Then $Hom(\Gamma, G)$ is the *representation variety* of Γ into G . To see that $Hom(\Gamma, G)$ can be given the structure of an algebraic set, take a finite generating set $\{\gamma_1, \dots, \gamma_n\}$ for Γ . A representation of Γ into G is determined by the image of the generating set. To parameterize all possible representations of Γ into G , one can begin by assigning to each generator γ_i a k by k matrix A_i of indeterminates, where k is the dimension of the definition representation of G . By substituting each A_i into the relations satisfied by the generating set $\{\gamma_1, \dots, \gamma_n\}$, one obtains a system of polynomial equations. Each solution to this system is a k dimensional representation of Γ . By adjoining to this system the algebraic relations defining G , one gets a system of equations where the solutions correspond to all representations of Γ into G . This gives $Hom(\Gamma, G)$ the structure of an algebraic set. And it is classical that different presenta-

tions of Γ correspond to bi-rationally equivalent algebraic sets. Of course different points in $\text{Hom}(\Gamma, G)$ can correspond to conjugate representations. Often it is better to pass to *character variety* $X(\Gamma, G)$, an algebraic set parametrizing characters of representations of Γ into G .

There are many natural geometric situations where one wishes to understand the totality of representations of Γ into G . One classical situation is where the domain group Γ is the fundamental group of a closed genus g hyperbolic surface $\pi_1(\Sigma_g)$, and the target group G is $PSL(2, \mathbb{R})$. This is the setting of Teichmüller theory, and much effort in the past century has been devoted to understanding the relationship between the algebraic structure of $X(\pi_1(\Sigma_g), PSL(2, \mathbb{R}))$, the geometry of $\mathbb{H}^2/\rho(\pi_1(\Sigma_g))$, and topology of Σ_g . For example, Goldman showed that the number of components of $X(\pi_1(\Sigma_g), PSL(2, \mathbb{R}))$ is $4g - 3$, where g is the genus of Σ_g . There is a distinguished component of real dimension $6g - 6$ parametrizing discrete and faithful representations $\pi_1(\Sigma_g) \rightarrow PSL(2, \mathbb{R})$, and these representations determine complete hyperbolic structures on Σ_g (i.e. Teichmüller space). So one thinks of this component as parametrizing deformations of hyperbolic structures on Σ_g .

Another classical situation is when one takes the domain group Γ to be a hyperbolic knot group $\pi_1(S^3 \setminus K)$ and target group to be $SL(2, \mathbb{C})$. In this case, there is a unique (up to conjugacy and conjugation) discrete and faithful representation of $\pi_1(S^3 \setminus K)$ into $PSL(2, \mathbb{C}) \equiv \text{Isom}^+(\mathbb{H}^3)$ corresponding to the complete hyperbolic structure on $S^3 \setminus K$. Thurston showed this representation always lifts to $SL(2, \mathbb{C})$, and the irreducible component of $X(\pi_1(S^3 \setminus K), SL(2, \mathbb{C}))$ (i.e. the Canonical component) containing this lift is of great geometric significance. This is the setting of Culler-Shalen theory. One of the highlights is that the ideal points of the canonical component correspond to splittings of $\pi_1(S^3 \setminus K)$, which in turn correspond to incompressible surfaces in $S^3 \setminus K$.

More recently, there has been great effort to understand $X(\Gamma, G)$ where G is a higher

rank Lie group like $SL(n, \mathbb{R})$, and Γ is the fundamental group of a manifold. This is the setting of higher Teichmüller theory, and its goal is to find higher rank analogues of phenomena in hyperbolic geometry. For example, work of Benoist implies that when Γ is the fundamental group of a hyperbolic 3-manifold, the component of $X(\Gamma, SL(4, \mathbb{R}))$ containing the lattice embedding $\Gamma \rightarrow SO^+(3, 1) \subset SL(4, \mathbb{R})$ consists solely of discrete faithful representations, which are holonomies of projective structures on the manifold.

1.0.2 Summary of Results

Character variety methods can be useful in situations where one is trying to understand certain phenomena but are unsure of how to produce explicit examples. As a simple analogy, one can compare how relatively simple it is to show most real numbers are transcendental, with how tricky it can be to show π or e is transcendental.

In the following chapters, we will use character variety methods to gain insight and train intuition on several open problems.

In chapter 3, we will discuss *strong density* and show the following:

- The real Lie groups $SU(3, 1)$ and $SL(4, \mathbb{R})$ contain strongly dense hyperbolic 3-manifold subgroups. As a result, these Lie groups contain non-Hitchin strongly dense surface subgroups.
- The figure 8 knot group admits embeddings into $SL(2, \mathbb{C})$ where no non-identity element has real trace.

Chapter 4 analyzes Schottky space and shows the following:

- Most Schottky groups have the following property: every pair of non-commuting elements have translations axes that do not lie on a common plane.

The results of this thesis have appeared in the papers [Lee26; Lee25].

Chapter 2

Preliminaries

In this chapter we review some basics about geometry in real hyperbolic space, complex hyperbolic space, and projective space, as well as geometric structures on 3-manifolds modeled on these geometries.

2.1 Real Hyperbolic Geometry

Arguments in future chapters will required knowledge about the hyperboloid and upper half space models of hyperbolic space. We begin this chapter by reviewing some facts about these models and how they are related. Then we present some basics about arithmetic hyperbolic manifolds. The material in this review comes from [MR03; Mar16].

2.1.1 The Hyperboloid Model of Hyperbolic 3-Space

This discussion could be carried out using any symmetric form of signature $(n, 1)$, but for concreteness set $J := \text{diag}(1, \dots, 1, -1)$ and let $\langle, \rangle$ be the bilinear form on $\mathbb{R}^{n+1}$ determined by J via

$$\langle x, y \rangle = x^T J y.$$

Let $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be the scalar function

$$f(x) := \langle x, x \rangle.$$

One computes that $df_x : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is given by $df_x(y) = 2\langle x, y \rangle$. That is, df_x is surjective for all $x \in \mathbb{R}^{n+1} \setminus \{0\}$, so $f^{-1}(r)$ is an n -dimensional smooth manifold. In particular, the 2-sheeted hyperboloid

$$f^{-1}(-1) = \{x \in \mathbb{R}^{n+1} : \langle x, x \rangle = -1\}$$

is a smooth n -dimensional manifold. Put

$$\mathcal{H}^n := \{x \in f^{-1}(-1) : x_{n+1} > 0\}.$$

That is, $\mathcal{H}^n$ is the upper sheet of the hyperboloid $f^{-1}(-1)$. We now describe a Riemannian structure on $\mathcal{H}^n$ that gives the hyperboloid model of $\mathbb{H}^n$.

Recall that tangent spaces to level sets are given by kernels of differentials. Since $df_x(y) = 2\langle x, y \rangle$, then we have

$$T_x \mathcal{H}^n = \ker(df_x) = \{y \in \mathbb{R}^{n+1} : \langle x, y \rangle = 0\}$$

for all $x \in \mathcal{H}^n$. By definition, we have $\langle x, x \rangle = -1$ for all $x \in \mathcal{H}^n$, and this can be used to show J restricts to a positive definite bilinear form on $T_x \mathcal{H}^n$. The smooth manifold $\mathcal{H}^n$ with the bilinear form on each $T_x \mathcal{H}^n$ given by restriction of J is the Riemannian manifold giving the hyperboloid model of $\mathbb{H}^n$.

In the hyperboloid model, the boundary at infinity $\partial_\infty \mathbb{H}^n$ is given by

$$\partial_\infty \mathcal{H}^n = \{x \in \mathbb{R}^{n+1} : \langle x, x \rangle = 0\}.$$

For any pair of distinct points $x, y \in \partial_\infty \mathcal{H}^n$, the unique geodesic through x and y is given by $\text{span}_{\mathbb{R}}\{x, y\} \cap \mathcal{H}^n$. In general, if $W < \mathbb{R}^{n+1}$ is a k dimensional linear subspace of $\mathbb{R}^{n+1}$ that intersects $\mathcal{H}^n$ non-trivially, then $W \cap \mathcal{H}^n$ is an isometric copy of $\mathbb{H}^k$ in $\mathcal{H}^n$.

The orientation preserving isometries of $\mathcal{H}^n$ can be identified with $SO^+(n, 1)$, which is the identity component of the real Lie group

$$SO(n, 1) = \{g \in SL(n+1, \mathbb{R}) : g^T J g = J\}$$

Recall that any $g \in \text{Isom}(\mathbb{H}^n)$ is classified by its fixed points:

- g is *elliptic* if it has a fixed point in $\mathbb{H}^n$.
- g is *parabolic* if it has no fixed points in $\mathbb{H}^n$ but one fixed point in $\partial_\infty \mathbb{H}^n$.
- g is *loxodromic* if it has no fixed points in $\mathbb{H}^n$ and two fixed points in $\partial_\infty \mathbb{H}^n$.

Loxodromic isometries act by translation along the unique geodesic through its fixed points, with possible rotation about its orthogonal hyperplanes. If $g \in SO^+(3, 1)$ is a loxodromic isometry of the hyperboloid model $\mathcal{H}^3$, then the eigenvalues of g are of the form $r, 1/r, e^{i\theta}$, and $e^{-i\theta}$ for real numbers $r > 1$ and θ , where r corresponds to the translation length of g along the geodesic connecting its fixed points, and θ corresponds to its rotation part.

2.1.2 The Conformal Model

In this subsection we introduce another model for $\mathbb{H}^3$ that will be relevant in future chapters, and discuss how it relates to the hyperboloid model. Recall the Poincare disk:

$$\mathcal{D} := \{x \in \mathbb{R}^3 : \|x\| < 1\}.$$

The metric tensor at $x \in \mathcal{D}$ given by

$$g_x = \left(\frac{2}{1 - \|x\|^2} \right)^2 g_x^E,$$

where g_x^E denotes the standard Euclidean metric tensor, gives $\mathcal{D}$ the structure of a Riemannian manifold that serves as the disk model of $\mathbb{H}^3$. In this model, the boundary at infinity $\partial_\infty \mathbb{H}^3$ is the Riemann sphere $\partial_\infty \mathcal{D} = S^2$.

Every isometry $g : \mathbb{H}^n \rightarrow \mathbb{H}^n$ extends uniquely to a homeomorphism $\partial_\infty \mathbb{H}^n \rightarrow \partial_\infty \mathbb{H}^n$. In dimension three, this allows for the identification of $Isom^+(\mathcal{D})$ with the group

$$PSL(2, \mathbb{C}) = SL(2, \mathbb{C}) / \{\pm Id\} = PGL(2, \mathbb{C})$$

of invertible 2x2 matrices up to scalar multiplication. The action of $PSL(2, \mathbb{C})$ is on $\partial_\infty \mathcal{D}$ by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

Explicitly, the Möbius action of $PSL(2, \mathbb{C})$ is not on $\mathcal{D}$, each $g \in PSL(2, \mathbb{C})$ represents (uniquely) the extension to $\partial_\infty \mathcal{D} = S^2$ of an isometry of $\mathcal{D}$ with respect to the metric tensor previously mentioned.

To see how the conformal and hyperboloid models of $\mathbb{H}^3$ relate, we review the famous Lorentz isomorphism between $PSL(2, \mathbb{C})$ and $SO^+(3, 1)$ (see section 2 of [CLT06]). Write

$$\mathcal{H} := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \right\}. \quad (2.1)$$

Then $\text{span}_{\mathbb{R}} \mathcal{H}$ is the real vector space of 2×2 Hermitian matrices. An explicit isomorphism from $\mathbb{R}^4$ to $\text{span}_{\mathbb{R}} \mathcal{H}$ is given by

$$(t, x, y, z) \mapsto \begin{bmatrix} t + z & x + iy \\ x - iy & t - z \end{bmatrix}.$$

The function $H \mapsto -\det(H)$ gives a quadratic form on $\text{span}_{\mathbb{R}} \mathcal{H}$ that is preserved by the following action of $PSL(2, \mathbb{C})$: given $[A] \in PSL(2, \mathbb{C})$, let

$$H.[A] = A^* H A$$

where $A \in SL(2, \mathbb{C})$ is any representative of $[A] \in PSL(2, \mathbb{C})$ and A^* is the Hermitian transpose of A . For any $A \in SL(2, \mathbb{C})$, let

$$\tau(A) : \text{span}_{\mathbb{R}} \mathcal{H} \rightarrow \text{span}_{\mathbb{R}} \mathcal{H}$$

denote the $\mathbb{R}$ -linear map defined by $\tau(A)(H) := A^* H A$. Then an explicit isomorphism from $PSL(2, \mathbb{C})$ to $SO^+(3, 1)$ is given by mapping each $[A] \in PSL(2, \mathbb{C})$ to the matrix representative of $\tau(A)$ with respect to $\mathcal{H}$, for any representative A of $[A]$.

2.1.3 Arithmetic Hyperbolic Manifolds

Recall that a *lattice* is a discrete subgroup $\Gamma < Isom(\mathbb{H}^n)$ such that $\mathbb{H}^n/\Gamma$ has finite volume. A hyperbolic manifold is the quotient of $\mathbb{H}^n$ by a torsion free lattice. Lattices in semisimple Lie groups come in two different types, *arithmetic* and *non-arithmetic*. There has been a lot of recent breakthroughs in our understanding of totally geodesic sub-manifolds of arithmetic hyperbolic manifolds, and the literature in this arena is vast. In this subsection, we will survey just the results we need to motivate and understand our future considerations, with some extra detail paid to dimension 3.

Arithmetic Kleinian Groups

A Kleinian group is a discrete subgroup of $PSL(2, \mathbb{C})$. A Fuchsian subgroup is a discrete subgroup of $PSL(2, \mathbb{C})$ whose action on $\mathbb{H}^3$ preserves an isometric copy of $\mathbb{H}^2$. In dimension 3, one has extra tools from the theory of quaternion algebras for understanding the Fuchsian subgroup structure of Kleinian groups. As this will be important for us later, we review some of the theory here.

Recall a *quaternion algebra* A over a field k is a four dimensional central simple algebra. Any quaternion algebra is either isomorphic to the matrix algebra $M(2, k)$ or to a division algebra. Let k be a number field and ν a *place* of k , or equivalence class of valuation. If A is a quaternion algebra over the number field k and ν is any place of k , A is said to *split* at ν if under extension of scalars, we have $A \otimes_k k_\nu \cong M(2, k_\nu)$. If $A \otimes_k k_\nu$ is a division algebra, then A is said to be *ramified* at ν .

