HYPERSURFACES IN HYPERBOLIC POINCARÉ MANIFOLDS AND CONFORMALLY INVARIANT PDES

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Abstract. We derive a relationship between the eigenvalues of the Weyl-Schouten tensor of a conformal representative of the conformal infinity of a hyperbolic Poincaré manifold and the principal curvatures on the level sets of its uniquely associated defining function with calculations based on [9] [10]. This relationship generalizes the result for hypersurfaces in $\mathbb{H}^{n+1}$ and their connection to the conformal geometry of $\mathbb{S}^n$ as exhibited in [7] and gives a correspondence between Weingarten hypersurfaces in hyperbolic Poincaré manifolds and conformally invariant equations on the conformal infinity. In particular, we generalize an equivalence exhibited in [7] between Christoffel-type problems for hypersurfaces in $\mathbb{H}^{n+1}$ and scalar curvature problems on the conformal infinity $\mathbb{S}^n$ to hyperbolic Poincaré manifolds.

1. Introduction

The relationship between the geometry of a conformally compact space and the geometry of its conformal infinity has been of recent interest in both physical and mathematical communities. The interest in such association is motivated primarily by the AdS/CFT correspondence where a conformal field theory on a compact manifold M^n correlates to the string theory on a negatively curved conformally compact Einstein manifold X^{n+1}, which has M as its conformal infinity. One can view such connections as originating from the identification between the group of isometries of hyperbolic space $\mathbb{H}^{n+1}$ and the group of conformal transformations of the round sphere $\mathbb{S}^n$. In fact, the study of such connections date back to the 1980's in the seminal paper of Fefferman and Graham [9].

Recently, an explicit example connecting the geometry of hyperbolic space $\mathbb{H}^{n+1}$ to the conformal geometry of the round sphere $\mathbb{S}^n$ was realized in [7] in the context of the hypersurface geometry of $\mathbb{H}^{n+1}$ and curvature prescription problems on $\mathbb{S}^n$ in the conformal class of the round metric. The question

Date: October 2009.
2000 *Mathematics Subject Classification.* Primary 53C30 53C40; Secondary 58J05.
Key words and phrases. hyperbolic poincaré manifold, hypersurfaces, second fundamental form, Schouten tensor, conformally invariant PDE.

of the existence a hypersurface Σ^n in hyperbolic space $\mathbb{H}^{n+1}$ with prescribed Weingarten functional of the principal curvatures of Σ is a natural extension of the classical problem in the Euclidean setting. Of particular interest is the Christoffel problem for hypersurfaces in hyperbolic space $\mathbb{H}^{n+1}$ where one is asked to find a hypersurface $\Sigma^n \subset \mathbb{H}^{n+1}$ with prescribed mean of the curvature radii. One of the initial difficulties of the Christoffel problem in $\mathbb{H}^{n+1}$ is to provide the appropriate formulation of the Gauss map and the principal curvature radii in the context of hyperbolic space. In [7] the relevant notions of the hyperbolic Gauss map and the hyperbolic principal curvature radii are developed using the ambient structure of the hyperboloid model of $\mathbb{H}^{n+1}$ where hyperbolic space is realized as a hypersurface in Minkowski spacetime. Moreover, [7] exhibits a strikingly precise relation between Christoffel-type problems for immersed hypersurfaces in $\mathbb{H}^{n+1}$ and scalar curvature prescription problems of conformal geometry on $\mathbb{S}^n$, viewed as the boundary of $\mathbb{H}^{n+1}$ at infinity. See also a related work of Mazzeo and Pacard [18].

In this note we take a viewpoint more reflective of conformal geometry and we generalize the correspondences exhibited in [7] between Christoffel-type problems and scalar curvature prescription problems of conformal geometry. For $n \geq 2$, let X^{n+1} denote the interior of a smooth compact manifold $\bar{X}^{n+1}$ with boundary $\partial X = M^n$. A Riemannian metric g on X is then said to be conformally compact if, for a defining function r for M, the conformal metric $\bar{g} = r^2 g$ extends to a metric on $\bar{X}$. The metric $\bar{g}$ restricted to TM induces a metric $\hat{g}$ on M, which rescales by conformal factor upon change in defining function and therefore defines a conformal structure $(M, [\hat{g}])$ on M called the conformal infinity of (X, g).

A *hyperbolic Poincaré manifold* is a conformally compact hyperbolic manifold. From the work of [10], given a representative $\gamma \in [\hat{g}]$ of the conformal infinity of a hyperbolic Poincaré manifold (X^{n+1}, g) and its associated geodesic defining function r, we may write the metric in the normal form

$$g = r^{-2}(dr^2 + g_r) \tag{1}$$

where

$$g_r = \gamma - r^2 P_\gamma + \frac{r^4}{4} Q(P_\gamma), \tag{2}$$

$$Q(P_\gamma)_{ij} = \gamma^{kl}(P_\gamma)_{ik}(P_\gamma)_{jl}$$

and for $n \geq 3$,

$$P_\gamma = \frac{1}{n-2}\left(Ric_\gamma - \frac{R_\gamma}{2(n-1)}\gamma\right)$$

is the Schouten tensor of γ with Ric_γ and R_γ denoting the Ricci and the scalar curvature of γ, respectively (please refer to §2 for more details). For

