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Killing fields generated by multiple solutions to the Fischer–Marsden equation

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In Memory of Professor S. Kobayashi

In the process of finding Einstein metrics in dimension $n \geq 3$, we can search critical metrics for the scalar curvature functional in the space of the fixed-volume metrics of constant scalar curvature on a closed oriented manifold. This leads to a system of PDEs (which we call the Fischer–Marsden Equation, after a conjecture concerning this system) for scalar functions, involving the linearization of the scalar curvature. The Fischer–Marsden conjecture said that if the equation admits a solution, the underlying Riemannian manifold is Einstein. Counter-examples are known by O. Kobayashi and J. Lafontaine. However, almost all the counter-examples are homogeneous. Multiple solutions to this system yield Killing vector fields. We show that the dimension of the solution space W can be at most $n + 1$, with equality implying that (M, g) is a sphere with constant sectional curvatures. Moreover, we show that the identity component of the isometry group has a factor $SO(W)$. We also show that geometries admitting Fischer–Marsden solutions are closed under products with Einstein manifolds after a rescaling. Therefore, we obtain a lot of non-homogeneous counter-examples to the Fischer–Marsden conjecture. We then prove that all the homogeneous manifold M with a solution are in this case. Furthermore, we also proved that a related Besse conjecture is true for the compact homogeneous manifolds.

Keywords: Critical metrics; metrics with constant scalar curvature; Einstein metrics; elliptic PDE system; killing vector fields; isometry group; Fischer–Marsden solutions; eigenfunctions; scalar curvature functional; linearization of scalar curvature.

Mathematics Subject Classification 2010: 53B21, 53C10, 53C12, 53C21, 53C22, 53C24, 53C25

1. Introduction and Summary of Results

Let M be a closed, connected, orientable manifold of dimension $n \geq 3$. Consider the scalar curvature s as a function on the space \mathfrak{S} of Riemannian metrics of fixed (unit) volume and constant scalar curvature. Define the Laplacian as the trace of the Hessian $\Delta = g^{ij}\nabla_i\nabla_j$. Eigenvalues of the Laplacian are (necessarily non-negative) constants $\lambda \geq 0$ for which there exist functions $u \in C^\infty(M)$, not identically zero, such that

$$\Delta u + \lambda u = 0 \tag{1}$$

(beware that in Besse [1], for instance, the opposite sign convention is used for Δ). From Koiso [14], we can conclude that, for any $g \in \mathfrak{S}$, if $s/(n-1)$ is not a positive eigenvalue of the Laplacian, then, for any symmetric bilinear 2-tensor h such that

$$Lh := \nabla^i\nabla^j h_{ij} - \Delta(h_{ij}g^{ij}) - h_{ij}R^{ij} = 0 \quad \text{and} \quad \int_M h_{ij}g^{ij}d\mu = 0 \tag{2}$$

we can find a one-parameter family $g(t)$ in \mathfrak{S} with $g'(0) = h$. Thus, for generic $g \in \mathfrak{S}$, the set of these h can be thought of as the tangent space of \mathfrak{S} . L is in fact the linearization of the scalar curvature, so that

$$\frac{\partial}{\partial t}(s_{g+th+O(t^2)})_{t=0} = Lh. \tag{3}$$

Following [1, p. 128], suppose g is a metric with $s/(n-1)$ not a positive eigenvalue of the Laplacian (so $s = 0$ is allowed). Define a metric $g \in \mathfrak{S}$ to be *critical* for the Einstein–Hilbert action $\mathcal{E}(g) = \int_M s_g d\mu$ if, given any one-parameter family $g(t)$ in \mathfrak{S} with derivative $g'(0) = h$ as above, we have $\frac{d}{dt}\mathcal{E}(g)(0) = 0$. Then g is critical in this sense if and only if there exists some function $f \in C^\infty(M)$ such that

$$(L^*f)_{ij} := \nabla_i\nabla_j f - (\Delta f)g_{ij} - fR_{ij} = R_{ij} - \frac{s}{n}g_{ij}, \tag{4}$$

where L^* denotes the L^2 -adjoint of L . For completeness, we outline a proof in the appendix. Now, taking the trace of Eq. (4), we obtain

$$\Delta f + \frac{s}{n-1}f = 0 \tag{5}$$

so that, since $s/(n-1)$ is not a positive eigenvalue, we must have f a constant (in fact, zero) and g must be an *Einstein manifold* $R_{ij} = (s/n)g_{ij}$. Besse [1, 4.48] goes further and asks, what if $s/(n-1)$ is in the spectrum? If g obeys Eq. (4) (and so is formally critical), must g be Einstein? If g is not Einstein, f cannot be a constant.

In this work, we choose to focus on what happens if there are multiple solutions f_1 and f_2 to (4). Indeed, since f is an eigenfunction of the Laplacian, we can write $u := 1 + f$ and rewrite (4) as the *critical metric equation*

$$\nabla_i\nabla_j u = uR_{ij} - \frac{s}{n-1}\left(u - \frac{1}{n}\right)g_{ij}. \tag{6}$$

If $u = 1 + f_1$ and $v = 1 + f_2$ are solutions to the above, then their difference $x = f_1 - f_2$ solves the linear equation

$$\nabla_i \nabla_j x = x \left(R_{ij} - \frac{s}{n-1} g_{ij} \right) \tag{7}$$

and x is an eigenfunction of the Laplacian with eigenvalue $s/(n-1)$. The *Fischer–Marsden Conjecture* asked whether g that satisfy (7) are Einstein. Counterexamples to that have been found (see, for instance, Kobayashi [11] and Lafontaine [15]). We notice that almost all the known examples are homogeneous and the dimension of the solution space of (7) is at least 2.

We will show that in general any product metric of the form $S^m \times N$ where N is Einstein yields a counterexample. We will call an x satisfying (7) a Fischer–Marsden solution.