Definition 2.1. Let k be a number field with exactly one complex place. Let A be a quaternion algebra over k that is ramified at all real places of k . Let $\mathcal{O}$ be an $\mathcal{O}_k$ order in A , and $\mathcal{O}^1 \subset \mathcal{O}$ the subgroup of units in $\mathcal{O}$ with reduced norm 1. Let $\rho : A \rightarrow M(2, \mathbb{C})$ be a k embedding induced by the complex embedding of k . If $\Gamma < PSL(2, \mathbb{C})$

is commensurable with $P\rho(\mathcal{O}^1)$, then Γ is an *arithmetic Kleinian group* and $\mathbb{H}^3/\Gamma$ is an arithmetic hyperbolic 3-manifold.

Remark 2.1. The definition of arithmetic Fuchsian group in $PSL(2, \mathbb{R})$ is similar. The required modification is that the number field k is totally real and A is a quaternion algebra over k that is split at exactly one real place and ramified at all other real embeddings of k .

We can also start with a discrete $\Gamma < PSL(2, K)$, $K \in \{\mathbb{R}, \mathbb{C}\}$, and consider recovering its arithmetic invariants.

Definition 2.2. Let $\Gamma < PSL(2, K)$, $K \in \{\mathbb{R}, \mathbb{C}\}$ be a non-elementary subgroup. The *invariant trace field* $k\Gamma$ of Γ is

$$k\Gamma := \mathbb{Q}(\text{tr}(\gamma^2) : \gamma \in \Gamma).$$

The *invariant quaternion algebra* $A\Gamma$ of Γ is

$$A\Gamma = \{\sum a_i \gamma_i^2 : a_i \in k\Gamma, \gamma_i \in \Gamma\}.$$

In the definition of $A\Gamma$, the sum is finite, and it is a theorem of Maclachlan and Reid that $A\Gamma$ is a quaternion algebra over $k\Gamma$.

Remark 2.2. Traces of elements in $PSL(2, K)$ are defined only up to sign so $k\Gamma$ is well defined. When $\Gamma < PSL(2, \mathbb{C})$ is a lattice, a consequence of Mostow rigidity is that $k\Gamma$ is a number field. It is a Theorem of Maclachlan and Reid that $A\Gamma$ is a quaternion algebra over $k\Gamma$ (for any non-elementary subgroup $\Gamma < PSL(2, K)$) and that both $k\Gamma$ and $A\Gamma$ are commensurability invariants.

This machinery allows one to have a very detailed understanding of the Fuchsian

subgroup structure of arithmetic Kleinian groups. It was proven by Maclachlan and Reid that if Γ is an arithmetic Kleinian group and $F < \Gamma$ is a non-elementary Fuchsian subgroup, then F is a subgroup of an arithmetic Fuchsian group. Moreover, the arithmetic invariants of F and Γ fit together in the way one would expect.

Theorem 2.1. (Maclachlan, Reid) If F is a non-elementary Fuchsian subgroup of the arithmetic Kleinian group Γ , then F is a subgroup of an arithmetic Fuchsian group and

1. $kF = k\Gamma \cap \mathbb{R}$.
2. $A\Gamma = AF \otimes_{kF} k\Gamma$.

This can be used to find lattices in $PSL(2, \mathbb{C})$ without any non-elementary Fuchsian subgroups. We explore this in the following example.

The Manifold Vol3

In this example we follow [JR01]. The manifold with third lowest volume in the census is frequently referred to as Vol3. It is a closed arithmetic hyperbolic manifold, and we will explain in this section why it does not have any totally geodesic surfaces. We have

$$\pi_1(\text{Vol3}) = \langle a, b | aabbABAbb, aBaBabaaab \rangle$$

where $A = a^{-1}$ and $B = b^{-1}$. The work in page 6 of [JR01] shows that $\rho_\infty : \pi_1(\text{Vol3}) \rightarrow PGL(2, \mathbb{C})$, defined by

$$\rho_\infty(a) = \begin{pmatrix} (\frac{1}{4})(i\sqrt{2} + \sqrt{6}) & (\frac{1}{8})(6 - 2i\sqrt{3}) \\ -1 & (\frac{1}{4})(i\sqrt{2} + \sqrt{6}) \end{pmatrix},$$

$$\rho_\infty(b) = \begin{pmatrix} (\frac{1+i}{4})(2 + \sqrt{3} + i) & (\frac{-1-i}{2})(\sqrt{1.5}) \\ (\frac{1+i}{4})(\sqrt{2} - i\sqrt{6}) & (\frac{-1+i}{4})(-2 + \sqrt{3} + i) \end{pmatrix},$$

is a lattice embedding of $\pi_1(\text{Vol}3)$ into $PGL(2, \mathbb{C})$. Put $\Gamma_3 := \text{Imp}_\infty$. By direct computation, one has that the invariant trace field $k\Gamma_3 = \mathbb{Q}(\sqrt{-3})$, and the invariant quaternion algebra $A\Gamma_3$ ramifies at the primes associated to 2 and $\sqrt{-3}$.

Lemma 2.1. Γ_3 does not have any non-elementary Fuchsian subgroups.

Proof. Suppose for contradiction $F < \Gamma_3$ is a non-elementary Fuchsian subgroup of Γ_3 . Then $kF = k\Gamma_3 \cap \mathbb{R} = \mathbb{Q}$. Moreover, F is a subgroup of an arithmetic Fuchsian group, so AF is a quaternion algebra over $kF = \mathbb{Q}$ that is split. That is, $AF = M(2, \mathbb{Q})$. The zero divisors in $AF = M(2, \mathbb{Q})$ would persist as zero divisors upon extension of scalars, so $AF \otimes_{kF} k\Gamma_3 = A\Gamma_3$ cannot have any finite ramification. Since direct computation shows $A\Gamma_3$ ramifies at the prime 2, this gives the desired contradiction. $\square$

We will see now that the existence of hyperbolic manifolds without any geodesic submanifolds of dimension greater than 1, is currently special to dimension 3.

Geodesic Submanifolds in Hyperbolic Manifolds

We now rapidly review what is known about geodesic submanifolds in hyperbolic manifolds, following [Bel+21].

Arithmetic hyperbolic manifolds come in three different types:

1. Type I arithmetic manifolds exist in all dimensions and contain infinitely many codimension 1 totally geodesic submanifolds.
2. Type II arithmetic manifolds exist in all odd dimensions ≥ 3 , and contain infinitely many geodesic submanifolds in all even codimensions (but none of codimension 1).

3. Type III arithmetic manifolds exist only in dimensions 3 and 7 and should be viewed as consequences of low dimensional accidents in Lie theory. In dimension 3, they correspond to lattices with no pure translations i.e. Kleinian groups $\Gamma < PSL(2, \mathbb{C})$ where for all $\gamma \in \Gamma \setminus \{1\}$, we have that $tr\gamma$ is not real. In dimension 7, they are called *trilatarian* lattices, and trilatarians contain totally geodesic hyperbolic 3-submanifolds.

In dimension 3, there are examples of non-arithmetic manifolds as well which do not contain any totally geodesic submanifolds of dimension > 1 . In dimensions ≥ 4 , the only known constructions of non-arithmetic manifolds come from reflection groups, Gromov and Piatetski-Shapiro's *hybrids*, Agol's *interbreeding*, and variations of those ideas. All such constructions contain codimension 1 totally geodesic submanifolds, and it is an important open problem whether or not one can find a hyperbolic manifold in dimension > 3 which does not contain any geodesic hyperbolic submanifolds. Such examples would be necessarily non-arithmetic.

Remark 2.3. It is explained in chapter 5 of [MR03] that type III arithmetic lattices have invariant quaternion algebras that contain real ramification. In the previous example with Vol3, the trace field $\mathbb{Q}(\sqrt{-3})$ does not admit any real embeddings. This means Γ_3 is a type II arithmetic lattice.

2.2 Complex Hyperbolic Geometry

In this section, we give a rapid review of the necessary background in complex hyperbolic geometry for our future arguments. A standard and comprehensive reference for this material is [Gol99].

Let $\mathbb{V}$ be the vector space $\mathbb{C}^4$ equipped with the Hermitian form defined by

$$\langle z, w \rangle = w^* H z,$$

where $H = \text{diag}(-1, 1, 1, 1)$ is the diagonal matrix representing the Hermitian form. If H' is any other Hermitian matrix with signature (3,1), then there is a 4×4 matrix C so that $\bar{C}^T H' C = H$. Consider the following subspaces:

$$\mathbb{V}_- = \{z \in \mathbb{V} : \langle z, z \rangle < 0\}, \mathbb{V}_0 = \{z \in \mathbb{V} : \langle z, z \rangle = 0\}.$$

Complex hyperbolic space $\mathbb{CH}^3$ is defined to be the image of $\mathbb{V}_-$ under the canonical projection $\mathbb{V} \setminus \{0\} \rightarrow \mathbb{CP}^3$ onto complex projective space. The image of $\mathbb{V}_0$ under this projection gives the ideal boundary $\partial\mathbb{CH}^3$.

The holomorphic isometry group of $\mathbb{CH}^3$ is the projective unitary group $PSU(3, 1)$, which by definition is $SU(3, 1)/\{\lambda I | \lambda \in \mathbb{C}^*\}$, although for convenience we will lift to the four fold cover $SU(3, 1)$ when analyzing isometries. The classification of isometries of $\mathbb{CH}^3$ mirrors that of $\mathbb{H}^3$. An isometry is *elliptic* if it fixes at least one point of $\mathbb{CH}^3$. It is *parabolic* if it fixes exactly one point, and that point lies in $\partial\mathbb{CH}^3$. It is *hyperbolic* if it fixes exactly two points, both of which lie in $\partial\mathbb{CH}^3$.

The eigenvalues of elliptic and parabolic elements all have norm 1. Hyperbolic elements have a pair of eigenvalues of the form $re^{i\theta}, r^{-1}e^{i\theta}$, $r > 1$, with associated one dimensional eigenspaces corresponding to its two fixed points in $\partial\mathbb{CH}^3$.

A *complex hyperbolic Kleinian group* is a discrete subgroup of $SU(3, 1)$. If $\Gamma < SU(3, 1)$ is a complex hyperbolic Kleinian group and $x \in \mathbb{CH}^3$, then its *limit set* Λ_Γ is the set of accumulation points of the orbit $\Gamma(x)$ in $\partial\mathbb{CH}^3$. It is known that Λ_Γ is independent of the choice of x . A complex hyperbolic Kleinian group Γ is called *elementary* if $|\Lambda_\Gamma| \leq 2$,

and *non-elementary* otherwise.

Chapter 3

Strong Density

A subgroup Γ of an algebraic group $\mathcal{G}$ is *strongly dense* if every pair of non-commuting elements in Γ generates a Zariski dense subgroup of $\mathcal{G}$. Strongly density was first introduced in [Bre+12] during their study of expansion properties of Cayley graphs of finite simple groups of Lie type. In their work, they show that many different algebraic groups contain an abundance of strongly dense free subgroups. Recently, ideas from geometry (in particular, higher Teichmüller Theory) have been used to show that many real Lie groups contain strongly dense surface groups [LR25; DW25; Jun24]. In this chapter, we present our work on strongly dense representations of hyperbolic 3-manifold groups.

3.1 Motivation

We begin with some motivation on why one would be interested in strongly dense embeddings of hyperbolic 3-manifold groups. First, recall that a group G is *residually free* if for all $g \in G \setminus \{1\}$, there exists a surjection of G onto a rank 2 free group which does not contain g in its kernel. From the perspective of those interested in strong density, it is desirable to have examples of strongly dense groups that are not residually free. This

is one of the main difference between the search for strongly dense surface groups versus strongly dense hyperbolic 3-manifold groups. Surface groups are all residually free while a Theorem of Wilton shows that hyperbolic 3-manifold groups are never residually free [Wil08].

The various arguments establishing the existence of strongly dense surface groups in various algebraic groups [BGL24; LR25; DW25; Jun24] are each unique to their respective circumstances. But the following technical step has shown up many times. Say one wishes to find a strongly dense embedding of the surface group $\pi_1(\Sigma)$ into G . To start off, one often needs the following Lemma for whatever target group G is being considered:

Lemma 3.1. For every pair of non-commuting elements $a, b \in \pi_1(\Sigma)$, there exists a representation $\rho : \pi_1(\Sigma) \rightarrow G$ such that $\langle \rho(a), \rho(b) \rangle$ is Zariski dense in G .

This is typically resolved for surface groups by appealing to residual freeness, which established the existence of a surjection $\pi_1(\Sigma) \rightarrow F_2$ onto the rank 2 free group F_2 whose kernel does not contain the commutator $[a, b] \neq 1$ of the pair of non-commuting words a, b . From here, one can use [Bre+12] to get a strongly dense embedding $F_2 \rightarrow G$.

Of course the representation varieties of 3-manifold groups are much less understood than that of surface groups. But the dependence of previous arguments on residual freeness shows there are fundamental differences between these cases as well.

Another motivation comes from recent advances in our understanding of superrigidity. The totally geodesic submanifolds of hyperbolic manifolds have recently received a lot of attention due to work of [Bad+21], which shows that they can be used as a geometric characterization of arithmeticity. They showed that if $\Gamma < SO(n, 1)$ is arithmetic, then $\mathbb{H}^n/\Gamma$ has either infinitely many maximal totally geodesic submanifolds, or none. But if Γ is non-arithmetic, then $\mathbb{H}^n/\Gamma$ has finitely many maximal totally geodesic submanifolds, or none.

From this perspective, dimension 3 is special in that it is the only dimension where we have rigidity, and where we have examples of strongly dense lattices due to the existence of Kleinian groups with non non-elementary Fuchsian subgroups. And so their representation theory should be more carefully studied. As mentioned in the preliminaries, for $n > 3$, no arithmetic lattices in $SO(n, 1)$ can be strongly dense. There are examples of non-arithmetic strongly dense lattices in $SO(3, 1)$ as well, but for $n > 3$, it is an important open question whether one can find a non-arithmetic lattices without codimension 1 totally geodesic submanifolds.

Even more recently, the presence of geodesic subspace stabilisers has been used to understand the failure of superrigidity in $SO(n, 1)$ and $SU(n, 1)$. Despite superrigidity not holding for these Lie groups, representations of lattices that are *geodesically rich* still admit continuous extensions to their respective ambient groups. As mentioned in [Bal+25], it is open whether an arithmetic lattice of simplest type in $SU(n, 1)$ can admit embeddings which are not geodesically rich. Geodesic richness is weaker than strongly dense, but it is still natural to wonder if the more pathological situation can occur: can a simply type arithmetic lattice admit a strongly dense embedding? Later in this chapter, we will see an example of this phenomena in real hyperbolic space via SL_2 deformations of the lattice embedding of the figure 8 knot group.

