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From 3-manifolds to modular data

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From 3-manifolds to modular data

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From 3-manifolds to modular data

To my parents, for always loving and supporting me.

To my teachers, for changing my life.

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Abstract

From 3-manifolds to modular data

by

Yang Qiu

The progress of TQFT has revealed connections between the algebraic world of tensor categories and the topological world of 3-manifolds, such as Reshetikhin-Turaev and Turaev-Viro theories. Motivated by M-theory in physics, Cho-Gang-Kim recently proposed another relation by outlining a program to construct modular data from certain classes of closed oriented 3-manifolds. In this thesis, we will talk about our mathematical exploration of this program. The main results in this thesis is based on the joint works:

- [10] Shawn X Cui, Yang Qiu, and Zhenghan Wang. From three dimensional manifolds to modular tensor categories. *arXiv preprint arXiv:2101.01674*, 2021.
- [9] Shawn X Cui, Paul Gustafson, Yang Qiu, and Qing Zhang. From torus bundles to particle-hole equivariantization. *Letters in Mathematical Physics, 112(15)*, 2022.

Contents

Chapter 1

Introduction

Quantum topology emerged from the discovery of the Jones polynomial [19] and the formulation of topological quantum field theory (TQFT) [2, 34] in the 1980s. Since then, rapid progress of the subject has revealed deep connections between the algebraic/quantum world of tensor categories and the topological/classical world of 3-manifolds. One bridge connecting these two worlds is given by TQFTs. More precisely, quantum invariants of 3-manifolds and $(2+1)$-dimensional TQFTs can be constructed from modular tensor categories, a special class of tensor categories. Two fundamental families in (2+1)-dimensions are the Reshetikhin-Turaev [26] and Turaev-Viro [31] TQFTs, both of which are based on certain tensor categories. Both families serve as vast generalizations of the Jones polynomial to knots in arbitrary 3-manifolds. Quantum invariants induced by TQFTs provide insights to understand 3-manifolds. For example, they can distinguish some homotopically equivalent but non-homeomorphic manifolds.

Recently M-theory in physics suggests another surprising different connection: classical topological invariants such as Chern-Simons invariants of $\mathrm{SL}(2,\mathbb{C})$-flat connections and $\mathrm{SL}(2,\mathbb{C})$-adjoint Reidemeister torsions of a 3-manifold X can be packaged together to produce a $(2+1)$-topological quantum field theory (TQFT) [8], which is essentially equivalent to a modular tensor category [30]. It is further conjectured in [8] that every modular tensor category can be obtained from a 3-manifold and a semi-simple Lie group. In this thesis, we study this program mathematically, and provide strong support for such a program. The program as outlined in [8] produces an algorithm to generate the potential modular T-matrix and the quantum dimensions of a candidate modular data. The modular S-matrix follows from essentially a trial-and-error procedure. Our main result is a mathematical construction of the modular data of a premodular category from each Seifert fiber space over $\mathbb{S}^2$ with three singular fibers and torus bundle over the circle with Sol geometry. The modular data constructed from Seifert fiber spaces are related to Temperley-Lieb-Jones categories and $\mathrm{SU}(2)_k$ categories, and the ones from torus bundles can be realized by $\mathbb{Z}_2$-equivariantization of certain pointed categories [10, 9]. A resulting premodular category is modular if and only if the three manifold is a $\mathbb{Z}_2$-homology sphere.

The program from 3-manifolds to modular tensor categories is a far-reaching progeny of the mysterious six-dimensional super-symmetric conformal field theories (SCFTs) spawned by M-theory. Our strong support for the program indirectly provides evidence for these 6d SCFTs. The dimension reduction or compactification of these 6d SCFTs to 3d depends on a 3-manifold X, and in general the resulting theory $T(X)$ is a super-conformal field theory. When X is non-hyperbolic, it is argued in [8] that $T(X)$ flows to a TQFT in the infrared limit and super-symmetry is decoupled, thus we obtain a (2+1)-TQFT labeled by X, hence a MTC $\mathcal{C}_X$. The program outlined in [8] centers on an algorithm to produce the quantum dimensions and topological twists of a MTC, and a trial-and-error algorithm for the modular S-matrix. The assumption on the three manifolds X in [8] includes that X is non-hyperbolic and the $\mathrm{SL}(2,\mathbb{C})$ representation variety of the fundamental group $\pi_1(X)$ consists of finitely many conjugacy classes that all could be conjugated into either $SU(2)$ or $\mathrm{SL}(2,\mathbb{R})$ subgroups of $\mathrm{SL}(2,\mathbb{C})$. Our examples show that all but the non-hyperbolic assumption can be dropped. One subtlety is that we need to use indecomposable reducible representations in our torus bundle over the circle examples and certain Seifert fiber spaces examples.

The efforts in [8] and [10, 9] suggest a far-reaching connection between 3-manifolds and (pre)modular tensor categories. However, this program is still at its infancy, and there remain many questions to be resolved. First and foremost, the program currently only provides an algorithm to compute the modular S- and T-matrices. Other data such as the F-symbols and R-symbols, which specify the associators and braidings, respectively [32], are still missing. Secondly, even for the modular data, the computation for the S-matrix essentially follows a trial-and-error procedure. A definite algorithm to achieve that is in demand. Thirdly, there are also a number of subtleties in choosing the correct set of characters as simple objects, determining the proper unit object, etc. We hope the insights obtained will lead to an intrinsic understanding of how and why this program works.

The content of the thesis is as follows. In Chapter 2, we review the ingredients about 3-manifolds contained in the program. In Chapter 3, we recall the notions about premodular tensor categories. In Chapter 4, we outline our version of the program and show its application on Seifert fiber spaces and torus bundles as mentioned before. In chapter 5, we discuss the related future questions that we will work on in the future.

Chapter 2

Aspects about 3-manifolds

2.1 Non-hyperbolic 3-manifolds

In this thesis, we mainly consider oriented closed compact 3-manifolds with 6 of 8 Thurston's geometries including $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}(2, \mathbb{R})}$, Nil, and Sol. Especially, we focus on Seifert fiber spaces over $\mathbb{S}^2$ with three singular fibers which refer to the first 5 geometries, and torus bundles over $\mathbb{S}^1$ with Sol geometry.

2.1.1 Seifert fiber spaces

We will recall some basics about Seifert fiber spaces from [28].

Definition 2.1.1. A Seifert fiber space (SFS) M is a closed 3-manifold together with a decomposition into a disjoint union of circles (called fibers) such that each fiber has a tubular neighborhood that forms a standard fibered torus.

We can denote the SFSs by the notation

$$M = \{b; (o, g); (p_1, q_1), (p_2, q_2), \cdots, (p_n, q_n)\}$$

as explained below. The quotient space of a SFS M, called the base orbifold B, by sending each circle, called a fiber, to a point is a topological surface. The symbol (o, g) means that the base topological surface B is an orientable closed surface of genus g.

Remark 2.1.2. Here we only consider the orientable Seifert fiber spaces M with an orientable base surface B. Generally, neither of M and B is necessarily orientable.

Each fiber has a product neighbourhood $D^2 \times S^1$ in the SFS M except n singular fibers labeled by $(p_i, q_i), i = 1, \cdots, n$. The neighborhood of the i-th singular fiber is obtained from $D^2 \times [0,1]$ by identifying the point $(x, 0), x \in D^2$ with the point $(r_{a_i,p_i}(x), 1)$, where r_{a_i,p_i} is the rotation of the disk D^2 by the angle $2\pi a_i/p_i$, where $a_i \in \mathbb{Z}$ satisfies $a_i q_i = 1 \mod p_i$. The pair of coprime integers (p_i, q_i) are the corresponding surgery coefficient. The fundamental group of M fits into a short exact sequence

$$1 \to \pi_1(F) \to \pi_1(M) \to \pi_1^{orb}(B) \to 1,$$

where $\pi_1(F) \cong \mathbb{Z}$ for a regular fiber $F \cong \mathbb{S}^1$ and $\pi_1^{orb}(B)$ is the orbifold fundamental group of B (not the same as the fundamental group $\pi_1(B)$ of the topological surface B in general). The integer b in the notation is the obstruction class, which is also the order of the generator of $\pi_1(F)$ in $\pi_1^{orb}(B)$.

The fundamental group of $M = \{b; (o, g); (p_1, q_1), (p_2, q_2), \cdots, (p_n, q_n)\}$ has a presentation

$$\begin{aligned}\pi_1(M) = \langle a_j, b_j, x_i, h,\ j = 1, \cdots, g, i = 1, \cdots, n \mid \\ [a_j, h] = [b_j, h] = [x_i, h] = x_i^{p_i} h^{q_i} = 1,\ x_1 \cdots x_n [a_1, b_1] \cdots [a_g, b_g] = h^b \rangle. \quad (2.1)\end{aligned}$$

In particular, the fundamental group of $M = \{0; (o, 0); (p_1, q_1), (p_2, q_2), (p_3, q_3)\}$ over base $\mathbb{S}^2$ and with three singular fibers, denoted simply as $\{(p_1, q_1), (p_2, q_2), (p_3, q_3)\}$ sometimes, is

$$\pi_1(M) = \langle x_1, x_2, x_3, h | x_i^{p_i} h^{q_i} = 1, x_i h = h x_i, x_1 x_2 x_3 = 1 \rangle. \tag{2.2}$$

The following changes for the symbol will not change the homeomorphism type of M.

(1) Change the sign of both p_i, q_i.

(2) Add 1 to b and subtract p_i from q_i.

(3) Add a fiber of type $(1, 0)$.

Since we consider SFSs as 3-manifolds up to homeomorphism rather than as fibered spaces, we may always set b to 0.

Given $M = \{b; (o, g); (p_1, q_1), (p_2, q_2), \cdots, (p_n, q_n)\}$, define the Euler number $e(M)$ of M and the Euler characteristic $\chi(B)$ of its base orbifold B by

$$e(M) = b + \sum_{i=1}^{n} \frac{q_i}{p_i},$$

$$\chi(B) = \chi(B_0) - \sum_{i=1}^{n} (1 - \frac{1}{p_i}),$$

where $\chi(B_0)$ is the usual Euler characteristic of the underlying topological surface B_0 of the orbifold B. The behavior of M depends on the sign of $e(M)$ and $\chi(B)$.

We can give a surgery diagram for $\{0; (o, 0); (p_1, q_1), (p_2, q_2), \cdots, (p_n, q_n)\}$ as shown in Fig. 2.1. Here we remove the regular neighbourhood of each component of the link and reattach a solid torus with the corresponding coefficient besides it. For the presentation of $\pi_1(M)$ in (2.2), x_i corresponds to the meridian of i-th vertical circle and h corresponds to the meridian of the horizontal circle.