If u and v are solutions to the critical metric equation (6), then $(udv - vdu)^\sharp$ is a *conformal Killing field*. Even nicer, if x and y are Fischer–Marsden solutions, then

$$Y = x\nabla y - y\nabla x \tag{8}$$

is a *Killing field* (as observed in Lafontaine [16] where the situation in dimension $n = 3$ is studied). We show that such a Killing field satisfies the equations

$$R_{iljk} Y^l = R_{ij} Y_k - R_{ik} Y_j - \frac{s}{n-1} (g_{ij} Y_k - g_{ik} Y_j), \tag{9}$$

where R_{ijkl} are components of the Riemann curvature tensor such that $R_{ij} = -R_{kikj}$, and

$$\text{Ric}(Y) = \rho Y \tag{10}$$

for some smooth function ρ defined where $Y \neq 0$ (depending on g but not on choice of Y). Furthermore, if w is any Fischer–Marsden solution, then so is $dw(Y) = Y^i \nabla_i w$. There is a constant $\beta < 0$ such that, if x and y are L^2 -orthonormal Fischer–Marsden solutions, then

$$\frac{|\nabla x|^2 + \beta}{x^2} = \frac{|\nabla y|^2 + \beta}{y^2} = \frac{\nabla_i x \nabla^i y}{xy} = \rho - \frac{s}{n-1}. \tag{11}$$

We use this to prove that the space of Fischer–Marsden solutions has dimension less than or equal to $(n+1)$ with equality only if (M, g) has constant curvature. In fact, we prove the stronger statement.

Theorem A. *Let W be the space of Fischer–Marsden solutions of (7), and I be the identity component of the isometry group of (M, g) . Then I is locally $SO(W) \times G_1$ with a compact Lie group G_1 which is the kernel of the action of I on W . Moreover, all the $SO(W)$ orbits are either $S^{\dim W - 1}$ or its fixed points.*

We then show the following.

Theorem B. *If (M, g) admits a Fischer–Marsden solution u of (7), then (after a possible rescaling) its product with a positive Einstein manifold (with a constant Einstein function) also has u as a Fischer–Marsden solution of (7).*

Thus we exhibit many nonhomogeneous geometries admitting solutions of (7). In the homogeneous case, we obtain a converse:

Theorem C. *If (M, g) is closed homogeneous manifold admitting a nontrivial Fischer–Marsden solution of (7), then we can write $M = S^{\dim(W)-1} \times N$ with N a homogeneous Einstein manifold (with a constant Einstein function).*

We shall study the nonhomogeneous case and equation of (6) from [1] in the near future. Here, we like to mention that if there is a solution for (6) on a non-Einstein manifold (on Einstein manifold (6) and (7) are essentially the same), we can always choose u to be invariant under the isometry group and $W \oplus \mathbf{R}(u - 1)$ is an invariant subspace of eigenfunctions since any difference of two solutions of (6) is a solution of (7). In particular, there is no homogeneous manifold M for (6) except that M is Einstein. This is because if (M, g) is homogeneous and there is an invariant solution u of (6), it must be a constant. The left side of (6) is zero and the right side of (6) implies that g is Einstein. That is, we have the following.

Theorem D. *The Besse conjecture is true for compact homogeneous Riemannian manifolds.*

This is possibly the first result for the Besse conjecture.

Throughout this paper, normal coordinates at considered points are used. See [13, part I, p. 148] for a reference.

In Sec. 2 and thereafter, all Riemann metrics are assumed to have constant scalar curvatures.

2. The Killing Fields and the Induced Map

First we remark that if u and v are solutions to

$$\nabla_i \nabla_j u = u R_{ij} - \frac{s}{n-1} (u - \alpha) g_{ij} \tag{12}$$

for some constant α , then

$$\nabla_j (u \nabla_i v - v \nabla_i u) = \nabla_j u \nabla_i v - \nabla_i u \nabla_j v - \frac{s\alpha}{n-1} (u - v) g_{ij}, \tag{13}$$

$$\nabla_i (u \nabla_j v - v \nabla_j u) + \nabla_j (u \nabla_i v - v \nabla_i u) = \frac{2s\alpha}{n-1} (v - u) g_{ij} \tag{14}$$

so that $(udv - vdu)^\sharp$ is a conformal Killing field if $\alpha \neq 0$ and a Killing field if $\alpha = 0$ (see [1, p. 40] for basic properties for Killing vector fields). Moreover, any

two scalar-function solutions u and v to (12) differ by a Fischer–Marsden solution, so we henceforth restrict our attention to the equation

$$\nabla_i \nabla_j x = x \left(R_{ij} - \frac{s}{n-1} g_{ij} \right) \tag{15}$$

and denote solutions to the above by x and y , with the Killing field they generate by

$$Y^i = \{x, y\}^i = x \nabla^i y - y \nabla^i x. \tag{16}$$

Our most important tool will be the map $f \mapsto A_Y f = Y^i \nabla_i f = df(Y)$ with $f \in C^\infty(M)$. This *induced map* is skew-symmetric with respect to the L^2 -inner product:

$$\int_M u A_Y v d\mu + \int_M v A_Y u d\mu = \int_M Y^i \nabla_i (uv) d\mu = \int_M \nabla^i (Y^i uv) d\mu = 0 \tag{17}$$

if Y is divergence-free. This is also because the isometry group is compact and therefore, any finite-dimension representation of it is skew-symmetric. Furthermore, if Y is a Killing vector field, A_Y takes eigenfunctions of the Laplacian to eigenfunctions of the Laplacian with the same eigenvalue.

The induced map A_Y also takes Fischer–Marsden solutions to Fischer–Marsden solutions since Y is a Killing vector field. This is because, if φ is an isometry and u is a Fischer–Marsden solution, then $\varphi^* u$ is a Fischer–Marsden solution (since the defining equation is in terms of Riemannian invariants). Thus, given a Killing field Y and its one-parameter family of isometries φ_t , we have that $\varphi_t^* u$ are solutions. Taking the partial derivative with respect to t , we have that $Y^i \nabla_i u = A_Y u$ is a solution.

Henceforth, we write Au for $A_Y u$ if there is no possibility of confusion. Now we study the induced map in more depth.

Proposition 1. *Let $Y = \{x, y\}$ be a Killing field generated by Fischer–Marsden solutions x and y . Let A be the map induced by Y . Then $|Y|^2 = xAy - yAx$, and $A^2x = -\beta^2x$ and $A^2y = -\beta^2y$ for some constant β .*

Proof. We have

$$|Y|^2 = (x \nabla_i y - y \nabla_i x) Y^i = xAy - yAx. \tag{18}$$

Next

$$\begin{aligned} 0 &= 2Y^i Y^j \nabla_i Y_j = Y^i \nabla_i |Y|^2 = Y^i (\nabla_i x (Ay) - \nabla_i y (Ax) \\ &\quad + x \nabla_i (Ay) - y \nabla_i (Ax)) = xA^2y - yA^2x. \end{aligned} \tag{19}$$

If $x = 0$ at a point, then $y = 0$ or $A^2x = 0$ there. But if $x = y = 0$ at a point, then $Y = 0$ there so that A^2x must also equal zero there. So the nodal (vanishing) set of x is contained in the nodal set of A^2x . It follows from an observation in Gichev [4] that the eigenfunctions x and A^2x must be linearly dependent.