In summary, the search for strongly dense embeddings of lattices may shed light on the mysteries of superrigidity, and its failure in $SO(n, 1)$ and $SU(n, 1)$. In this chapter, we warm up by showing lattice embeddings of closed hyperbolic surface groups have strongly dense image in $PSL(2, \mathbb{R})$. Then we give our first project, which shows that projective and complex hyperbolic deformations of the lattice embedding of $\pi_1(\text{Vol}3)$ give rise to strongly dense hyperbolic 3-manifold subgroups in $SL(4, \mathbb{R})$ and $SU(3, 1)$. Finally, we discuss ongoing work on the search for strongly dense embeddings of the figure 8 knot group.

3.2 Warm-up: Strongly Density Hyperbolic Surfaces

We perform a warm-up example by showing that $PSL(2, \mathbb{R})$ contains an abundance of strongly dense lattices. Since $PSL(2, \mathbb{R})$ is just 3-dimensional, there are not that many possible proper algebraic subgroups. Standard results from the theory of linear algebraic groups [Bor91] give the following:

Fact: If $G < PSL(2, \mathbb{R})$ is a non-elementary subgroup, then either G is solvable, or Zariski dense.

This means if $\langle a, b \rangle < PSL(2, \mathbb{R})$ is a rank 2 free group generated by two hyperbolic isometries a, b , then either a and b can be simultaneously upper triangulated, or $\langle a, b \rangle$ is Zariski dense in $PSL(2, \mathbb{R})$. The following lemma will prove useful in verifying certain non-elementary subgroups of hyperbolic isometries generated by a pair of non-commuting loxodromics (and in particular hyperbolics) are Zariski dense.

Lemma 3.2. For $K \in \{\mathbb{R}, \mathbb{C}\}$, if $a, b \in PSL(2, K)$ are hyperbolics (or loxodromics) with exactly one common fixed point at infinity, then $tr[a, b] = 2$, where $[a, b] = aba^{-1}b^{-1}$.

Proof. Conjugating the common fixed point to infinity, we can write

$$a = \pm \begin{pmatrix} x & 1 \\ 0 & 1/x \end{pmatrix}, \quad b = \pm \begin{pmatrix} y & t \\ 0 & 1/y \end{pmatrix}$$

for some $x, y, t \in K$ where $|x|, |y| > 1$. Direct computation then yields $tr[a, b] = 2$. $\square$

Proposition 3.1. If $\rho : \pi_1(\Sigma) \rightarrow PSL(2, \mathbb{R})$ is a lattice embedding of the closed hyperbolic surface group $\pi_1(\Sigma)$, then $\rho(\pi_1(\Sigma))$ is strongly dense in $PSL(2, \mathbb{R})$.

Proof. Let $x, y \in \pi_1(\Sigma)$ be a pair of non-commuting elements. We must show that $\rho(x), \rho(y)$ generate a Zariski dense subgroup of $PSL(2, \mathbb{R})$. This is equivalent to showing that $\rho(x)$ and $\rho(y)$ cannot be simultaneously upper triangulated.

Since ρ is a lattice embedding of the closed hyperbolic surface group $\pi_1(\Sigma)$, then $\rho(\pi_1(\Sigma))$ contains no parabolic isometries. If it were possible to simultaneously upper triangulate $\rho(x)$ and $\rho(y)$, then $\rho(x)$ and $\rho(y)$ would have to be a pair of hyperbolic isometries that share exactly one common fixed point at infinity. By Lemma 3.2, $\rho([x, y])$ would be a parabolic, as it has trace 2 and x and y do not commute. As previously mentioned, this is not possible, so we conclude $\langle \rho(x), \rho(y) \rangle$ is Zariski dense in $PSL(2, \mathbb{R})$.

□

3.3 Strongly Dense Embeddings of Vol3

We now give details on the first project of this thesis, which produces the first examples of strongly dense representations of hyperbolic 3-manifold groups into $SL(4, \mathbb{R})$ and $SU(3, 1)$.

In order to discuss our results more carefully, we review some background. Recall that for $n \geq 3$, if a closed n -manifold admits a complete hyperbolic structure then this structure is unique by Mostow rigidity. However, one can regard hyperbolic structure as an instance of real projective structure, and in the projective setting there is no notion of Mostow rigidity. For some hyperbolic 3-manifolds M , the holonomy representation $\pi_1(M) \rightarrow SO^+(3, 1)$ from the complete hyperbolic structure can be continuously deformed into representations $\pi_1(M) \rightarrow SL(4, \mathbb{R})$ that are holonomies of convex projective structures on M . See for instance [Bal15; CLT06; CLT07].

We focus on the closed hyperbolic manifold Vol3, which is perfect for probing the strongly dense subgroup structure of various target groups. This is because, as mentioned in the previous chapter, Vol3 is a closed arithmetic hyperbolic 3-manifold of second type, meaning that the lattice embedding of $\pi_1(\text{Vol3})$ contains no non-elementary Fuchsian subgroups. That is, the lattice embedding of $\pi_1(\text{Vol3})$ has strongly dense im-

age in $SO(3, 1)$. Moreover, its projective and complex hyperbolic deformation theory has been extensively studied in [CLT06; CLT07; LR14]. Recall its fundamental group has presentation

$$\langle a, b | aabbABAbb, aBaBabaaab \rangle,$$

where $A = a^{-1}$ and $B = b^{-1}$.

A one complex parameter family of representations of Vol3 is given in Section 2.4 of [CLT07]. In order to get a family of representations with simpler expressions for the matrix entries, we instead take an orbifold that is four-fold covered by Vol3 . The orbifold is obtained from an order 4 rotational symmetry of Vol3 that fixes the axis of $aBaba$ and induces an automorphism of $\pi_1(\text{Vol3})$ such that $a \mapsto BA$ and $b \mapsto aba$. See remark 7.2 in [CLT06] for more details. Let Γ_3 denote the resulting orbifold fundamental group. Then Γ_3 is generated by two elements u and c , where $u^4 = 1$ and $c = u^2a$. Note that $\pi_1(\text{Vol3})$ can be recovered from $a = u^2c$ and $b = (aua)^{-1}u$.

We take the one complex parameter family of representations of Γ_3 found in [LR14]. The representation is given by

$$\rho_v(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{v^2-4}{v^2+8}} & 1 \\ 0 & 0 & -\frac{2(v^2+2)}{v^2+8} & -\sqrt{\frac{v^2-4}{v^2+8}} \end{pmatrix}$$

and by mapping $\rho_v(c)$ to

$$\begin{pmatrix} (v + \sqrt{v^2 + 8})/4 & 0 & (4 - v^2 - v\sqrt{v^2 + 8})/8 & 0 \\ 0 & (v - \sqrt{v^2 + 8})/4 & 0 & (-4 + v^2 - v\sqrt{v^2 + 8})/8 \\ 1 & 0 & (-v - \sqrt{v^2 + 8})/4 & 0 \\ 0 & -1 & 0 & (-v + \sqrt{v^2 + 8})/4 \end{pmatrix}$$

At the specialization $v = 2$, $\rho_2 : \Gamma_3 \rightarrow SO^+(3, 1)$ is the discrete faithful representation coming from the hyperbolic structure. As shown in [LR14], for all $v \in (2, \infty)$, deep work of Koszul [Kos68] and Benoist [Ben03; Ben05] can be used to show that the representations ρ_v are holonomies of strictly convex projective structures. All such ρ_v are discrete, faithful, and *biproximal*. That is, for every infinite order $\gamma \in \Gamma_3$, $\rho_v(\gamma)$ has a unique eigenvalue of strictly largest modulus, and a unique eigenvalue of strictly lowest modulus. For $v \in (-2, 2)$, $\rho_v : \Gamma_3 \rightarrow SU(3, 1)$ (see Section 2.4 of [CLT07]). Theorem 2.3 of [CLT07] implies there exists $\epsilon > 0$ such that for all $v \in (2 - \epsilon, 2)$, ρ_v is discrete and faithful. With this notation, we can state our first main result that “most” projective deformations of the hyperbolic structure on Vol3 give rise to holonomy representations that are strongly dense.

Theorem 3.1. The representation $\rho_v : \pi_1(\text{Vol}3) \rightarrow SL(4, \mathbb{R})$ is strongly dense for all but a possibly countable subset of $v \in (2, \infty)$.

An immediate byproduct of our efforts is that many complex hyperbolic deformations of Vol3 also give strongly dense representations:

Theorem 3.2. For all $\epsilon > 0$, there exists $t \in (2 - \epsilon, 2)$, with $\rho_t : \pi_1(\text{Vol}3) \rightarrow SU(3, 1)$ discrete and faithful, such that $\rho_t(\Gamma_3)$ is strongly dense in $SU(3, 1)$.

All currently known strongly dense representations of surface groups into $SL(n, \mathbb{R})$ have been *Hitchin*. As shown in Section 3 of [LR13], $\pi_1(\text{Vol}3)$ contains closed hyper-

bolic surface subgroups $\pi_1(F) < \pi_1(\text{Vol}3)$ such that for all $v \in (2, \infty)$, $\rho_v|_{\pi_1(F)}$ is not Hitchin. Since strongly dense representations remain strongly dense upon restriction to non-abelian subgroups, we have the following Corollary:

Corollary 3.1. $SL(4, \mathbb{R})$ contains non-Hitchin strongly dense surface subgroups.

3.3.1 Irreducibility

A subgroup of $GL(4, \mathbb{R})$ is *irreducible* if its action on $\mathbb{R}^4$ has no invariant subspaces. Say a subgroup of $GL(4, \mathbb{R})$ is *absolutely irreducible* if its action on $\mathbb{C}^4$ has no invariant subspaces. A subgroup of $GL(4, \mathbb{R})$ is *strongly irreducible* if all of its finite index subgroups are irreducible. Our first task toward the proof of our main Theorems will be to show ρ_v is irreducible for all but finitely many specializations of v , and strongly irreducible for all but countably many specializations of v , when restricted to a non-abelian free subgroup of Γ_3 .

Recall from the previous chapter that $PSL(2, \mathbb{C})$ and $SO^+(3, 1)$ are isomorphic, with the isomorphism induced by the action of $PSL(2, \mathbb{C})$ on the real four dimensional vector space of Hermitian matrices via conjugate transpose. From this isomorphism one can deduce the following facts about hyperbolic isometries in $SO^+(3, 1)$. One can consult Section 2 of [CLT06] for more detail. A hyperbolic $g \in SO^+(3, 1)$ has eigenvalues λ , $\frac{1}{\lambda}$, $e^{i\theta}$, and $e^{-i\theta}$ with $\lambda \in \mathbb{R}$ and $|\lambda| > 1$. We refer to the $\mathbb{R}$ -span of any λ and $\frac{1}{\lambda}$ eigenvector of g as the attracting and repelling eigenspaces, respectively, of g . These are the 1-dimensional subspaces of $\mathbb{R}^4$ that contain the ray in the light cone corresponding to the attracting and repelling fixed point at infinity of g , respectively. We refer to the $\mathbb{C}$ -span of any $e^{\pm i\theta}$ eigenvector of g as a rotation eigenspace of g . If a hyperbolic in $SO^+(3, 1)$ has a 1-eigenvalue, then it must mean $\theta = 0$ and the hyperbolic is a pure translation, coming from a hyperbolic in $PSL(2, \mathbb{C})$ with real trace.

Lemma 3.3. Let $G = \langle \gamma_1, \gamma_2 \rangle$ be a free subgroup of Γ_3 . Put $g_j = \rho_2(\gamma_j)$. If $W \subseteq \mathbb{R}^4$ is a $\rho_2(G)$ -invariant subspace generated by light like vectors, then $W = \mathbb{R}^4$.

Proof. Let $\{w_j\}$ be a basis for W where each w_j is a light like vector. If $\dim_{\mathbb{R}} W = 1$, then w_1 corresponds to a point at infinity fixed by the action of $\rho_2(G)$ on hyperbolic space. But no such point can exist because g_1 and g_2 are non-commuting hyperbolics and $\langle g_1, g_2 \rangle$ is discrete.

If $\dim_{\mathbb{R}} W = 2$, then $W = \text{span}_{\mathbb{R}}\{w_1, w_2\}$ is the unique geodesic with endpoints corresponding to w_1 and w_2 . The $\rho_2(G)$ -invariance of W means this geodesic is fixed by the action of H on hyperbolic space. But no such geodesic can exist because g_1 and g_2 are non-commuting hyperbolics and $\langle g_1, g_2 \rangle$ is discrete.

If $\dim_{\mathbb{R}} W = 3$, then $\langle g_1^T, g_2^T \rangle$ has an invariant 1 dimensional subspace. That is, g_1^T and g_2^T have a common eigenvector. This means all the elements in the commutator subgroup $[\langle g_1^T, g_2^T \rangle, \langle g_1^T, g_2^T \rangle]$ have a 1 eigenvalue. Hence, all hyperbolics in $[\rho_2(G), \rho_2(G)]$ have a 1 eigenvalue. But hyperbolics in $SO^+(3, 1)$ with a 1 eigenvalue are pure translations and correspond to hyperbolics in $PSL(2, \mathbb{C})$ with real trace. So this would imply $[\rho_2(G), \rho_2(G)]$ contains a non-elementary Fuchsian subgroup of Γ_3 . But Γ_3 does not contain any non-elementary Fuchsian subgroups.

We conclude that $\dim_{\mathbb{R}} W = 4$ as desired. $\square$

Lemma 3.4. Let $G = \langle \gamma_1, \gamma_2 \rangle$ be a free subgroup of Γ_3 . Then there exists a pair of hyperbolics in $\rho_2(G)$ with attracting and repelling eigenspaces that span $\mathbb{R}^4$.

Proof. The subspace of $\mathbb{R}^4$ generated by all attracting and repelling eigenspaces of elements in $\rho_2(G)$ is a $\rho_2(G)$ -invariant subspace generated by light like vectors. By Lemma 3.3, this means we can obtain light like vectors $\{w_j\}_{j=1}^4$ forming a basis of $\mathbb{R}^4$ such that w_j is an attracting eigenvector of $g_j \in \rho_2(G)$.

For $k \in \mathbb{N}$, consider the hyperbolics $g_1^k g_2^{-k}$ and $g_3^k g_4^{-k}$. As k increases, the fixed points of $g_1^k g_2^{-k}$ converge projectively to w_1 and w_2 , and the fixed points of $g_3^k g_4^{-k}$ converge projectively to w_3 and w_4 . Linear independence is an open condition. Since $\{w_j\}_{j=1}^4$ is a basis for $\mathbb{R}^4$, this means for all large k the attracting and repelling eigenspaces of $g_1^k g_2^{-k}$ and $g_3^k g_4^{-k}$ span $\mathbb{R}^4$. $\square$

Proposition 3.2. If $G = \langle \gamma_1, \gamma_2 \rangle$ is a free subgroup of Γ_3 , then $\rho_2(G) = \langle g_1, g_2 \rangle$ is absolutely irreducible.

