UC Berkeley
UC Berkeley Electronic Theses and Dissertations

Title
Some Minorants and Majorants of Random Walks and Lévy Processes

Permalink
https://escholarship.org/uc/item/49c231p6

Author
Abramson, Joshua Simon

Publication Date
2012

Peer reviewed|Thesis/dissertation

eScholarship.org Powered by the California Digital Library
University of California

Some Minorants and Majorants of Random Walks and Lévy Processes

by

Joshua Simon Abramson

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Statistics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Steven Neil Evans, Chair
Professor James Pitman
Professor Fraydoun Rezakhanlou

Fall 2012

Some Minorants and Majorants of Random Walks and Lévy Processes

Copyright 2012
by
Joshua Simon Abramson

Abstract

Some Minorants and Majorants of Random Walks and Lévy Processes

by

Joshua Simon Abramson

Doctor of Philosophy in Statistics

University of California, Berkeley

Professor Steven Neil Evans, Chair

This thesis consists of four chapters, all relating to some sort of minorant or majorant of random walks or Lévy processes.

In Chapter 1 we provide an overview of recent work on descriptions and properties of the convex minorant of random walks and Lévy processes as detailed in Chapter 2, [72] and [73]. This work rejuvenated the field of minorants, and led to the work in all the subsequent chapters. The results surveyed include point process descriptions of the convex minorant of random walks and Lévy processes on a fixed finite interval, up to an independent exponential time, and in the infinite horizon case. These descriptions follow from the invariance of these processes under an adequate path transformation. In the case of Brownian motion, we note how further special properties of this process, including time-inversion, imply a sequential description for the convex minorant of the Brownian meander. This chapter is based on [3], which was co-written with Jim Pitman, Nathan Ross and Geronimo Uribe Bravo.

Chapter 1 serves as a long introduction to Chapter 2, in which we offer a unified approach to the theory of concave majorants of random walks. The reasons for the switch from convex minorants to concave majorants are discussed in Section 1.1, but the results are all equivalent. This unified theory is arrived at by providing a path transformation for a walk of finite length that leaves the law of the walk unchanged whilst providing complete information about the concave majorant – the path transformation is different from the one discussed in Chapter 1, but this is necessary to deal with a more general case than the standard one as done in Section 2.6. The path transformation of Chapter 1, which is discussed in detail in Section 2.8, is more relevant to the limiting results for Lévy processes that are of interest in Chapter 1. Our results lead to a description of a walk of random geometric length as a Poisson point process of excursions away from its concave majorant, which

is then used to find a complete description of the concave majorant of a walk of infinite length. In the case where subsets of increments may have the same arithmetic mean (the more general case mentioned above), we investigate three nested compositions that naturally arise from our construction of the concave majorant. This chapter is based on [4], which was co-written with Jim Pitman.

In Chapter 3, we study the Lipschitz minorant of a Lévy process. For $\alpha > 0$, the α-Lipschitz minorant of a function $f : \mathbb{R} \to \mathbb{R}$ is the greatest function $m : \mathbb{R} \to \mathbb{R}$ such that $m \le f$ and $|m(s) - m(t)| \le \alpha|s - t|$ for all $s, t \in \mathbb{R}$, should such a function exist. If $X = (X_t)_{t\in\mathbb{R}}$ is a real-valued Lévy process that is not pure linear drift with slope $\pm\alpha$, then the sample paths of X have an α-Lipschitz minorant almost surely if and only if $|\mathbb{E}[X_1]| < \alpha$. Denoting the minorant by M, we investigate properties of the random closed set $\mathcal{Z} := \{t \in \mathbb{R} : M_t = X_t \wedge X_{t-}\}$, which, since it is regenerative and stationary, has the distribution of the closed range of some subordinator "made stationary" in a suitable sense. We give conditions for the contact set $\mathcal{Z}$ to be countable or to have zero Lebesgue measure, and we obtain formulas that characterize the Lévy measure of the associated subordinator. We study the limit of $\mathcal{Z}$ as $\alpha \to \infty$ and find for the so-called abrupt Lévy processes introduced by Vigon that this limit is the set of local infima of X. When X is a Brownian motion with drift β such that $|\beta| < \alpha$, we calculate explicitly the densities of various random variables related to the minorant. This chapter is based on [2], which was co-written with Steven N. Evans.

Finally, in Chapter 4 we study the structure of the shocks for the inviscid Burgers equation in dimension 1 when the initial velocity is given by Lévy noise, or equivalently when the initial potential is a two-sided Lévy process ψ_0 (the change in notation for a Lévy process from Chapter 3 is necessitated by the change from a temporal to a spatial independent variable). This shock structure turns out to give rise to a parabolic minorant of the Lévy process – see Section 4.2 for details. The main results are that when ψ_0 is abrupt in the sense of Vigon or has bounded variation with $\limsup_{|h|\downarrow 0} h^{-2}\psi_0(h) = \infty$, the set of points with zero velocity is regenerative, and that in the latter case this set is equal to the set of Lagrangian regular points, which is non-empty. When ψ_0 is abrupt the shock structure is discrete and when ψ_0 is eroded there are no rarefaction intervals. This chapter is based on [1].

To Sarah Steventon,

whose patience never paid off, but whose love, support, care, friendship and person I will forever treasure, and without whom so many of the events leading to the inception and successful completion of this thesis would not have happened.

Contents

Acknowledgments

Firstly, I would like to thank my advisor Steve Evans. Right from my initial contact with Steve during the application process for Berkeley, and throughout the many hours of concentrated staring at the board in his office and the countless lunches at sunny tables, it has been a pleasure and an honor to work with him. Steve is easy to be around and full of interest in the world around him. He is patient and humble yet worldly and exciting. In research, Steve showed me that you cannot rush or else something will be missed, and that you must learn of the tools available to you, or else your route will be impassable. In the classroom, Steve taught me for six semesters in a row, and the variety of content and thoroughness of presentation was nothing short of spectacular. Much of the work we did together over my first two years was in the end unfruitful, but the experience was essential to my future success; for that, and for much more, I am greatly indebted to Steve.

Next, I would like to acknowledge Jim Pitman, who has been like a second advisor to me. Jim's passion for stochastic processes is infectious, and his cheery nature makes him a delight to work with. Moreover Jim's mastery over the core of stochastic process theory means that every interaction with him leads to an insight gained. I am very thankful to him for letting me work with him on the convex minorant project, during which I not only learnt how to properly write a paper, but also picked up an interest in the topic of this thesis: minorants and majorants of stochastic processes. Thank you Jim, and I wish you all the best with your project to reform and greatly improve the way we publish, archive and disseminate scientific papers.

I would also like to thank my other co-authors in the convex minorant project, Nathan Ross and Geronimo Uribe Bravo. The research was made considerably easier and more exciting by having such knowledgeable people to comfortably bounce ideas off. Nathan deserves a special mention for being a pillar of friendship and support within the department right from day one, when we set off as a rookie lecturer and a rookie GSI into the unknown depths of a graduate course. Teaching for Nathan, working with Nathan and being lectured by Nathan were all equally enjoyable and I'm sure he has a bright future ahead of him.

My path to Berkeley was made possible by ESPRC funding for my masters program in the UK, and Berkeley itself would not have been possible without the Loeve fellowship for my first year, for which I am very grateful to David Aldous and the rest of the fellowship committee. I am also indebted to the department for their continued support in the form of GSI positions. The department was a stimulating and pleasant environment to work, and in particular I will long remember coffee hour discussions with luminaries such as Roger Purves and Howard d'Abrera.

The town of Berkeley has been a wonderful place to spend the last few years, and its impact on my life cannot be underestimated. I value so much the support and friendship from my housemates and others over this time and look forward to the time we will no doubt spend together in the future. I also am thankful for the continued friendship of everyone back in London, especially to my patient parents.

Two professors from my time at Cambridge prior to Berkeley deserve special mention. Julia Gog, who gave me the confidence to go on with mathematics, and Nathanael Berestycki, whose exciting masters thesis project and general enthusiasm led me to study probability rather than statistics.

Finally, I am very grateful for the detailed comments and helpful suggestions of the various anonymous referees that have had the difficult task of correcting my work, and Fraydoun Rezakhanlou for agreeing to be the external member of my thesis committee.

Chapter 1

Convex Minorants of Random Walks and Lévy Processes

1.1 Introduction

The *greatest convex minorant* (or simply convex minorant for short) of a real-valued function $(x_u, u \in U)$ with domain U contained in the real line is the maximal convex function $(\underline{c}_u, u \in I)$ defined on a closed interval I containing U with $\underline{c}_u \leq x_u$ for all $u \in U$. A number of authors have provided descriptions of certain features of the convex minorant for various stochastic processes such as random walks [6, 21, 42, 86], Brownian motion [23, 26, 45, 71, 89, 11], Cauchy processes [12], Markov Processes [55], and Lévy processes (Chapter XI of [65]). Figure 1.1 illustrates an instance of the convex minorant for each of a random walk, a Brownian motion, and a Cauchy process on a finite interval.

The recent articles [4, 72, 73] provide a relatively complete description of the convex minorant of random walks, Brownian Motion, and Lévy processes respectively which not only encompass much of the somewhat ad hoc previous work on convex minorants, but also provide new tools to derive previously unknown properties of such convex minorants. Chapter 2 of this thesis covers the random walk case and is the basis for one of those three articles ([4]). The purpose of this chapter is to provide an overview of all three works in order to demonstrate the strength of the recent revival in investigations into minorants and in order to put the work of Chapter 2 in context. As it is a summary, we will focus on stating results in this chapter in a streamlined fashion, referring to Chapter 2, [73] and [72] where needed to furnish details and proofs.

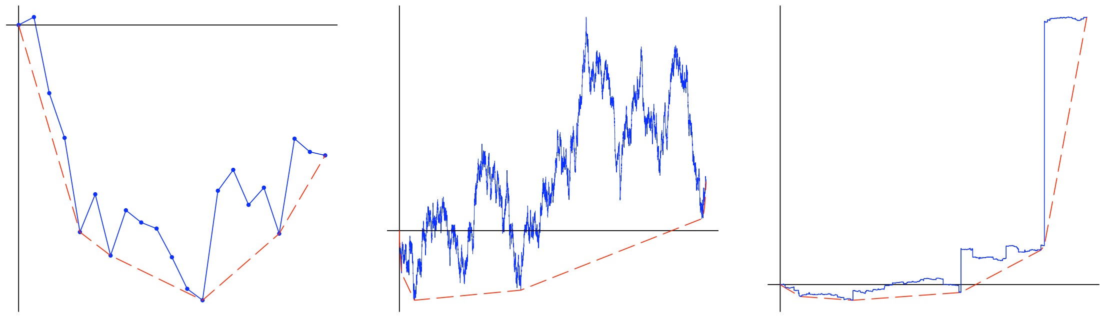

Figure 1.1: Illustration of the convex minorant of a random walk, a Brownian motion, and a Cauchy process on a finite interval.

Note that although the results of the random walk case will be repeated in Chapter 2, they are presented in a slightly different fashion here in order to emphasize which elements of Chapter 2 were important in the development of [72] and [73]. Specifically, the so called '3214' transformation is emphasized, as well as the connection with *uniform stick breaking*, since this provides the cleanest link between the random walk case and the limiting cases of Brownian motion and Lévy processes (studied in [72] and [73] respectively). Note further that Chapter 2 works with concave majorants, rather than convex minorants, however the results are clearly equivalent. The reason for the switch is that much of the historical work for random walks was done for the concave majorant case, whilst the work for Brownian motion was traditionally done for the convex minorant case. Moreover, in the random walk case the details of the point process description of the convex minorant or concave majorant described below are easier to understand in the concave majorant case.

The layout of this chapter is as follows. First, we conclude the introduction by providing a brief literature review of work related to convex minorants of various stochastic processes prior to the three recent articles mentioned above. In Section 1.2 we discuss the convex minorant of a random walk. In Section 1.3 we describe the limiting case of the results of Section 1.2 - the convex minorant of a Lévy process. In Section 1.4 we provide an overview of additional results in the special case of Brownian motion. We conclude in Section 1.5 by presenting a selection of open problems relating to convex minorants of stochastic processes.

1.1.1 History

Let $S_0 = 0$ and $S_j = \sum_{i=1}^{j} X_i$ for $1 \le j \le n$, where $X_1, \dots, X_n$ are exchangeable random variables such that almost surely no two subsets of $X_1, \dots, X_n$ have the same arithmetic mean (satisfied for example if the marginal X_i have continuous distribution). Let $S^{[0,n]} := \{(j, S_j) : 0 \le j \le n\}$, so that $S^{[0,n]}$ is the random walk of length n with increments distributed as $X_1, \dots, X_n$.

As illustrated by Figure 1.1, the convex minorant of $S^{[0,n]}$ is a piecewise linear function comprised of a finite number of linear segments. We refer to the linear segments as *faces*, the *length* of a face is as projected onto the horizontal time axis, and the slope of a face is the slope of the corresponding segment. We also refer to the points where the convex minorant equals the process as vertices; note that these points are also the endpoints of the linear segments.

In the 1950's, E. Sparre Andersen [6] discovered the following remarkable result: for F_n the number of faces of the convex minorant of $S^{[0,n]}$, there is the equality in distribution

$$F_n \overset{d}{=} K_n = \sum_{j=1}^{n} I_j, \tag{1.1.1}$$

where K_n is the number of cycles in a uniformly distributed random permutation of the set $[n] := \{1, \dots, n\}$, and $I_j, j = 1, 2, \dots$ is a sequence of independent Bernoulli variables with $\mathbb{P}(I_j = 1) = 1/j$ and $\mathbb{P}(I_j = 0) = 1 - 1/j$ for each j. The second equality in (1.1.1) is an elementary and well known representation of K_n which holds for a number of natural constructions of uniform random permutations of n simultaneously for all n, including both the construction from records of the X_i [42], and the Chinese Restaurant Process [70].

A further result that seems to have been known by Spitzer [86], and shown explicitly by Goldie [42] using a generalization by Brunk of Spitzer's Lemma [21], is that the distribution of the *partition of* n generated by the lengths of the faces of the convex minorant on $[0, n]$, which may be encoded by these lengths in non-increasing order, has the same distribution as the partition of n generated by the cycles of a uniform random permutation. Thus the partition of n induced by the lengths of the faces of the convex minorant may be generated by a discrete *uniform stick breaking process* on $[0, n]$ [70].

Another case that has been considered is that of a random walk run for a geometrically distributed amount of time. In [44], the authors consider decomposition at the minimum (obviously a vertex of the convex minorant) of such a walk, and due to (1.1.1), some results in this case can be gleaned from the cycle structure of a random permutation of random length which was studied in [83]. Note that at this point, nothing has been said about the convex minorant of a random walk run to infinity which is a case our results cover.

The convex minorant of Brownian motion has also been considered in the literature, where again we must divide efforts into the three time cases: finite, random, and infinite. Descriptions of the convex minorant of Brownian motion run for an infinite time were first provided in [45] and [71]. In [45], the author studies the decreasing process which has jumps occurring at values of slope achieved by the convex minorant with increments equal to the lengths of the faces at those slopes. In [71], many of the results of [45] are recovered using a sequential description of the convex minorant derived from Williams decomposition [96]. Embellishment and elaboration of both these efforts can be found in [26, 24, 23].

Williams decomposition can also be used to derive some facts about the convex minorant of Brownian motion run for an exponential time, as the decomposition relates to the time and value of the minimum of Brownian motion with drift.

The convex minorant of Brownian motion run for a fixed time has also been considered in [89] where particular attention is paid to the interval partition of $[0, 1]$ derived by cutting at times of the vertices of the convex minorant. It is also important to note that the convex minorant of Brownian motion run for a fixed time is easily related to the convex minorant of Brownian bridge which can be related to the convex minorant of Brownian motion run for an infinite time via a Doob transform. This was observed in [45] and additional details of this relationship were fleshed out in [9].

Finally, the convex minorant of Lévy processes in general have also been considered. For the convex minorant up to a random time, the limiting case of the random walk of [44] was worked out in [43]. The convex minorant of a Lévy process run for an infinite time is discussed in [65], see also [11]. The convex minorant of a Cauchy process on the unit interval (an especially nice case, as we shall see below) was discussed in the note [12].

The descriptions contained in [4, 72, 73] encompass all three finite, random, and infinite time horizons in all three of the cases mentioned above (random walk, Brownian motion, Lévy processes). We also note here that although some properties of the convex minorant of Brownian motion in [72] can be read from [73], more can be said in the Brownian case than for Lévy processes in general.

1.2 Random Walks

As in Section 1.1.1, let $S_0 = 0$ and $S_j = \sum_{i=1}^{j} X_i$ for $1 \leq j \leq n$, where $X_1, \ldots, X_n$ are exchangeable random variables such that almost surely no two subsets of $X_1, \ldots, X_n$ have the same arithmetic mean (satisfied for example if the X_i are i.i.d. with continuous distribution). Let $S^{[0,n]} := \{(j, S_j) : 0 \leq j \leq n\}$, so that $S^{[0,n]}$ is the random walk of length n with increments distributed as $X_1, \ldots, X_n$. As already mentioned in Section 1.1.1, the convex minorant of $S^{[0,n]}$ consists of piecewise linear segments which we refer to as *faces*. Let F_n be the number of faces of $\underline{C}^{[0,n]}$, the convex minorant of $S^{[0,n]}$, and define

$$0 < N_{n,1} < N_{n,1} + N_{n,2} < \cdots < N_{n,1} + \cdots + N_{n,F_n} = n$$

to be the successive indices j with $0 \leq j \leq n$ such that $S_j = \underline{C}_j$; we refer to $N_{n,i}$ as the *length* of the ith face of $\underline{C}^{[0,n]}$. Finally, let $L_{n,1}, \ldots, L_{n,F_n}$ be the lengths of the faces of $\underline{C}^{[0,n]}$ arranged in non-increasing order. We refer to this sequence as the *partition* of n generated by the convex minorant of $S^{[0,n]}$. Recall the following classical result.

Theorem 1.2.1 ([6, 21, 42, 86])**.** *The sequence $L_{n,1}, \ldots, L_{n,F_n}$ of ranked lengths of faces of the convex minorant of $S^{[0,n]}$, a random walk with exchangeable increments with almost surely no subset average ties has the same distribution as the ranked cycle lengths of a uniformly chosen permutation of n elements:*

$$\mathbb{P}(F_n = k, L_i = n_i, 1 \leq i \leq k) = \prod_{j=1}^{n} \frac{1}{j^{a_j} a_j!}$$

where $a_j := \#\{i : n_i = j\}$, and $n_1 \geq \ldots \geq n_k$ with $n_1 + \cdots + n_k = n$.

The following natural question was the starting point of our study of convex minorants.

> Given the partition of n generated by the faces of the convex minorant of $S^{[0,n]}$, how are the lengths ordered to form the *composition* of n generated by the convex minorant of $S^{[0,n]}$?

In the notation above, the sequence of variables $(N_{n,1}, \ldots, N_{n,F_n})$ is the composition of n generated by the convex minorant.

In the case that the X_i are i.i.d. the answer to this question is easy to describe. For $j = 1, \ldots, n$ each face of length j is assigned an increment distributed as S_j, independently of all other increments, and then the faces are ordered according to increasing slope. Formally, we have the following result.

Theorem 1.2.2 (Corollary 2.2.3). *Let $(N_{n,1}, \ldots, N_{n,F_n})$ be the composition of n induced by the lengths of the faces of the convex minorant of $S^{[0,n]}$. Assuming no subset average ties, the joint distribution of $N_{n,1}, \ldots, N_{n,F_n}$ is given by the formula*

$$\mathbb{P}(F_n = k, N_{n,i} = n_i, 1 \le i \le k) = \mathbb{P}\left(\frac{S_{n_1}^{(1)}}{n_1} < \frac{S_{n_2}^{(2)}}{n_2} < \cdots < \frac{S_{n_k}^{(k)}}{n_k}\right)\prod_{i=1}^{k}\frac{1}{n_i}$$

for all $n_1, \ldots, n_k$ with $n_1 + \cdots + n_k = n$, and where for $1 \le i \le k$

$$S_{n_i}^{(i)} := S_{n_1+\cdots+n_i} - S_{n_1+\cdots+n_{i-1}} \stackrel{d}{=} S_{n_i}.$$

In particular, if the X_i are independent, then so are the $S_{n_j}^{(i)}$ for $1 \le i \le k$.

The special case of Cauchy increments gives rise to the following appealing version of Theorem 1.2.2.

Corollary 1.2.3 (Corollary 2.3.1). *Suppose that the X_i are independent and such that S_k/k has the same distribution for every k, as when the X_i have a Cauchy distribution. Then*

$$\mathbb{P}(F_n = k; N_{n,i} = n_i, 1 \le i \le k) = \frac{1}{k!}\prod_{i=1}^{k}\frac{1}{n_i},$$

and hence $\{N_{n,i} : 1 \le i \le F_n\}$ has the same distribution as the composition of n created by first choosing a random permutation of n and then putting the cycle lengths in uniform random order.

Note that the continuum limit of this result can be read from Bertoin's work [12] and follows from the description provided in [73] as discussed below.

In order to proceed further, it is crucial that we introduce the representation of the convex minorant as a point process of lengths and increments of the faces, where the lengths are chosen according to the cycle structure of a random permutation of n elements and the increments are chosen according to Theorem 1.2.2 (independently if the X_i are). Figure 1.2 illustrates this representation.

From this point, we can use Theorem 1.2.2 to provide a construction of the convex minorant of a random walk of a random length in the case of independent increments. We already have some description in this case since we have a construction conditional on the length, but more can be said. The work of Shepp and Lloyd [83] on the cycle structure of permutations combined with the forthcoming Proposition 1.2.9 yield the following result.

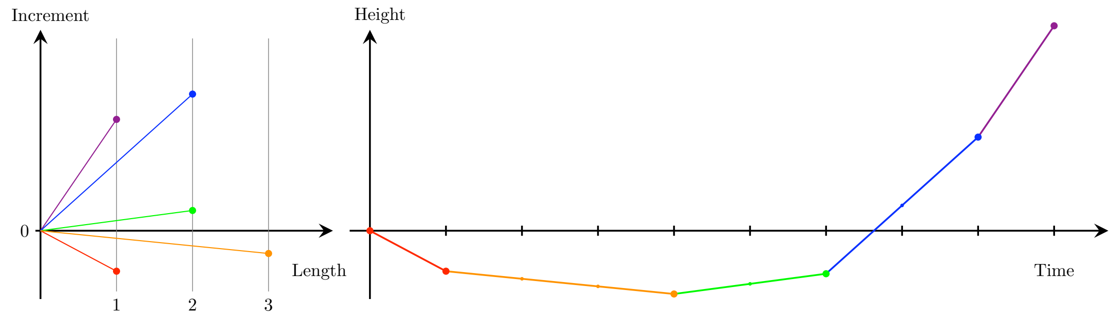

Figure 1.2: Illustration of the convex minorant of a random walk as a point process.

Theorem 1.2.4 (Theorem 2.4.2)**.** *Let $n(q)$ be a geometric random variable with parameter $1-q$; that is $\mathbb{P}(n(q) \geq n) = q^n, n = 0, 1, \ldots$. If $X_1, X_2, \ldots$ are independent with common continuous distribution, then the point process of lengths and increments of faces the convex minorant of $S^{[0,n(q)]}$ is a Poisson point process on $\{1, 2, \ldots\} \times \mathbb{R}$ with intensity*

$$j^{-1} q^j \mathbb{P}(S_j \in dx), \quad j = 1, 2, \ldots, \quad x \in \mathbb{R}.$$

Moreover, let $T_i = \sum_{l=1}^{i} N_{n(q),l}$, $0 \leq i \leq F_{n(q)}$, be the consecutive indices at which $S^{[0,n(q)]}$ meets its convex minorant, so that $T_0 = 0$ and $T_{F_{n(q)}} = n(q)$. Then the sequence of path segments

$$\{(S_{T_i+k} - S_{T_i}, 0 \leq k \leq N_{n(q),i+1}), i = 0, \ldots, F_{n(q)} - 1\},$$

is a list of the points of a Poisson point process in the space of finite random walk segments

$$\{(s_1, \ldots, s_j) \text{ for some } j = 1, 2, \ldots\}$$

whose intensity measure on paths of length j is $q^j j^{-1}$ times the conditional distribution of the path $(S_1, \ldots, S_j)$ given that $S_k > (k/j)S_j$ for all $1 \leq k \leq j-1$.

An important facet of the Poisson point process description is that it provides a decomposition of a random walk up to the index of its minimum. For example, the description of Theorem 1.2.4 is a more complete description of the convex minorant of a random walk which was the basis for Spitzer's combinatorial identity [86].

Theorem 1.2.5 ([86]). *Let $X_1, X_2, \ldots$ be independent with common continuous distribution, $S_0 = 0$, $S_k = \sum_{i=1}^k X_i$ for $k \geq 1$, and $M_n := \min_{0 \leq k \leq n} S_k$. Then*

$$\sum_{n=0}^{\infty} q^n \mathbb{E} e^{itM_n} = \exp\left(\sum_{k=1}^{\infty} \frac{q^k}{k} \mathbb{E} e^{itS_k^-}\right),$$

where $S_k^- = \min\{S_k, 0\}$.

Now, by letting q tend to one in Theorem 1.2.4, we obtain a description of the convex minorant of $S^{[0,\infty)}$, a random walk on $[0, \infty)$.

Theorem 1.2.6 (Theorem 2.5.1). *If $X_1, X_2, \ldots$ are independent with common continuous distribution with $\mathbb{E}X_1 \in (-\infty, \infty]$, then the point process of lengths and increments of faces the convex minorant of $S^{[0,\infty)}$ is a Poisson point process on $\{1, 2, \ldots\} \times \mathbb{R}$ with intensity*

$$j^{-1}\mathbb{P}(S_j \in dx), \quad j = 1, 2, \ldots, \quad \frac{x}{j} < \mathbb{E}X_1.$$

Similar to Theorem 1.2.4, there is a companion path space statement which we omit for the sake of brevity.

The key to the results above is a certain property of a transformation of the walk $S^{[0,n]}$, which not only yields the results above, but also provides a construction of the walk jointly with its convex minorant. We will call this transformation the '3214' transformation, as it is described by first dividing the walk $S^{[0,n]}$ into four consecutive paths, and then reordering these four pieces with the third one first, the second one second, and so on.

The '3214' transform of $S^{[0,n]}$ is generated by a random variable U which is uniform on $\{1, \ldots, n\}$ and is independent of $S^{[0,n]}$. Given $U = u$, we then define g and d as the indices of the left and right endpoints of the face of the convex minorant of $S^{[0,n]}$ straddling the index u. Note that g and d are almost surely well defined by this description due to the no subset average ties assumption. Consider the four paths of the random walk on the intervals $[0, g]$, $[g, u]$, $[u, d]$, and $[d, n]$. With this setup, the '3214' transform is defined by reordering the four path fragments of $S^{[0,n]}$ described above to form a new walk path $S_U^{[0,n]}$ in the order $3-2-1-4$. Figure 1.3 illustrates the notation and provides an example of the transformation.

The following lemma summarizes the crucial feature of this transform.

Lemma 1.2.7 (Theorem 2.8.1). *Let $X_1, \ldots, X_n$ be exchangeable random variables with no subset average ties and $S^{[0,n]}$ the random walk generated by the X_i. Let U uniform on $\{1, \ldots, n\}$, independent of the X_i and $S_U^{[0,n]}$ the '3214' transform of $S^{[0,n]}$*

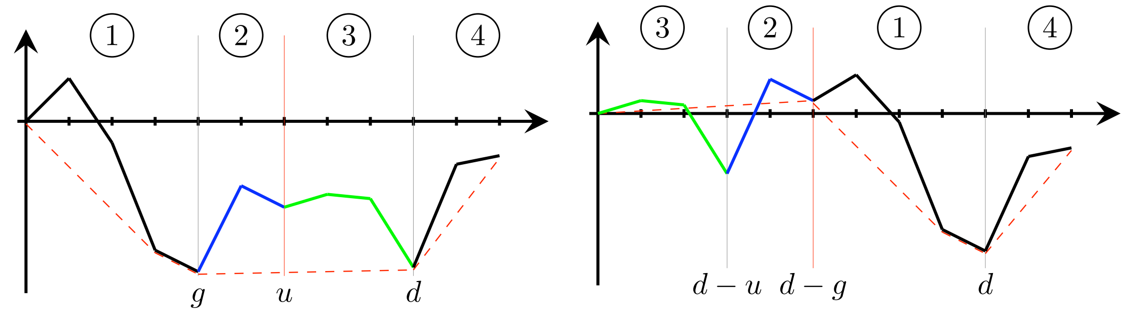

Figure 1.3: Notation and application of the '3214' transformation.

generated by U*. If* g *and* d *are the indices of the endpoints of the face of the convex minorant of* $S^{[0,n]}$ *to the left and right of* U*, then*

$$\left(U, S^{[0,n]}\right) \stackrel{d}{=} \left(d-g, S_U^{[0,n]}\right).$$

To see how Lemma 1.2.7 corroborates the story above, we introduce discrete *uniform stick breaking* on $[0,n]$, one of the many well-known representations of the distribution of the cycle lengths of a uniformly chosen permutation on n elements.

Definition 1.2.8. For an integer n, define the discrete uniform stick breaking sequence of random variables $M_{n,1}, \ldots, M_{n,K_n}$ as follows.

- $M_{n,1}$ is uniform on $\{1, \ldots, n\}$.
- For $i \geq 1$, if $\sum_{j=1}^{i} M_{n,j} < n$, then $M_{n,i+1}$ is uniform on $\left\{1, n - \sum_{j=1}^{i} M_{n,j}\right\}$.
- For $i \geq 1$, if $\sum_{j=1}^{i} M_{n,j} = n$, then set $K_n = i$, and end the process.

We refer to the variables $L_{n,1}, \ldots, L_{n,K_n}$ defined to be $M_{n,1}, \ldots, M_{n,K_n}$ rearranged in non-increasing order as the *partition* of n generated by uniform stick breaking.

To be explicit, we state the following well-known proposition (see [70]).

Proposition 1.2.9. *The partition of* n *generated by uniform stick breaking has the same distribution as the ranked cycle lengths of a uniformly chosen permutation of* n *elements.*

From this point, some consideration yields the following implications of Lemma 1.2.7 for a walk with i.i.d. increments and no subset average ties:

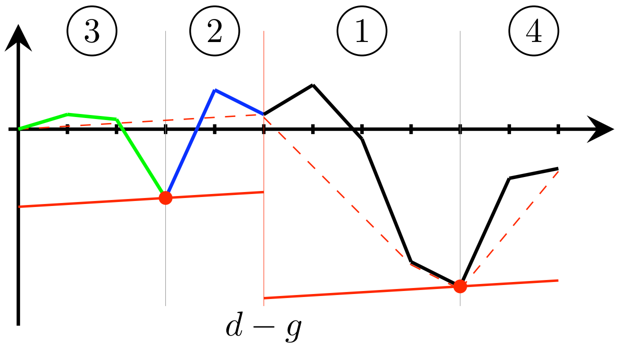

Figure 1.4: Illustration of the inverse '3214' transform for the walk of Figure 1.3.

- The lengths of the faces of the convex minorant of $S^{[0,n]}$ are distributed as discrete stick breaking.
- Conditional on the lengths of the faces of the convex minorant of $S^{[0,n]}$, the excursions above the segments are independent.
- Given a segment of length j, the excursion above the segment can be realized as the unique cyclic permutation of a random walk of length j equal in distribution to $S^{[0,j]}$ which yields a convex minorant of exactly one segment.

The last item is similar in spirit to Vervaat's transform of a Brownian bridge to an excursion [91]. As this transformation is not well developed for random walks and Lévy processes in general (some statements for Lévy processes are found in [90]), this last item carries real content.

The proof of Lemma 1.2.7 essentially follows from two observations. The first is that given the values of the increments $X_1 = x_1, \ldots, X_n = x_n$, S_j is distributed as $\sum_{i=1}^{j} x_{\sigma_i}$ for $j = 1, \ldots, n$ and where σ is a permutation chosen uniformly at random. From this point we only need to show that the '3214' transformation is a bijection between $\{1, \ldots, n\}\times$ 'paths generated from permutations of $x_1, \ldots, x_n$' for fixed increments x_i having no subset average ties. The bijection is easily verified after noting that for a given value of $d-g$, the indices at which Segment 1 meets Segment 4 and Segment 3 meets Segment 2 are found by raising a line with slope equal to the mean of the first $d-g$ increments. Figure 1.4 illustrates this inverse transformation.

If we remove the assumption that almost surely, no two subsets of the X_i have the same mean, then the process of generating excursions described above may generate excursions that meet the corresponding face of the convex minorant at points other than the end points, and excursions that have the same slope. This implies that there is not necessarily a unique cyclic permutation transforming a walk into an excursion, and neither is there necessarily a unique ordering of the excursions that puts them in non-decreasing order of slope. Such technical issues can be dealt with, and we do so in Section 2.6. However, the '3214' transformation cannot be used for the proof in this case so a different methodology to the one outlines above is used to prove the results of Chapter 2.

1.3 Lévy processes

A real valued process X is a Lévy process on $[0, \infty)$ if $X_0 = 0$, X is cadlag (right continuous with left limits), and X has independent and stationary increments. As is well known, Lévy processes are the continuous scaling limits of discrete time random walks generated by i.i.d. increments, so it is not surprising that continuous analogs of the results of Section 1.2 hold for Lévy processes. However, there are a few interesting wrinkles not present in the discrete case and many technical details to be considered in pushing the discrete results to the limit. We restrict our analysis to the case that X_t has continuous distribution for all $t > 0$, which is equivalent to the assumption that X is not a compound Poisson process with drift.

In analogy to the case of random walk, we can view the intervals that a Lévy process is strictly greater than its convex minorant on $[0, 1]$ as an interval partition of the unit interval. The formal statement of this last fact is proved in [73], but also intersects with the work [55].

Proposition 1.3.1 ([73])**.** *Let X be a Lévy process with continuous distribution and $\underline{C}$ the convex minorant of X on $[0, 1]$. Let $\mathcal{O} = \{s \in (0, t) : \underline{C}_s < X_s \wedge X_{s-}\}$ and $\mathcal{I}$ be the set of connected components of $\mathcal{O}$. The following conditions hold almost surely:*

1. *The open set $\mathcal{O} = \{s \in (0, t) : \underline{C}_s < X_s \wedge X_{s-}\}$ has Lebesgue measure 1.*
2. *$\mathcal{I}$ is a set of disjoint intervals and the closure of its union is $[0, 1]$.*
3. *If (g_1, d_1) and (g_2, d_2) are distinct intervals of $\mathcal{I}$, then the slopes of $\underline{C}$ over those intervals differ:*

$$\frac{\underline{C}_{d_1} - \underline{C}_{g_1}}{d_1 - g_1} \neq \frac{\underline{C}_{d_2} - \underline{C}_{g_2}}{d_2 - g_2}.$$

For each $(g, d) \in \mathcal{I}$, we refer to g and d as *vertices*, the *length* is $d - g$, the *increment* is $\underline{C}_d - \underline{C}_g$, and the *slope* is $(\underline{C}_d - \underline{C}_g)/(d - g)$.

Because the partition of n generated by the convex minorant of an i.i.d. generated random walk of n steps is distributed as the partition of n generated by the cycles of a random permutation for any increment distribution, we might hope that a similar universal result holds for Lévy processes and also that this universal result might be a limiting continuous distribution of the cycle structure of a random permutation. This is indeed the case, but before stating our result we define this continuous limit.

Definition 1.3.2. Define the continuous uniform stick breaking sequence of random variables as the sequence $L_1, L_2, \ldots$ defined as follows.

- L_1 is uniform on $[0, 1]$.
- For $i \geq 1$, L_{i+1} is uniform on $\left[0, 1 - \sum_{j=1}^{i} L_j\right]$.

We refer to the variables $L_1, L_2, \ldots,$ rearranged in non-increasing order as the *partition* of $[0, 1]$ generated by uniform stick breaking.

The variables $L_1, L_2, \ldots$ almost surely sum to one and their law once arranged in decreasing order is referred to as the *Poisson-Dirichlet distribution with parameter one* which is the limiting distribution of the cycle structure of a permutation chosen uniformly at random (see [70]). We can now state the following result and note that a proof in the special case of Brownian motion was sketched in [89].

Theorem 1.3.3 ([73]). *The sequence of ranked lengths of faces of the convex minorant of a Lévy process with continuous distributions has the Poisson-Dirichlet distribution with parameter one.*

In light of Theorem 1.3.3, the following natural question arises. Given the interval partition of $[0, 1]$ generated by the convex minorant of a Lévy process X with continuous distribution, how are the intervals ordered to form the interval *composition* of $[0, 1]$ generated by the convex minorant of X?

In total analogy with the answer for i.i.d. random walks, the answer to this question is easy to describe. Given the interval of length l, the increment is distributed as X_l independent of all other increments, and the faces are ordered according to increasing slope.

Theorem 1.3.4 ([73]). *Let X be a Lévy process with continuous distribution and let $\{(g_i, d_i)\}_{i\geq 1}$ denote the intervals of $\underline{C}$, the convex minorant of X on $[0, 1]$. Let $L_1, L_2, \ldots$ be generated by uniform stick breaking, $S_0 := 0$, and for $i \geq 1$, define $S_i := \sum_{j=1}^{i} L_i$. Then we have the following equality in distribution between sequences:*

$$\left(\left(d_i - g_i, \underline{C}_{d_i} - \underline{C}_{g_i}\right), i \geq 1\right) \stackrel{d}{=} \left(\left(L_i, X_{S_i} - X_{S_{i-1}}\right), i \geq 1\right).$$

We note here that applying the theorem to a Cauchy process yields the main result of Bertoin [12] and also shows the composition generated by the convex minorant on $[0,1]$ is a uniform ordering of the generated partition; c.f. Corollary 1.2.3.

We can also consider the convex minorant of a Lévy process X on an interval of a random exponential length independent of X to obtain the following analog of Theorem 1.2.4.

Theorem 1.3.5 ([73]). *Let T be a rate θ exponential random variable, X a Lévy process with continuous distribution which is independent of T and let $\underline{C}^T$ denote the convex minorant of X on $[0,T]$. The point process*

$$\left\{ \left(d_i - g_i, \underline{C}^T_{d_i} - \underline{C}^T_{g_i}\right), i \geq 1 \right\}$$

generated by the lengths t and increments x of $\underline{C}^T$ has the same distribution as a Poisson point process with intensity measure

$$\mu(dt, dx) = \frac{e^{-\theta t}}{t} dt\mathbb{P}(X_t \in dx).$$

By integrating out the independent exponential variable, we can also use Theorem 1.3.5 gain insight into the structure of the convex minorant of a Lévy process on $[0,1]$.

For example, this program yields the following neat dichotomy for stable Lévy processes.

Proposition 1.3.6 ([73]). *Let X be a symmetric stable process with parameter α; that is $\mathbb{E}e^{iuX_t} = e^{-t|u|^\alpha}$, and let $\mathcal{S}$ be the set of slopes and $\mathcal{T}$ be the set of times of vertices of the convex minorant of X on $[0,1]$.*

- *If $1 < \alpha \leq 2$, then $\mathcal{S}$ has no accumulation points, $\mathcal{S} \cap (a, \infty)$ and $\mathcal{S} \cap (-\infty, -a)$ are infinite for $a > 0$, and $\mathcal{T}$ has accumulation points at zero and one only.*
- *If $0 < \alpha \leq 1$, then $\mathcal{S}$ is dense $\mathbb{R}$ and every point of $\mathcal{T}$ is an accumulation point.*

By letting θ tend to zero in Theorem 1.3.5, we obtain a description of the convex minorant of X on $[0,\infty)$ which was also derived in [65].

Theorem 1.3.7 ([65, 73]). *If X is a Lévy process with continuous distribution and and*

$$I := \liminf_{t\to\infty} \frac{X_t}{t} \in (-\infty, \infty],$$

then the lengths t and increments x of the convex minorant of X on $[0,\infty)$ is a Poisson point process with intensity

$$\frac{\mathbb{P}(X_t \in dx)}{t}, \quad x < It.$$

Both of the previous theorems carry an Itô type excursion theory analogous to that of Theorem 1.2.4 for random walks, see [73, Thm. 4].

Theorems 1.3.3 and 1.3.4 follow from a direct analog of Lemma 1.2.7 for a '3214' transform for Lévy processes. The proof uses limiting arguments which crucially hinge on certain regularity conditions for Lévy processes governing the behavior of the process at the vertices of the convex minorant.

1.4 Brownian Motion

Since Brownian motion is a Lévy process (and stable with index 2), the results of the previous section apply to the convex minorant of Brownian motion, and as mentioned in Section 1.1.1 some of these results were known (from [23, 26, 45, 71, 89]). However, Brownian motion offers extra analysis due to its special properties among Lévy processes (e.g. continuity and time inversion). We begin by noting the following special case of Theorem 1.3.5.

Theorem 1.4.1 ([45]). *Let Γ_1 be an exponential random variable with rate one. The lengths x and slopes s of the faces of the convex minorant of a Brownian motion on $[0,\Gamma_1]$ form a Poisson point process on $\mathbb{R}^+ \times \mathbb{R}$ with intensity measure*

$$\frac{\exp\{-\frac{x}{2}(2+s^2)\}}{\sqrt{2\pi x}}\, ds\, dx, \quad x \geq 0, s \in \mathbb{R}.$$

As with random walks and Lévy processes, the minimum on $[0,T]$ of a Brownian motion is a distinguished point of the convex minorant and the process after the minimum can be described by restricting the point process of slopes and increments to those points with positive slopes. Due to Proposition 1.3.6, we can define

$$\alpha_0 < \alpha_1 < \alpha_2 < \cdots < 1$$

with $\alpha_n \uparrow 1$ as $n \to \infty$ to be times of vertices of the convex minorant of a Brownian motion B on $[0,1]$, arranged relative to

$$\alpha_0 := \text{argmin}_{0 \leq t \leq 1} B_t.$$

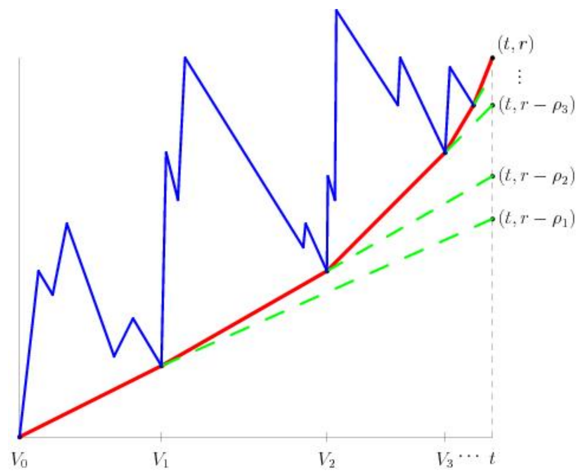

Figure 1.5: An illustration of the notation of Theorem 1.4.3. The blue line represents a Brownian meander of length t, and the red line its convex minorant. Note also that $V_i := t - \tau_i$ for $i = 0, 1, \ldots$

Brownian scaling and Theorem 1.4.1 yield an implicit description of the distribution of the sequence $(\alpha_i)_{i\geq 0}$. Moreover, Denisov's decomposition for Brownian motion at the minimum [29] implies that the process after the minimum is a Brownian meander, for which we now provide an alternate description. First we make the following definition.

Definition 1.4.2. We say that a sequence of random variables $(\tau_n, \rho_n)_{n\geq 0}$ satisfies the (τ, ρ) recursion if for all $n \geq 0$:

$$\rho_{n+1} = U_n \rho_n \quad \text{and} \quad \tau_{n+1} = \frac{\tau_n \rho_{n+1}^2}{\tau_n Z_{n+1}^2 + \rho_{n+1}^2}$$

for the two independent sequences of i.i.d. uniform $(0, 1)$ variables U_n and i.i.d. squares of standard normal random variables Z_n^2, both independent of (τ_0, ρ_0).

Theorem 1.4.3 ([72])**.** *Let $(X(v), 0 \leq v \leq t)$ be a Brownian meander of length t, and let $(\underline{C}(v), 0 \leq v \leq t)$ be its convex minorant. The vertices of $(\underline{C}(v), 0 \leq v \leq t)$ occur at times $0 = V_0 < V_1 < V_2 < \cdots$ with $\lim_n V_n = t$. Let $\tau_n := t - V_n$ so $\tau_0 = t > \tau_1 > \tau_2 > \cdots$ with $\lim_n \tau_n = 0$. Let $\rho_0 = X(t)$ and for $n \geq 1$ let $\rho_0 - \rho_n$ denote the intercept at time t of the line extending the segment of the convex minorant of X on the interval (V_{n-1}, V_n). The convex minorant $\underline{C}$ of X is uniquely determined* *by the sequence of pairs* (τ_n, ρ_n) *for* $n = 1, 2, \ldots$ *which satisfies the* (τ, ρ) *recursion with*

$$\rho_0 \stackrel{d}{=} \sqrt{2t\Gamma_1} \text{ and } \tau_0 = t,$$

where Γ_1 *is an exponential random variable with rate one.*

In [72], we use the descriptions provided by Theorems 1.4.1 and 1.4.3 (and the interplay between them as per Denisov's decomposition [29]) to derive various properties about the convex minorant of Brownian motion on $[0, 1]$, such as formulas for densities of the α_i. We also use the equivalence of the two descriptions to discover new identities between related quantities in each description. We conclude this section with an elementary example of such an identity; direct proofs of this and similar results remain elusive.

Corollary 1.4.4 ([72])**.** *Let* W *and* Z *be standard normal random variables,* U *uniform on* $(0, 1)$*, and* R *Rayleigh distributed having density* $re^{-r^2/2}$*,* $r > 0$*. If all of these variables are independent, then*

$$\left(\frac{W^2 + (1-U)^2R^2}{1 + U^2R^2/Z^2}, \frac{(1-U)R}{\sqrt{T}}\right) \stackrel{d}{=} \left(Z^2, \frac{(1-U)R}{\sqrt{T}}\right).$$

Note that the two coordinate variables on the right are independent.

1.5 Open Problems

We end this chapter with a list of open problems.

- Under what conditions is the right derivative of the convex minorant of a Lévy process with continuous distribution discrete, continuous, or mixed?
- Provide a description of the convex minorant of a continuous time process with *exchangeable* increments.
- Provide a framework independent of the convex minorant of Brownian motion that explains the equivalence of the Poisson point process of Theorem 1.4.1 with the sequential description of Theorem 1.4.3.
- Is there a version of the sequential description of Theorem 1.4.3 for random walks or Lévy processes?

Chapter 2

Concave Majorants of Random Walks and Related Poisson Processes

2.1 Introduction

Let $S_0 = 0$ and $S_j = \sum_{i=1}^{j} X_i$ for $1 \le j \le n$, where $X_1, \dots, X_n$ are exchangeable random variables. Let $\mathbf{A}$ be the assumption that almost surely no two subsets of $X_1, \dots, X_n$ have the same arithmetic mean, and assume for now that $\mathbf{A}$ holds. Let $S^{[0,n]} := \{(j, S_j) \,:\, 0 \le j \le n\}$, so that $S^{[0,n]}$ is the random walk of length n with increments distributed like $X_1, \dots, X_n$. Let

$$0 < N_{n,1} < N_{n,1} + N_{n,2} < \cdots < N_{n,1} + \cdots + N_{n,F_n} = n$$

be the successive times j with $0 \le j \le n$ such that $S_j = \bar{C}^{[0,n]}(j)$, where $\bar{C}^{[0,n]}$ is the *concave majorant* of the walk $S^{[0,n]}$, i.e. the least concave function C on $[0, n]$ such $C(j) \ge S_j$ for $1 \le j \le n$. The random variable F_n is the *number of faces* of the concave majorant. Without assumption $\mathbf{A}$, more care needs to be taken in defining the faces of the concave majorant; this will be discussed further in Section 2.6.

The ith face of the concave majorant is a chord from $(N_{n,1}+\cdots+N_{n,i-1}, S_{N_{n,1}+\cdots+N_{n,i-1}})$ to $(N_{n,1} + \cdots + N_{n,i}, S_{N_{n,1}+\cdots+N_{n,i}})$. We define the *length*, *increment* and *slope* of the ith face to be N_i, $\Delta_{n,i}$ and $\frac{\Delta_{n,i}}{N_i}$ respectively, where

$$\Delta_{n,i} := (S_{N_{n,1}+\cdots+N_{n,i}} - S_{N_{n,1}+\cdots+N_{n,i-1}}), \qquad \text{for } 1 \le i \le F_n.$$

As mentioned in Section 1.1.1, Sparre Andersen [6] gave the following result: for any exchangeable $X_1, \dots, X_n$ satisfying assumption $\mathbf{A}$, there is the equality in distribution

$$F_n \stackrel{d}{=} K_n = \sum_{j=1}^{n} I_j \tag{2.1.1}$$

where K_n is the number of cycles in a uniformly distributed random permutation of the set $[n] := \{1, \ldots, n\}$, and $I_j, j = 1, 2, \ldots$ is a sequence of independent Bernoulli variables with $\mathbb{P}(I_j = 1) = 1/j$ and $\mathbb{P}(I_j = 0) = 1 - 1/j$ for each j. Further, Goldie [42] showed that under assumption **A** the distribution of the *partition of* n generated by the lengths of the faces of the concave majorant on $[0, n]$, which may be encoded by these lengths in non-increasing order, has the same distribution as the partition of n generated by the cycles of a uniform random permutation (we will in fact prove this result as a corollary of our main theorem). The Goldie result raises the following problem:

The rearrangement problem. Conditionally given that the partition of n generated by the lengths of the faces of the concave majorant of the random walk $S^{[0,n]}$ has segment lengths $n_1, \ldots, n_k$ with $n_1 \geq n_2 \geq \ldots \geq n_k > 0$,

- in what order and with what increments should the faces $f_1, \ldots, f_k$ of the concave majorant with lengths $n_1, \ldots, n_k$ respectively be arranged to recreate the concave majorant of the random walk $S^{[0,n]}$?
- given the concave majorant, what is the distribution of values of the random walk $S^{[0,n]}$ between vertices of the concave majorant?

We answer this question by giving in Theorem 2.1.1 a simultaneous construction of the walk and its concave majorant conditional on the partition generated by the lengths of the faces of the concave majorant. The theorem will be proved under assumption **A** in Section 2.2, and in the general case in Section 2.6, with the key idea of both proofs being that it is enough to show that the theorem is true when $X_1, \ldots, X_n$ are samples without replacement from a set of n real numbers. Since the construction given in the theorem applies to general exchangeable $X_1, \ldots, X_n$ it allows us to investigate in Section 2.6 the structure of the concave majorant in the general case. The statement of the theorem is complicated, but easy to describe informally, particularly under assumption **A**, in which case the construction is as follows. Conditional on the lengths of the blocks of the partition generated by the concave majorant being $(n_1, \ldots, n_k)$:

- Split $X_1, \ldots, X_n$ into k blocks

$$(X_1, \ldots, X_{n_1})(X_{n_1+1}, \ldots, X_{n_1+n_1}) \cdots (X_{\sum_{i=1}^{k-1} n_i+1}, \ldots, X_{\sum_{i=1}^{k} n_i})$$

- Arrange the blocks in order of decreasing arithmetic means.
- Perform the unique cyclic permutations of the increments within each block such that the walk with those cyclically permuted increments remains below the line joining its start and end points.

This process defines a permutation of the original increments which leaves the distribution of the walk $S^{[0,n]}$ unchanged and at the same time provides us with information about the concave majorant. In the case where $X_1, \ldots, X_n$ are independent, then we may just generate independent walks of length $n_1, \ldots, n_k$, cyclically permute the increments of each walk appropriately, and then arrange the walks in order of decreasing slope. The idea of using cyclic permutations to transform random walk bridges into excursions is due to Vervaat [91].

When assumption **A** is not satisfied there are two more complications. Some of the blocks may have the same arithmetic mean, in which case their ordering is chosen uniformly, and within a block there may be more than one cyclic permutation of increments that leaves the walk with those increments below the line joining its start and end points, in which case the cyclic permutation is chosen uniformly from the possible options. By exchangeability, it would also work to take the blocks with the same arithmetic mean in order of appearance rather than randomly ordering them, but this makes the statement of the theorem harder and in fact does not make the proof any easier.

To facilitate the statement of the theorem, it is necessary to define the set of all permutations that cyclically permute increments within certain blocks and then arrange those blocks in some order.

Definition. Let Σ_n be the set of permutations of $[n]$, and let $\mathcal{P}_n$ be the set of partitions of n, encoded in non-increasing order. For $(n_1, \ldots, n_k) \in \mathcal{P}_n$ let $\Sigma_{(n_1,\ldots,n_k)} \subseteq \Sigma_n$ be such that $\sigma \in \Sigma_{(n_1,\ldots,n_k)}$ if and only if for some $\tau \in \Sigma_k$ and $(r_1, \ldots, r_k) \in \mathbb{Z}^k$ we have

$$\sigma\left(\sum_{l=1}^{i-1} n_{\tau(l)} + j\right) = \sum_{l=1}^{\tau(i)-1} n_l + \left((j - 1 + r_i) \bmod n_{\tau(i)}\right) + 1$$

for $1 \le j \le n_{\tau(i)}$, $1 \le i \le k$.