Actually, what [4] asserts is the following. Let u and v be eigenfunctions corresponding to the same eigenvalue. If a nodal domain (connected component of the complement of the nodal set) of u is contained in that of v , then $u = cv$ for c a constant. Now let $\mathcal{N}[u]$ denote the nodal set of u and suppose $\mathcal{N}[v] \subseteq \mathcal{N}[u]$. Then $\mathcal{N}[u]^c \subseteq \mathcal{N}[v]^c$. If a connected component of $\mathcal{N}[u]^c$ is not contained in a connected component of $\mathcal{N}[v]^c$, then the boundary of the latter, namely $\mathcal{N}[v]$, intersects $\mathcal{N}[u]^c$. But that is impossible since $\mathcal{N}[v] \subseteq \mathcal{N}[u]$. Thus if one nodal set is contained in another, we can conclude that the eigenfunctions are linearly dependent.

We see that x and y are eigenvectors of A^2 with the same eigenvalue. Since A is skew-symmetric by (17), we have

$$\int_M xA^2x d\mu = - \int_M (Ax)^2 d\mu \leq 0 \tag{20}$$

so we can write $A^2x = -\beta^2x$ and $A^2y = -\beta^2y$ for some constant β . This also follows from (19).

This proves the proposition. □

Proposition 2 (Special Form of the Curvature). *Let Y be a Killing field generated by Fischer–Marsden solutions x and y . We have*

$$R_{iljk}Y^l = R_{ij}Y_k - R_{ik}Y_j - \frac{s}{n-1}(g_{ij}Y_k - g_{ik}Y_j) \tag{21}$$

and $R_i^j Y_j = \rho_Y Y_i$ for some function ρ_Y defined where $|Y| \neq 0$ for which $d\rho_Y(Y) = 0$.

Proof. Let $Y = \{x, y\}$. Taking its covariant derivative, we have by (15)

$$\begin{aligned} \nabla_j Y_k &= \nabla_j x \nabla_k y - xy \left(R_{jk} - \frac{s}{n-1} g_{jk} \right) \\ &\quad - \nabla_k x \nabla_j y + xy \left(R_{jk} - \frac{s}{n-1} g_{jk} \right), \end{aligned} \tag{22}$$

$$\nabla_j Y_k = \nabla_j x \nabla_k y - \nabla_k x \nabla_j y. \tag{23}$$

Since Killing fields Y satisfy $\nabla_i \nabla_j Y_k = R_{iljk}Y^l$ (see, for instance, Besse [1, 1.81, p. 40]), again taking the covariant derivative and applying (15) we obtain

$$R_{iljk}Y^l = \left(R_{ij} - \frac{s}{n-1} g_{ij} \right) Y_k - \left(R_{ik} - \frac{s}{n-1} g_{ik} \right) Y_j. \tag{24}$$

Then

$$0 = R_{iljk}Y^i Y^l Y^k = R_{ij}Y^i |Y|^2 - \frac{s}{n-1} Y_j |Y|^2 - \left(\text{Rc}(Y, Y) - \frac{s}{n-1} |Y|^2 \right) Y_j \tag{25}$$

so that $R_{ij}Y^j = R_i^k Y_k = \rho_Y Y_i$ for some function ρ_Y . Then by divergence-freeness of Y , symmetry of Ric , and skew-symmetry of ∇Y ,

$$d\rho_Y(Y) = Y^i \nabla_i \rho_Y = \nabla_i (\rho_Y Y^i) = \nabla^i (R_{ij}Y^j) = 0 + R_{ij} \nabla^i Y^j = 0 \tag{26}$$

since s is a constant, by the second Bianchi $\nabla^i R_{ij} = 0$ (see [1, p. 120, 4.19]) and the proposition follows. \square

3. The Fischer–Marsden Solution Space and the Isometry Group

At this point we see, given $Y = \{x, y\}$, that x and y are special eigenfunctions for A_Y . However, it is not clear how y and Ax are related. Let us explore this now.

Proposition 3. *Let \bar{Y} and Y be two Killing fields that are pointwise linearly dependent. Then they are linearly dependent as vector fields.*

Proof. If they are pointwise linearly dependent, then

$$\bar{Y}_i Y_j = Y_i \bar{Y}_j, \quad \bar{Y}_i = \frac{g(\bar{Y}, Y)}{|Y|^2} Y_i = f Y_i. \tag{27}$$

Then where $|Y| \neq 0$,

$$\nabla_j \bar{Y}_i = \nabla_j f Y_i + f \nabla_j Y_i. \tag{28}$$

Therefore,

$$\nabla_i f Y_j = -\nabla_j f Y_i, \quad \nabla^j f Y_j = 0. \tag{29}$$

It follows that

$$|\nabla f|^2 |Y|^2 = -(\nabla^j f Y_j)^2 = 0 \tag{30}$$

everywhere, which is only possible if f is constant, and \bar{Y} and Y are linearly dependent. \square

Proposition 4. *Let $Y = \{x, y\}$ be a Killing field generated by L^2 -orthonormal Fischer–Marsden solutions x and y . Let A be the induced map. Then there is a constant $\beta < 0$ such that*

$$Ax = \beta y, \quad Ay = -\beta x, \tag{31}$$

$$\frac{|\nabla x|^2 + \beta}{x^2} = \frac{|\nabla y|^2 + \beta}{y^2} = \frac{\nabla_i x \nabla^i y}{xy} = \rho_Y - \frac{s}{n-1}. \tag{32}$$

Proof. Since Ax is also Fischer–Marsden, we know we can get another Killing field by

$$\bar{Y}^i = x \nabla^i Ax - Ax \nabla^i x. \tag{33}$$

But

$$Ax = x \nabla_j x \nabla^j y - y |\nabla x|^2. \tag{34}$$

So

$$\begin{aligned} \nabla^i Ax &= \nabla^i x \nabla_j x \nabla^j y - \nabla^i y |\nabla x|^2 + x \left(\rho_Y - \frac{s}{n-1} \right) Y^i \\ &+ (xy - yx) \left(R^{ij} - \frac{s}{n-1} g^{ij} \right) \nabla_j x. \end{aligned} \tag{35}$$