Proof. By Lemma 3.4, passing to a subgroup if necessary, we can assume without loss of generality that the $\mathbb{C}$ -span of the attracting and repelling eigenspaces of g_1 and g_2 is $\mathbb{C}^4$. Let $W \subseteq \mathbb{C}^4$ be a $\rho_2(G)$ -invariant subspace. As in the proof of Lemma 3.3, if $\dim_{\mathbb{C}} W$ is 1 or 3, then $[\rho_2(G), \rho_2(G)]$ would contain non-elementary Fuchsian subgroups. But it is known $\rho_2(G)$ contains no such subgroup.

Suppose for contradiction $\dim_{\mathbb{C}} W = 2$. Since g_1 and g_2 have 4 distinct eigenvalues and W is G -invariant, W must contain two eigenspaces in $\mathbb{C}^4$ of g_1 and g_2 .

Claim 1. W contains both rotation eigenspaces of g_1 and g_2 .

(*Proof of Claim 1.*) Let x be an attracting or repelling eigenvector of g_1 or g_2 . Then $\text{span}_{\mathbb{R}}\{\rho_2(G)(x)\}$, the subspace of $\mathbb{R}^4$ generated by the $\rho_2(G)$ -orbit of x , is $\rho_2(G)$ -invariant. It is also generated by light like vectors. By Lemma 3.3, there exists $\{x_j\}_{j=1}^4 \subseteq \rho_2(G)(x)$ such that $\text{span}_{\mathbb{R}}\{x_j\}_{j=1}^4 = \mathbb{R}^4$. Then $\text{span}_{\mathbb{C}}\{x_j\}_{j=1}^4 = \mathbb{C}^4$. So we conclude $x \notin W$ because otherwise, we would have $\{x_j\}_{j=1}^4 \subseteq \rho_2(G)(x) \subseteq W$, which cannot hold since we are assuming $\dim_{\mathbb{C}} W = 2$. Therefore, the only eigenspaces of g_1 and g_2 that W can contain are their rotation eigenspaces. $\square$

In what follows, let J denote the symmetric form defining the $SO(3, 1; \mathbb{R})$ version of

$\mathbb{H}^3$. Let $\langle, \rangle$ denote the form on $\mathbb{C}^4$ defined by

$$\langle x, w \rangle = w^* J x.$$

Then $\langle, \rangle$ is a Hermitian form on $\mathbb{C}^4$ that is preserved by $\rho_2(G)$. For any subspace $V \subseteq \mathbb{C}^4$, put

$$V^\perp := \{w \in \mathbb{C}^4 : \forall v \in V, \langle v, w \rangle = 0\}.$$

For any subspace V , $V^\perp$ is also a subspace and we have

$$\dim_{\mathbb{C}} V + \dim_{\mathbb{C}} V^\perp = 4.$$

Claim 2. For $j = 1, 2$, if w is an $e^{i\theta}$ eigenvector of g_j , then $\text{span}_{\mathbb{C}}\{w\}^\perp$ contains the attracting and repelling eigenspaces of g_j .

(Proof of Claim 2). Let x be a $\lambda \in \mathbb{R}$ eigenvector of g_j , where $|\lambda| \neq 1$. For any k , we have

$$|\langle w, x \rangle| = |\langle g_j^k(w), g_j^k(x) \rangle| = |e^{ik\theta} \lambda^k \langle w, x \rangle| = |\lambda|^k |\langle w, x \rangle|. \quad (3.1)$$

Since (3.1) holds for any $k \in \mathbb{Z}$ and $|\lambda| \neq 1$, we must have $\langle w, x \rangle = 0$. $\square$

By Claim 1, the rotation eigenspaces of g_j generate W , for $j = 1$ or 2 . In particular, g_j acts on W by rotations. This means Claim 2 implies that for any $w \in W$, $\text{span}_{\mathbb{C}}\{w\}^\perp$ contains the attracting and repelling eigenspaces of g_1 and g_2 . But the $\mathbb{C}$ -span of the attracting and repelling eigenspaces of g_1 and g_2 is $\mathbb{C}^4$, while $\dim_{\mathbb{C}}\{w\}^\perp = 3$. This gives the desired contradiction, and we conclude $\dim_{\mathbb{C}} W = 4$ as desired. $\square$

Proposition 3.3. (1) If $G = \langle \gamma_1, \gamma_2 \rangle$ is a non-abelian free subgroup of Γ_3 , then $\rho_v(G)$ is irreducible for all but finitely many specializations of v .

- (2) There exists a subset $\mathcal{S} \subset [2, \infty)$ such that $[2, \infty) \setminus \mathcal{S}$ is at most countable and the image of all rank 2 free subgroups of Γ_3 under ρ_v is strongly irreducible for all $v \in \mathcal{S}$.

Proof. (1) Proposition 3.2 implies $\rho_2(G)$ is absolutely irreducible. Burnside's lemma (see [HR80]) implies that the group algebra $\mathbb{C}\rho_v(G)$ contains elements $A_1(v), \dots, A_{16}(v) \in \mathbb{C}\rho_v(G)$ such that $\{A_1(2), \dots, A_{16}(2)\}$ is a basis for $M(4, \mathbb{C})$. Put

$$\delta(v) := \det[A_1(v), \dots, A_{16}(v)].$$

All the matrix entries of elements in $\rho_v(G)$ lie in $\mathbb{Q}(v, \sqrt{v^2 - 4})$. This means we can regard δ as an algebraic function of the single complex parameter v .

The failure of absolute irreducibility of $\rho_t(G)$ at the specialization $v = t$ implies either $\delta(t) = 0$, or t is a pole of δ . The absolute irreducibility of $\rho_2(G)$ implies $\delta(2) \neq 0$, so δ has at most finitely many zeros or poles. We conclude that $\rho_v(G)$ is irreducible for all but finitely many v .

- (2) Since Γ_3 contains countably many rank 2 free groups, and all finite index subgroups of any non-abelian free $G < \Gamma_3$ contain a rank 2 free group, part (2) of this proposition follows immediately from part (1). $\square$

3.3.2 Complex Hyperbolic Geometry

The main result we need from complex hyperbolic geometry is the following theorem of Kim and Kim in [KK14]:

Theorem 3.3. Let $\Gamma < SU(3, 1)$ be a non-elementary complex hyperbolic Kleinian group. Then $\text{tr}(\gamma) \in \mathbb{R}$ for all $\gamma \in \Gamma$ if and only if Γ is conjugate to a subgroup of $SO(3, 1)$ or $SU(1, 1) \times SU(2)$.

We conclude this section with a basic observation that will aid in our future applications of Theorem 3.3.

Lemma 3.5. Let $G = \langle \gamma_1, \gamma_2 \rangle$ be a non-abelian free subgroup of Γ_3 . Then there exists $\epsilon > 0$ such that for all $v \in (2 - \epsilon, 2)$, $\rho_v(G)$ is a non-elementary complex hyperbolic Kleinian group.

Proof. By Proposition 2.4 of [CLT07], we can choose $\epsilon > 0$ such that ρ_v is discrete and faithful for all $v \in (2 - \epsilon, 2)$, meaning $\rho_v(G)$ will be a complex hyperbolic Kleinian group for all such v . Since $\rho_2(\gamma_j) \in SO(3, 1)$ are loxodromics of real hyperbolic space for $j = 1, 2$, they both have a pair of real eigenvalues $\lambda_j, \lambda_j^{-1}$ for some $\lambda_j > 1$, each of which corresponding to distinct fixed points at infinity. The eigenvalues and eigenspaces of $\rho_v(\gamma_j)$ depend continuously on v so we can choose $\epsilon > 0$ such that $\rho_v(\gamma_j)$ has a pair of eigenvalues that are not of unit norm, for all $v \in (2 - \epsilon, 2)$. That is we can choose $\epsilon > 0$ so that for all $v \in (2 - \epsilon, 2)$, $\rho_v(\gamma_j) \in SU(3, 1)$ will be a pair of non-commuting complex hyperbolics. This means $\rho_v(\gamma_1)$ and $\rho_v(\gamma_2)$ will have disjoint fixed point sets. Since $\Lambda_{\rho_v(G)}$ contains the attracting and repelling fixed points of $\rho_v(\gamma_j)$, we conclude $\rho_v(G)$ is also non-elementary for all such v . $\square$

3.3.3 Proof of Theorems

Our first goal of this section will be to show that if $G < \Gamma_3$ is a free group of rank 2 and $v \in (2, \infty)$ is such that $\rho_v(G)$ is strongly irreducible, then $\rho_v(G)$ is Zariski dense in $SL(4, \mathbb{R})$, except for possibly finitely many exceptional specializations of v . Once this is established, we then show $\rho_v(\Gamma_3)$ is strongly dense in $SL(4, \mathbb{R})$, with possibly countably many exceptional specializations of $v \in (2, \infty)$. Analogous results for $SU(3, 1)$ deformations will then be clear once our techniques are established.

Our analysis of Zariski closures is based on the following result of Crampon and

Marquis ([CM14], Proposition 6.5), whose translation can be found in Proposition 3.6 of [Bra20]:

Theorem 3.4. Suppose Γ is a strongly irreducible subgroup of $SL(n+1, \mathbb{R})$ which preserves a properly convex $\Omega \subset \mathbb{RP}^n$. Let $\mathcal{G}$ be the Zariski closure of Γ . Then $\mathcal{G}$ is a Zariski-connected real semi-simple Lie group.

Recall that for $v \in (2, \infty)$, $\rho_v : \Gamma_3 \rightarrow SL(4, \mathbb{R})$ is a discrete faithful representation that is the holonomy of a strictly convex projective structure on the orbifold. This means Theorem 3.4 and Proposition 3.3 imply that for all $v \in (2, \infty)$, except for possibly finitely many exceptional specializations of v , $\rho_v(G)$ has Zariski closure a semi-simple Lie subgroup of $SL(4, \mathbb{R})$. All such subgroups have been classified, one can check [GT16] for a reference. So to show a given strongly irreducible $\rho_v(G)$ is Zariski dense in $SL(4, \mathbb{R})$, we will show no other Lie subgroups of $SL(4, \mathbb{R})$ can be a possible Zariski closure.

Since G is a free group of rank 2 and ρ_v is discrete, faithful, and irreducible, most Lie groups can be easily ruled out as possible Zariski closures by general algebraic reasons. The main obstruction to Zariski density in our specific context is the possibility that $\rho_v(G)$ preserves either a symmetric or skew-symmetric form. This is handled by the following series of Lemmas and Proposition.

Lemma 3.6. Let G be a non-abelian free subgroup of Γ_3 . Then there is no non-zero real symmetric form preserved by $\rho_{i\sqrt{2}}(G)$.

Proof. Recall that Γ_3 is the orbifold fundamental group of an orbifold that is four-fold covered by Vol3. This means, by passing to a finite index subgroup if necessary, we can assume without loss of generality that $G < \pi_1(\text{Vol3})$.

At the specialization $v = i\sqrt{2}$, one checks that the representation is reducible, and

that we can conjugate so that

$$\rho_{i\sqrt{2}}(\gamma) = \begin{bmatrix} \overline{\rho_\infty(\gamma)} & 0 \\ \star & \rho_\infty(\gamma) \end{bmatrix},$$

where $\rho_\infty : \pi_1(\text{Vol3}) \rightarrow SL(2, \mathbb{C})$ is a lift of the holonomy representation of $\pi_1(\text{Vol3})$ into $PSL(2, \mathbb{C})$ coming from the hyperbolic structure.

Indeed, letting

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

direct calculation yields

$$M^{-1}\rho_{i\sqrt{2}}(u)M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \end{bmatrix}$$

and

$$M^{-1}\rho_{i\sqrt{2}}(c)M = \begin{bmatrix} \frac{1}{4}(i\sqrt{2} - \sqrt{6}) & \frac{1}{8}(-6 - 2i\sqrt{3}) & 0 & 0 \\ -1 & \frac{1}{4}(-i\sqrt{2} + \sqrt{6}) & 0 & 0 \\ 0 & 0 & \frac{1}{4}(i\sqrt{2} + \sqrt{6}) & \frac{1}{8}(6 - 2i\sqrt{3}) \\ 0 & 0 & 1 & \frac{1}{4}(-i\sqrt{2} - \sqrt{6}) \end{bmatrix},$$

where u and c were the generators of Γ_3 . Let

$$J = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix},$$

be a symmetric form preserved by $\rho_{i\sqrt{2}}(G)$. That is, $\rho_{i\sqrt{2}}(\gamma)^T J \rho_{i\sqrt{2}}(\gamma) = J$ for all $\gamma \in G$.

This means the system of equations

$$\rho_\infty(\gamma)^T J_4 \rho_\infty(\gamma) = J_4 \tag{3.2}$$

must hold for all $\gamma \in G$.

But (3.2) can only be satisfied if J_4 is a real symmetric form preserved by $\rho_\infty(G)$. But $\rho_\infty(G)$ is Zariski dense in $SL(2, \mathbb{C})$ because $\pi_1(\text{Vol3})$ contains no non-elementary Fuchsian subgroups. Since (3.2) implies that the algebraic subgroup

$$\{A \in SL(2, \mathbb{C}) : A^T J_4 A = J_4\}$$

of $SL(2, \mathbb{C})$ must contain $\rho_\infty(G)$, the Zariski density of $\rho_\infty(G)$ in $SL(2, \mathbb{C})$ means that every $A \in SL(2, \mathbb{C})$ must satisfy (3.2). Therefore, J_4 must be symplectic. Since J_4 must also be symmetric, we conclude that $J_4 = 0$.

Since $J_4 = 0$, then (3.2) holds only if

$$\rho_\infty(\gamma)^* J_3 \rho_\infty(\gamma) = J_3$$

for all $\gamma \in G$.

Claim 3. There is no non-zero $M \in M(2, \mathbb{C})$ such that $\rho_\infty^*(\gamma) M \rho_\infty(\gamma) = M$ for all $\gamma \in G$.