$n = 2$, P_γ is a symmetric 2-tensor on M satisfying

$$\gamma^{ij}(P_\gamma)_{ij} = \frac{R_\gamma}{2} \quad \text{and} \quad \gamma^{jk}(P_\gamma)_{ij,k} = (R_\gamma)_{,i}.$$

We will show that the horospherical metric associated to a horospherical ovaloid in hyperbolic space $\mathbb{H}^{n+1}$ can be realized as a representative of the conformal infinity $(\mathbb{S}^n, [g_0])$ of hyperbolic space $(\mathbb{H}^{n+1}, g_{\mathbb{H}})$. This is because a horospherical ovaloid in hyperbolic space $\mathbb{H}^{n+1}$ determines a geodesic defining function r for the infinity $\mathbb{S}^n$ of $\mathbb{H}^{n+1}$, where $r = e^{-s}$ and s is the hyperbolic distance to the horospherical ovaloid. In general, on an asymptotically hyperbolic manifold X, one should replace the notion of a horospherical ovaloid by that of an *essential set*. As defined in [2], the exponential map from the normal bundle of an essential set is a diffeomorphism to the outside of the essential set in X. Hence, an essential set provides a geodesic defining function $r = e^{-s}$ where s is the distance to the essential set in X. A similar idea was realized in early works of Epstein [5] [6].

A straightforward calculation based on (2) yields a generalization of the relation in [7] between the eigenvalues of the Schouten tensor of the horospherical metric and the hyperbolic principal curvature radii of the level sets of the associated geodesic defining function. To avoid any possible sign confusion of the principal curvature of a hypersurface we recall that, the second fundamental of a hypersurface Σ in X^{n+1} with respect to an orientation induced by a choice a normal direction N is defined to be

$$II = -\frac{1}{2}\mathcal{L}_N g, \tag{3}$$

where $\mathcal{L}$ is the Lie derivative. In our convention, for instance, the principal curvature of a unit sphere in Euclidean space with the orientation induced by the inward normal direction is 1.

Theorem 1.1. *Suppose that (X^{n+1}, g) is a hyperbolic Poincaré manifold and let γ be a representative of its conformal infinity $(M^n, [\hat{g}])$ with associated geodesic defining function r. Then the eigenvalues λ_i of the tensor P_γ in the expansion (2) satisfy*

$$1 - \frac{r^2}{2}\lambda_i = \frac{2}{1 - \kappa_i} \tag{4}$$

where $\kappa_i = \kappa_i(r)$ denotes the i^{th} outward principal curvature on the level sets of the geodesic defining function r and $\frac{2}{1-\kappa_i}$ is considered to be the i^{th} hyperbolic principal curvature radius.

As studied in [7], when $n \geq 3$, the relationship (4) in Theorem 1.1 can be used to turn questions regarding foliations near the conformal infinity by particular classes of hypersurfaces in hyperbolic Poincaré manifolds into questions regarding the conformal geometry of the conformal infinity and visa versa. For example, taking the trace of (4), one finds that

$$R_\gamma = \frac{4(n-1)}{r^2}\left(n - \sum_{i=1}^{n} \frac{2}{1-\kappa_i}\right). \tag{5}$$

Therefore, finding a foliation by hypersurfaces with constant mean of the hyperbolic curvature radii is equivalent to finding a constant scalar curvature metric on the conformal infinity. Hence, due to the resolution of the Yamabe problem we have the following Corollary.

Corollary 1.1. *Suppose that (X^{n+1}, g) is a hyperbolic Poincaré manifold. Then there always exists a foliation of hypersurfaces of constant mean of the hyperbolic curvature radii near the infinity. Such foliations are parameterized by geodesic defining functions r associated with constant scalar curvature S representatives of the conformal infinity and the mean of the hyperbolic curvature radii of the foliation is given by*

$$\frac{1}{n}\sum_{i=1}^{n} \frac{2}{1-\kappa_i} = 1 - \frac{r^2}{4n(n-1)}S. \tag{6}$$

Moreover, if the conformal infinity $(M, [\hat{g}])$ of (X^{n+1}, g) has negative Yamabe invariant, then such foliations are unique.