In the definition of $\Sigma_{(n_1,\ldots,n_k)}$ just given, the cyclic shift chosen for the $\tau(i)$th block is given by r_i and the ordering of the k blocks is given by τ.

Theorem 2.1.1. *Let $S_0 = 0$ and $S_j = \sum_{\ell=1}^{j} X_\ell$ for $1 \le j \le n$, where $X_1, \ldots, X_n$ are random variables with any exchangeable joint distribution. Let $S^{[0,n]} = \{(j, S_j) : 0 \le j \le n\}$. Independently of $X_1, \ldots, X_n$, let $L_{n,1}, L_{n,2}, \ldots, L_{n,K_n}$ be a sequence of random variables distributed like the lengths of cycles of a random permutation of $[n]$*

arranged in non-increasing order. Conditionally given $\{K_n = k\}$ *and* $\{L_{n,i} = n_i : 1 \le i \le k\}$*, let* B *be the random subset of* Σ_n *defined by the following relation.* σ *is in* B *if and only if* $\sigma \in \Sigma_{(n_1,\dots,n_k)}$ *and there exists* $\tau \in \Sigma_k$ *such that the function defined on* $[k]$ *by*

$$i \mapsto \Delta_{n,i}^{\sigma,\tau} := \frac{1}{n_{\tau(i)}} \left(\sum_{\ell=n_{\tau(1)}+\cdots+n_{\tau(i-1)}+1}^{n_{\tau(1)}+\cdots+n_{\tau(i)}} X_{\sigma(\ell)} \right) \tag{2.1.2}$$

is non-increasing in i *and for each* $1 \le i \le k$ *we have*

$$\frac{1}{m} \left(\sum_{\ell=n_{\tau(1)}+\cdots+n_{\tau(i-1)}+1}^{n_{\tau(1)}+\cdots+n_{\tau(i-1)}+m} X_{\sigma(\ell)} \right) \le \Delta_{n,i}^{\sigma,\tau} \quad \text{for } 1 \le m \le n_{\tau(i)}. \tag{2.1.3}$$

Conditionally given B*, let* ρ *be a uniform random element of* B*, independently of all previously introduced random variables. For* $1 \le j \le n$ *let* $S_j^\rho = \sum_{\ell=1}^{j} X_{\rho(\ell)}$ *and let* $S_\rho^{[0,n]} = \{(j, S_j^\rho) : 0 \le j \le n\}$*. Then* $S_\rho^{[0,n]} \stackrel{d}{=} S^{[0,n]}$*.*

The condition involving (2.1.2) ensures that the permutation that we end up choosing puts the blocks of increments in non-increasing order of arithmetic mean, i.e. in non-increasing order of slope, and the condition involving (2.1.3) ensures that the cyclic permutation chosen for each block makes the walk stay below the line joining the start and end points of the increments of that block. In the case where $X_1, \dots, X_n$ satisfy assumption **A**, the random set B almost surely only consists of one element and thus the additional random variable ρ is not needed.

Some of the ideas of our construction are contained within the work of Spitzer [86], who observed that if $\Delta_{n,i}$ is the increment of the walk over the ith face of the concave majorant, then for the maximum

$$M_n := \max_{0 \le k \le n} S_k$$

there is the almost sure representation

$$M_n = \sum_{i=1}^{F_n} \Delta_{n,i} 1(\Delta_{n,i} \ge 0). \tag{2.1.4}$$

Spitzer showed the much simpler representation in distribution

$$M_n \stackrel{d}{=} \sum_{i=1}^{K_n} \Delta_{n,i}^* 1(\Delta_{n,i}^* \ge 0) \tag{2.1.5}$$

where K_n is the number of cycles of a random permutation independent of the random walk $S^{[0,n]} = \{(j, S_j) : 0 \leq j \leq n\}$, and given $K_n = k$ and that the permutation has cycles of lengths say $L_{n,1}, \ldots, L_{n,k}$, the $\Delta^*_{n,i}$ are conditionally independent, with

$$(\Delta^*_{n,i} \,|\, K_n = k, L_{n,i} = \ell) \stackrel{d}{=} S_\ell, \qquad \text{for } 1 \leq i \leq k, \text{ and } 1 \leq \ell \leq n.$$

This is an immediate corollary of our theorem, and something we investigate further in Section 2.5.3. Some consequences of this result lead to other ideas which arise in this paper. Let $S^+_\ell = S_\ell \vee 0$. As pointed out by Spitzer, Hunt's remarkable identity [53, Theorem 4.1]

$$\mathbb{E}(M_n) = \sum_{\ell=1}^{n} \frac{\mathbb{E}(S^+_\ell)}{\ell} \tag{2.1.6}$$

follows easily from (2.1.5), along with the following complete description of the distribution of M_n for every $n = 1, 2, \ldots$ (this description is known as Spitzer's Identity): for $|q| < 1$

$$\sum_{n=0}^{\infty} q^n \mathbb{E} e^{itM_n} = \exp\left(\sum_{k=1}^{\infty} \frac{q^k}{k} \mathbb{E} e^{itS^+_k}\right) \tag{2.1.7}$$

To indicate how (2.1.6) follows from (2.1.5), recall that the expected number of cycles of length ℓ in a random permutation of $[n]$ is ℓ^{-1}. So (2.1.6) decomposes the expectation of the sum in (2.1.5) according the contributions from cycles of various sizes ℓ. To provide a similar interpretation of (2.1.7), let $n(q)$ denote a random variable with geometric distribution with parameter $1 - q$, so $\mathbb{P}(n(q) \geq n) = q^n$ for $n = 0, 1, \ldots$, and assume $n(q)$ is independent of the random walk. Then multiplying (2.1.7) by $1 - q$ and using the expansion $-\log(1 - q) = \sum_{k=1}^{\infty} q^k/k$ allows (2.1.7) to be rewritten [44]:

$$\mathbb{E} e^{itM_{n(q)}} = \exp\left(\sum_{k=1}^{\infty} \frac{q^k}{k} (\mathbb{E} e^{itS^+_k} - 1)\right) \tag{2.1.8}$$

Otherwise put, the maximum $M_{n(q)}$ of the walk up to the independent geometric time $n(q)$ has a compound Poisson distribution:

$$M_{n(q)} \stackrel{d}{=} \sum_{k=1}^{\infty} \sum_{i=1}^{N(q^k/k)} S^+_{k,i} \tag{2.1.9}$$

where for fixed q the $N(q^k/k)$ are independent Poisson variables with parameters q^k/k for $k = 1, 2, \ldots$, and given these variables the $S_{k,i}$ for $1 \leq i \leq N(q^k/k)$ are independent with $S_{k,i} \stackrel{d}{=} S_k$. As observed by Greenwood and Pitman [44], the identity in distribution (2.1.9), and the companion result which determines the common distribution of $S_n - M_n$ and $\min_{0\leq k\leq n} S_k$ for every n, can be derived, along with other results of fluctuation theory for the distribution of ladder heights and ladder times, from the decomposition

$$S_{n(q)} = M_{n(q)} + (S_{n(q)} - M_{n(q)}) \tag{2.1.10}$$

which expresses the compound Poisson variable $S_{n(q)}$ as the sum of two independent compound Poisson variables with with positive and negative ranges respectively. Moreover, as shown in [43], this discussion can be passed to a continuous time limit to derive the companion circle of fluctuation identities for maxima, minima and ladder processes associated with Lévy processes. In section 2.5.3 we give new explanations for the compound Poisson distributions mentioned above.

The rest of this chapter is structured as follows. In Section 2.2 we will prove Theorem 2.1.1 under assumption **A** and give corollaries relating to the partition and composition induced by the concave majorant. In Section 2.3 we will analyze some specific examples of composition probabilities, including the Cauchy increment case, which turns out to be particularly simple. In Section 2.4 we extend the description to the case where n is replaced by $n(q)$, a geometric random variable with parameter $1 - q$, which results in a description of the concave majorant and the excursions under each face as a Poisson point process. In Section 2.5 we apply the Poissonian theory. First, by letting $q \to 1$ we find a description of the concave majorant for the random walk on $[0, \infty)$, and the associated excursions under each face. Then we analyze the behaviour of the concave majorant as n grows. As a final application we investigate the pre and post maximum parts of the walk. In Section 2.5.3 we investigate the two concave majorants that result from decomposing the random walk at its maximum, and their associated partitions. In Section 2.6 we extend the theory to $X_1, \ldots, X_n$ not satisfying assumption **A**. Also in Section 2.6 we investigate three nested compositions of integers that arise naturally. At the end of this Section 2.6 some examples of how the general theory can be applied are given. In Section 2.7 we finish answering the rearrangement problem mentioned above by describing the law of a random walk conditional on the value of its concave majorant. Finally, in Section 2.8, we describe an important path transformation (the '3214' transformation discussed in Chapter 1) that as discussed in Chapter 1 provides Pitman and Uribe Bravo with the basis for a full investigation into the concave majorant of a Lévy process [73].

2.2 Proof of Theorem 2.1.1 under assumption A and the partition and composition laws

We begin with a simple Lemma due to Spitzer relating to cyclic permutations of increments of walks that shows that under assumption **A** the appropriate cyclic permutations discussed in the introduction are almost surely unique.

Lemma 2.2.1. *[86, Theorem 2.1] Let $x = (x_1, \dots, x_n)$ be a vector such that no two subsets of the coordinates have the same arithmetic mean. For $1 \le k \le n$ let $x_{k+n} = x_k$, and let $x(k) = (x_k, x_{k+1}, \dots, x_{k+n})$. Then there is a unique $1 \le k^* \le n$ such that the walk with increments $x(k^*) = (x_{k^*}, x_{k^*+1}, \dots, x_{k^*+n})$ lies below the chord joining its start and end points.*

Proof. (Theorem 2.1.1 under assumption A) By conditioning on the set of values that $X_1, \dots, X_n$ take it is enough to show that $S_\rho^{[0,n]} \stackrel{d}{=} S^{[0,n]}$ in the case where $X_1, \dots, X_n$ are samples without replacement from n real numbers $x_1, \dots, x_n$ such that no two subsets of $x_1, \dots, x_n$ have the same arithmetic mean. Thus it is enough to show that for every permutation $\sigma \in \Sigma_n$ we have

$$\mathbb{P}(X_{\rho(1)} = x_{\sigma(1)}, \dots, X_{\rho(n)} = x_{\sigma(n)}) = \frac{1}{n!}$$

and without loss of generality it is enough to show this for σ the identity permutation. Suppose the concave majorant of the deterministic walk with increments $(x_1, \dots, x_n)$ has k faces whose lengths *in order of appearance* are $(m_1, \dots, m_k)$, so that the composition induced by the lengths of the faces of the concave majorant is $(m_1, \dots, m_k)$. Let $\tau \in \Sigma_k$ be such that

$$(n_1, \dots, n_k) := (m_{\tau(1)}, \dots, m_{\tau(k)})$$

are the lengths of the k faces in *non-increasing order*, so that the partition induced by the lengths of the faces of the concave majorant is $(n_1, \dots, n_k)$.

First suppose that each element of $(n_1, \dots, n_k)$ is distinct. Then the event $\{X_{\rho(\ell)} = x_\ell \,:\, 1 \le \ell \le n\}$ occurs if and only if

(i) the partition chosen according to the lengths of the cycles of a random permutation is $(n_1, \dots, n_k)$;

(ii) for each $1 \le i \le k$, the ordered list $(X_{n_1+\dots+n_{i-1}+1}, \dots, X_{n_1+\dots+n_i})$ is one of the n_i cyclic permutations of the ordered list $(x_{m_1+m_2+\dots+m_{\tau(i)-1}+1}, \dots, x_{m_1+m_2+\dots+m_{\tau(i)}})$.

According to the Ewens Sampling Formula, the event in (i) has probability $\prod_{i=1}^{k} \frac{1}{n_i}$. The event in (ii) is independent of the event in (i), and has probability $\frac{1}{n!} \prod_{i=1}^{k} n_i$.

Now suppose that the elements of $(n_1, \ldots, n_k)$ are not distinct. For $1 \le j \le n$ let $I_j = \{i : n_i = j\}$ and let $a_j = |I_j|$. The event $\{X_{\rho(\ell)} = x_\ell : 1 \le \ell \le n\}$ occurs if and only if

(i) the partition chosen according to the lengths of the cycles of a random permutation is $(n_1, \ldots, n_k)$;

(ii) for each $1 \le j \le n$, for each $i \in I_j$ the ordered list $(X_{n_1+\cdots+n_{i-1}+1}, \ldots, X_{n_1+\cdots+n_i})$ is one of the $n_i = j$ cyclic permutations of the ordered list $(x_{m_1+m_2+\cdots+m_{\tau(i')-1}+1}, \ldots, x_{m_1+m_2+\cdots+m_{\tau(i')}})$ for some $i' \in I_j$.

By the Ewens Sampling Formula, the event in (i) has probability $\left(\prod_{i=1}^{k} \frac{1}{n_i}\right)\left(\prod_{j=1}^{n} \frac{1}{a_j!}\right)$. The event in (ii) is independent of the event in (i), and has probability $\frac{1}{n!}\left(\prod_{i=1}^{k} n_i\right)\left(\prod_{j=1}^{n} a_j!\right)$. Hence $\mathbb{P}(X_{\rho(\ell)} = x_\ell : 1 \le \ell \le n) = \frac{1}{n!}$. □

As a direct consequence of Theorem 2.1.1 we have the result of Goldie [42] mentioned in the introduction.

Corollary 2.2.2. *Let $M_{n,1}, \ldots, M_{n,F_n}$ be the lengths of the faces of the concave majorant of $S^{[0,n]}$ arranged in non-increasing order. Then under assumption **A** the joint distribution of $M_{n,1}, \ldots, M_{n,F_n}$ is given by the formula*

$$\mathbb{P}(F_n = k, M_{n,i} = n_i, 1 \le i \le k) = \prod_{j=1}^{n} \frac{1}{j^{a_j} a_j!}$$

for all $(n_1, \ldots, n_k) \in \mathcal{P}_n$, where $a_j = \#\{i : 1 \le i \le k, n_i = j\}$ for $1 \le j \le n$. I.e. The partition of n induced by the lengths of the faces of the concave majorant of $S^{[0,n]}$ has the law of a partition of n induced by the cycle lengths of a random permutation.

Proof. Following the construction in Theorem 2.1.1, the lengths $L_{n,1}, \ldots, L_{n,K_n}$ are exactly the lengths of the faces of the concave majorant of $S_\rho^{[0,n]}$, and the conclusion follows since $S^{[0,n]} \stackrel{d}{=} S_\rho^{[0,n]}$. □

Further, Theorem 2.1.1 allows us to describe the law of the composition induced by the lengths of the faces of the concave majorant.

Corollary 2.2.3. *Let $(N_{n,1}, \ldots, N_{n,F_n})$ be the composition of n induced by the lengths of the faces of the concave majorant of $S^{[0,n]}$. Then under assumption* **A** *the joint distribution of $N_{n,1}, \ldots, N_{n,F_n}$ is given by the formula*

$$\mathbb{P}(F_n = k, N_{n,i} = n_i, 1 \leq i \leq k) = \mathbb{P}\left(\frac{S_{n_1}^{(1)}}{n_1} > \frac{S_{n_2}^{(2)}}{n_2} > \cdots > \frac{S_{n_k}^{(k)}}{n_k}\right) \prod_{i=1}^{k} \frac{1}{n_i}$$

for all compositions $(n_1, \ldots, n_k)$ of $[n]$ into k parts, where for $1 \leq i \leq k$

$$S_{n_i}^{(i)} := S_{n_1+\cdots+n_i} - S_{n_1+\cdots+n_{i-1}} \stackrel{d}{=} S_{n_i}$$

In particular, if the X_i are independent, then so are the $S_{n_i}^{(i)}$ for $1 \leq i \leq k$.

Proof. Fix a composition $(n_1, \ldots, n_k)$ and let $(\overrightarrow{n}_{\tau(1)}, \ldots, \overrightarrow{n}_{\tau(k)})$ be $(n_1, \ldots, n_k)$ in non-increasing order. Let T be the set of $\tau \in \Sigma_k$ such that $(\overrightarrow{n}_{\tau(1)}, \ldots, \overrightarrow{n}_{\tau(k)}) = (n_1, \ldots, n_k)$. Then $|T| = \prod_{j=1}^{n} a_j$, where $a_j = \#\{i : 1 \leq i \leq k, n_i = j\}$ for $1 \leq j \leq n$. We are interested in comparing the slopes of the faces of the concave majorant that result from the construction in Theorem 2.1.1. In this direction, for $1 \leq i \leq k$ let

$$S_{\overrightarrow{n}_{\tau(i)}}^{(\tau(i))} = S_{\overrightarrow{n}_1+\cdots+\overrightarrow{n}_{\tau(i)}} - S_{\overrightarrow{n}_1+\cdots+\overrightarrow{n}_{\tau(i)-1}} \stackrel{d}{=} S_{\overrightarrow{n}_{\tau(i)}} = S_{n_i}$$

Under the construction in Theorem 2.1.1, the events $\{F_n = k\}$ and $\{N_{n,i} = n_i : 1 \leq i \leq k\}$ occur if and only if

(i) $(L_{n,1}, \ldots, L_{n,K_n}) = (\overrightarrow{n}_1, \ldots, \overrightarrow{n}_k)$;

(ii) $\frac{S_{\overrightarrow{n}_{\tau(1)}}^{(\tau(1))}}{n_1} > \frac{S_{\overrightarrow{n}_{\tau(2)}}^{(\tau(2))}}{n_2} > \cdots > \frac{S_{\overrightarrow{n}_{\tau(k)}}^{(\tau(k))}}{n_k}$ for some $\tau \in T$.

As before, the event in (i) has probability $\left(\prod_{i=1}^{k} \frac{1}{n_i}\right)\left(\prod_{j=1}^{n} \frac{1}{a_j!}\right)$. The event in (ii) is independent of the event in (i), and by exchangeability the probability that it occurs for one particular element of T is

$$\mathbb{P}\left(\frac{S_{n_1}^{(1)}}{n_1} > \frac{S_{n_2}^{(2)}}{n_2} > \cdots > \frac{S_{n_k}^{(k)}}{n_k}\right)$$

Recalling that $|T| = \prod_{j=1}^{n} a_j$ completes the proof. □

2.3 Examples of composition probabilities

The special case of Cauchy increments gives rise to the following appealing version of Corollary 2.2.3.

Corollary 2.3.1. *Suppose that the X_i are independent and such that S_k/k has the same distribution for every k, as when the X_i have a Cauchy distribution. Then*

$$\mathbb{P}(F_n = k; N_{n,i} = n_i, 1 \le i \le k) = \frac{1}{k!}\prod_{i=1}^{k}\frac{1}{n_i}$$

and hence $\{N_{n,i} : 1 \le i \le F_n\}$ has the same distribution as the composition of n created by first choosing a random permutation of n and then putting the cycle lengths in uniform random order.

Proof. Since $\frac{S^{(1)}_{n_1}}{n_1}, \ldots, \frac{S^{(k)}_{n_k}}{n_k}$ is an i.i.d. sequence each of the $k!$ orderings is equally likely, and hence $\mathbb{P}(\frac{S^{(1)}_{n_1}}{n_1} > \cdots > \frac{S^{(k)}_{n_k}}{n_k}) = \frac{1}{k!}$. □

Note that the continuum limit of this result can be read from Bertoin's work [12]. The above result shows that the Cauchy discrete model is the same as that derived by random sampling from the continuum Cauchy model, as per Gnedin's theory of sampling consistent compositions of positive integers [41]. That is, let $U_1, \ldots, U_n$ be independent identically distributed uniform random variables on $[0,1]$ and let X be a Cauchy process on $[0,1]$. Generate a composition of n by putting i in the same block as j if and only if U_i and U_j fall in the same segment of the composition of $[0,1]$ induced by the lengths of the faces of the concave majorant of X, and then ordering blocks according to the ordering of the faces of the concave majorant of X. Then the composition of n that is generated will have the same distribution as $(N_{n,1}, \ldots, N_{n,F_n})$ in Corollary 2.3.1. This does not seem at all obvious a priori, and according to simulation is not true in the Brownian case, suggesting that it is not true in general.

Now let $X_1, \ldots, X_n$ be any exchangeable sequence of random variables satisfying assumption **A**, as in Corollary 2.2.3. We now give some numerical examples of composition probabilities when n is small. Let

$$p(n_1, \ldots, n_k) := \mathbb{P}(F_n = k, N_{n,i} = n_i, 1 \le i \le k)$$

Using symmetry and the partition probabilities given in Corollary 2.2.2, universal values are

$$p(1,1) = 1/2, \quad p(2) = 1/2$$

$$p(3) = 1/3, \quad p(2,1) = p(1,2) = 1/4, \quad p(1,1,1) = 1/6$$
$$p(4) = 1/4, \quad p(1,3) = p(3,1) = 1/6, \quad p(2,2) = 1/8, \quad p(1,1,1,1) = 1/24$$

As n increases, the first values that depend on the particular choice of increment distributions are

$$p(1,1,2) = p(2,1,1) = \frac{1}{2}\mathbb{P}(X_1 > X_2 > \tfrac{1}{2}(X_3 + X_4))$$

$$p(1,2,1) = \frac{1}{2}\mathbb{P}(X_1 > \tfrac{1}{2}(X_2 + X_3) > X_4)$$

where according to the partition probabilities we must have

$$p(1,1,2) + p(2,1,1) + p(1,2,1) = 1/4$$

We consider two special cases - independent Cauchy increments and independent Gaussian increments. When the increments are independent and Cauchy, the 3 probabilities above are equal, with

$$2p(1,2,1) = \mathbb{P}(X_1 > \tfrac{1}{2}(X_2 + X_3) > X_4) = 1/6 = 0.1666666...$$

Note that

$$\mathbb{P}(X_1 > \tfrac{1}{2}(X_2 + X_3) > X_4) = \mathbb{P}(\tfrac{1}{2}(X_2 + X_3) - X_1 < 0 \text{ and } X_4 - \tfrac{1}{2}(X_2 + X_3) < 0).$$

In the centered Gaussian case with $Var(X_1) = 1$ this is the probability of the negative quadrant for a centered bivariate normal with equal variances $3/2$ and covariance $-1/2$ and thus correlation $\rho = -1/3$. That probability is given by

$$\frac{1}{4} + \frac{\arcsin(-1/3)}{2\pi} = 0.195913276$$

The difference with the Cauchy case is quite small. The fact that it is larger is consistent with the known differences in behaviour of the limit partitions for large n after scaling; it is known that the concave majorant of Brownian motion is more likely to have longer faces in its central region than the concave majorant of a Cauchy process. We conclude this section by conjecturing that $p(1,2,1)$ is a monotonic function of the stability index α for symmetric stable laws.

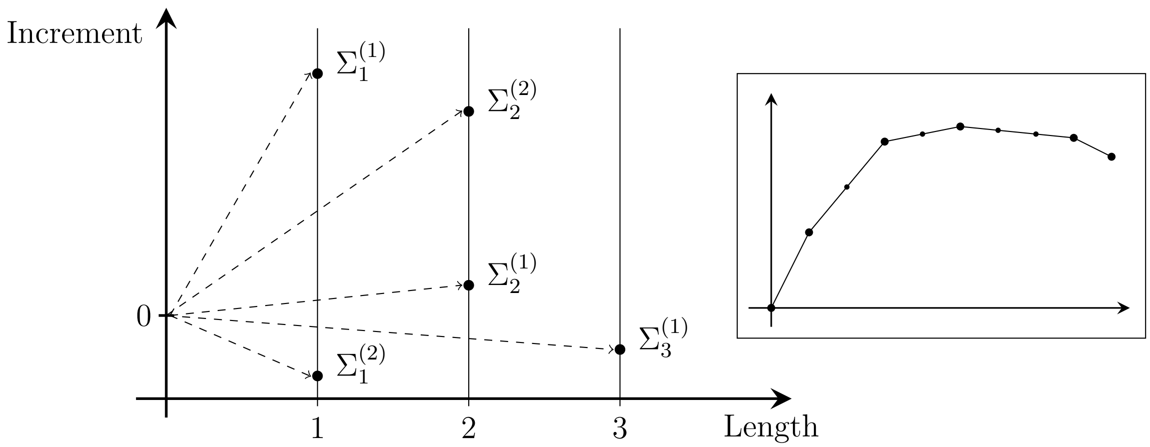

Figure 2.1: An example point process and the resulting concave majorant. The dashed lines show the slope of each face, and these faces are arranged in decreasing order of slope.

2.4 A Poisson point process description

The concave majorant of $S^{[0,n]}$ can be viewed as a random point process on $\{1,\ldots,n\}\times\mathbb{R}$, where a point at (j,s) means that one of the faces of the concave majorant has length j and increment s. Let $A_n(j)$ be the number of faces of the concave majorant of $S^{[0,n]}$ that have length j for $1\le j\le n$, and let $\Sigma_j^{(1)},\ldots,\Sigma_j^{(A_n(j))}$ be the increments of the faces with length j in uniform random order. Thus if $X_1,\ldots,X_n$ are independent then for each $1\le j\le n$, conditionally given $A_n(j)=a_j$, $\Sigma_j^{(\ell)}$ is an independent copy of S_j for each $1\le \ell\le a_j$. Figure 2.1 shows an example of such a point process. To construct the concave majorant from this point process the faces with lengths and increments indicated by the points are arranged in decreasing order of slope.

Now suppose we have an infinite sequence of exchangeable random variables $X_1,X_2,\ldots$, such that almost surely no two subsets have the same arithmetic mean. As before let $S_0=0$ and $S_j=\sum_{i=1}^{j}X_i$ for $j\ge 1$. Following ideas from the fluctuation theory of Greenwood and Pitman [44] we now randomise the length of the walk by setting the number of steps of the random walk equal to $n(q)$, where $n(q)$ is a geometric random variable with parameter $1-q$, so that

$$\mathbb{P}(n(q)\ge n)=q^n \quad \text{for } n=0,1,2,\ldots$$

Let $S^{[0,n(q)]} = \{(j, S_j) : 0 \leq j \leq n(q)\}$, and let

$$0 < N_{n(q),1} < N_{n(q),1} + N_{n(q),2} < \cdots < N_{n(q),1} + \cdots + N_{n(q),F_{n(q)}} = n(q)$$

be the successive times that $S^{[0,n(q)]}$ meets its concave majorant, where $F_{n(q)}$ is the number of faces of the concave majorant of $S^{[0,n(q)]}$. The following Lemma, which involves a fundamental Poisson representation of the geometric distribution, is due to Shepp and Lloyd [83], who were just working with partitions generated by random permutations, not concave majorants.

Lemma 2.4.1. *Let $A_j = \#\{i : 1 \leq i \leq F_{n(q)}, N_{n(q),i} = j\}$ for $j \geq 1$. Then A_j has the Poisson distribution with mean q^j/j, independently for each $j \geq 1$.*

Proof. Noting that $\log(1-q) = -\sum_j q^j/j$, we have that

$$\begin{aligned} \mathbb{P}(A_j = a_j, j \geq 1) &= \mathbb{P}(n(q) = \textstyle\sum_{j\geq 1} j a_j)\mathbb{P}(A_j = a_j, j \geq 1 | n(q) = \sum_{j \geq 1} j a_j) \\ &= (1-q) q^{\sum_j j a_j} \frac{1}{\prod_j j^{a_j} a_j!} \\ &= \prod_j \frac{\left(\frac{q^j}{j}\right)^{a_j} e^{-\frac{q^j}{j}}}{a_j!} \end{aligned}$$

where the second equality comes from Corollary 2.2.2. ☐

For the next theorem, and in fact the rest of this section, it is important that we assume $X_1, X_2, \ldots$ are independent with common continuous distribution. The theorem asserts that the point process discussed above is a Poisson point process under this assumption.

Theorem 2.4.2. *If $X_1, X_2, \ldots$ are independent with common continuous distribution, then the point process of lengths and increments of faces of the concave majorant of $S^{[0,n(q)]}$ is a Poisson point process on $\{1, 2, \ldots\} \times \mathbb{R}$ with intensity $j^{-1} q^j \mathbb{P}(S_j \in dx)$ for $j = 1, 2, \ldots$, $x \in \mathbb{R}$. Moreover, let $T_i = \sum_{l=1}^{i} N_{n(q),l}$, $0 \leq i \leq F_{n(q)}$, be the consecutive times at which $S^{[0,n(q)]}$ meets its concave majorant, so that $T_0 = 0$ and $T_{F_{n(q)}} = n(q)$. Then the sequence of path segments*

$$\{(S_{T_i+k} - S_{T_i}, 0 \leq k \leq N_{n(q),i}), i = 0, \ldots, F_{n(q)} - 1\},$$

is a list of the points of a Poisson point process in the space of finite random walk segments

$$\{(s_1, \ldots, s_j) \text{ for some } j = 1, 2, \ldots\}$$

whose intensity measure on paths of length j is $q^j j^{-1}$ times the conditional distribution of $(S_1, \ldots, S_j)$ given that $S_k < (k/j) S_j$ for all $1 \leq k \leq j-1$.

Proof. Conditionally given $A_j = a_j$ the increment for each face of length j is an independent copy of S_j by Theorem 2.1.1. Combined with Lemma 2.4.1 this proves the first statement.

Conditional on the concave majorant of $S^{[0,n(q)]}$ having a face of length j and increment s, the increments of $S^{[0,n(q)]}$ over that face of the concave majorant have the distribution of $(X_1, \dots, X_j)$ given that $\sum_{\ell=1}^{k} X_\ell < (k/j)s$ for all $1 \leq k \leq j-1$ and $\sum_{\ell=1}^{j} X_\ell = s$, and this law is independent for each face of $S^{[0,n(q)]}$. This implies the second statement. $\square$

A simple but important corollary of Theorem 2.4.2 is the following.

Corollary 2.4.3. *$(n(q), S_{n(q)})$ has a compound Poisson distribution, and the total number of faces $F_{n(q)}$ of the concave majorant of $S^{[0,n(q)]}$ has Poisson distribution with mean*

$$\sum_{j=1}^{\infty} j^{-1} q^j = -\log(1-q).$$

The first assertion of Corollary 2.4.3 can in fact be seen directly since $(n(q), S_{n(q)}) = \sum_{i=1}^{n(q)} (1, X_i)$ and $n(q)$ is itself compound Poisson. Explicitly, $n(q)$ is a Poisson compound of a log-series law: $n(q)$ has probability generating function $\mathbb{E}z^{n(q)} = (1-q)/(1-qz)$ which can be expressed as $e^{-\lambda(1-\phi(z))}$ where $\lambda = -\ln(1-q)$ and ϕ is the probability generating function of the log-series law with parameter q. This well known decomposition of a geometric random variable reappears later in Lemma 2.6.8.

2.5 Applications of the Poissonian description

2.5.1 The random walk on $[0, \infty)$

By letting $q \to 1$ it is possible to deduce the structure of the concave majorant of the random walk on $[0, \infty)$ using Theorem 2.4.2. Groeneboom [45] gave a Poissonian description of the concave majorant of BM on $[0, \infty)$; that there is a closely parallel description for random walks does not seem to have been pointed out before. The case of Lévy processes will be covered in the forthcoming paper by Pitman and Uribe Bravo [73].

Suppose $\mathbb{E}(X_1) = \mu \in [-\infty, \infty)$. Informally, as $q \to 1$ the intensity measure of the Poisson point process of face lengths and increments approaches $j^{-1}\mathbb{P}(S_j \in dx)$, but since the slope of the concave majorant converges downwards to μ but does not reach it, only the faces with slope greater than μ will contribute to the concave majorant in the limit. Therefore by Poisson thinning we get a new intensity measure $j^{-1}\mathbb{P}(S_j \in dx)1(x > j\mu)$. Moreover, we can also describe path segments of the walk below each face of the concave majorant as a Poisson point process.

Theorem 2.5.1. *Let $S_0 = 0$ and $S_j = \sum_{i=1}^{j} X_i$ for $j \geq 1$, where $X_1, X_2, \ldots$ are independent random variables with common continuous distribution that has a well defined mean $\mu := \mathbb{E}(X_1) \in [-\infty, \infty)$. Let $S^{[0,\infty)} = \{(j, S_j) \,:\, j \geq 0\}$. Let $0 = T_0 < T_1 < T_2 < \cdots$ be the successive times that $S^{[0,\infty)}$ meets its concave majorant, and let $N_i = T_i - T_{i-1}$ for $i \geq 1$. Then the sequence of path segments*

$$\{(S_{T_i+k} - S_{T_i}, 0 \leq k \leq N_i), i = 0, 2, \ldots\}$$

is a list of the points of a Poisson point process in the space of finite random walk segments

$$\{(s_1, \ldots, s_j) \text{ for some } j = 1, 2, \ldots\}$$

whose intensity measure on paths of length j is j^{-1} times the restriction to $S_j \in (j\mu, \infty)$ of the conditional distribution of $(S_1, \ldots, S_j)$ given that $S_k < (k/j)S_j$ for all $1 \leq k < j$.

Proof. The combination of the following four facts is enough to prove the theorem:

(i) the number of faces of length j has a Poisson distribution with mean $j^{-1}\mathbb{P}(S_j > j\mu)$;

(ii) these numbers are independent as j varies;

(iii) given all of these numbers, and with n faces of length j, the n walks on the associated faces, when listed in a uniform random order independently of the walks on the faces, are n independent processes each distributed according to $(S_1, \ldots, S_j)$ given that $S_k < (k/j)S_j$ for all $1 \leq k < j$ and $S_j > j\mu$.

(iv) given n faces of length j, the increments of these faces, when listed in uniform random order, are distributed like n independent copies of S_j given $S_j > j\mu$.

The main thing to check is that (i) and (ii) are true, i.e. that the counts

$$A_\infty(j) := \#\{j : N_i = j\}$$

are independent Poisson variables with mean $j^{-1}\mathbb{P}(S_j \geq j\mu)$. Once we have shown this, (iii) and (iv) follow from Poisson thinning and previous discussions relating to the independence of the walks below each segment.

Let $n(q)$ be a geometric random variable with parameter $1 - q$. Let $S^{[0,n(q)]} = \{(j, S_j) \,:\, 0 \leq j \leq n(q)\}$, so that the concave majorant of $S^{[0,n(q)]}$ and $S^{[0,\infty)}$ agree up until some random time $T^*_{n(q)}$.

Lemma 2.5.2. *$T^*_{n(q)}$ is the maximal T_i with $T_i \le n(q)$.*

Proof. To see this, let i be such that $T_i \le n(q)$. Since the concave majorant of $S^{[0,n(q)]}$ is everywhere less than or equal to the concave majorant of $S^{[0,\infty)}$, if they did not agree at time T_i then the concave majorant of $S^{[0,n(q))}$ would go beneath the point (T_i, S_{T_i}), but this is a contradiction since (T_i, S_{T_i}) is in $S^{[0,n(q)]}$. □

Let

$$A_{n(q)}(j) := \#\{i : N_{n(q),i} = j\}$$

where $N_{n(q),1}, \ldots, N_{n(q),F_{n(q)}}$ are the lengths of faces of the concave majorant of $S^{[0,n(q)]}$. There are the obvious decompositions

$$A_\infty(j) = A_\infty(j)(0, T^*_{n(q)}] + A_\infty(j)(T^*_{n(q)}, \infty] \tag{2.5.1}$$

$$A_{n(q)}(j) = A_{n(q)}(j)(0, T^*_{n(q)}] + A_{n(q)}(j)(T^*_{n(q)}, \infty] \tag{2.5.2}$$

where e.g. $A_\infty(j)(0, T^*_{n(q)}]$ is the number of faces of the concave majorant of $S^{[0,\infty)}$ of length j up to and including the face ending at time $T^*_{n(q)}$, and the other terms are defined similarly. Moreover, since $T^*_{n(q)}$ is by definition the maximal common vertex of the concave majorants of $S^{[0,n(q)]}$ and $S^{[0,\infty)}$, it is clear that

$$\begin{aligned} A_\infty(j)(0, T^*_{n(q)}] &= A_{n(q)}(j)(0, T^*_{n(q)}] \\ &= \#\{i : N_{n(q),i} = j,\ S_{T_i} - S_{T_{i-1}} > j\alpha_{n(q)}\} \end{aligned} \tag{2.5.3}$$

where $\alpha_{n(q)}$ is the right derivative of the concave majorant of $S^{[0,\infty)}$ at time $T^*_{n(q)}$. Conditionally given $\alpha_{n(q)}$, by Poisson thinning and Theorem 2.4.2 the distribution of the right hand side of (2.5.3) is Poisson with mean $q^j j^{-1}\mathbb{P}(S_j > j\alpha_{n(q)})$, independently for each j. The strategy at this point is to let $q \to 1$, so that $T_{n(q)} \to \infty$ and $\alpha_{n(q)} \to \mu$, resulting in $A_\infty(j)$ having Poisson distribution with mean $j^{-1}\mathbb{P}(S_j > j\mu)$, independently for each j, i.e. resulting in (i) and (ii).

Let $\{q_m\}_{m\ge1}$ be any sequence such that if $\{n(q_m)\}_{m\ge1}$ is a sequence of independent geometric random variables with parameters $1 - q_m$ then $n(q_m) \to \infty$ almost surely as $m \to \infty$ (so that necessarily $q_m \to 1$). Suppose that $T_{(n(q_m))} \to \infty$ and $\alpha_{n(q_m)} \to \mu$ almost surely, so that

$$\begin{aligned} A_\infty(j) &= \lim_{m\to\infty} A_\infty(j)(0, T_{(n(q_m))}] \\ &= \lim_{m\to\infty} \#\{i : N_{n(q_m),i} = j,\ S_{T_i} - S_{T_{i-1}} > j\alpha_{n(q_m)}\} \end{aligned} \tag{2.5.4}$$

$$\tag{2.5.5}$$

where the first equality is from (2.5.1) and the second is from (2.5.3). Since $\alpha_{n(q_m)} \to \mu$ almost surely, by continuity of the function $x \mapsto \mathbb{P}(S_j > jx)$ the distribution of the right hand side of (2.5.4) is Poisson with parameter $j^{-1}\mathbb{P}(S_j > j\mu)$, independently for each j. This proves (i) and (ii).

It remains to prove that $T_{(n(q_m))} \to \infty$ and $\alpha_{n(q_m)} \to \mu$ almost surely as $m \to \infty$. For every $i \geq 1$, since $T_i < \infty$ we will have $n(q_m) > T_i$ eventually, and hence by Lemma 2.5.2 for every $i \geq 1$ we will have $T_{(n(q_m))} \geq T_i$ eventually. Since $T_i \to \infty$ this implies that $T_{(n(q_m))} \to \infty$ almost surely.

Lemma 2.5.3. *Almost surely no face of the concave majorant of $S^{[0,\infty)}$ can have slope less than μ.*

Proof. If $\mu = -\infty$ then the conclusion is clear. Suppose $\mu \in (-\infty, \infty)$, then since $S_n - n\mu$ is a mean zero random walk and hence recurrent, for every $i \geq 1$ there will almost surely be some $n_i > T_i$ such that $S_{n_i} > S_{T_i} + (n_i - T_i)\mu$, and hence for any vertex of the concave majorant the slope of the face to the right must be greater than μ. □

Lemma 2.5.4. *For every $\epsilon > 0$ there will almost surely be a face of the concave majorant with slope x such that $\mu < x < \mu + \epsilon$.*

Proof. For any $\mu \in [-\infty, \infty)$ by the strong law of large numbers $S_n/n \to \mu$ almost surely as $n \to \infty$. But if there was no slope of the concave majorant on $[0, \infty)$ with slope $x < \mu + \epsilon$ then we would have $\limsup_n S_n/n > \mu$. Combined with Lemma 2.5.3 this gives the conclusion. □

We already have that $T_{(n(q_m))} \to \infty$ almost surely. Since $\alpha_{n(q_m)}$ is the right derivative of the concave majorant of $S^{[0,\infty)}$ at $T_{(n(q_m))}$, Lemma 2.5.4 implies that $\alpha_{n(q_m)} \to \mu$ almost surely as $m \to \infty$. This concludes the proof of Theorem 2.5.1. □

2.5.2 The structure of the concave majorant of $S^{[0,n]}$ as n varies

Theorem 2.1.1 relates to the structure of the concave majorant of a random walk of fixed length, and the Theorems 2.4.2 and 2.5.1 allow randomized lengths or infinite length. So far though, we have not discussed how the structure changes as the number of steps of the walk increases, but theorem 2.5.1 and its proof now allow us to make some comments. Recall that F_n is the number of faces of the concave majorant of $S^{[0,n]} = \{(j, S_j) \ : \ 0 \leq j \leq n)\}$, and in the case where $X_1, \ldots, X_n$ are independent with common continuous distribution we know from (2.1.1) that for each fixed n there is the equality in distribution

$$F_n \stackrel{d}{=} K_n := \sum_{j=1}^{n} I_j$$

where the I_j are independent Bernoulli variables with $\mathbb{P}(I_j = 1) = 1/j$. However, as observed by Steele [88] the identity in law between F_n and K_n does not hold jointly as n varies, and as pointed out by Qiao and Steele [75] the asymptotic behaviour of F_n and K_n as $n \to \infty$ may be quite different. They provide an example of a continuous distribution of X_i such that for each $m = 1, 2, \dots$

$$\mathbb{P}(F_n = m \text{ infinitely often }) = 1$$

It is an easy consequence of theorem 2.5.1 that

$$\mathbb{P}(F_n = 1 \text{ infinitely often }) = 1$$

if and only if $\mathbb{E}(X^+) = \infty$. It appears that the Poisson analysis of $F_{n(q)}$ can be used to provide a more thorough description of the possible asymptotic behaviours of F_n as n varies. In particular, as a consequence of the argument of the proof of Lemma 2.5.2, if $\mathbb{E}(X^+) < \infty$ then F_n is bounded below by the number of faces of the majorant on $[0, n]$ which are part of the majorant on $[0, \infty)$, and this number is increasing in n, with limit ∞.

2.5.3 Decomposition at the maximum

Theorem 2.4.2 provides tools for analyzing the behaviour of the random walk $S^{[0,n(q)]}$ before and after the time it achieves its maximum. By conditioning on $n(q) = n$, we can then do the same for $S^{[0,n]}$. The key idea is that by taking the faces of the concave majorant that have positive slope we get only those faces that lie in the region up to where the random walk achieves its maximum, and by taking the faces with negative slope we get only those faces that lie in the region after the time when the random walk achieves its maximum. This approach was used by Spitzer to find identities involving the maximum of a random walk [86], as indicated in Section 2.1.

Let $X_1, X_2, \dots$ be a sequence of independent random variables with common continuous distribution, and let $S_0 = 0$ and $S_j = \sum_{i=1}^{j} X_i$ for $j \geq 1$. Let $S^{[0,n]} = \{(j, S_j) : 0 \leq j \leq n\}$ and $S^{[0,n(q)]} = \{(j, S_j) : 0 \leq j \leq n(q)\}$. Let L_n be the almost surely unique time at which $S^{[0,n]}$ achieves its maximum, and let the value of the maximum be M_n. Let F_n denote the number of faces of the concave majorant of the walk $S^{[0,n]}$, with the convention $F_0 = 0$, and let $(N_{n,i}, \Delta_{n,i})$ denote the length and increment associated with the ith of these faces. We make similar definitions when n is randomized to $n(q)$.

Theorem 2.5.5. *$(L_{n(q)}, M_{n(q)})$ and $(n(q) - L_{n(q)}, S_{n(q)} - M_{n(q)})$ are independent and both have compound Poisson distributions.*

Proof. By construction

$$\Delta_{n,i} = S_{N_{n,1}+\cdots N_{n,i-1}+N_{n,i}} - S_{N_{n,1}+\cdots N_{n,i-1}}$$

and

$$\begin{aligned} (L_n, M_n) &= \textstyle\sum_{i=1}^{K_n} (N_{n,i}, \Delta_{n,i}) 1(\Delta_{n,i} > 0) \\ (n - L_n, S_n - M_n) &= \textstyle\sum_{i=1}^{K_n} (N_{n,i}, \Delta_{n,i}) 1(\Delta_{n,i} \le 0) \end{aligned}$$

From Theorem 2.4.2 the $(N_{n(q),i}, \Delta_{n(q),i})$ are the points of a Poisson point process on $\{1, 2 \ldots\} \times \mathbb{R}$ with intensity $j^{-1} q^j \mathbb{P}(S_j \in dx), j \in \{1, 2, \ldots\}, x \in \mathbb{R}$, and thus the conclusion follows. ☐

Remark 2.5.6. As discussed in Section 2.1 the compound Poisson nature of $M_{n(q)}$ and $S_{n(q)} - M_{n(q)}$ and their independence was discovered by Greenwood and Pitman [44], but this section gives a more explicit explanation of their distribution.

In the special case where $\mathbb{P}(S_j > 0)$ is constant for $1 \le j \le n$, by conditioning on the event $n(q) = n$ and $L_{n(q)} = \ell$ we can deduce results about the concave majorant of $S^{[0,n]}$ either side of its maximum.

Theorem 2.5.7. *Let $X_1, \ldots, X_n$ be independent with common continuous distribution. Let $S_0 = 0$ and $S_j = \sum_{i=1}^j X_i$ for $1 \le j \le n$, and let $S^{[0,n]} = \{(j, S_j) : 0 \le j \le n\}$. Suppose that $\mathbb{P}(S_j > 0) = p_+$ for $1 \le j \le n$. Then conditionally given $L_n := \arg\max_{0 \le j \le n} S_j = \ell$, the partition generated by the lengths of the faces of the concave majorant of $S^{[0,n]}$ on the interval $[0, \ell]$ is distributed according to the Ewens sampling formula with parameter p_+. That is, if A_j^+ is the number of faces of the concave majorant with positive slope of length j, then for any $\{a_j : j \ge 1\}$ such that $\sum_j j a_j = \ell \le n$,*

$$\mathbb{P}(A_j^+ = a_j, j \ge 1 | L_n = \ell) = \frac{\Gamma(p_+)\ell!}{\Gamma(p_+ + \ell)} \prod_{j=1}^{\ell} \frac{(p_+)^{a_j}}{j^{a_j} a_j!} \tag{2.5.6}$$

The partition generated by the lengths of the faces of the concave majorant of $S^{[0,n]}$ on the interval $[\ell, n]$ is also distributed according to the Ewens sampling formula but with parameter $p_- = 1 - p_+$.

Proof. Let $A_{n(q),j}^+$ be the number of faces of the concave majorant of $S^{[0,n(q)]}$ with positive slope of length j. From the proof of Theorem 2.5.5 it is easy to see that

$A^+_{n(q),j}$ has a Poisson distribution with parameter $j^{-1}q^j p_-$, independently for each j, and independently of $S^{[0,n(q)]}$ after time $L_{n(q)}$. Thus for any $\{a_j : j \geq 1\}$ such that $\sum_j ja_j = \ell$,

$$\begin{aligned}
\mathbb{P}(A^+_j = a_j, j \geq 1 | L_n = \ell) &= \mathbb{P}(A^+_{n(q),j} = a_j, j \geq 1 | L_{n(q)} = \ell, n(q) = n) \\
&= \mathbb{P}(A^+_{n(q),j} = a_j, j \geq 1 | L_{n(q)} = \ell) \\
&= \frac{\mathbb{P}(A^+_{n(q),j} = a_j, j \geq 1)}{\mathbb{P}(L_{n(q)} = \ell)} \\
&= \frac{\prod_j \frac{(p_+)^{a_j} q^{ja_j}}{j^{a_j} a_j!} \exp\{-\frac{p_+ q^j}{j}\}}{\mathbb{P}(L_{n(q)} = \ell)} \qquad (2.5.7)
\end{aligned}$$

Under the assumption $\mathbb{P}(S_j > 0) = p_+$ for $1 \leq j \leq n$, it is known [35, Chapter XII, (8.12)] that for the random walk $S^{[0,n]}$, the almost surely unique index L_n such that $S_{L_n} = \max_{0 \leq j \leq n} S_j$ has the beta-binomial distribution

$$\mathbb{P}(L_n = \ell) = (-1)^n \binom{p_- - 1}{\ell} \binom{p_+ - 1}{n - \ell} \qquad (0 \leq \ell \leq n)$$

which is the mixture of binomial(n, p) distributions for p with beta(p_+, p_-) distribution on $[0, 1]$. Thus

$$\mathbb{P}(L_{n(q)} = \ell) = \frac{\Gamma(p_+ + \ell) q^\ell (1 - q)^{p_+}}{\Gamma(p_+) \ell!}$$

Thus (2.5.7) reduces to (2.5.6). The partition after the maximum is proved similarly. □

2.6 The general case

Let $S_j = \sum_{i=1}^j X_i$ for $1 \leq j \leq n$, where $X_1, X_2, \ldots$ is a sequence of exchangeable random variables. Let $S^{[0,n]} = \{(j, S_j) : 1 \leq j \leq n\}$, and let $\bar{C}^{[0,n]}$ be the concave majorant of $S^{[0,n]}$. The concave majorant in this case, where there may some subsets of $X_1, \ldots, X_n$ that have the same arithmetic mean, is less well studied. However, the literature does contain some results for the case where $X_1, X_2, \ldots$ are also assumed to be independent.

Sparre Andersen [5] introduced the random variable H_n, the number of $1 \leq j \leq n$ such that $S_j = \bar{C}^{[0,n]}(j)$, and F_n, the number of faces of the concave majorant, i.e. the number of distinct slopes in the concave majorant (note that Andersen uses K_n instead of F_n, but we will always use K_n to represent the number of cycles in a random permutation of $[n]$). Figure 2.2 shows an example of a random walk with $F_n = 3$ and $H_n = 8$. Clearly, $F_n \leq H_n$, and in the case of continuous distributions we have $F_n = H_n$ almost surely. Sparre Andersen derived the generating function

$$H(s,t) := \sum_{n=0}^{\infty} \sum_{m=0}^{n} \mathbb{P}(H_n = m) s^n t^m \tag{2.6.1}$$

for all distributions of X_1. As will be shown in Theorem 2.6.5 the theory presented in this section provides a powerful new method of deriving this formula, and in addition a formula for a similar generating function involving F_n.

Sherman [84] introduced a further variable J_n relating to the concave majorant with $H_n \leq J_n \leq F_n$. Sherman deduces a Spitzer identity which relates the generating functions of J_n and Φ_n, the periodicity of $(X_1, \ldots, X_n)$, that is, the maximal number ϕ such that $(X_1, \ldots, X_n) = (X_1, \ldots, X_{n/\phi}, \; \ldots \; , X_1, \ldots, X_{n/\phi})$.

In this section it will be important to make a distinction between *excursions*, *segments* and *faces*, and between their associated compositions of n. The following definitions are illustrated in Figure 2.2.

- An *excursion* is a section of a walk between two integer valued times with the property that the walk touches its concave majorant at the end points of the excursion but lies strictly below it between the end points. The number of distinct excursions of $S^{[0,n]}$ is equal to H_n. Let $\Xi^H_{[0,n]}$ be the composition of n induced by the lengths of the excursions of $S^{[0,n]}_\rho$, the transformed walk of Theorem 2.1.1. Although this has the same distribution as the composition induced by the lengths of the excursions of $S^{[0,n]}$, the forthcoming discussion about *segment* compositions only makes sense for $S^{[0,n]}_\rho$. We say that the *slope* of an excursion is the slope of the line joining its start and end points.

- A *segment* will always refer to one segment of a partition. That is, if $(n_1, \ldots, n_k)$ a partition of n then we say it has k segments with associated lengths $n_1, \ldots, n_k$. As we described in the introduction, to generate a walk with the law of $S^{[0,n]}$ whilst simultaneously getting information about its concave majorant, i.e. to generate $S^{[0,n]}_\rho$, we first choose a random partition induced by the cycle lengths of a uniform random permutation. If we are just interested in the concave majorant of $S^{[0,n]}_\rho$, then we only need to associate a slope with each segment of that partition and then arrange the segments in order of non-increasing slope, where the ordering of any segments with the same slope is chosen uniformly randomly. Keeping track of the end points of the segments results in another induced composition of n, which we call $\Xi^K_{[0,n]}$. This composition arises from our construction and cannot be read off from a given random walk.