(*proof of Claim 3*). First, we recall the following notation from Section 3.3.1. As before, $\mathcal{H}$ denotes the basis fixed in (2.1) of Section 3.3.1, of the 2×2 Hermitian matrices as a real vector space. For any $A \in SL(2, \mathbb{C})$,

$$\tau(A) : \text{span}_{\mathbb{R}} \mathcal{H} \rightarrow \text{span}_{\mathbb{R}} \mathcal{H}$$

is the $\mathbb{R}$ -linear map defined by $\tau(A)(H) = A^*HA$. An isomorphism from $PSL(2, \mathbb{C})$ to $SO^+(3, 1)$ is given by sending each $[A] \in PSL(2, \mathbb{C})$ to the matrix representation of $\tau(A)$ with respect to $\mathcal{H}$ for any representative A of $[A] \in PSL(2, \mathbb{C})$. That is, in the notation of the proof of this Proposition, we have that $\rho_2(\gamma)$ is the matrix representation of the $\mathbb{R}$ linear map $\tau(\rho_\infty(\gamma))$. Therefore, $\rho_2(\gamma)$ is also the matrix representation of the complexification

$$\tau(A)_{\mathbb{C}} : \text{span}_{\mathbb{C}} \mathcal{H} \rightarrow \text{span}_{\mathbb{C}} \mathcal{H}.$$

Since $\text{span}_{\mathbb{C}} \mathcal{H} = M(2, \mathbb{C})$, the absolute irreducibility of $\rho_2(G)$ (see Proposition 3.2) gives the desired claim, as any $M \in M(2, \mathbb{C})$ satisfying $\rho_\infty^*(\gamma)M\rho_\infty(\gamma) = M$ for all $\gamma \in G$ would correspond to a vector in $\mathbb{C}^4$ fixed by $\rho_2(G)$. $\square$

Claim 3 implies $J_3 = 0$. Hence $J_2 = J_3^T = 0$. Then similar reasoning will show $J_1 = 0$. $\square$

Remark 3.1. At the specialization $v = -i\sqrt{2}$ the representation $\rho_{-i\sqrt{2}}$ is also irreducible. It has a two dimensional invariant subspace, and restriction to that subspace gives the complex conjugate of ρ_∞ , so an analogue of Lemma 3.6 also holds for $\rho_{-i\sqrt{2}}$.

Lemma 3.7. Let G be a rank 2 free subgroup of Γ_3 . There is no interval $I \subset \mathbb{R}$ containing 2 with non-empty interior such that $\rho_v(G)$ preserves a symmetric form for all $v \in I$.

Proof. Suppose for contradiction there exists such an interval I . For all $v \in I$, let J_v denote the non-zero symmetric form preserved by $\rho_v(G)$. Up to scaling, J_v is the unique

form satisfying the system of equations

$$\rho_v(\gamma)^T J_v \rho_v(\gamma) = J_v$$

for all $v \in I$ and $\gamma \in G$. Therefore, we can regard the entries of J_v as functions with the following shape:

$$p(v) + q(v)\sqrt{v^2 - 4} + r(v)\sqrt{v^2 + 8} + s(v)\sqrt{v^2 - 4}\sqrt{v^2 + 8}, \quad (3.3)$$

where we have scaled J_v so that for all the entries, the functions p, q, r, s as in equation 3.3 are in $\mathbb{Q}[v]$ with no common factors.

Our strategy at this point is to reach a contradiction by using Lemma 3.6, which implies $J_{i\sqrt{2}}$ must be the zero matrix, to show that the functions p, q, r, s do indeed have a common factor. This is done by considering some Galois actions on the field in which the entries of $J_{i\sqrt{2}}$ live.

Toward this end, put $F = \mathbb{Q}(i\sqrt{2})$ and let $K = F(\sqrt{6}, i)$. At the specialization $v = i\sqrt{2}$, the entries of $J_{i\sqrt{2}}$ lie in the field K . The Galois group $Gal(K/F)$ is the Klein 4 group generated by the maps ϕ_1 and ϕ_2 , whose action on K is given by

$$\phi_1 : \begin{cases} \sqrt{6} & \mapsto -\sqrt{6} \\ i & \mapsto i \end{cases} \quad \phi_2 : \begin{cases} \sqrt{6} & \mapsto \sqrt{6} \\ i & \mapsto -i \end{cases} .$$

Let

$$\begin{aligned}\alpha_0 &= p(i\sqrt{2}) + q(i\sqrt{2})i\sqrt{6} + r(i\sqrt{2})\sqrt{6} + s(i\sqrt{2})6i, \\ \alpha_1 &= p(i\sqrt{2}) - q(i\sqrt{2})i\sqrt{6} - r(i\sqrt{2})\sqrt{6} + s(i\sqrt{2})6i, \\ \alpha_2 &= p(i\sqrt{2}) - q(i\sqrt{2})i\sqrt{6} + r(i\sqrt{2})\sqrt{6} - s(i\sqrt{2})6i, \\ \alpha_3 &= p(i\sqrt{2}) + q(i\sqrt{2})i\sqrt{6} - r(i\sqrt{2})\sqrt{6} - s(i\sqrt{2})6i.\end{aligned}$$

That is, α_0 is the general shape of equations for the entries of J_v as in (3.3), specialized at $v = i\sqrt{2}$. Then $\alpha_1 = \phi_1(\alpha_0)$, $\alpha_2 = \phi_2(\alpha_0)$, and $\alpha_3 = \phi_1(\phi_2(\alpha_0))$ are the Galois conjugates of α_0 .

By Lemma 3.6, we have $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$. Direct calculation gives the following equations:

$$\begin{aligned}0 &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 4p(i\sqrt{2}), \\ 0 &= \alpha_0 - \alpha_1 - \alpha_2 + \alpha_3 = 4q(i\sqrt{2})i\sqrt{6}, \\ 0 &= \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 = 4r(i\sqrt{2})\sqrt{6}, \\ 0 &= \alpha_0 + \alpha_1 - \alpha_2 - \alpha_3 = 4s(i\sqrt{2})6i.\end{aligned}$$

From this we conclude $i\sqrt{2}$ is a root of all such equations p, q, r, s . Since we have $p, q, r, s \in \mathbb{Q}[v]$, this implies that $-i\sqrt{2}$ is also a root of all such equations. Therefore, $v^2 + 2$ must be a common factor of all entries of J_v , giving the desired contradiction. $\square$

Proposition 3.4. Let G be a rank 2 free subgroup of Γ_3 . There exists $\gamma \in G$ such that the characteristic polynomial $\chi_{\rho_v(\gamma)}$ is not palindromic for all but finitely many $v \in \mathbb{C}$.

Proof. In [CLT07], it is shown that the characteristic polynomial $\chi_{\rho_v(\gamma)}(Q)$ of $\rho_v(\gamma)$ has

the following shape:

$$1 - (p_\gamma(v) - q_\gamma(v)\sqrt{v^2 - 4})Q + r_\gamma(v)Q^2 - (p_\gamma(v) + q_\gamma(v)\sqrt{v^2 - 4})Q^3 + Q^4.$$

Here, $p_\gamma, q_\gamma, r_\gamma \in \mathbb{Q}[v]$. This means $\chi_{\rho_v(\gamma)}$ is palindromic if and only if either $v \in \{\pm 2\}$ or $q_\gamma(v) \equiv 0$.

Assume for contradiction $\chi_{\rho_v(\gamma)}$ is palindromic for infinitely many $v \in \mathbb{C}$ and all $\gamma \in G$. This means $q_\gamma \equiv 0$, and hence $\text{tr} \rho_v(\gamma) = -p_\gamma(v)$ for all $\gamma \in G$. In particular, we have $\text{tr} \rho_v(\gamma) \in \mathbb{R}$ for all $\gamma \in G$ and $v \in (-2, 2)$.

Recall that the image of ρ_v lies in $SU(3, 1)$ for all $v \in (-2, 2)$. By Proposition 3.5, there exists $\epsilon > 0$ such that $\rho_v(G)$ is a non-elementary complex hyperbolic Kleinian group for all $v \in (2 - \epsilon, 2)$. By Lemma 3.3, we can take ϵ so that $\rho_v(G)$ is also irreducible. Since $\rho_v(G)$ is an irreducible complex hyperbolic Kleinian group such that $\text{tr} \rho_v(\gamma) \in \mathbb{R}$ for all $\gamma \in G$, Theorem 3.3 implies $\rho_v(G)$ can be conjugated into $SO(3, 1)$ for all $v \in (2 - \epsilon, 2)$. This contradicts Lemma 3.7. $\square$

We now put together the previous results to prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1. By Proposition 3.3, we can obtain a set $\mathcal{S} \subset [2, \infty)$ such that $[2, \infty) \setminus \mathcal{S}$ is at most countable and $\rho_v(G)$ is strongly irreducible for all $v \in \mathcal{S}$ and all rank 2 free subgroups $G < \Gamma_3$. Fix G a rank 2 free subgroup of Γ_3 . We first show that for all but a possibly finite subset of $v \in \mathcal{S}$, $\rho_v(G)$ is Zariski dense in $SL(4, \mathbb{R})$.

Let $\overline{\rho_v(G)}$ denote the Zariski closure of $\rho_v(G)$ in $SL(4, \mathbb{R})$ and $\overline{\rho_v(G)}^0$ denote its identity component. By the strong irreducibility of $\rho_v(G)$ for $v \in \mathcal{S}$, Theorem 3.4 implies $\overline{\rho_v(G)}$ is a semi-simple Lie subgroup of $SL(4, \mathbb{R})$. But ρ_v is also discrete and faithful, meaning it discretely embeds a free group of rank 2 into $\overline{\rho_v(G)}$. Hence, $\overline{\rho_v(G)}$ cannot be compact or solvable. By the classification of Lie sub-algebras of $sl(4, \mathbb{R})$ (see [GT16]), if

we can show $\overline{\rho_v(G)}^0$ does not preserve any form, then we can conclude $\rho_v(G)$ is Zariski dense in $SL(4, \mathbb{R})$.

By Proposition 3.4, there exists $\rho_v(\gamma) \in \rho_v(G)$ such that its characteristic polynomial $\chi_{\rho_v(\gamma)}$ is not palindromic for all but finitely many specializations of v .

Claim 4. If $v \in \mathcal{S}$ is such that $\chi_{\rho_v(\gamma)}$ is not palindromic, then $\rho_v(G)$ is Zariski dense in $SL(4, \mathbb{R})$.

(*proof of Claim 4*). Recall that $\rho_v(\gamma)$ is biproximal, meaning it has a unique real eigenvalue of largest modulus, and a unique real eigenvalue of smallest modulus. Let

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| > |\lambda_4|$$

denote the eigenvalues of $\rho_v(\gamma)$. Also recall that $\chi_{\rho_v(\gamma)}$ being non-palindromic is equivalent to the eigenvalues of $\rho_v(\gamma)$ not being multiplicatively symmetric about 1. These facts, along with $\det \rho_v(\gamma) = 1$ and $\lambda_1, \lambda_4 \in \mathbb{R}$, implies that $|\lambda_2|^n |\lambda_3|^n \neq 1$ for all $n \in \mathbb{N}$. That is, $\rho_v^n(\gamma)$ has non-palindromic characteristic polynomial for all n because the eigenvalues of $\rho_v^n(\gamma)$ can never be multiplicatively symmetric about 1. In particular, no power of $\rho_v(\gamma)$ can preserve any form. Since $\overline{\rho_v(G)}^0$ is a finite index subgroup of $\overline{\rho_v(G)}$, this implies $\rho_v(G)$ is Zariski dense in $SL(4, \mathbb{R})$. $\square$

Claim 4 implies that every rank 2 free subgroup of Γ_3 has image under ρ_v a Zariski dense subgroup of $SL(4, \mathbb{R})$, for all but possibly countably many $v \in \mathcal{S}$. A Ping-Pong argument shows that every pair of non-commuting elements in Γ_3 generates a subgroup that contains a rank 2 free subgroup. Hence, we can conclude that $\rho_v : \Gamma_3 \rightarrow SL(4, \mathbb{R})$ is strongly dense for all but possibly countably many $v \in \mathcal{S}$. $\square$

Proof of Theorem 3.2. By theorem 2.3 of [CLT07], we can take $\epsilon > 0$ such that for all $v \in (2 - \epsilon, 2)$, ρ_v is discrete and faithful. Proposition 3.3 implies there exists $t \in (2 - \epsilon, 2)$

such that the image of every rank 2 free subgroup of Γ_3 under ρ_t is irreducible. By Proposition 3.4, we can choose this $t \in (2 - \epsilon, 2)$ so that every rank 2 free subgroup of Γ_3 has an element whose image under ρ_t has characteristic polynomial that is not palindromic.

A Ping-Pong argument shows that every pair of non-commuting elements in Γ_3 generates a subgroup that contains a rank 2 free group. So we will get our desired result if we can show that every rank 2 free subgroup of Γ_3 has image under ρ_t a Zariski dense subgroup of $SU(3, 1)$.

Let $G < \Gamma_3$ be a rank 2 free group. Let $\overline{\rho_t(G)}$ denote the Zariski closure of $\rho_t(G)$ in $SU(3, 1)$. Then $\overline{\rho_t(G)}$ is a closed subgroup of $SU(3, 1)$ with its classical topology. Theorem 4.4.2 of [CG74] implies that either $\overline{\rho_t(G)} = SU(3, 1)$, or $\overline{\rho_t(G)}$ preserves a fixed point in $\partial\mathbb{CH}^3$, or $\overline{\rho_t(G)}$ preserves a totally geodesic submanifold of $\mathbb{CH}^3$. Since $\rho_t(G)$ is irreducible by our construction of t , $\overline{\rho_t(G)}$ cannot have a global fixed point. If $\overline{\rho_t(G)}$ preserves a totally geodesic submanifold of $\mathbb{CH}^3$, then Proposition 2.5.1 of [CG74] on the classification of totally geodesic subspaces implies that $\overline{\rho_t(G)}$ is contained in a copy of $SO(3, 1)$. But $\rho_t(G)$ contains an element with non-palindromic characteristic polynomial, and hence cannot preserve a symmetric form. We conclude $\overline{\rho_t(G)} = SU(3, 1)$, and our desired result follows. $\square$

3.4 The Figure 8 Knot

Let $S^3 \setminus K$ denote the figure 8 knot complement. In this section we discuss strongly dense representations of $\pi_1(S^3 \setminus K)$. Recall that $\pi_1(S^3 \setminus K)$ admits a lattice embedding into $PSL(2, \mathbb{C})$. The image of its lattice embedding lies in a conjugate of an index 12 subgroup of the Bianchi group $PSL(2, \mathcal{O}_3)$ (see [MR03]). This means it is a type I arithmetic lattice, and hence contains infinitely many commensurability classes of non-

elementary Fuchsian subgroups. In particular, the figure 8 knot group's lattice embedding does not have strongly dense image. However, it can still potentially admit strongly dense embeddings into $SL(2, \mathbb{C})$. In the following subsections we will explain why this is so, and discuss some progress on the question of whether or not it admits a strongly dense embedding into $SL(4, \mathbb{R})$.

3.4.1 The Canonical Component

The material in this subsection is classical. We summarize the main ideas from Culler Shalen theory necessary to show why the figure 8 knot group admits strongly dense embeddings into $SL(2, \mathbb{C})$. While we focus on the figure 8, a lot of what we say is true for a much wider class of knots.