More generally, the relationship (4) can similarly be applied to the generalized Yamabe or σ_k curvature problem to give foliations of certain hyperbolic Poincaré manifolds by hypersurfaces with constant linear combinations or rational functions of generalized mean curvatures. For $1 \le k \le n$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, let

$$\sigma_k(\lambda) := \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

denote the k^{th} elementary symmetric function on $\mathbb{R}^n$. Let Γ_k denote the connected component of

$$\{\lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0\}$$

containing the positive cone $\{\lambda \in \mathbb{R}^n \mid \lambda_1, \ldots, \lambda_n > 0\}$. Given a representative g_0 of the conformal infinity $(M^n, [\hat{g}])$ of a hyperbolic Poincaré manifold (X^{n+1}, g), we denote the eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$ of the the Schouten tensor P_{g_0} by $\lambda(P_{g_0})$ and the k^{th} elementary symmetric function of the eigenvalues of the Schouten tensor P_{g_0} by $\sigma_k(P_{g_0})$. Moreover, if $\tilde{g}_0 = e^{2\phi_0} g_0$ is a conformally related metric on M, then we denote the k^{th} elementary symmetric function of the eigenvalues of the Schouten tensor $P_{\tilde{g}_0}$ corresponding to $\tilde{g}_0$ by $\sigma_k(P_{\tilde{g}_0})$. Applying the works of [14] [13] [16], it follows from that fact that

M is compact and locally conformally flat, that for $n \geq 3$, if $\lambda(P_{g_0}) \in \Gamma_k$, then there exists a smooth positive function ϕ_0 on M such that $\tilde{g}_0 = e^{2\phi_0} g_0$ with

$$\lambda(P_{\tilde{g}_0}) \in \Gamma_k \quad \text{and} \quad \sigma_k(P_{\tilde{g}_0}) = 1. \tag{7}$$

In light of (4), (7) and the observations above, we have the following Corollary.

Corollary 1.2. *For $n \geq 3$, let (X^{n+1}, g) be a hyperbolic Poincaré manifold with conformal infinity $(M^n, [\hat{g}])$. Suppose that there exists a metric $g_0 \in [\hat{g}]$ with $\lambda(P_{g_0}) \in \Gamma_k$ for some $1 \leq k \leq n$. Then there exists a foliation near M parameterized by a geodesic defining function r associated to a conformal metric $\tilde{g}_0 \in [g_0]$ with constant σ_k curvature $\sigma_k(P_{\tilde{g}_0}) = 1$ such that the level sets of r have outward principal curvatures $\kappa_i = \kappa_i(r)$ satisfying*

$$\sum_{i_1 < \cdots < i_k} \frac{1+\kappa_{i_1}}{1-\kappa_{i_1}} \cdot \frac{1+\kappa_{i_2}}{1-\kappa_{i_2}} \cdots \frac{1+\kappa_{i_k}}{1-\kappa_{i_k}} = \left(\frac{r^2}{2}\right)^k. \tag{8}$$

This paper is organized as follows. In Section 2 we introduce hyperbolic Poincaré manifolds and we recall several related geometric preliminaries and concepts. In addition, we recall an application of the ambient metric construction of Fefferman and Graham [10], which gives the asymptotic expansion (2) for the tangential component of hyperbolic Poincaré metrics in normal form (1). In Section 3 we introduce the notion of the horospherical metric associated to a horospherical ovaloid in $\mathbb{H}^{n+1}$ and we relate such horospherical metrics to representatives of the conformal infinity. This observation allows us to put the two constructions in [7] and [9] [10] in the same light. Finally, in Section 4 we prove Theorem 1.1.

2. Hyperbolic Poincaré Manifolds

In this section we introduce hyperbolic Poincaré manifolds and their properties mostly adopted from [10]. Readers are referred to [10] for details. Let X^{n+1} denote the interior of a smooth compact manifold $\bar{X}^{n+1}$ with boundary $\partial X = M^n$. A smooth function $r : \bar{X} \to \mathbb{R}$ is said to be a *defining function* for M if

(1) $r > 0$ in X;
(2) $r = 0$ on M; and
(3) $dr \neq 0$ on M.

A Riemannian metric g on X is then said to be *conformally compact* if for a defining function r for M, the conformal metric $\bar{g} = r^2 g$ extends to a metric on $\bar{X}$. The metric $\bar{g}$ restricted to TM induces a metric $\hat{g}$ on M, which rescales by conformal factor upon change in defining function and therefore defines a conformal structure $(M, [\hat{g}])$ on M called the *conformal infinity* of (X, g). A straightforward computation as in [17] shows that the sectional curvatures of g approach $-|dr|^2_{\bar{g}}$ near M. Accordingly, we have the following definition for asymptotically hyperbolic manifolds.

Definition 2.1. *A complete Riemannian manifold (X^{n+1}, g) is said to be* asymptotically hyperbolic *if g is conformally compact and $|dr|^2_{\bar{g}} = 1$ on M for a defining function r for M in X.*

We recall the following lemma from [11] [15] concerning geodesic defining functions.