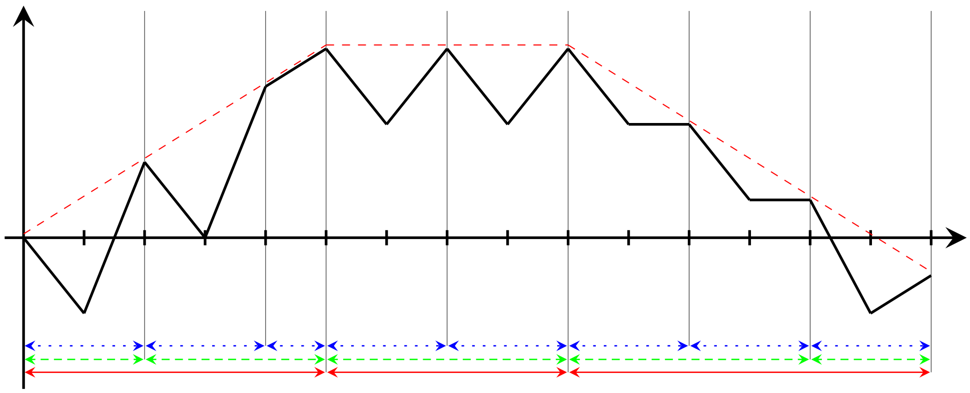

Figure 2.2: An example of a random walk with non-continuous increment distribution for $n = 15$, with $F_n = 3$ and $H_n = 8$. The concave majorant is shown with dashed line. The compositions induced by the excursion lengths and face lengths are fixed by the values of the walk, and an example of a possible composition induced by the lengths of the chords associated with the partition segments is shown. The compositions going from top to bottom are $\Xi^H_{[0,15]} = (2, 2, 1, 2, 2, 2, 2, 2,)$, $\Xi^K_{[0,15]} = (2, 3, 4, 4, 2)$ and $\Xi^F_{[0,15]} = (5, 4, 6)$.

- A *face* will mean one face of the concave majorant. The number of distinct faces is equal to F_n. Let $\Xi^F_{[0,n]}$ be the composition of n induced by the lengths of the faces of $S^{[0,n]}_\rho$. Again, this has the same distribution as the composition of n induced by the lengths of the faces of $S^{[0,n]}$.

- The terms *excursion block*, *segment block* and *face block* will mean blocks of the compositions $\Xi^H_{[0,n]}$, $\Xi^K_{[0,n]}$ and $\Xi^F_{[0,n]}$ respectively, where for example the *blocks* of the composition $(3, 4, 1)$ of 8 in order are defined to be $[0, 3]$, $[3, 7]$ and $[7, 8]$. The slope associated with any block $[a, b]$ is defined by $(S^\rho_b - S^\rho_a)/(b - a)$.

Since the values of any walk on $[0, n]$ between two vertices of its concave majorant, i.e. between the start and end points of some face, are composed of one or many consecutive excursions, $\Xi^H_{[n]}$ is some refinement of $\Xi^F_{[n]}$, which we write as $\Xi^H_{[n]} \preceq \Xi^F_{[n]}$. For $S^{[0,n]}_\rho$ constructed as in Theorem 2.1.1, define H^ρ_n and F^ρ_n similarly to H_n and F_n, and note that $H_n \stackrel{d}{=} H^\rho_n$ and $F_n \stackrel{d}{=} F^\rho_n$. Recall that K_n is the number of segments in the partition chosen at the beginning of the construction. We will have $H^\rho_n \le K_n \le F^\rho_n$, and moreover $\Xi^K_{[0,n]}$ will be such that $\Xi^H_{[0,n]} \preceq \Xi^K_{[0,n]} \preceq \Xi^F_{[0,n]}$. We will discuss these nested compositions further after proving Theorem 2.1.1 in the general case.

Proof. (Theorem 2.1.1) As in the proof of Theorem 2.1.1 under assumption **A**, it is enough to show that if $X_1, \ldots, X_n$ are samples without replacement from a list $x_1, \ldots, x_n$ of real numbers, where now each number is labelled but no longer necessarily distinct in value, then

$$\mathbb{P}(X_{\rho(1)} = x_1, \ldots, X_{\rho(n)} = x_n) = \frac{1}{n!}$$

Let $x = (x_1, \ldots, x_n)$, and suppose this is fixed throughout the proof of the theorem. Let $\bar{c}^{[0,n]}$ be the concave majorant of the deterministic walk with increments $x_1, \ldots, x_n$. Some notation and a couple of combinatorial lemmas are needed before continuing.

For any $n \in \mathbb{N}$, let $\mathcal{N}_n$ be the set of all compositions of n. Let $f \in \mathbb{N}$, $h \in \mathbb{N}$ and $(v_1, \ldots, v_f) \in \mathcal{N}_h$. Let $\mathcal{N}_{(v_1,\ldots,v_f),(k_1,\ldots,k_f)}$ be the set

$$\{(h_1, \ldots, h_{\sum_{i=1}^f k_i}) \in \mathcal{N}_h \ : \ (h_{\sum_{i=1}^{j-1} k_i}, \ldots, h_{\sum_{i=1}^{j} k_i}) \in \mathcal{N}_{v_j} \text{ for } 1 \le j \le f\}$$

Thus an element of $\mathcal{N}_{(v_1,\ldots,v_f),(k_1,\ldots,k_f)}$ is a composition of h formed by joining together compositions of $v_1, \ldots, v_f$ which contain $k_1, \ldots, k_f$ blocks respectively (and hence $\mathcal{N}_{(v_1,\ldots,v_f),(k_1,\ldots,k_f)}$ may be an empty set for some values of $(k_1, \ldots, k_f)$).

Lemma 2.6.1. *Let $f \in \mathbb{N}$, $h \in \mathbb{N}$ and $(v_1, \ldots, v_f) \in \mathcal{N}_h$. Then*

$$\sum_{k=f}^{h} \sum_{(k_1,\ldots,k_f) \in \mathcal{N}_k} \sum_{(h_1,\ldots,h_k) \in \mathcal{N}_{(v_1,\ldots,v_f),(k_1,\ldots,k_f)}} \prod_{i=1}^{k} \frac{1}{k_1! \cdots k_f!} \frac{1}{h_1 \cdots h_k} = 1 \qquad (2.6.2)$$

Proof. The numbers that are being summed over bear a strong resemblance to the unsigned Stirling numbers of the first kind $|S(n,k)|$, which enumerate the number of permutations of n with k cycles. Using this as a guide, consider a set A consisting of permutations of $v_1, \ldots, v_f$, where permutations corresponding to v_i and v_j with $i \neq j$ are considered distinct even if they are identical. The number of such sets where for each $1 \le j \le f$ the permutation of v_j has k_j cycles of sizes $h_{\sum_{i=1}^{j-1} k_i}, \ldots, h_{\sum_{i=1}^{j} k_i}$ is

$$\frac{v_1! \cdots v_f!}{k_1! \cdots k_f! \ \cdot \ h_1 \cdots h_k}$$

Since the total number of elements of A is $v_1! \cdots v_f!$, and the summation in (2.6.2) simplifies to be the sum over the subsets of A such that for each $1 \le j \le f$ the permutation of v_j has k_j cycles of sizes $h_{\sum_{i=1}^{j-1} k_i}, \ldots, h_{\sum_{i=1}^{j} k_i}$, the value of the sum must be 1. $\square$

Let $f(\bar{c}^{[0,n]})$ be the number of faces of $\bar{c}^{[0,n]}$, and let $\ell_1(\bar{c}^{[0,n]}),\ldots,\ell_{f(\bar{c}^{[0,n]})}(\bar{c}^{[0,n]})$ be the lengths of those faces, arranged in the order those faces appear in $\bar{c}^{[0,n]}$. Let $\mathcal{N}(\bar{c}^{[0,n]})$ be the set

$$\{(n_1,\ldots,n_k)\in\mathcal{N}_n\ :\ \exists\ k_1<\cdots<k_{f(\bar{c}^{[0,n]})}\text{ s.t. }\sum_{i=k_{j-1}}^{k_j}n_i=\ell_j(\bar{c}^{[0,n]}),1\le j\le f(\bar{c}^{[0,n]})\}$$

Loosely, $\mathcal{N}(\bar{c}^{[0,n]})$ is the set of possible values for $\Xi^K_{[0,n]}$ conditionally given that the concave majorant of $S^{[0,n]}_\rho$ is $\bar{c}^{[0,n]}$. For $(n_1,\ldots,n_k)\in\mathcal{N}(\bar{c}^{[0,n]})$, let

$$\{k_j(n_1,\ldots,n_k),1\le j\le f(\bar{c}^{[0,n]})\}=\{(k_1,\ldots,k_{f(\bar{c}^{[0,n]})})\ :\ \sum_{i=k_{j-1}}^{k_j}n_i=\ell_j(\bar{c}^{[0,n]})\}$$

Then $k_j(\Xi^K_{[0,n]})$ represents the number of blocks of $\Xi_{[0,n]}$ that lie in the jth face block, i.e. in the jth block of $\Xi^F_{[0,n]}$. Finally, let

$$\mathcal{N}_x(\bar{c}^{[0,n]})=\{(n_1,\ldots,n_k)\in\mathcal{N}(\bar{c}^{[0,n]})\ :\ \sum_{j=1}^{n_i}x_j=\bar{c}^{[0,n]}(n_i)\text{ for }1\le i\le k\}$$

Then $\mathcal{N}_x(\bar{c}^{[0,n]})$ is the set of possible values for $\Xi^K_{[0,n]}$ conditionally given that $\{X_{\rho(i)}=x_i\ :\ 1\le i\le n\}$.

Lemma 2.6.2. *For every composition* $(n_1,\ldots,n_k)\in\mathcal{N}_x(\bar{c}^{[0,n]})$, *for* $1\le i\le k$ *let*

$$h_i(x,n_1,\ldots,n_k)=\#\{j:n_1+\cdots+n_{i-1}<j\le n_1+\cdots+n_i\,,\,\textstyle\sum_{l=1}^j x_l=\bar{c}^{[0,n]}(j)\}$$

Then

$$\sum_{k=1}^n\sum_{(n_1,\ldots,n_k)\in\mathcal{N}_x(\bar{c}^{[0,n]})}\left(\prod_{i=1}^k\frac{1}{h_i(x,n_1,\ldots,n_k)}\right)\left(\prod_{j=1}^{f(\bar{c}^{[0,n]})}\frac{1}{k_j(n_1,\ldots,n_k)!}\right)=1\quad(2.6.3)$$

Proof. Let $h=\#\{j:1\le j\le n,\sum_{l=1}^j=\bar{c}^{[0,n]}(j)\}$ and for $1\le i\le f(\bar{c}^{[0,n]})$ let

$$v_i(x)=\\\#\{j:\ell_1(\bar{c}^{[0,n]})+\cdots+\ell_{i-1}(\bar{c}^{[0,n]})<j\le\ell_1(\bar{c}^{[0,n]})+\cdots+\ell_i(\bar{c}^{[0,n]})\,,\,\textstyle\sum_{l=1}^j x_l=\bar{c}^{[0,n]}(j)\}$$

Associate with each composition $(n_1,\ldots,n_k)\in\mathcal{N}_x(\bar{c}^{[0,n]})$ of length k a composition of h

$$(h_1(x,n_1,\ldots,n_k),h_2(x,n_1,\ldots,n_k),\ldots,h_k(x,n_1,\ldots,n_k))$$

so that there is a bijection between the elements of $\mathcal{N}_x(\bar{c}^{[0,n]})$ with k blocks and the set of compositions $(h_1, \ldots, h_k)$ of h with k blocks that are formed by joining together in order compositions of $v_1, \ldots, v_{f(\bar{c}^{[0,n]})}$ which have $k_1, \ldots, k_{f(\bar{c}^{[0,n]})}$ blocks respectively. Thus the term on the left hand side of (2.6.3) is

$$\sum_{k=f}^{h} \sum_{(k_1,\ldots,k_{f(\bar{c}^{[0,n]})})\in\mathcal{N}_k} \sum_{(h_1,\ldots h_k)\in\mathcal{N}_{(v_1,\ldots,v_f),(k_1,\ldots,k_f)}} \prod_{i=1}^{k} \frac{1}{k_1!\cdots k_f!}\frac{1}{h_1\cdots h_k}$$

which by Lemma 2.6.1 is 1. □

Fix a composition $(n_1, \ldots, n_k)$ of n. For $1 \le j \le n$ let $I_j = \{i : n_i = j\}$ and let $a_j = |I_j|$. Following the construction of $S_\rho^{[0,n]}$ described in the introduction, we see that the event $\{\Xi_{[0,n]}^K = (n_1, \ldots, n_k)$ and $X_{\rho(\ell)} = x_\ell, 1 \le \ell \le n\}$ occurs if and only if

(i) $L_{n,1}, \ldots, L_{n,K_n}$ is $(n_1, \ldots, n_k)$ in non-increasing order;

(ii) for each $1 \le j \le n$, for each $i \in I_j$ the ordered list $(X_{n_1+\cdots+n_{i-1}+1}, \ldots, X_{n_1+\cdots+n_i})$ is one of the $n_i = j$ cyclic permutations of the ordered list $(x_{m_1+m_2+\cdots+m_{\tau(i')-1}+1}, \ldots, x_{m_1+m_2+\cdots+m_{\tau(i')}})$ for some $i' \in I_j$;

(iii) for each $1 \le j \le n$, for each $i \in I_j$ the cyclic permutation that is chosen for the ordered list of increments $(X_{n_1+\cdots+n_{i-1}+1}, \ldots, X_{n_1+\cdots+n_i})$ is the unique cyclic permutation that results in the ordered list becoming exactly $(x_{m_1+m_2+\cdots+m_{\tau(i')-1}+1}, \ldots, x_{m_1+m_2+\cdots+m_{\tau(i')}})$;

(iv) for each $1 \le j \le f(\bar{c})^{[0,n]})$ the ordering of the $k_j(n_1, \ldots, n_k)$ segments within the jth face is chosen correctly out of the $k_j!$ possible orderings.

Recall that for $1 \le i \le k$ we have

$$h_i(x, n_1, \ldots, n_k) = \#\{j : n_1 + \cdots + n_{i-1} < j \le n_1 + \cdots + n_i\,,\ \textstyle\sum_{l=1}^{j} x_l = \bar{c}^{[0,n]}(j)\}$$

so that in (iii) there are $\prod_{i=1}^{k} h_i(x, n_1, \ldots, n_k)$ possible choices of combinations of cyclic permutations. Then the probability of the event $\{\Xi_{[0,n]}^K = (n_1, \ldots, n_k)$ and $X_{\rho(\ell)} = x_\ell, 1 \le \ell \le n\}$ is

$$\left(\prod_{j=1}^{n}\frac{1}{a_j!}\prod_{i=1}^{k}\frac{1}{n_i}\right)\left(\frac{1}{n!}\prod_{i=1}^{k} n_i \prod_{j=1}^{n} a_j!\right)\left(\prod_{i=1}^{k}\frac{1}{h_i(x,n_1,\ldots,n_k)}\right)\left(\prod_{j=1}^{f(\bar{c}^{[0,n]})}\frac{1}{k_j(n_1,\ldots,n_k)!}\right)$$

where the first two terms should be familiar from the proof of Theorem 2.1.1 under assumption **A**. Finally, by summing this probability over all possible compositions, we have that the probability of the event $\{X_{\rho(\ell)} = x_\ell, 1 \le \ell \le n\}$ is

$$\frac{1}{n!}\sum_{k=1}^{n} \sum_{(n_1,\ldots,n_k)\in\mathcal{N}_x(\bar{c}^{[0,n]})} \left(\prod_{i=1}^{k} \frac{1}{h_i(x,n_1,\ldots,n_k)}\right)\left(\prod_{j=1}^{f(\bar{c}^{[0,n]})} \frac{1}{k_j(n_1,\ldots,n_k)!}\right) = \frac{1}{n!}$$

where the equality is by Lemma 2.6.2. This completes the proof of Theorem 2.1.1. □

In the case where $X_1, X_2, \ldots$ are independent, the Poisson point process ideas of Section 2.4 lead to a simpler description of the concave majorant. For the rest of this section it is assumed that $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables and $n(q)$ is a geometric variable with parameter $1-q$. Let $S^{[0,n(q)]} = \{(j, S_j) : 0 \le \jmath \le n(q)\}$, where $S_0 = 0$ and $S_j = \sum_{i=1}^{j} X_i$ for $j \ge 1$. Let $\bar{C}^{[0,n]}$ be the concave majorant of $S^{[0,n(q)]}$. The following theorem is the extension to the non-continuous increment case of Theorem 2.4.2.

Theorem 2.6.3. *If $X_1, X_2, \ldots$ are independent with common distribution and $n(q)$ a geometric variable with parameter $1-q$, then the lengths and increments of the faces of the concave majorant of the random walk $S^{[0,n(q)]}$ have the following law. Let $\mathfrak{P}$ be a Poisson point process of on $\{1,2,\ldots\}\times\mathbb{R}$ with intensity $j^{-1}q^j\mathbb{P}(S_j \in dx)$ for $j = 1,2,\ldots$, $x \in \mathbb{R}$. Note that this process may result in multiple points at the same location. Each point of $\mathfrak{P}$ represents the length and increment of a chord associated with some segment of a partition of $n(q)$. Chords with the same slope are joined together in uniform random order, independently of their lengths, to form the faces of the concave majorant. Moreover, let $K_{n(q)}$ be the total number of chords associated with partition segments and for $1 \le i \le K_{n(q)}$ let $N_{n(q),i}$ be the length of the ith of these chords once they have been ordered by decreasing slope and uniform randomization of ties. Then the sequence of path segments*

$$\{(S_{\sum_{l=1}^{i-1} N_{n(q),l}+k} - S_{\sum_{l=1}^{i-1} N_{n(q),l}}, 0 \le k \le N_{n(q),i}), i = 1,\ldots,K_{n(q)}\}$$

is a list of the points of a Poisson point process in the space of finite random walk segments

$$\{(s_1,\ldots,s_j) \text{ for some } j = 1,2,\ldots\}$$

whose intensity measure on paths of length j is j^{-1} times the conditional distribution of $(S_1,\ldots,S_j)$ given that $S_k < (k/j)S_j$ for all $1 \le k < j$. Again, this Poisson point process may result in multiple points at the same location.

Proof. For any $n \in \mathbb{N}$, conditionally given $n(q) = n$, the projection of the points of $\mathfrak{P}$ onto $\{1, \dots, n\}$ has the law of a partition of n generated by the cycle lengths of a random permutation of $[n]$ by Lemma 2.4.1. Hence we know from Theorem 2.1.1 that for every $n \in \mathbb{N}$, conditionally given $n(q) = n$, the process described in the theorem gives the correct law for the concave majorant of $S^{[0,n]}$ and gives the correct law for $\Xi^K_{[0,n]}$, the composition induced by the lengths of the partition segments involved in creating $S^{[0,n]}_\rho$. The remaining assertions follow by independence of the walks associated with each partition segment. $\square$

We now move towards describing the joint law of the nested compositions $\Xi^H_{[0,n(q)]} \preceq \Xi^K_{[0,n(q)]} \preceq \Xi^F_{[0,n(q)]}$ in the case where $X_1, X_2, \dots$ are independent and the walk has geometric length. The full description of this law will be given in Theorem 2.6.9 at the end of this section, along with some applications of the theory. Let $S^{[0,n(q)]}_\rho$ be such that conditionally given $n(q) = n$, $S^{[0,n(q)]}_\rho$ is constructed in the same way as $S^{[0,n]}_\rho$ in Theorem 2.1.1, and let $\bar{C}^{[0,n(q)]}_\rho$ be the concave majorant of $S^{[0,n]}_\rho$. We begin by describing the laws of $H_{n(q)}$, $K_{n(q)}$ and $F_{n(q)}$, which are defined to be the number of excursions, segments and faces respectively of $\bar{C}^{[0,n(q)]}_\rho$.

We need some new notation, some of which is taken from Sparre Andersen [5]. Let $x_1, x_2, \dots$ be an enumeration of the set of real numbers x for which $\mathbb{P}(S_k = kx)$ is positive for some $k > 0$, and let

$$\begin{aligned}
\mu_j(q) &= \sum_{k=1}^{\infty} k^{-1} q^k \mathbb{P}(S_k = kx_j), \qquad \text{for } j = 1, 2, \dots \\
\mu_0(q) &= \sum_{k=1}^{\infty} k^{-1} q^k \mathbb{P}(S_k \neq kx_j \text{ for } j = 1, 2, \dots) \\
&= -\log(1-q) - \sum_{j=1}^{\infty} \mu_j(q)
\end{aligned}$$

Proposition 2.6.4. *Let $H_{q,j}$, $K_{q,j}$ and $F_{q,j}$ be the number of excursion, segments and faces in $\bar{C}^{[0,n(q)]}_\rho$ of slope x_j for $j \geq 1$. Then for each $j \geq 1$*

(i) $H_{q,j}$ *is a geometric random variable with parameter* $\exp(-\mu_j(q))$, *independently of* $\{H_{q,i} : i \neq j\}$.

(ii) $K_{q,j}$ *is a Poisson random variable with parameter* $\mu_j(q)$, *independently of* $\{K_{q,i} : i \neq j\}$.

(iii) $F_{q,j}$ *is a Bernoulli random variable with parameter* $1 - \exp(-\mu_j(q))$, *independently of* $\{F_{q,i} : i \neq j\}$.

Let $H_{q,0}$, $K_{q,0}$ and $F_{q,0}$ be the number of excursion, segments and faces with slope not equal to x_j for any $j \geq 1$. Then

(iv) $H_{q,0} = K_{q,0} = F_{q,0}$ *almost surely and their common distribution is Poisson with parameter $\mu_0(q)$, independently of $\{H_{q,j}, K_{q,j}, F_{q,j} : j \geq 1\}$.*

Proof. (ii) follows from Theorem 2.6.3, (iii) is implied by (ii) since a face of slope x exists if and only if there is at least one segment of slope x, and (iv) is also implied by Theorem 2.6.3 since it concerns the restriction of the Poisson point process to slopes which have zero probability, as in the case of continuous increment distributions.

Fix $j \geq 1$. (ii) implies that $\mathbb{P}(H_{q,j} \geq 1) = \mathbb{P}(K_{q,j} \geq 1) = 1 - \exp(-\mu_j(q))$. Given that there at least n excursions of slope x_j, by the memoryless property of the geometric distribution of $n(q)$, the law of the remaining values of the walk $S_\rho^{[0,n(q)]}$ is the same as the law of a walk generated by the Poisson process of path segments in Theorem 2.6.3 but thinned to only include segments with slope $x \geq x_j$. Thus

$$\mathbb{P}(H_{q,j} \geq n+1 | H_{q,j} \geq n) = \mathbb{P}(K_{q,j} \geq 1) = 1 - \exp(-\mu_j(q))$$

which proves (i). □

Theorem 2.6.5. *Let H_n and F_n be the number of excursions and faces for $S^{[0,n]}$, and let K_n be the number of segments for $S_\rho^{[0,n]}$. Then for $0 \leq s, t \leq 1$,*

$$\begin{aligned} H(s,t) &= e^{t\mu_0(s)} \prod_{j=1}^{\infty} \frac{1}{1 - t + te^{-\mu_j(s)}} \\ K(s,t) &= e^{t\mu_0(s)} \prod_{j=1}^{\infty} e^{t\mu_j(s)} = (1-s)^{-t} \\ F(s,t) &= e^{t\mu_0(s)} \prod_{j=1}^{\infty} (1 - t + te^{\mu_j(s)}) \end{aligned}$$

Proof. Recall first that $H_n^\rho \stackrel{d}{=} H_n$ and $F_n^\rho \stackrel{d}{=} F_n$. Let $n(s)$ be a geometric random variable with parameter $1 - s$ and consider the walk of $n(s)$ steps. We have by definition

$$H_{n(s)} = H_{s,0} + \sum_{j=1}^{\infty} H_{s,j}$$

Thus the generating function of $H_{n(s)}$ is the product of the generating functions of $H_{s,0}$ and $H_{s,j}$, $j \geq 1$. These are known from Proposition 2.6.4, thus

$$\begin{aligned}\sum_{m=0}^{\infty} t^m \mathbb{P}(H_{n(s)} = m) &= e^{(t-1)\mu_0(s)} \prod_{j=1}^{\infty} \frac{e^{-\mu_j(s)}}{1-t+te^{-\mu_j(s)}} \\ &= (1-s)e^{t\mu_0(s)} \prod_{j=1}^{\infty} \frac{1}{1-t+te^{-\mu_j(s)}}\end{aligned}$$

We can conclude that

$$\begin{aligned}H(s,t) &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \mathbb{P}(H_n = m)s^n t^m \\ &= (1-s)^{-1} \sum_{m=0}^{\infty} t^m \sum_{n=m}^{\infty} (1-s)s^n \mathbb{P}(H_n = m) \\ &= (1-s)^{-1} \sum_{m=0}^{\infty} t^m \mathbb{P}(H_{n(s)} = m) \\ &= e^{t\mu_0(s)} \prod_{j=1}^{\infty} \frac{1}{1-t+te^{-\mu_j(s)}}\end{aligned}$$

The deduction for $F(s,t)$ is similar.

The generating function $G_{K_n}(z) = \sum_{m=1}^{\infty} z^m \mathbb{P}(K_n = m)$ is well known from the equality in (2.1.1), which leads easily to the result for $K(s,t)$. □

Remark 2.6.6. $H(s,t)$ in Theorem 2.6.5 is as in (2.6.1) and agrees with Sparre Andersen's formula [5, Theorem 2].

In order to fully describe the joint law of the nested compositions, two more lemmas are necessary. The first contains information about the lengths of each segment or excursion, and the second describes how many excursions there are in each segment. We already know from the Poissonian description of the concave majorant the distribution of the number of segments with a given slope, and thus we already know the distribution of the number of segments within each face (see Theorem 2.6.9 for the full description).

Lemma 2.6.7. *Consider the walk of $n(q)$ steps. For $j \geq 1$, conditionally given $K_{q,j} = k_{q,j}$, let $L^K_{q,j,1}, \ldots, L^K_{q,j,k_{q,j}}$ be the lengths of the $k_{q,j}$ segments of $S^{[0,n(q)]}_{\rho}$ of slope x_j. Then $L^K_{q,j,1}, \ldots, L^K_{q,j,k_{q,j}}$ are independent from each other and the lengths of all* *other segments¿ Moreover they are identically distributed with common probability generating function* $G_{L^K_{q,j}}(z) = \mu_j(zq)/\mu_j(q)$.

For $j \geq 1$, *conditionally given* $H_{q,j} = h_{q,j}$, *let* $L^H_{q,j,1}, \ldots, L^H_{q,j,h_{q,j}}$ *be the lengths of the* $h_{q,j}$ *excursions of* $S_\rho^{[0,n(q)]}$ *of slope* x_j. *Then* $L^K_{q,j,1}, \ldots, L^K_{q,j,h_{q,j}}$ *are independent from each other and the lengths of all other segments. Moreover they are identically distributed with common probability generating function* $G_{L^H_{q,j}}(z) = (1-e^{-\mu_j(zq)})/(1-e^{-\mu_j(q)})$.

Furthermore, each excursion in the face of slope x_j *is independent and has the law of a random walk with increment distribution* X_1 *conditioned on making its first return to the line through the origin with slope* x_j *before* $n(q)$, *an independent geometric random variable with parameter* $1-q$, *and remaining below that line before its first return time – the excursion is taken to be that walk up to the time of its first return to the line with slope* x_j.

Proof. By Poisson process properties, each $L^K_{q,j,1}, \ldots, L^K_{q,j,h_{q,j}}$ are independent from each other and the lengths of all other segments. By Poisson thinning, $\mathbb{P}(L^K_{q,j,1} = l) = l^{-1}q^l\mathbb{P}(S_k = kx_j)$, which gives the claimed generating function.

By the memoryless property of the geometric distribution of $n(q)$, each excursion of slope x_j is independent, and is clearly independent from all excursions of other slopes. This gives the final assertion of the Lemma. By considering the total lengths of the face with slope x_j we see that

$$\sum_{i=1}^{H_{q,j}} L^H_{q,j,i} = \sum_{i=1}^{K_{q,j}} L^K_{q,j,i}$$

By comparing the generating functions of both sides and using Proposition 2.6.4 we can deduce the claimed generating function $G_{L^H_{q,j}}(z)$. □

Lemma 2.6.8. *Conditionally given there are* $k_{q,j}$ *segments of* $S_\rho^{[0,n(q)]}$ *of slope* x_j, *let* $E_{q,j,1}, \ldots, E_{q,j,k_{q,j}}$ *be the number of excursions in each of those* $k_{n(q)}$ *segments. Then* $E_{q,j,1}, \ldots, E_{q,j,k_{q,j}}$ *are independent of each other and all other excursions and are identically distributed. Their common distribution is the log-series distribution with parameter* $1 - e^{-\mu_j(q)}$, *that is*

$$\mathbb{P}(E_{q,j,1} = i) = \frac{(1-e^{-\mu_j(q)})^i}{i\mu_j(q)} \quad , i = 1, 2, \ldots$$

Proof. By Theorem 2.6.3 the values of the walk $S_\rho^{[0,n(q)]}$ over each segment are independent, which gives the independence of $E_{q,j,1}, \ldots, E_{q,j,k_{q,j}}$. By the independence of the excursions in the face of slope x_j and the independence of the walks over each segment of slope x_j, $L^H_{q,j,1}, \ldots, L^H_{q,j,E_{q,j}}$ are independent and identically distributed. By considering the total length of each segment of slope x_j, we have the identity in distribution

$$L^K_{q,j,1} \stackrel{d}{=} \sum_{i=1}^{E_{q,j,1}} L^H_{q,j,1}$$

which after applying generating function analysis reveals that

$$G_{E_{q,j,1}}(z) := \sum_{l=1}^{\infty} z^i \mathbb{P}(E_{q,j,1} = i) = \sum_{i=1}^{\infty} z^i \frac{(1 - e^{-\mu_j(q)})^i}{i\mu_j(q)}$$

□

We are now ready to describe the joint law of the three nested compositions $\Xi^H_{[0,n(q)]} \preceq \Xi^K_{[0,n(q)]} \preceq \Xi^F_{[0,n(q)]}$. The following theorem is a summary of most of the information from Theorem 2.6.3 to Lemma 2.6.8.

Theorem 2.6.9. *Let $n(q)$ be a geometric random variable with parameter $1-q$. Let $X_1, X_2, \ldots$ be independent and identically distributed. Let $S_j = \sum_{i=1}^{j} X_i$ for $j \geq 1$. Let $x_1, x_2, \ldots$ be an enumeration of the set of real numbers x for which $\mathbb{P}(S_k = kx)$ is positive for some $k > 0$, and for $j \geq 1$ let*

$$\mu_j(q) = \sum_{k=1}^{\infty} k^{-1} q^k \mathbb{P}(S_k = kx_j)$$

Let $S^{[0,n(q)]}_\rho$ be such that conditionally given $n(q) = n$, $S^{[0,n(q)]}_\rho$ is constructed in the same way as $S^{[0,n]}_\rho$ in Theorem 2.1.1. Let $\bar{C}^{[0,n(q)]}_\rho$ be the concave majorant of $S^{[0,n(q)]}_\rho$. Then independently for each $j \geq 1$:

- *There is a face of $\bar{C}^{[0,n(q)]}_\rho$ with slope x_j with probability $1 - e^{-\mu_j(q)}$.*
- *Conditionally given there is a face of slope x_j the number of blocks of $\Xi^K_{[0,n]}$ with associated slope x_j has the Poisson distribution with parameter $\mu_j(q)$, conditionally on the value being at least one.*
- *Conditionally given there are $k_{q,j}$ blocks of $\Xi^K_{[0,n]}$ with associated slope x_j, the number of excursions blocks in each of the $k_{q,j}$ segment blocks has the log-series distribution with parameter $1 - e^{-\mu_j(q)}$, independently for each segment.*

- *The length of each excursion of slope x_j is independent of all other excursions and has distribution with generating function*

$$G_{L^H_{q,j}}(z) = (1 - e^{-\mu_j(zq)})/(1 - e^{-\mu_j(q)})$$

Any face block with associated slope x such that $x \neq x_j$ for any $j \geq 1$ will be comprised of exactly one segment block, which will also be comprised of exactly one excursion block. The lengths and increments of faces with slope x such that $x \neq x_j$ for any $j \geq 1$ form a Poisson point process on $\{1, 2, \ldots\} \times \mathbb{R}$ with intensity $i^{-1}\mathbb{P}(S_i \in ds)$ for $i \geq 1, s \in \mathbb{R}$, but restricted to the region

$$\{(i, s) \in \{1, 2, \ldots\} \times \mathbb{R} : s \neq ix_j \text{ for any } j \geq 1\}$$

Three nested compositions with the joint law of $\Xi^H_{[0,n(q)]}$, $\Xi^K_{[0,n(q)]}$ and $\Xi^F_{[0,n(q)]}$ are created by uniformly randomly ordering the excursions within each segment, uniformly randomly ordering the segments within each face, arranging the faces in order of decreasing slope, and then looking at the induced compositions of excursion blocks, segment blocks and face blocks.

Theorem 2.6.9 implies that the compositions $\Xi^H_{[0,n(q)]} \preceq \Xi^K_{[0,n(q)]} \preceq \Xi^F_{[0,n(q)]}$ can be generated by nested renewal processes on $\mathbb{N}$ that terminate at some geometric time. There would be three types of renewal epochs. The first would be when a new face block started, which implies a new segment block and excursion block would also start. The second would be when only a new segment block and excursion block started, and the third would be when only a new excursion block started. Unlike in previous investigations into nested renewal sequences [18, 30], the distributions of the length until the next renewal may change with time, and after a renewal has occurred, the number of future renewals may depend on how many have already occurred.

Theorem 2.6.9 allows us to readily compute the probability of many fluctuation events for $S^{[0,n(q)]}$. Some examples are

- For each $j \geq 1$, the probability that $\bar{C}^{[0,n(q)]}$ consists of only one face of slope x_j is $(1-q)^{-1}e^{-\mu_j(q)}$.
- The probability that $S^{[0,n(q)]}$ has a unique minimum, i.e. the probability that $\bar{C}^{[0,n(q)]}$ has no face of slope zero, is $\exp[-\sum_{k=1}^{\infty} k^{-1}q^k\mathbb{P}(S_k = 0)]$.
- For each $j \geq 1$, the expected length of the face of $\bar{C}^{[0,n(q)]}$ of slope x_j is $\sum_{k=1}^{\infty} q^k\mathbb{P}(S_k = kx_j)$.

2.7 $S^{[0,n]}$ conditional on its concave majorant

To complete the rearrangement problem stated in the introduction, we now give a description of the law of $S^{[0,n]}$ conditional on $\bar{C}^{[0,n]} = \bar{c}^{[0,n]}$. It is a generalization of the well known Vervaat transform for turning a bridge of a random walk into an excursion [91, Theorem 5]. It relies on first choosing a segment composition $\Xi^K_{[0,n]}$ conditional on $\bar{C}^{[0,n]}_\rho = \bar{c}^{[0,n]}$ and then choosing a walk conditional on $\Xi^K_{[0,n]}$.

Let Supp$(\bar{C}^{[0,n]})$ be the support of the measure on concave functions on $[0,n]$ that represents the law of $\bar{C}^{[0,n]}$. For any composition $(n_1,\dots,n_k)$ of n we say that $\sigma \in \Sigma_n$ is a $(n_1,\dots,n_k)$-cyclic permutation of $[n]$ if its only action is to cyclically permute the first n_1 elements of $[n]$, cyclically permute the next n_2 elements of $[n]$ and so on. For example, 234175689 is a $(4,3,2)$-cyclic permutation of [9]. Recall that in Section 2.6 we defined $\mathcal{N}_n$ to be the set of compositions of n, and $\mathcal{N}(\bar{c}^{[0,n]}) \subseteq \mathcal{N}_n$ to be the set of possible values of $\Xi^K_{[0,n]}$ conditionally given $\bar{C}^{[0,n]}_\rho = \bar{c}^{[0,n]}$.

Theorem 2.7.1. *Let $S_0 = 0$ and $S_j = \sum_{\ell=1}^j X_\ell$ for $1 \le j \le n$, where $X_1,\dots,X_n$ are exchangeable random variables. Let $S^{[0,n]} = \{(j,S_j) : 0 \le j \le n\}$ and let $\bar{C}^{[0,n]}$ be the concave majorant of $S^{[0,n]}$. Suppose $\bar{c}^{[0,n]} \in Supp(\bar{C}^{[0,n]})$. Let $q(\cdot)$ be the probability density function on $\mathcal{N}_n$ that is the regular conditional distribution of $\Xi_{[0,n]}$ conditionally given $\bar{C}^{[0,n]}_\rho = \bar{c}^{[0,n]}$. Let $(N_{n,1}, N_{n,2},\dots,N_{n,K_n})$ be a composition of n chosen according to the density function $q(\cdot)$, independently of $\{X_j : 1 \le j \le n\}$.*

Conditionally given $\{K_n = k\}$ and $\{N_{n,i} = n_i : 1 \le i \le k\}$, let $Y_1,\dots,Y_n$ be random variables, independent of all previously introduced random variables, whose joint law that is the regular conditional joint distribution of $X_1,\dots,X_n$ conditionally given $\{S_j \in d\bar{c}^{[0,n]}(j), j = \sum_{i=1}^m n_i, 1 \le m \le k\}$.

Conditionally given $Y_1,\dots,Y_n$, let B be the random set of $(n_1,\dots,n_k)$-cyclic permutations of $[n]$ such that

$$Y_{\sigma(j)} \ge \bar{c}^{[0,n]}(j) \quad \text{for } 1 \le j \le n$$

if and only if $\sigma \in B$. Let $\hat{\rho}$ be an independently chosen uniform random element of B, and let $S^{\hat{\rho}}_j = \sum_{\ell=1}^j Y_{\hat{\rho}(\ell)}$ for $1 \le j \le n$. Then $S^{[0,n]}_{\hat{\rho}} := \{(j, S^{\hat{\rho}}_j) : 1 \le j \le n\}$ has the regular conditional distribution of $S^{[0,n]}$ conditionally given $\bar{C}^{[0,n]} = \bar{c}^{[0,n]}$.

The theorem is direct result of Bayes' rule and Theorem 2.1.1. Note that when $X_1,\dots,X_n$ satisfy assumption **A**, $\mathcal{N}(\bar{c}^{[0,n]})$ has only one element, the composition induced by the lengths of the faces of $\bar{c}^{[0,n]}$, and A also only contains one element by Lemma 2.2.1, so the theorem simplifies significantly. It remains to describe $q(\cdot)$.

Lemma 2.7.2. *Suppose $\bar{c}^{[0,n]} \in Supp(\bar{C}^{[0,n]})$ and that $X_1, \dots, X_n$ are exchangeable. The regular conditional distribution of $\Xi_{[0,n]}$ conditionally given $\bar{C}_\rho^{[0,n]} = \bar{c}^{[0,n]}$ is given by*

$$\mathbb{P}(\bar{C}^{[0,n]}(j) \in d\bar{c}^{[0,n]}(j), 1 \le j \le n)\mathbb{P}(\Xi^K_{[0,n]} = (n_1, \dots, n_k)|\bar{C}_\rho^{[0,n]} = \bar{c}^{[0,n]})$$

$$= 1_{(n_1,\dots,n_k)\in\mathcal{N}(\bar{c}^{[0,n]})} \frac{\prod_{i=1}^k n_i}{\prod_{j=1}^{f(\bar{c}^{[0,n]})} k_j(n_1, \dots, n_k)!} \mathbb{P}(S_j \in d\bar{c}^{[0,n]}(j), j = \textstyle\sum_{i=1}^l n_i, 1 \le l \le k)$$

where S_j, $1 \le j \le n$ is as in Theorem 2.7.1.

Proof. Let $(n_1, \dots, n_k) \in \mathcal{N}(\bar{c}^{[0,n]})$. Following the construction in Theorem 2.1.1, by the Ewens sampling formula the probability that $\{L_{n,1}, \dots, L_{n,K_n}\}$ is a list of the elements of $(n_1, \dots, n_k)$ in non-increasing order is $\left(\prod_{j=1}^n (a_j!)^{-1}\right)\left(\prod_{i=1}^k n_i^{-1}\right)$ where $a_j = \#\{i : 1 \le i \le k, n_i = j\}$ for $1 \le j \le n$. Conditionally given $\{L_{n,1}, \dots, L_{n,K_n}\}$ is a list of the elements of $(n_1, \dots, n_k)$ in non-increasing order the probability of the event $\{\Xi^K = (n_1, \dots, n_k), \bar{C}^{[0,n]} = \bar{c}^{[0,n]}\}$ is

$$\left(\frac{\prod_{j=1}^n a_j!}{\prod_{j=1}^{f(\bar{c}^{[0,n]})} k_j(n_1, \dots, n_k)!}\right) \mathbb{P}(S_j \in d\bar{c}^{[0,n]}(j), j = \sum_{i=1}^l n_i, 1 \le l \le k)$$

where the denominator in the multiplicative factor in the brackets is due to the restrictions on the orderings of partition segments within each face, and the numerator is because of repeated segment lengths. □

We say that the concave majorant of a walk is *trivial* if it has only one face. A particularly useful form of Theorem 2.7.1 arises from the special case when the increments $X_1, \dots, X_n$ are independent, the probability that the concave majorant of $S^{[0,n]}$ is trivial with slope zero is positive, and we want the conditional distribution of the walk $S^{[0,n]}$ given it has trivial concave majorant of slope zero. By subtraction of a line of constant slope, this gives us the conditional distribution of the walk $S^{[0,n]}$ given it has trivial concave majorant of any slope, as long as the probability that the concave majorant of $S^{[0,n]}$ is trivial with that slope is positive. In the case where we want the regular conditional distribution for $S^{[0,n]}$ conditional on having trivial concave majorant of a slope that has zero probability, then the only possible value for $\Xi_{[0,n]}$ is the trivial composition (n).

Corollary 2.7.3. *Let $S_0 = 0$ and $S_j = \sum_{i=1}^j X_i$ for $1 \le j \le n$, where $X_1, \dots, X_n$ are independent identically distributed random variables, and let $S^{[0,n]} = \{(j, S_j) : 0 \le j \le n\}$. Suppose that*

$$p_{triv} := \mathbb{P}(\textit{concave majorant of } S^{[0,n]} \textit{ is trivial with slope zero}) > 0$$

Define a probability density function $q(\cdot)$ on $\mathcal{N}_n$ by

$$q\left((n_1,\dots,n_k)\right) = \frac{1}{p_{triv}k!}\prod_{i=1}^{k} n_i u_{n_i}$$

where $u_j = \mathbb{P}(S_j = 0)$ for $1 \le j \le n$. Let $(N_{n,1}, N_{n,2}, \dots, N_{n,K_n})$ be a composition of n chosen according to the density function $q(\cdot)$, independently of $\{X_j : 1 \le j \le n\}$.

Conditionally given $\{K_n = k\}$ and $\{N_{n,i} = n_i \,:\, 1 \le i \le k\}$, independently for each $1 \le i \le k$ let $Y_{n_1+\cdots+n_{i-1}+1}, \dots, Y_{n_1+\cdots+n_i}$ be random variables, independent of all previously introduced random variables, whose joint law that is the regular conditional joint distribution of $X_1, \dots, X_{n_i}$ conditionally given $\sum_{\ell=1}^{n_i} X_\ell = 0$.

Conditionally given $Y_1, \dots, Y_n$, let B be the random set of $(n_1,\dots,n_k)$-cyclic permutations of $[n]$ such that

$$Y_{\sigma(j)} \le 0 \quad \text{for } 1 \le j \le n$$

if and only if $\sigma \in B$. Let $\hat\rho$ be an independently chosen uniform random element of B, and let $S_j^{\hat\rho} = \sum_{\ell=1}^{j} Y_{\hat\rho(\ell)}$ for $1 \le j \le n$. Then $S_{\hat\rho}^{[0,n]} := \{(j, S_j^{\hat\rho}) \,:\, 1 \le j \le n\}$ has the regular conditional distribution of $S^{[0,n]}$ conditionally given $S^{[0,n]}$ has trivial concave majorant with slope zero.

2.8 A path transformation

This section covers the '3214' path transformation discussed at length in Chapter 1. Essentially, the idea is that a uniformly sampled face of the concave majorant should have uniform length and the walk over it should be a Vervaat like transform of some walk of the same length.

Let $S_0 = 0$ and $S_j = \sum_{i=1}^{n} X_i$ for $1 \le j \le n$, where $X_i, i = 1, \dots, n$ are exchangeable random variables satisfying assumption **A**. We introduce the following path transformation for the random walk $S^{[0,n]} = \{(j, S_j)\,,\, 1 \le j \le n\}$. Let U be distributed uniformly on $[n]$. Let g and d be the left and right end points respectively of the face of the concave majorant of $S^{[0,n]}$ containing the Uth increment X_U. Define S_j^U for $1 \le j \le n$ by

$$S_j^U = \begin{cases} S_{U+j} - S_U & \text{for } 0 \le j < d - U \\ S_{g+j-(d-U)} + S_d - S_g - S_U & \text{for } d - U \le j < d - g \\ S_{j-(d-g)} + S_d - S_g & \text{for } d - g \le j < d \\ S_j & \text{for } d \le j \le n. \end{cases} \tag{2.8.1}$$

and let $S_U^{[0,n]} = \{(j, S_j^U)\,,\, 1 \le j \le n\}$.

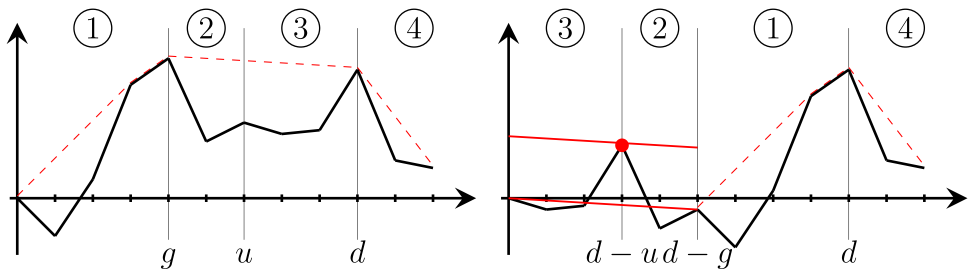

Figure 2.3: An example of the '3214' path transformation of Theorem 2.8.1. The walk on the right is the transformed version of the walk on the left. Note how given $d-g$ the transform is easily inverted - the index at which the first $d-g$ increments should start after cyclic permutation is marked, and can be found by lowering a line with the slope the mean of the first $d-g$ increments.

Theorem 2.8.1. *Under assumption* **A***,*

$$(U, S^{[0,n]}) \stackrel{d}{=} (d-g, S_U^{[0,n]})$$

Proof. As in the proof of Theorem 2.1.1 under assumption **A** in Section 2.2, it is enough to show that the equality in distribution holds when $X_1, \dots, X_n$ are samples without replacement from $x_1, \dots, x_n$ satisfying assumption **A**. $S^{[0,n]}$ and $S_U^{[0,n]}$ may thus be thought of as permutations of n, so we may think of the mapping $(U, S^{[0,n]}) \mapsto (d-g, S_U^{[0,n]})$ as a mapping from $[n] \times \Sigma_n$ to itself. Since U is uniform on $[n]$, and the ordering of $X_1, \dots, X_n$ is a uniform random permutation of $x_1, \dots, x_n$, it is enough to show that this mapping is a bijection. To do this, it suffices to show that the mapping is surjective. This can be seen visually in Figure 2.3 since it is clear from the figure and its description that the map is easily inverted. More formally, to show that the map is surjective it is sufficient to show that for $k \in [n]$ there exists $u \in [n]$ and $\sigma \in \Sigma_n$ such that

$$\left(u, \{(0,0), (1, x_{\sigma(1)}), (2, x_{\sigma(1)} + x_{\sigma_2}), \dots, (n, \sum_{i=1}^{n} x_{\sigma(i)})\}\right)$$

$$\mapsto \left(k, \{(0,0), (1, x_1), (2, x_1 + x_2), \dots, (n, \sum_{i=1}^{n} x_i)\}\right)$$

Let f be the number of faces of the concave majorant of the walk of length $n-k$ with increments $x_{k+1}, \dots, x_n$, and let the lengths and increments of these faces in order of appearance be $(\ell_1, s_1), \dots, (\ell_f, s_f)$. Let r be the unique $r \in [k]$ such that the walk with increments

$$(x_{r+1}, x_{(r+1) \bmod k+1}, x_{(r+2) \bmod k+1}, \dots, x_{(r+k-2) \bmod k+1}, x_r)$$

remains below its concave majorant. Let $s^* = \sum_{i=1}^k x_i$, and let m be the unique $m \in \{0, \dots, f\}$ such that

$$\frac{s_m}{\ell_m} > \frac{s^*}{k} > \frac{s_{m+1}}{\ell_{m+1}}$$

where we say that $s_0/\ell_0 = +\infty$ and $s_{f+1}/\ell_{f+1} = \infty$. The appropriate $(\sigma(1), \dots, \sigma(n))$ is given by

$$\begin{aligned}&(k+1, k+2, \dots, k + \textstyle\sum_{i=1}^m \ell_i,\\ &\qquad r+1, (r+1) \bmod k+1, (r+2) \bmod k+1, \dots, r, k + \textstyle\sum_{i=1}^m \ell_i + 1, \dots, n)\end{aligned}$$

□

Remark 2.8.2. As discussed in Chapter 1, Theorem 2.8.1 provides an alternative method of proving Theorem 2.1.1 under assumption $\mathbf{A}$, since by applying the transformation again to the $S_U^{[0,n]}$ restricted to the interval $[d-g, n]$, and then doing this repeatedly until there is nothing left to transform, we are actually performing the inverse of the transformation given in Theorem 2.1.1. However, this method does not extend to cover the general case as considered in Section 2.6.

Chapter 3

Lipschitz Minorants of Brownian Motion and Lévy Processes

3.1 Introduction

A function $g : \mathbb{R} \to \mathbb{R}$ is α-Lipschitz for some $\alpha > 0$ if $|g(s) - g(t)| \leq \alpha|s - t|$ for all $s, t \in \mathbb{R}$. If Γ is a set of α-Lipschitz functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\sup\{g(t_0) : g \in \Gamma\} < \infty$ for some $t_0 \in \mathbb{R}$, then the function $g^* : \mathbb{R} \to \mathbb{R}$ defined by $g^*(t) = \sup\{g(t) : g \in \Gamma\}$, $t \in \mathbb{R}$, is α-Lipschitz. Also, if $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function, then the set of α-Lipschitz functions dominated by f is non-empty if and only if f is bounded below on compact intervals and satisfies $\liminf_{t\to-\infty} f(t) - \alpha t > -\infty$ and $\liminf_{t\to+\infty} f(t) + \alpha t > -\infty$. Therefore, in this case there is a unique greatest α-Lipschitz function dominated by f, and we call this function the *α-Lipschitz minorant* of f.

Denoting the α-Lipschitz minorant of f by m, an explicit formula for m is

$$\begin{aligned} m(t) &= \sup\{h \in \mathbb{R} : h - \alpha|t - s| \leq f(s) \text{ for all } s \in \mathbb{R}\} \\ &= \inf\{f(s) + \alpha|t - s| : s \in \mathbb{R}\}. \end{aligned} \tag{3.1.1}$$

For the sake of completeness, we present a proof of these equalities in Lemma 3.8.1. The first equality says that for each $t \in \mathbb{R}$ we construct $m(t)$ by considering the set of "tent" functions $s \mapsto h - \alpha|t - s|$ that have a peak of height h at the position t and are dominated by f, and then taking the supremum of those peak heights – see Figure 3.2. The second equality is simply a rephrasing of the first.

The property that the pointwise supremum of a suitable family of α-Lipschitz functions is also α-Lipschitz is reminiscent of the fact that the pointwise supremum of a suitable family of convex functions is also convex, and so the notion of the α-Lipschitz minorant of a function is analogous to that of the convex minorant. Indeed, there is a well-developed theory of abstract or generalized convexity that subsumes both of these concepts and is used widely in nonlinear optimization, particularly in the theory of optimal mass transport – see [33, 10, 31, 56], Section 3.3 of [76] and Chapter 5 of [95]. Lipschitz minorants have also been studied in convex analysis for their Lipschitz regularization and Lipschitz extension properties, and in this area are known as Pasch-Hausdorff envelopes [48, 49, 77, 57].

Furthermore, the second expression in (3.1.1) can be thought of as producing a function analogous to the smoothing of the function f by an integral kernel (that is, a function of the form $t \mapsto \int_{\mathbb{R}} K(|t-s|)f(s)\,ds$ for some suitable kernel $K : \mathbb{R} \to \mathbb{R}$) where one has taken the "min-plus" or "tropical" point of view and replaced the algebraic operations of $+$ and $\times$ by, respectively, $\wedge$ and $+$, so that integrals are replaced by infima. Note that if f is a continuous function that possesses an α_0-Lipschitz minorant for some α_0 (and hence an α-Lipschitz minorant for all $\alpha \geq \alpha_0$), then the α-Lipschitz minorants converge pointwise monotonically up to f as $\alpha \to +\infty$. Standard methods in optimization theory involve approximating a general function by a Lipschitz function and then determining approximate optima of the original function by finding optima of its Lipschitz approximant [46, 47, 51, 68].

We investigate here the stochastic process $(M_t)_{t\in\mathbb{R}}$ obtained by taking the α-Lipschitz minorant of the sample path of a real-valued Lévy process $X = (X_t)_{t\in\mathbb{R}}$ for which the α-Lipschitz minorant almost surely exists, a condition that turns out to be equivalent to $|\mathbb{E}[X_1]| < \alpha$ when $X_0 = 0$ (excluding the trivial case where $X_t = \pm\alpha t$ for $t \in \mathbb{R}$) – see Proposition 3.2.1. See Figure 3.1 for an example of the minorant of a Brownian sample path. The original motivation for this undertaking was the abovementioned analogy between α-Lipschitz minorants and convex minorants and the rich (and growing) literature on convex minorants of Brownian motion and Lévy processes in general [45, 71, 11, 27, 12, 24, 89, 73], much of which has been discussed in detail in Chapter 1.