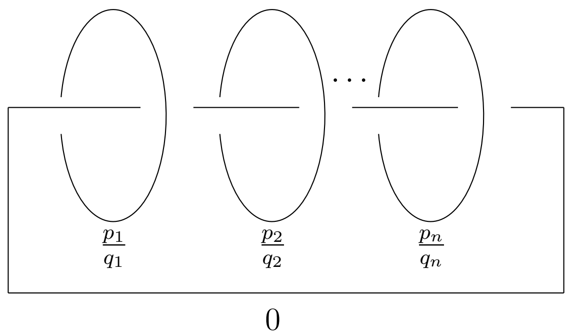

Figure 2.1

2.1.2 Torus bundles with Sol geometry

Definition 2.1.3. A torus bundle M over $\mathbb{S}^1$ is a mapping torus of 2-torus with a gluing map $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$.

The fundamental group of M has the presentation,

$$\pi_1(M) = \langle x, y, h \mid x^a y^c = h^{-1} x h,\ x^b y^d = h^{-1} y h,\ x y x^{-1} y^{-1} = 1 \rangle, \tag{2.3}$$

where x and y are the meridian and longitude, respectively, on the torus, and h corresponds to a loop around the base $\mathbb{S}^1$.

In this thesis, we pay attention to the torus bundles with Sol geometry, i.e. $|a+d| > 2$. We will briefly recall Thurston's geometries in next subsection.

2.1.3 Thurston's 8 geometries

Let X be a simply connected smooth manifold and G be a Lie group. A model geometry is a pair (X, G) together with a transitive action of G on X with compact stabilizers. A model geometry is called maximal if G is maximal among groups acting smoothly and transitively on X with compact stabilizers.

A geometric structure on a manifold M is a diffeomorphism from M to X/Γ for some model geometry (X, G), where Γ is a discrete subgroup of G acting freely on X. If a given manifold admits a geometric structure, then it admits one whose model is maximal.

Thurston has classified 3-dimensional geometries and there are 8 geometries as the following theorem shows.

Theorem 2.1.4 (Thurston)**.** Any maximal, simply connected, 3-dimensional geometry which admits a compact quotient is equivalent to one of the 8 geometries: $\mathbb{H}^3$, $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, Nil, and Sol.

Scott [27] has proved the uniqueness of the geometric structure on a closed 3-manifold.

Theorem 2.1.5 (Scott)**.** If M is a closed 3-manifold which admits a geometric structure modelled on one of the eight geometries, then the geometry involved is unique.

Moreover, Scott classified the closed 3-manifolds which admit a non-hyperbolic geometric structure.

Theorem 2.1.6 (Scott)**.** Let M be an oriented closed 3-manifold.

(1) M admits a geometric structure modelled on Sol if and only if M is finitely covered by a torus bundle over $\mathbb{S}^1$ with an Anosov gluing map which is an automorphism of the 2-torus given by an invertible 2 by 2 matrix whose eigenvalues are real and distinct.

(2) M admits a geometric structure modelled on one of $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, or Nil if and only if M is a Seifert fiber space. Furthermore the geometry for M is determined by $\chi(B)$ and $e(M)$ as follows:

Remark 2.1.7. Seifert fiber spaces account for all oriented closed manifolds in 6 of 8 Thurston geometries including $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ and Nil geometry. Since fundamental groups of manifolds with $\mathbb{S}^3$ geometry are Abelian and we focus on non-Abelian $\mathrm{SL}(2,\mathbb{C})$ representations of fundamental groups in this paper, our examples refer to all nonhyperbolic geometries except $\mathbb{S}^3$ geometry.

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$\mathbb{S}^2 \times \mathbb{R}$	$\mathbb{E}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$e \neq 0$	$\mathbb{S}^3$	Nil	$\widetilde{\mathrm{SL}(2, \mathbb{R})}$

2.2 SL$(2, \mathbb{C})$ character

2.2.1 Non-Abelian character

We recall some algebraic sets of $\mathrm{SL}(2, \mathbb{C})$ characters of 3-manifolds. Suppose M is an orientable connected closed compact 3-manifold. Then $\pi_1(M)$ is a finitely generated group.

Definition 2.2.1. A *$SL(2, \mathbb{C})$ representation* of M is a homomorphism $\rho : \pi_1(M) \longrightarrow \mathrm{SL}(2, \mathbb{C})$. A *$SL(2, \mathbb{C})$ character* is the function $\chi_\rho : \pi_1(M) \longrightarrow \mathrm{SL}(2, \mathbb{C})$ by $\chi_\rho(a) = \mathrm{Tr}(\rho(a))$.

Set $G = \mathrm{SL}(2, \mathbb{C})$. Denote by $R(M, G)$ the set of $\mathrm{SL}(2, \mathbb{C})$ representations, by $\chi(M, G))$ the set of $\mathrm{SL}(2, \mathbb{C})$ characters. There is a natural map $t : R(M, G) \longrightarrow \chi(M, G)$. Both $R(M, G)$ and $\chi(M, G)$ admit the structure of a affine algebraic variety such that $\chi(M, G)$ is an algebro-geometric quotient of $R(M, G)$. In this thesis, we do not consider this structure.

There are three obvious nontrivial automorhphisms of $\mathrm{SL}(2, \mathbb{C})$ by sending an element $g \in \mathrm{SL}(2, \mathbb{C})$ to its complex conjuagte g^*, its transpose followed by inverse $(g^t)^{-1}$, and the composition $(g^\dagger)^{-1}$ of the previous two operations. For each representation of $\pi_1(M)$ to $\mathrm{SL}(2, \mathbb{C})$, post-composing with one of the three automorhphisms of $\mathrm{SL}(2, \mathbb{C})$ gives rise to another representation, hence representations in $R(M, G)$ come in group of four in general. Another obvious way to change a representation ρ in $R(M, G)$ is to tensor ρ with a representation of $\pi_1(M)$ to the center $Z(G)$ of G. Representations of $\pi_1(M)$ to the center $Z(G)$ are in one-to-one correspondence with cohomology classes in the cohomology group $H^1(M, Z(G))$.

For our purpose, we consider the non-Abelian characters as follows.

Definition 2.2.2. Let $\chi \in \chi(M, G)$ be an $\mathrm{SL}(2, \mathbb{C})$-character of a 3-manifold M. We say χ is *non-Abelian* if at least one representation $\rho : \pi_1(M) \to \mathrm{SL}(2, \mathbb{C})$ with character χ is non-Abelian, i.e. ρ has non-Abelian image in $\mathrm{SL}(2, \mathbb{C})$. The set of all non-Abelian characters of M is denoted by $\chi^{\mathrm{nab}}(M)$.

Remark 2.2.3. Irreducible representations do not share their characters with reducible ones, and the characters of irreducible representations one-to-one correspond to the conjugacy classes of irreducible representations [11]. Thus the set of non-Abelian characters is the union of two disjoint parts: irreducible characters and reducible ones. For a reducible character, there can exist two representations within the same character, but different conjagacy classes.

2.2.2 Computation

We compute the non-Abelian characters of the examples we will consider.

Let $M = \{0; (o, 0); (p_1, q_1), (p_2, q_2), (p_3, q_3)\}$ be a Seifert fiber space over $\mathbb{S}^1$ with singular fibers, and its fundamental group has the following presentation,

$$\pi_1(M) \ = \ \langle\, x_1, x_2, x_3, h \,|\, x_k^{p_k} h^{q_k} = 1,\ x_k h = h x_k,\ x_1 x_2 x_3 = 1, k = 1, 2, 3 \,\rangle$$

Let $\rho : \pi_1(M) \to G$ be a non-Abelian representation. Since h is in the center of $\pi_1(M)$ and ρ is non-Abelian, $\rho(h)$ must be $\pm I$. It follows that each $\rho(x_k)$ has finite order, and is diagonalizable in particular. Moreover, any $\rho(x_k)$ does not commute with another $\rho(x_j)$. This implies neither $\rho(x_k)$ can be $\pm I$. Up to conjugation, we assume $\rho(x_k)$ take the following form (writing $\rho(x_k)$ simply as x_k),

$$x_1 = \begin{pmatrix} e^{i\alpha_1} & 0 \\ 0 & e^{-i\alpha_1} \end{pmatrix}, x_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} e^{i\alpha_2} & 0 \\ 0 & e^{-i\alpha_2} \end{pmatrix}, x_3 \sim \begin{pmatrix} e^{i\alpha_3} & 0 \\ 0 & e^{-i\alpha_3} \end{pmatrix} \tag{2.4}$$

where $0 < \alpha_k < \pi$, $ad - bc = 1$, and b and c are not simultaneously zero. We have the following linear equations for a and d.

$$\mathrm{Tr}(x_2) = e^{i\alpha_2} + e^{-i\alpha_2} = a + d \tag{2.5}$$

$$\mathrm{Tr}(x_3) = e^{i\alpha_3} + e^{-i\alpha_3} = ae^{i\alpha_1} + de^{-i\alpha_1} \tag{2.6}$$

Hence, given the α_k's, or equivalently $\mathrm{Tr}(x_k)$, a and d are uniquely determined, and $a = \bar{d}$. Moreover, when $|a| \neq 1$ implying $bc \neq 0$, this also determines ρ up to conjugacy. When $|a| = 1$ implying $bc = 0$, there are precisely two conjugacy classes with

$$x_2 = \begin{pmatrix} a & 1 \\ 0 & \bar{a} \end{pmatrix} \text{ or } x_2 = \begin{pmatrix} a & 0 \\ 1 & \bar{a} \end{pmatrix} \tag{2.7}$$

It can be checked that these two representations are complex conjugate to each other up to conjugacy, and that their characters take real values. They give rise to the same character. There are two types of non-Abelian representations. One type is irreducible satisfying $b, c \neq 0$. Characters of representations of this type one-to-one correspond to conjugacy classes of representations [11]. The other type is reducible with exactly one of b, c zero. Each character of this type corresponds to two conjugacy classes.

To summarize, the triple $(\alpha_1, \alpha_2, \alpha_3)$ and $\mathrm{Tr}(h)$ uniquely determine the character. Next, we find all possible such triples.

If $h = I$, each $e^{i\alpha_k}$ is a p_k-th root of 1. If $h = -I$, then $e^{i\alpha_k}$ is a p_k-th root of 1 if q_k is even, and a p_k-th root of -1 if q_k is odd. We claim all triples satisfying the above conditions can be realized by some representations. Indeed, given such a triple $(\alpha_1, \alpha_2, \alpha_3)$, we define $\rho(x_1)$ and $\rho(x_2)$ as in Equation 2.4 and let $\rho(x_3) := (\rho(x_1)\rho(x_2))^{-1}$. Equations 2.5, 2.6 determine a and d, and we arbitrarily choose b and c such that $ad-bc = 1$. Again, Equations 2.5, 2.6 guarantee that $\rho(x_k)$ so defined has eigenvalues $e^{\pm i\alpha_k}$, and therefore they satisfy all the relations in the presentation of $\pi_1(M)$.

Set $\alpha_k = \frac{2\pi n_k}{p_k}$ and $\rho(h) = e^{2\pi i\lambda} I$, $\lambda = 0, \frac{1}{2}$. If $\lambda = 0$ or if $\lambda = \frac{1}{2}$ and q_k is even, then n_k is an integer strictly between 0 and $\frac{p_k}{2}$. If $\lambda = \frac{1}{2}$ and q_k is odd, then n_k is a proper half integer strictly between 0 and $\frac{p_k}{2}$. The quadruple (n_1, n_2, n_3, λ) completely characterizes a character.