Lemma 2.1. *Let (X, g) be an asymptotic hyperbolic manifold. Then any representative g_0 in the conformal infinity of g determines a unique defining function r such that $r^2 g$ extends to a metric on $\bar{X}$, $r^2 g|_{TM} = g_0$ and $|dr|^2_{r^2 g} \equiv 1$ in a neighborhood U of M in $\bar{X}$.*

We will call such special defining function a geodesic defining function associated with the representative g_0. Given a representative g_0 of the conformal infinity $(M^n, [\hat{g}])$ of an asymptotic hyperbolic manifold (X^{n+1}, g), the product structure $M \times [0, \epsilon)$ in a neighborhood of M induced by the geodesic defining function r from Lemma 2.1 yields the *normal form*

$$g = r^{-2}(dr^2 + g_r)$$

with formal asymptotic expansion

$$g_r = g_0 + r g_1 + r^2 g_2 + \cdots$$

where the coefficients g_j are symmetric 2-tensors on M. Decomposing the Einstein tensor $Ric_g + ng$ with respect to the product structure $M \times [0, \epsilon)$ as in [8] yields differential equations that can be successively differentiated and inductively solved at $r = 0$ to derive the expansions for n odd

$$g_r = g_0 + r^2 g_2 + (\text{even powers}) + r^{n-1} g_{n-1} + r^n g_n + \cdots \tag{9}$$

while for n even

$$g_r = g_0 + r^2 g_2 + (\text{even powers}) + h r^n \log r + r^n g_n + \cdots \tag{10}$$

provided sufficient regularity is assumed and $Ric_g + ng$ vanishes to sufficiently high order at infinity.

For $0 \le j < n$, the terms g_j in the expansions (9) and (10) are tensors on M that are locally determined by the particular representative g_0 of the conformal infinity. For n odd g_n is trace-free but formally undetermined and for n even h is locally determined and trace-free while the trace of g_n is locally determined but the trace-free part of g_n is formally undetermined (see [8]). One can explicitly compute the tensors g_j for $0 \le j < n$ in the expansions

(9) and (10) using the aforementioned differential equation resulting from the Einstein condition at infinity. Of particular interest, for $n \geq 3$ one finds that $g_2 = -P_{g_0}$ where

$$P_{g_0} = \frac{1}{n-2}\left(Ric_{g_0} - \frac{R_{g_0}}{2(n-1)}g_0\right)$$

is the Schouten tensor of the conformal representative g_0. The asymptotic expansions described above are fundamental in many works concerning the geometry and topology of conformally compact manifolds as well as in the exploration of properties of submanifold observables in the AdS/CFT corresponce (see for example [1] [3] [4] [8] [12] [19]) .

In this note we focus on a class of manifolds that serve as the prototypical models of asymptotically hyperbolic manifolds known as *hyperbolic Poincaré manifolds*. Such manifolds are conformally compact hyperbolic manifolds obtained from quotients of hyperbolic space $\mathbb{H}^{n+1}$ by discrete groups of isometries. Similar to [8], given a representative g_0 of the conformal infinity $(M^n, [\hat{g}])$ of a hyperbolic Poincaré manifold (X^{n+1}, g) one can use the fact that (X, g) has constant sectional curvature $K_g = -1$ to decompose the tensor

$$R_{\alpha\beta\gamma\mu} + (g_{\alpha\gamma}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\gamma}) = 0$$

with respect to the product structure $M \times [0, \epsilon)$ induced by the geodesic defining function r to yield the differential equation

$$0 = rR^{g^r}_{ijkl} - \frac{1}{2}(g_{il}g'_{jk} + g'_{il}g_{jk} - g'_{ik}g_{jl} - g_{ik}g'_{jl}) + \frac{r}{4}(g'_{il}g'_{jk} - g'_{ik}g'_{jl}) \tag{11}$$

where latin letters denote tangential directions to r level sets, $g_{ij} = (g_r)_{ij}$, and $g'_{ij} = \partial_r g^r_{ij}$ for simplicity here. Taking successive derivatives of equation (11) and solving at $r = 0$ one finds that the tangential metric

$$g_r = g_0 - r^2 P_{g_0} + \frac{r^4}{4}Q(P_{g_0}) \tag{12}$$

where

$$Q(P_{g_0})_{ij} = g_0^{kl}(P_{g_0})_{ik}(P_{g_0})_{jl}.$$

The asymptotic expansion (12) for a hyperbolic Poincaré metric is perhaps easier to recognize using the ambient metric construction of Fefferman and Graham [10]. We summarize the application of the work [10] to derive the expansion (12) for a hyperbolic Poincaré metric below.

Let g_0 be a representative of the conformal infinity $(M^n, [\hat{g}])$ of a hyperbolic Poincaré manifold (X^{n+1}, g) and let r be the geodesic defining function associated to g_0 so that g has the *normal form*

$$g = r^{-2}(dr^2 + g_r)$$

in a neighborhood $M \times [0, \epsilon)$ of M. Consider the ambient metric

$$\tilde{g} = s^2 g - ds^2 \tag{13}$$

on $M \times [0, \epsilon) \times R_+$. Then (X, g) is isometrically identified with $\{s = 1\}$ in the ambient spacetime and a straightforward calculation shows that the curvature tensor of the ambient metric $\tilde{g}$ satisfies

$$Riem_{\tilde{g}} = s^2 [Riem_g + g \wedge g]$$

where $(g \wedge g)_{\alpha\beta\gamma\mu} = g_{\alpha\gamma} g_{\beta\mu} - g_{\alpha\mu} g_{\beta\gamma}$. Hence, it follows that the ambient metric (13) of a hyperbolic Poincaré metric is necessarily flat. Under the change of variables