Consider the $SL(2, \mathbb{C})$ character variety of the figure 8 knot group, denoted $X(\pi_1(S^3 \setminus K), SL(2, \mathbb{C}))$. Mostow rigidity implies that up to conjugation, the lattice embedding is the unique discrete and faithful representation of $\pi_1(S^3 \setminus K)$ into $PSL(2, \mathbb{C})$. The lattice embedding of $\pi_1(S^3 \setminus K)$ into $PSL(2, \mathbb{C})$ was shown by Thurston to lift to $SL(2, \mathbb{C})$. And the character of the lift of discrete faithful representations into $SL(2, \mathbb{C})$ lie on a single irreducible component of $X(\pi_1(S^3 \setminus K), SL(2, \mathbb{C}))$. This component is called the *canonical component*, which we denote by X_0 .

Thurston showed that X_0 is a curve of complex dimension 1. The complex curve X_0 admits a projective completion $\hat{X}_0$, and the points in $\hat{X}_0 \setminus X_0$ are called ideal points of X_0 .

Culler and Shalen showed that associated to each ideal point of X_0 is a splitting of the group $\pi_1(S^3 \setminus K)$. Moreover, this splitting is realized by an incompressible surface in $S^3 \setminus K$. That is, if p is an ideal point of X_0 , then associated to p is an incompressible surface $\Sigma_p \subset S^3 \setminus K$. If Σ_p is separating, then $\pi_1(S^3 \setminus K)$ decomposes as a free product

with amalgamation $A \star_{\pi_1(\Sigma_p)} B$. If Σ_p is non-separating, then $\pi_1(S^3 \setminus K)$ decomposes as an HNN extension, $A \star_{\pi_1(\Sigma_p)}$.

To compute what surfaces are detected by the ideal points of the canonical component, one can use the *A-polynomial* of K . The A-polynomial, roughly speaking, is a 2 variable polynomial associated to K defined in [Coo+94] that represents a parameterization of X_0 via the eigenvalues of the standard meridian and longitude curves under the $SL(2, \mathbb{C})$ representations associated to points on X_0 .

In [Coo+94], they showed that the slopes of edges of the Newton polygon of the A-polynomial are boundary slopes of incompressible surfaces detected by the ideal points of X_0 in Culler Shalen theory. The figure 8 knot has A polynomial

$$-M^4 + L(M^8 - M^6 - 2M^4 - M^2 + 1) - L^2 M^4.$$

The Newton polygon has sides of slope ± 4 . This means X_0 contains two ideal points, which we will denote by b, w , corresponding to two different splittings of $\pi_1(S^3 \setminus K)$ that are realized by incompressible surfaces Σ_b, Σ_w where $\partial\Sigma_b$ and $\partial\Sigma_w$ are peripheral loops with boundary slope ± 4 . As it turns out, the black and white checkerboard surfaces of $S^3 \setminus K$ also have boundary slopes ± 4 , and so the incompressible surfaces detected by the ideal points b, w are the black and white checkerboard surfaces.

To each $\gamma \in \pi_1(S^3 \setminus K)$, let $\tau_\gamma : X_0 \rightarrow \mathbb{C}$ denote the trace function

$$\tau_\gamma(\rho) := \text{tr}(\rho(\gamma)).$$

Each τ_γ is a regular function on X_0 . Theorem 2.2.1 of [CS83] implies that if $\gamma \in \pi_1(S^3 \setminus K)$ does not have a representative that can be homotoped into Σ_b (resp. Σ_w), then $|\tau_\gamma(\rho_t)| \rightarrow \infty$ for any sequence $(\rho_t)_{t=1}^\infty$ in X_0 approaching b (resp. w). No $\gamma \in \pi_1(S^3 \setminus K)$ has

representative curves that can be homotoped into both black and white checkerboard surfaces. Hence we get the following Proposition:

Proposition 3.5. For all $\gamma \in \pi_1(S^3 \setminus K)$, the trace function τ_γ is non-constant.

We now show that there are representations of $\pi_1(S^3 \setminus K)$ into $SL(2, \mathbb{C})$ where no non-identity element has real trace. In particular, these embeddings will be strongly dense in $SL(2, \mathbb{C})$, where $SL(2, \mathbb{C})$ is viewed as a real algebraic group.

Proposition 3.6. There exists strongly dense embeddings of $\pi_1(S^3 \setminus K)$ into $SL(2, \mathbb{C})$.

Proof. By Proposition 3.5, for all $\gamma \in \pi_1(S^3 \setminus K)$, τ_γ is a non-constant regular function. Hence $\tau_\gamma^{-1}(\mathbb{R})$ is a closed subset of X_0 with empty interior. The Baire category theorem then implies the set

$$X_0 \setminus \bigcup_{\gamma \in \pi_1(S^3 \setminus K) \setminus \{1\}} \tau_\gamma^{-1}(\mathbb{R})$$

is not empty. Let ρ be a representation associated to a character in

$$X_0 \setminus \bigcup_{\gamma \in \pi_1(S^3 \setminus K) \setminus \{1\}} \tau_\gamma^{-1}(\mathbb{R}).$$

Note that ρ is faithful because if $\gamma \in \pi_1(S^3 \setminus K)$ is not the identity, then $\rho(\gamma)$ cannot be the identity as it has non-real trace.

We now show $\rho(\pi_1(S^3 \setminus K))$ is a strongly dense subgroup of $SL(2, \mathbb{C})$ as a real algebraic group. Let $\gamma_1, \gamma_2 \in \pi_1(S^3 \setminus K)$ be a pair of non-commuting elements. Then $\rho(\gamma_1), \rho(\gamma_2)$, and $\rho([\gamma_1, \gamma_2])$ all have non-real trace. This means $\rho(\gamma_1)$ and $\rho(\gamma_2)$ is a pair of non-commuting loxodromics, and Lemma 3.2 implies that they do not have a common fixed point at $\partial_\infty \mathbb{H}^3$. Therefore $\rho(\langle \gamma_1, \gamma_2 \rangle)$ is Zariski dense in $SL(2, \mathbb{C})$.

□

3.4.2 Projective Deformations of the Figure 8 Knot

Ballas (in [Bal15]) and Thistlethwaite (independently) computed projective deformations of the hyperbolic structure on the figure 8 knot complement:

$$\rho_t(x) = \begin{pmatrix} 1 & 0 & 1 & t-1 \\ 0 & 1 & 1 & t \\ 0 & 0 & 1 & t+1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \rho_t(y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2+1/t & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Here the lattice embedding into $SO(3,1)$ occurs at the specialization $t = 1/2$. For $t \in (1/2, \infty)$, the representation $\rho_t : \pi_1(S^3 \setminus K) \rightarrow SL(4, \mathbb{R})$ is discrete and faithful; it is the holonomy of a properly convex projective structure on $S^3 \setminus K$. Given that most projective deformations of the hyperbolic structure on $Vol3$ give rise to strongly dense embeddings of $\pi_1(Vol3)$ into $SL(4, \mathbb{R})$, it is natural to wonder whether or not something similar happens for the figure 8 knot complement. In the $Vol3$ argument, we used multiple times that the lattice embedding of $\pi_1(Vol3)$ has strongly dense image into $PSL(2, \mathbb{C}) \cong SO^+(3,1)$, due to the fact that it does not contain any non-elementary Fuchsian subgroups.

This is the essential difference between $Vol3$ and the figure 8 knot complement: the lattice embedding of $\pi_1(S^3 \setminus K)$ does not have strongly dense image because it contains infinitely many non-elementary Fuchsian subgroups. So it is natural to wonder, is strong density a generic property of geometric deformations of hyperbolic structure, or does it depend on the lattice embedding being strongly dense?

Our analysis in the previous section showed that most SL_2 deformations of $\pi_1(S^3 \setminus K)$'s lattice embedding do indeed have strongly dense image in $SL(2, \mathbb{C})$. So despite the presence of non-elementary Fuchsian subgroups, one could expect that SL_4 deformations of the lattice embedding $\pi_1(S^3 \setminus K) \rightarrow SO(3,1)$ also give rise to strongly dense embeddings.

At this time, the presence of non-elementary Fuchsian subgroups are still a major obstacle that is stopping us from using the same arguments in the *Vol3* case to produce strongly dense embeddings of $\pi_1(S^3 \setminus K)$ into $SL(4, \mathbb{R})$. But there is evidence to suspect that projective deformations are still strongly dense. In the argument for *Vol3*, we made great use of certain reducible specializations that gave rise to lower dimensional representations of geometric significance. Similar facts hold for the component of $Hom(\pi_1(S^3 \setminus K), SL(4, \mathbb{C}))$ computed by Ballas and Thistlethwaite. At $t = -1/2$, the representation has a 2D invariant subspace. Restriction to it corresponds to a lattice embedding into $PSL(2, \mathbb{C})$. Indeed, letting

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & -i\sqrt{3} & 0 & i\sqrt{3} \\ -i(2i + \sqrt{3}) & 0 & i(-2i + \sqrt{3}) & 0 \\ -i(i + \sqrt{3}) & 0 & i(-i + \sqrt{3}) & 0 \end{pmatrix},$$

one gets

$$M^{-1}\rho_{-1/2}(x)M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1+i\sqrt{3}}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1-i\sqrt{3}}{2} & 1 \end{pmatrix}, M^{-1}\rho_{-1/2}(y)M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

At $t = \frac{-1+i\sqrt{3}}{4}$, the representation has a 3D invariant subspace. Restriction to it corresponds to the holonomy of a branched spherical CR structure (i.e. a $(PU(2, 1), S^3)$ structure) on $S^3 \setminus K$, as computed by Falbel in section 6 of [Fal08].

Indeed, letting

$$M = \begin{pmatrix} -1 + i\sqrt{3} & \frac{-i(-i+\sqrt{3})}{4} & 0 & 0 \\ 0 & \frac{-i(-i+\sqrt{3})}{4} & -1 & 0 \\ 0 & 0 & \frac{-i(-i+\sqrt{3})}{4} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

one gets

$$M^{-1}\rho_{\frac{-1+i\sqrt{3}}{4}}(x)M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \frac{-1-i\sqrt{3}}{2} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M^{-1}\rho_{\frac{-1+i\sqrt{3}}{4}}(y)M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 - 2i\sqrt{3} & 1 & 1 & 0 \\ -1 + i\sqrt{3} & \frac{-1-i\sqrt{3}}{2} & -1 & 1 \end{pmatrix}.$$

From these computations, we get the following analogue of Proposition 3.5 for projective deformations.

Proposition 3.7. For all $\gamma \in \pi_1(S^3 \setminus K)$, the trace function $t \mapsto \text{tr}(\rho_t(\gamma))$ is non-constant.

But unlike $SL(2, \mathbb{C})$, non-constant trace functions is not enough to guarantee Zariski density in $SL(4, \mathbb{R})$. At this time, if $F < \pi_1(S^3 \setminus K)$ is such that $\rho_{1/2}(F)$ is a non-elementary Fuchsian subgroup in the lattice embedding, it is unknown whether or not $\rho_t(F)$ can preserve a symmetric form for all $t \in (1/2, \infty)$. We suspect this not to be the case.

Current methods can show that if $H < \pi_1(S^3 \setminus K)$ is such that under the lattice embedding, $\rho_{1/2}(H)$ is strongly dense in $SO(3, 1)$, then $\rho_t(H)$ will be strongly dense in $SL(4, \mathbb{R})$ for almost all $t \in (1/2, \infty)$. But whether or not $\pi_1(S^3 \setminus K)$ contains such a

subgroup is a special case of an open problem first mentioned in [Bre+12].

Chapter 4

Schottky

The project in this chapter was originally motivated by the following question of Hee Oh, mentioned during the lecture in the conference "Visions in Arithmetic and Beyond: Celebrating Peter Sarnak's Work and Impact":

Is there a Schottky group whose limit set contains no four points lying in a circle?

We briefly give some context for the question. Given a finitely generated torsion free Kleinian group $\Gamma < PSL_2(\mathbb{C})$, one wants to determine criterion for when a discrete faithful representation $\rho : \Gamma \rightarrow PSL_2(\mathbb{C})$ must be a Möbius conjugation. Earlier work on rigidity theory, for example by Mostow [Mos68; Mos73], and later Sullivan [Sul81], deals with Γ whose limit set Λ_Γ is all of $S^2 = \partial_\infty \mathbb{H}^3$.

Kim and Oh provide the first results for Kleinian groups of the second kind [KO23]. They show when Γ is a Zariski dense finitely generated Kleinian group with disconnected domain of discontinuity, and $\rho : \Gamma \rightarrow PSL_2(\mathbb{C})$ is a discrete, faithful, Zariski dense representation with ρ equivariant boundary map $f : \Lambda_\Gamma \rightarrow S^2$, if f maps "circular slices" of Λ_Γ into circles, then ρ is a Möbius conjugation.

Their work is a major advance in rigidity theory for being able to address Kleinian groups of the second kind. But for Γ with a connected domain of discontinuity, not much is known about their rigidity properties and it is difficult to know what to expect. Schottky groups are discrete subgroups whose limit sets are homeomorphic to Cantor sets, and hence have a connected domain of discontinuity (more precise definitions will be given in later sections). They are classical objects and among the first examples of discrete subgroups of $PSL_2(\mathbb{C})$, so this is a natural class of examples to consider if one wishes to push the theory farther. An affirmative answer to the above question would mean there can be no analogue of the Theorem of Kim and Oh for Kleinian groups with connected domain of discontinuity.

In this project we provide evidence that such a Schottky group exists. In what follows, G is a free group of rank $n > 1$ and $S(G)$ is *Schottky space*. The Theorem colloquially states that for “most” $PSL_2(\mathbb{C})$ Schottky embeddings of G , the resulting groups have no pair of distinct translation axes that lie on a common plane $\mathbb{H}^2$.

Theorem 4.1. *There exists a subset $R \subset S(G)$ such that $\text{int}(R) = \emptyset$ and for all $\sigma \in S(G) \setminus R$ and any pair of non-commuting words $w_1, w_2 \in G$, $\text{Fix}(\sigma(w_1)) \cup \text{Fix}(\sigma(w_2))$ is not contained in a circle.*

In next two subsections we review some basics about Schottky groups and Schottky space. Then we classify pairs of words in the free group based on a geometric criterion. Finally, we prove Theorem 4.1.