Thus

$$\begin{aligned} \bar{Y}^i &= x\nabla^i x \nabla_j x \nabla^j y - x\nabla^i y |\nabla x|^2 + x^2 \left(\rho_Y - \frac{s}{n-1} \right) Y^i \\ &\quad - x\nabla^i x \nabla_j x \nabla^j y + y\nabla^i x |\nabla x|^2, \end{aligned} \tag{36}$$

$$\bar{Y}^i = \left[x^2 \left(\rho_Y - \frac{s}{n-1} \right) - |\nabla x|^2 \right] Y^i. \tag{37}$$

By Proposition 3, we see that there is some constant $\bar{\beta}$ such that

$$x^2 \left(\rho_Y - \frac{s}{n-1} \right) - |\nabla x|^2 = \bar{\beta}. \tag{38}$$

We must have $\bar{\beta} \neq 0$. Otherwise $\bar{Y} = 0$, and $x\nabla^i Ax = Ax\nabla^i x$. Contracting with Y now yields $x A^2 x = (Ax)^2$. Actually, $\bar{\beta}$ has nothing to do with y and only depends on x . We could always make it to be -1 by rescaling as we shall see later on, on the unit sphere. Integrating, we have $-\int_M (Ax)^2 d\mu = \int_M (Ax)^2 d\mu$ and so $Ax = 0$ and hence $A^2 x = 0$. But from Proposition 1, $A^2 x/x = A^2 y/y$ so that also $A^2 y = 0$. But then $0 = \int_M y A^2 y d\mu = -\int_M (Ay)^2 d\mu$ so that $Ay = 0$. Since $|Y|^2 = xAy - yAx$ by (18), we have Y identically 0. But then $dx/x = dy/y$ so that $y = Cx$, which contradicts the orthogonality of x and y . So indeed $\bar{\beta} \neq 0$. Moving forward, from $\bar{Y} = \bar{\beta}Y$ we have

$$x\nabla_i (Ax - \bar{\beta}y) = (Ax - \bar{\beta}y)\nabla_i x. \tag{39}$$

Separating variables and solving the differential equation, there is a constant C such that

$$Ax = \bar{\beta}y + Cx. \tag{40}$$

Integrating,

$$0 = \int_M xAx d\mu = \bar{\beta} \int_M xy d\mu + C \int_M x^2 d\mu. \tag{41}$$

Thus $C = 0$ since $\int_M xy d\mu$ by the assumption of L^2 -orthogonality. So $Ax = \bar{\beta}y$. Then $Ay = -(\beta^2/\bar{\beta})x$ by Proposition 1. We conclude that $\beta \neq 0$, by reasoning similar to that which implied $\bar{\beta} \neq 0$.

Now write $c = |\beta/\bar{\beta}|$ and $\tilde{x} = x\sqrt{c}$ and $\tilde{y} = y/\sqrt{c}$. Then we have $A\tilde{x} = \beta\tilde{y}$ and $A\tilde{y} = -\beta\tilde{x}$. Integrating

$$\tilde{y}A\tilde{x} + \tilde{x}A\tilde{y} = \beta(\tilde{y}^2 - \tilde{x}^2) \tag{42}$$

we see that $\int_M \tilde{x}^2 d\mu = \int_M \tilde{y}^2 d\mu$, which is the same as saying $\bar{\beta}^2 = \beta^2$ (because x and y are orthonormal by assumption). Since we have not yet fixed the sign of β , we will fix $\beta = \bar{\beta}$.

By considering the system of equations for Ax and Ay :

$$x\nabla_i x \nabla^i y - y|\nabla x|^2 = \beta y, \quad x|\nabla y|^2 - y\nabla_i x \nabla^i y = -\beta x \tag{43}$$

and (38), we deduce

$$\frac{|\nabla x|^2 + \beta}{x^2} = \frac{|\nabla y|^2 + \beta}{y^2} = \frac{\nabla_i x \nabla^i y}{xy} = \rho_Y - \frac{s}{n-1}. \tag{44}$$

Consider the first equality in (44), and examine a point where $x = 0$ and $y \neq 0$ (such a point must exist, or else x and y are linearly dependent). We see that $\beta < 0$, and this finishes the proof. \square

Proposition 5. *The constant $\beta < 0$ and ρ_Y do not depend on the choice of Killing field Y generated by Fischer–Marsden solutions x and y , and we write simply ρ . Moreover, if A is the induced map and u is a Fischer–Marsden solution orthogonal to both x and y , we have $Au = 0$.*

Proof. Let x, y , and z be orthonormal Fischer–Marsden solutions. Consider $Y = \{x, y\}, Z = \{x, z\}$ and $U = \{y, z\}$. Let β, β' , and β'' be their respective constants. Since Ric is symmetric, we have

$$(\rho_Y - \rho_Z)Y_i Z^i = (R_{ij} - R_{ij})Y^i Z^j = 0. \tag{45}$$

But at a point where $Y \neq 0, Z \neq 0$ and $x \neq 0$, we have

$$\rho_Y - \rho_Z = \frac{|\nabla x|^2 + \beta}{x^2} - \frac{|\nabla x|^2 + \beta'}{x^2} = \frac{\beta - \beta'}{x^2} \tag{46}$$

by the first and last expressions in (44). Thus if $\rho_Y = \rho_Z$ we have $\beta = \beta'$. Otherwise $Y_i Z^i = 0$ on an open dense subset, so that in fact $Y_i Z^i = 0$ everywhere. But that is

$$x^2 \nabla_i y \nabla^i z - xy \nabla_i x \nabla^i z - xz \nabla_i x \nabla^i y + yz |\nabla x|^2 = 0. \tag{47}$$

Dividing by $x^2 y z$ we get

$$\frac{\nabla_i y \nabla^i z}{yz} - \frac{\nabla_i x \nabla^i z}{xz} - \frac{\nabla_i x \nabla^i y}{xy} + \frac{|\nabla x|^2}{x^2} = 0. \tag{48}$$

Substituting expressions involving the β constants from (44) into (48) yields

$$\frac{|\nabla x|^2 + \beta + \beta'}{x^2} = \frac{|\nabla y|^2 + \beta''}{y^2}. \tag{49}$$

By considering (44) and (49) together, we see that if $x = 0$ and $y \neq 0$, then $\beta + \beta' = -|\nabla x|^2 = \beta$. So in fact, $\beta' = 0$, a contradiction. So actually $\rho_Y = \rho_Z$ and $\beta = \beta'$. If w is yet another solution, we have that ρ and β are the same for $\{x, y\}, \{x, w\} = \{-w, x\}$ and $\{-w, z\} = \{z, w\}$.