$$-2\rho = r^2, \quad s = rt \quad \text{for} \quad \rho \leq 0$$

the ambient metric (13) takes the *normal form*

$$\tilde{g} = 2\rho dt^2 + 2t dt d\rho + t^2 g_\rho$$

where g_ρ is a 1-parameter family of metrics on M. Straightforward computations give the equations

$$\tilde{R}_{ijkl} = t^2 [R^{g_\rho}_{ijkl} + \frac{1}{2}(g_{il} g'_{jk} + g_{jk} g'_{il} - g_{ik} g'_{jl} - g_{jl} g'_{ik}) + \frac{\rho}{2}(g'_{ik} g'_{il} - g'_{il} g'_{jk})]$$

and

$$\tilde{R}_{\rho ik\rho} = \frac{1}{2} t^2 [g''_{ik} - \frac{1}{2} g^{jl} g'_{ij} g'_{kl}]$$

where $g_{ij} = (g_\rho)_{ij}$, $g'_{ij} = \partial_\rho (g_\rho)_{ij}$ and $g''_{ij} = \partial_\rho \partial_\rho (g_\rho)_{ij}$ for simplicity. Therefore, we may derive as in the proof of Theorem 7.4 in [10] that

$$g'_{ik}\big|_{\rho=0} = 2P^{g_0}_{ik}$$

and

$$g''_{ik}\big|_{\rho=0} = 2g_0^{jl} P^{g_0}_{ij} P^{g_0}_{kl}$$

and $g'''_{ij} = 0$.

To illustrate the above notions and definitions, we consider the hyperboloid model of hyperbolic space $(\mathbb{H}^{n+1}, g_{\mathbb{H}})$. Here

$$\mathbb{H}^{n+1} = \{(x, t) \in \mathbb{R}^{n+1,1} \mid |x|^2 - t^2 = -1, \ t > 0\}$$

is realized as a hypersurface in Minkowski spacetime $\mathbb{R}^{n+1,1}$ equipped with the Lorentz metric

$$g_{\mathbb{L}} = -dt^2 + |dx|^2.$$

The hyperbolic metric is then given by

$$g_{\mathbb{H}} = \frac{1}{1 + |x|^2} (d|x|)^2 + |x|^2 g_0,$$

where g_0 is the standard round metric on $\mathbb{S}^n$. Letting $d_{g_{\mathbb{H}}}$ denote the hyperbolic geodesic distance from the vertex $e_{n+2} \in \mathbb{H}^{n+1} \subset \mathbb{R}^{n+1,1}$, the function

$$r := 2e^{-d_{g_{\mathbb{H}}}} = \frac{2}{|x| + \sqrt{1+|x|^2}}$$

determines the geodesic defining function associated with the standard round metric as a representative of the conformal infinity $(\mathbb{S}^n, [g_0])$ of hyperbolic space $(\mathbb{H}^{n+1}, g_{\mathbb{H}})$. We then have the metric expansion

$$g_{\mathbb{H}} = r^{-2}\left(dr^2 + \left(1 - \frac{r^2}{4}\right)^2 g_0\right).$$

Notice that $P_{g_0} = \frac{1}{2}g_0$ for the standard round sphere so that the expansion above is of the form (12).

3. Horospherical Metrics

In this section we introduce the horospherical metric on the space of all horospheres as a parametrization of a neighborhood of the infinity of hyperbolic space and we present the induced horospherical metrics on horospherial ovaloids in $\mathbb{H}^{n+1}$. Readers are referred to the paper [7] for more details. Our goal is to relate horospherical metrics to representatives of the conformal infinity and to put the two constructions in [7] and [9] [10] in the same light.

Consider the hyperboloid model of hyperbolic space

$$\mathbb{H}^{n+1} = \{(x,t) \in \mathbb{R}^{n+1,1} \mid |x|^2 - t^2 = -1,\ t > 0\},$$

where $\mathbb{R}^{n+1,1}$ denotes Minkowski spacetime equipped with the Lorentz metric $g_{\mathbb{L}} = -|dt|^2 + |dx|^2$. Horospheres in $\mathbb{H}^{n+1}$ are intersections of degenerate affine hyperplanes of $\mathbb{R}^{n+1,1}$ with $\mathbb{H}^{n+1}$ and can be uniquely characterized by their points at infinity $x \in \mathbb{S}^n$, which are the null directions inside the hyperplanes, and the signed hyperbolic distance α from the horosphere to the vertex $e_{n+2} \in \mathbb{H}^{n+1}$, where α is positive if e_{n+2} is inside a given horosphere and negative otherwise. Accordingly, one can identify the space of horospheres in $\mathbb{H}^{n+1}$ with $\mathbb{S}^n \times \mathbb{R}$ and endow the space of horospheres with a natural degenerate metric $\langle \cdot\,, \cdot \rangle_\infty = e^{2\alpha} g_0$ in the conformal class of the round metric g_0 on $\mathbb{S}^n$.