For an integer $p > 0$, denote by $[0 \cdots p]$ the set of integers $\{0, 1, \cdots, p\}$, and by $[0 \cdots p]^e$ (resp. $[0 \cdots p]^o$) the subset of even (resp. odd) integers in $[0 \cdots p]$. The non-Abelian character variety of M is given as follows,

$$\begin{aligned} \chi^{\text{nab}}(M) = & \left\{ \left(\frac{j_1+1}{2}, \frac{j_2+1}{2}, \frac{j_3+1}{2}, \frac{1}{2} \right) \mid j_k \in [0 \cdots p_k - 2]^{\epsilon_k} \right\} \\ & \sqcup \left\{ \left(\frac{j_1+1}{2}, \frac{j_2+1}{2}, \frac{j_3+1}{2}, 0 \right) \mid j_k \in [0 \cdots p_k - 2]^{o} \right\}, \end{aligned} \tag{2.8}$$

where $\epsilon_k =$ 'e' if q_k is odd, and $\epsilon_k =$ 'o' otherwise. For $(n_1, n_2, n_3, \lambda) \in \chi^{\text{nab}}(M)$, a corresponding representation ρ has $e^{\pm \frac{2\pi i n_k}{p_k}}$ as the eigenvalue of $\rho(x_k)$ and $\rho(h) = e^{2\pi i\lambda} I$.

The size of $\chi^{\text{nab}}(M)$ is

$$|\chi^{\text{nab}}(M)| = \lfloor \frac{p_1}{2} \rfloor \lfloor \frac{p_2}{2} \rfloor \lfloor \frac{p_3}{2} \rfloor + \lfloor \frac{p_1-1}{2} \rfloor \lfloor \frac{p_2-1}{2} \rfloor \lfloor \frac{p_3-1}{2} \rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

For instance, if all the q_k's are odd, then $\chi^{\mathrm{nab}}(M)$ can also be written as,

$$\chi^{\mathrm{nab}} = \Big\{ \big(\frac{j_1+1}{2}, \frac{j_2+1}{2}, \frac{j_3+1}{2}, \frac{(j_1+1) \mod 2}{2}\big) \mid j_k \in [0 \cdots p_k - 2],\ j_1 = j_2 = j_3 \mod 2 \Big\}$$

Remark 2.2.4. M has reducible characters if and only if the greatest common divisor of p_1, p_2, p_3 is 1.

Let M be a torus bundle over $\mathbb{S}^1$ with monodromy map $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ such that $|a+d| > 2$. For simplicity, we assume that $a, b, c, d > 0$ and set $N = a + d + 2$. Its fundamental group has the presentation

$$\pi_1(M) = \langle x, y, h \mid x^a y^c = h^{-1} x h,\ x^b y^d = h^{-1} y h,\ x y x^{-1} y^{-1} = 1 \rangle, \tag{2.9}$$

Let $\rho : \pi_1(M) \longrightarrow \mathrm{SL}(2, \mathbb{C})$ be a non-Abelian representation. First, we consider the case where $\rho(x)$ is diagonalizable. Up to conjugation, assume $\rho(x)$ is diagonal. Since y commutes with x, $\rho(y)$ is also diagonal, and moreover, $\rho(x)$ and $\rho(y)$ cannot be both contained in the center $\{\pm I\}$. (Otherwise, the image of ρ would be Abelian.) If $\rho(x) \neq \pm I$, it follows from the relation $x^a y^c = h^{-1} x h$ that $\rho(h)$, up to conjugation, simply permutes the two eigenvectors of $\rho(x)$. The same conclusion is obtained if $\rho(y) \neq \pm I$. Hence, we may assume ρ takes the following form (abbreviating $\rho(x)$ simply as x),

$$x = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad y = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.10}$$

where $\mathrm{Im}(\alpha) \geq 0$ and either $\alpha \neq \pm 1$ or $\beta \neq \pm 1$. The presentation of $\pi_1(M)$ yields the following equations for ρ,

$$\alpha^{a+1}\beta^{c} \;=\; \alpha^{b}\beta^{d+1} \;=\; 1, \tag{2.11}$$

from which we deduce the relations,

$$\alpha^{a+d+2} \;=\; \beta^{a+d+2} \;=\; 1. \tag{2.12}$$

Let $N = |a+d+2|$. Hence α and β are both N-th root of unity. Set $\alpha = e^{\frac{2\pi i\,k}{N}},\ \beta = e^{\frac{2\pi i\,l}{N}}$ such that $0 \le k \le \frac{N}{2},\ 0 \le l < N$, and either $k \ne 0, \frac{N}{2}$ or $l \ne 0, \frac{N}{2}$. Then, Equation 2.11 can be equivalently written as,

$$\begin{aligned} (a+1)\,k + c\,l &= \mu N \\ b\,k + (d+1)\,l &= \nu N \end{aligned} \tag{2.13}$$

Since the coefficient matrix for Equation 2.13 is nonsingular (its determinant is $\pm N$), each irreducible representation is determined by the pair (μ, ν) and denoted $Y(\mu, \nu)$.

Next we consider the case where $\rho(x)$ is not diagonalizable. Then neither is $\rho(y)$ diagonalizable. Up to conjugation, we may assume that $\rho(x)$ and $\rho(y)$ are both upper triangular, each have a single eigenvalue $+1$ or -1 lying on the diagonal, and $\rho(h)$ is diagonal. Thus, ρ takes the form,

$$x = (-1)^{\epsilon_x}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y = (-1)^{\epsilon_y}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}, \tag{2.14}$$

where $\epsilon_x, \epsilon_y \in \{0, 1\}$ and $u \ne 0$. From the presentation of $\pi_1(M)$, we deduce the equations to be satisfied,

$$\begin{aligned}(a+1)\,\epsilon_x + c\,\epsilon_y &= 0 \mod 2\\ b\,\epsilon_x + (d+1)\,\epsilon_y &= 0 \mod 2\end{aligned} \tag{2.15}$$

$$c\,u^2 + (a-d)u - b = 0, \quad v^2 = \frac{1}{cu+a}. \tag{2.16}$$

Equation 2.16 is equivalent to,

$$(v+v^{-1})^2 = a+d+2, \quad u = \frac{v^{-2}-a}{c}. \tag{2.17}$$

From Equation 2.17, we see that for each fixed ϵ_x and ϵ_y, there are four inequivalent representations, but only two characters. We choose a representative for each character by setting,

$$u = \frac{d-a+\sqrt{(a+d)^2-4}}{2c}, \quad v^2 = \frac{1}{cu+a} = \frac{a+d-\sqrt{(a+d)^2-4}}{2}. \tag{2.18}$$

The solution set to Equation 2.15 depends on the parity of the entries of the monodromy matrix. Let P be the quadruple that records the parity of the entries $(a,d;b,c)$ and we use 'e' to denote for 'even' and 'o' for 'odd'. For instance, $P=(e,e;o,e)$ means b is odd and the rest are even. The solutions contain the following possible values for ϵ_x and ϵ_y,

- $\epsilon_x = 0,\ \epsilon_y = 0$;
- $\epsilon_x = 1,\ \epsilon_y = 1$, only if $P=(e,e;o,o)$ or $P=(o,o;e,e)$;
- $\epsilon_x = 0,\ \epsilon_y = 1$, only if $P=(o,o;o,e)$ or $P=(o,o;e,e)$;

- $\epsilon_x = 1,\ \epsilon_y = 0$, only if $P = (o, o; e, o)$ or $P = (o, o; e, e)$.

We can also refer to pairs (ϵ_x, ϵ_y) in (μ, ν)-coordinates using Equation 2.13 and defining $k = \epsilon_x(N/2)$ and $l = \epsilon_y(N/2)$. From Equation 2.17, we see that for each fixed ϵ_x and ϵ_y, there are four inequivalent representations but only two characters, which we denote by $X^{\pm}(\mu, \nu)$.

To summarize, the non-Abelian characters of M contain two types, the irreducible and the reducible ones. The irreducible characters take the form of Equation 2.10 and are determined by Equation 2.13. The reducible characters take the form of Equation 2.14 and are determined by Equation 2.18 and the possible values of ϵ_x and ϵ_y discussed above.

We consider solutions (k, l) of Equation 2.13 in $\mathbb{Z}_N \times \mathbb{Z}_N$. Note that, for now we do not place any additional restrictions on the solutions. We denote this solution space by G.

Lemma 2.2.5. G is a subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$ isomorphic to $\mathbb{Z}_r \times \mathbb{Z}_{\frac{N}{r}}$, where $r = gcd(a+1, c, b, d+1)$.

Proof: Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_N \times \mathbb{Z}_N$ be the group homomorphism given by

$$f\begin{pmatrix}\mu\\ \nu\end{pmatrix} = \begin{pmatrix} d+1 & -c \\ -b & a+1 \end{pmatrix}\begin{pmatrix}\mu\\ \nu\end{pmatrix}$$

The solution space G is the image of f and a subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$.

Define the chain complex $\mathbb{Z} \times \mathbb{Z} \xrightarrow{g} \mathbb{Z} \times \mathbb{Z} \xrightarrow{f} \mathbb{Z}_N \times \mathbb{Z}_N$ where $g = \begin{pmatrix} a+1 & c \\ b & d+1 \end{pmatrix}$. Then $\mathrm{Im}(f) \cong \mathbb{Z} \times \mathbb{Z}/\mathrm{Ker}(f)$ and $\mathrm{Ker}(f) = \mathrm{Im}(g)$. By considering the Smith normal form of g, we obtain an isomorphism $G \cong \mathbb{Z}_r \times \mathbb{Z}_{N/r}$ where $r = gcd(a+1, c, b, d+1)$. $\square$

We can use G to characterize non-Abelian characters of M by the following lemma.

Lemma 2.2.6. The irreducible characters $Y(\mu,\nu)$ of M are in one-to-one correspondence with subsets $\{g,-g\}\subset G$ where $2g\neq 0$. In addition, the pairs $X^{\pm}(\mu,\nu)$ of reducible non-Abelian characters are in one-to-one correspondence with elements $g\in G$ such that $2g=0$.

Proof: Suppose that $(\mu,\nu)\in G$ corresponds to a representation ρ as in Equation 2.10 which is not necessarily non-Abelian. We first show that ρ is non-Abelian if and only if $2(\mu,\nu)\neq 0$. According to the previous subsection, ρ is non-Abelian if and only if $\rho(x),\rho(y)$ do not both take values in $\{I,-I\}$, which is equivalent to the statement that $\rho(x^2),\rho(y^2)$ are not both I. Since $2(\mu,\nu)$ corresponds to the representation $(x\mapsto \rho(x^2), y\mapsto \rho(y^2), h\mapsto \rho(h))$, the claim follows from the fact that the representation $(x\mapsto I, y\mapsto I, h\mapsto \rho(h))$ corresponds to $0\in G$.