4.1 Schottky Groups

Let D^- and D^+ be the closure of two disjoint simply connected domains in $\hat{\mathbb{C}}$, where $\partial D^\pm$ is a Jordan curve. A loxodromic $\gamma \in PSL_2(\mathbb{C})$ pairs D^- to D^+ if the following two conditions hold:

- $\gamma(\partial D^-) = \partial D^+$
- $\gamma(int(D^-)) = \hat{\mathbb{C}} \setminus D^+$

If $\{D_i^\pm\}_{i=1}^n$ is a collection of closures of mutually disjoint simply connected domains, and γ_i is a loxodromic that pairs D_i^- to D_i^+ , then $\langle \gamma_1, \dots, \gamma_n \rangle$ is a *Schottky group*. If each $D_i^\pm$ in the previous sentence is a geometric disk, then $\langle \gamma_1, \dots, \gamma_n \rangle$ is a *classical Schottky group*.

It is a simple matter to construct classical Schottky groups. If D^- and D^+ are disks with centers d^- and d^+ , and radii r and s , then

$$z \mapsto \frac{rs}{z - d^-} + d^+$$

is a Möbius transformation taking ∂D^- to ∂D^+ and mapping $int(D^-)$ onto $\hat{\mathbb{C}} \setminus D^+$. Then

$$\frac{-1}{\sqrt{rs}} \begin{pmatrix} d^+ & rs - d^- d^+ \\ 1 & -d^- \end{pmatrix} \in PSL_2(\mathbb{C})$$

will be a loxodromic pairing D^- to D^+ when $|d^- - d^+| > 2\sqrt{rs}$. So if $\{D_i^\pm\}_{i=1}^n$ is any collection of pairwise disjoint disks in $\mathbb{C}$ such that the distance between the centers is much larger than their respective radii, then there exists a classical Schottky group $\langle \gamma_1, \dots, \gamma_n \rangle$ where γ_i pairs D_i^- to D_i^+ . In particular, for all $n \in \mathbb{N}$, there are classical Schottky groups whose generators pair disks with radii less than $1/n$.

Schottky groups are free, discrete, and purely loxodromic. For any loxodromic γ , let γ^+ and γ^- denote the attracting and repelling fixed points of γ and let $Fix(\gamma) := \{\gamma^\pm\}$. From the definitions it is clear that if γ pairs D^- to D^+ , then $\gamma^\pm \in D^\pm$. In later arguments, for words ω in a Schottky group $\langle \gamma_1, \dots, \gamma_n \rangle$ where γ_i pairs D_i^- to D_i^+ , we will need to specify which disk in the collection $\{D_i^\pm\}_{i=1}^n$ contains ω^+ and ω^- . For this

reason, we introduced the following notation.

Let $\Gamma = \langle \gamma_1, \dots, \gamma_n \rangle$ be a Schottky group where γ_i pairs D_i^- to D_i^+ . We let $D^\pm(\gamma_i) := D_i^\pm$. Put $D^+(\gamma_i^{-1}) := D^-(\gamma_i)$ and $D^-(\gamma_i^{-1}) := D^+(\gamma_i)$. The collection $\{D_i\}_{i=1}^n$ is not unique to Γ . So $D^\pm(\gamma_i)$ just refers to any collection of disks paired by the Schottky generators.

Since the attracting/repelling fixed point of γ is the repelling/attracting fixed point of γ^{-1} , one can verify via induction the following basic but crucial observation:

Lemma 4.1. *Let $\omega = \gamma_{i_1}^{m_1} \dots \gamma_{i_k}^{m_k} \in \Gamma$, with $m_i \in \mathbb{Z} \setminus \{0\}$, be a freely reduced word. Let $\text{sgn}(m) = +1$ if $m > 0$ and $\text{sgn}(m) = -1$ if $m < 0$. Then we have $\omega^+ \in D^+(\gamma_{i_1}^{\text{sgn}(m_1)})$ and $\omega^- \in D^+(\gamma_{i_k}^{-\text{sgn}(m_k)})$.*

While there is no canonical choice of disks for Γ , all statements and arguments referring to $D^\pm(\gamma_{i_k})$ will hold for any collection of disks satisfying the Schottky definition, and will not require choices.

4.2 Schottky Space

From now on, let $G = \langle g_1, \dots, g_n \rangle$ denote a fixed rank n free group with $\{g_1, \dots, g_n\}$ a fixed ordered set of generators. *Schottky space*, denoted $S(G)$, is the set of all $PSL_2(\mathbb{C})$ conjugacy classes of embeddings $\sigma : G \rightarrow PSL_2(\mathbb{C})$ such that its image

$$\sigma(G) = \langle \sigma(g_1), \dots, \sigma(g_n) \rangle$$

is a Schottky group. It is a classical result of Bers [Ber75] that Schottky space can be identified with a connected open subset of $\mathbb{C}^{3n-3}$. We review some details of this identification.

For any point $[\sigma]$ in Schottky space, we can take a representative σ where $\sigma(g_1)^+ = -1$, $\sigma(g_2)^- = 0$, and $\sigma(g_2)^+ = 1$. Then σ is determined by the squared traces of the generators, and the remaining fixed points of the generators which have yet to be specified. Then the following embedding identifies Schottky space with a domain in $\mathbb{C}^{3n-3}$:

$$[\sigma] \mapsto (tr^2(\sigma(g_1)), tr^2(\sigma(g_2)), \sigma(g_2)^-, \dots, tr^2(\sigma(g_n)), \sigma(g_n)^+, \sigma(g_n)^-).$$

From this point forward, we identify Schottky space $S(G)$ with a domain in $\mathbb{C}^{3n-3}$ as described above. We abuse notation so that $\sigma \in S(G)$ refers to both a point in Schottky space under this identification, and an embedding $\sigma \rightarrow PSL_2(\mathbb{C})$ such that $\langle \sigma(g_1), \dots, \sigma(g_n) \rangle$ is a Schottky group with $\sigma(g_1)^+ = -1$, $\sigma(g_1)^- = 0$, and $\sigma(g_2)^+ = 1$.

We conclude this section with some remarks on holomorphic functions defined on simply connected open subsets of $S(G)$. As a warm-up, recall that for any simply connected domain $\Omega \subset \mathbb{C}^k$, if $f : \Omega \rightarrow \mathbb{C}$ is holomorphic and non-vanishing, then $\ln f$ is also holomorphic. For such f , $e^{\ln f/m}$ will also be holomorphic, so that f has holomorphic m th roots for all m . In particular, $\sqrt{f}$ is a holomorphic function on Ω .

Now suppose $\Omega \subset S(G)$ is a simply connected open subset of $S(G)$ (note that Schottky space itself is not simply connected [IY06]). Let g_i be one of the fixed generators of G . The entries of $\sigma(g_i)$ can be written as rational functions in $\sigma^\pm(g_i)$ and $\sqrt{tr^2\sigma(g_i)}$. Notice that for all $\sigma \in S(G)$, the quantities $(\sigma(g_i)^+ - \sigma(g_i)^-)$, $tr\sigma(g_i)$, and $tr^2\sigma(g_i) - 4$ are never 0 because $\sigma(G)$ is purely loxodromic and discrete.

For any non-identity element $w \in G$, we conclude $\sigma \mapsto \sigma^+(w) - \sigma^-(w)$ defines a (necessarily non-vanishing) holomorphic function on Ω , as the entries of $\sigma(w)$ are $\mathbb{Z}$ -polynomials in the entries of the generators $\sigma(g_i)$, and the fixed points of any

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{C})$$

are given by

$$\frac{a - d \pm \sqrt{(a + d)^2 - 4}}{2c}.$$

Later, for fixed pairs of non-commuting words $w_1, w_2 \in G$, we will define functions mapping $\sigma \in \Omega$ to the cross ratio of the fixed points of $\sigma(w_1), \sigma(w_2)$. By this discussion, such functions are holomorphic. See [YT13] for more details on how fixed point cross ratios have been used to study Schottky space.

4.3 Classifying Pairs of Words

Given a $\sigma \in S(G)$ and two freely reduced non-commuting words $w_i, w_j \in G$, we will need to know which $D^\pm(g_k)$ contains which fixed points of $\sigma(w_i)$ and $\sigma(w_j)$, and whether or not some $D^\pm(g_k)$ contains multiple points in $Fix(\sigma(w_i)) \cup Fix(\sigma(w_j))$. By Lemma 4.1, this depends only on what the left-most and right-most letters of the words are. We now classify pairs of freely reduced non-commuting words in G with this goal in mind. Afterwards, we explain more explicitly the geometric intuition behind the classification. Recall that a word is *cyclically reduced* if its left-most and right-most letters are not inverses of each other.

Definition 4.1. Let $w_i, w_j \in G$ be two freely reduced non-commuting words.

1. Say w_i, w_j is a *type 1 pair* if both words are cyclically reduced and both w_i and w_i^{-1} have different left-most letters with w_j and w_j^{-1} .
2. Say w_i, w_j is a *type 2 pair* if either of the following occurs:

- (a) w_i is not cyclically reduced and has a different left-most letter from w_j and w_j^{-1} .
 - (b) w_i and w_j are both cyclically reduced and the pair is not of type 1.
3. Say w_i, w_j is a *type 3 pair* if w_i is not cyclically reduced, w_j is cyclically reduced, and w_i has the same left-most letter as w_j or w_j^{-1} (but not both).
 4. Say w_i, w_j is a *type 4 pair* if neither word is cyclically reduced and w_i has the same left-most (and hence same right-most letters) with either w_j or w_j^{-1} .

We now explain the geometric intuition behind the classification. Let $\sigma \in S(G)$. Say $w_1, w_2 \in G$ is a pair of freely reduced non-commuting words. If w_1, w_2 is a

- type 1 pair, then each $D^\pm(\sigma(g_i))$ contains at most one point in $Fix(\sigma(w_1)) \cup Fix(\sigma(w_2))$.
- type 2 pair, then some $D^\pm(\sigma(g_k))$ contains exactly two points in $Fix(\sigma(w_1)) \cup Fix(\sigma(w_2))$.
- type 3 pair, then some $D^\pm(\sigma(g_k))$ contains exactly three points in $Fix(\sigma(w_1)) \cup Fix(\sigma(w_2))$.
- type 4 pair, then some $D^\pm(\sigma(g_k))$ contains $Fix(\sigma(w_1)) \cup Fix(\sigma(w_2))$.

While the disks $D^\pm(\sigma(g_k))$ are not unique to each $\sigma(G)$, any collection of disks satisfying the Schottky group definition will also satisfy the containment properties listed above. Moreover, for type 2 pairs, when $D^\pm(\sigma(g_k))$ contains two points in $Fix(\sigma(w_1)) \cup Fix(\sigma(w_2))$ it will contain the same 2 fixed points for all $\sigma \in S(G)$. For example, if w_1, w_2 is a type 2 pair and (say) for some $\sigma_0 \in S(G)$ we have

$$\{\sigma_0(w_1)^+, \sigma_0(w_2)^+\} = (Fix(\sigma_0(w_1)) \cup Fix(\sigma_0(w_2))) \cap D^+(\sigma_0(g_1)),$$

then for all $\sigma \in S(G)$ we will have

$$\{\sigma(w_1)^+, \sigma(w_2)^+\} = (Fix(\sigma(w_1)) \cup Fix(\sigma(w_2))) \cap D^+(\sigma(g_1)).$$

This geometric picture helps one see that all pairs of non-commuting freely reduced words are indeed of one of the types listed above, since under a Schottky embedding the fixed points of any word must be in one of the Schottky disks. The four types account for all possible ways the 4 fixed points $Fix(\sigma(w_1)) \cup Fix(\sigma(w_2))$ can be distributed among the disks $\{D^\pm(\sigma(g_i))\}$.

4.4 Proofs

Let $w_1, w_2 \in G$ be a pair of freely reduced non-commuting words. We will show there exists $\sigma \in S(G)$ such that $Fix(\sigma(w_1)) \cup Fix(\sigma(w_2))$ is not contained in a line. Our argument will be divided into cases based on what type of pair w_1, w_2 is. The next lemma will help us reduce the argument to word pairs of type 1 and 2.

Lemma 4.2. *If $w_1, w_2 \in G$ is a pair of type 3 or 4, then there exists $g \in G$ such that gw_1g^{-1}, gw_2g^{-1} is a pair of type 1 or 2 after free reduction.*

Proof. First suppose the pair is of type 3. Say w_1 is not cyclically reduced, so $w_1 = gw'_1g^{-1}$ for some $g, w'_1 \in G$, where gw'_1g^{-1} is freely reduced and w'_1 is cyclically reduced. Then w_2 is cyclically reduced. Without loss of generality we suppose w_1 and w_2 have the same left-most letter.

Write $g = g_{i_1}^{m_1} \cdots g_{i_r}^{m_r}$, $m_j \in \mathbb{Z} \setminus \{0\}$. Let $sgn(m_j) = +1$ if $m_j > 0$ and $sgn(m_j) = -1$ if $m_j < 0$. We claim that after free reduction of $g^{-1}w_2g$, the pair $w'_1, g^{-1}w_2g$ is of type 1 or 2. It cannot be a type 4 pair because w'_1 is cyclically reduced, so it suffices to show the pair is not of type 3.

Suppose for contradiction that $w'_1, g^{-1}w_2g$ is a type 3 pair. Since w'_1 is cyclically reduced and the pair is of type 3, then after free reduction $g^{-1}w_2g$ is not a cyclically reduced word, and it has either the same left-most letter as w'_1 , or the same right-most letter as w'_1 . We start by arguing the pair does not have the same right-most letter.

Recall that w_2 is cyclically reduced. Also $w_1 = gw'_1g^{-1}$ and w_2 share the same left-most letter, which is $g_{i_1}^{sgn(m_1)}$. This means $g_{i_1}^{-sgn(m_1)}$ cannot be the right-most letter of w_2 . Hence the word

$$w_2g = w_2g_{i_1}^{m_1} \cdots g_{i_r}^{m_r}$$

is freely reduced. But the right-most letter of w'_1 is not $g_{i_r}^{sgn(m_r)}$ because $w_1 = gw'_1g^{-1}$ is freely reduced.

Therefore, w'_1 and $g^{-1}w_2g$ must share the same left-most letter after free reduction of $g^{-1}w_2g$. But the left-most letter of w'_1 is not $g_{i_r}^{-sgn(m_r)}$ because $w_1 = gw'_1g^{-1}$ is freely reduced. This means after free reduction of $g^{-1}w_2g$, the entire g^{-1} word must have been eliminated. That is, after free reduction we have $g^{-1}w_2g = w'_2g$ for some freely reduced word w'_2 . As argued in the previous paragraph, w'_2g is freely reduced. By assumption, w'_2g is not cyclically reduced, so the left-most letter of w'_2g is $g_{i_r}^{-sgn(m_r)}$. But again, the left-most letter of w'_1 cannot be $g_{i_r}^{-sgn(m_r)}$. This gives the desired contradiction.