Now suppose

$$\phi : \Sigma^n \to \mathbb{H}^{n+1}$$

is an immersed oriented hypersurface and let

$$\eta : \Sigma^n \to \mathbb{S}^{n+1}_1$$

denote the Lorentzian unit normal map taking values in de-Sitter spacetime

$$\mathbb{S}^{n+1}_1 = \{(x,t) \in \mathbb{R}^{n+1,1} \mid |x|^2 - t^2 = 1\}.$$

From the map

$$\psi := \phi + \eta : \Sigma^n \to \mathbb{N}^{n+1}_+ \tag{14}$$

taking values in the positive light-cone

$$\mathbb{N}^{n+1}_+ = \{(x,t) \in \mathbb{R}^{n+1,1} \mid |x|^2 - t^2 = 0,\ t > 0\},$$

one defines the hyperbolic Gauss map as the direction of the light-cone map (14) in $\mathbb{S}^n$. One finds that the light-cone map of horospheres is constant for the inward orientation and that parallel horospheres correspond to collinear vectors in $\mathbb{N}^{n+1}_+$. Hence, one also can identify the space of horospheres in $\mathbb{H}^{n+1}$ with $\mathbb{N}^{n+1}_+$. Moreover, it is easily seen that the horospherical metric on the space of all horospheres is exactly the same as the induced metric on the light-cone from the Lorentz metric $g_{\mathbb{L}}$ of Minkowski spacetime.

One therefore can realize the *horospherical metric* associated to a horospherical ovaloid in $\mathbb{H}^{n+1}$, that is a compact hypersurface $\Sigma^n \subseteq \mathbb{H}^{n+1}$ for which the Gauss map is regular, as the pull-back by the light-cone map ψ of the induced metric on the hypersurface as viewed in the positive light-cone $\mathbb{N}^{n+1}_+$. We recall from [7] that a compact immersed hypersurface is said to be a horospherical ovaloid in $\mathbb{H}^{n+1}$ if it can be oriented so that it is horospherically convex at every point and that an oriented hypersurface in $\mathbb{H}^{n+1}$ is horospherically convex at a point if and only if all the principal curvatures of at the point verify simultaneously less than 1 or greater than 1.

Alternatively, one can define the horospherical metric as in [20] by

$$h := I_\Sigma - 2II_\Sigma + III_\Sigma \tag{15}$$

where I_Σ, II_Σ and III_Σ are respectively the first, second and third fundamental forms of Σ in $\mathbb{H}^{n+1}$. In [7] Espinar, Gálvez and Mira view the image of the light cone map (14) as a co-dimension 2 hypersurface in Minkowoski spacetime and derive a relation between the principal curvatures of an immersed hypersurface in $\mathbb{H}^{n+1}$ and the eigenvalues of the Schouten tensor of its associated horospherical metric. In order to connect the work of [7] with ours in the context of conformal geometry, we compute the horospherical metric as defined in (15). Given a hyperbolic Poincaré manifold (X^{n+1}, g) and a respresentative γ of its conformal infinity $(M^n, [\hat{g}])$ we first compute the third fundamental form on level sets determined by the associated geodesic defining function r. A straightforward computation gives

$$\begin{aligned}
III_r(\partial_i, \partial_j) &= I_r(\nabla_{\partial_i} N_r, \nabla_{\partial_j} N_r) = I_r(\nabla_{\partial_i} r\partial_r, \nabla_{\partial_j} r\partial_r) \\
&= r^{-2} g^r_{ij} - r^{-1}\partial_r g^r_{ij} + \frac{1}{4} g_r^{pq} \partial_r g^r_{ip} \partial_r g^r_{jq} \\
&= \left(r^{-2}\gamma_{ij} - P^\gamma_{ij} + \frac{r^2}{4}\gamma^{kl} P^\gamma_{ik} P^\gamma_{jl} \right) + \left(2P^\gamma_{ij} - r^2 \gamma^{kl} P^\gamma_{ik} P^\gamma_{jl} \right) \\
&+ \frac{1}{4} g_r^{pq} \left(-2rP^\gamma_{ip} + r^3 \gamma^{kl} P^\gamma_{ik} P^\gamma_{pl} \right) \left(-2rP^\gamma_{jq} + r^3\gamma^{kl} P^\gamma_{jk} P^\gamma_{ql} \right).
\end{aligned}$$

In terms of an orthonormal basis $\{e_1, \ldots, e_n\}$ with respect to γ that diagonalizes the tensor P_γ, it follows

$$\begin{aligned}
III_r(e_i, e_j) &= \left(r^{-2}\delta_{ij} - \lambda_i \delta_{ij} + \frac{r^2}{4}\lambda_i^2 \delta_{ij} \right) + \left(2\lambda_i \delta_{ij} - r^2 \lambda_i^2 \delta_{ij} \right) \\
&\quad + \frac{1}{4} \left(1 - \frac{r^2}{2}\lambda_k \right)^{-2} \delta^{kl} \left(-2r\lambda_i \delta_{ik} + r^3 \lambda_i^2 \delta_{ik} \right) \left(-2r\lambda_j \delta_{jl} + r^3 \lambda_j^2 \delta_{jl} \right) \\
&\quad = r^{-2} \left(1 + \frac{r^2}{2}\lambda_i \right)^2 \delta_{ij}.
\end{aligned}$$