Now consider a Fischer–Marsden solution u orthogonal to x and y if any such exists. Then

$$Au = (x \nabla_i y - y \nabla_i x) \nabla^i u = (xyu - yxu) \left(\rho - \frac{s}{n-1} \right) = 0 \tag{50}$$

which proves the proposition. \square

With all the machinery in place, we can establish the following.

Theorem 1. *Let W be the Fischer–Marsden solution space, and I the identity component of the isometry group of (M, g) . Then I is locally isomorphic to $SO(W) \times G_1$, where G_1 is the kernel of the representation of I on W . In particular, W has dimension at most $n + 1$, with equality implying that (M, g) is the round sphere. Moreover, all the $SO(W)$ orbits are either spheres or its fixed points.*

Proof. We continue our argument in the proof of Proposition 5. With $Y = \{x, y\}, Z = \{x, z\}$, and $U = \{y, z\}$, we calculate

$$[Y, Z]^j = Y^i \nabla_i Z^j - Z^i \nabla_i Y^j = \beta U^j. \tag{51}$$

Therefore, $x \wedge y \rightarrow Y$ is a Lie algebra isomorphism from $W \wedge W = so(W)$ to its image in the Killing fields. One could also use the following elementary argument for a proof of the upper bound on dimension:

Let x_i be an L^2 -orthonormal basis for the Fischer–Marsden solution space. If there are at least $(n + 1)$ of the x_i , then choose these and consider $\{x_i, x_j\}$. These are $n(n + 1)/2$ Killing fields. Let us show that they are linearly independent. To see this, suppose to the contrary that

$$\sum_{1 \leq i < j \leq n+1} \alpha_{ij} (x_i \nabla_k x_j - x_j \nabla_k x_i) = 0 \tag{52}$$

for constants α_{ij} not all zero. In particular, suppose that some specific α_{ij} is nonzero. Relabeling indices if necessary, we can suppose $\alpha_{12} \neq 0$. Then

$$0 = \sum_{1 \leq i < j \leq n+1} \alpha_{ij} (x_i \nabla_k x_j - x_j \nabla_k x_i) \nabla^k x_1 = \beta \sum_{j=1}^{n+1} \alpha_{1j} x_j \tag{53}$$

by Propositions 4 and 5. But the x_j are linearly independent and $\beta \neq 0$, so this would mean $\alpha_{1j} = 0$ for every j : a contradiction. Thus (M, g) has $n(n + 1)/2$ linearly independent Killing fields, and so is maximally symmetric. There cannot be more linearly independent Fischer–Marsden solutions, or that would induce an even higher degree of symmetry. Being maximally symmetric, (M, g) must have constant curvature. Since (M, g) is a closed Einstein manifold, we have for some Fischer–Marsden solution x :

$$\nabla_i \nabla_j x + \frac{s}{n(n - 1)} x g_{ij} = 0. \tag{54}$$

So that (M, g) is isometric to the round sphere by Obata’s Theorem [17] Theorem A.

In general, the tangent space of $SO(W)$ orbit at a given point is a subspace of the space generated by $\nabla x_i / x_i - \nabla x_j / x_j$ and therefore, the dimension is $\leq k - 1$. The orbit passed through that point is a sphere or a point. \square

4. Examples and the Homogeneous Cases

Since the round sphere is an example of a Riemannian manifold with nontrivial solutions to the Fischer–Marsden equation, it may be asked if there are others.

Lafontaine in [16] shows that $S^1 \times N$ is also an example, where N is a surface of constant positive curvature. We show that our structure results fit quite well with these known examples. For example, if $M = S^1 \times N$, we could let $\beta = -1$ and $x = \cos \theta$, then $dx = -\sin \theta d\theta$. The standard metrics gives $\frac{|\nabla x|^2 - 1}{x^2} = -1$. Since S^1 is totally geodesic, we have $\rho = R_{11} = 0$, we could let $\frac{s}{n-1} = 1$ and $R_{ii} = 1$ with $i > 1$.

If $n > 1$, let x be one of the Euclidean coordinate for S^n in \mathbf{R}^{n+1} , we can also consider $x = \cos \theta$ and $dx = -\sin \theta d\theta$. Then, ∇x generates geodesics. $|dx|^2 = \sin^2 \theta$ with the standard metrics for the unit sphere. $\frac{|dx|^2 - 1}{x^2} = -1$. $\rho = n - 1$, $s = n(n - 1)$ and therefore, $\rho - \frac{s}{n-1} = n - 1 - n = -1$ also. For the coordinates x, y , we have $|d(ax + by)|^2 - 1 = -(ax + by)^2$ with (a, b) an unit vector. We then have $(dx, dy) = -xy$ as in (44).

As a generalization, we exhibit the following examples.

Theorem 2 (Product with Einstein Manifolds). *If (V, g') is a manifold admitting a Fischer–Marsden solution u , and (N, g'') is a closed oriented Einstein manifold with Einstein constant $c^2 > 0$, then we can always rescale V so that u is a solution to the Fischer–Marsden equation on $M = V \times N$ with metric $g = g' + g''$. If $(V, g') = (S^1, d\theta^2)$, we do not rescale, but rather we set u to be $\cos(c\theta)$ or $\sin(c\theta)$. If there are multiple solutions on (V, g') , the quantities β and ρ on (M, g) will be the same as those on (V, g') . Conversely, if (M, g) is a product of (V, g') and (N, g'') and admits a Fischer–Marsden solution u , then u must be the pull-back of a solution on one of the factors, and the other factor must be Einstein (with a constant Einstein function).*

Proof. If g' and g'' are Riemannian metrics, then the product metric $g = g' + g''$ on $V \times N$ has Ricci curvature $\text{Ric} = \text{Ric}' + \text{Ric}''$. The scalar curvature is $s = s' + s''$. The Hessian $Dd(fh)$ of functions $f \in C^\infty(V)$ and $h \in C^\infty(N)$ satisfies

$$Dd(fh) = hDdf, \quad Dd(fh) = fDdh, \quad Dd(fh) = df \otimes dh \tag{55}$$

for vectors tangent to V , vectors tangent to N , and the case where one vector is tangent to V and the other to N , respectively.