In particular, we study properties of the *contact set* $\mathcal{Z} := \{t \in \mathbb{R} : M_t = X_t \wedge X_{t-}\}$. This random set is clearly stationary and, as we show in Theorem 3.2.6, it is also regenerative. Consequently, its distribution is that of the closed range of a subordinator "made stationary" in a suitable manner. For a broad class of Lévy processes we are able to identify the associated subordinator in the sense that we can determined its Laplace exponent – see Theorem 3.3.13.

We then consider the Lebesgue measure of the random set $\mathcal{Z}$ in Theorem 3.3.1 and Remark 3.3.2. If the paths of the Lévy process have either unbounded variation or bounded variation with drift d satisfying $|d| > \alpha$, then the associated subordinator has zero drift, and hence the random set $\mathcal{Z}$ has zero Lebesgue measure almost surely. Conversely, if the paths of the Lévy process have bounded variation and drift d satisfying $|d| < \alpha$, then the associated subordinator has positive drift, and hence the

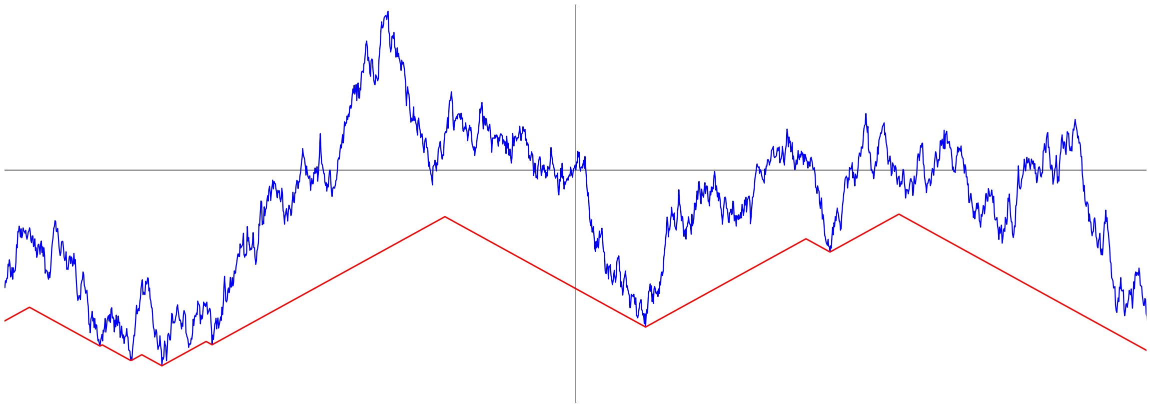

Figure 3.1: A typical Brownian motion sample path and its associated Lipschitz minorant.

random set $\mathcal{Z}$ has infinite Lebesgue measure almost surely. In Theorem 3.3.8 we give conditions under which the Lévy measure of the subordinator associated to the set $\mathcal{Z}$ has finite total mass, which implies that $\mathcal{Z}$ is a discrete set in the case where it has zero Lebesgue measure. Using the methodologies developed to investigate the Lebesgue measure of $\mathcal{Z}$ we give in Theorem 3.3.11 an interesting result relating to the local behavior of a Lévy process at its local extrema.

If for the moment we write $\mathcal{Z}_\alpha$ instead of $\mathcal{Z}$ to stress the dependence on α, then it is clear that $\mathcal{Z}_{\alpha'} \subseteq \mathcal{Z}_{\alpha''}$ for $\alpha' \leq \alpha''$. We find in Theorem 3.4.5 that if the Lévy process is *abrupt*, that is, its paths have unbounded variation and "sharp" local extrema in a suitable sense (see Definition 3.4.1 for a precise definition), then the set $\bigcup_\alpha \mathcal{Z}_\alpha$ is almost surely the set of local infima of the Lévy process.

Lastly, when the Lévy process is a Brownian motion with drift, we can compute explicitly the distributions of a number of functionals of the α-Lipschitz minorant process. In order to describe these results, we first note that it follows from Lemma 3.8.3 below that the graph of the α-Lipschitz minorant M over one of the connected components of the complement of $\mathcal{Z}$ is almost surely a "sawtooth" that consists of a line of slope $+\alpha$ followed by a line of slope $-\alpha$. Set $G := \sup\{t < 0 : t \in \mathcal{Z}\}$, $D := \inf\{t > 0 : t \in \mathcal{Z}\}$, and put $K := D - G$. Let T be the unique $t \in [G, D]$ such that $M(t) = \max\{M(s) : s \in [G, D]\}$. That is, T is place where the peak of the sawtooth occurs. Further, let $H := X_T - M_T$ be the distance between the Brownian path and the α-Lipschitz minorant at the time where the peak occurs.

The following theorem summarizes a series of results that we establish in Section 3.7.

Theorem 3.1.1. *Suppose that X is a Brownian motion with drift β, where $|\beta| < \alpha$. Then, the following hold.*

(a) The Lévy measure Λ of the subordinator associated to the contact set $\mathcal{Z}$ has finite mass and is characterized up to a scalar multiple by

$$\frac{\int_{\mathbb{R}_+} 1 - e^{-\theta x}\,\Lambda(dx)}{\int_{\mathbb{R}_+} x\,\Lambda(dx)} = \frac{4(\alpha^2-\beta^2)\theta}{\left(\sqrt{2\theta+(\alpha-\beta)^2}+\alpha-\beta\right)\left(\sqrt{2\theta+(\alpha+\beta)^2}+\alpha+\beta\right)}$$

(b) When $\beta = 0$ the measure Λ is absolutely continuous with respect to Lebesgue measure with

$$\frac{\Lambda(dx)}{\Lambda(\mathbb{R}_+)} = \frac{2\alpha}{\sqrt{2\pi}}\left[x^{-\frac{1}{2}}e^{-\frac{\alpha^2 x}{2}} - 2\alpha^2\Phi(-\alpha x^{\frac{1}{2}})\right]\,dx,$$

where Φ is the standard normal cumulative distribution function (that is, $\Phi(z) := \int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,dt$).

(c) The distribution of T is characterized by

$$\mathbb{E}[e^{-\theta T}] = 8\alpha(\alpha^2-\beta^2)\frac{1}{\theta}\left(\frac{1}{\sqrt{(\alpha+\beta)^2-2\theta}+3\alpha-\beta} - \frac{1}{\sqrt{(\alpha-\beta)^2+2\theta}+3\alpha+\beta}\right)$$

for $-\frac{(\alpha-\beta)^2}{2} \le \theta \le \frac{(\alpha+\beta)^2}{2}$. Also,

$$\mathbb{P}\{T > 0\} = \frac{1}{2}\left(1+\frac{\beta}{\alpha}\right).$$

(d) The random variable H has a Gamma$(2, 4\alpha)$ *distribution; that is, the distribution of H is absolutely continuous with respect to Lebesgue measure with density $h \mapsto (4\alpha)^2 h e^{-4\alpha h}$, $h \ge 0$.*

The rest of this chapter is organized as follows. In Section 3.2 we provide precise definitions and give some preliminary results relating to the nature of the contact set. In Section 3.3 we describe the subordinator associated with the contact set, and in Section 3.4 we describe the limit of the contact set as $\alpha \to \infty$. In order to prove Theorem 3.3.13 we need some preliminary results relating to the future infimum of a Lévy process, which we give in Section 3.5, and then we prove Theorem 3.3.13 in Section 3.6. In Section 3.7 we cover the special case when X is a two sided Brownian motion with drift in detail. Finally, in Section 3.8 we give some basic facts about the α-Lipschitz minorant of a function that are helpful throughout the paper.

3.2 Definitions and Preliminary Results

3.2.1 Basic definitions

Let $X = (X_t)_{t\in\mathbb{R}}$ be a real-valued Lévy process. That is, X has *càdlàg* sample paths, $X_0 = 0$, and $X_t - X_s$ is independent of $\{X_r : r \le s\}$ with the same distribution as X_{t-s} for $s, t \in \mathbb{R}$ with $s < t$.

The Lévy-Khintchine formula says that for $t \ge 0$ the characteristic function of X_t is given by $\mathbb{E}[e^{i\theta X_t}] = e^{-t\Psi(\theta)}$ for $\theta \in \mathbb{R}$, where

$$\Psi(\theta) = -ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x 1_{\{|x|<1\}})\,\Pi(dx)$$

with $a \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$, and Π a σ-finite measure concentrated on $\mathbb{R}\setminus\{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\,\Pi(dx) < \infty$. We call σ^2 the *infinitesimal variance* of the Brownian component of X and Π the *Lévy measure* of X.

The sample paths of X have bounded variation almost surely if and only if $\sigma = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|)\,\Pi(dx) < \infty$. In this case Ψ can be rewritten as

$$\Psi(\theta) = -id\theta + \int_{\mathbb{R}} (1 - e^{i\theta x})\,\Pi(dx).$$

We call $d \in \mathbb{R}$ the drift coefficient. For full details of these definitions see [15].

We will often need the result of Štatland [87] that if X has paths of bounded variation with drift d, then

$$\lim_{t\downarrow 0} t^{-1} X_t = d \quad \text{a.s.} \tag{3.2.1}$$

The counterpart of Štatland's result when X has paths of unbounded variation is Rogozin's result

$$\liminf_{t\downarrow 0} t^{-1} X_t = -\infty \quad \text{and} \quad \limsup_{t\downarrow 0} t^{-1} X_t = +\infty \quad \text{a.s.} \tag{3.2.2}$$

For the sake of reference, we also record here a regularity criterion due to Rogozin [79] (see also [15, Proposition VI.11]) that we use frequently:

$$\begin{gathered} \text{zero is regular for } (-\infty, 0] \\ \Longleftrightarrow \\ \int_0^1 t^{-1}\mathbb{P}\{X_t \le 0\}\,dt = \infty. \end{gathered} \tag{3.2.3}$$

Of course, (3.2.3) has an obvious analogue that determines when zero is regular for $[0, \infty)$.

3.2.2 Existence of a minorant

Proposition 3.2.1. *Let X be a Lévy process. The α-Lipschitz minorant of X exists almost surely if and only if either $\sigma = 0$, $\Pi = 0$ and $|d| = \alpha$ (equivalently, $X_t = \alpha t$ for all $t \in \mathbb{R}$ or $X_t = -\alpha t$ for all $t \in \mathbb{R}$), or $\mathbb{E}[|X_1|] < \infty$ and $|\mathbb{E}[X_1]| < \alpha$.*

Proof. As we remarked in the Introduction, the α-Lipschitz minorant of a function $f : \mathbb{R} \to \mathbb{R}$ exists if and only if f is bounded below on compact intervals and satisfies $\liminf_{t\to-\infty} f(t) - \alpha t > -\infty$ and $\liminf_{t\to+\infty} f(t) + \alpha t > -\infty$.

Since the sample paths of a Lévy process are almost surely bounded on compact intervals, we need necessary and sufficient conditions for $\liminf_{t\to-\infty} X_t - \alpha t > -\infty$ and $\liminf_{t\to+\infty} X_t + \alpha t > -\infty$ to hold almost surely. This is equivalent to requiring that

$$\limsup_{t\to+\infty} X_t - \alpha t < +\infty \quad \text{a.s.} \qquad \text{and} \qquad \liminf_{t\to+\infty} X_t + \alpha t > -\infty \quad \text{a.s.} \tag{3.2.4}$$

It is obvious that that the two conditions in (3.2.4) hold if $\sigma = 0$, $\Pi = 0$ and $|d| = \alpha$. It is clear from the strong law of large numbers that they also hold if $\mathbb{E}[|X_1|] < \infty$ and $|\mathbb{E}[X_1]| < \alpha$.

Consider the converse. Writing $x^+ := x \vee 0$ and $x^- := -(x \wedge 0)$ for $x \in \mathbb{R}$, the strong law of large numbers precludes any case where either $\mathbb{E}[X_1^+] = +\infty$ and $\mathbb{E}[X_1^-] < +\infty$ or $\mathbb{E}[X_1^+] < +\infty$ and $\mathbb{E}[X_1^-] = +\infty$. A result of Erickson [32, Chapter 4, Theorem 15] rules out the possibility $\mathbb{E}[X_1^+] = \mathbb{E}[X_1^-] = +\infty$, and so $\mathbb{E}[|X_1|] < \infty$. It now follows from the strong law of large numbers that $\lim_{t\to\infty} t^{-1} X_t = \mathbb{E}[X_1]$ and so $|\mathbb{E}[X_1]| \le \alpha$. Suppose that X_t is non-degenerate for $t \neq 0$ (that is, that $\sigma \neq 0$ or $\Pi \neq 0$). Then, $\limsup_{t\to\infty} X_t - \mathbb{E}[X_1]t = +\infty$ a.s. and $\liminf_{t\to\infty} X_t - \mathbb{E}[X_1]t = -\infty$ a.s. (see, for example, [54, Corollary 9.14]), and so $|\mathbb{E}[X_1]| < \alpha$ in this case. □

Hypothesis 3.2.2. From now on we assume, unless we note otherwise, that the Lévy process $X = (X_t)_{t\in\mathbb{R}}$ has the properties:

- $X_0 = 0$;
- X_t is non-degenerate for $t \neq 0$;
- $\mathbb{E}[|X_1|] < \infty$;
- $|\mathbb{E}[X_1]| < \alpha$.

Notation 3.2.3. As in the Introduction, let $M = (M_t)_{t\in\mathbb{R}}$ be the α-Lipschitz minorant of X. Put $\mathcal{Z} = \{t \in \mathbb{R} : M_t = X_t \wedge X_{t-}\}$.

3.2.3 The contact set is regenerative

It follows fairly directly from our standing assumptions Hypothesis 3.2.2 that the random set $\mathcal{Z}$ is almost surely unbounded above and below. (Alternatively, it follows even more easily from Hypothesis 3.2.2 that $\mathcal{Z}$ is non-empty almost surely. We show below that $\mathcal{Z}$ is stationary, and any non-empty stationary random set is necessarily almost surely unbounded above and below.)

We now show that the contact set $\mathcal{Z}$ is stationary and that it is also regenerative in the sense of Fitzsimmons and Taksar [36]. For simplicity, we specialize the definition in [36] somewhat as follows by only considering random sets defined on probability spaces (rather than general σ-finite measure spaces).

Let Ω^0 denote the class of closed subsets of $\mathbb{R}$. For $t \in \mathbb{R}$ and $\omega^0 \in \Omega^0$, define

$$d_t(\omega^0) := \inf\{s > t : s \in \omega^0\}, \quad r_t(\omega^0) := d_t(\omega^0) - t,$$

and

$$\tau_t(\omega^0) = \mathbf{cl}\{s - t : s \in \omega^0 \cap (t, \infty)\} = \mathbf{cl}\left((\omega^0 - t) \cap (0, \infty)\right).$$

Here **cl** denotes closure and we adopt the convention $\inf \emptyset = +\infty$. Note that $t \in \omega^0$ if and only if $\lim_{s \uparrow t} r_s(\omega^0) = 0$, and so $\omega^0 \cap (-\infty, t]$ can be reconstructed from $r_s(\omega^0)$, $s \le t$, for any $t \in \mathbb{R}$. Set $\mathcal{G}^0 = \sigma\{r_s : s \in \mathbb{R}\}$ and $\mathcal{G}^0_t = \sigma\{r_s : s \le t\}$. Clearly, $(d_t)_{t \in \mathbb{R}}$ is an increasing càdlàg process adapted to the filtration $(\mathcal{G}^0_t)_{t \in \mathbb{R}}$, and $d_t \ge t$ for all $t \in \mathbb{R}$.

A *random set* is a measurable mapping S from a measurable space $(\Omega, \mathcal{F})$ into $(\Omega^0, \mathcal{G}^0)$.

Definition 3.2.4. A probability measure $\mathbb{Q}$ on $(\Omega^0, \mathcal{G}^0)$ is regenerative with regeneration law $\mathbb{Q}^0$ if

(i) $\mathbb{Q}\{d_t = +\infty\} = 0$, for all $t \in \mathbb{R}$;

(ii) for all $t \in \mathbb{R}$ and for all $\mathcal{G}^0$-measurable nonnegative functions F,

$$\mathbb{Q}\left[F(\tau_{d_t}) \,|\, \mathcal{G}^0_{t+}\right] = \mathbb{Q}^0[F], \tag{3.2.5}$$

where we write $\mathbb{Q}[\cdot]$ and $\mathbb{Q}^0[\cdot]$ for expectations with respect to $\mathbb{Q}$ and $\mathbb{Q}^0$. A random set S defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a regenerative set if the push-forward of $\mathbb{P}$ by the map S (that is, the distribution of S) is a regenerative probability measure.

Remark 3.2.5. Suppose that the probability measure $\mathbb{Q}$ on $(\Omega^0, \mathcal{G}^0)$ is stationary; that is, if S^0 is the identity map on Ω^0, then the random set S^0 on $(\Omega^0, \mathcal{G}^0, \mathbb{Q})$ has the same distribution as $u + S^0$ for any $u \in \mathbb{R}$ or, equivalently, that the process $(r_t)_{t\in\mathbb{R}}$ has the same distribution as $(r_{t-u})_{t\in\mathbb{R}}$ for any $u \in \mathbb{R}$. Then, in order to check conditions (i) and (ii) of Definition 3.2.4 it suffices to check them for the case $t = 0$.

The probability measure $\mathbb{Q}^0$ is itself regenerative. It assigns all of its mass to the collection of closed subsets of $\mathbb{R}_+$. As remarked in [36], it is well known that any regenerative probability measure with this property arises as the distribution of a random set of the form $\mathbf{cl}\{Y_t : Y_t > Y_0,\ t \geq 0\}$, where $(Y_t)_{t\geq 0}$ is a subordinator (that is, a non-decreasing, real-valued Lévy process) with $Y_0 = 0$ – see [59, 58]. Note that $\mathbf{cl}\{Y_t : Y_t > Y_0,\ t \geq 0\}$ has the same distribution as $\mathbf{cl}\{Y_{ct} : Y_{ct} > Y_{c0},\ t \geq 0\}$, and the distribution of the subordinator associated with a regeneration law can at most be determined up to linear time change (equivalently, the corresponding drift and Lévy measure can at most be determined up to a common constant multiple). It turns out that the distribution of the subordinator is unique except for this ambiguity – again see [59, 58].

We refer the reader to [36] for a description of the sense in which a stationary regenerative probability measure $\mathbb{Q}$ with regeneration law $\mathbb{Q}^0$ can be regarded as $\mathbb{Q}^0$ "made stationary". Note that if Λ is the Lévy measure of the subordinator associated with $\mathbb{Q}$ in this way, then, by stationarity, it must be the case that $\int_{\mathbb{R}_+} y\,\Lambda(dy) < \infty$.

Theorem 3.2.6. *The random (closed) set $\mathcal{Z}$ is stationary and regenerative.*

Proof. It follows from Lemma 3.8.3 that $\mathcal{Z}$ is a.s. closed.

We next show that $\mathcal{Z}$ is stationary. Note for $u \in \mathbb{R}$ that $u + \mathcal{Z} = \{t \in \mathbb{R} : X_{(t-u)} \wedge X_{(t-u)-} = M_{(t-u)}\}$. Define $(\check{X}_t)_{t\in\mathbb{R}}$ by $\check{X}_t = X_{t-u} - X_{(-u)}$ for $t \in \mathbb{R}$ and let $\check{M}$ be the α-Lipschitz minorant of $\check{X}$. Note that $\check{M}_t = M_{t-u} - X_{(-u)}$ for $t \in \mathbb{R}$. Therefore, $u + \mathcal{Z} = \{t \in \mathbb{R} : \check{X}_t \vee \check{X}_{t-} = \check{M}_t\}$ and hence $u + \mathcal{Z}$ has the same distribution as $\mathcal{Z}$ because $\check{X}$ has the same distribution as X.

We now show that $\mathcal{Z}$ is regenerative. For $t \in \mathbb{R}$ set

$$D_t := \inf\{s > t : X_s \wedge X_{s-} = M_s\} = d_t \circ \mathcal{Z},$$

$$R_t := D_t - t,$$

$$S_t := \inf\left\{s > t : X_s \wedge X_{s-} - \alpha(s-t) \leq \inf\{X_u - \alpha(u-t) : u \leq t\}\right\},$$

and $\mathcal{F}_t := \bigcap_{s>t} \sigma\{X_u : u \leq s\}$.

We claim that $X_{S_t} \leq X_{S_t-}$ a.s. Suppose to the contrary that the event $A := \{X_{S_t} > X_{S_t-}\}$ has positive probability. On the event A, $X_s > X_{S_t} - (X_{S_t} - X_{S_t-})/2$ for $s \in (S_t, S_t + \delta)$ for some (random) $\delta > 0$. Hence, $X_{s-} \geq X_{S_t} - (X_{S_t} - X_{S_t-})/2$ for $s \in (S_t, S_t + \delta)$, and so $X_s \wedge X_{s-} - \alpha(s-t) > X_{S_t} \wedge X_{S_t-} - \alpha(S_t - t) = X_{S_t-} - \alpha(S_t - t)$

for $s \in (S_t, S_t + \delta)$ on the event A provided δ is sufficiently small. It follows that, on the event A, $X_{S_t-} - \alpha(S_t - t) \le \inf\{X_u - \alpha(u-t) : u \le t\}$ and $X_s \wedge X_{s-} - \alpha(s-t) > \inf\{X_u - \alpha(u-t) : u \le t\}$ for $t \le s < S_t$. Define a non-decreasing sequence of stopping times $\{U_n\}_{n\in\mathbb{N}}$ by

$$U_n := \inf\left\{s > t : X_s \wedge X_{s-} - \alpha(s-t) \le \inf\{X_u - \alpha(u-t) : u \le t\} + \frac{1}{n}\right\},$$

and set $U_\infty := \sup_{n\in\mathbb{N}} U_n$. We have shown that, on the event A, $U_n < S_t$ for all $n \in \mathbb{N}$ and $U_\infty = S_t$. By the quasi-left-continuity of X, $\lim_{n\to\infty} X_{U_n} = X_{U_\infty}$ a.s. In particular, $X_{S_t} = X_{S_t-}$ a.s. on the event A, and so A cannot have positive probability.

Lemma 3.8.4 now gives that

$$D_t = \inf\{s \ge S_t : X_t \wedge X_{t-} + \alpha(s - S_t) = \inf\{X_u + \alpha(u - S_t) : u \ge S_t\}\}$$

almost surely.

We have already remarked that $\mathcal{Z}$ is almost surely unbounded above and below, and hence condition (i) of Definition 3.2.4 holds. By Remark 3.2.5, in order to check condition (ii) of Definition 3.2.4, it suffices to consider the case $t = 0$.

For notational simplicity, set $S := S_0$ and $D := D_0$ – see Figure 3.3 for two illustrations of the construction of S and D from a sample path. For a random time U, let $\mathcal{F}_U$ be the σ-field generated by random variables of the form ξ_U, where $(\xi_t)_{t\in\mathbb{R}}$ is some optional process for the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ (cf. Millar [63, 60]). It follows from Corollary 3.8.2 (where we are thinking intuitively of removing the process to the right of D rather than to the right of zero) that $\bigcap_{\epsilon>0} \sigma\{R_s : s \le \epsilon\} \subseteq \mathcal{F}_D$.

Put

$$\tilde{X} = (\tilde{X}_s)_{s\ge 0} := ((X_{S+s} - X_S) + \alpha s)_{s\ge 0}.$$

By the strong Markov property at the stopping time S and the spatial homogeneity of X, the process $\tilde{X}$ is independent of $\mathcal{F}_S$ with the same distribution as the Lévy process $(X_t + \alpha t)_{t\ge 0}$. Suppose for the Lévy process $(X_t + \alpha t)_{t\ge 0}$ that zero is regular for the interval $(0, \infty)$. A result of Millar [62, Proposition 2.4] implies that almost surely there is a unique time $\tilde{T}$ such that $\tilde{X}_{\tilde{T}} = \inf\{\tilde{X}_s : s \ge 0\}$ and that if $\bar{T}$ is such that $\tilde{X}_{\bar{T}-} = \inf\{\tilde{X}_s : s \ge 0\}$, then $\bar{T} = \tilde{T}$. Thus, $\tilde{T} = \sup\{t \ge 0 : \tilde{X}_t \wedge \tilde{X}_{t-} = \inf\{\tilde{X}_s : s \ge 0\}\}$ and $D = S + \tilde{T}$. Combining this observation with the main result of Millar [60] (see Remark 3.2.7 below) and the fact that $\tilde{X}_{\tilde{T}} = \inf\{\tilde{X}_s : s \ge 0\}$ gives that $(\tilde{X}_{\tilde{T}+t})_{t\ge 0}$ is conditionally independent of $\mathcal{F}_D$ given $\tilde{X}_{\tilde{T}}$. Thus, again by the spatial homogeneity of $\tilde{X}$, $(\tilde{X}_{\tilde{T}+t} - \tilde{X}_{\tilde{T}})_{t\ge 0}$ is independent of $\mathcal{F}_D$. This establishes condition (ii) of Definition 3.2.4 for $t = 0$.

If zero is not regular for the interval $(0, \infty)$ for the Lévy process $(X_t + \alpha t)_{t \geq 0}$, then zero is necessarily regular for the interval $(0, \infty)$ for the Lévy process $(X_{-t-} + \alpha t)_{t \geq 0}$ because this latter process the same distribution as $(-(X_t + \alpha t) + 2\alpha t)_{t \geq 0}$. The argument above then establishes that the random set $-\mathcal{Z}$ is regenerative. It follows from [36, Theorem 4.1] that $\mathcal{Z}$ is regenerative with the same distribution as $-\mathcal{Z}$. $\square$

Remark 3.2.7. A key ingredient in the proof of Theorem 3.2.6 was the result of Millar from [60] which says that, under suitable conditions, the future evolution of a càdlàg strong Markov process after the time it attains its global minimum is conditionally independent of the past up to that time given the value of the process and its left limit at that time. That result follows in turn from results in [38] on last exit decompositions or results in [74] on analogues of the strong Markov property at general coterminal times. We did not apply Millar's result directly; rather, we considered a random time $D = D_0$ that was the last time after a stopping time that a strong Markov process attained its infimum over times greater than the stopping time and combined Millar's result with the strong Markov property at the stopping time. An alternative route would have been to observe that the random time D is a *randomized coterminal time* in the sense of [63] for a suitable strong Markov process.

3.3 Identification of the associated subordinator

Let $Y = (Y_t)_{t \geq 0}$ be "the" subordinator associated with the regenerative set $\mathcal{Z}$. Write δ and Λ for the drift coefficient and Lévy measure of Y. Recall that these quantities are unique up to a common scalar multiple. The closed range of Y either has zero Lebesgue measure almost surely or infinite Lebesgue measure almost surely according to whether δ is zero or positive [32, Chapter 2, Theorem 3]. Consequently, the same dichotomy holds for the contact set $\mathcal{Z}$, and the following result gives necessary and sufficient conditions for each alternative.

Theorem 3.3.1. *If $\sigma = 0$, $\Pi(\mathbb{R}) < \infty$, and $|d| = \alpha$, then the Lebesgue measure of $\mathcal{Z}$ is almost surely infinite. If X is not of this form, then the Lebesgue measure of $\mathcal{Z}$ is almost surely zero if and only if zero is regular for the interval $(-\infty, 0]$ for at least one of the Lévy processes $(X_t + \alpha t)_{t \geq 0}$ and $(-X_t + \alpha t)_{t \geq 0}$.*

Proof. Suppose first that $\sigma = 0$, $\Pi(\mathbb{R}) < \infty$ and $|d| = \alpha$. In this case, the paths of X are piecewise linear with slope d. Our standing assumption $|\mathbb{E}[X_1]| < \alpha$ and the strong law of large numbers give $\lim_{t \to -\infty} t^{-1} X_t = \lim_{t \to +\infty} t^{-1} X_t = \mathbb{E}[X_1]$. It is now clear that $\mathcal{Z}$ has positive Lebesgue measure with positive probability and hence infinite Lebesgue measure almost surely.

Suppose now that X is not of this special form. It suffices by Fubini's theorem and the stationarity of $\mathcal{Z}$ to show that $\mathbb{P}\{0 \in \mathcal{Z}\} > 0$ if and only if zero is not regular for $(-\infty, 0]$ for both of the Lévy processes $(X_t + \alpha t)_{t \geq 0}$ and $(-X_t + \alpha t)_{t \geq 0}$.

Set $I^- := \inf\{X_t - \alpha t : t \leq 0\}$ and $I^+ := \inf\{X_t + \alpha t : t \geq 0\}$. Recall from (3.1.1) that $M_0 = I^- \wedge I^+$. Therefore,

$$\begin{aligned}\mathbb{P}\{0 \in \mathcal{Z}\} &= \mathbb{P}\{I^- \wedge I^+ = X_0 \wedge X_{0-} = 0\} \\ &= \mathbb{P}\{I^- = I^+ = 0\} \\ &= \mathbb{P}\{I^- = 0\}\mathbb{P}\{I^+ = 0\},\end{aligned}$$

and so $\mathbb{P}\{0 \in \mathcal{Z}\} > 0$ if and only if $\mathbb{P}\{I^- = 0\} > 0$ and $\mathbb{P}\{I^+ = 0\} > 0$.

Note that I^- has the same distribution as $\inf\{-X_t + \alpha t : t \geq 0\}$. From the formulas of Pecherskii and Rogozin [69] (or [15, Theorem VI.5]),

$$\mathbb{E}[e^{\theta I^-}] = \exp\left(\int_0^\infty \int_{(-\infty,0]} (e^{\theta x} - 1)t^{-1}\mathbb{P}\{-X_t + \alpha t \in dx\}\, dt\right) \tag{3.3.1}$$

and

$$\mathbb{E}[e^{\theta I^+}] = \exp\left(\int_0^\infty \int_{(-\infty,0]} (e^{\theta x} - 1)t^{-1}\mathbb{P}\{X_t + \alpha t \in dx\}\, dt\right). \tag{3.3.2}$$

Taking the limit as $\theta \to \infty$ and applying monotone convergence in (3.3.1) and in (3.3.2) gives

$$\mathbb{P}\{I^- = 0\} = \exp\left(-\int_0^\infty t^{-1}\mathbb{P}\{-X_t + \alpha t < 0\}\, dt\right) \tag{3.3.3}$$

and

$$\mathbb{P}\{I^+ = 0\} = \exp\left(-\int_0^\infty t^{-1}\mathbb{P}\{X_t + \alpha t < 0\}\, dt\right). \tag{3.3.4}$$

Since we are assuming that it is not the case that $\sigma = 0$, $\Pi(\mathbb{R}) < \infty$ and $|d| = \alpha$, we have $\mathbb{P}\{X_t + \alpha t = 0\} = \mathbb{P}\{-X_t + \alpha t = 0\} = 0$ for all $t > 0$. Moreover, by our standing assumption $|\mathbb{E}[X_1]| < \alpha$ it certainly follows that both $X_t + \alpha t$ and $-X_t + \alpha t$ drift to $+\infty$. Hence, by a result of Rogozin [79] (or see [15, Theorem VI.12])

$$\int_1^\infty t^{-1}\mathbb{P}\{X_t + \alpha t \leq 0\}\, dt < \infty \quad \text{and} \quad \int_1^\infty t^{-1}\mathbb{P}\{-X_t + \alpha t \leq 0\}\, dt < \infty. \tag{3.3.5}$$

The result now follows from (3.2.3) which implies that zero is not regular for the interval $(-\infty, 0]$ for both $(-X_t + \alpha t)_{t \geq 0}$ and $(X_t + \alpha t)_{t \geq 0}$ if and only if

$$\int_0^1 t^{-1}\mathbb{P}\{-X_t + \alpha t \leq 0\}\, dt < \infty \quad \text{and} \quad \int_0^1 t^{-1}\mathbb{P}\{X_t + \alpha t \leq 0\}\, dt < \infty.$$

□

Remark 3.3.2. (i) Note that zero is regular for the interval $(-\infty, 0]$ for both $(X_t + \alpha t)_{t\geq 0}$ and $(-X_t + \alpha t)_{t\geq 0}$ when X has paths of unbounded variation, since then $\liminf_{t\to 0} t^{-1}X_t = -\infty$ by (3.2.2).

(ii) If X has paths of bounded variation and drift coefficient d, then $\lim_{t\downarrow 0} t^{-1}(X_t + \alpha t) = d+\alpha$ and $\lim_{t\downarrow 0} t^{-1}(-X_t+\alpha t) = -d+\alpha$ by (3.2.1). Thus, if $|d| < \alpha$, then zero is regular for $(-\infty, 0]$ for neither $(X_t+\alpha t)_{t\geq 0}$ or $(-X_t+\alpha t)_{t\geq 0}$, whereas if $|d| > \alpha$, then zero is regular for $(-\infty, 0]$ for exactly one of those two processes.

(iii) If X has paths of bounded variation and $|d| = \alpha$, then an integral condition due to Bertoin involving the Lévy measure Π determines whether zero is regular for the interval $(-\infty, 0]$ for whichever of the processes $(X_t+\alpha t)_{t\geq 0}$ or $(-X_t+\alpha t)_{t\geq 0}$ has zero drift coefficient [17].

Remark 3.3.3. Recall the notation $G = \sup\{t < 0 : t \in \mathcal{Z}\}$, $D = \inf\{t > 0 : t \in \mathcal{Z}\}$ and $K = D - G$ (note that $D = d_0 \circ \mathcal{Z}$). If the Lebesgue measure of $\mathcal{Z}$ is almost surely zero (equivalently when $\delta = 0$ [32, Chapter 2, Theorem 3]), then $0 \notin \mathcal{Z}$ and $G < 0 < D$, and the distribution of K is obtained by size-biasing the Lévy measure Λ; that is,

$$\mathbb{P}\{K \in dx\} = \frac{x\,\Lambda(dx)}{\int_{\mathbb{R}_+} y\,\Lambda(dy)} \tag{3.3.6}$$

(recall that $\int_{\mathbb{R}_+} y\,\Lambda(dy) < \infty$ since $\mathcal{Z}$ is stationary).

If the Lebesgue measure of $\mathcal{Z}$ is positive almost surely (and hence $\delta > 0$), then $\mathbb{P}\{K = 0\} > 0$ and we see by multiplying together (3.3.3) and (3.3.4) that

$$\begin{aligned}\mathbb{P}\{K = 0\} &= \exp\left(-\int_0^\infty t^{-1}\left(\mathbb{P}\{X_t + \alpha t < 0\} + \mathbb{P}\{-X_t + \alpha t < 0\}\right) dt\right) \\ &= \exp\left(-\int_0^\infty t^{-1}\mathbb{P}\{X_t \notin [-\alpha t, \alpha t]\}\, dt\right).\end{aligned} \tag{3.3.7}$$

In this latter case, the conditional distribution of K given $K > 0$ is the size-biasing of Λ. The relationship between δ and Λ is easily deduced since $\mathbb{P}\{K = 0\}$ is the expected proportion of the real line that is covered by the range of the subordinator associated with $\mathcal{Z}$. Thus

$$\mathbb{P}\{K = 0\} = \frac{\delta}{\delta + \int_{\mathbb{R}_+} y\Lambda(dy)}.$$

Remark 3.3.4. Theorem 3.3.1 and Remark 3.3.2 provide conditions for deciding whether the Lebesgue measure of the contact set $\mathcal{Z}$ is zero almost surely or positive almost surely, i.e. whether $\delta = 0$ or $\delta > 0$. In Theorem 3.3.8 we provide conditions for deciding whether $\Lambda(\mathbb{R}_+) < \infty$ or $\Lambda(\mathbb{R}_+) = \infty$ for the case $\delta = 0$ and $\Pi(\mathbb{R}) = \infty$. Since $\delta = 0$, these conditions determine whether the contact set $\mathcal{Z}$ is a discrete set or not. On the way, we provide in Propositions 3.3.5, 3.3.6, and 3.3.7 descriptions of the local behavior of a Lévy process at its local extrema.

Clearly, if $\sigma = 0$ and $\Pi(\mathbb{R}) < \infty$, then $\Lambda(\mathbb{R}_+) < \infty$. Consider the remaining case $\sigma = 0$, $\Pi(\mathbb{R}) = \infty$, and $\delta > 0$. We claim that $\Lambda(\mathbb{R}_+) = \infty$. To see this, suppose to the contrary that $\Lambda(\mathbb{R}_+) < \infty$, then there almost surely exists $t_1 < t_2$ such that $X_t \wedge X_{t-} = M_t$ for all $t_1 < t < t_2$. Because $t \mapsto M_t$ is continuous, the paths of X cannot jump between times t_1 and t_2. However, when $\Pi(\mathbb{R}) = \infty$ the jump times of X are almost surely dense in $\mathbb{R}$.

Write

$$\mathcal{M}^- := \bigcup_{\epsilon>0} \{t \in \mathbb{R} : X_t \wedge X_{t-} = \inf\{X_s : s \in (t-\epsilon, t+\epsilon)\}\} \tag{3.3.8}$$

for the set of local infima of the path of X and

$$\mathcal{M}^+ := \bigcup_{\epsilon>0} \{t \in \mathbb{R} : X_t \wedge X_{t-} = \sup\{X_s : s \in (t-\epsilon, t+\epsilon)\}\} \tag{3.3.9}$$

for the set of local suprema. The following result is essentially due to Vigon [92].

Proposition 3.3.5. *Let X be a Lévy process with paths of unbounded variation. Then, $X_t = X_{t-}$ for all $t \in \mathcal{M}^-$ and all $t \in \mathcal{M}^+$ almost surely. Moreover, for any $r > 0$, $\liminf_{\varepsilon\downarrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_t) \geq r$ for all $t \in \mathcal{M}^-$ almost surely if and only if*

$$\int_0^1 t^{-1}\mathbb{P}\{X_t \in [0, rt]\}\, dt < \infty, \tag{3.3.10}$$

and $\limsup_{\varepsilon\downarrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_t) \leq -r$ for all $t \in \mathcal{M}^+$ almost surely if and only if

$$\int_0^1 t^{-1}\mathbb{P}\{X_t \in [-rt, 0]\}\, dt < \infty. \tag{3.3.11}$$

Proof. We show the equivalence involving local infima. The equivalence involving local suprema then follows by a time reversal argument.

Let $(X_t^q)_{t\geq 0}$ be a copy of $(X_t)_{t\geq 0}$ killed at an independent exponential time ξ with parameter $0 < q < \infty$. Define

$$\rho := \arg\inf_{0<t<\xi} X_t^q \wedge X_{t-}^q \quad \text{and} \quad \sigma := \arg\sup_{0<t<\xi} X_t^q \wedge X_{t-}^q.$$

By a localization argument such as the one indicated in the proof of [92, Theorem 1.3], it is sufficient to show

$$X_\rho = X_{\rho-} \tag{3.3.12}$$

and

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{\rho+\varepsilon} - X_\rho) \geq r \text{ if and only if (3.3.10) holds.} \tag{3.3.13}$$

Because X has paths of unbounded variation, $\liminf_{t\downarrow 0} t^{-1}X_t = -\infty$ and $\limsup_{t\downarrow 0} t^{-1}X_t = +\infty$ almost surely by (3.2.2), and hence zero is regular for both $(-\infty, 0]$ and $[0,\infty)$. Equation (3.3.12) then follows from [62, Theorem 3.1]. The inequality (3.3.13) is exactly [92, Proposition 3.6]. □

Proposition 3.3.6. *Let X be any Lévy process with paths of bounded variation with drift $d \neq 0$. Then, $X_t \neq X_{t-}$ for all $t \in \mathcal{M}^-$ and all $t \in \mathcal{M}^+$ almost surely. Moreover, $\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_t) = d$ for all $t \in \mathcal{M}^-$ and all $t \in \mathcal{M}^+$ almost surely.*

Proof. Using the same notation and arguments as in Proposition 3.3.5, it suffices to show that $X_\rho \neq X_{\rho-}$ almost surely and that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{\rho+\varepsilon} - X_\rho) = d. \tag{3.3.14}$$

A result of Millar states that any Lévy process for which zero is not regular for $[0,\infty)$ must jump out of its global infimum and that any Lévy process for which zero is not regular for $(-\infty, 0]$ must jump into its global infimum – see [62, Theorem 3.1]. By (3.2.1), $\lim_{t\downarrow 0} t^{-1}X_t = d$ almost surely, and so one of these alternatives must hold. Hence, in either case, $X_\rho \neq X_{\rho-}$.

Moreover, the fact that ρ is a jump time of X implies (3.3.14). To see this, we argue as in [73]. For $\delta > 0$, let $0 < J_1^\delta < J_2^\delta < \ldots$ be the successive nonnegative times at which X has jumps of size greater than δ in absolute value. The strong Markov property applied at the stopping time J_i^δ and (3.2.1) gives that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{J_i^\delta+\varepsilon} - X_{J_i^\delta}) = d.$$

Hence, at any random time V such that $X_V \neq X_{V-}$ almost surely we have

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{V+\varepsilon} - X_V) = d.$$

□

Proposition 3.3.7. *Let X be a Lévy process with paths of bounded variation, drift $d = 0$, and $\Pi(\mathbb{R}) = \infty$. Then, $\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_t) = 0$ for all $t \in \mathcal{M}^-$ and all $t \in \mathcal{M}^+$ almost surely. Moreover,* *(i) If zero is not regular for $[0,\infty)$, then $X_t > X_{t-}$ for all $t \in \mathcal{M}^-$ and $X_t = X_{t-}$ for all $t \in \mathcal{M}^+$ almost surely.*

(ii) If zero is not regular for $(-\infty, 0]$, then $X_t = X_{t-}$ for all $t \in \mathcal{M}^-$ and $X_t < X_{t-}$ for all $t \in \mathcal{M}^+$ almost surely.

(iii) If zero is regular for both $(-\infty, 0]$ and $[0,\infty)$, then $X_t = X_{t-}$ for all $t \in \mathcal{M}^-$ and all $t \in \mathcal{M}^+$ almost surely.

Proof. Note that since $\Pi(\mathbb{R}) = \infty$, zero must be regular for at least one of $(-\infty, 0]$ and $[0,\infty)$. Results (i), (ii) and (iii) are direct consequences of [62, Theorem 3.1].

Arguing as in Proposition 3.3.6, we get that $\lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_t) = 0$ for any time t such that $X_t \neq X_{t-}$. Using the notation of Propositions 3.3.5 and 3.3.6, suppose that $X_\rho = X_{\rho-}$ and $\liminf_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{\rho+\varepsilon} - X_\rho) > \gamma > 0$. Then, for all $0 < \gamma' < \gamma$ the time ρ is the time of a local infimum of the process $(X_t - \gamma' t)_{t \geq 0}$. Since the drift coefficient of this modified process is less than zero for all such γ', the path of X must jump at time ρ, which is a contradiction. Hence, $\liminf_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{\rho+\varepsilon} - X_\rho) = 0$. □

Theorem 3.3.8. *Let X be a Lévy process that satisfies our standing assumptions Hypothesis 3.2.2 and $\Pi(\mathbb{R}) = \infty$. Then, $\Lambda(\mathbb{R}_+) < \infty$ if and only if*

$$\int_0^1 t^{-1}\mathbb{P}\{X_t \in [-\alpha t, \alpha t]\}\, dt < \infty. \tag{3.3.15}$$

Proof. We break the proof up into the consideration of a number of cases. All of the cases except the first rely on the facts that D is the time of a local infimum of the process $(X_t + \alpha t)_{t \geq 0}$ and that G is the time of a local infimum of the process $(X_{-t} + \alpha t)_{t \geq 0}$.

Case 1: *The process X has paths of bounded variation almost surely and $|d| < \alpha$.*

Suppose that $0 \leq d < \alpha$. By (3.2.1), $\lim_{t \downarrow 0} t^{-1} X_t = d$. Therefore, zero is regular for $(-\infty, 0]$ for the modified process $(X_t - \alpha t)_{t \geq 0}$ but not for the modified process $(X_t + \alpha t)_{t \geq 0}$. Rogozin's regularity criterion (3.2.3) gives that

$$\int_0^1 t^{-1}\mathbb{P}\{X_t - \alpha t \leq 0\}\, dt = \infty$$

and

$$\int_0^1 t^{-1}\mathbb{P}\{X_t + \alpha t \leq 0\}\, dt < \infty.$$

Hence,

$$\int_0^1 t^{-1}\mathbb{P}\{X_t \in [-\alpha t, \alpha t]\,dt$$
$$= \int_0^1 t^{-1}\mathbb{P}\{X_t \le \alpha t\}\,dt - \int_0^1 t^{-1}\mathbb{P}\{X_t \le -\alpha t\}\,dt$$
$$= \infty,$$

and the inequality (3.3.15) fails. Note by Theorem 3.3.1 that $\delta > 0$, and hence $\Lambda(\mathbb{R}_+) = \infty$ – see Remark 3.3.4. The proof for $-\alpha < d \le 0$ is similar.

Case 2: *The process X has paths of bounded variation almost surely and $|d| > \alpha$.*

Suppose that $d > \alpha$. By (3.2.1), $\lim_{t\downarrow 0} t^{-1}X_t = d$, and so zero is not regular for $(-\infty, 0]$ for the modified process $(X_t - \alpha t)_{t\ge 0}$. It follows from (3.2.3) that

$$\int_0^1 t^{-1}\mathbb{P}\{X_t - \alpha t \le 0\}\,dt < \infty.$$

Hence,

$$\int_0^1 t^{-1}\mathbb{P}\{X_t \le \alpha t\}\,dt < \infty$$

and the inequality (3.3.15) certainly holds. It follows from Proposition 3.3.6 that $\liminf_{\varepsilon\downarrow 0} \varepsilon^{-1}((X_{D+\varepsilon} + \alpha\varepsilon) - X_D) > 2\alpha$ a.s. Thus, if

$$D' := \inf\{t > D : t \in \mathcal{Z}\},$$

then $D' > D$ a.s. The regenerative property of $\mathcal{Z}$ implies that $\mathcal{Z}$ is discrete, and hence $\Lambda(\mathbb{R}_+) < \infty$. The proof for $d < -\alpha$ is similar.

Case 3: *The process X has paths of bounded variation almost surely, $d = -\alpha$, and zero is not regular for $(-\infty, 0]$ for the modified process $(X_t + \alpha t)_{t\ge 0}$.*

By (3.2.1), $\lim_{\epsilon\downarrow 0} \varepsilon^{-1}(X_t - \alpha t) = -2\alpha$, and so zero is not regular for $[0,\infty)$ for the modified process $(X_t - \alpha t)_{t\ge 0}$. It follows from (3.2.3) that

$$\int_0^1 t^{-1}\mathbb{P}\{X_t + \alpha t \le 0\}\,dt < \infty$$

and

$$\int_0^1 t^{-1}\mathbb{P}\{X_t - \alpha t \ge 0\}\,dt < \infty.$$

Hence,

$$\int_0^1 t^{-1}\mathbb{P}\{X_t \in [-\alpha t, \alpha t]\, dt$$
$$= \int_0^t t^{-1}\, dt - \int_0^1 t^{-1}\mathbb{P}\{X_t \leq -\alpha t\}\, dt - \int_0^1 t^{-1}\mathbb{P}\{X_t \geq \alpha t\}\, dt$$
$$= \infty,$$

and the inequality (3.3.15) fails. Theorem 3.3.1 implies that $\delta > 0$ and hence $\Lambda(\mathbb{R}_+) = \infty$ – see Remark 3.3.4.

Case 4: *The process X has paths of bounded variation almost surely, $d = -\alpha$, and zero is regular for both $(-\infty, 0]$ and $[0, \infty)$ for the modified process $(X_t + \alpha t)_{t\geq 0}$.*

As in Case 3,

$$\int_0^1 t^{-1}\mathbb{P}\{X_t - \alpha t \geq 0\}\, dt < \infty.$$

Also,

$$\int_0^1 t^{-1}\mathbb{P}\{X_t + \alpha t \leq 0\}\, dt = \infty$$

and

$$\int_0^1 t^{-1}\mathbb{P}\{X_t + \alpha t \geq 0\}\, dt = \infty.$$

Hence,

$$\int_0^1 t^{-1}\mathbb{P}\{X_t \in [-\alpha t, \alpha t]\, dt$$
$$= \int_0^1 t^{-1}\mathbb{P}\{X_t \geq -\alpha t\}\, dt - \int_0^1 t^{-1}\mathbb{P}\{X_t \geq \alpha t\}\, dt$$
$$= \infty,$$

and inequality (3.3.15) fails. Note that $\delta = 0$ by Theorem 3.3.1, so if $\Lambda(\mathbb{R}_+) < \infty$, then $M_t = X_D + \alpha(t - D)$ for $0 \leq t \leq \varepsilon$ for some $\varepsilon > 0$, but $\liminf_{\varepsilon\downarrow 0} \varepsilon^{-1}((X_{D+\varepsilon} + \alpha\varepsilon) - X_D) = 0$ a.s. by Proposition 3.3.7. Thus, we must have $\Lambda(\mathbb{R}_+) = \infty$.

Case 5: *The process X has paths of bounded variation almost surely, $d = -\alpha$, zero is regular for $(-\infty, 0]$ and not regular for $[0, \infty)$ for the modified process $(X_t + \alpha t)_{t\geq 0}$.*

Similarly to Case 4,

$$\int_0^1 t^{-1}\mathbb{P}\{X_t - \alpha t \geq 0\}\, dt < \infty,$$

$$\int_0^1 t^{-1}\mathbb{P}\{X_t + \alpha t \leq 0\}\, dt = \infty,$$

and

$$\int_0^1 t^{-1}\mathbb{P}\{X_t + \alpha t \geq 0\}\, dt < \infty.$$

In particular,

$$\int_0^1 t^{-1}\mathbb{P}\{X_t \geq -\alpha t\}\, dt < \infty,$$

and the inequality (3.3.15) holds. Proposition 3.3.7 gives that $X_D > X_{D-}$ a.s. Thus, $D' > D$ a.s. The regenerative property of $\mathcal{Z}$ implies that $\mathcal{Z}$ is discrete, and hence $\Lambda(\mathbb{R}_+) < \infty$.

Case 6: *The process X has paths of bounded variation almost surely and $d = \alpha$.*

This is handled by considering the behavior at G for the time reversed process in the manner of Cases 3,4, and 5.

Case 7: *The process X has paths of unbounded variation almost surely and $\int_0^1 t^{-1}\mathbb{P}\{X_t \in [-\alpha t, (\alpha+\gamma)t]\}\, dt < \infty$ for some $\gamma > 0$.*

Proposition 3.3.5 gives that $\liminf_{\varepsilon\downarrow 0} \varepsilon^{-1}((X_{D+\varepsilon}+\alpha\varepsilon)-X_D) \geq 2\alpha+\gamma$ a.s., which, as in Case 2, implies that $D' > D$ a.s. and hence $\Lambda(\mathbb{R}_+) < \infty$.

Case 8: *The process X has paths of unbounded variation, (3.3.15) holds, but $\int_0^1 t^{-1}\mathbb{P}\{X_t \in [-\alpha t, (\alpha+\gamma)t]\}\, dt = \infty$ for every $\gamma > 0$.*

To get to the desired result that $\Lambda(\mathbb{R}_+) < \infty$ we introduce a new technique involving convex minorants of Lévy processes.

As before, let $(X_t^q)_{t\geq 0}$ be a copy of $(X_t)_{t\geq 0}$ but killed at an independent exponential time ξ with parameter $0 < q < \infty$, and let $\rho = \arg\inf_{0<t<\xi} X_t \wedge X_{t-}$.

By results of Pitman and Uribe Bravo on the convex minorant of the path of a Lévy process [73, Corollary 2] discussed in Section 1.3, the linear segments of the convex minorant of the process $(X_t + \alpha t)_{0\leq t\leq \xi}$ define a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}$, where a point at (t, x) represents a segment with length t and increment x. The intensity measure of the Poisson point process is the measure on $\mathbb{R}_+ \times \mathbb{R}$ given by

$$e^{-qt}t^{-1}\mathbb{P}\{X_t + \alpha t \in dx\}dt. \tag{3.3.16}$$

In order to recreate the convex minorant from the point process, the segments are arranged in increasing order of slope. Note that the convex minorant after time ρ can be recreated by only piecing together the segments of positive slope.

Let $\mathcal{I}$ be the infimum of the slopes of all segments of the convex minorant of $(X_t + \alpha t)_{0\leq t\leq \xi}$ that have positive slope. Under the assumption (3.3.15), it follows from (3.3.16) that

$$\mathbb{P}\{\mathcal{I} \geq 2\alpha\} = \exp\left(-\int_0^\infty e^{-qt}t^{-1}\mathbb{P}\{X_t + \alpha t \in [0, 2\alpha t]\}\, dt\right) > 0.$$

Thus, with positive probability, there exists $\varepsilon > 0$ such that $(X_{\rho+t} + \alpha t) - X_\rho \geq 2\alpha t$ for all $0 \leq t \leq \varepsilon$. Hence by Millar's zero-one law at the infimum of a Lévy process, such an ε exists almost surely. By the almost sure uniqueness of the value of the infimum of a Lévy process that is not a compound Poisson process with zero drift [15, Proposition VI.4], almost surely there exists $\varepsilon > 0$ such that $(X_{\rho+t} + \alpha t) - X_\rho > 2\alpha t$ for all $0 < t \leq \varepsilon$.