Therefore, the horospherical metric associated to a level set of a geodesic defining function r is

$$\begin{aligned}
h(e_i, e_j) &= I_r(e_i, e_j) - 2II_r(e_i, e_j) + III_r(e_i, e_j) \\
&= r^{-2} \left(1 - \frac{r^2}{2}\lambda_i \right)^2 \delta_{ij} + 2r^{-2} \left(1 - \frac{r^2}{2}\lambda_i \right) \left(1 + \frac{r^2}{2}\lambda_i \right) \delta_{ij} \\
&\quad + r^{-2} \left(1 + \frac{r^2}{2}\lambda_i \right)^2 \delta_{ij} \\
&= r^{-2} \left(1 - r^2\lambda_i + \frac{r^4}{4}\lambda_i^2 + 2 - \frac{r^4}{2}\lambda_i^2 + 1 + r^2\lambda_i + \frac{r^4}{4}\lambda_i^2 \right) \delta_{ij} \\
&= 4r^{-2}\delta_{ij}.
\end{aligned}$$

Thus, given a conformal representative γ of the conformal infinity $(M^n, [\hat{g}])$ of a hyperbolic Poincaré manifold (X^{n+1}, g) and its associated geodesic defining function r, the horospherical metrics associated to the level sets of r are given by $h = 4r^{-2}\gamma$. On the other hand, given an outwardly convex smooth hypersurface $\Sigma^n \subset X$, from which the exponential map is a diffeomorphism from the normal bundle to the outside, we find from the associated geodesic defining function $\tilde{r} = e^{-d_\Sigma}$, that the horospherical metric on $\Sigma = \{\tilde{r} = 1\}$ is given by $h \in [\hat{g}]$. Hence, in the context of conformal geometry one may regard horospherical metrics associated to the hypersurfaces given in [7] simply as conformal representatives of the conformal infinity.

Now we would like to illustrate that the ambient metric construction in [10] somehow gives a nice extension to the notions of the horospherical metrics in [7]. As in [10], given a hyperbolic Poincaré manifold (X^{n+1}, g) with conformal infinity $(M, [\hat{g}])$ we define the metric bundle $\mathcal{G}$ over M to be the space of pairs (h, x) with $x \in M$ and $h = s^2\hat{g}(x)$ for some $s \in \mathbb{R}_+$ where $\mathcal{G}$ is equipped with the projection

$$\pi : \mathcal{G} \to M \quad \text{defined by} \quad (h, x) \overset{\pi}{\mapsto} x$$

and dilations

$$\delta_s : \mathcal{G} \to \mathcal{G} \quad \text{defined by} \quad (h, x) \overset{\delta_s}{\mapsto} (s^2 h, x)$$

for $s \in \mathbb{R}_+$. The metric bundle $\mathcal{G}$ assumes the role of the light cone, that is, the space of all horospheres, and the metric bundle is similarly equipped with a tautological degenerate metric defined at $z = (h, x) \in \mathcal{G}$ by $g_{\mathcal{G}} = \pi^* h$, which is homogeneous of degree 2 with respect to dilations and therefore depends only on the conformal class $[\hat{g}]$.

Fixing a representative g_0 of the conformal infinity $(M, [\hat{g}])$, one obtains a trivialization of metric bundle $\mathcal{G} \cong \mathbb{R}_+ \times M$ by identifying

$$(t, x) \in \mathbb{R}_+ \times M \quad \text{with} \quad (t^2 g_0(x), x) \in \mathcal{G}.$$

Given local coordinates $(x) = (x^1, \ldots, x^n)$ on $\mathcal{U} \subset M$ we obtain local coordinates (t, x) on $\pi^{-1}(\mathcal{U})$ where

$$g_{ij}^{\mathcal{G}} = t^2 g_{ij}^0 dx^i dx^j$$

so that the representative g_0 of the conformal infinity of (X, g) can be considered as the section of the bundle $\mathcal{G} \cong \mathbb{R}_+ \times M$ determined by the level submanifold $\{t = 1\}$. On the ambient space $(\mathbb{R}_+ \times M) \times \mathbb{R}$ with coordinates (t, x, ρ) the ambient or cone metric $\tilde{g} = s^2 g - ds^2$ from (13) with $(X, g) = \{s = 1\}$ takes the normal form

$$\tilde{g} = 2\rho dt^2 + 2t dt d\rho + t^2 g_\rho$$

where

$$-2\rho = r^2, \quad s = rt \quad \text{for} \quad \rho \leq 0$$

and r is the geodesic defining function uniquely associated to g_0. Therefore, given an outwardly convex hypersurface $\Sigma^n \subset X^{n+1}$ and letting $\alpha = d_\Sigma$ denote the signed geodesic distance from Σ, which is positive outside Σ, one finds that under the change of variables

$$t = e^\alpha$$

that the ambient metric restricted to $(X, g) = \{s = 1\}$ takes the form

$$\tilde{g}\big|_X = d\alpha^2 + e^{2\alpha} g_\alpha.$$

Hence, one may view the change of variables $t = e^\alpha$ with respect to a given hypersurface as straightening out the hypersurface and giving a new coordinate on the metric bundle, which results in determining a new representative of the conformal infinity.