Let V have dimension m . Rescale (V, g') so that its scalar curvature s_V satisfies $s_V = (m - 1)c^2$, where $c^2 = s''/(n - m)$ is the Einstein constant of (N, g'') , and $n = \dim M$. Thus the scalar curvature of (M, g) satisfies

$$\frac{s}{n - 1} = \frac{s_V + (n - m)c^2}{n - 1} = \frac{m - 1 + n - m}{n - 1}c^2 = c^2. \tag{56}$$

Let u be a Fischer–Marsden solution on V if $m > 1$, or if $m = 1$, choose u to be $\cos(c\theta)$ or $\sin(c\theta)$. Working on (M, g) , for vectors tangent to N , we have

$$\nabla_i \nabla_j u = 0 = (R''_{ij} - c^2 g''_{ij})u = \left(R_{ij} - \frac{s}{n - 1} g_{ij} \right) u \tag{57}$$

since the Einstein constant of (N, g'') satisfies $c^2 = s''/(n - m)$. For vectors tangent to V , we have, if $m > 1$,

$$\nabla_i \nabla_j u = \left(R'_{ij} - \frac{s_V}{m - 1} g'_{ij} \right) u = \left(R_{ij} - \frac{s}{n - 1} g_{ij} \right) u \tag{58}$$

since $s_V/(m - 1) = c^2 = s/(n - 1)$. If $m = 1$, we have

$$\nabla_i \nabla_j u = \frac{d^2 u}{d\theta^2} = 0 - c^2 u = \left(R_{ij} - \frac{s}{n - 1} g_{ij} \right) u. \tag{59}$$

If one vector is tangent to V and the other is tangent to N , the cross-terms vanish. In all cases we have

$$\nabla_i \nabla_j u = \left(R_{ij} - \frac{s}{n - 1} g_{ij} \right) u. \tag{60}$$

Thus u is a solution on M to the Fischer–Marsden equation. Now $|\nabla u|^2$ is the same whether ∇ and g are taken with respect to V or with respect to M . Also, if x and y are two solutions, $(x \nabla_i y - y \nabla_i x) \nabla^i x$ will be the same on V and M , so that β will not have changed. Therefore

$$\rho_V - \frac{s_V}{m - 1} = \frac{|\nabla x|^2 + \beta}{x^2} = \rho_M - \frac{s}{n - 1}. \tag{61}$$

Since $s_V/(m - 1) = s/(n - 1)$, we also have that ρ is invariant.

For the converse, suppose that (M, g) is a product of (V, g') and (N, g'') and admits a Fischer–Marsden solution u . The equation for the cross-terms is

$$\frac{\partial}{\partial x^i} \frac{\partial u}{\partial x^j} = 0, \tag{62}$$

where i indices are tangent to V and j indices are tangent to N . This is because in such a case, the Christoffel symbols Γ_{ij}^k vanish, as do the terms g_{ij} and R_{ij} . The only way this equation can hold is if $u = fh + f_1 + h_1$, where f and f_1 are functions on V , and h and h_1 are functions on N . However, then the equation is $\nabla_i f \nabla_j h + \nabla_j f \nabla_i h = 0$, which implies

$$|\nabla f|^2 |\nabla h|^2 = -(\nabla_i f \nabla^i h)^2. \tag{63}$$

The only way that can hold is if one of the functions, say without loss of generality h , is constant (say equal to h_0) on a nonempty open set $\mathcal{U} \subseteq M$. On the complement of \mathcal{U} , f must be constant. But then $h - h_0$ is an eigenfunction of the Laplacian on all of M . Since it vanishes on a nonempty open set, we must have $h = h_0$ on all of M . So we can write $u = f + h$ without loss of generality. Plugging this u in for the equations with indices tangent to V and N , respectively, we see that either f or h must be constant. Without loss of generality, let h be constant, so that $u = f$.

Now that we know u is the pull-back of a function on V , we must have

$$0 = \nabla_i \nabla_j u = \left(R''_{ij} - \frac{s}{n - 1} g''_{ij} \right) u \tag{64}$$

for i and j tangent to N . It follows that (N, g'') is Einstein with positive Einstein constant $s/(n - 1)$. On the other hand, for some constant τ , we have

$$\nabla'_i \nabla'_j u = (R'_{ij} - \tau g'_{ij}) u \tag{65}$$

for indices i and j tangent to V . Tracing gives us $\Delta' u = (s' - \tau m)u$, where m is the dimension of V . Taking the divergence yields

$$\nabla'_j \Delta' u = \Delta' \nabla'_j u - R'_{ij} \nabla'^j u = -\tau \nabla'_j u \tag{66}$$

whence we conclude that $\tau = s'/(m - 1)$. Thus u is a Fischer–Marsden solution on V , as desired. \square

By looking at any example of the form $S^1 \times N$, we have $x = \cos(c\theta)$ and $y = \sin(c\theta)$, and so $\rho = 0$. Thus there are nontrivial examples besides the sphere where ρ is constant, and it becomes pertinent to prove

Proposition 6. *Let (M, g) be a Riemannian manifold admitting k orthonormal Fischer–Marsden solutions x_1, x_2, \dots, x_k and suppose ρ is a constant. If P is a homogeneous harmonic polynomial of degree α in k variables, then $P(x_1, x_2, \dots, x_k)$ is an eigenfunction of the Laplacian with the corresponding eigenvalue equal to $-\alpha[(\alpha - 1)\rho - \frac{\alpha s}{n-1}]$.*

Proof. If P is a real-valued function of the x_i , we have

$$\Delta(P(x)) = \sum_i \frac{\partial P}{\partial x_i} \Delta x_i + \sum_{i,j} \frac{\partial^2 P}{\partial x_i \partial x_j} \nabla_i x_i \nabla^j x_j \tag{67}$$

$$= -\frac{s}{n-1} \sum_i \frac{\partial P}{\partial x_i} x_i + \left(\rho - \frac{s}{n-1}\right) \sum_{i,j} \frac{\partial^2 P}{\partial x_i \partial x_j} x_i x_j - \beta(\Delta P)(x), \tag{68}$$

where we have used Eq. (44) to obtain the rightmost two terms. Because P is homogeneous, this simplifies to

$$\Delta(P(x)) = -\frac{s\alpha}{n-1} P(x) + \left(\rho - \frac{s}{n-1}\right) \alpha(\alpha - 1) P(x) - \beta(\Delta P)(x). \tag{69}$$