Using the same localization argument as before, this behavior extends to all local infima almost surely, and hence is true at time D for the process $(X_t + \alpha t)_{t \geq 0}$, which allows us to conclude that $D' > D$ a.s. Since $\delta = 0$, this implies $\Lambda(\mathbb{R}_+) < \infty$. □

Remark 3.3.9. An example of a process satisfying our standing assumptions for which (3.3.15) fails for all $\alpha > 0$ is given by truncating the Lévy measure of the symmetric Cauchy process to remove all jumps with magnitude greater than m, so that the Lévy measure becomes $1_{|x| \leq m} x^{-2}\, dx$. To see this, first note that (3.3.15) fails for the symmetric Cauchy process because, by the self-similarity properties of this process, the probability that it lies in an interval of the form (at, bt) at time $t > 0$ does not depend on t and $\int_0^1 t^{-1}\, dt = \infty$. Then observe that the difference between the probabilities that the truncated process and the symmetric Cauchy processes lie in some interval at time t is at most the probability that the symmetric Cauchy process has at least one jump of size greater than m before time t. The latter probability is $1 - e^{-\lambda t}$ with $\lambda = 2\int_m^\infty x^{-2}\, dx < \infty$, and $\int_0^1 t^{-1}(1 - e^{-\lambda t})\, dt < \infty$.

Remark 3.3.10. If X is a symmetric β-stable process for $1 < \beta \leq 2$, then (3.3.15) holds for all $\alpha > 0$. To see this, first note that X_1 has a bounded density. Hence, by scaling, $\mathbb{P}\{X_t \in [-\alpha t, \alpha t]\} = \mathbb{P}\{t^{1/\beta}X_1 \in [-\alpha t, \alpha t]\} \leq ct^{1-1/\beta}$ for some constant c depending on α, and then observe that $\int_0^1 t^{-1}t^{1-1/\beta}\, dt < \infty$. Further cases when (3.3.15) holds for all $\alpha > 0$ are discussed in Remark 3.4.3.

The technique introduced at the end of the previous proof allows a strengthening of Propositions 3.3.5 and 3.3.6, which we state only for local infima but has a clear counterpart for local suprema. The result also covers much of the information from Proposition 3.3.7 but does not strengthen it.

Theorem 3.3.11. *Let X be an Lévy process such that $\sigma \neq 0$ or $\Pi(\mathbb{R}) < \infty$. Define*

$$r^* := \sup\{r \geq 0 : \int_0^1 t^{-1}\mathbb{P}\{X_t \in [0, rt])dt < \infty\}$$

Then,

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_t \wedge X_{t-}) = r^*$$

for all $t \in \mathcal{M}^-$ almost surely. Moreover, define, for $r \geq 0$ and $t \geq 0$,

$$T_t^{(r)} := \inf\{s > 0 : X_{t+s} - X_t \wedge X_{t-} \leq rs\}.$$

If $\int_0^1 t^{-1}\mathbb{P}\{X_t \in [0, r^*t]\}\, dt < \infty$, then $T_t^{(r^*)} > 0$ for all $t \in \mathcal{M}^-$ almost surely.

Proof. As usual, we need only show that the given properties hold at time $\rho = \arg\inf_{0<t<\xi} X_t^q \wedge X_{t-}^q$. Suppose first that $\int_0^1 t^{-1}\mathbb{P}\{X_t \in [0, r^*t]\}\, dt < \infty$. Let $\mathcal{I}$ be the infimum of the slopes of all segments of the convex minorant of $(X_t)_{0 \leq t \leq \xi}$ that have positive slope. It follows from (3.3.16) that

$$\mathbb{P}\{\mathcal{I} \geq r^*\} = \exp\left(-\int_0^\infty e^{-qt} t^{-1}\mathbb{P}\{X_t \in [0, r^*t]\}\, dt\right) > 0.$$

Thus, with positive probability, there exists $\varepsilon > 0$ such that $X_{\rho+t} - X_\rho \geq r^*t$ for all $0 \leq t \leq \varepsilon$. Hence, by Millar's zero-one law at the infimum of a Lévy process, such an ε exists almost surely. By the almost sure uniqueness of the value of the infima of a Lévy process that is not a compound Poisson process with zero drift [15, Proposition VI.4], there exists almost surely $\varepsilon > 0$ such that $X_{\rho+t} - X_\rho > r^*t$ for all $0 < t \leq \varepsilon$. Hence, $T_\rho^{(r^*)} > 0$ a.s.

For any $0 \leq r < r^*$ we have $\int_0^1 t^{-1}\mathbb{P}\{X_t \in [0, rt]\}\, dt < \infty$. Applying the above argument gives that $T_\rho^{(r)} = 0$ almost surely, and thus

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_t \wedge X_{t-}) \geq r$$

for all $t \in \mathcal{M}^-$ almost surely, for all $0 \leq r < r^*$.

For any $r > r^*$ we have $\int_0^1 t^{-1}\mathbb{P}\{X_t \in [0, rt]\}\, dt = \infty$, and hence $\mathbb{P}\{\mathcal{I} \geq r\} = 0$. Since the convex minorant of $(X_t)_{0 \leq t \leq \xi}$ almost surely contains linear segments with positive slope less that or equal to r, it follows that $T_t^{(r)} = 0$ almost surely. Hence, for all $r > r^*$,

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_t \wedge X_{t-}) \leq r$$

for all $t \in \mathcal{M}^-$ almost surely. □

Remark 3.3.12. The value of r^* is infinite when X has non-zero Brownian component or is a stable process with stability parameter in the interval $(1, 2]$ – see the discussion of *abrupt* processes in Section 3.4. Vigon provides in an unpublished work

[93] a practical method for determining whether the integral in Theorem 3.3.11 is finite or not for processes with paths of unbounded variation:

$$\int_0^\infty t^{-1} e^{-qt} \mathbb{P}\{X_t \in [at, bt]\}\, dt = \frac{1}{2\pi} \int_a^b \left(\int_0^\infty \mathfrak{R} \frac{1}{\Psi(-u) + iur}\, du \right) dr,$$

where Ψ is as defined in Section 3.2.1.

In Section 3.6 we prove the following result, which characterizes the subordinator associated with $\mathcal{Z}$ when X has paths of unbounded variation and satisfies certain extra conditions. Note that the conclusion $\delta = 0$ in the result follows from Remark 3.3.2(i).

Theorem 3.3.13. *Let X be a Lévy process with paths of unbounded variation almost surely that satisfies our standing assumptions Hypothesis 3.2.2. Suppose further that X_t has absolutely continuous distribution for all $t \neq 0$, and that the either*

(i) X creeps upward or downward, or

(ii) the densities of the random variables $\inf_{t \geq 0}\{X_t + \alpha t\}$ and $\inf_{t \geq 0}\{X_{-t} + \alpha t\}$ are square integrable.

Then $\delta = 0$ and Λ is characterized by

$$\begin{aligned}
&\frac{\int_{\mathbb{R}_+} (1 - e^{-\theta x})\, \Lambda(dx)}{\int_{\mathbb{R}_+} x\, \Lambda(dx)} \\
&\quad = 4\pi\alpha \int_{-\infty}^{\infty} \Bigg\{ \exp\Bigg(\int_0^\infty t^{-1} \mathbb{E}\Big[\big(e^{izX_t - iz\alpha t} - 1\big) \mathbf{1}\{X_t \geq +\alpha t\} \\
&\qquad\qquad + \big(e^{izX_t + iz\alpha t} - 1\big) \mathbf{1}\{X_t \leq -\alpha t\}\Big]\, dt \Bigg) \\
&\qquad - \exp\Bigg(\int_0^\infty t^{-1} \mathbb{E}\Big[\big(e^{-\theta t + izX_t - iz\alpha t} - 1\big) \mathbf{1}\{X_t \geq +\alpha t\} \\
&\qquad\qquad + \big(e^{-\theta t + izX_t + iz\alpha t} - 1\big) \mathbf{1}\{X_t \leq -\alpha t\}\Big]\, dt \Bigg) \Bigg\}\, dz
\end{aligned}$$

for $\theta \geq 0$, and moreover $\Lambda(\mathbb{R}_+) < \infty$.

Note that the existence of the densities of the infima in the hypothesis (ii) of Theorem 3.3.13 comes from the assumption that X_t has absolutely continuous distribution for all $t \neq 0$ – see Lemma 3.5.1. In Corollary 3.5.4 we show that condition

(ii) holds when X has non-zero Brownian component, although it is already well known that (i) holds that case.

When the conditions of Theorem 3.3.13 are not satisfied, we are able to give a characterization of Λ as a limit of integrals in the following way. Let $X^\varepsilon = X + \varepsilon B$, with B a (two-sided) standard Brownian motion independent of X, and let Λ^ε be the Lévy measure of the subordinator associated with the contact set for X^ε. Then in the case $\delta = 0$ we have the representation

$$\frac{\int_{\mathbb{R}_+}(1-e^{-\theta x})\,\Lambda(dx)}{\int_{\mathbb{R}_+} x\,\Lambda(dx)} = \lim_{\varepsilon\downarrow 0}\frac{\int_{\mathbb{R}_+}(1-e^{-\theta x})\,\Lambda^\varepsilon(dx)}{\int_{\mathbb{R}_+} x\,\Lambda^\varepsilon(dx)}.$$

See Lemma 3.6.4 in Section 3.6 for details of this limit and (3.6.10) for a proof of the above equality.

Theorem 3.3.8 together with the conclusion $\Lambda(\mathbb{R}_+) < \infty$ of Theorem 3.3.13 result in

Corollary 3.3.14. *Suppose the conditions of Theorem 3.3.13 are satisfied, then* $\int_0^1 t^{-1}\mathbb{P}\{X_t \in [-\alpha t, \alpha t]\}\,dt < \infty$*.*

3.4 The limit of the contact set for increasing slopes

We now investigate how $\mathcal{Z}$ changes as α increases. For the sake of clarity, let X be a fixed Lévy process with $X_0 = 0$ and $\mathbb{E}[|X_1|] < \infty$. Write $M^{(\alpha)} = (M_t^{(\alpha)})_{t\in\mathbb{R}}$ for the α-Lipschitz minorant of X for $\alpha > |\mathbb{E}[X_1]|$, and put $\mathcal{Z}_\alpha := \{t \in \mathbb{R} : X_t \wedge X_{t-} = M_t^{(\alpha)}\}$. For $|\mathbb{E}[X_1]| < \alpha' \le \alpha''$, we have $M_t^{(\alpha')} \le M_t^{(\alpha'')} \le X_t$ for all $t \in \mathbb{R}$ (because any α'-Lipschitz function is also α''-Lipschitz), and so $\mathcal{Z}_{\alpha'} \subseteq \mathcal{Z}_{\alpha''}$. We note in passing that $\mathcal{Z}_{\alpha'}$ is *regeneratively embedded* in $\mathcal{Z}_{\alpha''}$ in the sense of Bertoin [16].

If X has paths of bounded variation and drift coefficient d, then $|d| < \alpha$ for all α large enough. Since $\lim_{t\downarrow 0} t^{-1}X_t = -\lim_{t\downarrow 0} t^{-1}X_{-t} = d$, the law of large numbers implies that

$$\lim_{\alpha\to\infty}\mathbb{P}\{0\in\mathcal{Z}_\alpha\} = \lim_{\alpha\to\infty}\mathbb{P}\{\inf_{t\ge 0}(X_t+\alpha t) = \inf_{t\le 0}(X_t-\alpha t) = 0\} = 1,$$

and thus the set $\bigcup_{\alpha>|\mathbb{E}[X_1]|} Z_\alpha$ has full Lebesgue measure.

We now consider the case where X has paths of unbounded variation. In order to state our result, we need to recall the definition of the so-called *abrupt* Lévy processes introduced by Vigon [92]. Recall from (3.3.8) that $\mathcal{M}^-$ is the set of local infima of the path of X, and that as noted in [92], if the paths of X have unbounded variation, then almost surely $X_{t-} = X_t$ for all $t \in \mathcal{M}^-$.

Definition 3.4.1. A Lévy process X is *abrupt* if its paths have unbounded variation and almost surely for all $t \in \mathcal{M}^-$

$$\limsup_{\varepsilon \uparrow 0} \frac{X_{t+\varepsilon} - X_{t-}}{\varepsilon} = -\infty \quad \text{and} \quad \liminf_{\varepsilon \downarrow 0} \frac{X_{t+\varepsilon} - X_t}{\varepsilon} = +\infty.$$

Remark 3.4.2. An equivalent definition may be made in terms of local suprema [92, Remark 1.2]: a Lévy process X with paths of unbounded variation is abrupt if almost surely for any t that is the time of a local supremum,

$$\liminf_{\varepsilon \uparrow 0} \frac{X_{t+\varepsilon} - X_{t-}}{\varepsilon} = +\infty \quad \text{and} \quad \limsup_{\varepsilon \downarrow 0} \frac{X_{t+\varepsilon} - X_t}{\varepsilon} = -\infty.$$

Remark 3.4.3. A Lévy process X with paths of unbounded variation is abrupt if and only if

$$\int_0^1 t^{-1}\mathbb{P}\{X_t \in [at, bt]\}\, dt < \infty, \quad \forall a < b, \tag{3.4.1}$$

(see [92, Theorem 1.3]). Examples of abrupt Lévy processes include stable processes with stability parameter in the interval $(1, 2]$, processes with non-zero Brownian component, and any processes that creep upwards or downwards. An example of an unbounded variation process that is not abrupt is the symmetric Cauchy process (however this process is *eroded* in the sense of the upcoming Definition 4.3.6).

Remark 3.4.4. The analytic condition given in Remark 3.4.3 (3.4.1) for a Lévy process X to be abrupt has an interpretation in terms of the smoothness of the convex minorant of X over a finite interval. The results of Pitman and Uribe Bravo [73] (in particular, Theorem 1.3.7) imply that the number of segments of the convex minorant of X over a finite interval with slope between a and b is finite for all $a < b$ if and only if (3.4.1) holds.

We now return to the question of the limit of $\mathcal{Z}_\alpha$.

Theorem 3.4.5. *Let X be a Lévy process with $X_0 = 0$ and $|\mathbb{E}[X_1]| < \infty$. Then $\bigcup_{\alpha > |\mathbb{E}[X_1]|} \mathcal{Z}_\alpha \supseteq \mathcal{M}^-$. Furthermore, if X is abrupt, then $\bigcup_{\alpha > |\mathbb{E}[X_1]|} \mathcal{Z}_\alpha = \mathcal{M}^-$.*

Proof. Suppose that $t \in \mathcal{M}^-$ so that there exists $\epsilon > 0$ such that $\inf\{X_s : t - \epsilon < s < t + \epsilon\} = X_t = X_{t-}$. Fix any $\beta > |\mathbb{E}[X_1]|$. Then, by the strong law of large numbers, $\inf\{X_s + \beta s : s \geq 0\} > -\infty$ and $\inf\{X_s - \beta s : s \leq 0\} > -\infty$. It is clear that if $\alpha \in \mathbb{R}$ is such that

$$\alpha > -\frac{\inf\{X_s + \beta s : s \geq 0\} \vee \inf\{X_s - \beta s : s \leq 0\},}{\epsilon}$$

then $X_t = X_{t-} = M_t^{(\alpha)}$ and $t \in \mathcal{Z}_\alpha$. Hence $\bigcup_{\alpha > |\mathbb{E}[X_1]|} \mathcal{Z}_\alpha \supseteq \mathcal{M}^-$.

Now suppose that X is abrupt, and let $t \in \mathcal{Z}_\alpha$ for some $\alpha > |\mathbb{E}[X_1]|$. Then, one of the following three possibilities must occur:

(a) $X_t > X_{t-}$ and $\limsup_{\varepsilon \uparrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_{t-}) \leq \alpha$;

(b) $X_{t-} > X_t$ and $\liminf_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_t) \geq -\alpha$;

(c) $X_{t-} = X_t$ and $\limsup_{\varepsilon \uparrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_{t-}) \leq \alpha$, $\liminf_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_t) \geq -\alpha$.

We discount options (a) and (b) by assuming that t is a jump time of X and then showing that the lim inf or lim sup part of the statements cannot occur. Our argument borrows heavily from the proof of Property 2 in [73, Proposition 1] (stated without proof here as Proposition 1.3.1), which itself is based on the proof of [62, Proposition 2.4], but is more detailed.

Arguing as in the proof of Proposition 3.3.6, for $\delta > 0$, let $0 < J_1^\delta < J_2^\delta < \dots$ be the successive nonnegative times at which X has jumps of size greater than δ in absolute value. The strong Markov property applied at the stopping time J_i^δ and (3.2.2) gives that

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{J_i^\delta + \varepsilon} - X_{J_i^\delta}) = -\infty \quad \text{and} \quad \limsup_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{J_i^\delta + \varepsilon} - X_{J_i^\delta}) = +\infty.$$

Hence, at any random time V such that $X_V \neq X_{V-}$ almost surely we have

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{V+\varepsilon} - X_V) = -\infty,$$

and, by a time reversal,

$$\limsup_{\varepsilon \uparrow 0} \varepsilon^{-1}(X_{V+\varepsilon} - X_{V-}) = +\infty.$$

Thus, neither of the possibilities (a) or (b) hold, and so (c) must hold. It then follows from Theorem 3.4.6 below that X must have a local infimum or supremum at t. However, X cannot have a local supremum at t by Remark 3.4.2, and so X must have a local infimum at t. $\square$

The key to proving Theorem 3.4.5 in the abrupt case was the following theorem that describes the local behavior of an abrupt Lévy process at arbitrary times. This result is an immediate corollary of [92, Theorem 2.6] once we use the fact that almost surely the paths of a Lévy processes cannot have both points of increase and points of decrease [37].

Theorem 3.4.6. *Let X be an abrupt Lévy process. Then, almost surely for all t one of the following possibilities must hold:*

(i) $\limsup_{\varepsilon\uparrow 0}\varepsilon^{-1}(X_{t+\varepsilon}-X_{t-})=+\infty$ *and* $\liminf_{\varepsilon\downarrow 0}\varepsilon^{-1}(X_{t+\varepsilon}-X_t)=-\infty$*;*

(ii) $\limsup_{\varepsilon\uparrow 0}\varepsilon^{-1}(X_{t+\varepsilon}-X_{t-})<+\infty$ *and* $\liminf_{\varepsilon\downarrow 0}\varepsilon^{-1}(X_{t+\varepsilon}-X_t)=-\infty$*;*

(iii) $\limsup_{\varepsilon\uparrow 0}\varepsilon^{-1}(X_{t+\varepsilon}-X_{t-})=+\infty$ *and* $\liminf_{\varepsilon\downarrow 0}\varepsilon^{-1}(X_{t+\varepsilon}-X_t)>-\infty$*;*

(iv) *X has a local infimum or supremum at t.*

Remark 3.4.7. Theorem 3.4.5 shows that the α-Lipschitz minorant provides a method for "sieving out" a certain discrete set of times of local infima of an abrupt process. This method has the property that if we let $\alpha\to\infty$, then eventually we collect all the times of local infima. Alternative methods for sieving out the local minima of Brownian motion are discussed in [66, 67]. One method is to take all local infima times t such that $X_{t+s}-X_t>0$ for all $s\in(-h,h)$ for some fixed h, and then let $h\to 0$. Another is to take all local infima times t such that $X_{s_+}-X_t\geq h$ for some time $s_+\in(0,\inf\{s>0:X_s-X_t=0\})$ and such that $X_{s_-}-X_t\geq h$ for some time $s_-\in(0,\inf\{s<0:X_s-X_t=0\})$, and then again let $h\to 0$. This work is extended to Brownian motion with drift in [34].

3.5 Future infimum of a Lévy process

For future use, we collect together in this section some preliminary results concerning the distribution of the infimum of a Lévy process $(Z_t)_{t\geq 0}$ and the time at which the infimum is attained.

Let $Z=(Z_t)_{t\geq 0}$ be a Lévy process such that $Z_0=0$. Set $\underline{Z}_t:=\inf\{Z_s:0\leq s\leq t\}$, $t\geq 0$. If Z is not a compound Poisson process (that is, either Z has a non-zero Brownian component or the Lévy measure of Z has infinite total mass or the Lévy measure has finite total mass but there is a non-zero drift coefficient), then

$$\mathbb{P}\{\exists 0\leq s<t<u:\underline{Z}_s=\underline{Z}_t=Z_t\wedge Z_{t-}=\underline{Z}_u\}=0 \tag{3.5.1}$$

– see, for example, [15, Proposition VI.4]. Hence, almost surely for each $t\geq 0$ there is a unique time U_t such that $Z_{U_t}\wedge Z_{U_t-}=\underline{Z}_t$. If, in addition, $\lim_{t\to\infty}Z_t=+\infty$, then almost surely there is a unique time U_∞ such that $Z_{U_\infty} \wedge Z_{U_\infty -} = \underline{Z}_\infty := \inf\{Z_s : s \geq 0\}$.

Lemma 3.5.1. *Let Z be a Lévy process such that $Z_0 = 0$, Z_t has an absolutely continuous distribution for each $t > 0$, and $\lim_{t\to\infty} Z_t = +\infty$. Then, the distribution of $(U_\infty, \underline{Z}_\infty)$ restricted to $(0,\infty) \times (-\infty, 0]$ is absolutely continuous with respect to Lebesgue measure. Moreover, $\mathbb{P}\{(U_\infty, \underline{Z}_\infty) = (0,0)\} > 0$ if and only if zero is not regular for $(-\infty, 0)$.*

Proof. Because the random variable Z_t has an absolutely continuous distribution for each $t > 0$, it follows from [73, Theorem 2] that for all $t > 0$ the restriction of the distribution of the random vector $(U_t, \underline{Z}_t)$ is absolutely continuous with respect to Lebesgue measure on the set $(0,t] \times (-\infty, 0]$. Observe that

$$\mathbb{P}\left\{\exists s : (U_t, \underline{Z}_t) = (U_\infty, \underline{Z}_\infty)\ \forall t \geq s\right\} = 1.$$

Thus, if $A \subseteq (0,\infty) \times (-\infty, 0]$ is Borel with zero Lebesgue measure, then

$$\mathbb{P}\left\{(U_\infty, \underline{Z}_\infty) \in A\right\} = \lim_{t\to\infty} \mathbb{P}\left\{(U_t, \underline{Z}_t) \in A\right\} = 0.$$

The proof the claim concerning the atom at $(0,0)$ follows from the above formula, the fact that $\mathbb{P}\{(U_t, \underline{Z}_t) = (0,0)\} > 0$ if and only if zero is not regular for the interval $(-\infty, 0)$ [73, Theorem 2], and the hypothesis that $\lim_{t\to\infty} Z_t = +\infty$. □

Remark 3.5.2. Note that if the process Z has a non-zero Brownian component, then the random variable Z_t has an absolutely continuous distribution for all $t > 0$. Moreover, in this case zero is regular for the interval $(-\infty, 0)$

Let $\tau = (\tau_t)_{t\geq 0}$ be the local time at zero for the process $Z - \underline{Z}$. Write τ^{-1} for the inverse local time process. Set $\underline{H}_t := \underline{Z}_{\tau^{-1}(t)}$ The process $\underline{H} := (\underline{H}_t)_{t\geq 0}$ is the *descending ladder height process* for Z. If $\lim_{t\to\infty} Z_t = +\infty$, then $\hat{\underline{H}} := -\underline{H}$ is a subordinator killed at an independent exponential time (see, for example, [15, Lemma VI.2]).

For the sake of completeness, we include the following observation that combines well-known results and probably already exists in the literature – it can be easily concluded from Theorem 19 and the remarks at the top of page 172 of [15].

Lemma 3.5.3. *Let Z be a Lévy process such that $Z_0 = 0$ and $\lim_{t\to\infty} Z_t = +\infty$. Then, the distribution of random variable $\underline{Z}_\infty$ is absolutely continuous with a bounded density if and only if the (killed) subordinator $\hat{\underline{H}}$ has a positive drift coefficient.*

Proof. Let $S = (S_t)_{t\geq 0}$ be an (unkilled) subordinator with the same drift coefficient and Lévy measure as $\underline{\hat{H}}$, so that $-\underline{Z}_\infty$ has the same distribution as S_ζ, where ζ is an independent, exponentially distributed random time. Therefore, for some $q > 0$,

$$\mathbb{P}\{-\underline{Z}_\infty \in A\} = \int_0^\infty q e^{-qt} \mathbb{P}\{S_t \in A\}\, dt$$

for any Borel set $A \subseteq \mathbb{R}$. By a result of Kesten for general Lévy processes (see, for example, [15, Theorem II.16]) the q-resolvent measure $\int_0^\infty e^{-qt}\mathbb{P}\{S_t \in \cdot\}\, dt$ of S is absolutely continuous with a bounded density for all $q > 0$ (equivalently, for some $q > 0$) if and only if points are not essentially polar for S. Moreover, points are not essentially polar for a Lévy process with paths of bounded variation (and, in particular, for a subordinator) if and only if the process has a non-zero drift coefficient [15, Corollary II.20]. □

Corollary 3.5.4. *Let X be a Lévy process that satisfies our standing assumptions Hypothesis 3.2.2 and which has paths of unbounded variation almost surely. Then, the random variables $\inf\{X_t + \alpha t : t \geq 0\}$ and $\inf\{X_t - \alpha t : t \leq 0\}$ both have absolutely continuous distributions with bounded densities if and only if X has a non-zero Brownian component.*

Proof. By Lemma 3.5.3, the distributions in question are absolutely continuous with bounded densities if and only if the drift coefficients of the descending ladder processes for the two Lévy processes $(X_t + \alpha t)_{t\geq 0}$ and $(-X_t + \alpha t)_{t\geq 0}$ are non-zero. By the results of [61] (see also [15, Theorem VI.19]), this occurs if and only if both $(X_t + \alpha t)_{t\geq 0}$ and $(-X_t + \alpha t)_{t\geq 0}$ have positive probability of creeping down across x for some (equivalently, all) $x < 0$, where we recall that a Lévy process creeps down across $x < 0$ if the first passage time in $(-\infty, x)$ is not a jump time for the path of the process. Equivalently, both densities exist and are bounded if and only if the Lévy process $(X_t+\alpha t)_{t\geq 0}$ creeps downwards and the Lévy process $(X_t-\alpha t)_{t\geq 0}$ creeps upwards, where the latter notion is defined in the obvious way.

A result of Vigon [94] (see also [32, Chapter 6, Corollary 9]) states that when the paths of X have unbounded variation, $(X_t+\alpha t)_{t\geq 0}$ creeps downward if and only if X creeps downward, and hence, in turn, if and only if $(X_t - \alpha t)_{t\geq 0}$ creeps downwards. A similar result applies to creeping upwards.

Thus, both densities exist and are bounded if and only if X creeps downwards and upwards. This occurs if and only if the ascending and descending ladder processes of X have positive drifts [15, Theorem VI.19], which happens if and only if X has a non-zero Brownian component [32, Chapter 4, Corollary 4(i)] (or see the remark after the proof of [15, Theorem VI.19]). □

3.6 The complementary interval straddling zero

3.6.1 Distributions in the case of a non-zero Brownian component

Suppose that $X = (X_t)_{t\in\mathbb{R}}$ is a Lévy process that satisfies our standing assumptions Hypothesis 3.2.2. Also, suppose until further notice that X has a non-zero Brownian component.

Recall that $M = (M_t)_{t\in\mathbb{R}}$ is the α-Lipschitz minorant of X and $\mathcal{Z}$ is the stationary regenerative set $\{t \in \mathbb{R} : X_t \wedge X_{t-} = M_t\}$. Recall also that $K = D - G$, where $G = \sup\{t < 0 : X_t \wedge X_{t-} = M_t\} = \sup\{t < 0 : t \in \mathcal{Z}\}$ and $D = \inf\{t > 0 : X_t \wedge X_{t-} = M_t\} = \inf\{t > 0 : t \in \mathcal{Z}\}$. Lastly, recall that T is the unique $t \in [G, D]$ such that $M_t = \max\{M_s : s \in [G, D]\}$.

Let f^+ (respectively, f^-) be the joint density of the random variables we denoted by $(U_\infty, \underline{Z}_\infty)$ in Lemma 3.5.1 in the case where the Lévy process Z is $(X_t + \alpha t)_{t\geq 0}$ (respectively, $(-X_t + \alpha t)_{t\geq 0}$).

Proposition 3.6.1. *Let X be a Lévy process that satisfies our standing assumptions Hypothesis 3.2.2. Suppose, moreover, that X has a non-zero Brownian component. Set $L := T - G$ and $R := D - T$. Then, the random vector (T, L, R) has a distribution that is absolutely continuous with respect to Lebesgue measure with joint density*

$$(\tau, \lambda, \rho) \mapsto 2\alpha \int_{-\infty}^{0} f^-(\lambda, h) f^+(\rho, h)\, dh, \quad \lambda, \rho > 0 \text{ and } \tau - \lambda < 0 < \tau + \rho.$$

Therefore, (T, G, D) also has an absolutely continuous distribution with joint density

$$(\tau, \gamma, \delta) \mapsto 2\alpha \int_{-\infty}^{0} f^-(\tau - \gamma, h) f^+(\delta - \tau, h)\, dh, \quad \gamma < 0 < \delta \text{ and } \gamma < \tau < \delta,$$

and K has an absolutely continuous distribution with density

$$\kappa \mapsto 2\alpha\kappa \int_0^\kappa \int_{-\infty}^0 f^-(\xi, h) f^+(\kappa - \xi, h)\, dh\, d\xi, \quad \kappa > 0.$$

Proof. Observe that X is abrupt and so, by Theorem 3.3.8 and Remark 3.4.3, $\mathcal{Z}$ is a stationary discrete random set with intensity

$$\left(\frac{\int_{\mathbb{R}_+} x\Lambda(dx)}{\Lambda(\mathbb{R}_+)}\right)^{-1} = \frac{\Lambda(\mathbb{R}_+)}{\int_{\mathbb{R}_+} x\Lambda(dx)} < \infty.$$

Hence, the set of times of peaks of the α-Lipschitz minorant M is also a stationary discrete random set with the same finite intensity. The point process consisting of a single point at time T is included in the set of times of peaks of M, and so for A a Borel set with Lebesgue measure $|A|$ we have

$$\begin{aligned}\mathbb{P}\{T \in A\} &\le \mathbb{P}\{\text{at least one peak of } M \text{ at a time } t \in A\}\\ &\le \mathbb{E}\left[\text{number of times of peaks in } A\right]\\ &= \frac{\Lambda(\mathbb{R}_+)}{\int_{\mathbb{R}_+} x\Lambda(dx)}\,|A|.\end{aligned}$$

Thus, the distribution of T is absolutely continuous with respect to Lebesgue measure with density bounded above by $\Lambda(\mathbb{R}_+)/\int_{\mathbb{R}_+} x\Lambda(dx)$.

It follows from the observations made in the proof of Theorem 3.2.6 about the nature of the global infimum of the process $\tilde{X}$ that under our hypotheses, almost surely $X_G = X_{G-} = M_T - \alpha|G - T| = M_T - \alpha L$, $X_D = X_{D-} = M_T - \alpha|D - T| = M_T - \alpha R$, and $X_t \wedge X_{t-} > M_T - \alpha|t - T|$ for $t \notin \{G, D\}$. Thus,

$$0 = \inf\{X_{T+t} - (M_T - \alpha t) : t \ge 0\} = X_{T+R} - (M_T - \alpha R)$$

and

$$\begin{aligned}0 &= \inf\{X_{T+t} - (M_T + \alpha t) : t \le 0\}\\ &= \inf\{X_{T-t} - (M_T - \alpha t) : t \ge 0\} = X_{T-L} - (M_T - \alpha L).\end{aligned}$$

Consequently,

$$\begin{aligned}X_{T-L} - X_T + \alpha L &= \inf\{X_{T-t} - X_T + \alpha t : t \ge 0\}\\ &= \inf\{X_{T+t} - X_T + \alpha t : t \ge 0)\} = X_{T+R} - X_T + \alpha R.\end{aligned} \tag{3.6.1}$$

Conversely, (T, L, R) is the unique triple with $T - L < 0 < T + R$ such that (3.6.1) holds.

Fix $\tau \in \mathbb{R}$ and $\lambda, \rho \in \mathbb{R}_+$ such that $\tau - \lambda < 0 < \tau + \rho$. Set

$$\begin{aligned}Z_t^- &:= X_{\tau-t} - X_\tau + \alpha t, \quad t \ge 0,\\ \underline{Z}^- &:= \inf\{Z_t^- : t \ge 0\},\\ U^- &:= \inf\{t \ge 0 : Z_t^- = \underline{Z}^-\}.\end{aligned}$$

For $0 < \Delta\tau < \rho$ set

$$\begin{aligned}Z_t^+ &:= X_{t+\tau+\Delta\tau} - X_{\tau+\Delta\tau} + \alpha t, \quad t \ge 0,\\ \underline{Z}^+ &:= \inf\{Z_t^+ : t \ge 0\},\\ U^+ &:= \inf\{t \ge 0 : Z_t^+ = \underline{Z}^+\}.\end{aligned}$$

From (3.6.1) we have that

$$\begin{aligned}
&\mathbb{P}\{T \in [\tau, \tau + \Delta\tau],\, L > \lambda,\, R > \rho\} \\
&\quad \le \mathbb{P}(\{U^- > \lambda - \Delta\tau,\, U^+ > \rho - \Delta\tau\} \\
&\qquad \cap \{\exists 0 \le s \le \Delta\tau : X_\tau + \underline{Z}^- + \alpha s = X_{\tau+\Delta\tau} + \underline{Z}^+ + \alpha(\Delta\tau - s)\}).
\end{aligned} \tag{3.6.2}$$

Similarly,

$$\begin{aligned}
&\mathbb{P}\{T \in [\tau, \tau + \Delta\tau],\, L > \lambda,\, R > \rho\} \\
&\quad \ge \mathbb{P}(\{U^- > \lambda,\, U^+ > \rho\} \\
&\qquad \cap \{\exists 0 \le s \le \Delta\tau : X_\tau + \underline{Z}^- + \alpha s = X_{\tau+\Delta\tau} + \underline{Z}^+ + \alpha(\Delta\tau - s)\} \\
&\qquad \cap \{\inf\{X_{\tau+s} - (X_\tau + \underline{Z}^- + \alpha s) : 0 \le s \le \Delta\tau\} > 0\} \\
&\qquad \cap \{\inf\{X_{\tau+\Delta\tau-s} - (X_{\tau+\Delta\tau} + \underline{Z}^+ + \alpha s) : 0 \le s \le \Delta\tau\} > 0\}) \\
&\quad \ge \mathbb{P}(\{U^- > \lambda,\, U^+ > \rho\} \\
&\qquad \cap \{\exists 0 \le s \le \Delta\tau : X_\tau + \underline{Z}^- + \alpha s = X_{\tau+\Delta\tau} + \underline{Z}^+ + \alpha(\Delta\tau - s)\}) \\
&\qquad - \mathbb{P}(\{\exists 0 \le s \le \Delta\tau : X_\tau + \underline{Z}^- + \alpha s = X_{\tau+\Delta\tau} + \underline{Z}^+ + \alpha(\Delta\tau - s)\} \\
&\qquad\quad \cap \{\inf\{X_{\tau+s} - (X_\tau + \underline{Z}^- + \alpha s) : 0 \le s \le \Delta\tau\} \le 0\}) \\
&\qquad - \mathbb{P}(\{\exists 0 \le s \le \Delta\tau : X_\tau + \underline{Z}^- + \alpha s = X_{\tau+\Delta\tau} + \underline{Z}^+ + \alpha(\Delta\tau - s)\} \\
&\qquad\quad \cap \{\inf\{X_{\tau+\Delta\tau-s} - (X_{\tau+\Delta\tau} + \underline{Z}^+ + \alpha s) : 0 \le s \le \Delta\tau\} \le 0\}).
\end{aligned} \tag{3.6.3}$$

Observe that

$$\begin{aligned}
&\mathbb{P}(\{\exists 0 \le s \le \Delta\tau : X_\tau + \underline{Z}^- + \alpha s = X_{\tau+\Delta\tau} + \underline{Z}^+ + \alpha(\Delta\tau - s)\} \\
&\qquad \cap \{\inf\{X_{\tau+s} - (X_\tau + \underline{Z}^- + \alpha s) : 0 \le s \le \Delta\tau\} \le 0\}) \\
&\quad = \mathbb{P}(\{\{\exists 0 \le s \le \Delta\tau : (\underline{Z}^+ - \underline{Z}^-) + (X_{\tau+\Delta\tau} - X_\tau) = 2\alpha s - \alpha\Delta\tau\} \\
&\qquad\quad \cap \{\underline{Z}^- \ge \inf_{0 \le s \le \Delta\tau} (X_{\tau+s} - X_\tau - \alpha s)\}) \\
&\quad = \mathbb{P}(\{\{(\underline{Z}^+ - \underline{Z}^-) + (X_{\tau+\Delta\tau} - X_\tau) \in [-\alpha\Delta\tau, \alpha\Delta\tau]\} \\
&\qquad\quad \cap \{\underline{Z}^- \ge \inf_{0 \le s \le \Delta\tau} (X_{\tau+s} - X_\tau - \alpha s)\}).
\end{aligned}$$

By Corollary 3.5.4, the independent random variables $\underline{Z}^-$ and $\underline{Z}^+$ have densities bounded by some constant c. Moreover, they are independent of $(X_{\tau+s})_{0\le s\le\Delta\tau}$. Conditioning on $\underline{Z}^-$ and $(X_{\tau+s})_{0\le s\le\Delta\tau}$, we see that the last probability is, using $|\cdot|$ to denote Lebesgue measure, at most

$$\begin{aligned}
&\mathbb{E}[c|[\underline{Z}^- - (X_{\tau+\Delta\tau} - X_\tau) - \alpha\Delta\tau, \underline{Z}^- - (X_{\tau+\Delta\tau} - X_\tau) + \alpha\Delta\tau]| \\
&\qquad \times \mathbf{1}\{\underline{Z}^- \ge \inf_{0 \le s \le \Delta\tau} (X_{\tau+s} - X_\tau - \alpha s)\}] \\
&\quad = 2c\alpha\Delta\tau\mathbb{P}\{\underline{Z}^- \ge \inf_{0 \le s \le \Delta\tau} (X_{\tau+s} - X_\tau - \alpha s)\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}\mathbb{P}(\{\exists 0 \le s \le \Delta\tau : X_\tau + \underline{Z}^- + \alpha s = X_{\tau+\Delta\tau} + \underline{Z}^+ + \alpha(\Delta\tau - s)\} \\ \cap \{\inf\{X_{\tau+s} - (X_\tau + \underline{Z}^- + \alpha s) : 0 \le s \le \Delta\tau\} \le 0\}) \\ = o(\Delta\tau)\end{aligned} \tag{3.6.4}$$

as $\Delta\tau \downarrow 0$.

The same argument shows that

$$\begin{aligned}\mathbb{P}(\{\exists 0 \le s \le \Delta\tau : X_\tau + \underline{Z}^- + \alpha s = X_{\tau+\Delta\tau} + \underline{Z}^+ + \alpha(\Delta\tau - s)\} \\ \cap \{\inf\{X_{\tau+\Delta\tau-s} - (X_{\tau+\Delta\tau} + \underline{Z}^+ + \alpha s) : 0 \le s \le \Delta\tau\} \le 0\}) \\ = o(\Delta\tau)\end{aligned} \tag{3.6.5}$$

as $\Delta\tau \downarrow 0$.

Now,

$$\begin{aligned}&\mathbb{P}(\{U^- > \lambda,\, U^+ > \rho\} \\ &\quad \cap \{\exists 0 \le s \le \Delta\tau : X_\tau + \underline{Z}^- + \alpha s = X_{\tau+\Delta\tau} + \underline{Z}^+ + \alpha(\Delta\tau - s)\}) \\ &= \mathbb{P}(\{U^- > \lambda,\, U^+ > \rho\} \\ &\quad \cap \{\exists 0 \le s \le \Delta\tau : (\underline{Z}^+ - \underline{Z}^-) + (X_{\tau+\Delta\tau} - X_\tau) = 2\alpha s - \alpha\Delta\tau\}) \\ &= \mathbb{P}(\{U^- > \lambda,\, U^+ > \rho\} \\ &\quad \cap \{(\underline{Z}^+ - \underline{Z}^-) + (X_{\tau+\Delta\tau} - X_\tau) \in [-\alpha\Delta\tau, +\alpha\Delta\tau]\}).\end{aligned}$$

The random vectors $(U^-, \underline{Z}^-)$ and $(U^+, \underline{Z}^+)$ are independent with respective densities f^- and f^+, and so the joint density of $(U^-, U^+, \underline{Z}^+ - \underline{Z}^-)$ is

$$(u, v, w) \mapsto \int_{-\infty}^{\infty} f^-(u, h - w) f^+(v, h)\, dh.$$

Thus, using the facts that the random variable $\underline{Z}^+ - \underline{Z}^-$ is independent of $X_{\tau+\Delta\tau} - X_\tau$ and the latter random variable has the same distribution as $X_{\Delta\tau}$,

$$\begin{aligned}&\mathbb{P}(\{U^- > \lambda,\, U^+ > \rho\} \\ &\quad \cap \{(\underline{Z}^+ - \underline{Z}^-) + (X_{\tau+\Delta\tau} - X_\tau) \in [-\alpha\Delta\tau, +\alpha\Delta\tau]\}) \\ &= \int_\lambda^\infty du \int_\rho^\infty dv \int_{-\infty}^\infty dw \int_{-\infty}^\infty dh \\ &\quad \times \mathbb{P}\{-w - \alpha\Delta\tau < X_{\Delta\tau} < -w + \alpha\Delta\tau\}\, f^-(u, h - w) f^+(v, h).\end{aligned}$$

By Fubini's theorem,

$$\begin{aligned}\int_{-\infty}^{\infty} dw\, \mathbb{P}\{-w - \alpha\Delta\tau < X_{\Delta\tau} < -w + \alpha\Delta\tau\} \\ = \mathbb{E}\left[\int_{-\infty}^{\infty} dw\, \mathbf{1}\{-X_{\Delta\tau} - \alpha\Delta\tau < w < -X_{\Delta\tau} + \alpha\Delta\tau\}\right] \\ = \mathbb{E}\left[2\alpha\Delta\tau\right] = 2\alpha\Delta\tau.\end{aligned}$$

Moreover, for any $\epsilon > \Delta\tau$,

$$\begin{aligned}\int_{-\infty}^{\infty} dw\, \mathbb{P}\{-w - \alpha\Delta\tau < X_{\Delta\tau} < -w + \alpha\Delta\tau\}\mathbf{1}\{|w| > \epsilon\} \\ = \mathbb{E}\left[\int_{-\infty}^{\infty} dw\, \mathbf{1}\{-X_{\Delta\tau} - \alpha\Delta\tau < w < -X_{\Delta\tau} + \alpha\Delta\tau,\, |w| > \epsilon\}\right] \\ = \mathbb{E}\left[(|X_{\Delta\tau}| - (\epsilon - \Delta\tau))_+ \wedge (2\Delta\tau)\right].\end{aligned}$$

Note that $(\Delta\tau)^{-1}[(|X_{\Delta\tau}| - (\epsilon - \Delta\tau))_+ \wedge (2\Delta\tau)] \le 2$ and that the random variable on the left of this inequality converges to 0 almost surely as $\Delta\tau \downarrow 0$. Hence, by bounded convergence,

$$\lim_{\Delta\tau\downarrow 0}\int_{-\infty}^{\infty} dw\, (\Delta\tau)^{-1}\mathbb{P}\{-w - \alpha\Delta\tau < X_{\Delta\tau} < -w + \alpha\Delta\tau\}\mathbf{1}\{|w| > \epsilon\} = 0.$$

Furthermore, the independent random variables $\underline{Z}^-$ and $\underline{Z}^+$ both have bounded densities by Corollary 3.5.4; that is, the functions $h \mapsto \int_0^\infty du\, f^-(u,h)$ and $h \mapsto \int_0^\infty dv\, f^+(v,h)$ both belong to $L^1 \cap L^\infty$. Therefore, the functions $h \mapsto \int_\lambda^\infty du\, f^-(u,h)$ and $h \mapsto \int_\rho^\infty dv\, f^+(v,h)$ both certainly belong to $L^1 \cap L^\infty$.

It now follows from the Lebesgue differentiation theorem that

$$\begin{aligned}\lim_{\Delta\tau\downarrow 0}(\Delta\tau)^{-1}\int_{-\infty}^{\infty} dw\, \mathbb{P}\{-w - \alpha\Delta\tau < X_{\Delta\tau} < -w + \alpha\Delta\tau\}\int_\lambda^\infty du\, f^-(u, h-w) \\ = 2\alpha\int_\lambda^\infty du\, f^-(u,h)\end{aligned}$$

for Lebesgue almost every $h \in \mathbb{R}$. Moreover, the quantity on the left is bounded by $\sup_{h\in\mathbb{R}} 2\alpha\int_\lambda^\infty du\, f^-(u,h) < \infty$. Therefore, by (3.6.2), (3.6.3), (3.6.4), (3.6.5), and bounded convergence,

$$\begin{aligned}\lim_{\Delta\tau\downarrow 0}(\Delta\tau)^{-1}\mathbb{P}\{T \in [\tau, \tau + \Delta\tau],\, L > \lambda,\, R > \rho\} \\ = 2\alpha\int_\lambda^\infty du\int_\rho^\infty dv\int_{-\infty}^{\infty} dh\, f^-(u,h)f^+(v,h).\end{aligned} \tag{3.6.6}$$

As we observed above, the measure $\mathbb{P}\{T \in d\tau\}$ is absolutely continuous with density bounded above by $\Lambda(\mathbb{R}_+) < \infty$, and so the same is certainly true of the measure $\mathbb{P}\{T \in d\tau,\, L > \lambda,\, R > \rho\}$ for fixed λ and ρ. Therefore, by (3.6.6) and the Lebesgue differentiation theorem,

$$\begin{aligned}&\mathbb{P}\{T \in A,\, L > \lambda,\, R > \rho\}\\&\quad = 2\alpha \int_{-\infty}^{\infty} d\tau \int_{\lambda}^{\infty} du \int_{\rho}^{\infty} dv \int_{-\infty}^{\infty} dh\, f^-(u,h) f^+(v,h) \mathbf{1}\{\tau \in A\}\end{aligned}$$

for any Borel set $A \subseteq (-\rho, \lambda)$, and this establishes that (T, L, R) has the claimed density.

The remaining two claims follow immediately. □

Corollary 3.6.2. *Under the assumptions of Proposition 3.6.1,*

$$\begin{aligned}\mathbb{E}[e^{-\theta K}] = -4\pi\alpha \frac{d}{d\theta} \int_{-\infty}^{\infty} \Bigg(&\exp\left\{ \int_0^\infty dt \int_0^\infty [e^{-\theta t + izx} - 1] t^{-1} \mathbb{P}\{X_t - \alpha t \in dx\} \right\} \\ &\times \exp\left\{ \int_0^\infty dt \int_0^\infty [e^{-\theta t - izx} - 1] t^{-1} \mathbb{P}\{-X_t - \alpha t \in dx\} \right\} \Bigg) dz.\end{aligned}$$

Proof. From Proposition 3.6.1,

$$\begin{aligned}&\mathbb{E}[e^{-\theta K}]\\
&\quad = 2\alpha \int_{-\infty}^{0} \int_0^\infty \kappa \int_0^\kappa f^-(\kappa - \xi, h) f^+(\xi, h) e^{-\theta\kappa}\, d\xi\, d\kappa\, dh\\
&\quad = 2\alpha \int_{-\infty}^{0} \int_0^\infty \int_\xi^\infty \kappa f^-(\kappa - \xi, h) f^+(\xi, h) e^{-\theta\kappa}\, d\kappa\, d\xi\, dh\\
&\quad = 2\alpha \int_{-\infty}^{0} \Bigg(\int_0^\infty f^+(\xi, h) e^{-\theta\xi} \int_\xi^\infty (\kappa - \xi) f^-(\kappa - \xi, h) e^{-\theta(\kappa - \xi)}\, d\kappa\, d\xi\\
&\qquad + \int_0^\infty \xi f^+(h, \xi) e^{-\theta\xi} \int_\xi^\infty f^-(h, \kappa - \xi) e^{-\theta(\kappa-\xi)}\, d\kappa\, d\xi \Bigg) dh \qquad (3.6.7)\\
&= 2\alpha \int_{-\infty}^{0} \Bigg(\int_0^\infty f^+(\xi, h) e^{-\theta\xi} \int_0^\infty \kappa f^-(\kappa, h) e^{-\theta\kappa}\, d\kappa\, d\xi\\
&\qquad + \int_0^\infty \xi f^+(\xi, h) e^{-\theta\xi} \int_0^\infty f^-(\kappa, h) e^{-\theta\kappa}\, d\kappa\, d\xi \Bigg) dh\\
&\quad = -2\alpha \frac{d}{d\theta} \Bigg(\int_{-\infty}^{0} \Big(\int_0^\infty f^+(\xi, h) e^{-\theta\xi}\, d\xi \Big) \Big(\int_0^\infty f^-(\kappa, h) e^{-\theta\kappa}\, d\kappa \Big)\, dh \Bigg).\end{aligned}$$

Viewing $\int_0^\infty f^+(\xi,h)e^{-\theta\xi}\,d\xi$ and $\int_0^\infty f^-(\kappa,h)e^{-\theta\kappa}\,d\kappa$ as functions of h that belong to $L^1 \cap L^\infty \subset L^2$, we can use Plancherel's Theorem and then the Pecherskii-Rogozin formulas [32, p. 28] again to get that $\mathbb{E}[e^{-\theta K}]$ is

$$\begin{aligned}
&-2\alpha\frac{d}{d\theta}2\pi\int_{-\infty}^{\infty}\Bigg(\int_0^\infty\int_0^\infty f^+(\xi,-h)e^{izh-\theta\xi}\,d\xi\,dh \\
&\qquad\times\overline{\int_0^\infty\int_0^\infty f^-(\kappa,-h)e^{izh-\theta\kappa}\,d\kappa\,dh}\,dz\Bigg) \\
&=-4\pi\alpha\frac{d}{d\theta}\int_{-\infty}^{\infty}\Bigg(\exp\left\{\int_0^\infty dt\int_0^\infty[e^{-\theta t+izx}-1]t^{-1}\mathbb{P}\{X_t-\alpha t\in dx\}\right\} \\
&\qquad\times\overline{\exp\left\{\int_0^\infty dt\int_0^\infty[e^{-\theta t+izx}-1]t^{-1}\mathbb{P}\{-X_t-\alpha t\in dx\}\right\}}\Bigg)\,dz \\
&=-4\pi\alpha\frac{d}{d\theta}\int_{-\infty}^{\infty}\Bigg(\exp\left\{\int_0^\infty dt\int_0^\infty[e^{-\theta t+izx}-1]t^{-1}\mathbb{P}\{X_t-\alpha t\in dx\}\right\} \\
&\qquad\times\exp\left\{\int_0^\infty dt\int_0^\infty[e^{-\theta t-izx}-1]t^{-1}\mathbb{P}\{-X_t-\alpha t\in dx\}\right\}\Bigg)\,dz.
\end{aligned}$$

□

3.6.2 Extension to more general Lévy processes

Corollary 3.6.2 establishes Theorem 3.3.13 when X has a non-zero Brownian component. The next few results allow us establish the latter result for the class of Lévy processes described in its statement.

Recall the definitions

$$\begin{aligned}
G &:= \sup\{t<0 : X_t \wedge X_{t-} = M_t\} = \sup\{t<0 : t\in\mathcal{Z}\}, \\
D &:= \inf\{t>0 : X_t \wedge X_{t-} = M_t\} = \inf\{t>0 : t\in\mathcal{Z}\}, \\
T &:= \arg\max\{M_t : G\le t\le D\}, \\
S &:= \inf\{t>0 : X_t\wedge X_{t-}-\alpha t\le \inf\{X_s-\alpha s : s\le 0\}\}.
\end{aligned}$$

As in the proof of Theorem 3.2.6, it follows from Lemma 3.8.4 that almost surely

$$D = \inf\{t\ge S : X_t\wedge X_{t-}+\alpha(t-S) = \inf\{X_u+\alpha(u-S) : u\ge S\}\}.$$

Proposition 3.6.3. *Suppose that X is Lévy process satisfying our standing assumptions Hypothesis 3.2.2. Then, $\mathbb{P}\{0\notin\mathcal{Z},\, S=0\}=0$. In addition,* *(a) If X has paths of unbounded variation, then $G < T < S < D$ a.s.*

(b) If X has paths of bounded variation and drift coefficient d satisfying $d < -\alpha$, then $G < T < S < D$ a.s., and if X has paths of bounded variation and drift coefficient d satisfying $d > \alpha$, then $G < T < S \leq D$ a.s.