Suppose that $(\mu_1,\nu_1),(\mu_2,\nu_2)\in G$ correspond to the same irreducible character. Let (k_1,l_1) and (k_2,l_2) be the corresponding solutions to Equation 2.13, and ρ_1 and ρ_2 be the corresponding representations as defined in Equation 2.10. Then either $\rho_1(x)=\rho_2(x)$ and $\rho_1(y)=\rho_2(y)$, or $\rho_1(x)=\rho_2(x^{-1})$ and $\rho_1(y)=\rho_2(y^{-1})$, which implies that $(\mu_1,\nu_1)=\pm(\mu_2,\nu_2)$. This proves the first part of the lemma.

For the second part, let ρ denote a reducible non-Abelian representation, and let $\epsilon_x,\epsilon_y\in\{0,1\}$ be the corresponding sign exponents as defined in Equation 2.14. By considering the diagonal entries of $\rho(x)$ and $\rho(y)$, such a representation ρ exists if and only if the following equations are satisfied.

$$(a+1)\epsilon_x\frac{N}{2}+c\epsilon_y\frac{N}{2}=\mu N$$

$$b\epsilon_x\frac{N}{2}+(d+1)\epsilon_y\frac{N}{2}=\nu N$$

The solutions of above equations are in one-to-one correspondence with elements in G of order 1 or 2. Fixing (ϵ_x, ϵ_y), the corresponding characters occur in pairs $X^{\pm}(\mu, \nu)$. This proves the second part of the lemma. $\square$

2.3 Adjoint Reidemeister torsion

The Reidemeister torsion (R-torsion) $\tau(M)$ of a celluation K_M of a manifold M uses the action of the fundamental group $\pi_1(M)$ on the universal cover $\widetilde{K_M}$ to measure the complexity of the celluation of M. It is a topological invariant of M from determinants of matrices obtained from the incidences of the cells of $\widetilde{K_M}$. The R-torsion makes essential use of the bases in the chain complex of the universal cover, while the homology and homotopy groups do not see the geometric information encoded in the based chain complex. For our purpose, we need the non-Abelian generalization of R-torsion twisted by a representation $\rho : \pi_1(X) \to G$ for some semi-simple Lie group G, in particular the Reidemeister torsion for the adjoint representation of $\mathrm{SL}(2, \mathbb{C})$, introduced in [25]. We recall some basics here, for more details, please refer to [23, 25, 29].

Let

$$C_* = (0 \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \longrightarrow 0)$$

be a chain complex of finite dimensional vector spaces over the field $\mathbb{C}$. Choose a basis c_i of C_i and a basis h_i of the i-th homology group $H_i(C_*)$. The torsion of C_* with respect to these choices of bases is defined as follows. For each i, let b_i be a set of vectors in C_i such that $\partial_i(b_i)$ is a basis of $\mathrm{Im}(\partial_i)$ and let $\tilde{h}_i$ denote a lift of h_i in $\mathrm{Ker}(\partial_i)$. Then the set of vectors $\tilde{b}_i := \partial_{i+1}(b_{i+1}) \sqcup \tilde{h}_i \sqcup b_i$ is a basis of C_i. Let D_i be the transition matrix from c_i to $\tilde{b}_i$. To be specific, each column of D_i corresponds to a vector in $\tilde{b}_i$ being expressed as a linear combination of vectors in c_i. Define the torsion

$$\tau(C_*, c_*, h_*) := \left| \prod_{i=0}^{n} \det(D_i)^{(-1)^{i+1}} \right|$$

Remark 2.3.1.

(1) The torsion, as it is denoted, does not depend on the choice of b_i and the lifting of h_i.

(2) Here we define the torsion as the norm of the usual torsion, thus we do not need to deal with sign ambiguities.

Let M be a finite CW-complex and (V, ρ) be a homomorphism $\rho : \pi_1(M) \longrightarrow \mathrm{SL}(V)$. The vector space V turns into a left $\mathbb{Z}[\pi_1(X)]$-module. The universal cover $\tilde{M}$ has a natural CW structure from M, and its chain complex $C_*(\widetilde{M})$ is a free *left* $\mathbb{Z}[\pi_1(M)]$-module via the action of $\pi_1(M)$ as covering transformations. View $C_*(\tilde{M})$ as a *right* $\mathbb{Z}[\pi_1(M)]$-module by $\sigma.g := g^{-1}.\sigma$ for $\sigma \in C_*(\widetilde{M})$ and $g \in \pi_1(M)$. We define the twisted chain complex $C_*(M; \rho) := C_*(\widetilde{M}) \otimes_{\mathbb{Z}[\pi_1(M)]} V$. Let $\{e^i_\alpha\}_\alpha$ be the set of i-cells of M ordered in an arbitrary way. Choose a lifting $\tilde{e}^i_\alpha$ of e^i_α in $\widetilde{M}$. It follows that $C_i(\widetilde{M})$ is generated by $\{\tilde{e}^i_\alpha\}_\alpha$ as a free $\mathbb{Z}[\pi_1(M)]$-module (left or right). Choose a basis of $\{v_\gamma\}_\gamma$ of V. Then $c_i(\rho) := \{\tilde{e}^i_\alpha \otimes v_\gamma\}$ is a $\mathbb{C}$-basis of $C_i(M; \rho)$.

Definition 2.3.2. Let $\rho : \pi_1(M) \longrightarrow \mathrm{SL}(2, \mathbb{C})$ be a homomorphism, $\mathrm{Adj} : \mathrm{SL}(2, \mathbb{C}) \longrightarrow \mathrm{GL}(\mathfrak{sl}_2(\mathbb{C}))$ be the adjoint representation of $\mathrm{SL}(2, \mathbb{C})$ on its Lie algebra. We call $\rho_{\mathrm{ad}} = \mathrm{Adj} \circ \rho$ an *adjoint $SL(2, \mathbb{C})$ representation* of M. If $C_*(M, \rho_{\mathrm{ad}})$ is acyclic, we call ρ is *adjoint acyclic*. Define the *adjoint Reidemeister torsoin* of ρ to be

$$\mathrm{Tor}(M, \rho) := \tau(M, \rho_{\mathrm{ad}}).$$

Remark 2.3.3. In this thesis, we will only deal with the adjoint Reidemeister torsion of ρ. For that matter, we simply call it the torsion of ρ. When no confusion arises, we abbreviate $\mathrm{Tor}(M, \rho)$ as $\mathrm{Tor}(\rho)$.

The following result will be useful in computing torsions.

Multiplicativity Lemma Let $0 \longrightarrow C'_* \longrightarrow C_* \longrightarrow C''_* \longrightarrow 0$ be an exact sequence of chain complexes. Assume that C_*, C'_*, C''_* are based by c_*, c'_*, c''_*, respectively, and their homology groups based by h_*, h'_*, h''_*, respectively. Associated to the short exact sequence is the long exact sequence H_* in homology

$$\cdots \longrightarrow H_j(C'_*) \longrightarrow H_j(C_*) \longrightarrow H_j(C''_*) \longrightarrow\longrightarrow H_{j-1}(C'_*) \longrightarrow \cdots$$

with the reference bases. For each i, identify c'_i with its image in C_i and arbitrarily choose a preimage $\tilde{c}''_i$ of c''_i in C_i. If the transition matrix between the bases c_i and $c'_i \sqcup \tilde{c}''_i$ has determinant ± 1, we call c_*, c'_*, c''_* compatible. In this case, we have

$$\tau(C_*, c_*, h_*) \ = \ \tau(C'_*, c'_*, h'_*) \ \tau(C''_*, c''_*, h''_*) \ \tau(H_*, \{h_* \sqcup h'_* \sqcup h''_*\}).$$

2.4 Chern-Simons invariant

In this section, we recall some basics about Chern-Simons invariants. Given an orientable connected closed three manifold M, a morphism ρ of its fundamental group $\pi_1(X)$ to a simply connected semi-simple Lie group G can be identified as the holonomy representation of a flat connection A_ρ on the trivial principal G-bundle over X. Therefore, in the following we will use the terms a representation ρ and a flat connection A_ρ interchangeably via such an identification.

Definition 2.4.1. Let M be an oriented closed 3-manifold and $\rho : \pi_1(M) \longrightarrow \mathrm{SL}(2, \mathbb{C})$

be a $\mathrm{SL}(2,\mathbb{C})$ representation. Denote by A_ρ the corresponding Lie algebra $\mathfrak{sl}_2(\mathbb{C})$-valued 1-form on M. The *Chern-Simons (CS) invariant* of (M,ρ) is defined as

$$\mathrm{CS}(\rho) = \frac{1}{8\pi^2}\int_M \mathrm{Tr}(dA_\rho \wedge A_\rho + \frac{2}{3}A_\rho \wedge A_\rho \wedge A_\rho) \mod 1, \tag{2.19}$$

where the integral with its coefficient in the front is well-defined up to integers.

Remark 2.4.2.

(1) The CS invariant depends on the orientation on M. To be specific, let $\overline{M}$ be the same manifold as M with reverse orientation, then $\mathrm{CS}(M) = -\mathrm{CS}(\overline{M}) \mod 1$.

(2) The CS invariant $\mathrm{CS}(\rho)$ depends only on the character $\chi(\rho)$ of ρ [21], hence it descends from the representation variety $R(M)$ to the character variety $\chi(M)$.

(3) Generally, the CS invariant of a representation can be a complex number, such as the one of the holonomy representation of the hyperbolic structure on a hyperbolic manifold. However, the CS invariants of the examples we consider in this thesis are all rational numbers.

CS invariants can be computed by cutting manifold M into several pieces. For our purpose, we recall the method proposed by Klassen and Kirk in [21].

Let T be a torus and consider $\chi(T)$, the character variety of T to $\mathrm{SL}(2,\mathbb{C})$. It is direct to see that $\chi(T)$ can be identified with $\mathrm{Hom}(\pi_1(T),\mathbb{C}^*)/\sim$ where $f \sim g$ if $f(\cdot) = g(\cdot)^{-1}$. We now describe a 'coordinate-version' of $\chi(T)$.