4. Principal Curvatures

In this section we carry out a straightforward calculation to prove our main theorem. Suppose that (X^{n+1}, g) is a hyperbolic Poincaré manifold and $(M^n, [\hat{g}])$ is its conformal infinity. Let γ be a representative of the conformal infinity and let r be the geodesic defining function associated to γ so that g has the normal form

$$g = r^{-2}(dr^2 + g_r)$$

near M with

$$g_r = \gamma - r^2 P_\gamma + \frac{r^4}{4} Q(P_\gamma)$$

where

$$Q(P_\gamma)_{ij} = \gamma^{kl}(P_\gamma)_{ik}(P_\gamma)_{jl}.$$

Then the level sets of r give a foliation near M with induced metric

$$I_r = r^{-2} g_r = r^{-2}\gamma - P_\gamma + \frac{r^2}{4} Q(P_\gamma)$$

and outward pointing normal $N_r = -r\partial_r$ where $\partial_r := \nabla_{\bar{g}} r$. Hence, the level sets of r have second fundamental form, according to our definition (3),

$$\begin{aligned} II_r &= \frac{1}{2} r\partial_r \left(r^{-2} g_r\right) = -r^{-2} g_r + \frac{1}{2} r^{-1}\partial_r g_r \\ &= -r^{-2}\gamma + P_\gamma - \frac{r^2}{4} Q(P_\gamma) - P_\gamma + \frac{r^2}{2} Q(P_\gamma) \\ &= -r^{-2}\gamma + \frac{r^2}{4} Q(P_\gamma). \end{aligned}$$

Now let $\{e_1, \ldots, e_n\}$ denote an orthonormal basis with respect to γ that diagonalizes the tensor P_γ. Then

$$\gamma(e_i, e_j) = \delta_{ij} \quad \text{and} \quad P_\gamma(e_i, e_j) = \lambda_i \delta_{ij}$$

where λ_i denotes the i^{th} eigenvalue of the tensor P_γ. Moreover,

$$\begin{aligned} I_r(e_i,e_j) &= r^{-2}\gamma(e_i,e_j) - P_\gamma(e_i,e_j) + \frac{r^2}{4}\gamma^{-1}(e_k,e_l)P_\gamma(e_i,e_k)P_\gamma(e_j,e_l) \\ &= r^{-2}\delta_{ij} - \lambda_i\delta_{ij} + \frac{r^2}{4}\delta^{kl}\lambda_i\delta_{ik}\lambda_j\delta_{jl} \\ &= r^{-2}\left(1 - r^2\lambda_i + \frac{r^4}{4}\lambda_i^2\right)\delta_{ij} \\ &= r^{-2}\left(1 - \frac{r^2}{2}\lambda_i\right)^2\delta_{ij} \end{aligned}$$

and

$$\begin{aligned} II_r(e_i,e_j) &= -r^{-2}\gamma(e_i,e_j) + \frac{r^2}{4}\gamma^{-1}(e_k,e_l)P_\gamma(e_i,e_k)P_\gamma(e_j,e_l) \\ &= -r^{-2}\delta_{ij} + \frac{r^2}{4}\delta^{kl}\lambda_i\delta_{ik}\lambda_j\delta_{jl} \\ &= -r^{-2}\left(1 - \frac{r^4}{4}\lambda_i^2\right)\delta_{ij} \\ &= -r^{-2}\left(1 - \frac{r^2}{2}\lambda_i\right)\left(1 + \frac{r^2}{2}\lambda_i\right)\delta_{ij}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left(I_r^{-1}II_r\right)(e_i,e_j) = I_r^{-1}(e_i,e_k)II_r(e_k,e_j) \\ &= -r^2\left(1 - \frac{r^2}{2}\lambda_i\right)^{-2}\delta^{ik}r^{-2}\left(1 - \frac{r^2}{2}\lambda_k\right)\left(1 + \frac{r^2}{2}\lambda_k\right)\delta_{kj} = -\frac{1 + \frac{r^2}{2}\lambda_i}{1 - \frac{r^2}{2}\lambda_i}\delta_{ij}. \end{aligned}$$

But the Weingarten matrix $I_r^{-1}II_r$ on the level sets of r satisfies

$$\left(I_r^{-1}II_r\right)(e_i,e_j) = \kappa_i^r\delta_{ij}$$

where κ_i^r denotes the i^{th} principal curvature of a level set of r with respect to the outward direction. Hence,

$$\kappa_i^r = -\frac{1 + \frac{r^2}{2}\lambda_i}{1 - \frac{r^2}{2}\lambda_i} = -\frac{2}{1 - \frac{r^2}{2}\lambda_i} + 1$$

so that

$$1 - \frac{r^2}{2}\lambda_i = \frac{2}{1 - \kappa_i^r},$$

which establishes Theorem 1.1.

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