(c) If X has paths of bounded variation and drift coefficient d satisfying $|d| < \alpha$, then almost surely either $0 \in \mathcal{Z}$ and $G = T = S = D = 0$, or $0 \notin \mathcal{Z}$ and $G \leq T \leq S \leq D$. Furthermore, $T = S = D$ almost surely on the event $\{T = S\}$.

Proof. Firstly, if $0 \notin \mathcal{Z}$, then $\inf\{X_u - \alpha u : u \leq 0\} < 0$, and thus $S > 0$ a.s. on the event $\{0 \notin \mathcal{Z}\}$.

(a) Suppose that X has paths of unbounded variation. We have from Theorem 3.3.1 (see Remark 3.3.2 (i)) that $0 \notin \mathcal{Z}$ almost surely. It follows from (3.2.2) that at the stopping time S

$$-\liminf_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{S+\varepsilon} - X_S) = \limsup_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{S+\varepsilon} - X_S) = \infty,$$

and hence it is not possible for the α-Lipschitz minorant to meet the path of X at time S. Thus, $T < S < D$ almost surely by Corollary 3.8.5. By time reversal, $G < T$ almost surely.

(b) Suppose X has paths of bounded variation and drift coefficient d satisfying $|d| > \alpha$, then we have from Theorem 3.3.1 (see Remark 3.3.2 (ii)) that $0 \notin \mathcal{Z}$ almost surely. Therefore, by Corollary 3.8.5, if $T = S$ then $T = S = D$.

Suppose that $d < -\alpha$. It follows from (3.2.1) that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{S+\varepsilon} - X_S) = d, \quad \text{a.s.}$$

Thus, $S \notin \mathcal{Z}$ and, in particular, $S < D$, so that $T < S < D$ a.s.

On the other hand, if $d > \alpha$, then the Lévy process $(X_t - \alpha t)_{t \geq 0}$ has positive drift and so the associated descending ladder process has zero drift coefficient [32, p. 56]. In that case, for any $x < 0$ we have $X_V < x$ almost surely, where $V := \inf\{t \geq 0 : X_t - \alpha t \leq x\}$ [15, Theorem III.4]. Therefore,

$$X_S - \alpha S < \inf\{X_u - \alpha u : u \leq 0\} \quad \text{a.s.}$$

If $T = S$, then $T = S = D$ by Corollary 3.8.5, and then

$$\begin{aligned} X_S &= X_D \wedge X_{D-} \\ &= X_G \wedge X_{G-} + \alpha(D - G) \\ &= X_G \wedge X_{G-} + \alpha(S - G), \end{aligned}$$

which results in the contradiction

$$X_G \wedge X_{G-} - \alpha G = X_S - \alpha S < \inf\{X_u - \alpha u : u \le 0\}.$$

Thus, $T < S \le D$ a.s.

The results for G now follow by a time reversal argument.

(c) Suppose X has paths of bounded variation and drift coefficient d satisfying $|d| > \alpha$. We know from Theorem 3.3.1 and Remark 3.3.2 that the subordinator associated with $\mathcal{Z}$ has non-zero drift and so $\mathcal{Z}$ has positive Lebesgue measure almost surely. The subset of points of $\mathcal{Z}$ that are isolated on either the left or the right is countable and hence has zero Lebesgue measure. It follows from the stationarity of $\mathcal{Z}$ that $G = T = S = D = 0$ almost surely on the event $\{0 \in \mathcal{Z}\}$. The remaining statements can be read from Corollary 3.8.5. □

Lemma 3.6.4. *Let X be a Lévy process that satisfies our standing assumptions Hypothesis 3.2.2. Suppose, moreover, that if X has paths of bounded variation, then $|d| \neq \alpha$. For $\varepsilon > 0$ set $X^\varepsilon = X + \varepsilon B$, where B is a standard Brownian motion on $\mathbb{R}$, independent of X. Define G^ε, D^ε and $K^\varepsilon = D^\varepsilon - G^\varepsilon$ to be the analogues of G, D and K with X replaced by X^ε. Then, $(G^\varepsilon, D^\varepsilon)$ converges almost surely to (G, D) as $\varepsilon \downarrow 0$, and so K^ε converges almost surely to K as $\varepsilon \downarrow 0$.*

Proof. By symmetry, it suffices to show that D^ε converges almost surely to D as $\varepsilon \downarrow 0$. We first show the convergence on the event $\{S > 0\}$.

Let S^ε be the analogue of the stopping time S with X replaced by X^ε. As we observed in the proof of Theorem 3.2.6, $X_S - \alpha S = X_S \wedge X_{S-} - \alpha S \le \inf\{X_u - \alpha u : u \le 0\}$. If X has paths of unbounded variation or bounded variation with drift satisfying $d < \alpha$, then, since S is a stopping time, $\liminf_{u \downarrow S}(X_u - X_S - \alpha(u - S))/(u - S) < 0$. If X has paths of bounded variation with drift satisfying $d > \alpha$, then by the remarks at the top of page 56 of [32], the downwards ladder height process of the process $(X_u - \alpha u)_{u \ge 0}$ (resp. $(-X_u + \alpha u)_{u \ge 0}$) has zero drift (resp. non-zero drift). By Lemma 3.5.3, the distribution of $\inf\{X_u - \alpha u : u \le 0\}$ is absolutely continuous with a bounded density, and hence

$$\mathbb{P}\{X_S - \alpha S = \inf\{X_u - \alpha u : u \le 0\}\} = 0$$

by Fubini's theorem and the fact that the range of a subordinator with zero drift has zero Lebesgue measure almost surely.

For all three of these cases, given any $\delta > 0$ we can, with probability one, thus find a time $t \in (S, S + \delta)$ such that

$$X_t \wedge X_{t-} - \alpha t < \inf\{X_u - \alpha u : u \le 0\}.$$

By the strong law of large numbers for the Brownian motion B,

$$\lim_{\varepsilon \downarrow 0} \inf\{X_u^\varepsilon - \alpha u : u \leq 0\} = \inf\{X_u - \alpha u : u \leq 0\}.$$

Hence, $X_t^\varepsilon \wedge X_{t-}^\varepsilon - \alpha t \leq \inf\{X_u^\varepsilon - \alpha u : u \leq 0\}$ for ε sufficiently small, and so $S^\varepsilon \leq S + \delta$ for such an ε. Therefore, $\limsup_{\varepsilon \downarrow 0} S^\varepsilon \leq S$.

On the other hand, for any $\delta > 0$ we have

$$\inf\{X_t \wedge X_{t-} - \alpha t - \inf\{X_u - \alpha u : u \leq 0\} : t \in [0, (S-\delta)_+]\} > 0.$$

Thus, $X_t^\varepsilon \wedge X_{t-}^\varepsilon - \alpha t > \inf\{X_u^\varepsilon - \alpha u : u \leq 0\}$ for all $t \in [0, (S-\delta)_+]$ for ε sufficiently small, so that $S^\varepsilon \geq (S-\delta)_+$. Therefore, $\liminf_{\varepsilon \downarrow 0} S^\varepsilon \geq S$. Consequently, $\lim_{\varepsilon \downarrow 0} S^\varepsilon = S$.

Now, as a result of the uniqueness of the global infima of Lévy processes that are not compound Poisson processes with zero drift [15, Proposition VI.4], and the law of large numbers applied to B, we have

$$\lim_{\varepsilon \downarrow 0} \arg \inf_{u \geq S^\varepsilon} \{X_u^\varepsilon + \alpha(u - S^\varepsilon)\} = \arg \inf_{u \geq S} \{X_u + \alpha(u - S)\}.$$

It follows readily that D^ε converges to D almost surely as $\epsilon \downarrow 0$ on the event $\{S > 0\}$.

Suppose now that we are on the event $\{S = 0\}$. Then, by Proposition 3.6.3, $0 \in \mathcal{Z}$ almost surely, and we may suppose that X satisfies the conditions of part (c) of that result, so that $G = T = S = D = 0$ almost surely. Then, by the strong law of large numbers for the Brownian motion B, almost surely

$$\lim_{\varepsilon \downarrow 0} \inf_{u \leq 0} \{X_u^\varepsilon - \alpha u\} = \lim_{\varepsilon \downarrow 0} \inf_{u \geq 0} \{X_u^\varepsilon + \alpha u\} = 0.$$

Therefore, D^ε also converges to D almost surely as $\epsilon \downarrow 0$ on the event $\{S = 0\}$. □

We are finally in a position to give the proof of Theorem 3.3.13. Suppose for the moment that X has a non-zero Brownian component. It follows from Theorem 3.3.1 that $\delta = 0$, and hence from (3.3.6) we have that

$$\begin{aligned}\frac{\int_{\mathbb{R}_+} (1 - e^{-\theta x})\, \Lambda(dx)}{\int_{\mathbb{R}_+} x\, \Lambda(dx)} &= \frac{\int_{\mathbb{R}_+} \left(\int_0^\theta x e^{-\varphi x}\, d\varphi\right) \Lambda(dx)}{\int_{\mathbb{R}_+} x\, \Lambda(dx)} \\ &= \int_0^\theta \left(\frac{\int_{\mathbb{R}_+} x e^{-\varphi x}\, \Lambda(dx)}{\int_{\mathbb{R}_+} x\, \Lambda(dx)}\right) d\varphi = \int_0^\theta \mathbb{E}[e^{-\varphi K}]\, d\varphi.\end{aligned} \tag{3.6.8}$$

By Corollary 3.6.2, this last integral is

$$\begin{aligned}4\pi\alpha \int_{-\infty}^{\infty} \Bigg\{ &\exp\Bigg(\int_0^\infty t^{-1}\mathbb{E}\Big[\big(e^{izX_t - iz\alpha t} - 1\big)\,\mathbf{1}\{X_t \ge +\alpha t\} \\ &\quad + \big(e^{izX_t + iz\alpha t} - 1\big)\,\mathbf{1}\{X_t \le -\alpha t\}\Big]\,dt\Bigg) \\ &- \exp\Bigg(\int_0^\infty t^{-1}\mathbb{E}\Big[\big(e^{-\theta t + izX_t - iz\alpha t} - 1\big)\,\mathbf{1}\{X_t \ge +\alpha t\} \\ &\quad + \big(e^{-\theta t + izX_t + iz\alpha t} - 1\big)\,\mathbf{1}\{X_t \le -\alpha t\}\Big]\,dt\Bigg)\Bigg\}\,dz,\end{aligned} \tag{3.6.9}$$

as claimed in the theorem.

Now suppose X has zero Brownian component, but satisfies the conditions of Theorem 3.3.13. Since X has paths of unbounded variation almost surely, it follows from Remark 3.3.2 that $\delta = 0$. Let $X^\varepsilon = X + \varepsilon B$ and K^ε be as in Lemma 3.6.4, and let Λ^ε be the Lévy measure of the subordinator associated with the set of points where X^ε meets its α-Lipschitz minorant. By Lemma 3.6.4 we know that $K^\varepsilon \to K$ almost surely, and thus since $\delta = 0$, arguing as in (3.6.8) we have

$$\begin{aligned}\frac{\int_{\mathbb{R}_+}(1-e^{-\theta x})\,\Lambda(dx)}{\int_{\mathbb{R}_+} x\,\Lambda(dx)} &= \int_0^\theta \mathbb{E}[e^{-\varphi K}]\,d\varphi \\ &= \lim_{\varepsilon\downarrow 0}\int_0^\theta \mathbb{E}[e^{-\varphi K^\varepsilon}]\,d\varphi = \lim_{\varepsilon\downarrow 0}\frac{\int_{\mathbb{R}_+}(1-e^{-\theta x})\,\Lambda^\varepsilon(dx)}{\int_{\mathbb{R}_+} x\,\Lambda^\varepsilon(dx)}.\end{aligned} \tag{3.6.10}$$

Suppose hypothesis (i) of Theorem 3.3.13 holds, i.e. X creeps upward or downward. We will show that this hypothesis implies that the density of at least one of the random variables $\inf_{t\ge 0}\{X_t + \alpha t\}$ or $\inf_{t\ge 0}\{-X_t + \alpha t\}$ is bounded. Hence, in the notation of of the proof of Corollary 3.6.2, it can then be seen that

$$\int_{-\infty}^0 \Big(\int_0^\infty f^+(\xi,h)e^{-\theta\xi}\,d\xi\Big)\Big(\int_0^\infty f^-(\kappa,h)e^{-\theta\kappa}\,d\kappa\Big)\,dh < \infty \tag{3.6.11}$$

for all $\theta \ge 0$.

Suppose first that X creeps downward. A result of Vigon [94] (see also [32, Chapter 6, Corollary 9]) states that when the paths of X have unbounded variation, X creeps downward if and only if $(X_t + \alpha t)_{t\ge 0}$ creeps downward. By the results of [61] (see also [15, Theorem VI.19]), this occurs if and only if the drift coefficient of the descending ladder processes for the process $(X_t + \alpha t)_{t\ge 0}$ is non-zero. By Lemma

3.5.3, this occurs if and only if the density of the random variable $\inf_{t\geq 0}\{X_t + \alpha t\}$ exists and is bounded. A similar argument shows that X creeps upward if and only if the density of $\inf_{t\geq 0}\{-X_t + \alpha t\}$ exists and is bounded.

It can also be seen that (3.6.11) holds for all $\theta \geq 0$ in the alternative case when hypothesis (ii) of Theorem 3.3.13 holds since under that hypothesis the densities of $\inf_{t\geq 0}\{X_t + \alpha t\}$ and $\inf_{t\geq 0}\{-X_t + \alpha t\}$ are square integrable.

By the same methods used in the proof of Corollary 3.6.2 from the last line of (3.6.7) onwards, it follows from the finiteness of (3.6.11) that (3.6.9) is finite. Then, since for each fixed value of z the integrand in (3.6.9) is a product of characteristic functions of certain infima, and hence not equal to zero, we can apply Fubini's theorem to swap the order of the integrals within the exponentials (here we are using the absolute continuity of the distribution of X_t for all $t > 0$). We now have that the integrand for each fixed value of z with X_t replaced by X_t^ε converges to the integrand with just X_t as $\varepsilon \to 0$. Then, by finiteness of (3.6.9), we have that (3.6.9) with X_t replaced by X_t^ε converges to (3.6.9).

It remains to show that $\Lambda(\mathbb{R}_+) < \infty$. For all $\theta \geq 0$ we have from (3.6.9) that

$$\frac{\int_{\mathbb{R}_+}(1-e^{-\theta x})\,\Lambda(dx)}{\int_{\mathbb{R}_+} x\,\Lambda(dx)} \leq 4\pi\alpha \int_{-\infty}^{\infty} \exp\biggl(\int_0^\infty t^{-1}\mathbb{E}\Bigl[\bigl(e^{izX_t - iz\alpha t} - 1\bigr)\,\mathbf{1}\{X_t \geq +\alpha t\} + \bigl(e^{izX_t + iz\alpha t} - 1\bigr)\,\mathbf{1}\{X_t \leq -\alpha t\}\Bigr]\,dt\biggr)\,dz\,.$$

By the same methods used in the proof of Corollary 3.6.2 from the last line of (3.6.7) onwards the right hand side of this inequality is equal to the integral in (3.6.11) evaluated at $\theta = 0$, and since this is finite we can conclude that $\Lambda(\mathbb{R}_+) < \infty$. □

3.7 Lipschitz Minorants of Brownian Motion

3.7.1 Williams' path decomposition for Brownian motion with drift

We recall for later use a path composition due to David Williams that describes the distribution of a Brownian motion with positive drift in terms of the segment of the path up to the time the process achieves its global minimum and the segment of the path after that time – see [78, p. 436] or, for a concise description, [20, Section IV.5].

For $\mu \in \mathbb{R}$, let $Z^{(\mu)} = (Z_t^{(\mu)})_{t\geq 0}$ be a Brownian motion with drift μ started at zero. Take $\beta > 0$ and let E be a random variable that is independent of $Z^{(-\beta)}$ and has an exponential distribution with mean $(2\beta)^{-1}$. Set

$$T_E := \inf\{t \geq 0 : Z^{(-\beta)} = -E\}.$$

Then, there is a diffusion $W = (W_t)_{t\geq 0}$ with the properties

(i) W is independent of $Z^{(-\beta)}$ and E;

(ii) $W_0 = 0$;

(iii) $W_t > 0$ for all $t > 0$ a.s.;

such that if we define a process $(\tilde{Z}_t)_{t\geq 0}$ by

$$\tilde{Z}_t := \begin{cases} Z_t^{(-\beta)}, & 0 \leq t < T_E, \\ Z_{T_E}^{(-\beta)} + W_{t-T_E}, & t \geq T_E, \end{cases} \tag{3.7.1}$$

then $\tilde{Z}$ has the same distribution as $Z^{(\beta)}$. Thus, in particular,

$$-\inf\{Z_t^{(\beta)} : t \geq 0\} \sim \mathrm{Exp}(2\beta) \tag{3.7.2}$$

and the unique time that $Z^{(\beta)}$ achieves its global minimum is distributed as T_E. Recall also that

$$\mathbb{E}[\inf\{t \geq 0 : Z_t^{(-\beta)} = h\}] = \frac{h}{\beta} \tag{3.7.3}$$

for $h \leq 0$ (see, for example, [20, page 295, equation 2.2.0.1]).

3.7.2 Random variables related to the Brownian Lipschitz minorant

Proposition 3.7.1. *Let X be a Brownian motion with drift β, where $|\beta| < \alpha$. Then, the distribution of K is characterized by*

$$\mathbb{E}[e^{-\theta K}] = \frac{8\alpha(\alpha^2 - \beta^2)\left(\frac{1}{\sqrt{2\theta+(\alpha+\beta)^2}} + \frac{1}{\sqrt{2\theta+(\alpha-\beta)^2}}\right)}{\left(\sqrt{2\theta + (\alpha+\beta)^2} + \sqrt{2\theta + (\alpha-\beta)^2} + 2\alpha\right)^2}$$

for $\theta \geq 0$, and hence Λ is characterized by

$$\frac{\int_{\mathbb{R}_+} (1 - e^{-\theta x})\, \Lambda(dx)}{\int_{\mathbb{R}_+} x\, \Lambda(dx)} = \frac{4(\alpha^2 - \beta^2)\theta}{\left(\sqrt{2\theta + (\alpha-\beta)^2} + \alpha - \beta\right)\left(\sqrt{2\theta + (\alpha+\beta)^2} + \alpha + \beta\right)}$$

for $\theta \geq 0$.

Proof. We have from [20, page 269, equation 1.14.3(1)] that

$$\int_{-\infty}^{0} f^-(\xi, h)e^{-\theta\xi}\, d\xi = 2(\alpha - \beta)e^{h(\sqrt{2\theta+(\alpha-\beta)^2}+(\alpha-\beta))}$$

and

$$\int_{0}^{\infty} f^+(\xi, h)e^{-\theta\xi}\, d\xi = 2(\alpha + \beta)e^{h(\sqrt{2\theta+(\alpha+\beta)^2}+(\alpha+\beta))}.$$

Thus, from (3.6.7),

$$\begin{aligned}
\mathbb{E}[e^{-\theta K}] &= -2\alpha\frac{d}{d\theta}\left(\int_{-\infty}^{0} 4(\alpha^2-\beta^2)e^{h(\sqrt{2\theta+(\alpha+\beta)^2}+\sqrt{2\theta+(\alpha-\beta)^2}+2\alpha)}\, dh\right)\\
&= 8\alpha(\alpha^2-\beta^2)\frac{d}{d\theta}\left(\frac{1}{\sqrt{2\theta+(\alpha+\beta)^2}+\sqrt{2\theta+(\alpha-\beta)^2}+2\alpha}\right)\\
&= \frac{8\alpha(\alpha^2-\beta^2)\left(\frac{1}{\sqrt{2\theta+(\alpha+\beta)^2}}+\frac{1}{\sqrt{2\theta+(\alpha-\beta)^2}}\right)}{\left(\sqrt{2\theta+(\alpha+\beta)^2}+\sqrt{2\theta+(\alpha-\beta)^2}+2\alpha\right)^2},
\end{aligned}$$

as required.

Now, by (3.6.8),

$$\begin{aligned}
\frac{\int_{\mathbb{R}_+}(1-e^{-\theta x})\,\Lambda(dx)}{\int_{\mathbb{R}_+} x\,\Lambda(dx)} &= \int_0^\theta \mathbb{E}[e^{-\varphi K}]\, d\varphi\\
&= 8\alpha(\alpha^2-\beta^2)\Bigg[\frac{1}{\sqrt{(\alpha+\beta)^2}+\sqrt{(\alpha-\beta)^2}+2\alpha}\\
&\qquad - \frac{1}{\sqrt{2\theta+(\alpha+\beta)^2}+\sqrt{2\theta+(\alpha-\beta)^2}+2\alpha}\Bigg]\\
&= 8\alpha(\alpha^2-\beta^2)\Bigg[\frac{1}{4\alpha}\\
&\qquad - \frac{1}{\sqrt{2\theta+(\alpha+\beta)^2}+\sqrt{2\theta+(\alpha-\beta)^2}+2\alpha}\Bigg]\\
&= \frac{4(\alpha^2-\beta^2)\theta}{\left(\sqrt{2\theta+(\alpha-\beta)^2}+\alpha-\beta\right)\left(\sqrt{2\theta+(\alpha+\beta)^2}+\alpha+\beta\right)}
\end{aligned}$$

after a little algebra. □

Remark 3.7.2. There is an alternative way to verify that the Laplace transform for K presented in Proposition 3.7.1 is correct. Recall from the proof of Theorem 3.2.6 that $D = S + \tilde{T}$, where the independent random variables S and $\tilde{T}$ are defined by

$$S = \inf\{s > 0 : X_s - \alpha s = \inf\{X_u - \alpha u : u \leq 0\}\}$$

and

$$\tilde{T} = \sup\{t \geq 0 : \tilde{X}_t = \inf\{\tilde{X}_s : s \geq 0\}\}$$

with

$$(\tilde{X}_s)_{s\geq 0} := ((X_{S+s} - X_S) + \alpha s)_{s\geq 0}\,.$$

Set $I^- := \inf\{X_u - \alpha u : u \leq 0\}$. Because $(X_{-t} + \alpha t)_{t\geq 0}$ is a Brownian motion with drift $\alpha - \beta$, we know from Subsection 3.7.1 that $-I^-$ has an exponential distribution with mean $(2(\alpha-\beta))^{-1}$. Now $(X_t - \alpha t)_{t\geq 0}$ is a Brownian motion with drift $\beta - \alpha$, and so, again from Subsection 3.7.1, S is distributed as the time until this process achieves its global minimum. It follows that

$$\mathbb{E}[e^{-\theta S}] = \frac{2(\alpha-\beta)}{\sqrt{2\theta + (\alpha-\beta)^2} + \alpha - \beta}$$

and

$$\mathbb{E}[e^{-\theta \tilde{T}}] = \frac{2(\alpha+\beta)}{\sqrt{2\theta + (\alpha+\beta)^2} + \alpha + \beta}$$

– see, for example, [20, page 266, equation 1.12.3(2)].

By stationarity, D has the same distribution as $U(D-G) = UK$, where U is an independent random variable that is uniformly distributed on $[0,1]$. Thus,

$$\mathbb{E}[e^{-\theta D}] = \int_0^1 \mathbb{E}[e^{-u\theta K}]\,du = \mathbb{E}\left[\frac{1}{\theta K}\left(1 - e^{-\theta K}\right)\right] = \frac{1}{\theta}\frac{\int_{\mathbb{R}_+}(1-e^{-\theta x})\,\Lambda(dx)}{\int_{\mathbb{R}_+} x\,\Lambda(dx)},$$

and

$$\begin{aligned}\frac{\int_{\mathbb{R}_+}(1-e^{-\theta x})\,\Lambda(dx)}{\int_{\mathbb{R}_+} x\,\Lambda(dx)} &= \theta\mathbb{E}[e^{-\theta D}] \\ &= \theta\mathbb{E}[e^{-\theta S}]\mathbb{E}[e^{-\theta\tilde{T}}] \\ &= \frac{4(\alpha^2-\beta^2)\theta}{\left(\sqrt{2\theta+(\alpha-\beta)^2}+\alpha-\beta\right)\left(\sqrt{2\theta+(\alpha+\beta)^2}+\alpha+\beta\right)}.\end{aligned}$$

This equality agrees with the one found in Proposition 3.7.1. Differentiating the expression on the right with respect to θ and recalling the observation (3.6.8), we arrive at the the expression for the Laplace transform of K in Proposition 3.7.1.

Proposition 3.7.3. *Let X be a Brownian motion with zero drift. Then,*

$$\mathbb{P}\{K \in d\kappa\} = \left(\frac{4\alpha^3}{\sqrt{2\pi}}\kappa^{1/2}e^{-\alpha^2\kappa/2} - 4\alpha^4\kappa\Phi(-\alpha\kappa^{1/2})\right) d\kappa,$$

where Φ is the standard normal cumulative distribution function. Thus,

$$\frac{\Lambda(dx)}{\Lambda(\mathbb{R}_+)} = \frac{2\alpha}{\sqrt{2\pi}}x^{-1/2}e^{-\alpha^2x/2} - 2\alpha^2\Phi(-\alpha x^{1/2})$$

Proof. We have from [20, page 269, equation 1.14.4(1)] that

$$f^-(\xi,h) = f^+(\xi,h) = \frac{-2\alpha h}{\sqrt{2\pi}\xi^{3/2}} \exp\left\{-\frac{(\alpha\xi - h)^2}{2\xi}\right\}.$$

Thus, by Proposition 3.6.1,

$$\begin{aligned}\frac{\mathbb{P}\{K \in d\kappa\}}{d\kappa} &= \frac{4\alpha^3\kappa e^{-\alpha^2\kappa/2}}{\pi}\int_0^\kappa\int_{-\infty}^0 \frac{h^2}{\xi^{3/2}(\kappa-\xi)^{3/2}} \exp\left\{2\alpha h - \frac{\kappa h^2}{2\xi(\kappa-\xi)}\right\} dh\, d\xi \\ &= \frac{4\alpha^3 e^{-\alpha^2\kappa/2}}{\pi\kappa}\int_{-\infty}^0 h^2e^{2\alpha h}\left(\int_0^1 \frac{1}{\xi^{3/2}(1-\xi)^{3/2}} \exp\left\{-\frac{h^2/2\kappa}{\xi(1-\xi)}\right\} d\xi\right) dh.\end{aligned}$$

The change of variable $y = \frac{1}{\xi(1-\xi)} - 4$ gives that

$$\int_0^{1/2} \frac{1}{\xi^{3/2}(1-\xi)^{3/2}} \exp\left\{-\frac{c}{\xi(1-\xi)}\right\} d\xi = e^{-4c}\int_0^\infty z^{-1/2}e^{-cz}dz = \frac{e^{-4c}\sqrt{\pi}}{\sqrt{c}}$$

for any $c > 0$, and hence

$$\begin{aligned}\frac{\mathbb{P}\{K \in d\kappa\}}{d\kappa} &= \frac{4\alpha^3e^{-\alpha^2\kappa/2}}{\pi\kappa}\int_{-\infty}^0 h^2e^{2\alpha h}\frac{2e^{-2h^2/\kappa}\sqrt{\pi}}{\sqrt{h^2/2\kappa}}\, dh \\ &= -\frac{8\sqrt{2}\alpha^3e^{-\alpha^2\kappa/2}}{\sqrt{\pi\kappa}}\int_{-\infty}^0 he^{2\alpha h - 2h^2/\kappa}\, dh.\end{aligned}$$

The further change of variable $z = 2\kappa^{-1/2}h - \alpha\kappa^{1/2}$ leads to

$$\begin{aligned}\frac{\mathbb{P}\{K \in d\kappa\}}{d\kappa} &= -4\alpha^3\int_{-\infty}^{-\alpha\kappa^{1/2}} (\kappa^{1/2}z + \alpha\kappa)\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\, dz \\ &= -4\alpha^3\left(-\frac{\kappa^{1/2}}{\sqrt{2\pi}}e^{-\alpha^2\kappa/2} + \alpha\kappa\Phi(-\alpha\kappa^{1/2})\right) \\ &= \frac{4\alpha^3}{\sqrt{2\pi}}\kappa^{1/2}e^{-\alpha^2\kappa/2} - 4\alpha^4\kappa\Phi(-\alpha\kappa^{1/2}).\end{aligned}$$

Because $\Lambda(dx)$ is proportional to $x^{-1}\mathbb{P}\{K \in dx\}$, we need only find $\int_{\mathbb{R}_+} x^{-1}\mathbb{P}\{K \in dx\}$ to establish the claim for Λ, and this can be done using methods of integration similar to those used in Remark 3.7.4 below to check that the density of K integrates to one. □

Remark 3.7.4. We can check directly that the density given for K integrates to one. For the first term, we use the substitution $\eta = \alpha^2\kappa/2$, and for the second we use the substitution $\eta = \alpha^2\kappa$ and then change the order of integration to get that the integral of the claimed density is

$$\begin{aligned}
&\frac{4}{\Gamma(3/2)}\int_0^\infty \eta^{1/2}e^{-\eta}\,d\eta - 4\int_0^\infty \eta\Phi(-\eta^{1/2})\,d\eta \\
&\quad = 4 - \frac{4}{\sqrt{2\pi}}\int_0^\infty \int_{\eta^{1/2}}^\infty \eta e^{-y^2/2}\,dy\,d\eta \\
&\quad = 4 - \frac{4}{\sqrt{2\pi}}\int_0^\infty \left(\int_0^{y^2} \eta\,d\eta\right) e^{-y^2/2}\,dy \\
&\quad = 4 - \frac{2}{\sqrt{2\pi}}\int_0^\infty y^4 e^{-y^2/2}\,dy \\
&\quad = 4 - \frac{3}{\Gamma(5/2)}\int_0^\infty x^{3/2}e^{-x}\,dx \;=\; 1.
\end{aligned}$$

Proposition 3.7.5. *Let X be a Brownian motion with drift β, where $|\beta| < \alpha$. Recall that $T := \arg\max\{M_t : G \le t \le D\}$ and $H := X_T - M_T$. Then, H has a* Gamma$(2, 4\alpha)$ *distribution; that is, the distribution of H is absolutely continuous with respect to Lebesgue measure with density $h \mapsto (4\alpha)^2 he^{-4\alpha h}$, $h \ge 0$. Also,*

$$\mathbb{P}\{T > 0\} = \frac{1}{2}\left(1 + \frac{\beta}{\alpha}\right),$$

and the distribution of T is characterized by

$$\mathbb{E}\left[e^{-\theta T}\right] = 8\alpha(\alpha^2 - \beta^2)\frac{1}{\theta}\left(\frac{1}{\sqrt{(\alpha+\beta)^2 - 2\theta} + 3\alpha - \beta} - \frac{1}{\sqrt{(\alpha-\beta)^2 + 2\theta} + 3\alpha + \beta}\right)$$

for $-\frac{(\alpha-\beta)^2}{2} \le \theta \le \frac{(\alpha+\beta)^2}{2}$.

Proof. Consider the claim regarding the distribution of H. A slight elaboration of the proof of Proposition 3.6.1 shows, in the notation of that result, that the random vector $(T, L, R, -H)$ has a distribution that is absolutely continuous with respect to Lebesgue measure with joint density $(\tau, \lambda, \rho, \eta) \mapsto 2\alpha f^-(\lambda, \eta) f^+(\rho, \eta)$, $\lambda, \rho > 0$, $\tau - \lambda < 0 < \tau + \rho$, $\eta < 0$. Therefore,

$$\mathbb{P}\{H \in dh\} = 2\alpha \int_0^\infty \int_0^\infty (\lambda + \rho) f^-(\lambda, -h) f^+(\rho, -h)\, d\lambda d\rho d\eta.$$

By (3.7.2),

$$\int_0^\infty f^-(\lambda, -h)\, d\lambda = 2(\alpha - \beta) e^{-2(\alpha-\beta)h}. \tag{3.7.4}$$

Combining this with (3.7.3) gives

$$\int_{-\infty}^0 \lambda f^-(\lambda, -h)\, d\lambda = \frac{-\eta}{\alpha - \beta} \times 2(\alpha - \beta) e^{-2(\alpha-\beta)h} = 2h\eta e^{-2(\alpha-\beta)h}. \tag{3.7.5}$$

Similarly,

$$\int_0^\infty f^+(\rho, -h)\, d\rho = 2(\alpha + \beta) e^{-2(\alpha+\beta)h} \tag{3.7.6}$$

and

$$\int_{-\infty}^0 \rho f^+(\rho, \eta)\, d\rho = 2h e^{-2(\alpha+\beta)h}. \tag{3.7.7}$$

Thus,

$$\begin{aligned}\mathbb{P}\{H \in dh\} = 2\alpha \Big[& 2he^{-2(\alpha-\beta)h} \times 2(\alpha+\beta) e^{-2(\alpha+\beta)h} \\ & + 2(\alpha-\beta) e^{-2(\alpha-\beta)h} \times 2he^{-2(\alpha+\beta)h} \Big]\, dh \\ = {} & (4\alpha)^2 h e^{-4\alpha h}\, dh.\end{aligned}$$

Note that $T > 0$ if and only if $I^+ > I^-$, where

$$I^+ := \inf\{X_t + \alpha t : t \geq 0\} \quad \text{and} \quad I^- := \inf\{X_t - \alpha t : t \leq 0\}.$$

Recall from Subsection 3.7.1 that the independent random variables I^+ and I^- are exponentially distributed with respective means $(2(\alpha+\beta))^{-1}$ and $(2(\alpha-\beta))^{-1}$. It follows that

$$\mathbb{P}\{T > 0\} = \frac{2(\alpha+\beta)}{2(\alpha+\beta) + 2(\alpha-\beta)} = \frac{1}{2}\left(1 + \frac{\beta}{\alpha}\right).$$

We can also derive this last result from Proposition 3.6.1 as follows.

$$\begin{aligned}
\mathbb{P}\{T > 0\} &= 2\alpha \int_{-\infty}^{0} \int_{0}^{\infty} \int_{-\infty}^{0} \int_{\tau}^{\infty} f^{-}(\tau - \gamma, h) f^{+}(\delta - \tau, h) \, d\delta d\gamma d\tau dh \\
&= 2\alpha \int_{-\infty}^{0} \int_{0}^{\infty} \int_{-\infty}^{0} f^{-}(\tau - \gamma, h) \left(\int_{0}^{\infty} f^{+}(\eta, h) \, d\eta \right) dh d\tau d\gamma \\
&= 2\alpha \int_{-\infty}^{0} \left(\int_{0}^{\infty} \int_{-\infty}^{0} f^{+}(\tau - \gamma, h) \, d\gamma \, d\tau \right) \left(\int_{0}^{\infty} f^{+}(\eta, h) \, d\eta \right) \, dh \\
&= 2\alpha \int_{-\infty}^{0} \left(\int_{0}^{\infty} \eta f^{-}(\eta, h) \, d\eta \right) \left(\int_{0}^{\infty} f^{+}(\eta, h) \, d\eta \right) \, dh.
\end{aligned}$$

Substituting in (3.7.5) and (3.7.6), and then evaluating the resulting straightforward integral establishes the result.

The Laplace transform of T may be calculated using very similar methods. □

3.8 Some facts about Lipschitz minorants

The following is a restatement of (3.1.1) accompanied by a proof.

Lemma 3.8.1. *Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has α-Lipschitz minorant $m : \mathbb{R} \to \mathbb{R}$. Then,*

$$\begin{aligned}
m(t) &= \sup\{h \in \mathbb{R} : h - \alpha|t - s| \le f(s) \text{ for all } s \in \mathbb{R}\} \\
&= \inf\{f(s) + \alpha|t - s| : s \in \mathbb{R}\}.
\end{aligned}$$

Proof. Consider the first equality. Fix $t \in \mathbb{R}$. Because m is α-Lipschitz, if $h \le m(t)$, then $h - \alpha|t-s| \le m(t) - \alpha|t-s| \le m(s) \le f(s)$ for all $s \in \mathbb{R}$. On the other hand, if $h > m(t)$, then $s \mapsto (h - \alpha|t-s|) \vee m(s)$ is an α-Lipschitz function that dominates m (strictly at t), and so $(h - \alpha|t-s|) \vee m(s) > f(s)$ for some $s \in \mathbb{R}$. This implies that $h - \alpha|t-s| > f(s)$, since $m(s) \le f(s)$. The second equality is simply a rephrasing of the first. □

We leave the proof of the following straightforward consequence of Lemma 3.8.1 to the reader.

Corollary 3.8.2. *Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has α-Lipschitz minorant $m : \mathbb{R} \to \mathbb{R}$. Define functions $f^{\leftarrow} : \mathbb{R} \to \mathbb{R}$ and $f^{\rightarrow} : \mathbb{R} \to \mathbb{R}$ by*

$$f^{\leftarrow}(t) := \begin{cases} f(t), & t < 0, \\ m(0) - \alpha t, & t \ge 0, \end{cases}$$

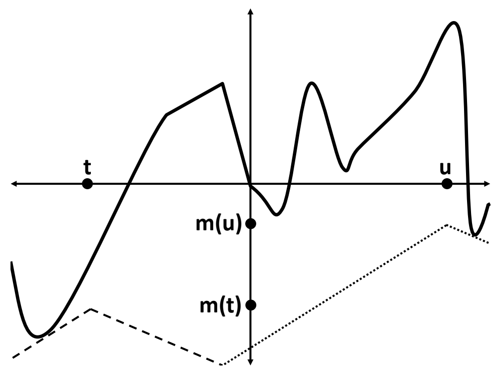

Figure 3.2: Lemma 3.8.1 shows that the height of the α-Lipschitz minorant of a function f at a fixed time t is given by $\sup\{h \in \mathbb{R} : h - \alpha|t-s| \le f(s) \text{ for all } s \in \mathbb{R}\}$.

and

$$f^{\rightarrow}(t) := \begin{cases} m(0) + \alpha t, & t \le 0, \\ f(t), & t > 0. \end{cases}$$

Denote the α-Lipschitz minorants of $f^{\leftarrow}$ and $f^{\rightarrow}$ by $m^{\leftarrow}$ and $m^{\rightarrow}$, respectively. Then, $m^{\leftarrow}(t) = m(t)$ for all $t \le 0$ and $m^{\rightarrow}(t) = m(t)$ for all $t \ge 0$.

The next result says that if f is a càdlàg function with α-Lipschitz minorant m, then on an open interval in the complement of the closed set $\{t \in \mathbb{R} : m(t) = f(t) \wedge f(t-)\}$ the graph of the function m is either a straight line or a "sawtooth".

Lemma 3.8.3. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ be a càdlàg function with α-Lipschitz minorant $m : \mathbb{R} \to \mathbb{R}$. The set $\{t \in \mathbb{R} : m(t) = f(t) \wedge f(t-)\}$ is closed. If $t' < t''$ are such that $f(t') \wedge f(t'-) = m(t')$, $f(t'') \wedge f(t''-) = m(t'')$, and $f(t) \wedge f(t-) > m(t)$ for $t' < t < t''$, then, setting $t^* = (f(t'') \wedge f(t''-) - f(t') \wedge f(t'-) + \alpha(t'' + t'))/(2\alpha)$,*

$$m(t) = \begin{cases} f(t') \wedge f(t'-) + \alpha(t - t'), & t' \le t \le t^*, \\ f(t'') \wedge f(t''-) + \alpha(t'' - t), & t^* \le t \le t''. \end{cases}$$

Proof. We first show that the set $\{t \in \mathbb{R} : m(t) = f(t) \wedge f(t-)\}$ is closed by showing that its complement is open. Suppose t is in the complement, so that $f(t) \wedge f(t-) - m(t) =: \epsilon > 0$. Because f is càdlàg and m is continuous, there exists $\delta > 0$ such that if $|s - t| < \delta$, then $f(s) > f(t) \wedge f(t-) - \epsilon/3$ and $m(s) < m(t) + \epsilon/3$. Hence, $f(s-) \ge f(t) \wedge f(t-) - \epsilon/3$ and $f(s) \wedge f(s-) - m(s) > \epsilon/3$ for $|s - t| < \delta$, showing that a neighborhood of t is also in the complement.

Turning to the second claim, define a function $\tilde{m} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{m}(t) := \begin{cases} f(t') \wedge f(t'-) + \alpha(t - t'), & t \le t^*, \\ f(t'') \wedge f(t''-) + \alpha(t'' - t), & t^* \le t. \end{cases}$$

That is, $\tilde{m}(t) = h^* - \alpha|t - t^*|$, where

$$h^* = (f(t'') \wedge f(t''-) + f(t') \wedge f(t'-) + \alpha(t'' - t'))/2.$$

Because $m(t') = \tilde{m}(t')$ $m(t'') = \tilde{m}(t'')$, and m is α-Lipschitz, we have $m(t) \le \tilde{m}(t)$ for $t \in [t', t'']$ and $m(t) \ge \tilde{m}(t)$ for $t \notin [t', t'']$. Suppose for some $t_0 \in (t', t'')$ that $m(t_0) < \tilde{m}(t_0)$. We must have that $m(t_0) - \alpha|t' - t_0| \le m(t') \le f(t') \wedge f(t'-)$ and $m(t_0) - \alpha|t'' - t_0| \le m(t'') \le f(t'') \wedge f(t''-)$. Moreover, both of these inequalities must be strict, because otherwise we would conclude that $m(t_0) \ge \tilde{m}(t_0)$.

We can therefore choose $\epsilon > 0$ sufficiently small so that $m(t_0) + \epsilon - \alpha|t - t_0| < f(t) \wedge f(t-)$ for $t \in [t', t'']$. This implies that $m(t_0) + \epsilon - \alpha|t - t_0| < \tilde{m}(t) \le m(t) \le f(t) \wedge f(t-)$ for $t \notin [t', t'']$. Thus, $t \mapsto (m(t_0) + \epsilon - \alpha|t - t_0|) \vee m(t)$ is an α-Lipschitz function that is dominated everywhere by f and strictly dominates m at the point t_0, contradicting the definition of m. □

We have a recipe for finding $\inf\{t > 0 : f(t) \wedge f(t-) = m(t)\}$ when f is a càdlàg function with α-Lipschitz minorant m. Figure 3.3 gives two examples of how the recipe applies to different paths (note that the value of α differs for the two examples).

Lemma 3.8.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a càdlàg function with α-Lipschitz minorant $m : \mathbb{R} \to \mathbb{R}$. Set*

$$\mathbf{d} := \inf\{t > 0 : f(t) \wedge f(t-) = m(t)\},$$

$$\mathbf{s} := \inf\left\{t > 0 : f(t) \wedge f(t-) - \alpha t \le \inf\{f(u) - \alpha u : u \le 0\}\right\},$$

and

$$\mathbf{e} := \inf\left\{t \ge \mathbf{s} : f(t) \wedge f(t-) + \alpha(t - \mathbf{s}) = \inf\{f(u) + \alpha(u - \mathbf{s}) : u \ge \mathbf{s}\}\right\}.$$

Suppose that $f(\mathbf{s}) \le f(\mathbf{s}-)$. Then, $\mathbf{e} = \mathbf{d}$.

Proof. It suffices to show the following:

$$f(t) \wedge f(t-) > m(t) \text{ for } 0 < t < \mathbf{e}, \tag{3.8.1}$$

$$f(\mathbf{e}) \wedge f(\mathbf{e}-) \le m(\mathbf{e}), \tag{3.8.2}$$

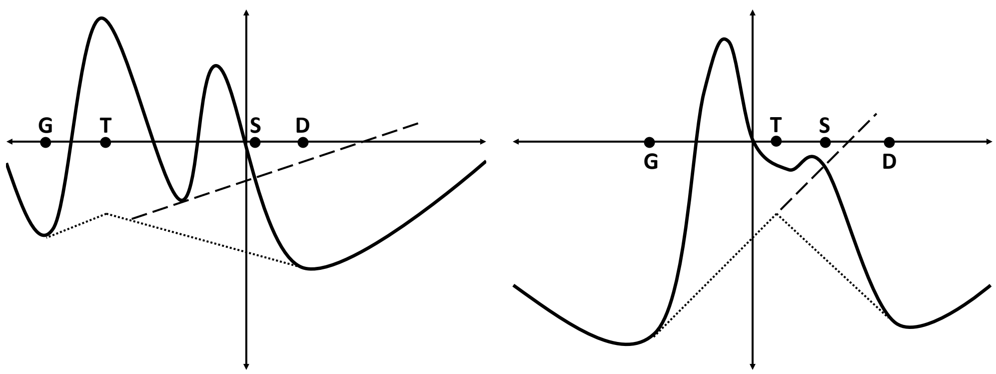

Figure 3.3: Two instances of the construction of Lemma 3.8.4.

$$\mathbf{d} > 0 \Longrightarrow \mathbf{e} > 0. \tag{3.8.3}$$

For $0 < t < \mathbf{s}$, it follows from the definition of $\mathbf{s}$ that

$$\begin{aligned} f(t) \wedge f(t-) &> \inf\{f(u) - \alpha u : u \le 0\} + \alpha t \\ &= \inf\{f(u) + \alpha(t-u) : u \le 0\} \\ &\ge \inf\{f(u) + \alpha|t-u| : u \in \mathbb{R}\} = m(t). \end{aligned}$$

For $\mathbf{s} \le t < \mathbf{e}$, it follows from the definition of $\mathbf{e}$ that

$$f(t) \wedge f(t-) + \alpha(t - \mathbf{s}) > \inf\{f(u) + \alpha(u - \mathbf{s}) : u \ge \mathbf{s}\},$$

and hence

$$\begin{aligned} f(t) \wedge f(t-) &> \inf\{f(u) + \alpha(u - \mathbf{s}) : u \ge \mathbf{s}\} - \alpha(t - \mathbf{s}) \\ &= \inf\{f(u) + \alpha(u-t) : u \ge \mathbf{s}\} \\ &\ge \inf\{f(u) + \alpha|t-u| : u \in \mathbb{R}\} = m(t). \end{aligned}$$

This completes the proof of (3.8.1)

Now $f(\mathbf{e}) \wedge f(\mathbf{e}-) + \alpha(\mathbf{e} - \mathbf{s}) = \inf\{f(u) + \alpha(u - \mathbf{s}) : u \ge \mathbf{s}\}$, and so $f(\mathbf{e}) \wedge f(\mathbf{e}-) = \inf\{f(u) + \alpha(u - \mathbf{e}) : u \ge \mathbf{s}\}$ This certainly gives

$$f(\mathbf{e}) \wedge f(\mathbf{e}-) \le \inf\{f(u) + \alpha|\mathbf{e} - u| : u \ge \mathbf{s}\}. \tag{3.8.4}$$

Combined with the definition of $\mathbf{s}$, it also gives

$$\begin{aligned} f(\mathbf{e}) \wedge f(\mathbf{e}-) + \alpha(\mathbf{e} - \mathbf{s}) &\le f(\mathbf{s}) + \alpha(\mathbf{s} - \mathbf{s}) \\ &\le \inf\{f(s) - \alpha s : s \le 0\} + \alpha \mathbf{s}. \end{aligned}$$

Thus, $f(\mathbf{e}) \wedge f(\mathbf{e}-) + 2\alpha(\mathbf{e} - \mathbf{s}) \leq \inf\{f(s) + \alpha(\mathbf{e} - s) : s \leq 0\}$ and hence, *a fortiori*,

$$f(\mathbf{e}) \wedge f(\mathbf{e}-) \leq \inf\{f(s) + \alpha|\mathbf{e} - s| : s \leq 0\}. \tag{3.8.5}$$

For $0 < s < \mathbf{s}$, $f(s) - s > \inf\{f(r) - \alpha r : r \leq 0\}$, and so

$$\begin{aligned} \inf\{f(s) + \alpha|\mathbf{e} - s| : 0 \leq s < \mathbf{s}\} &= \inf\{f(s) + \alpha(\mathbf{e} - s) : 0 \leq s < \mathbf{s}\} \\ &= \inf\{f(s) - \alpha s : 0 \leq s < \mathbf{s}\} + \alpha\mathbf{e} \\ &\geq \inf\{f(r) - \alpha r : r \leq 0\} + \alpha\mathbf{e} \qquad (3.8.6) \\ &= \inf\{f(r) + \alpha(\mathbf{e} - r) : r \leq 0\} \\ &= \inf\{f(r) + \alpha|\mathbf{e} - r| : r \leq 0\}. \end{aligned}$$

Combining (3.8.4), (3.8.5) and (3.8.6) gives (3.8.2).

The proof of (3.8.3) is a straightforward consequence of Lemma 3.8.3 and we leave it to the reader. □

Corollary 3.8.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a càdlàg function with α-Lipschitz minorant $m : \mathbb{R} \to \mathbb{R}$. Define $\mathbf{d}$, $\mathbf{s}$, and $\mathbf{e}$ as in Lemma 3.8.4. Assume that $f(\mathbf{s}) \leq f(\mathbf{s}-)$, so that $\mathbf{e} = \mathbf{d}$. Put $\mathbf{g} := \sup\{t < 0 : f(t) \wedge f(t-) = m(t)\}$ and assume that $f(0) \wedge f(0-) > m(0)$, so that $f(t) \wedge f(t-) > m(t)$ for $t \in (\mathbf{g}, \mathbf{d})$. Let $\mathbf{t} := (f(\mathbf{d}) \wedge f(\mathbf{d}-) - f(\mathbf{g}) \wedge f(\mathbf{g}-) + \alpha(\mathbf{d} + \mathbf{g}))/(2\alpha)$ be the point in $[\mathbf{g}, \mathbf{d}]$ at which the function m achieves its maximum. Then, $\mathbf{g} \leq \mathbf{t} \leq \mathbf{s} \leq \mathbf{d}$. Moreover, if $\mathbf{t} = \mathbf{s}$, then $\mathbf{t} = \mathbf{s} = \mathbf{d}$.*

Proof. We first show that $\mathbf{g} \leq \mathbf{t} \leq \mathbf{s} \leq \mathbf{d}$. We certainly have $\mathbf{g} \leq \mathbf{s} \leq \mathbf{d}$ and $\mathbf{g} \leq \mathbf{t} \leq \mathbf{d}$, so it suffices to prove that $\mathbf{t} \leq \mathbf{s}$. Because $\mathbf{s} \geq 0$, this is clear when $\mathbf{t} < 0$, so it further suffices to consider the case where $\mathbf{t} \geq 0$. Suppose, then, that $\mathbf{g} \leq 0 \leq \mathbf{s} < \mathbf{t} \leq \mathbf{d}$.

From Lemma 3.8.3 we have $m(u) = f(\mathbf{g}) \wedge f(\mathbf{g}-) + \alpha(u - \mathbf{g})$ for $\mathbf{g} \leq u \leq \mathbf{t}$ and $f(u) \wedge f(u-) \geq f(\mathbf{g}) \wedge f(\mathbf{g}-) + \alpha(u - \mathbf{g})$ for $u \leq \mathbf{t}$. Therefore, $\inf\{f(u) \wedge f(u-) - \alpha u : u \leq 0\} \geq f(\mathbf{g}) \wedge f(\mathbf{g}-) - \mathbf{g}$, and hence $\inf\{f(u) \wedge f(u-) - \alpha u : u \leq 0\} = f(\mathbf{g}) \wedge f(\mathbf{g}-) - \alpha\mathbf{g}$. Now, by definition of $\mathbf{s}$, $f(\mathbf{s}) \wedge f(\mathbf{s}-) - \alpha\mathbf{s} \leq \inf\{f(u) \wedge f(u-) - \alpha u : u \leq 0\}$, and so

$$\begin{aligned} f(\mathbf{s}) \wedge f(\mathbf{s}-) &\leq f(\mathbf{g}) \wedge f(\mathbf{g}-) - \alpha\mathbf{g} + \alpha\mathbf{s} \\ &= f(\mathbf{g}) \wedge f(\mathbf{g}-) + \alpha(\mathbf{s} - \mathbf{g}) \\ &= m(\mathbf{s}), \end{aligned}$$

which contradicts $\mathbf{d} = \inf\{u > 0 : f(u) \wedge f(u-) = m(u)\} = \inf\{u > 0 : f(u) \wedge f(u-) \leq m(u)\}$ unless $\mathbf{s} = 0$ and $f(0) \wedge f(0-) = m(0)$, but we have assumed that this is not the case.

A similar argument shows that if $\mathbf{t} = \mathbf{s}$, then $\mathbf{t} = \mathbf{s} = \mathbf{d}$. □

Chapter 4

Structure of Shocks in Burgers Turbulence with Lévy Noise Initial Data

4.1 Introduction

Burgers introduced the equation

$$\partial_t u + \partial_x(u^2/2) = \varepsilon \partial_{xx}^2 u$$

as a simple model of hydrodynamic turbulence for compressible fluids, where the parameter $\varepsilon > 0$ describes the viscosity of the fluid and the solution represents the velocity of a fluid particle located at x at time t [22]. It can be seen as a simplification of the Navier-Stokes equation arrived at by neglecting pressure and force terms, but also arises in other physical problems, such as the formation of the superstructure of the universe [98].