Let H be a group with the presentation,

$$H = \langle x, y, b \mid [x,y] = bxbx = byby = b^2 = 1\rangle,$$

and define an action of H on $\mathbb{C}^2$ by

$$x(\alpha,\beta) = (\alpha+1,\beta),\ y(\alpha,\beta) = (\alpha,\beta+1),\ b(\alpha,\beta) = (-\alpha,-\beta).$$

Denote the image of $(\alpha,\beta) \in \mathbb{C}^2$ in the quotient space $\mathbb{C}^2/H$ by $[\alpha,\beta]$. Let $\vec{v} = (v_1, v_2)$ be any $\mathbb{Z}$-basis of $H_1(T)$, and define the map,

$$f_{\vec{v}} \colon \mathbb{C}^2/H \to \chi(T),$$

such that $f_{\vec{v}}[\alpha,\beta] \in \chi(T)$ sends

$$v_1 \mapsto e^{2\pi i\alpha}, \quad v_2 \mapsto e^{2\pi i\beta}.$$

It can be checked that $f_{\vec{v}}$ is a homeomorphism. A representation of $\pi_1(T)$ that induces the character $f_{\vec{v}}[\alpha,\beta]$ is given by,

$$v_1 \mapsto \begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha} \end{pmatrix}, \quad v_2 \mapsto \begin{pmatrix} e^{2\pi i\beta} & 0 \\ 0 & e^{-2\pi i\beta} \end{pmatrix}.$$

Furthermore, the homeomorphism $f_{\vec{v}}$ is natural in the following sense. Let $\vec{w}$ be another basis such that $\vec{w} = \vec{v}A$ for some $A \in \mathrm{GL}(2,\mathbb{Z})$ (viewing $\vec{w}$ and $\vec{v}$ as row vectors), and define the map $\Phi_{\vec{v},\vec{w}} \colon \mathbb{C}^2 \to \mathbb{C}^2$ by right multiplying (row) vectors of $\mathbb{C}^2$ by A on the right. Then $\Phi_{\vec{v},\vec{w}}$ induces a homeomorphism, still denoted by $\Phi_{\vec{v},\vec{w}}$, from $\mathbb{C}^2/H$ to $\mathbb{C}^2/H$, and the following diagram commutes,

$$\begin{tikzcd} \mathbb{C}^2/H \arrow[r, "\Phi_{\vec{v},\vec{w}}"] \arrow[d, "f_{\vec{v}}"] & \mathbb{C}^2/H \arrow[dl, "f_{\vec{w}}"] \\ \chi(T) & \end{tikzcd}$$

Hence, we think of each $\mathbb{C}^2/H$ with a choice of basis $\vec{v}$ as a coordinate realization of $\chi(T)$. In fact, $\chi(T)$ is isomorphic to the direct limit[1] of $\{(\mathbb{C}^2/H)_{\vec{v}},\ \Phi_{\vec{v},\vec{w}}\}$,

$$\chi(T) \simeq \varinjlim (\mathbb{C}^2/H)_{\vec{v}},$$

where $(\mathbb{C}^2/H)_{\vec{v}}$ is a copy of $\mathbb{C}^2/H$ indexed by $\vec{v}$.

Next, we introduce a $\mathbb{C}^*$ bundle over $\chi(T)$. Define an action of H on $\mathbb{C}^2 \times \mathbb{C}^*$ lifting that on $\mathbb{C}^2$ by

$$\begin{aligned} x(\alpha, \beta; z) &= (\alpha + 1, \beta; ze^{2\pi i\beta}), \\ y(\alpha, \beta; z) &= (\alpha, \beta + 1; ze^{-2\pi i\alpha}), \\ b(\alpha, \beta; z) &= (-\alpha, -\beta; z). \end{aligned}$$

The canonical projection $\mathbb{C}^2 \times \mathbb{C}^* \to \mathbb{C}^2$ induces a projection

$$p \colon \mathbb{C}^2 \times \mathbb{C}^*/H \to \mathbb{C}^2/H,$$

which makes $\mathbb{C}^2 \times \mathbb{C}^*/H$ a $\mathbb{C}^*$ bundle over $\mathbb{C}^2/H$. Given two bases $\vec{v},\ \vec{w}$ of $H_1(T)$ with $\vec{w} = \vec{v}A$, $\Phi_{\vec{v},\vec{w}}$ can be covered by a bundle isomorphism. Explicitly, define $\tilde{\Phi}_{\vec{v},\vec{w}} \colon \mathbb{C}^2 \times \mathbb{C}^*/H \to \mathbb{C}^2 \times \mathbb{C}^*/H$ which maps $[\alpha, \beta; z]$ to $[(\alpha, \beta)A; z^{\det(A)}]$. Then the following diagram commutes,

$$\begin{tikzcd} (\mathbb{C}^2 \times \mathbb{C}^*/H)_{\vec{v}} \arrow[r, "\tilde{\Phi}_{\vec{v},\vec{w}}"] \arrow[d, "p"] & (\mathbb{C}^2 \times \mathbb{C}^*/H)_{\vec{w}} \arrow[d, "p"] \\ (\mathbb{C}^2/H)_{\vec{v}} \arrow[r, "\Phi_{\vec{v},\vec{w}}"] & (\mathbb{C}^2/H)_{\vec{w}} \end{tikzcd} \tag{2.20}$$

Let $\tilde{E}(T)$ be the direct limit of $\{(\mathbb{C}^2 \times \mathbb{C}^*/H)_{\vec{v}},\ \tilde{\Phi}_{\vec{v},\vec{w}}\}$. Then Equation 2.20 induces a

[1]Here all maps involved are isomorphisms, so the notion of direct limit and inverse limit do not make a difference.

map $p\colon \tilde{E}(T) \to \chi(T)$ which makes $\tilde{E}(T)$ a $\mathbb{C}^*$ bundle over $\chi(T)$, and the diagram below commutes,

$$\begin{tikzcd} \tilde{E}(T) \arrow[d, "p"] & (\mathbb{C}^2 \times \mathbb{C}^*/H)_{\vec{v}} \arrow[l] \arrow[d, "p"] \\ \chi(T) & (\mathbb{C}^2/H)_{\vec{v}} \arrow[l, "f_{\vec{v}}"'] \end{tikzcd}$$

We often represent an element of $\tilde{E}(T)$ by a 'coordinate' $[\alpha, \beta; z]_{\vec{v}}$ with respect to a basis $\vec{v}$. Changing the basis to $\vec{w} = \vec{v}A$ induces the equality

$$[\alpha, \beta; z]_{\vec{v}} = [(\alpha, \beta)A; z^{\det(A)}]_{\vec{w}},$$

and when the bases involved are clear from the context, we will omit them.

We also need an 'orientation-version' of $\tilde{E}(T)$. Now assume T is *oriented*, and define $E(T)$ to be the direct limit of $\{(\mathbb{C}^2 \times \mathbb{C}^*/H)_{\vec{v}},\ \tilde{\Phi}_{\vec{v},\vec{w}}\}$ where the limit is taken only over positive bases $\vec{v}$ of $H_1(T)$, namely, those $\vec{v}$ such that $v_1 \wedge v_2$ matches the orientation of T. Apparently, $E(T)$ and $E(-T)$ are both bundles over $\chi(T)$, and are both isomorphic to $\tilde{E}(T)$. However, it will be of conceptual convenience for latter calculations to distinguish $E(T)$ from $E(-T)$.

There is a fiber-wise pairing $\langle\ ,\ \rangle$ defined on $E(T) \times E(-T)$ as follows. Given $e \in E(T)$, $e' \in E(-T)$ such that $p(e) = p(e')$, choose an arbitrary positive basis $\vec{v} = (v_1, v_2)$ of $H_1(T)$ and hence $\vec{v}' := (-v_1, v_2)$ is a positive basis of $H_1(-T)$, and write $e = [\alpha, \beta; z]_{\vec{v}}$, $e' = [-\alpha, \beta; z']_{\vec{v}'}$ (or $e' = [\alpha, -\beta; z']_{-\vec{v}'}$). Then $\langle e, e'\rangle := zz'$. It can be checked that the pairing is well defined.

Lastly, the above notions can be generalized to multiple tori in a natural way. Let $S = \sqcup_{i=1}^k T_i$ be a disjoint union of k oriented tori. Then $\chi(S) = \chi(T_1) \times \cdots \times \chi(T_k)$. The group H^k acts on $(\mathbb{C}^2)^k$ component-wise and the quotient is a 'coordinate-version' of $\chi(S)$. The action of H^k can also be lifted to $(\mathbb{C}^2)^k \times \mathbb{C}^*$ where the i-th component H_i in

H^k acts on the i-th copy of $\mathbb{C}^2$ in $(\mathbb{C}^2)^k$ times $\mathbb{C}^*$, and $E(T)$ is the quotient of $(\mathbb{C}^2)^k \times \mathbb{C}^*$ by this action. For $n \leq k$, similar to the pairing above, there is a generalized 'pairing':

$$E(T_1 \sqcup \cdots \sqcup T_k) \times E(-T_1 \sqcup \cdots \sqcup -T_m) \to E(T_{m+1} \sqcup \cdots \sqcup T_k).$$

With the above notations, we recall several theorems in [21]. Let M be an oriented compact 3-manifold with toral boundaries $\partial M = \sqcup_{i=1}^k T_i$ and $\rho\colon \pi_1(M) \to \mathrm{SL}(2, \mathbb{C})$ be a holonomy representation. It is well-known that $\mathrm{CS}(\rho)$ in Equation 2.19 is not well defined since M has boundary. Let

$$c_M(\rho) = e^{2\pi i\,\mathrm{CS}(\rho)}.$$

Theorem 2.4.3 (Theorem 3.2 of [21])**.** The Chern-Simons invariant defines a lifting $c_M : \chi(M) \longrightarrow E(\partial M)$ of the restriction map r from the character variety of M to the character variety of ∂M,

$$\begin{tikzcd} & E(\partial M) \arrow[d, "p"] \\ \chi(M) \arrow[ur, "c_M"] \arrow[r, "r"] & \chi(\partial M) \end{tikzcd}$$

Moreover, if $Y = M_1 \cup M_2$ is a closed oriented 3-manifold such that M_1 and M_2 are glued along toral boundaries $\partial M_1 = -\partial M_2$, then for $\chi \in \chi(Y)$, we have

$$e^{2\pi i\,\mathrm{CS}(\chi)} = \langle c_{M_1}(\chi_1), c_{M_2}(\chi_2)\rangle,$$

where χ_i denotes the restriction of χ on M_i.

The following theorem is also due to [21] which the authors proved for the case of SU(2) representations (Theorem 2.7), but an almost identical proof also works for $\mathrm{SL}(2, \mathbb{C})$ representations.

Theorem 2.4.4. Let M be an oriented 3-manifold with toral boundaries $\partial M = \sqcup_{i=1}^k T_i$

and $\rho(t)\colon \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$ be a path of representations. Let $(\alpha_i(t), \beta_i(t))$ be a lift of $\chi \circ \rho(t)|_{T_i}$ to $\mathbb{C}^2$ with respect to some basis of $H_1(T_i)$. Suppose

$$c_X(\rho(t)) = [\alpha_1(t), \beta_1(t), \cdots, \alpha_k(t), \beta_k(t); z(t)]$$

Then

$$z(1)z(0)^{-1} = \exp\left(2\pi i \sum_{j=1}^{k} \int_0^1 (\alpha_j \frac{d\beta_j}{dt} - \beta_j \frac{d\alpha_j}{dt})\right)$$

In particular, if $\rho(1)$ is the trivial representation, then

$$c_X(\rho(0)) = \left[\alpha_1(0), \beta_1(0), \cdots, \alpha_k(0), \beta_k(0); \exp\bigl(-2\pi i \sum_{j=1}^{k} \int_0^1 (\alpha_j \frac{d\beta_j}{dt} - \beta_j \frac{d\alpha_j}{dt})\bigr)\right]$$

The following two facts are proved for SU(2) representations in [21] (Theorems 4.1 and 4.2, respectively). Similar methods combined with Theorems 2.4.3 and 2.4.4 above show that they also hold for $\mathrm{SL}(2,\mathbb{C})$ representations.