Finally, if the pair is of type 4, we can let $g \in G$ be such that gw_1g^{-1} is cyclically reduced after free reduction. This means gw_1g^{-1}, gw_2g^{-1} is not a type 4 pair after free reduction. If it is type 1 or 2, then we are done. If it is of type 3, then we conjugate further as in the previous case. $\square$

For w_1, w_2 a pair of freely reduced non-commuting words in G , put

$$R(w_1, w_2) := \{\sigma \in S(G) : \text{Fix}(\sigma(w_1)) \cup \text{Fix}(\sigma(w_2)) \text{ is contained in a circle}\}.$$

Our next task is to show $R(w_1, w_2)$ has empty interior. To do this, we will need to exhibit two Schottky embeddings $\sigma', \sigma'' \in S(G)$ where the cross ratio of the fixed points of $\sigma'(w_1), \sigma'(w_2)$ is different from the cross ratio of the fixed points of $\sigma''(w_1), \sigma''(w_2)$. The candidate Schottky embeddings we will consider will be a sequence of classical Schottky groups whose generators pair disks with successively smaller diameter, shrinking to zero.

To construct this sequence, fix $2n$ distinct points $\{d_i^\pm\}_{i=1}^{2n}$ in $\mathbb{C}$ which are pairwise distance (say) at least 100 apart. For $t \in \mathbb{N}$, let $D_i^\pm[t]$ be a disk centered at $d_i^\pm$ with diameter less than $1/t$. Let $\sigma_t : G \rightarrow PSL_2(\mathbb{C})$ be an embedding such that $\sigma_t(G)$ is a classical Schottky group where $\sigma_t(g_i)$ pairs $D_i^-[t]$ to $D_i^+[t]$. Put $D_t^\pm(\sigma_t(g_i)) := D_i^\pm[t]$ and $D_t^\pm(\sigma_t(g_i^{-1})) := D_i^\mp[t]$. The representations σ_t are not yet points in our model of Schottky space $S(G)$ because the fixed points of $\sigma_t(g_1)$ and $\sigma_t(g_2)$ have not yet been normalized, but as will be clear later on, we can conjugate the representations to get the desired points in $S(G)$. We summarize the properties about $\{\sigma_t\}_{t=1}^\infty$ that we need for future arguments in the following Lemma. The absolute value refers to the standard Euclidean distance as points in $\mathbb{C}$.

Lemma 4.3. *The sequence $\{\sigma_t(G)\}$ of classical Schottky groups satisfies the following two properties:*

1. *There are universal constants $c_1, c_2 \in (0, \infty)$ such that if $x \in D_t^\pm(\sigma_t(g_i))$, $y \in D_t^\pm(\sigma_t(g_j))$, and $D_t^\pm(\sigma_t(g_i)) \neq D_t^\pm(\sigma_t(g_j))$, then $c_1 < |x - y| < c_2$.*
2. *For all generators g_i , we have $\lim_{t \rightarrow \infty} \text{Diam}(D_t^\pm(\sigma_t(g_i))) = 0$*

The first property of Lemma 4.3 comes from the fact the centers are fixed during the construction of $\{\sigma_t\}_{t=1}^\infty$.

Proposition 4.1. *For any pair of freely reduced non-commuting words $w_1, w_2 \in G$, $R(w_1, w_2)$ has empty interior in $S(G)$.*

Proof. By Lemma 4.2, pairs of type 3 or 4 are conjugate in G to a pair of type 1 or 2. For any $w, g \in G$ and $\sigma \in S(G)$, we have $\text{Fix}(\sigma(gwg^{-1})) = \sigma(g)\text{Fix}(\sigma(w))$, so it suffices to consider pairs of type 1 or 2. We consider cases based on the pair type.

Case 1: Say w_1, w_2 is a type 2 pair where (say) w_1 is not cyclically reduced. For any $\sigma : G \rightarrow PSL_2(\mathbb{C})$ where $\langle \sigma(g_1), \dots, \sigma(g_n) \rangle$ is a Schottky group, put

$$\begin{aligned} \chi_{w_1, w_2}(\sigma) &:= [\sigma^+(w_1), \sigma^+(w_2), \sigma^-(w_1), \sigma^-(w_2)] \\ &= \frac{(\sigma^-(w_1) - \sigma^+(w_1))(\sigma^-(w_2) - \sigma^+(w_2))}{(\sigma^-(w_1) - \sigma^+(w_2))(\sigma^-(w_2) - \sigma^+(w_1))}. \end{aligned}$$

That is, $\chi_{w_1, w_2}(\sigma)$ is the cross ratio of the fixed points of $\sigma(w_1), \sigma(w_2)$. We will now show there exists $\sigma', \sigma'' \in S(G)$ such that $\chi_{w_1, w_2}(\sigma') \neq \chi_{w_1, w_2}(\sigma'')$.

Let $\{\sigma_t\}_{t=1}^\infty$ be the sequence of Schottky embeddings as in Lemma 4.3. Since w_1, w_2 is a type 2 pair with w_1 not cyclically reduced, there is a g_j in the fixed set of generators $\{g_1, \dots, g_n\}$ of G , and $s \in \{\pm 1\}$, such that $\sigma_t^\pm(w_1) \in D_t^+(\sigma_t(g_j^s))$ for all t . Moreover, $\sigma^\pm(w_2) \notin D_t^+(\sigma_t(g_j^s))$. Hence, property 2 of the sequence $\{\sigma_t\}_{t=1}^\infty$ implies

$$\lim_{t \rightarrow \infty} |\sigma_t^-(w_1) - \sigma_t^+(w_1)| = 0.$$

The term $|\sigma_t^-(w_2) - \sigma_t^+(w_2)|$ either converges to 0 or remains bounded above as $t \rightarrow \infty$, depending on whether or not w_2 is also cyclically reduced. Meanwhile, by property 1 of the sequence $\{\sigma_t\}_{t=1}^\infty$, there are constants $c_1, c_2 \in (0, \infty)$ such that

$$c_1 < |\sigma_t^-(w_1) - \sigma_t^+(w_2)|, |\sigma_t^-(w_2) - \sigma_t^+(w_1)| < c_2$$

for all t . We conclude

$$\lim_{t \rightarrow \infty} |\chi_{w_1, w_2}(\sigma_t)| = 0$$

so there must be $\sigma_{t'}, \sigma_{t''}$ in $\{\sigma_t\}_{t=1}^\infty$ with $\chi_{w_1, w_2}(\sigma_{t'}) \neq \chi_{w_1, w_2}(\sigma_{t''})$. There exists $\gamma \in PSL_2(\mathbb{C})$ such that $\gamma\sigma_{t'}(g_1)^+\gamma^{-1} = -1$, $\gamma\sigma_{t'}(g_1)^-\gamma^{-1} = 0$, and $\gamma\sigma_{t'}(g_2)^+\gamma^{-1} = 1$. Let $\sigma' \in S(G)$ be such that $\sigma'(w) = \gamma\sigma_{t'}(w)\gamma^{-1}$ for all $w \in G$. Cross ratios are invariant under Möbius transformations, and $(\gamma\sigma_{t'}(w)\gamma^{-1})^\pm = \gamma(\sigma_{t'}(w))^\pm$, so we have

$$\chi_{w_1, w_2}(\sigma') = [\gamma\sigma_{t'}^+(w_1), \gamma\sigma_{t'}^+(w_2), \gamma\sigma_{t'}^-(w_1), \gamma\sigma_{t'}^-(w_2)] = \chi_{w_1, w_2}(\sigma_{t'}).$$

Obtain σ'' from a $PSL_2(\mathbb{C})$ conjugation of $\sigma_{t''}$ in the same way. Then σ', σ'' are points in $S(G)$ such that $\chi_{w_1, w_2}(\sigma') \neq \chi_{w_1, w_2}(\sigma'')$, as desired.

Now for any $\sigma \in S(G)$, since $S(G)$ is a connected open subset of $\mathbb{C}^{3n-3}$, there exists a smoothly embedded arc joining σ, σ' and σ'' . Let Ω_σ be a tubular neighborhood of such an arc. Then we see $\{\Omega_\sigma\}_{\sigma \in S(G)}$ is an open cover of $S(G)$ by simply connected subsets, each of which contains σ' and σ'' . Manifolds are Lindelof so we can pass to a countable sub-cover, which we denoted by $\{\Omega_m\}_{m=1}^\infty$.

On each Ω_m , define $\chi_{w_1, w_2}|_{\Omega_m} : \Omega_m \rightarrow \mathbb{C}$ by $\chi_{w_1, w_2}|_{\Omega_m}(\sigma) := \chi_{w_1, w_2}(\sigma)$. Then $\chi_{w_1, w_2}|_{\Omega_m}$ is a non-constant holomorphic function. By the Open Mapping Theorem for several complex variables (see [KW17]), $\chi_{w_1, w_2}|_{\Omega_m}^{-1}(\mathbb{R})$ has empty interior. Since the cross ratio of four points is real if and only if they lie on a circle, we have that for all $\sigma \in \Omega_m \setminus \chi_{w_1, w_2}|_{\Omega_m}^{-1}(\mathbb{R})$, $Fix(\sigma(w_1)) \cup Fix(\sigma(w_2))$ does not lie on a circle. That is,

$$R(w_1, w_2) = \bigcup_{m=1}^\infty \chi_{w_1, w_2}|_{\Omega_m}^{-1}(\mathbb{R}),$$

and since each $\chi_{w_1, w_2}|_{\Omega_m}^{-1}(\mathbb{R})$ is closed and has empty interior, the Baire Category Theorem (for locally compact Hausdorff spaces) implies $R(w_1, w_2)$ has empty interior as desired.

Case 2: Now suppose w_1, w_2 is a type 2 pair with both words cyclically reduced. Furthermore, replacing either word with their inverses if necessary, we assume they have the

same left-most letter. Put

$$\begin{aligned}\chi_{w_1, w_2}(\sigma) &:= [\sigma^+(w_1), \sigma^-(w_1), \sigma^+(w_2), \sigma^-(w_2)] \\ &= \frac{(\sigma^+(w_2) - \sigma^+(w_1)) (\sigma^-(w_2) - \sigma^-(w_1))}{(\sigma^+(w_2) - \sigma^-(w_1)) (\sigma^-(w_2) - \sigma^+(w_1))}.\end{aligned}$$

Let σ_t denote the same sequence of representations from Lemma 4.3 as in case 1. We will show there exists $\sigma_{t'}, \sigma_{t''} \in \{\sigma_t\}_{t=1}^\infty$ such that $\chi_{w_1, w_2}(\sigma_{t'}) \neq \chi_{w_1, w_2}(\sigma_{t''})$. This suffices to give the desired result because the same reasoning as in case 1, in the part after $\sigma_{t'}$ and $\sigma_{t''}$ were found, applies verbatim.

Under the current assumptions of this case, there is a g_l in the fixed set of generators $\{g_1, \dots, g_n\}$ for G and $s \in \{\pm 1\}$ such that $\sigma_t^+(w_1), \sigma_t^+(w_2) \in D_t^+(g_l^s)$ for all t . This means

$$\lim_{t \rightarrow \infty} |\sigma_t^+(w_2) - \sigma_t^+(w_1)| = 0,$$

while $|\sigma_t^-(w_2) - \sigma_t^-(w_1)|$ either remains bounded above or decays to 0 as $t \rightarrow \infty$. Neither $\{\sigma^+(w_2), \sigma^-(w_1)\}$ nor $\{\sigma^-(w_2), \sigma^+(w_1)\}$ can be contained in a single $D_t^\pm(\sigma_t(g_l))$ for any σ_t or generator g_l because w_1, w_2 is a type 2 pair and $\{\sigma^+(w_2), \sigma^+(w_1)\}$ always lies in $D_t^+(g_l^s)$ for all t all σ_t . Hence the terms $|\sigma^+(w_2) - \sigma^-(w_1)|$ and $|\sigma^-(w_2) - \sigma^+(w_1)|$ must be bounded below strictly away from 0 for all t . From this we conclude

$$\lim_{t \rightarrow \infty} |\chi_{w_1, w_2}(\sigma_t)| = 0,$$

so there must be $\sigma', \sigma'' \in S(G)$ with $\chi_{w_1, w_2}(\sigma') \neq \chi_{w_1, w_2}(\sigma'')$.

Case 3: Finally, suppose w_1, w_2 is a type 1 pair. Put

$$\begin{aligned}\chi_{w_1, w_2}(\sigma) &:= [\sigma^+(w_1), \sigma^-(w_1), \sigma^+(w_2), \sigma^-(w_2)] \\ &= \frac{(\sigma^+(w_2) - \sigma^+(w_1)) (\sigma^-(w_2) - \sigma^-(w_1))}{(\sigma^+(w_2) - \sigma^-(w_1)) (\sigma^-(w_2) - \sigma^+(w_1))}.\end{aligned}$$

As before, it suffices to find $\sigma', \sigma'' \in S(G)$ such that $\chi_{w_1, w_2}(\sigma') \neq \chi_{w_1, w_2}(\sigma'')$. Observe that if $\sigma \in S(G)$ is such that $\sigma(G)$ is Fuchsian, then $\chi_{w_1, w_2}(\sigma) \in \mathbb{R}$ and $\chi_{w_1, w_2}(\sigma) \in (-\infty, 0)$ if and only if $Fix(\sigma(w_1))$ separates $Fix(\sigma(w_2))$ i.e. if and only if the translation axes of $\sigma(w_1)$ has non-empty intersection with the translation axes of $\sigma(w_2)$.

Since w_1, w_2 is a type 1 pair, for all $\sigma \in S(G)$, each point in $Fix(\sigma(w_1)) \cup Fix(\sigma(w_2))$ always lies in a different Schottky disk of $\sigma(G)$. Then it is easy to use a Ping Pong argument to obtain $\sigma', \sigma'' \in S(G)$ where $\chi_{w_1, w_2}(\sigma') \in (-\infty, 0)$ and $\chi_{w_1, w_2}(\sigma'') \in (0, \infty)$.

□

Now we prove the main Theorem.

Proof. Let

$$R := \bigcup_{w_1 w_2 \neq w_2 w_1} R(w_1, w_2),$$

where $R(w_1, w_2)$ is defined as in Proposition 4.1 and the union is over all pairs of non-commuting words in G . This is a countable union. It is clear each $R(w_1, w_2)$ is closed, and each $R(w_1, w_2)$ has empty interior by Proposition 4.1, so R also has empty interior by the Baire category theorem for locally compact Hausdorff spaces. For any $\sigma \in S(G) \setminus R$, $Fix(\sigma(w_1)) \cup Fix(\sigma(w_2))$ is not contained in a line for any pair of non-commuting words $w_1, w_2 \in G$, by construction of R .

□

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