It is known that under certain conditions, as $\varepsilon \to 0$ the solution converges to the unique entropy condition satisfying weak solution of the *inviscid Burgers equation*

$$\partial_t u + \partial_x(u^2/2) = 0. \tag{4.1.1}$$

A physical interpretation of the weak entropy condition satisfying solution to (4.1.1) is that at time zero, infinitesimal particles are uniformly spread on the line, with initial velocity $u(\cdot, 0)$, and these particles evolve according to the dynamics of completely inelastic shocks. That is, the velocity of a particle changes only when the cluster of particles it is in collides with another cluster, in which case the clusters stick together and form a heavier cluster, with conservation mass and momentum determining the mass and velocity of the new cluster.

There is an abundant literature on the solution to 4.1.1 when the initial velocity $u(\cdot, 0)$ is a random process. See for example [8, 7, 40, 39, 19, 14, 22, 25, 52, 80, 81, 82, 85, 98, 55, 97]. We will investigate the solution when $u(\cdot, 0)$ is a Lévy noise, i.e. when the *potential* process $\psi_0 = (\psi_0(x))_{x\in\mathbb{R}}$, defined by $\psi_0(x) - \psi_0(y) = \int_x^y u(z, 0)\, dz$, has stationary independent increments (note that the change from the independent variable in the Lévy process being spatial rather than temporal necessitates a change in notation from the Lévy processes of Chapter 3). In particular, we investigate qualitative features of the shock structure of the solution, and thus extend the work of Bertoin [19] (ψ_0 a stable Lévy process with stability index $\alpha \in (1/2, 2]$), Giraud [40] (extensive results for the case $\alpha \in (1/2, 1)$) and Lachièze-Rey [55] (ψ_0 a bounded variation Lévy process).

In order to explain our results, we must first discuss the general solution to (4.1.1) and some related concepts. We follow [40, Section 2.1] closely. Suppose that ψ_0 has discontinuities only of the first kind and satisfies $\psi_0(x) = o(x^2)$ as $|x| \to \infty$. Then as $\varepsilon \to 0$ the unique solution of Burgers equation with viscosity $\varepsilon > 0$ converges (except on a countable set) to a weak solution of (4.1.1), referred to as the Hopf-Cole solution (see [50, 28]). The right continuous version of this solution is

$$u(x, t) = t^{-1}(x - a(x, t)),$$

where, taking the supremum over all possible arguments if necessary,

$$a(x, t) := \arg\sup \left\{ \psi_0(y) - \tfrac{1}{2t}(y - x)^2 \,:\, y \in \mathbb{R} \right\},$$

or more precisely,

$$a(x, t) := \sup \left\{ z \in \mathbb{R} \,:\, \psi_0(z) - \tfrac{1}{2t}(z - x)^2 \geq \psi_0(y) - \tfrac{1}{2t}(y - x)^2 \;\forall\, y \in \mathbb{R} \right\}.$$

The function $x \mapsto a(x, t)$ is non-decreasing and right continuous and its right continuous inverse $a \mapsto x(a, t)$ is known as the *Lagrangian function*, and gives the position at time t of the particle initially located at a.

A discontinuity of $x \mapsto u(x, t)$ is called a shock and occurs when $x \mapsto a(x, t)$ jumps, i.e. when $a(x, t) \neq a(x-, t) := \lim_{y\uparrow x} a(y, t)$. From the point of view of the particle description, the location of a shock corresponds to the location of a cluster at time t. This cluster results from the aggregation of the particles initially located in $[a(x-, t), a(x, t)]$; its velocity is (according to the conservation of masses and momenta)

$$v(x, t) = -\frac{\psi_0(a(x, t)) - \psi_0(a(x-, t))}{a(x, t) - a(x-, t)} = \tfrac{1}{2} \left[u(a(x, t)) + u(a(x-, t)) \right].$$

The interval $[a(x-,t), a(x,t)]$ is called a *shock interval* and x a *Eulerian shock point*. We define the *shock structure* of the solution at time t to be the closed range of $a(\cdot,t)$. Of particular interests are points which are not isolated on the left or the right in that closed range, since they represent the initial locations of particles that have not been involved in any collisions by time t. We call any such point a *Lagrangian regular point*. Finally, we call (x, y) a *rarefaction* interval if $a(\cdot,t)$ stays constant on $[x, y)$. A rarefaction interval represents an interval where there are no fluid particles at time t.

Our results concern qualitative features of the shock structure, the regenerativity of the process $(u(x,t))_{x\in\mathbb{R}}$ at points where $u(x,t) = 0$, and the relationship between such points and the Lagrangian regular points. For our arguments, there is no loss of generality to assume $t = 1$ – the properties we show will be true for any $t > 0$. Thus we restrict our attention to the case $t = 1$ and set $a(x) = a(x,1)$, $u(x) = u(x,1)$ for all $x \in \mathbb{R}$. The shock structure is then

$$\mathcal{A} := \mathbf{cl}\{y \in \mathbb{R} : a(x) = y \text{ for some } x \in \mathbb{R}\},$$

i.e. the closure of the range of $a(\cdot)$, and Lagrangian regular points are the subset of points of $\mathcal{A}$ that are neither left nor right isolated. We also define $\mathcal{A}_0 \subset \mathcal{A}$ by

$$\mathcal{A}_0 := \mathbf{cl}\{x \in \mathbb{R} : a(x) = x\} = \mathbf{cl}\{x \in \mathbb{R} : u(x) = 0\},$$

Note that both $\mathcal{A}$ and $\mathcal{A}_0$ are stationary sets when ψ_0 is a Lévy process, and since adding a drift term has no affect on the distributions of these random sets, *we will assume throughout that if ψ_0 has bounded variation then it has zero drift coefficient.*

To ensure that $\mathcal{A}$ is non-empty we will always assume that $\lim_{|x|\to\infty} x^{-2}\psi_0(x) = 0$, and in the bounded variation case we mostly assume that $\limsup_{|h|\downarrow 0} h^{-2}\psi_0(h) = \infty$ to ensure that $\mathcal{A}$ has a nice structure. Most of our results in the bounded variation case also require a further assumption relating to overshoots at hitting times – see Assumption **B** in Section 4.3.3.

In all cases we show that the Lebesgue measure of $\{y \in \mathbb{R} : a(x) = y \text{ for some } x \in \mathbb{R}\}$ is zero (see Lemma 4.4.1) and in the bounded variation case we show that this set is closed (see Theorem 4.4.14). For ψ_0 in an interesting class of unbounded variation Lévy processes called *abrupt* Lévy processes (see Section 4.3.4 for a definition), we also show that this set is closed and that moreover $\mathcal{A}$ is a discrete set (see Theorem 4.4.6 and Corollary 4.4.7), extending the result of [19] that this is true when ψ_0 is a stable process with $\alpha \in (1, 2]$. A result from [19] relating to Cauchy processes is also extended to a more general class of unbounded variation processes, the *eroded* Lévy processes (again, see Section 4.3.4 for a definition). For these eroded processes, there are no rarefaction intervals.

We show that if ψ_0 is of unbounded variation and abrupt, or of bounded variation and satisfying Assumption **B**, then the process $u = (u(x))_{x\in\mathbb{R}}$ is regenerative at points y such that $u(y) = 0$, that between any two consecutive such points it must first be positive and then negative, and that the only accumulation points of jump times of u are at such points (see Theorem 4.4.3, Theorem 4.4.6, Proposition 4.4.12 and Theorem 4.4.15). For ψ_0 a stable processes with $\alpha \in (1/2, 1)$, this is the main result of [40], hence our work generalizes that result to a wider class of bounded variation processes (it is also shown in [40] that for those stable processes $\mathcal{A}$ is a discrete set – we could not generalize this result to our wider class of bounded variation processes). Key to proving this result is the theory of *randomized coterminal times* due to Millar (see Section 4.3.5), which allows us to decompose the process at $T := \inf\{x \geq 0 : x \in \mathcal{A}_0\}$, i.e. at the first non-negative element of $\mathcal{A}_0$. The results of Lachièze-Rey [55] also form an indispensable part of our arguments in the bounded variation case.

Another important result of [40] is that when ψ_0 is a stable processes with $\alpha \in (1/2, 1)$, $\mathcal{A}_0$ is exactly equal to the set of points of $\mathcal{A}$ at which ψ_0 is continuous, which is in turn equal to the set of Lagrangian regular points. We extend this result to our more general class of bounded variation processes (see Proposition 4.4.12 and Theorem 4.4.15) again using the results of Lachièze-Rey [55].

The rest of this chapter is organized as follows. In Section 4.2 we discuss geometric interpretations of $a(x)$ that make the proofs easier to read and which explain how the shock structure gives us a *parabolic majorant* of the Lévy process. We also introduce in Section 4.2 the important connection between $\mathcal{A}$ and the concave majorant of $(\psi_0(x) - \frac{1}{2}x^2)_{x\in\mathbb{R}}$. In Section 4.4 we present and prove all of our results, with the exception of the proof of the regenerativity property of $\mathcal{A}_0$ mentioned above, which we prove in Section 4.5.

We conclude the introduction by noting that $\mathcal{A}_0$ is the set of fixed points of the proximal mapping for the Moreau envelope of ψ_0 [57, 77] and thus may be of interest in convex analysis.

4.2 Geometric Interpretations and Relation to Parabolic and Concave Majorants

Recall from Section 4.1 that

$$a(x) = \arg\sup\left\{\psi_0(y) - \tfrac{1}{2}(y-x)^2 \,:\, y \in \mathbb{R}\right\},$$

i.e. $a(x)$ is the (largest) location of the supremum of $y \mapsto \psi_0(y) - \frac{1}{2}(y-x)^2$. One has the following geometric interpretation: consider a realization of the initial potential

ψ_0 and a parabola $y \mapsto \frac{1}{2}(z-x)^2 + C$, where C is chosen such that the parabola is strictly above the path of ψ_0. Let C decrease until this parabola touches the graph of ψ_0. Then $a(x)$ is the largest abscissa of the contact points.

Now consider what happens to $a(x)$ as x increases. Suppose for example that $x < a(x)$, then the center of the parabola will move forward, and C will increase so that the largest abscissa of the contact points between the parabola and ψ_0 remains at $a(x)$. This will keep going until for some $z > x$, the location of the largest supremum of $y \mapsto \psi_0(y) - \frac{1}{2}(y-z)^2$ is no longer at $a(x)$, that is, the parabola centered at z passing through the point $(a(x), \psi_0(a(x)))$ will touch ψ_0 again at $(a(z), \psi_0(a(z)))$, where $a(z) > a(x)$. This creates a jump in a, and hence in u, at the location z. The story is similar when $x > a(x)$, except that now C will decrease in order to keep the parabola touching ψ_0 as the center of the parabola moves forward.

The above discussion leads to the idea of the *parabolic majorant* of ψ_0. As x varies, there will be values of x for which the parabola $y \mapsto \psi_0(y) - \frac{1}{2}(y-z)^2$ will touch ψ_0 at more than one point. For such values of x we get a parabolic curve segment between the first and last of those points that is at least as large as ψ_0 at all points. Thus we define the parabolic majorant to be the collection of those parabolic curve segments. An alternative but equivalent definition would be to take the collection of parabolas of the form $y \mapsto \psi_0(y) - \frac{1}{2}(y-z)^2$ that touched ψ_0 at two consecutive shock points, and then form the parabolic majorant by taking only the segments of those parabolas between the two consecutive shock points at which each parabola touches ψ_0.

Another important geometric property of the Hopf-Cole solution relates to concave majorants. Similarly to the concave majorant of a function on a finite interval as in Chapter 2, for any $f : \mathbb{R} \to \mathbb{R}$, the concave majorant of f is the minimal concave function $\bar{C}_f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ such that $\bar{C}_f(x) \geq f(x) \vee f(x-)$ for every $x \in \mathbb{R}$.

Let $\bar{C} : \mathbb{R} \to \mathbb{R}$ denote the concave majorant of $(\psi_0(x) - \frac{1}{2}x^2)_{x \in \mathbb{R}}$, and denote its right continuous derivative by $\bar{c} = \bar{C}'$. Since $\bar{c}(\cdot)$ is non-increasing, we can consider the Stieltjes measure $-d\bar{c}$. The connection with $\mathcal{A}$ is the following.

Lemma 4.2.1. *For any* ψ_0*,*

(i) $\mathrm{Supp}(d\bar{c}) \subseteq \boldsymbol{cl}\{y : \exists\, x \text{ s.t. } a(x) = y\} = \mathcal{A}$*;*

(ii) $\{y : \exists\, x \text{ s.t. } a(x) = y\} \subseteq \mathrm{Supp}(d\bar{c})$.

Hence if $\{y : \exists\, x \text{ s.t. } a(x) = y\}$ *is closed then* $Supp(d\bar{c}) = \mathcal{A}$.

Proof. (i) Suppose first that $y \in \mathrm{Supp}(d\bar{c})$ is isolated on both sides in $\mathrm{Supp}(d\bar{c})$ or is in the interior of $\mathrm{Supp}(d\bar{c})$. Then there exists $x \in \mathbb{R}$ such that

$$\left(\psi_0(y+z) \vee \psi_0((y+z)-) - \tfrac{1}{2}(y+z)^2\right) - \left(\psi_0(y) \vee \psi_0(y-) - \tfrac{1}{2}y^2\right) < -xz$$

for all $z \neq 0$. But then

$$\psi_0(y+z) \vee \psi_0((y+z)-) - \tfrac{1}{2}(y+z-x)^2 \le \psi_0(y) \vee \psi_0(y-) - \tfrac{1}{2}(y-x)^2$$

with equality only if $z = 0$. Hence

$$a(x) = y + \arg\sup\left\{\psi_0(y+z) - \tfrac{1}{2}((y+z)-x)^2 \,:\, z \in \mathbb{R}\right\} = y + 0 = y,$$

and thus $y \in \mathcal{A}$.

Now suppose y is not isolated in $\mathrm{Supp}(d\bar{c})$. Then there exists a sequence of points $\{y_n\}_{n\ge 0}$ such that $y_n \to y$ with each y_n either isolated on both sides in $\mathrm{Supp}(d\bar{c})$ or in the interior of $\mathrm{Supp}(d\bar{c})$. Let $\{x_n\}_{n\ge 0}$ be such that $a(x_n) = y_n$ for each $n \ge 0$. Then $a(x_n) \to y$ and hence $y \in \mathcal{A}$ since $\mathcal{A}$ is closed.

(ii) Suppose there exists x such that $a(x) = y$. From the definition of $a(x)$ it follows that

$$\psi_0(y-z) \vee \psi_0((y-z)-) - \tfrac{1}{2}((y-z)-x)^2 \le \psi_0(y) \vee \psi_0(y-) - \tfrac{1}{2}(y-x)^2$$

for all $z \ge 0$ and

$$\psi_0(y+z) \vee \psi_0((y+z)-) - \tfrac{1}{2}((y+z)-x)^2 < \psi_0(y) \vee \psi_0(y-) - \tfrac{1}{2}(y-x)^2$$

for all $z > 0$. Thus

$$\left(\psi_0(y-z) \vee \psi_0((y-z)-) - \tfrac{1}{2}(y-z)^2\right) - \left(\psi_0(y) \vee \psi_0(y-) - \tfrac{1}{2}y^2\right) \le zx \tag{4.2.1}$$

for all $z \ge 0$ and

$$\left(\psi_0(y+z) \vee \psi_0((y+z)-) - \tfrac{1}{2}(y+z)^2\right) - \left(\psi_0(y) \vee \psi_0(y-) - \tfrac{1}{2}y^2\right) < -zx \tag{4.2.2}$$

for all $z > 0$.

(4.2.1) implies that $\bar{c}(y-) \ge -z$ and (4.2.2) implies that $\bar{c}(y) < -z$. Hence $y \in \mathrm{Supp}(d\bar{c})$. □

4.3 Definitions and Background Material

Many of the definitions and results in this section are repeated from Chapter 3, but due to the change in notation (necessitated by the change from a temporal to spatial independent variable as mentioned in the introduction) we repeat them here for ease of reference.

4.3.1 Lévy processes

Let $\psi_0 = (\psi_0(x))_{x\in\mathbb{R}}$ be a real-valued Lévy process. That is, ψ_0 has *càdlàg* sample paths, $\psi_0(0) = 0$, and $\psi_0(y) - \psi_0(x)$ is independent of $(\psi_0(z))_{z\le x}$ with the same distribution as $\psi_0(y-x)$ for all $x, y \in \mathbb{R}$ with $x < y$.

The Lévy-Khintchine formula says that for $x \ge 0$ the characteristic function of $\psi_0(x)$ is given by $\mathbb{E}[e^{i\theta\psi_0(x)}] = e^{-x\Psi(\theta)}$ for $\theta \in \mathbb{R}$, where

$$\Psi(\theta) = -ic\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta y} + i\theta y 1_{\{|y|<1\}})\, \Pi(dy)$$

with $c \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$, and Π a σ-finite measure concentrated on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}}(1\wedge y^2)\,\Pi(dy) < \infty$. We call σ^2 the *infinitesimal variance* of the Brownian component of ψ_0 and Π the *Lévy measure* of X.

The sample paths of ψ_0 have bounded variation almost surely if and only if $\sigma = 0$ and $\int_{\mathbb{R}}(1 \wedge |y|)\,\Pi(dy) < \infty$. In this case Ψ can be rewritten as

$$\Psi(\theta) = -id\theta + \int_{\mathbb{R}} (1 - e^{i\theta y})\, \Pi(dy).$$

We call $d \in \mathbb{R}$ the drift coefficient. Recall from the introduction to this chapter that we will assume $d = 0$ throughout without affecting our results. For full details of these definitions see [15].

4.3.2 Fluctuation theory

We will often make use of some basic results from fluctuation theory for Lévy processes.

The first is due to Štatland [87]. If ψ_0 has paths of bounded variation with drift d, then

$$\lim_{h\downarrow 0} h^{-1}\psi_0(h) = d \quad \text{a.s.} \tag{4.3.1}$$

Since the jump times of ψ_0 form a countable set of stopping times, by the strong Markov property it follows that for all y such that $\psi_0(y) \neq \psi_0(y-)$, i.e. at all jump times y of ψ_0, we have

$$\lim_{h\downarrow 0} h^{-1}(\psi_0(y+h) - \psi_0(y)) = d \quad \text{a.s.} \tag{4.3.2}$$

The counterpart of Štatland's result when ψ_0 has paths of unbounded variation is Rogozin's result

$$\liminf_{h\downarrow 0} h^{-1}\psi_0(h) = -\infty \quad \text{and} \quad \limsup_{h\downarrow 0} h^{-1}\psi_0(h) = +\infty \quad \text{a.s.} \tag{4.3.3}$$

By the strong Markov property, it again follows that for all y such that $\psi_0(y) \neq \psi_0(y-)$, i.e. at all jump times y of ψ_0, we have

$$\begin{aligned} \liminf_{h\downarrow 0} h^{-1}(\psi_0(y+h) - \psi_0(y)) = -\infty \quad \text{a.s.} \quad \text{and} \\ \limsup_{h\downarrow 0} h^{-1}(\psi_0(y+h) - \psi_0(y)) = +\infty \quad \text{a.s.} \end{aligned} \tag{4.3.4}$$

4.3.3 Hypotheses on ψ_0

We now define some hypotheses on ψ_0. We will always assume the first and the second ensures that the shock structure is nice when ψ_0 has paths of bounded variation. Let $\bar{C} : \mathbb{R} \to \mathbb{R}$ denote the concave majorant of $(\psi_0(x) - \frac{1}{2}x^2)_{x\in\mathbb{R}}$, and denote its right continuous derivative by $\bar{c} = \bar{C}'$. Since $\bar{c}(\cdot)$ is non-increasing, we can consider the Stieltjes measure $-d\bar{c}$.

Hypothesis A. Let ψ_0 be such that almost surely $\lim_{|x|\to\infty} x^{-2}\psi_0(x) = 0$.

Remark 4.3.1. (i) Hypothesis **A** implies $\bar{C}$ is finite and $\sup_{x\in\mathbb{R}}\{\psi_0(x) - \frac{1}{2}x^2\} < \infty$.

(ii) Hypothesis **A** holds for stable processes with stability index $\alpha \in (1/2, 2]$.

Hypothesis B. If ψ_0 has paths of bounded variation then let ψ_0 be such that

$$\limsup_{h\downarrow 0} h^{-2}\psi_0(h) = +\infty \quad \text{and} \quad \liminf_{h\downarrow 0} h^{-2}\psi_0(h) = -\infty \quad \text{a.s.} \tag{4.3.5}$$

Remark 4.3.2. (i) If ψ_0 has paths of unbounded variation then (4.3.5) always holds by (4.3.3).

(ii) If ψ_0 has paths of bounded variation then by (4.3.1) Hypothesis **B** implies that ψ_0 has zero drift coefficient (but we are already assuming this is true throughout). In fact, Bertoin et al. have fully characterized which bounded variation Lévy processes satisfy (4.3.5) [13, Theorem 3.2] (clearly it is necessary to at least have $\Pi((-\infty, 0)) = \Pi((0, \infty)) = \infty$).

(iii) Hypothesis **B** holds for stable processes with stability index $\alpha \in (1/2, 2]$.

(iv) Again by the strong Markov property, under Hypothesis **B** it follows that for all y such that $\psi_0(y) \neq \psi_0(y-)$,

$$\begin{aligned} \limsup_{h\downarrow 0} h^{-2}(\psi_0(y+h) - \psi_0(y)) = +\infty \quad \text{a.s.} \quad \text{and} \\ \liminf_{h\downarrow 0} h^{-2}(\psi_0(y+h) - \psi_0(y)) = -\infty \quad \text{a.s.} \end{aligned} \tag{4.3.6}$$

The following assumption will be necessary for the advanced results in the bounded variation case. Recall that we have already assumed ψ_0 to have zero drift coefficient.

Assumption B. Suppose ψ_0 has paths of bounded variation.

(I) Let $T = \inf_{x\geq 0}\{\psi_0(x) - bx - \frac{1}{2}x^2 \geq s\}$ for some $b > 0$ and $s > 0$. Then on the set $\{T < \infty\}$ we have $\psi_0(T) - bT - \frac{1}{2}T^2 > s$ almost surely.

(II) Let $T = \inf_{x\geq 0}\{\psi_0(x) + bx - \frac{1}{2}x^2 \leq -s\}$ for some $b > 0$ and $s > 0$. Then on the set $\{T < \infty\}$ we have $\psi_0(T) + bT - \frac{1}{2}T^2 < s$ almost surely.

Remark 4.3.3. (i) By quasi-continuity of Lévy processes the conclusion still holds when $\psi_0(x)$ is replaced by $\psi_0(x) \vee \psi_0(x-)$ in the definitions of T.

(ii) (II) has an equivalent time reversed version: let $T = \inf_{x\leq 0}\{\psi_0(x) + bx - \frac{1}{2}x^2 \geq s\}$ for some $b > 0$ and $s > 0$. Then $\psi_0(T) + bT - \frac{1}{2}T^2 > s$ a.s.

4.3.4 Abrupt and eroded Lévy processes

Abrupt Lévy processes were introduced in Section 3.4, but for the reasons discussed at the start of this section we will reintroduce them her. We will also introduce their counterparts, eroded Lévy processes.

Definition 4.3.4. A Lévy process ψ_0 is *abrupt* if its paths have unbounded variation and almost surely for all m such that ψ_0 has a local maximum at m,

$$\liminf_{h\downarrow 0} h^{-1}(\psi_0(m-h) - \psi_0(m)) = +\infty \quad \text{and} \quad \limsup_{h\downarrow 0} h^{-1}(\psi_0(m+h) - \psi_0(m)) = -\infty.$$

The following theorem describes the local behavior of an abrupt Lévy process at arbitrary times (in a slightly different manner from Theorem 3.4.6). This result is an immediate corollary of the more general result [92, Theorem 2.6] once we use the fact that almost surely the paths of a Lévy processes cannot have both points of increase and points of decrease [37].

Theorem 4.3.5. *Let ψ_0 be a two sided abrupt Lévy process. Then, almost surely for all $x \in \mathbb{R}$, if*

$$\limsup_{h\downarrow 0} h^{-1}(\psi_0(x-h) - \psi_0(x-)) < \infty \quad \textit{and} \quad \limsup_{h\downarrow 0} h^{-1}(\psi_0(x+h) - \psi_0(x)) < \infty$$

then ψ_0 has a local supremum at x.

At the other end of the scale from abrupt processes are eroded processes, also introduced by Vigon [93].

Definition 4.3.6. A Lévy process ψ_0 is *eroded* if its paths have unbounded variation and almost surely for all m such that ψ_0 has a local maximum at m,

$$\liminf_{h\downarrow 0} h^{-1}(\psi_0(m-h)-\psi_0(m)) = 0 \quad \text{and} \quad \limsup_{h\downarrow 0} h^{-1}(\psi_0(m+h)-\psi_0(m)) = 0.$$

Vigon [93, Theorem 1.4] gives the following characterization of eroded processes (the result may also be deduced as a special case of Theorem 3.3.11).

Remark 4.3.7. A Lévy process ψ_0 with paths of unbounded variation is eroded if and only if

$$\int_0^1 x^{-1}\mathbb{P}\{\psi_0(x) \in [ax, bx]\}\, dx = \infty, \quad \forall a < 0 < b. \tag{4.3.7}$$

4.3.5 Randomized coterminal times

Randomized coterminal times were introduced by Millar in order to extend the set of times at which some sort of decomposition into two independent processes could take place [64]. Essentially they are last exit times from randomized sets. For example, the largest time at which the supremum of a Markov process $(\phi(x))_{x\geq 0}$ is achieved is the last exit time from the random interval $[\sup_x \phi(x), \infty)$.

In this subsection we assume $(\phi(x))_{x\geq 0}$ is a càdlàg strong Markov process with state space $(E, \mathcal{E})$, a locally compact metric space (in fact we will only use state space $([0,\infty)\times\mathbb{R}, \mathcal{B}([0,\infty)\times\mathbb{R}))$. Denote by $\mathcal{F}_x$ the sigma fields that are the right continuous completions of the natural sigma fields $\mathcal{F}_x^0 = \sigma\{\phi(y), y \leq x\}$, and let $\mathcal{F} = \bigvee_{x\geq 0}\mathcal{F}_x$. Let θ_x be the standard shift operator, so that $\phi(y)(\theta_x\omega) = \phi(x+y)(\omega)$ for every $y \geq 0$. Recall that a *random time* R is a $[0,\infty]$-valued $\mathcal{F}$-measurable random variable, and that a random time T is a *terminal time* if it is optional and $T = x + T\circ\theta_x$ on $\{T > x\}$. For a random time R, define

$$\mathcal{F}(R+) := \big\{F \in \mathcal{F} : \text{for all } x > 0, \text{ there exists } F_x \in \mathcal{F}_x \\ \text{such that } F \cap \{R < x\} = F_x \cap \{R < x\}\big\}.$$

Definition 4.3.8. Suppose we are given

- a measure space $(A, \mathfrak{U})$,
- a family of terminal times $\{T_a\}_{a\in A}$ such that $(a,\omega) \to T_a(\omega)$ is $\mathfrak{U}\times\mathcal{F}$-measurable,

· a measurable mapping Z from $(\Omega, \mathcal{F})$ to $(A, \mathfrak{U})$.

A random time R is a *randomized coterminal time* based on $(A, \mathfrak{U})$, $\{T_a\}_{a \in A}$, Z if

(I) for each $x \geq 0$ there is an $\mathcal{F}_x$-measurable A-valued random variable Z_x such that $Z = Z_x$ on the set $\{R \leq x\}$,

(II) for each $0 \leq y < x$ there exists $B(y, x) \in \mathcal{F}_x$ such that

$$\{y \leq R < x\} = B(y, x) \cap \{T_{Z(\omega)}(\theta_x \omega) = +\infty\}.$$

Note that by (I) the Z in (II) can be replaced by Z_x.

Example 4.3.9. Suppose $\lim_{x\to\infty} \psi_0(x) = -\infty$, then $R = \arg\sup\{\psi_0(x) : x \geq 0\}$ is a randomized coterminal time with $(A, \mathfrak{U}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $T_a = \inf\{x > 0 : \psi_0(x) \vee \psi_0(x-) \geq a\}$, $Z = \sup_{x \geq 0} \psi_0(x)$ and $Z_x = \sup_{y \leq x} \psi_0(y)$. Property (I) is immediate since if the supremum occurs before x, then it is equal to the supremum attained by $\psi(y)$ on $[0, x]$, and to see that property (II) holds note that

$$\begin{aligned} \{y < R \leq x\} &= \{ \text{ the supremum of } \psi_0 \text{ occurs in } (y, x] \ \} \\ &= \{ \ \psi_0 \text{ goes at least as high in } (y, x] \text{ as it did before time } y, \\ &\qquad \text{and never after } x \text{ goes as high as it did during } (y, x] \ \} \\ &= \{Z_{yx} \leq Z_x\} \cap \{T_{Z_x(\omega)}(\theta_x \omega) = +\infty\}, \end{aligned}$$

where $Z_{yx} = \sup_{y < w \leq x} \psi_0(w)$, so $B(y, x) = \{Z_{yx} \leq Z_x\}$ here.

The following result is [64, Theorem 3.4] and essentially says that for a randomized coterminal time R, conditional on $Z = z$, the post R process is 'just' the original process conditioned on $\{T_z = +\infty\}$, and is still Markovian. Note that Z is $\mathcal{F}(R+)$ measurable by (I).

Theorem 4.3.10. *Let $(\phi(x))_{x \geq 0}$ be a Hunt process, and R a randomized coterminal time based on $(A, \mathfrak{U})$, $\{T_a\}_{a \in A}$, Z. Then for bounded Borel f,*

$$\mathbb{E}\left(f(\phi(R + x)) | \mathcal{F}((R + y)+)\right) = \int f(b)\, H_{x-y}(Z; \phi(R + y), db), \quad 0 < y < x,$$

where $H_x(z; a, db) := \mathbb{P}^a(\phi(x) \in db)\mathbb{P}^b(T_z = \infty)/\mathbb{P}^a(T_z = \infty)$.

The next result is a combination of [64, Proposition 5.4] and [64, (a) following Proposition 5.4], where we have trivially extended the state space to include a deterministic element as well as a Lévy process. It gives conditions under which, conditionally given Z and $\phi(R)$, the post R process is independent of $\mathcal{F}(R+)$. The proof relies on the zero-one property of Lévy processes at local maxima or jump times.

Proposition 4.3.11. *Let $(\psi(x))_{x\geq 0}$ be a Lévy process and let $\phi(x) = (x, \psi(x))$ for $x \geq 0$. Let R be a randomized coterminal time for $(\phi(x))_{x\geq 0}$ based on $(A, \mathfrak{U})$, $\{T_a\}_{a\in A}$, Z. Suppose that $\mathbb{P}(R$ is the time of a local maximum of $\psi) = \mathbb{P}(R < \infty)$ or $\mathbb{P}(R$ is a jump time of $\psi) = \mathbb{P}(R < \infty)$. Then conditional on Z and $\phi(R)$, the post R process is independent of $\mathcal{F}(R+)$, and it is Markov with transitions $H_x(Z; a, db)$.*

4.4 Main Results

In this section we first present results in a general setting and then treat processes with paths of unbounded and bounded variation separately. Recall from Remark 4.3.2 that Hypothesis **B** is automatically satisfied when ψ_0 has paths of unbounded variation, and when ψ_0 has paths of bounded variation then by assumption the drift coefficient of ψ_0 is zero.

4.4.1 Unbounded and Bounded Variation

Lemma 4.4.1. *Let ψ_0 be a two-sided Lévy process satisfying satisfying Hypotheses **A** and **B**. The Lebesgue measure of $\{y \in \mathbb{R} : a(x) = y$ for some $x \in \mathbb{R}\}$ is zero a.s.*

Proof. By application of Fubini's theorem and stationarity it suffices to show that $\mathbb{P}(a(x) = 0$ for some $x \in \mathbb{R}) = 0$. Suppose there exists $x > 0$ such that $a(x) = 0$, then $\limsup_{h\downarrow 0} h^{-1}\psi_0(h) \leq -x < 0$. But this happens only on an event of probability zero by (4.3.1) or (4.3.3) for processes with bounded or unbounded variation respectively.

Similarly, if there exists $x < 0$ such that $a(x) = 0$, then $-\liminf_{h\uparrow 0} h^{-1}\psi_0(h) \leq -x < 0$. But this happens only on an event of probability zero by the time reversed versions of (4.3.1) and (4.3.3).

Finally, if $a(0) = 0$ then $\limsup_{h\downarrow 0} h^{-2}\psi_0(h) \leq 1 < \infty$, which by (4.3.5) only occurs on a set of probability zero. □

Recall that $u(x) = x - a(x)$ and that $\mathcal{A}_0 = \mathbf{cl}\{x \in \mathbb{R} : u(x) = 0\}$.

Lemma 4.4.2. *Let ψ_0 be a two-sided Lévy process satisfying Hypotheses **A** and **B**. Suppose there exists $a^-, a^+ \in \mathcal{A}_0$ such that $a^- < a^+$ and $a \notin \mathcal{A}_0$ for $a^- < a < a^+$. Then almost surely there exists $a^- < x_0 < a^+$ such that $u(x) > 0$ for all $a^- < x < x_0$ and $u(x) < 0$ for all $x_0 < x < a^+$.*

Proof. From (4.3.5) we have $a(0) > 0$ a.s. and hence

$$a_0^+ := \inf\{x \geq 0 : x \in \mathcal{A}_0\} > 0 \quad \text{a.s., and}$$
$$a_0^- := \inf\{x \geq 0 : -x \in \mathcal{A}_0\} > 0 \quad \text{a.s.,}$$

where we have applied time reversal to get the second inequality.

By stationarity, it suffices to show that the claim is true for $a^+ = a_0^+$ and $a^- = a_0^-$, since any almost sure behaviour of u over the interval (a_0^-, a_0^+) must be shared by u over (a^-, a^+) for any two consecutive members $a^- < a^+$ of $\mathcal{A}_0$. Define

$$R := \inf\left\{y \geq 0 \,:\, \psi_0(y-x) \vee \psi_0((y-x)-) - \psi_0(y) \vee \psi_0(y-) \leq \tfrac{1}{2}x^2 \text{ for all } x > 0\right\}.$$

Since R is a stopping time, (4.3.5) implies that $R < a(R)$ a.s., and hence $u(R) < 0$ a.s. So we cannot have $u(x) \geq 0$ for all $x \in (a_0^-, a_0^+)$. A time reversal argument then implies that we cannot have $u(x) \leq 0$ for all $x \in (a_0^-, a_0^+)$.

Since $a(x)$ is non-decreasing, u has only downwards jumps, and thus u cannot go from being negative to positive without passing through zero. Hence we have the a.s. existence of the x_0 in the claim. □

The proof of the following Theorem is in Section 4.5.

Theorem 4.4.3. *Let ψ_0 be an abrupt two-sided Lévy process with paths of unbounded variation satisfying Hypothesis **A** or a two-sided Lévy process with paths of bounded variation satisfying Hypotheses **A** and **B** and Assumption **B**. Define*

$$T := \inf\{x \geq 0 \,:\, x \in \mathcal{A}_0\}.$$

Then $(\psi_0(T+x)-\psi_0(T))_{x\geq 0}$ is independent of $(\psi_0(T-x))_{x\geq 0}$. As a consequence, the processes $(u(T+x))_{x\geq 0}$ and $(u(T-x))_{x\geq 0}$ are independent and $\mathcal{A}_0$ is a regenerative set.

It is important to relate $\mathcal{A}_0$ to the set of Lagrangian regular points, when such points exist. As we shall see in Theorem 4.4.15, when ψ_0 is a two-sided Lévy process with paths of bounded variation satisfying Hypotheses **A** and **B** and Assumption **B**, $\mathcal{A}_0$ is exactly equal to the set of Lagrangian regular points.

4.4.2 Unbounded variation

Lemma 4.4.4. *Let ψ_0 be a two-sided Lévy process satisfying Hypothesis **A** with paths of unbounded variation. Then ψ_0 is continuous at every point in the set $\{y \in \mathbb{R} : a(x) = y$ for some $x \in \mathbb{R}\}$ a.s.*

Proof. From (4.3.4) and a time reversal argument, it follows that almost surely for every y such that y is a jump time of ψ_0, i.e. $\psi_0(y) \neq \psi_0(y-)$,

$$\limsup_{h\downarrow 0} h^{-1}(\psi_0(y+h)-\psi_0(y)) = +\infty \quad \text{and} \quad \limsup_{h\downarrow 0} h^{-1}(\psi_0(y-h)-\psi_0(y-)) = +\infty.$$

If $y = a(x)$ or $y = a(x-)$ for some x, and if y is such that say $\psi_0(y) > \psi_0(y-)$, then for every $h > 0$ we have

$$\psi_0(y) - \tfrac{1}{2}(x-y)^2 \geq \psi_0(y+h) - \tfrac{1}{2}(x-y-h)^2.$$

Therefore we would have

$$\limsup_{h \downarrow 0} h^{-1}(\psi_0(y+h) - \psi_0(y)) \leq y - x < \infty,$$

which is impossible, except on an event with probability zero. The case of a negative jump is similar, working now at the left of the jump. □

Corollary 4.4.5. *Let ψ_0 be a two-sided abrupt Lévy process satisfying Hypothesis $\boldsymbol{A}$. Then ψ_0 has a local supremum at every point in $\{y \in \mathbb{R} : a(x) = y$ for some $x \in \mathbb{R}\}$ a.s.*

Proof. Take any point $y \in \mathcal{A}$ and let x be such that $a(x) = y$. Then for every $z \geq 0$ we have

$$\psi_0(y+z) \vee \psi_0((y+z)-) - \psi_0(y) \vee \psi_0(y-) \leq \tfrac{1}{2}(z-(x-y))^2 - \tfrac{1}{2}(x-y)^2.$$

Recall from Lemma 4.4.4 that ψ_0 is a.s. continuous at y, thus almost surely

$$\limsup_{h \downarrow 0} h^{-1}(\psi_0(x-h) - \psi_0(x-)) \leq (x-y)$$

and

$$\limsup_{h \downarrow 0} h^{-1}(\psi_0(x+h) - \psi_0(x)) \leq -(x-y)\,.$$

Theorem 4.3.5 then implies that ψ_0 must have a local supremum at y. □

For abrupt Lévy processes, the shock structure is discrete.

Theorem 4.4.6. *Let ψ_0 be a two-sided abrupt Lévy process satisfying Hypothesis $\boldsymbol{A}$. Then $\mathcal{A}$ is a discrete set a.s.*

Proof. Because the random set $\mathcal{A}$ is stationary (i.e. its law is invariant by translation), we have to prove that $\#\{[1,2] \cap \mathcal{A}\} < \infty$ a.s. It is easy to verify that the probability that $a(x) \in [1,2]$ for some x with $|x| > n$ goes to zero as $n \to \infty$, so it suffices in fact to establish that for each fixed n larger than some n_0,

$$\#\{a(x) \in [1,2] : |x| \leq n\} < \infty \quad \text{a.s.} \tag{4.4.1}$$

Suppose first that $\mathbb{E}|\psi_0(1)| < \infty$. Let n_0 be large enough such that $|\mathbb{E}\psi_0(1)| < 2n_0$. Now, if a point $y \in [1,2]$ can be expressed as $y = a(x)$ for some $x \in [-n, n]$, then

$$\psi_0(y \pm h) \vee \psi_0((y \pm h)-) < \psi_0(y) \vee \psi_0(y-) + 2nh \quad \text{for every } h \in (0,2],$$

and since by Lemma 4.4.4 ψ_0 is continuous at y,

$$\psi_0(y \pm h) < \psi_0(y) + 2nh \quad \text{for every } h \in (0,2].$$

Defining the Lipschitz majorant of ψ_0 equivalently to how the Lipschitz minorant of ψ_0 is defined in Chapter 3, we have that if ψ_0 is continuous at y, then y is in the contact set of the $2n$-Lipschitz majorant of ψ_0 if and only if $\psi_0(y \pm h) - \psi_0(y) < 2nh$ for all $h \in \mathbb{R}$ (note that the existence of the Lipschitz majorant follows from our assumption that $|\mathbb{E}\psi_0(1)| < 2n_0 \leq 2n$). From Theorem 3.3.8 it follows that there are only finitely many such contact points y in the interval $[1,2]$ almost surely. Suppose it were the case that

$$\mathbb{P}(\#\{a(x) \in [1,2] \,:\, |x| \leq n\} = \infty) > 0.$$

Then with positive probability there would exist $y_\ell, y_r \in [1,2]$ such that $y_\ell < y_r$ and $\#\{a(x) \in [y_\ell, y_r] \,:\, |x| \leq n\} = \infty$. Moreover, by the law of large numbers applied to the left and to the right, with positive probability there would exist such a pair with both y_ℓ and y_r in the contact set of the $2n$-Lipschitz majorant of ψ_0. If both y_ℓ and y_r were in the contact set of the majorant, then every element of the infinite set $\{a(x) \in [y_\ell, y_r] \,:\, |x| \leq n\}$ would also be in the contact set of the majorant, but that is an event with zero probability. Hence $\#\{a(x) \in [1,2] \,:\, |x| \leq n\} < \infty$ a.s.

Now remove the assumption that $\mathbb{E}|\psi_0(1)| < \infty$. For each $N \in \mathbb{N}$ define the two-sided Lévy process $\tilde{\psi_0}^N$ by

$$\tilde{\psi_0}^N(x) = \left\{ \begin{array}{ll} \psi_0(x) - \displaystyle\sum_{\substack{0 \leq y \leq x: \\ \psi_0(y) \neq \psi_0(y-)}} (\psi_0(y) - \psi_0(y-)) 1_{|\psi_0(y) - \psi_0(y-)| > N} & \text{for } x \geq 0 \\ \psi_0(x) + \displaystyle\sum_{\substack{0 \leq y \leq x: \\ \psi_0(y) \neq \psi_0(y-)}} (\psi_0(y) - \psi_0(y-)) 1_{|\psi_0(y) - \psi_0(y-)| > N} & \text{for } x < 0 \end{array} \right\}$$

so that $\tilde{\psi_0}^N$ is identical to ψ_0 but with all the jumps of magnitude greater than N removed. Let $\tilde{\mathcal{A}}^N$ be defined in the same way that $\mathcal{A}$ is for the original process ψ_0. Since $\mathbb{E}|\tilde{\psi_0}^N| < \infty$ the above arguments imply that $\tilde{\mathcal{A}}^N \cap [1,2]$ is a finite set almost surely for every N.

From the fact that $\Pi(N, \infty) < \infty$ for every $N \in \mathbb{N}$, and the hypothesis that $\psi_0(x) = o(x^2)$ a.s. as $|x| \to \infty$, it follows that almost surely there exists a random $\tilde{N} \in \mathbb{N}$ such that $\mathcal{A} \cap [1,2] = \tilde{\mathcal{A}}^{\tilde{N}} \cap [1,2]$. Hence $\mathcal{A} \cap [1,2]$ is a finite set almost surely. □

Corollary 4.4.7. *Let ψ_0 be a two-sided abrupt Lévy process satisfying Hypothesis $\boldsymbol{A}$. Then $\{y \in \mathbb{R} : a(x) = y$ for some $x \in \mathbb{R}\}$ is closed a.s.*

The proof of the following theorem closely follows the proof of [19, Theorem 5] with Cauchy processes replaced by eroded processes.

Theorem 4.4.8. *Let ψ_0 be a two-sided eroded Lévy process satisfying Hypothesis $\boldsymbol{A}$. Then with probability one there are no rarefaction intervals.*

Proof. Recall from Lemma 4.4.4 that jump times of ψ_0 do not belong to $\mathcal{A}$ almost surely. Now suppose (x, x') is a rarefaction interval, that is $a(\cdot)$ stays constant on $[x, x')$; denote its value by y. As y is not a jump time of ψ_0, we have for all $h > 0$,

$$\psi_0(y) - \tfrac{1}{2}(x-y)^2 \geq \psi_0(y-h) - \tfrac{1}{2}(x-y+h)^2,$$
$$\psi_0(y) - \tfrac{1}{2}(x'-y)^2 \geq \psi_0(y+h) - \tfrac{1}{2}(x'-y-h)^2.$$

We deduce that

$$\liminf_{h \downarrow 0} h^{-1}(\psi_0(y) - \psi_0(y-h)) \geq y - x,$$
$$\limsup_{h \downarrow 0} h^{-1}(\psi_0(y+h) - \psi_0(y)) \leq y - x'.$$

Since $x < x'$, we can find a rational number $q \in (y - x', y - x)$. Then y is the location of a local maximum of $(\psi_0^{(q)}(x))_{x \in \mathbb{R}}$, where $\psi_0^{(q)}(x) := \psi_0(x) - qx$, and moreover

$$\liminf_{h \downarrow 0} h^{-1}(\psi_0^{(q)}(y) - \psi_0^{(q)}(y+h)) > 0. \tag{4.4.2}$$

On the other hand, the family $\left(\psi_0^{(s)}, s \in \mathbb{Q}\right)$ is a countable family of eroded processes. For each of these processes, with probability one, for any $s \in \mathbb{Q}$ and any location μ of a local maximum for $\psi_0^{(s)}$,

$$\liminf_{h \downarrow 0} h^{-1}(\psi_0^{(q)}(\mu) - \psi_0^{(q)}(\mu+h)) = 0.$$

We conclude that (4.4.2) is impossible, except on an event of probability zero, and therefore almost surely there are no rarefaction intervals. □

4.4.3 Bounded variation

Theorem 4.4.9. *Let ψ_0 be a two-sided Lévy process satisfying Hypothesis **A** with paths of bounded variation. Suppose zero is regular for $[0,\infty)$ and $(-\infty,0]$ for $(\psi_0(x))_{x\geq 0}$, then a.s. Lagrangian regular points exist.*

Proof. We shall prove that the time of the maximum of $(\psi_0(x))_{0\leq x\leq 1}$ has positive probability of being Lagrangian regular, and as pointed out by Bertoin in a comment before the proof of Theorem 3 of [19], it is easy to deduce from this fact that Lagrangian regular points exist with probability one. This is because of stationarity and the asymptotic independence of the events A_0 and A_n as $n\to\infty$, where $A_n := \{\arg\sup_{n\leq x\leq n+1}\psi_0(x)$ is Lagrangian regular$\}$ for $n\geq 0$.

Let μ be the almost surely unique location of the maximum of $(\psi_0(x))_{0\leq x\leq 1}$. It follows from the concave majorant theory of Pitman and Uribe-Bravo [73] that $\mu\in(0,1)$, and that if $\bar{B}:[0,1]\to\mathbb{R}$ denotes the concave majorant of $(\psi_0(x))_{0\leq x\leq 1}$ then its derivative $\bar{b}=\bar{B}'$ is continuous at μ and

$$\bar{b}(\mu+h)<\bar{b}(\mu)=0<\bar{b}(\mu-h) \tag{4.4.3}$$

for every sufficiently small $h>0$.

The rest of the argument is exactly as in the proof of Theorem 3 of [19].

(4.4.3) implies that the support of the Stieltjes measure $-d\bar{b}$ contains μ, and more precisely μ is neither isolated to the left nor to the right in $\mathrm{Supp}(d\bar{b})$. Pick any $y\in\mathrm{Supp}(d\bar{b})$ arbitrarily close to μ. Clearly, the graph of $\bar{B}$ touches that of $(\psi_0(x))_{0\leq x\leq 1}$ at y, so we must have $\bar{B}(y)=\psi_0(y)$ or $\bar{B}(y)=\psi_0(y-)$. In both cases, y is the location of a maximum of $x\to\psi_0(x)-\bar{b}(y)x$ on $[0,1]$, and a fortiori y is then the unique location of the maximum of $x\to\psi_0(x)-\frac{1}{2}(y-\bar{b}(y)-x)^2$ on $[0,1]$. Plainly, μ is also the unique location of the maximum of $x\to\psi_0(x)-\frac{1}{2}(\mu-x)^2$ on $[0,1]$. Because $\psi_0(\mu)>\max(\psi_0(0),\psi_0(1))$, there is a positive probability that the preceding two maxima are global (i.e. on $\mathbb{R}$) and not only local (i.e. on $[0,1]$). We conclude that with positive probability, $\mu\in\mathcal{A}$ and is neither isolated on its right nor on its left, and therefore is a Lagrangian regular point. □

The next two results are due to Lachiéze-Rey [55, Theorem 4.3, Proposition 5.3] and allow us to find the behaviour of ψ_0 around points of $\mathcal{A}$ in Proposition 4.4.12.

Theorem 4.4.10. *Let ψ_0 be a two-sided Lévy process with paths of bounded variation. Let $\bar{C}:[0,1]\to\mathbb{R}$ denote the concave majorant of $(\psi_0(x)-\frac{1}{2}x^2)_{x\in\mathbb{R}}$ and denote its derivative by $\bar{c}=\bar{C}'$. Then for all $a\in\mathcal{A}$, a is left isolated (resp. right isolated) in $\mathcal{A}$ if $\bar{c}(a-)\neq -a$ (resp. $\bar{c}(a)\neq -a$).*

Proposition 4.4.11. *Let ψ_0 be a two-sided Lévy process with paths of bounded variation. Suppose $y \in \mathcal{A}$ and x is such that $a(x) = y$. Then almost surely if $x < y$ then $\psi_0(y-) < \psi_0(y)$ and if $x > y$ then $\psi_0(y-) > \psi_0(y)$.*

Proposition 4.4.12. *Let ψ_0 be a two-sided Lévy process satisfying Hypothesis **A** with paths of bounded variation. Then for every $y \in \mathcal{A}$, almost surely*

(i) if $\psi_0(y-) < \psi_0(y)$ then y is left isolated in $\mathcal{A}$, and if $\psi_0(y-) > \psi_0(y)$ then y is right isolated in $\mathcal{A}$;

(ii) if y is Lagrangian regular then ψ_0 is continuous at y;

(iii) if $y \neq a(y)$ then y is isolated in $\mathcal{A}$.

Proof. (i) Suppose $\psi_0(y-) < \psi_0(y)$. Then y will be left isolated in the support of the Stieltjes measure $-d\bar{c}$, and hence will be left isolated in $\mathcal{A}$ by Lemma 4.2.1. The argument is similar for the case $\psi_0(y-) > \psi_0(y)$.

(ii) Suppose ψ_0 is not continuous at y. Then either $\psi_0(y-) < \psi_0(y)$ or $\psi_0(y-) > \psi_0(y)$ and hence (i) implies that y cannot be Lagrangian regular.

(iii) By hypothesis, for any x such that $a(x) = y$, we must have $x > y$ or $x < y$. Suppose $x < y$. Proposition 4.4.11 implies that $\psi_0(y-) < \psi_0(y)$ and thus y will be left isolated in $\mathcal{A}$ by (i). Moreover, $a(x) = y$ implies that $-\bar{c}(y) \leq x < y$, thus y will be right isolated in $\mathcal{A}$ by Theorem 4.4.10. The argument is similar in the alternative case $x > y$. □

Proposition 4.4.12 (iii) immediately leads to the following corollary.

Corollary 4.4.13. *Let ψ_0 be a two-sided Lévy process satisfying Hypothesis **A** with paths of bounded variation. Then the set $\{x \in \mathbb{R} : a(x) = x\}$ is closed a.s.*

Theorem 4.4.10 also allows us to prove the following two theorems.

Theorem 4.4.14. *Let ψ_0 be a two-sided Lévy process with paths of bounded variation satisfying Hypothesis **A**, Hypothesis **B** and Assumption **B**(I). Then the set $\{y \in \mathbb{R} : a(x) = y \text{ for some } x \in \mathbb{R}\}$ is closed a.s. and hence is equal to $\mathcal{A}$ a.s.*

Proof. Since in the definition of $a(x)$ we take the supremum over all possible arg sups, we have that

$$\bar{c}(y-) > \bar{c}(y+h) \ \forall\ h > 0 \quad \Longleftrightarrow \quad \exists\ x \text{ s.t. } a(x) = y\ . \tag{4.4.4}$$

Suppose y is a right accumulation point of the set $\{y \in \mathbb{R} : a(x) = y \text{ for some } x \in \mathbb{R}\}$ so that there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ with $y_n \downarrow y$ and $\bar{c}(y_n-) > \bar{c}(y_n + h)$ for all $h > 0$ and hence $y \in \{y \in \mathbb{R} : a(x) = y \text{ for some } x \in \mathbb{R}\}$.

Now suppose y is a left but not a right accumulation point. Then by Lemma 4.2.1 there exists $\hat{y}$ such that $\bar{c}(y+h) = \bar{c}(y+)$ for all $0 \le h < \hat{y} - y$ and such that

$$\psi_0(\hat{y}) \vee \psi_0(\hat{y}-) - \frac{1}{2}\hat{y}^2 = \bar{C}(\hat{y}) \tag{4.4.5}$$

i.e. $\hat{y}$ is the next contact point after y for the concave majorant of $(\psi(x) - \frac{1}{2}x^2)_{x\in\mathbb{R}}$.