Fact 1 Let M be an oriented 3-manifold with toral boundaries $\partial M = \sqcup_{i=1}^{n} T_i$. Assume $H_1(M)$ is torsion free. Choose a positive basis (μ_i, λ_i) for $H_1(T_i)$. Let $\{x_j \mid j = 1, \cdots, m\}$ be a basis of $H_1(M)$ and $\mu_i = \sum a_{ij} x_j, \lambda_i = \sum b_{ij} x_j$. Suppose that $\rho : \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$ is an Abelian representation and $\mathrm{Tr}(\rho(x_j)) = e^{2\pi i \gamma_j} + e^{-2\pi i \gamma_j}$ for some $\gamma_j \in \mathbb{C}$. Then

$$c_M(\rho) = \Big[\ \sum a_{1j}\gamma_j,\ \sum b_{1j}\gamma_j,\ \cdots,\ \sum a_{nj}\gamma_j,\ \sum b_{nj}\gamma_j;\ 1\ \Big]$$

Fact 2 Let F be a genus g oriented surface with k punctures. The fundamental group of F has the presentation,

$$\pi_1(F) = \langle a_1, b_1, \cdots, a_g, b_g, x_1, \cdots, x_k \mid [a_1, b_1] \cdots [a_g, b_g] x_1 \cdots x_k = 1 \rangle,$$

where x_j corresponds to the oriented boundary (induced from F) of the j-th puncture. Let $Y = F \times S^1$ be endowed with the product orientation and let $\tilde{h} = * \times S^1$ be the central element of $\pi_1(Y)$ corresponding to the oriented S^1 component. Then $\partial Y = \sqcup_{j=1}^{k} T_j$ with T_j the torus corresponding to the j-th puncture and $(x_j, \tilde{h})$ is a positive basis for $H_1(T_j)$. Suppose $\rho : \pi_1(Y) \longrightarrow \mathrm{SL}(2,\mathbb{C})$ is a non-Abelian representation, which implies $\mathrm{Tr}(\rho(\tilde{h})) = 2\cos 2\pi\beta$ for some $\beta \in \{0, \frac{1}{2}\}$. Suppose $\mathrm{Tr}(\rho(x_j)) = e^{2\pi i \alpha_j} + e^{-2\pi i \alpha_j}$ for some $\alpha_j \in \mathbb{C}$. Then

$$c_Y(\rho) = \Big[\alpha_1, \beta, \cdots, \alpha_n, \beta; \exp\big(-2\pi i \beta \sum_{j=1}^{k} \alpha_j\big)\Big].$$

Note that $c_Y(\rho)$ does not change under the replacement of some α_j by $-\alpha_j$.

2.5 Computation about CS invariants and adjoint R-torsions

For our purpose, we compute the Chern-Simons invariants and adjoint Reidemeister torsions of Seifert fiber spaces and torus bundles as follows.

2.5.1 R-torsions of Seifert fiber spaces

Freed computed R-torsions of Brieskorn homology spheres for the adjoint representations of irreducible SU(2) representations in [16]. Kitano computed torsions of SFSs for standard irreducible $\mathrm{SL}(2,\mathbb{C})$ representations in [22]. We need to compute R-torsions of SFSs for the adjoint representations of nonAbelian $\mathrm{SL}(2,\mathbb{C})$ representations containing both irreducible and reducible ones. This may be known to experts, but we did not find a reference for explicitly doing so. To make the paper self-contained, we provide a detailed derivation of these torsions, generalizing the work of [16] and [22].

Let M be the SFS $\{0; (o, 0); (p_1, q_1), (p_2, q_2), (p_3, q_3)\}$. Decompose M as $\cup_{i=0}^{3} A_i \ \cup B$

along $\cup_{i=0}^{3} T_i$ where $B = (S^2 - 4pts) \times S^1$, and $B_0,\ B_i (i = 1, 2, 3)$ are solid tori attached to B by index 1, $\frac{p_i}{q_i}$ along $T_0,\ T_i$, respectively. Let $\rho : \pi_1(M) \longrightarrow \mathrm{SL}(2, \mathbb{C})$ be a non-Abelian representation, $V = \mathfrak{sl}(2, \mathbb{C})$ be the adjoint representation of ρ with the basis

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

From Section 2.2, ρ is parametrized by (n_1, n_2, n_3, h) where $0 < n_i < \frac{p_i}{2},\ n_i \in \frac{1}{2}\mathbb{Z}$, $h = 0, \frac{1}{2}$. Assume that $r_i, s_i \in \mathbb{Z}$, such that $p_i s_i - r_i q_i = 1$.

Proposition 2.5.1. When ρ is nonAbelian, $C_*(M) \otimes_{\mathbb{Z}[\pi_1(M)]} V$ is acyclic and

$$\mathrm{Tor}(M; \rho) = \frac{p_1 p_2 p_3}{\prod_{i=1}^{3} 4 \sin^2 \frac{2\pi r_i n_i}{p_i}}$$

Proof: Denote $C_* \otimes_{\mathbb{Z}[\pi_1]} V$ by $C_{*,\rho}$, twisted homology by H_*, and the matrix of element in π_1 under ρ by the same letter.

Given CW structure on M, we have the following exact chain sequence

$$0 \longrightarrow \bigoplus_{i=0}^{3} C_{*,\rho}(T_i) \longrightarrow \bigoplus_{i=0}^{3} C_{*,\rho}(A_i) \oplus C_{*,\rho}(B) \longrightarrow C_{*,\rho}(M) \longrightarrow 0$$

and long exact sequence

$$\begin{aligned} 0 \longrightarrow H_3(T_i) \longrightarrow H_3(A_i, B) \longrightarrow H_3(M) \longrightarrow \cdots \\ \longrightarrow H_0(T_i) \longrightarrow H_0(A_i, B) \longrightarrow H_0(M) \longrightarrow 0 \end{aligned}$$

Construct cell structure as follows.

$$C_0(B) =< v_B >, C_0(T_i) =< v_{T_i} >, C_0(A_i) =< v_{A_i} >$$

$$C_1(B) =< x_1, x_2, x_3, h >, C_1(T_i) =< m_i, l_i >, C_1(A_i) =< b_i >$$

$$C_2(B) =< u_{1,B}, u_{2,B}, u_{3,B} >, C_2(T_i) =< u_{T_i} >$$

where v_* are base points of connected spaces, x_i generate $\pi_1(S^3 - 4pts)$, $h = * \times S^1 \in \pi_1(S^3 - 4pts \times S^1)$, m_i, l_i are meridians and longitudes of T_i respectively, b_i are longitudes of boundary of A_i, $u_{i,B}$ are squares with boundary $x_i h x_i^{-1} h^{-1}$, u_{T_i} are squares with boundary $m_i l_i m_i^{-1} l_i^{-1}$. $T_i (i = 1, 2, 3)$ are attached to $x_i \times h$ by identity map and boundary of A_i by $\begin{pmatrix} s_i & -q_i \\ -r_i & p_i \end{pmatrix}$. T_0 is attached to $x_1 x_2 x_3 \times h$ and boundary of A_0 by identity map. x_1, x_2, x_3, h generate $\pi_1(M)$ as follows.

$$\pi_1(M) =< x_1, x_2, x_3, h | x^{p_i} h^{q_i} = 1, x_i h = h x_i, x_1 x_2 x_3 = 1 >$$

For matrix under ρ, we have

$$x_i \sim \begin{pmatrix} \zeta_i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_i^{-1} \end{pmatrix}, h = I$$

where ζ_i is a p_i-th root of unity. $m_i = x_i$, $b_i = x_i^{r_i}$, $l_i = h$. Here we use 1-cell with ends points attached as element in π_1.

The work of [16] can be generalized to irreducible representations of $SL(2, \mathbb{C})$. Thus we focus on reducible and nonAbelian representations. According to 2.7, taking upper triangular ones for example, they have the following form.

$$x_1 = \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}, x_2 = \begin{pmatrix} a_2 & 1 \\ 0 & a_2^{-1} \end{pmatrix}, x_3 = \begin{pmatrix} a_1^{-1} a_2^{-1} & -a_1 \\ 0 & a_1 a_2 \end{pmatrix}$$

where $a_1, a_2, a_3 = a_1^{-1}a_2^{-1}$ are roots of 1 or -1.

For joint representation, we have

$$x_1 = \begin{pmatrix} a_1^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_1^2 \end{pmatrix}, x_2 = \begin{pmatrix} a_2^{-2} & 2a_2^{-1} & -1 \\ 0 & 1 & -a_2 \\ 0 & 0 & a_2^2 \end{pmatrix}$$
$$x_3 = \begin{pmatrix} a_1^2 a_2^2 & -2a_2 & -a_1^{-2} \\ 0 & 1 & a_1^{-2}a_2^{-1} \\ 0 & 0 & a_1^{-2}a_2^{-2} \end{pmatrix} \tag{2.21}$$

Let $w_i^{\pm}$ be the eigenvectors of x_i for eigenvalue $\zeta_i = a_i^{-2} = e^{\frac{4\pi i n_i}{p_i}}, \zeta_i^{-1} = e^{-\frac{4\pi i n_i}{p_i}}$ respectively and w_i^0 be the eigenvector of x_i for eigenvalue 1. Then $w_i^{\pm}$ are the eigenvectors of x_i^r for $\zeta_i^{r_i}$ and w_i^0 be the eigenvector of $x_i^{r_i}$ for 1. By scaling, assume that $|[w_i^{\pm} w_i^0]| = 1$ in V. According to 2.21 , $w_1^{\pm}, w_2^{-}$ is a basis of V. Similarly, for lower triangular ones in 2.7, $w_1^{\pm}, w_2^{+}$ is a basis of V.