Since P is harmonic and ρ is constant, we are left with

$$\Delta(P(x)) = \alpha \left[(\alpha - 1)\rho - \frac{\alpha s}{n-1} \right] P(x) \tag{70}$$

as desired. \square

Proposition 7. *Let (M, g) be a Riemannian manifold admitting k orthonormal Fischer–Marsden solutions x_1, x_2, \dots, x_k , then ρ is a constant if and only if ∇x_1 is an eigenvector for the Ricci curvature operator. In this case, ∇x_i generates geodesics, and either (A) all those x_i have a mutual zero, or (B) $x_1^2 + x_2^2 + \dots + x_k^2$ is constant. In the latter case (B), we have $\rho = s(k - 2)/((n - 1)(k - 1))$, and (x_1, x_2, \dots, x_k) is a harmonic map into S^{k-1} . And moreover, M is a product of*

S^{k-1} with a Einstein manifold. In the case (A), the common zero set of the solutions are totally geodesic and all the geodesics generated by ∇x_i have the same length.

Proof. By taking the derivative of (44), we see that ρ is constant if and only if ∇x_1 is an eigenvector of the Ricci operator.

Indeed, we have that $\nabla \rho = 0$ if and only if

$$2x^{-1} \left(R_{ik} - \frac{s}{n-1} g_{ik} \right) \nabla_k x - 2Cx^{-1} \nabla_i x = 0$$

for any i by applying (7) with a constant C .

If ∇x_1 is an eigenvector of the Ricci operator, then $\nabla_{\nabla x_1} \frac{\nabla x_1}{|\nabla x_1|} = 0$ by applying (7) again. Therefore, the gradients generate geodesics, just as in the case of sphere.

Define $r = \sqrt{x_1^2 + x_2^2 + \dots + x_k^2}$. If the x_i have no mutual zero, then r is smooth, and, for any $\sigma \in \mathbb{R}$, we can consider r^σ . We have

$$\Delta(r^\sigma) = \sigma \left((\sigma - 2)r^{\sigma-4} \sum_{i,j} x_i x_j \nabla_k x_i \nabla^k x_j + r^{\sigma-2} \sum_i \left(|\nabla x_i|^2 - \frac{s}{n-1} x_i^2 \right) \right). \tag{71}$$

Using again the equations in (44), we obtain

$$\Delta(r^\sigma) = \sigma \left((\sigma - 1) \left(\rho - \frac{s}{n-1} \right) - \frac{s}{n-1} \right) r^\sigma - \sigma \beta r^{\sigma-2} (k - 2 + \sigma). \tag{72}$$

For $k > 2$, setting $\sigma = 2 - k$, we see that r^σ is an eigenfunction of the Laplacian. But then it must change sign on a closed manifold, unless it is constant. So r is constant. This implies that the eigenfunctions map M harmonically into S^{k-1} (see [3]). Moreover, we must have $(\sigma - 1)(\rho - s/(n - 1)) = s/(n - 1)$, which implies $\rho = s(k - 2)/(n - 1)(k - 1)$. For example, if $k = n + 1$, we get $\rho = s/n$. If $k = 2$, we carry out a similar analysis with $\log r$, and obtain $\Delta \log r = -2\rho$, which implies $\rho = 0$ by the Divergence Theorem. Thus we still have $\rho = s(k - 2)/((n - 1)(k - 1))$.

In the case (B), the fibers of the map are totally geodesic by [7, p. 52]. And the orbits associated with the Killing vector fields Y_{ij} (obtained by (16) with x_i and x_j in the place of x and y) are the same as those of ∇x_i 's. Since ∇x_i generate geodesics, the orbits are totally geodesics also. Therefore, $M = S^{k-1} \times N$ with N the fibers of the map.

In the case (A), by [7, Proposition 3.1, p. 51], the common zero set of the solution is totally geodesic. Also, since ∇x_1 generates geodesics, from (44) we have at a maximal point of x_1 , $\rho - \frac{s}{n-1} = -a^{-2} < 0$ and $(x'_1)^2 - 1 = -a^2 x_1^2$. Therefore, $x = a \cos \frac{t}{a}$. □

In the last sentence of the proof, it is very possible that: Let N_a be maximal point set of x_1 and N_{-a} be the minimal point set of x_1 . Then both of them are submanifolds and there is one-to-one map between them introduced by the closest point from the other submanifold. Those two points are connected by the geodesics generated by ∇x_1 . The system of these geodesics generate a submanifold, which

might be the required sphere of radius a . And M then is the product $S^{k-1} \times N$ with $N = N_a = N_{-a}$. Then the question remains that:

- (a) Does this picture actually work out?
- (b) Is ρ always a constant?

Corollary. *If ρ is constant and the Fischer–Marsden solution space has multiplicity m , then any $m - 1$ of the solutions have a common zero.*

Proof. If the above formula for ρ holds, it uniquely defines k . □

A situation where ρ will be constant is when (M, g) is homogeneous, a case we can completely classify:

Theorem 3. *Let (M, g) be closed, homogeneous and admit a Fischer–Marsden solution. Then (M, g) must be of the form $S^m \times N$ where N is an Einstein manifold.*

Proof. We can apply Proposition 7 directly.

But here we can offer another more group involved proof. The Fischer–Marsden defining equation is written in terms of Riemannian invariants. Therefore, if W is the space of Fischer–Marsden solutions, then the isometry group G has W as an invariant subspace. Thus there is a Lie group homomorphism $G \rightarrow SO(W)$. Since G is compact, it is reductive, meaning that its Lie algebra can be written as $\mathfrak{s} \oplus \mathfrak{a}$, where \mathfrak{s} is semisimple and \mathfrak{a} is abelian. By the classification of simple Lie groups, this means that we can write G is locally isomorphic to a finite covering $\tilde{G} = SO(W) \times G'$. The first factor must be $SO(W)$ since its orbits are spheres or fixed points and sphere is simply connected. \tilde{G} acts transitively on (M, g) by isometries via $\tilde{\gamma} \cdot m = \pi(\tilde{\gamma}) \cdot m$, where π is the covering map $\tilde{G} \rightarrow G$. The isotropy group is π^{-1} of the isotropy subgroup of G . G' fixes all the solution functions. Therefore, the intersection of $G'm$ with each $SO(W)$ orbit is unique. This gives a product for M .