Take any $q \in \mathbb{Q}$ such that $y < q < \hat{y}$. Let $\bar{C}^q : (-\infty, q] \to \mathbb{R}$ be the concave majorant of $\left(\psi(x) - \frac{1}{2}x^2\right)_{x\le q}$ and let $\bar{c}^q$ be its right continuous derivative, which will agree with $\bar{c}$ on the set $(-\infty, y)$. Define

$$E^q := \{x \le q \,:\, \bar{c}^q(x-) = -x\}\,.$$

Since $x \in E^q$ implies that at least one of $\limsup_{h\downarrow 0} h^{-1}(\psi_0(x+h) - \psi_0(x))$ or $\limsup_{h\downarrow 0} h^{-1}(\psi_0(x-h) - \psi_0(x-))$ is finite, (4.3.5) and Fubini imply that E^q has measure zero almost surely. Also, by Theorem 4.4.10 we know that $y \in E^q$ a.s. since y is not isolated on the left.

The outline of the rest of the argument is as follows. For each $x \in E^q$ we will define a random time that essentially is the first time the process $(\psi_0(q+z) - \frac{1}{2}(q+z)^2)_{z\ge 0}$ is greater than or equal to the line extending out from x with slope $-x$, i.e. the same slope as the concave majorant at x. This time should be the next time the process $(\psi_0(y) - \frac{1}{2}y^2)_{y\in\mathbb{R}}$ meets its concave majorant after y, but using Assumption **B** we show that it goes strictly above that line at that time, which leads to a contradiction.

For every $x \in E^q$ define $T^q(x) :=$

$$\inf\left\{z \ge 0 : \psi_0(q+z) \vee \psi_0((q+z)-) - \tfrac{1}{2}(q+z)^2 \ge \psi_0(x) \vee \psi_0(x-) - x\left((q-x)+z\right)\right\}$$

and note that almost surely $T^q(x) > 0$ for every $x \in E^q$ since by (4.3.5) $\bar{C}(q) > \psi_0(q) \vee \psi_0(q-) - \frac{1}{2}q^2$ a.s. Also, by definition $T^q(y) = \hat{y}$.

Assumption **B**(I) and the fact that E^q has measure zero a.s. imply that a.s.

$$\psi_0(T^q(x)) - \tfrac{1}{2}(q + T^q(x))^2 \ge \psi_0(x) \vee \psi_0(x-) - y\left((q-x) + T^q(x)\right) \tag{4.4.6}$$

for every $x \in E^q$ such that $T^q(x) < \infty$. But $y \in E^q$ and $T^q(y) = \hat{y}$ a.s. hence (4.4.5) and (4.4.7) would imply that $\hat{y} = \infty$, and thus y cannot be as assumed a left accumulation point and isolated on the right. Since we have shown the points not isolated on the right are included in the set $y \in \{y \in \mathbb{R} : a(x) = y \text{ for some } x \in \mathbb{R}\}$, this concludes the proof. □

Theorem 4.4.15. *Let ψ_0 be a two-sided Lévy process with paths of bounded variation satisfying Hypothesis **A**, Hypothesis **B** and Assumption **B**(II). Then for every $y \in \mathbb{R}$, $y = a(y)$ if and only if y is a Lagrangian regular point. Hence $\mathcal{A}_0$ is exactly the set of Lagrangian regular points.*

Proof. Suppose $y = a(x)$ is a Lagrangian regular point, which from Theorem 4.4.10, is possible only if $\bar{c}(y-) = -y = \bar{c}(y)$. Since y is isolated neither on the left or the right in $\mathcal{A}$, Lemma 4.2.1 implies that

$$\bar{c}(y+h) < \bar{c}(y) = -y < \bar{c}(y-h)$$

for every $h > 0$. Thus

$$\left(\psi_0(y) \vee \psi_0(y-) - \tfrac{1}{2}y^2\right) - ys < \left(\psi_0(y+s) \vee \psi_0((y+s)-) - \tfrac{1}{2}(y+s)^2\right)$$

for all $s \neq 0$. Rearranging, we see that

$$\psi_0(y+s) \vee \psi_0((y+s)-) - \psi_0(y) \vee \psi_0(y-) - \tfrac{1}{2}s^2 > 0$$

for all $s \neq 0$. It follows that $y = a(y)$.

Conversely, suppose that $y = a(y)$. If y is right isolated in $\mathcal{A}$, then there exists $\hat{y} > y$ with $a(\hat{y}) = y$, and hence by Proposition 4.4.11 y is the time of a negative jump of ψ_0. However, this would imply that $y \notin \mathcal{A}$ by the time reversed version of (4.3.6), and hence y is not right isolated in $\mathcal{A}$ a.s.

If y is left isolated in $\mathcal{A}$ we do not yet know that there necessarily exists $\hat{y}$ such that $\hat{y} < y$ and $a(\hat{y}) = y$, because although $y = a(y)$ implies that $\arg\sup\{\psi_0(x) - \frac{1}{2}(x-y)^2 : x \in \mathbb{R}\} = y$, the supremum may not be achieved at a unique point. Once we have shown that the supremum is unique a.s. a similar argument to the right isolated case above would show that y is not left isolated in y a.s. and hence that $y = a(y)$ implies that y is a Lagrangian regular point.

Suppose then that the supremum is not achieved at a unique point. Define

$$\hat{y} := \sup\{z < y \,:\, \psi_0(z) \vee \psi_0(z-) - \tfrac{1}{2}(z-y)^2 = \psi_0(y) \vee \psi_0(y-)\}$$

so that $\hat{y}$ is supremal among points for where the supremum is attained other than y. Note that $\psi_0(\hat{y}) \vee \psi_0(\hat{y}-) - \frac{1}{2}(\hat{y}-y)^2 = \psi_0(y) \vee \psi_0(y-)$ (i.e. the supremum in the definition of $\hat{y}$ is attained). Also, since we have assumed that y is left isolated in $\mathcal{A}$ we must have $\hat{y} < y$.

Take any $q \in \mathbb{Q}$ with $\hat{y} < q < y$. The remainder of the argument is a time reversed analogue of the argument used in the proof of Theorem 4.4.14 with a slightly expanded definition of E^q. Let $\bar{C}_q : [q, \infty) \to \mathbb{R}$ be the concave majorant of $\left(\psi(x) - \frac{1}{2}x^2\right)_{x \geq q}$ and let $\bar{c}_q$ be its right continuous derivative, which will agree with $\bar{c}$ on the set (y, ∞). Define

$$E_q := \{x \geq q \,:\, \bar{c}_q(x-) \leq -x\,,\, \bar{c}_q(x) \geq x\}\,.$$

Since $x \in E_q$ implies that at least one of $\limsup_{h\downarrow 0} h^{-1}(\psi_0(x+h)-\psi_0(x))$ or $\limsup_{h\downarrow 0} h^{-1}(\psi_0(x-h)-\psi_0(x-))$ is finite, (4.3.5) and Fubini imply that E_q has measure zero almost surely. Also, since $a(y)=y$ it follows that $y \in E_q$.

For every $x \in E_q$ define $T_q(x) :=$

$$\inf\left\{z \geq 0 \,:\, \psi_0(q-z)\vee\psi_0((q-z)-)-\tfrac{1}{2}(q-z)^2 \geq \psi_0(x)\vee\psi_0(x-)-x\left((q-x)-z\right)\right\}$$

and note that almost surely $T_q(x)>0$ for every $x \in E_q$ since by (4.3.5) $\bar{C}(q) > \psi_0(q)\vee\psi_0(q-)-\frac{1}{2}q^2$ a.s. Also, by definition $T_q(y)=\hat{y}$.

By Assumption **B**(II) (its time reversed version – see Remark 4.3.3(ii)) and the fact that E_q has measure zero a.s. it follows that a.s.

$$\psi_0(T_q(x))-\tfrac{1}{2}(q-T_q(x))^2 \geq \psi_0(x)\vee\psi_0(x-)-y\left((q-x)-T_q(x)\right) \tag{4.4.7}$$

for every $x \in E_q$ such that $T_q(x)<\infty$. But $y \in E_q$ and $T_q(y)=\hat{y}$ a.s. hence (4.4.5) and (4.4.7) would imply that $\hat{y}=-\infty$, and thus $\hat{y}$ cannot exist as assumed. □

4.5 Proof of Theorem 4.4.3

4.5.1 Facts relating to the first non-negative element of $\mathcal{A}_0$

In this section, we prove some results relating to the first non-negative element of $\mathcal{A}_0$ when ψ_0 is a non-random càdlàg function satisfying $\lim_{|x|\to\infty} x^{-2}\psi_0(x)=0$. Define

$$\begin{aligned}
\mathbf{t} &:= \inf\{y \geq 0 \,:\, a(y)=y\}\\
&= \inf\left\{y \geq 0 \,:\, \arg\sup\left\{\psi_0(x)-\tfrac{1}{2}(x-y)^2 : x \in \mathbb{R}\right\}=y\right\},\\
&= \inf\Big\{y \geq 0 \,:\, \psi_0(y-x)\vee\psi_0((y-x)-)-\psi_0(y)\vee\psi_0(y-) \leq \tfrac{1}{2}x^2 \text{ for all } x>0 \text{ and}\\
&\qquad \psi_0(y+x)\vee\psi_0((y+x)-)-\psi_0(y)\vee\psi_0(y-) < \tfrac{1}{2}x^2 \text{ for all } x>0\Big\}
\end{aligned}$$

The last equality is because of the convention that if the arg sup above is not unique we take it to be the supremum over all suitable arguments. Define further

$$\begin{aligned}
\mathbf{r} &:= \inf\left\{y \geq 0 \,:\, \psi_0(y-x)\vee\psi_0((y-x)-)-\psi_0(y)\vee\psi_0(y-) \leq \tfrac{1}{2}x^2 \text{ for all } x>0\right\},\\
\mathbf{s} &:= \inf\left\{y \geq \mathbf{r} \,:\, \psi_0(y+x)\vee\psi_0((y+x)-)-\psi_0(y)\vee\psi_0(y-) < \tfrac{1}{2}x^2 \text{ for all } x>0\right\}.
\end{aligned}$$

Note that $0 \leq \mathbf{r} \leq \mathbf{s} \leq \mathbf{t}$.

Lemma 4.5.1. *Let ψ_0 be any càdlàg function with $\lim_{|x|\to\infty} x^{-2}\psi_0(x)=0$. Then the infimum in the definition of $\mathbf{s}$ is achieved, that is,*

$$\psi_0(\mathbf{s}+x)\vee\psi_0((\mathbf{s}+x)-)-\psi_0(\mathbf{s})\vee\psi_0(\mathbf{s}-) < \tfrac{1}{2}x^2 \textit{ for all } x>0\,. \tag{4.5.1}$$

Proof. Suppose that (4.5.1) did not hold. Then by the definition of $\mathbf{s}$ there would exist a strictly decreasing sequence $\{\mathbf{s}_n\}_{n\geq 0}$ such that $\lim_n \mathbf{s}_n = \mathbf{s}$ and

$$\psi_0(\mathbf{s}_n + x) \vee \psi_0((\mathbf{s}_n + x)-) - \psi_0(\mathbf{s}_n) \vee \psi_0(\mathbf{s}_n-) < \tfrac{1}{2}x^2 \text{ for all } x > 0 \tag{4.5.2}$$

for every $n \geq 0$.

For $n \geq 1$, (4.5.2) with $x = \mathbf{s}_{n-1} - \mathbf{s}_n$ gives

$$\psi_0(\mathbf{s}_n) \vee \psi_0(\mathbf{s}_n-) > \psi_0(\mathbf{s}_{n-1}) \vee \psi_0(\mathbf{s}_{n-1}-) - \tfrac{1}{2}(\mathbf{s}_{n-1} - \mathbf{s}_n)^2$$

and thus since $\sum_{m=1}^{n}(\mathbf{s}_{m-1} - \mathbf{s}_m)^2 \leq (\mathbf{s}_0 - \mathbf{s}_n)^2$ we have

$$\psi_0(\mathbf{s}_n) \vee \psi_0(\mathbf{s}_n-) > \psi_0(\mathbf{s}_0) \vee \psi_0(\mathbf{s}_0-) - \tfrac{1}{2}(\mathbf{s}_0 - \mathbf{s}_n)^2.$$

By right continuity of $\psi_0(\cdot)$ at $\mathbf{s}$, recalling that $\lim_n \mathbf{s}_n = \mathbf{s}$ we may take the limit as $n \to \infty$ to get that

$$\psi_0(\mathbf{s}) \geq \psi_0(\mathbf{s}_0) \vee \psi_0(\mathbf{s}_0-) - \tfrac{1}{2}(\mathbf{s}_0 - \mathbf{s})^2. \tag{4.5.3}$$

Now, since we have assumed that (4.5.1) does not hold, there exists $x^* > 0$ such that

$$\psi_0(\mathbf{s} + x^*) \vee \psi_0((\mathbf{s} + x^*)-) - \psi_0(\mathbf{s}) \vee \psi_0(\mathbf{s}-) \geq \tfrac{1}{2}(x^*)^2\,,$$

and moreover without loss of generality we can assume that $\mathbf{s}_0$ is such that $\mathbf{s}_0 < \mathbf{s}+x^*$. But then starting from (4.5.3) we get

$$\begin{aligned}
\psi_0(\mathbf{s}_0) \vee \psi_0(\mathbf{s}_0-) &\leq \psi_0(\mathbf{s}) \vee \psi_0(\mathbf{s}-) + \tfrac{1}{2}(\mathbf{s}_0 - \mathbf{s})^2 \\
&\leq \psi_0(\mathbf{s}) \vee \psi_0(\mathbf{s}-) + \tfrac{1}{2}(x^*)^2 - \tfrac{1}{2}((\mathbf{s} + x^*) - \mathbf{s}_0)^2 \\
&\leq \psi_0(\mathbf{s} + x^*) \vee \psi_0((\mathbf{s} + x^*)-) - \tfrac{1}{2}((\mathbf{s} + x^*) - \mathbf{s}_0)^2,
\end{aligned}$$

which contradicts (4.5.2) with $n = 0$ and $x = \mathbf{s} + x^* - \mathbf{s}_0$. □

Lemma 4.5.2. *Let ψ_0 be any càdlàg function with $\lim_{|x|\to\infty} x^{-2}\psi_0(x) = 0$. Then $\mathbf{s} = \mathbf{t}$.*

Proof. Recall that $0 \leq \mathbf{r} \leq \mathbf{s} \leq \mathbf{t}$. We will show that at $\mathbf{s}$ the conditions of $\mathbf{r}$ are still satisfied, i.e.

$$\psi_0(\mathbf{s} - x) \vee \psi_0((\mathbf{s} - x)-) - \psi_0(\mathbf{s}) \vee \psi_0(\mathbf{s}-) \leq \tfrac{1}{2}x^2 \text{ for all } x > 0 \tag{4.5.4}$$

which combined with (4.5.1) implies that $\mathbf{s} \geq \mathbf{t}$ and hence $\mathbf{s} = \mathbf{t}$.

Suppose first that $\mathbf{r} = \mathbf{s}$, then clearly (4.5.4) is satisfied and hence $\mathbf{s} = \mathbf{t}$. Assume therefore that $\mathbf{r} < \mathbf{s}$. We will begin by showing that (4.5.4) holds for all $0 < x \le \mathbf{s} - \mathbf{r}$. It suffices to show that if we define

$$\tau := \arg\sup\left\{\psi_0(\mathbf{s} - y) \vee \psi_0((\mathbf{s} - y)-) - \tfrac{1}{2}y^2 : 0 \le y \le \mathbf{s} - \mathbf{r}\right\} \tag{4.5.5}$$

then we must have $\tau = 0$. Well, (4.5.5) implies that

$$\psi_0(\mathbf{s} - \tau) \vee \psi_0((\mathbf{s} - \tau)-) - \tfrac{1}{2}\tau^2 \ge \psi_0(\mathbf{s} - y) \vee \psi_0((\mathbf{s} - y)-) - \tfrac{1}{2}y^2$$

for all $0 \le y \le \tau$. Making the change of variables $y = \tau - x$, we see that

$$\psi_0(\mathbf{s} - \tau + x) \vee \psi_0((\mathbf{s} - \tau + x)-) - \psi_0(\mathbf{s} - \tau) \vee \psi_0((\mathbf{s} - \tau)-) \le \tfrac{1}{2}x^2 - x\tau$$

for all $0 \le x \le \tau$. Suppose that $\tau > 0$, so that

$$\psi_0(\mathbf{s} - \tau + x) \vee \psi_0((\mathbf{s} - \tau + x)-) - \psi_0(\mathbf{s} - \tau) \vee \psi_0((\mathbf{s} - \tau)-) < \tfrac{1}{2}x^2$$

for all $0 < x \le \tau$. Combined with (4.5.1) this would imply that

$$\psi_0(\mathbf{s} - \tau) \vee \psi_0((\mathbf{s} - \tau)-) + \tfrac{1}{2}x^2 > \psi((\mathbf{s} - \tau) + x) \text{ for all } x > 0\,.$$

But then since $\mathbf{s} - \tau \ge \mathbf{r}$, the definition of $\mathbf{s}$ would then imply that $\mathbf{s} \le \mathbf{s} - \tau < \mathbf{s}$, a clear contradiction. Hence $\tau = 0$ as required.

It remains to show that (4.5.4) holds for all $x > \mathbf{s} - \mathbf{r}$. Applying (4.5.4) at $x = \mathbf{s} - \mathbf{r}$ we see that

$$\psi_0(\mathbf{r}) \vee \psi_0(\mathbf{r}-) - \psi_0(\mathbf{s}) \vee \psi_0(\mathbf{s}-) \le \tfrac{1}{2}(\mathbf{s} - \mathbf{r})^2.$$

From the definition of $\mathbf{r}$,

$$\psi_0(\mathbf{r} - y) \vee \psi_0((\mathbf{r} - y)-) - \psi_0(\mathbf{r}) \vee \psi_0(\mathbf{r}-) \le \tfrac{1}{2}y^2$$

for all $y > 0$, and hence

$$\psi_0(\mathbf{r} - y) \vee \psi_0((\mathbf{r} - y)-) - \psi_0(\mathbf{s}) \vee \psi_0(\mathbf{s}-) \le \tfrac{1}{2}y^2 + \tfrac{1}{2}(\mathbf{s} - \mathbf{r})^2 < \tfrac{1}{2}((\mathbf{s} - \mathbf{r}) + y)^2$$

for all $y > 0$. Applying the change of variables $x = (\mathbf{s} - \mathbf{r}) + y$ shows that (4.5.4) holds for all $x > \mathbf{s} - \mathbf{r}$ and hence completes the proof. □

Define $\mathbf{r}_0 := \mathbf{r}$, and for $k \ge 0$, define

$$\mathbf{r}_{k+1} := \mathbf{r}_k + \arg\sup\left\{\psi_0(\mathbf{r}_k + x) \vee \psi_0((\mathbf{r}_k + x)-) - \tfrac{1}{2}x^2 : x \ge 0\right\}\,,$$

where if the arg sup above is not unique we take it to be the supremum over all suitable arguments.

Lemma 4.5.3. *Let ψ_0 be any càdlàg function with $\lim_{|x|\to\infty} x^{-2}\psi_0(x) = 0$. Then $\mathbf{r}_k \to \mathbf{t}$.*

Proof. Note first that $\mathbf{r}^* := \lim_k \mathbf{r}_k$ exists since $\mathbf{r}_k$ is an increasing sequence. If there were a $k \geq 0$ such that $\mathbf{r}_k = \mathbf{t}$, then necessarily $\mathbf{r}_j = \mathbf{t}$ for all $j \geq k$, thus we henceforth assume there is no such k.

Suppose that there exists a $k \geq 0$ such that $\mathbf{r}_k < \mathbf{t} < \mathbf{r}_{k+1}$, then

$$\psi_0(\mathbf{t}) \vee \psi_0(\mathbf{t}-) - \tfrac{1}{2}(\mathbf{t} - \mathbf{r}_k)^2 \leq \psi_0(\mathbf{r}_{k+1}) \vee \psi_0(\mathbf{r}_{k+1}-) - \tfrac{1}{2}(\mathbf{r}_{k+1} - \mathbf{r}_k)^2. \tag{4.5.6}$$

From the definition of $\mathbf{t}$ it follows that

$$\psi_0(\mathbf{r}_{k+1}) \vee \psi_0(\mathbf{r}_{k+1}-) - \psi_0(\mathbf{t}) \vee \psi_0(\mathbf{t}-) - \tfrac{1}{2}(\mathbf{r}_{k+1} - \mathbf{t})^2 < 0.$$

Thus if equality held in (4.5.6) it would be the case that

$$(\mathbf{r}_{k+1} - \mathbf{r}_k)^2 < (\mathbf{r}_{k+1} - \mathbf{t})^2 + (\mathbf{t} - \mathbf{r}_k)^2 = (\mathbf{r}_{k+1} - \mathbf{r}_k)^2 - 2(\mathbf{r}_{k+1} - \mathbf{t})(\mathbf{t} - \mathbf{r}_k),$$

and hence the inequality in (4.5.6) must be strict. (4.5.6) then implies that

$$\begin{aligned} \psi_0(\mathbf{r}_{k+1}) \vee \psi_0(\mathbf{r}_{k+1}-) - \psi_0(\mathbf{t}) \vee \psi_0(\mathbf{t}-) - \tfrac{1}{2}(\mathbf{r}_{k+1} - \mathbf{t})^2 \\ > (\mathbf{r}_{k+1} - \mathbf{r}_k)^2 - (\mathbf{r}_{k+1} - \mathbf{t})^2 - (\mathbf{t} - \mathbf{r}_k)^2 > 0, \end{aligned}$$

which contradicts the definition of $\mathbf{t}$, and hence there is no k such that $\mathbf{r}_k < \mathbf{t} < \mathbf{r}_{k+1}$. Thus $\mathbf{r}^* \leq \mathbf{t}$.

Suppose $\mathbf{r}^* < \mathbf{t}$, then $\mathbf{r}^* < \mathbf{s}$ by Lemma 4.5.2, and hence there exists $r_+ > 0$ such that

$$\psi_0(\mathbf{r}^* + r_+) - \psi_0(\mathbf{r}^*) \vee \psi_0(\mathbf{r}^*-) - \tfrac{1}{2}r_+^2 > 0.$$

Let $r_- > 0$ be such that

$$\tfrac{1}{2}(r_+ + r_-)^2 = \psi_0(\mathbf{r}^* + r_+) - \psi_0(\mathbf{r}^*) \vee \psi_0(\mathbf{r}^*-).$$

Then for all k large enough such that $\mathbf{r}_k > \mathbf{r}^* - r_-$ we have

$$\tfrac{1}{2}((\mathbf{r}^* + r_+) - \mathbf{r}_k)^2 < \psi_0(\mathbf{r}^* + r_+) - \psi_0(\mathbf{r}^*) \vee \psi_0(\mathbf{r}^*-)$$

and hence

$$\mathbf{r}_{k+1} > \mathbf{r}_k + ((\mathbf{r}^* + r_+) - \mathbf{r}_k) = \mathbf{r}^* + r_+ > \mathbf{r}^*,$$

which is clearly a contradiction. Thus we can conclude that $\mathbf{r}^* = \mathbf{t}$. ∎

4.5.2 Randomized coterminal times relating first non-negative element of $\mathcal{A}_0$

In this section we will use the notation of Definition 4.3.8 when checking if a given random time is a randomized coterminal time.

Lemma 4.5.4. *Let ψ_0 be a real valued càdlàg strong Markov process. Define a sequence of random times by $R_0 = 0$, and for $k \geq 0$,*

$$\begin{aligned} R_{k+1} &:= R_k + \arg\sup\left\{\psi_0(R_k + x) \vee \psi_0((R_k + x)-) - \psi_0(R_k) \vee \psi_0(R_k-) - \tfrac{1}{2}x^2 : x \geq 0\right\} \\ &= R_k + \arg\sup\left\{\psi_0(R_k + x) \vee \psi_0((R_k + x)-) - \tfrac{1}{2}x^2 : x \geq 0\right\} , \end{aligned}$$

where if the $\arg\sup$ *above is not unique we take it to be the supremum over all suitable arguments. Define ϕ to be the process $(\phi(x))_{x\geq 0}$, with*

$$\phi(x) = (\phi_1(x), \phi_2(x)) := (x, \psi_0(x))$$

for all $x \geq 0$. Then R_k is a randomized coterminal time for ϕ for each $k \geq 1$.

Proof. Let $A = \mathbb{R}^3$, $\mathfrak{U} = \mathcal{B}(\mathbb{R}^3)$ and let $Z = (R_{k-1}, R_k, \psi_0(R_k) \vee \psi_0(R_k-))$. Let $R_0^{(x)} = 0$ and for $k \geq 0$, if $R_k^{(x)} < x$ then let

$$R_{k+1}^{(x)} := R_k^{(x)} + \arg\sup\left\{\psi_0(R_k^{(x)} + y) \vee \psi_0((R_k^{(x)} + y)-) - \tfrac{1}{2}y^2 : 0 \leq y \leq x - R_k^{(x)}\right\},$$

but if $R_k^{(x)} = x$ then let $R_{k+1}^{(x)} = x$. Let $Z_x = (R_{k-1}^{(x)}, R_k^{(x)}, \psi_0(R_k^{(x)}) \vee \psi_0(R_k^{(x)}-))$, so that Z_x is an $\mathcal{F}_x$-measurable A-valued random variable as required. Finally, recalling that $(\phi_1(x), \phi_2(x)) = (x, \psi_0(x))$, define the family of terminal times $\{T_a\}_{a\in A}$ by

$$T_{(a_1,a_2,a_3)} := \inf\{x > 0 : \phi_2(x) \vee \phi_2(x-) - a_3 + \tfrac{1}{2}(a_1 - a_2)^2 \geq \tfrac{1}{2}(\phi_1(x) - a_1)^2\}.$$

(I) and (II) follow once we define $B(y, x) := \{y \leq R_k^{(x)} < x\}$. □

Lemma 4.5.5. *Let ψ_0 be a real valued càdlàg strong Markov process and define*

$$F := \inf\left\{x \geq 0 \,:\, \psi_0(x + s) \vee \psi_0((x + s)-) - \psi_0(x) \vee \psi_0(x-) < \tfrac{1}{2}s^2 \text{ for all } s > 0\right\}.$$

Define ϕ to be the process $(\phi(x))_{x\geq 0}$, with

$$\phi(x) = (\phi_1(x), \phi_2(x)) := (x, \psi_0(x))$$

for all $x \geq 0$. Then F is a randomized coterminal time for ϕ.

Proof. Let $A = \mathbb{R}^2$, $\mathfrak{U} = \mathcal{B}(\mathbb{R}^2)$, $Z = (F, \psi_0(F) \vee \psi_0(F-))$ and $Z_x = (F_x, \psi_0(F_x) \vee \psi_0(F_x-))$, where

$$F_x := \inf\{0 \leq y \leq x : \psi_0(y+s) \vee \psi_0((y+s)-) - \psi_0(y) \vee \psi_0(y-) \leq \tfrac{1}{2}s^2 \text{ for all } 0 < s \leq x-y\}.$$

It follows that Z_x is an $\mathcal{F}_x$-measurable A-valued random variable. Finally, recalling that $(\phi_1(x), \phi_2(x)) = (x, \psi_0(x))$, define the family of terminal times $\{T_a\}_{a \in A}$ by

$$T_{(a_1,a_2)} := \inf\{x > 0 : \phi_2(x) \vee \phi_2(x-) - a_2 \geq \tfrac{1}{2}(\phi_1(x) - a_1)^2\}.$$

By definition,

$$\psi_0(F+s) \vee \psi_0((F+s)-) - \psi_0(F) \vee \psi_0(F-) < \tfrac{1}{2}s^2$$

for all $s > 0$, and

$$\psi_0(F_x+s) \vee \psi_0((F_x+s)-) - \psi_0(F_x) \vee \psi_0(F_x-) < \tfrac{1}{2}s^2$$

for all $0 < s \leq x - F_x$. In particular, if $F \leq x$, then

$$\psi_0(F) \vee \psi_0(F-) - \psi_0(F_x) \vee \psi_0(F_x-) < \tfrac{1}{2}(F - F_x)^2.$$

Hence we see that on the set $\{F \leq x\}$,

$$\psi_0(F_x+s) \vee \psi_0((F_x+s)-) - \psi_0(F_x) \vee \psi_0(F_x-) < \tfrac{1}{2}s^2$$

for all $s > 0$, which implies that $F \leq F_x$. However, $F \geq F_x$ by definition, and therefore $F_x = F$ on the set $\{F \leq x\}$. Thus (I) is satisfied.

If we define $B(y,x) := \{y \leq F_x < x\}$, then clearly

$$\{y \leq F < x\} = B(y,x) \cap \{T_{Z_x(\omega)}(\theta_x\omega) = +\infty\} = B(y,x) \cap \{T_{Z(\omega)}(\theta_x\omega) = +\infty\},$$

and hence (II) is satisfied. □

Corollary 4.5.6. *Let ψ_0 be a Lévy process and define F as in Lemma 4.5.5. Suppose that ψ_0 is continuous at F. Then for any $(x_1, \ldots, x_n)$ with $x_i > 0$ for $i = 1, \ldots, n$, the joint law of $(\psi_0(F + x_i) - \psi_0(F))_{i=\{1,\ldots,n\}}$ depends only on $(x_1, \ldots, x_n)$.*

Proof. From Theorem 4.3.10 we know that the joint law of $(\psi_0(F + x_i))_{i=\{1,\ldots,n\}}$ depends only on $(x_1, \ldots, x_n)$ and $Z = (F, \psi_0(F))$. Moreover we can think of the post F process $(\psi_0(F + x))_{x \geq 0}$ as the original process started at $\psi_0(F)$ but conditioned to remain below a half parabola with its minimum at $\psi_0(F) \vee \psi_0(F-) = \psi_0(F)$. Then by the spatial homogeneity of Lévy processes, the joint law of $(\psi_0(F + x_i) - \psi_0(F))_{i=\{1,\ldots,n\}}$ cannot depend on $\psi_0(F)$, and by the temporal homogeneity of Lévy processes it cannot depend on F either. Thus the joint law of $(\psi_0(F + x_i) - \psi_0(F))_{i=\{1,\ldots,n\}}$ can depend only on $(x_1, \ldots, x_n)$. □

4.5.3 Proof of Theorem 4.4.3

Proof. (Theorem 4.4.3) Recall from the statement of the theorem that $T := \inf\{x \geq 0 : x \in \mathcal{A}_0\}$ and hence $T = \inf\{x \geq 0 : a(x) = x\}$. From Corollary 4.4.13 in the bounded variation case or Theorem 4.4.6 in the abrupt case, we know that the set $\{x \in \mathbb{R} : a(x) = x\}$ is closed a.s. and hence $a(T) = T$ a.s.

If ψ_0 has paths of unbounded variation, then Lemma 4.4.4 implies that ψ_0 is continuous at T a.s. If ψ_0 has paths of bounded variation Theorem 4.4.15 implies that T is a Lagrangian regular point a.s. and then Proposition 4.4.12(ii) implies that ψ_0 is continuous at T a.s.

Define two further random variables R and S by

$$R := \inf\left\{y \geq 0 : \psi_0(y-x) \vee \psi_0((y-x)-) - \psi_0(y) \vee \psi_0(y-) \leq \tfrac{1}{2}x^2 \text{ for all } x > 0\right\},$$
$$S := \inf\left\{y \geq R : \psi_0(y+x) \vee \psi_0((y+x)-) - \psi_0(y) \vee \psi_0(y-) < \tfrac{1}{2}x^2 \text{ for all } x > 0\right\}.$$

Note that $0 \leq R \leq S \leq T$. Note also that by the strong Markov property applied at the stopping time R, it follows that $S - R$ has the same law as F in Lemma 4.5.5.

Lemma 4.5.2 tells us that $S = T$ a.s., and thus $T = R + (S - R)$ a.s. Since R is a stopping time, $(\psi_0(R+x) - \psi_0(R))_{x\geq 0}$ is independent of $(\psi_0(R-x))_{x\geq 0}$ and has the same law as $(\psi_0(x))_{x\geq 0}$. Since $S - R$ has the same law as F in Lemma 4.5.5, we only need to show that the process $(\psi_0(F+x)-\psi_0(F))_{x\geq 0}$ is independent of $(\psi_0(x))_{0\leq x\leq F}$ when ψ_0 is a.s. continuous at F, and when we can further assume that

$$\psi_0(x) - \tfrac{1}{2}x^2 \leq 0 \text{ for all } x \leq 0. \tag{4.5.7}$$

By continuity of ψ_0 at F we only need to show that $(\psi_0(F + x) - \psi_0(F))_{x\geq 0}$ is independent of $(\psi_0(x))_{0\leq x<F}$. Moreover, Corollary 4.5.6 implies that the law of $(\psi_0(F + x) - \psi_0(F))_{x\geq 0}$ cannot depend on F or $\psi_0(F)$, hence it is enough to show that

$$(\psi_0(F + x) - \psi_0(F))_{x\geq 0} \text{ is independent of } (\psi_0(x))_{0\leq x<F}, \\ \text{conditionally given } F \text{ and } \psi_0(F). \tag{4.5.8}$$

Suppose first that ψ_0 has paths of unbounded variation and is abrupt. From Corollary 4.4.5 ψ_0 must have a local maximum at F, and from Lemma 4.5.5 we know that F is a randomized coterminal time for the process $(x, \psi_0(x))_{x\geq 0}$, hence by Proposition 4.3.11 it follows that $(F + x, \psi_0(F + x) - \psi_0(F))_{x\geq 0}$ is independent of $(x, \psi_0(x))_{0\leq x\leq F}$ conditionally given $(F, \psi_0(F))$. Hence we have (4.5.8).

Now suppose that ψ_0 has paths of bounded variation. Define a sequence of random times by $R_0 = 0$, and for $k \geq 0$ define

$$R_{k+1} := R_k + \arg\sup\left\{\psi_0(R_k + x) \vee \psi_0((R_k + x)-) - \tfrac{1}{2}x^2 : x \geq 0\right\}.$$

From Lemma 4.5.3 and the fact that $S - R$ has the same law as F, we have that $R_k \to F$ a.s. Suppose we have shown that for each $k \geq 1$, the process $(\psi_0(R_k+x))_{x\geq 0}$ is independent of $(\psi_0(x))_{0\leq x\leq R_k}$ conditionally given R_k and $\psi_0(R_k)$. If $R_k = R_{k+1}$ for some k, then $R_k = T$ and we are done. Thus assume that $R_k < R_{k+1}$ for every k. We have that $R_k \to F$ a.s., and the a.s. continuity of ψ_0 at F implies that $\psi_0(R_k) \to \psi_0(F)$. Thus the process $(\psi_0(F + x))_{x\geq 0}$ is independent of $(\psi_0(x))_{0\leq x<R_k}$ conditionally given F and $\psi_0(F)$, and (4.5.8) follows.

It remains to show that for each $k \geq 1$, the process $(\psi_0(R_k + x))_{x\geq 0}$ is independent of $(\psi_0(x))_{0\leq x\leq R_k}$ conditionally given R_k and $\psi_0(R_k)$. Note that under (4.5.7), $a(R_k) = R_{k+1}$ for every $k \geq 0$. Since we have assumed that $R_k < R_{k+1}$ for every $k \geq 0$, it follows from Proposition 4.4.11 that R_{k+1} is a positive jump time of ψ_0 for every $k \geq 0$. From Lemma 4.5.4 we know that R_k is a randomized coterminal time for the process $(x, \psi_0(x))_{x\geq 0}$, hence by Proposition 4.3.11 it follows that $(R_k + x, \psi_0(R_k + x) - \psi_0(R_k))_{x\geq 0}$ is independent of $(x, \psi_0(x))_{0\leq x\leq R_k}$ conditionally given $(R_k, \psi_0(R_k))$. □

Remark 4.5.7. For processes with bounded variation satisfying Hypotheses **A** and **B**, Giraud's proof of the regenerativity of the set of Lagrangian regular points [40, Theorem 2] when ψ_0 is a stable Lévy process with stability index $\alpha \in (1/2, 1)$ could also be used to prove Theorem 4.4.3. Hypothesis **B** ensures that equation (7) of [40] holds appropriately, and Theorem 4.4.15 ensures that the first sentence of Lemma 4 of [40] is true. Those are the only two results needed in that proof.

Bibliography

[1] J. Abramson. *Structure of Shocks in Burgers Turbulence with Lévy Noise Initial Data*. http://arxiv.org/abs/1204.4752. Under review by Journal of Statistical Physics. 2012.

[2] J. Abramson and S. N. Evans. *Lipschitz minorants of Brownian Motion and Lévy processes*. http://arxiv.org/abs/1110.1105. Under review by Probability Theory and Related Fields. 2012.

[3] J. Abramson et al. "Convex minorants of random walks and Lévy processes". *Electronic Communications in Probability* 16 (2011), pp. 423–434.

[4] J. Abramson and J. Pitman. "Concave majorants of random walks and related Poisson processes". *Combin. Probab. Comput.* 20.5 (2011), pp. 651–682.

[5] E. S. Andersen. *On the distribution of the random variable H_n. Tech. Sci. Note No. 1, Contract No. AF 61(052)-42, February 27, 1959.*

[6] E. S. Andersen. "On the fluctuations of sums of random variables II". *Math. Scand.* 2 (1954), pp. 195–223.

[7] M. Avellaneda. "Statistical properties of shocks in Burgers turbulence. II. Tail probabilities for velocities, shock-strengths and rarefaction intervals". *Comm. Math. Phys.* 169.1 (1995), pp. 45–59.

[8] M. Avellaneda and W. E. "Statistical properties of shocks in Burgers turbulence". *Comm. Math. Phys.* 172.1 (1995), pp. 13–38.

[9] F. Balabdaoui and J. Pitman. "The distribution of the maximal difference between a Brownian bridge and its concave majorant". *Bernoulli* 17.1 (2011), pp. 466–483.

[10] E. J. Balder. "An extension of duality-stability relations to nonconvex optimization problems". *SIAM J. Control Optimization* 15.2 (1977), pp. 329–343.

[11] R. F. Bass. "Markov processes and convex minorants". In: *Seminar on probability, XVIII.* Vol. 1059. Lecture Notes in Math. Berlin: Springer, 1984, pp. 29–41.

[12] J. Bertoin. “The convex minorant of the Cauchy process”. *Electron. Comm. Probab.* 5 (2000), 51–55 (electronic).

[13] J. Bertoin, R. A. Doney, and R. A. Maller. “Passage of Lévy processes across power law boundaries at small times”. *Ann. Probab.* 36.1 (2008), pp. 160–197.

[14] J. Bertoin. “Large-deviations estimates in Burgers turbulence with stable noise initial data”. *J. Statist. Phys.* 91.3-4 (1998), pp. 655–667.

[15] J. Bertoin. *Lévy processes.* Vol. 121. Cambridge Tracts in Mathematics. Cambridge: Cambridge University Press, 1996.

[16] J. Bertoin. “Regenerative embedding of Markov sets”. *Probab. Theory Related Fields* 108.4 (1997), pp. 559–571.

[17] J. Bertoin. “Regularity of the half-line for Lévy processes”. *Bull. Sci. Math.* 121.5 (1997), pp. 345–354.

[18] J. Bertoin. “Renewal theory for embedded regenerative sets”. *Ann. Probab.* 27.3 (1999), pp. 1523–1535.

[19] J. Bertoin. “Structure of shocks in Burgers turbulence with stable noise initial data”. *Comm. Math. Phys.* 203.3 (1999), pp. 729–741.

[20] A. N. Borodin and P. Salminen. *Handbook of Brownian motion—facts and formulae.* Second. Probability and its Applications. Basel: Birkhäuser Verlag, 2002, pp. xvi+672.

[21] H. D. Brunk. “A generalization of Spitzer’s combinatorial lemma”. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 2 (1964), 395–405 (1964).

[22] J. M. Burgers. *The Nonlinear Diffusion Equation.* Boston: Reidel, 1974.

[23] C. Carolan and R. Dykstra. “Characterization of the least concave majorant of Brownian motion, conditional on a vertex point, with application to construction”. *Ann. Inst. Statist. Math.* 55.3 (2003), pp. 487–497.

[24] C. Carolan and R. Dykstra. “Marginal densities of the least concave majorant of Brownian motion”. *Ann. Statist.* 29.6 (2001), pp. 1732–1750.

[25] A. J. Chorin. *Lectures on turbulence theory.* Mathematics Lecture Series, No. 5. Boston, Mass.: Publish or Perish Inc., 1975, pp. iii+159.

[26] E. Çinlar. “Sunset over Brownistan”. *Stochastic Process. Appl.* 40.1 (1992), pp. 45–53.

[27] E. Çinlar. “Sunset over Brownistan”. *Stochastic Process. Appl.* 40.1 (1992), pp. 45–53.

[28] J. D. Cole. "On a quasi-linear parabolic equation occurring in aerodynamics". *Quart. Appl. Math.* 9 (1951), pp. 225–236.

[29] I. V. Denisov. "Random walk and the Wiener process considered from a maximum point". *Teor. Veroyatnost. i Primenen.* 28.4 (1983), pp. 785–788.

[30] P. Diaconis et al. "Metrics on compositions and coincidences among renewal sequences". In: *Random discrete structures (Minneapolis, MN, 1993).* Vol. 76. IMA Vol. Math. Appl. New York: Springer, 1996, pp. 81–101.

[31] H. Dietrich. "Zur c-Konvexität und c-Subdifferenzierbarkeit von Funktionalen". *Optimization* 19.3 (1988), pp. 355–371.

[32] R. A. Doney. *Fluctuation theory for Lévy processes.* Vol. 1897. Lecture Notes in Mathematics. Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005, Edited and with a foreword by Jean Picard. Berlin: Springer, 2007.

[33] K.-H. Elster and R. Nehse. "Zur Theorie der Polarfunktionale". *Math. Operationsforsch. Statist.* 5.1 (1974), pp. 3–21.

[34] A. Faggionato. "The alternating marked point process of h-slopes of drifted Brownian motion". *Stochastic Process. Appl.* 119.6 (2009), pp. 1765–1791.

[35] W. Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. New York: John Wiley & Sons Inc., 1971, pp. xxiv+669.

[36] P. J. Fitzsimmons and M. Taksar. "Stationary regenerative sets and subordinators". *Ann. Probab.* 16.3 (1988), pp. 1299–1305.

[37] S. Fourati. "Points de croissance des processus de Lévy et théorie générale des processus". *Probab. Theory Related Fields* 110.1 (1998), pp. 13–49.

[38] R. K. Getoor and M. J. Sharpe. "Last exit decompositions and distributions". *Indiana Univ. Math. J.* 23 (1973/74), pp. 377–404.

[39] C. Giraud. "Genealogy of shocks in Burgers turbulence with white noise initial velocity". *Comm. Math. Phys.* 223.1 (2001), pp. 67–86.

[40] C. Giraud. "On regular points in Burgers turbulence with stable noise initial data". *Ann. Inst. H. Poincaré Probab. Statist.* 38.2 (2002), pp. 229–251.

[41] A. Gnedin and J. Pitman. "Regenerative composition structures". *Ann. Probab.* 33.2 (2005), pp. 445–479.

[42] C. M. Goldie. "Records, permutations and greatest convex minorants". *Math. Proc. Cambridge Philos. Soc.* 106.1 (1989), pp. 169–177.

[43] P. Greenwood and J. Pitman. "Fluctuation identities for Lévy processes and splitting at the maximum". *Adv. in Appl. Probab.* 12.4 (1980), pp. 893–902.

[44] P. Greenwood and J. Pitman. "Fluctuation Identities for Random Walk by Path Decomposition at the Maximum". *Advances in Applied Probability* 12.2 (1980), pp. 291–293.

[45] P. Groeneboom. "The concave majorant of Brownian motion". *Ann. Probab.* 11.4 (1983), pp. 1016–1027.

[46] P. Hansen, B. Jaumard, and S.-H. Lu. "Global optimization of univariate Lipschitz functions. I. Survey and properties". *Math. Programming* 55.3, Ser. A (1992), pp. 251–272.

[47] P. Hansen, B. Jaumard, and S.-H. Lu. "Global optimization of univariate Lipschitz functions. II. New algorithms and computational comparison". *Math. Programming* 55.3, Ser. A (1992), pp. 273–292.

[48] J.-B. Hiriart-Urruty. "Extension of Lipschitz functions". *J. Math. Anal. Appl.* 77.2 (1980), pp. 539–554.

[49] J.-B. Hiriart-Urruty. "Lipschitz r-continuity of the approximate subdifferential of a convex function". *Math. Scand.* 47.1 (1980), pp. 123–134.

[50] E. Hopf. "The partial differential equation $u_t + uu_x = \mu u_{xx}$". *Comm. Pure Appl. Math.* 3 (1950), pp. 201–230.

[51] R. Horst and H. Tuy. *Global optimization.* Second. Deterministic approaches. Berlin: Springer-Verlag, 1993, pp. xvi+698.

[52] A. W. Janicki and W. A. Woyczynski. "Hausdorff dimension of regular points in stochastic Burgers flows with Lévy α-stable initial data". *J. Statist. Phys.* 86.1-2 (1997), pp. 277–299.

[53] M. Kac. "Toeplitz matrices, translation kernels and a related problem in probability theory". *Duke Math. J.* 21 (1954), pp. 501–509.

[54] O. Kallenberg. *Foundations of modern probability.* Second. Probability and its Applications (New York). New York: Springer-Verlag, 2002.

[55] R. Lachièze-Rey. "Concave majorant of stochastic processes and Burgers turbulence". *J. Theoret. Probab.* 25.2 (2012), pp. 313–332.

[56] V. L. Levin. "Abstract convexity in measure theory and in convex analysis". *J. Math. Sci. (N. Y.)* 116.4 (2003). Optimization and related topics, 4, pp. 3432–3467.

[57] Y. Lucet. "What shape is your conjugate? A survey of computational convex analysis and its applications". *SIAM J. Optim.* 20.1 (2009), pp. 216–250.

[58] B. Maisonneuve. “Ensembles régénératifs de la droite”. *Z. Wahrsch. Verw. Gebiete* 63.4 (1983), pp. 501–510.

[59] B. Maisonneuve. “Ensembles régénératifs, temps locaux et subordinateurs”. In: *Séminaire de Probabilités, V (Univ. Strasbourg, année universitaire 1969–1970).* Berlin: Springer, 1971, 147–169. Lecture Notes in Math., Vol. 191.

[60] P. W. Millar. “A path decomposition for Markov processes”. *Ann. Probability* 6.2 (1978), pp. 345–348.

[61] P. W. Millar. “Exit properties of stochastic processes with stationary independent increments”. *Trans. Amer. Math. Soc.* 178 (1973), pp. 459–479.

[62] P. W. Millar. “Zero-one laws and the minimum of a Markov process”. *Trans. Amer. Math. Soc.* 226 (1977), pp. 365–391.

[63] P. W. Millar. “Random times and decomposition theorems”. In: *Probability (Proc. Sympos. Pure Math., Vol. XXXI, Univ. Illinois, Urbana, Ill., 1976).* Providence, R. I.: Amer. Math. Soc., 1977, pp. 91–103.

[64] P. W. Millar. “Random times and decomposition theorems”. In: *Probability (Proc. Sympos. Pure Math., Vol. XXXI, Univ. Illinois, Urbana, Ill., 1976).* Providence, R. I.: Amer. Math. Soc., 1977, pp. 91–103.

[65] M. Nagasawa. *Stochastic processes in quantum physics.* Vol. 94. Monographs in Mathematics. Basel: Birkhäuser Verlag, 2000, pp. xviii+598.

[66] J. Neveu and J. Pitman. “Renewal property of the extrema and tree property of the excursion of a one-dimensional Brownian motion”. In: *Séminaire de Probabilités, XXIII.* Vol. 1372. Lecture Notes in Math. Berlin: Springer, 1989, pp. 239–247.

[67] J. Neveu and J. W. Pitman. “The branching process in a Brownian excursion”. In: *Séminaire de Probabilités, XXIII.* Vol. 1372. Lecture Notes in Math. Berlin: Springer, 1989, pp. 248–257.

[68] V. I. Norkin and B. O. Onishchenko. “Minorant methods of stochastic global optimization”. *Kibernet. Sistem. Anal.* 41.2 (2005), pp. 56–70, 189.

[69] E. A. Pecherskii and B. A. Rogozin. “On Joint Distributions of Random Variables Associated with Fluctuations of a Process with Independent Increments”. *Theory of Probability and its Applications* 14.3 (1969), pp. 410–423.

[70] J. Pitman. *Combinatorial stochastic processes.* Vol. 1875. Lecture Notes in Mathematics. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002. Berlin: Springer-Verlag, 2006.

[71] J. Pitman. “Remarks on the convex minorant of Brownian motion”. In: *Seminar on Stochastic Processes, 1982*. Birkhäuser, Boston, 1983, pp. 219–227.

[72] J. Pitman and N. Ross. “The greatest convex minorant of Brownian motion, meander, and bridge”. English. *Probability Theory and Related Fields* 153 (3-4 2012), pp. 771–807.

[73] J. Pitman and G. Uribe Bravo. “The convex minorant of a Lévy process”. *Ann. Probab.* 40.4 (2012), pp. 1636–1674.

[74] A. O. Pittenger and C. T. Shih. “Coterminal families and the strong Markov property”. *Bull. Amer. Math. Soc.* 78 (1972), pp. 439–443.

[75] Z. Qiao and J. M. Steele. “Random walks whose concave majorants often have few faces”. *Statist. Probab. Lett.* 75.2 (2005), pp. 97–102.

[76] S. T. Rachev and L. Rüschendorf. *Mass transportation problems. Vol. I.* Probability and its Applications (New York). Theory. New York: Springer-Verlag, 1998, pp. xxvi+508.

[77] R. T. Rockafellar and R. J.-B. Wets. *Variational analysis.* Vol. 317. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Berlin: Springer-Verlag, 1998, pp. xiv+733.

[78] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes, and martingales. Vol. 2.* Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. Itô calculus. New York: John Wiley & Sons Inc., 1987, pp. xiv+475.

[79] B. A. Rogozin. “The local behavior of processes with independent increments”. *Teor. Verojatnost. i Primenen.* 13 (1968), pp. 507–512.

[80] R. Ryan. “Large-deviation analysis of Burgers turbulence with white-noise initial data”. *Comm. Pure Appl. Math.* 51.1 (1998), pp. 47–75.

[81] R. Ryan. “The statistics of Burgers turbulence initialized with fractional Brownian noise data”. *Comm. Math. Phys.* 191.1 (1998), pp. 71–86.

[82] Z.-S. She, E. Aurell, and U. Frisch. “The inviscid Burgers equation with initial data of Brownian type”. *Comm. Math. Phys.* 148.3 (1992), pp. 623–641.

[83] L. A. Shepp and S. P. Lloyd. “Ordered cycle lengths in a random permutation”. *Trans. Amer. Math. Soc.* 121 (1966), pp. 340–357.

[84] S. Sherman. “Fluctuation and periodicity”. *J. Math. Anal. Appl.* 9 (1964), pp. 468–476.

[85] Y. G. Sinaĭ. “Statistics of shocks in solutions of inviscid Burgers equation”. *Comm. Math. Phys.* 148.3 (1992), pp. 601–621.

[86] F. Spitzer. "A combinatorial lemma and its application to probability theory". *Trans. Amer. Math. Soc.* 82 (1956), pp. 323–339.

[87] E. S. Štatland. "On local properties of processes with independent increments". *Teor. Verojatnost. i Primenen.* 10 (1965), pp. 344–350.

[88] J. M. Steele. "The Bohnenblust-Spitzer algorithm and its applications". *J. Comput. Appl. Math.* 142.1 (2002). Probabilistic methods in combinatorics and combinatorial optimization, pp. 235–249.

[89] T. M. Suidan. "Convex minorants of random walks and Brownian motion". *Teor. Veroyatnost. i Primenen.* 46.3 (2001), pp. 498–512.

[90] G. Uribe Bravo. *Bridges of Lévy processes conditioned to stay positive.* http://arxiv.org/abs/1101.4184. 2011.

[91] W. Vervaat. "A relation between Brownian bridge and Brownian excursion". *Ann. Probab.* 7.1 (1979), pp. 143–149.

[92] V. Vigon. "Abrupt Lévy processes". *Stochastic Process. Appl.* 103.1 (2003), pp. 155–168.

[93] V. Vigon. *Dérivées de Dini des processus de Lévy.* http://www-irma.u-strasbg.fr/~vigon/boulot/pas_publication/fichiers/conjecture.pdf.

[94] V. Vigon. "Votre Lévy rampe-t-il?" *J. London Math. Soc. (2)* 65.1 (2002), pp. 243–256.

[95] C. Villani. *Optimal transport.* Vol. 338. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Old and new. Berlin: Springer-Verlag, 2009, pp. xxii+973.

[96] D. Williams. "Path decomposition and continuity of local time for one-dimensional diffusions. I". *Proc. London Math. Soc. (3)* 28 (1974), pp. 738–768.

[97] M. Winkel. "Limit clusters in the inviscid Burgers turbulence with certain random initial velocities". *J. Statist. Phys.* 107.3-4 (2002), pp. 893–917.

[98] W. A. Woyczyński. *Burgers-KPZ turbulence.* Vol. 1700. Lecture Notes in Mathematics. Göttingen lectures. Berlin: Springer-Verlag, 1998, pp. xii+318.