For $T_i (i = 1, 2, 3)$, we have

$$0 \longrightarrow C_{2,\rho}(T_i) \xrightarrow{\partial_2} C_{1,\rho}(T_i) \xrightarrow{\partial_1} C_{0,\rho}(T_i) \longrightarrow 0$$

where

$$\partial_2 = \begin{pmatrix} O \\ x_i - I \end{pmatrix}, \partial_1 = \begin{pmatrix} x_i - I & O \end{pmatrix}$$

We have

$$H_2(T_i) =< \tilde{u}_{T_i} \otimes w_i^0 >$$
$$H_1(T_i) =< \tilde{m}_i \otimes w_i^0, \tilde{l}_i \otimes w_i^0 >$$
$$H_0(T_i) =< \tilde{v}_{T_i} \otimes w_i^0 >$$

Choose preference basis h_* for $H_*(T_i)$ as above and similarly with others. Without confusion, we omit h_* in the expression as c_*.

$$\begin{aligned}
\tau(C_{*,\rho}(T_i)) &= |\frac{[\tilde{l}_i \otimes (x_i - I)w_i^{\pm}, \tilde{m}_i \otimes w_i^0, \tilde{l}_i \otimes w_i^0, \tilde{m}_i \otimes w_i^{\pm}]}{[\tilde{u}_{T_i} \otimes w_i^0, \tilde{u}_{T_i} \otimes w_i^{\pm}][\tilde{v}_{T_i} \otimes w_i^0, \tilde{v}_{T_i} \otimes (x_i - I)w_i^{\pm}]}| \\
&= |\frac{[\tilde{l}_i \otimes (\zeta_i^{\pm 1} - 1)w_i^{\pm}, \tilde{m}_i \otimes w_i^0, \tilde{l}_i \otimes w_i^0, \tilde{m}_i \otimes w_i^{\pm}]}{[\tilde{u}_{T_i} \otimes w_i^0, \tilde{u}_{T_i} \otimes w_i^{\pm}][\tilde{v}_{T_i} \otimes w_i^0, \tilde{v}_{T_i} \otimes (\zeta_i^{\pm 1} - 1)w_i^{\pm}]}| \\
&= 1
\end{aligned} \tag{2.22}$$

For T_0, we have $\partial_2 = 0, \partial_1 = 0$.

$$H_2(T_0) =< \tilde{u}_{T_0} \otimes e_i > (i = 1, 2, 3)$$
$$H_1(T_0) =< \tilde{m}_0 \otimes e_i, \tilde{l}_0 \otimes e_i >$$
$$H_0(T_0) =< \tilde{v}_{T_0} \otimes e_i >$$

$$\tau(C_{*\rho}(T_0)) = 1 \tag{2.23}$$

For $A_i (i = 1, 2, 3)$, we have

$$0 \longrightarrow C_{1,\rho}(A_i) \longrightarrow C_{0,\rho}(A_i) \longrightarrow 0$$

where $\partial_1 = x_i^{r_i} - I$.

We have

$$H_1(A_i) =< \tilde{b}_i \otimes w_i^0 >$$

$$H_0(A_i) =< \tilde{v}_{A_i} \otimes w_i^0 >$$

$$\begin{aligned}\tau(C_{*,\rho}(A_i)) &= |\frac{[\tilde{b}_i \otimes w_i^0, \tilde{b}_i \otimes w_i^{\pm}]}{[\tilde{v}_{A_i} \otimes (x_i^{r_i} - I)w_i^{\pm}, \tilde{v}_{A_i} \otimes w_i^0]}| \\ &= |\frac{[\tilde{b}_i \otimes w_i^0, \tilde{b}_i \otimes w_i^{\pm}]}{[\tilde{v}_{A_i} \otimes (\zeta_i^{\pm r_i} - 1)w_i^{\pm}, \tilde{v}_{A_i} \otimes w_i^0]}| \\ &= \frac{1}{|\zeta_i^{r_i} - 1||\zeta_i^{-r_i} - 1|} \end{aligned} \tag{2.24}$$

For A_0, we have $\partial_1 = 0$.

$$H_1(A_0) =< \tilde{b}_0 \otimes e_i > (i = 1, 2, 3)$$

$$H_0(A_0) =< \tilde{v}_{A_0} \otimes e_i >$$

$$\tau(C_{*\rho}(A_0)) = 1 \tag{2.25}$$

For B, we have

$$0 \longrightarrow C_{2,\rho}(B) \xrightarrow{\partial_2} C_{1,\rho}(B) \xrightarrow{\partial_1} C_{0,\rho}(B) \longrightarrow 0$$

where

$$\partial_2 = \begin{pmatrix} O & O & O \\ O & O & O \\ O & O & O \\ x_1 - I & x_2 - I & x_3 - I \end{pmatrix}, \partial_1 = \begin{pmatrix} x_1 - I & x_2 - I & x_3 - I & O \end{pmatrix}$$

We have

$$\begin{aligned}
&H_2(B) =< \tilde{u}_{i,B} \otimes w_i^0, (\tilde{u}_{1,B} + \tilde{u}_{2,B}x_1 + \tilde{u}_{3,B}x_2x_1) \otimes e_i > (i = 1, 2, 3) \\
&H_1(B) =< \tilde{x}_i \otimes w_i^0, (\tilde{x}_1 + \tilde{x}_2x_1 + \tilde{x}_3x_2x_1) \otimes e_i >
\end{aligned}$$

$$\begin{aligned}
&\tau(C_{*,\rho}(B)) \\
&= |[\tilde{u}_{i,B} \otimes w_i^0, \tilde{u} \otimes e_i, \tilde{u}_{1,B} \otimes w_1^{\pm}, \tilde{u}_{2,B} \otimes w_2^-]^{-1} \\
&\quad [\tilde{v}_B \otimes (x_1 - I)w_1^{\pm}, , \tilde{v}_B \otimes (x_2 - I)w_2^-]^{-1} \\
&\quad [\tilde{x}_i \otimes w_i^0, \tilde{x} \otimes e_i, \tilde{h} \otimes (x_1 - I)w_1^{\pm}, \tilde{h} \otimes (x_2 - I)w_2^-, \tilde{x}_1 \otimes w_1^{\pm}, \tilde{x}_2 \otimes w_2^-]| \\
&= |[\tilde{u}_{i,B} \otimes w_i^0, \tilde{u} \otimes e_i, \tilde{u}_{1,B} \otimes w_1^{\pm}, , \tilde{u}_{2,B} \otimes w_2^-]^{-1} \\
&\quad [\tilde{v}_B \otimes (\zeta_1^{\pm 1} - 1)w_1^{\pm}, \tilde{v}_B \otimes (\zeta_2^{-1} - 1)w_2^-]^{-1} \\
&\quad [\tilde{x}_i \otimes w_i^0, \tilde{x} \otimes e_i, \tilde{h} \otimes (\zeta_1^{\pm 1} - 1)w_1^{\pm}, \tilde{h} \otimes (\zeta_2^{-1} - 1)w_2^-, \tilde{x}_1 \otimes w_1^{\pm}, \tilde{x}_2 \otimes w_2^-]| \\
&= |[\tilde{u}_{i,B} \otimes w_i^0, \tilde{u} \otimes e_i, \tilde{u}_{1,B} \otimes w_1^{\pm}, , \tilde{u}_{2,B} \otimes w_2^-]^{-1}[\tilde{v}_B \otimes w_1^{\pm}, \tilde{v}_B \otimes w_2^-]^{-1} \\
&\quad [\tilde{x}_i \otimes w_i^0, \tilde{x} \otimes e_i, \tilde{h} \otimes w_1^{\pm}, \tilde{h} \otimes w_2^-, \tilde{x}_1 \otimes w_1^{\pm}, \tilde{x}_2 \otimes w_2^-]| \\
&= 1
\end{aligned} \tag{2.26}$$

where $\tilde{x} = \tilde{x}_1 + \tilde{x}_2x_1 + \tilde{x}_3x_2x_1$, $\tilde{u} = \tilde{u}_{1,B} + \tilde{u}_{2,B}x_1 + \tilde{u}_{3,B}x_2x_1$.

In the long exact sequence for twisted homology group, we have isomorphisms

$$0 \longrightarrow H_*(T_i) \longrightarrow H_*(A_i, B) \longrightarrow 0$$

Then $C_{*,\rho}(M)$ is acyclic as follows.

We have

$$0 \longrightarrow \bigoplus_{i=0}^{3} H_0(T_i) \longrightarrow \bigoplus_{i=0}^{3} H_0(A_i) \longrightarrow 0$$

where $\partial(\tilde{v}_{T_i} \otimes w_i^0) = \tilde{v}_{A_i} \otimes w_i^0$, $\partial(\tilde{v}_{T_0} \otimes e_i) = \tilde{v}_{A_0} \otimes e_i$, $det(\partial) = 1$.

$$0 \longrightarrow \bigoplus_{i=0}^{3} H_1(T_i) \longrightarrow \bigoplus_{i=0}^{3} H_1(A_i) \oplus H_1(B) \longrightarrow 0$$

where $\partial(\tilde{m}_i \otimes w_i^0) = (\tilde{x}_i - \tilde{b}_i Q_i) \otimes w_i^0$, $\partial(\tilde{l}_i \otimes w_i^0) = \tilde{b}_i P_i \otimes w_i^0$, $\partial(\tilde{m}_0 \otimes e_i) = (\tilde{x}_1 + \tilde{x}_2 x_1 + \tilde{x}_3 x_1 x_2) \otimes e_i$, $\partial(\tilde{l}_0 \otimes e_i) = \tilde{b}_0 \otimes e_i$, $Q_i = \sum_{j=1}^{q_i} x^{-jr_i}$, $P_j = \sum_{j=0}^{p_i-1} x^{jr_i}$, $det(\partial) = p_1 p_2 p_3$.

$$0 \longrightarrow \bigoplus_{i=0}^{3} H_2(T_i) \longrightarrow H_2(B) \longrightarrow 0$$

where $\partial(\tilde{u}_{T_i} \otimes w_i^0) = \tilde{u}_{i,B} \otimes w_i^0$, $\partial(\tilde{u}_0 \otimes e_i) = (\tilde{u}_{1,B} + \tilde{u}_{2,B} x_1 + \tilde{u}_{3,B} x_2 x_1) \otimes e_i$, $det(\partial) = 1$.

According to Multiplicativity lemma, Equations 2.22, 2.23, 2.24, 2.25, 2.26 and the calculations about homology above, we have

$$\mathrm{Tor}(C_{*,\rho}(M)) = \frac{p_1 p_2 p_3}{\prod_{i=1}^{3} 4 \sin^2 \frac{2\pi r_i n_i}{p_i}}$$

2.5.2 CS invariants of SFSs

Auckly computed the CS invariant of SFSs for SU(2) representations in [3]. The CS invariant of SFSs for $\mathrm{SL}(2, \mathbb{C})$ representations may be known to experts. However, to make the paper self-contained, we provide a proof to compute that using method from

[21].

Proposition 2.5.2. Let $M = \{0; (o, g); (p_1, q_1), (p_2, q_2), \cdots, (p_k, q_k)\}$ be an SFS with the presentation of $\pi_1(M)$ given in Equation 2.1 with $b = 0$. Choose integers s_j and r_j such that $p_j s_j - q_j r_j = 1$. Suppose $\rho : \pi_1(M) \to \mathrm{SL}(2, \mathbb{C})$ is non-Abelian such that $\mathrm{Tr}(\rho(x_j)) = 2\cos\frac{2\pi n_j}{p_j}$, then

$$\mathrm{CS}(\rho) = \begin{cases} \displaystyle\sum_{j=1}^{k} \frac{r_j n_j^2}{p_j} \mod 1, & \rho(h) = I \\ \displaystyle\sum_{j=1}^{k} (\frac{r_j n_j^2}{p_j} - \frac{q_j s_j}{4}) \mod 1, & \rho(h) = -I \end{cases}$$

Remark 2.5.3. The formula for the CS invariant in Proposition 2.5.2 differs from that in [3] with a negative sign. We believe this discrepancy is due to conventions.