Thus $M = S^m \times N$ with the product metric. By Theorem 2, N must be Einstein. □

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years before Guan came to United States, he had copies of the famous books [13] and never imagined that he himself would become one of the students of the famous differential geometer Kobayashi. In those three years in Berkeley, Guan had written and published a few papers under Professor Kobayashi. Paper [5] was Guan’s referee report for one of Professor Gang Tian’s paper and was published by Professor Kobayashi. When Guan finished [6] in the Spring 1992, Professor Kobayashi helped him smoothen the language. That eventually led to the solution of cohomogeneity one compact Kähler Einstein manifolds in, for example, [9]. That solution is closely related to the holomorphic vector bundle case presented in [12], see [8]. The second author dedicate this work to Professor Kobayashi. One can see the power of the Lie group actions.

We would also like to take this paper to honor Ms. Lu, Lingzi Dorothy, who died in the Boston Bombing April 2013 and got the first perfect score in Guan’s ordinary differential equation class MATH 46 during her final in University of California at Riverside in Fall 2010.

Appendix A

Here, we prove the assertion that tensors h satisfying

$$\nabla^i \nabla^j h_{ij} - \Delta(h_{ij} g^{ij}) - h_{ij} R^{ij} = 0 \quad \text{and} \quad \int_M h_{ij} g^{ij} d\mu = 0 \tag{A.1}$$

can always occur as first derivatives of one-parameter families $g(t)$ in \mathfrak{S} , provided $s/(n - 1)$ is not in the positive spectrum of $g(0)$. To see this, consider that from Theorem 2.5 in Koiso [14], we have that, given $g(0) \in \mathfrak{S}$, if we have a smooth perturbation $\tilde{g}(t)$, which might run outside of \mathfrak{S} , we can always write

$$g(t) = f(t)\tilde{g}(t) \tag{A.2}$$

with $f(0) = 1$ and $0 < f \in C^\infty(M)$, so that $g(t)$ actually lies in \mathfrak{S} for t close enough to zero. It only remains to show that we have enough control to make any h_{ij} as above the derivative of $g(t)$ at zero. Indeed, let

$$\tilde{g}_{ij} = g_{ij}(0) + th_{ij}. \tag{A.3}$$

Then $(g'(0))_{ij} = h_{ij} + f'(0)g_{ij}(0)$. Writing $g_{ij} = g_{ij}(0)$ for brevity, what we now need to show is that $f'(0) = 0$. What we know is that $f'(0)g_{ij}$ is in the kernel of $\Delta \circ L$, that is, the Laplacian composed with the linearization of the scalar curvature. This is because, on one hand, h_{ij} is in the kernel of L and, on the other hand, the linearization of Δs is $\Delta \circ L$ at g since the scalar curvature is constant. Thus $\Delta L(g'_{ij}(0)) = 0$. So $f'(0)g_{ij}$ satisfies

$$\Delta[\nabla^i \nabla^j (f'(0)g_{ij}) - \Delta(f'(0)g_{ij} g^{ij}) - (f'(0)g_{ij})R^{ij}] = 0. \tag{A.4}$$

Thus

$$\nabla^i \nabla^j (f'(0)g_{ij}) - \Delta(f'(0)g_{ij}g^{ij}) - (f'(0)g_{ij})R^{ij} = c \tag{A.5}$$

for some constant c , since harmonic functions are constant on a closed manifold. But this is just

$$\Delta(f'(0) - c_1) + \frac{s}{n-1}(f'(0) - c_1) = 0 \tag{A.6}$$

so that $f'(0)$ is equal to some constant c_1 . Thus $g'(0)_{ij} = h_{ij} + c_1g_{ij}$. However, we know

$$\int_M g^{ij}h_{ij}d\mu = \int_M g^{ij}(g'(0)_{ij})d\mu = 0. \tag{A.7}$$

So $c_1 = 0$ and $g'(0) = h$ as desired.

Now we show that, if $g \in \mathfrak{S}$ with $s/(n-1)$ not in the positive spectrum of g , then g is critical for the Einstein–Hilbert action $\mathcal{E}(g) = \int_M sd\mu$ if, and only if

$$\nabla_i \nabla_j f - (\Delta f)g_{ij} - fR_{ij} = R_{ij} - \frac{s}{n}g_{ij} \tag{A.8}$$

for some function $f \in C^\infty(M)$. For, indeed, the left-hand side L^*f is the adjoint of the linearization of scalar curvature applied to f , as can easily be checked. Now, given a path $g(t)$ in \mathfrak{S} with initial position g and initial velocity h , we have

$$\frac{d}{dt} \left(\int_M sd\mu \right)_{t=0} = \int_M (\nabla^i \nabla^j h_{ij} - \Delta(h_{ij}g^{ij}) - h_{ij}R^{ij})d\mu \tag{A.9}$$

$$= - \int_M R_{ij}h^{ij}d\mu = - \int_M \left(R_{ij} - \frac{s}{n}g_{ij} \right) h^{ij}d\mu. \tag{A.10}$$

Suppose now that $LL^*u = 0$. Then

$$\int_M (L^*u)^2d\mu = \int_M uLL^*ud\mu = 0 \tag{A.11}$$

We have

$$\int_M L \left(Rc - \frac{s}{n}g \right) ud\mu = \int_M \left(R_{ij} - \frac{s}{n}g_{ij} \right) (L^*u)_{ij}d\mu = 0. \tag{A.12}$$

Now, it can easily be checked that LL^* is elliptic. By the Fredholm alternative, we must always be able to solve $LL^*f = L(Rc - \frac{s}{n}g)$. Thus we have a Hodge-type decomposition

$$R_{ij} - \frac{s}{n}g_{ij} = (L^*f)_{ij} + v_{ij}, \tag{A.13}$$

where v is in the kernel of L . Suppose this equation can be solved with $v = 0$. Then, for any initial velocity h ,

$$\frac{d}{dt} \left(\int_M sd\mu \right)_{t=0} = - \int_M (L^*f)_{ij}h^{ij}d\mu = - \int_M f(Lh)d\mu = 0 \tag{A.14}$$

since h is in the kernel of L by definition. So g is critical. Otherwise, suppose that there exists a non-identically-zero solution v . Then take that v to be the initial

velocity. We have

$$\frac{d}{dt} \left(\int_M s d\mu \right)_{t=0} = - \int_M |v|^2 d\mu < 0 \quad (\text{A.15})$$

so that g cannot be critical. This finishes the proof.

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