Proof: Let $Y = F \times S^1$ be as in **Fact 2** above with the chosen generators x_j and $\tilde{h}$. Set $h = \tilde{h}^{-1}$. Then M is obtained from Y by gluing k solid tori where the j-th solid torus A_j is glued along T_j by sending the meridian to $x_j^{p_j} h^{q_j}$. The generators x_j and h match those as presented in Equation 2.1. Choose a meridian-longitude pair (μ_j, λ_j) for A_j such that (μ_j, λ_j) is a positive basis of $H_1(\partial A_j)$. The gluing of A_j to Y provides the transition of basis,

$$(\mu_j, \lambda_j) = (x_j, h) \begin{pmatrix} p_j & r_j \\ q_j & s_j \end{pmatrix}.$$

Since ρ is non-Abelian, $\rho(h)$ is $\pm I$. By assumption,

$$\mathrm{Tr}(\rho(x_j)) = \exp(\frac{2\pi i n_j}{p_j}) + \exp(-\frac{2\pi i n_j}{p_j}), \quad \mathrm{Tr}(\rho(h)) = 2\cos(2\pi m),\ m = 0, \frac{1}{2}.$$

Therefore,

$$
\begin{aligned}
c_Y(\rho) &= [\frac{n_1}{p_1}, -m, \cdots, \frac{n_k}{p_k}, -m; \exp(2\pi i\, m \sum_{j=1}^{k} \frac{n_j}{p_j})]_{(x_1,-h;\cdots;x_k,-h)} \\
c_{A_j}(\rho) &= [0, \frac{r_j n_j}{p_j} + s_j m; 1]_{(\mu_j,\lambda_j)} \\
&= [-q_j(\frac{r_j n_j}{p_j} + s_j m), r_j n_j + s_j p_j m; 1]_{(x_j,h)} \\
&= [\frac{n_j}{p_j} - s_j\alpha_j, m + r_j\alpha_j; 1],\ (\text{setting } \alpha_j = n_j + q_j m) \\
&= \left[\frac{n_j}{p_j} - s_j\alpha_j, m; \exp\big(2\pi i(r_j\alpha_j)(\frac{n_j}{p_j} - s_j\alpha_j)\big)\right] \\
&= \left[\frac{n_j}{p_j}, m; \exp\big(2\pi i(r_j\alpha_j)(\frac{n_j}{p_j} - s_j\alpha_j) + 2\pi i(s_j\alpha_j)m\big)\right]
\end{aligned}
$$

Note that the relation $x_j^{p_j} h^{q_j} = 1$ implies that α_j must be an integer. Applying the pairing on $c_Y(\rho)$ and each $c_{A_j}(\rho)$ one by one, we obtain,

$$
\begin{aligned}
\mathrm{CS}(\rho) &= \sum_{j=1}^{k} (r_j\alpha_j \frac{n_j}{p_j} + s_j\alpha_j m + m\frac{n_j}{p_j}) \\
&= \sum_{j=1}^{k} \big(\frac{r_j n_j^2}{p_j} + s_j m(n_j + \alpha_j)\big) \\
&= \sum_{j=1}^{k} (\frac{r_j n_j^2}{p_j} - s_j q_j m^2).
\end{aligned}
$$

2.5.3 R-torsions of torus bundles

In this subsection, we compute the adjoint R-torsions for the torus bundle over the circle M with the monodromy map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ where $|a+d| > 2$. Its fundamental group has a presentation given in Equation 2.9.

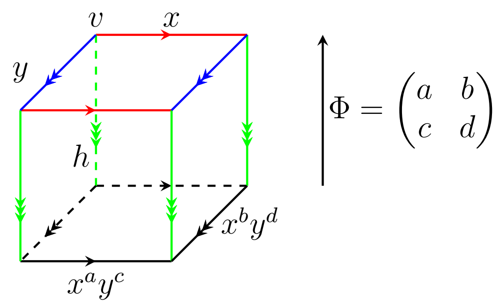

Figure 2.2. A cell structure for the torus bundle with monodromy matrix Φ For convenience but no other purposes, mark the vertical edges green, the horizontal on the top face red, and the 45^o-slope edges on the top face blue. Edges of the same color and the same arrow are identified. The front and back faces are identified by the obvious map, and so are the left and right side faces. The bottom face is identified to the top via the monodromy map Φ. Hence, the single-arrow edge and the double-arrow edge at the bottom face are homotopic to $x^a y^c$ and $x^b y^d$, respectively.

Construct a cell structure for M as follows. See Figure 2.2. The cell structure contains,

- a single 0-cell v;
- three 1-cells corresponding to the generators x, y, and h in the presentation of $\pi_1(M)$;
- three 2-cells corresponding to the three relations in the presentation of $\pi_1(M)$. Explicitly, denote them by s_1, s_2 and s_3 such that $\partial s_1 = yxy^{-1}x^{-1}$, $\partial s_2 = h^{-1}xh(x^a y^c)^{-1}$, and $\partial s_3 = h(x^b y^d)h^{-1}y^{-1}$. Graphically, s_1, s_2 and s_3 correspond to the top face, the back face, and the left face, respectively, in Figure 2.2 with the induced orientation of the cube.
- a single 3-cell t. Think of a 3-cell as a cube. Then the attaching map is determined by the identification of faces described in Figure 2.2.

Let V be a representation $\rho : \pi_1(M) \to GL(V)$, and let $\{v_j \mid j = 1, 2, \cdots\}$ be an arbitrary basis of V. We now construct the chain complex. For simplicity, assume that

$a, b, c, d \geq 0$, $a \geq c$, $b \geq d$. Other cases can be dealt similarly. Fix an arbitrary preimage $\tilde{v}$ of v. For each other cell σ, fix a lifting $\tilde{\sigma}$ starting at the base point $\tilde{v}$. We have the following chain complex,

$$0 \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

where $C_i = C_i(\widetilde{M}) \otimes_{\mathbb{Z}[\pi_1(M)]} V$. As a vector space, C_i has the following basis, $C_3 = \operatorname{span}\{\tilde{t} \otimes v_j \mid j = 1, 2, \cdots\}$, $C_2 = \operatorname{span}\{\tilde{s}_i \otimes v_j \mid i = 1, 2, 3,\ j = 1, 2, \cdots\}$, $C_1 = \operatorname{span}\{\tilde{\sigma} \otimes v_j \mid \sigma = x, y, h,\ j = 1, 2, \cdots\}$, $C_0 = \operatorname{span}\{\tilde{v} \otimes v_j \mid j = 1, 2, \cdots\}$. We present the boundary map ∂_i as a block matrix with each entry a $\dim(V) \times \dim(V)$ block. Also, denote $S : \mathbb{Z}[\pi_1(M)] \to \mathbb{Z}[\pi_1(M)]$ the antipode map that sends a group element $g \in \pi_1(M)$ to its inverse g^{-1} and linearly extends to the whole ring. Lastly, for a matrix A with entries in $\mathbb{Z}[\pi_1(M)]$, $\rho \circ S(A)$ is meant applying $\rho \circ S$ to every entry of A. With the above conventions, the boundary map is given by,

$$\partial_3 = \rho \circ S \begin{pmatrix} 1 - hw(x, y) \\ 1 - y \\ 1 - x \end{pmatrix}$$

$$\partial_2 = \rho \circ S \begin{pmatrix} y - 1 & 1 - h \sum_{i=1}^{a-1} x^i & h \sum_{i=1}^{b-1} x^i \\ 1 - x & -hx^a \sum_{i=1}^{c-1} y^i & hx^b \sum_{i=1}^{d-1} y^i - 1 \\ 0 & x - 1 & 1 - y \end{pmatrix}$$

$$\partial_1 = \rho \circ S \begin{pmatrix} x - 1 & y - 1 & h - 1 \end{pmatrix}$$

where w is a polynomial of x, y with the sum of its coefficients equal to 1.

For each of the non-Abelian characters of $\pi_1(M)$ to $\mathrm{SL}(2, \mathbb{C})$, we will compute its torsion below and show (implicitly) that the associated chain complex is always acyclic and the torsion does not depend on the representation chosen in the equivalence class of a character.

For an irreducible representation ρ given in Equation 2.10 that satisfies Equation 2.13, its adjoint representation has the form,

$$x = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^{-2} \end{pmatrix}, \; y = \begin{pmatrix} \beta^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta^{-2} \end{pmatrix}, \; h = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Denote by I and O and 3×3 identity matrix and zero matrix, respectively, and let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Define the block matrices,

$$K_1 = \begin{pmatrix} A \\ O \\ B \end{pmatrix}, \; K_2 = \begin{pmatrix} O & A \\ I & O \\ O & B \end{pmatrix}, \; K_3 = \begin{pmatrix} I \end{pmatrix}.$$

It can be checked directly that the columns (as vectors in C_{i-1}) of $\partial_i K_i$ is a basis of $\mathrm{Im}(\partial_i)$. Set $K_4 = K_0$ to be the empty matrix. Now for $i = 0, 1, 2, 3$, let

$$A_i = \left(\begin{array}{c:c} \partial_{i+1} K_{i+1} & K_i \end{array} \right),$$

then the columns of A_i give a basis for C_i. By direct calculations, we obtain the torsion,

$$\mathrm{Tor}(\rho) \;=\; \left|\frac{\det(A_1)\det(A_3)}{\det(A_0)\det(A_2)}\right| \;=\; \frac{|a+d+2|}{4}.$$

Now we compute the torsion of the reducible representations ρ given in Equation 2.14. The associated adjoint representation takes the form,

$$x = \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ y = \begin{pmatrix} 1 & -2u & -u^2 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix},\ h = \begin{pmatrix} v^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{v^2} \end{pmatrix},$$

which are clearly independent on the sign terms ϵ_x and ϵ_y. Let,

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Define the block matrices,

$$K_1 = \begin{pmatrix} E \\ O \\ F \end{pmatrix},\ K_2 = \begin{pmatrix} A & O \\ B & C \\ O & D \end{pmatrix},\ K_3 = \begin{pmatrix} I \end{pmatrix}.$$

The matrices K_i have the same properties as outlined in the case of irreducible representations above, and in the same way define the matrices A_i. It can be computed that,

$$\mathrm{Tor}(\rho) \;=\; \left|\frac{\det(A_1)\det(A_3)}{\det(A_0)\det(A_2)}\right| \;=\; |a+d+2|.$$

Some details for the derivation are as follows, where the condition $cu^2+(a-d)u-b=0$ is used to simplify expressions,

$$\begin{aligned}
\mathrm{Tor}(\rho) &= |\frac{(2cu+a-d)(b-u+du)(a-b+1+(c-d-1)u)}{u(1-cu-a)^2(u-1)}| \\
&= |\frac{(2cu+a-d)(b-u+du)(a-b+1+(c-d-1)u)}{(cu^2+(a-1)u)(cu^2+(a-1-c)u-a+1)}| \\
&= |\frac{(2cu+a-d)(b-u+du)(a-b+1+(c-d-1)u)}{((d-1)u+b)((d-c-1)u+b-a+1)}| \\
&= |\frac{(d-c-1)u+b-a+1}{2(c-d-1)cu^2+(2c(a-b+1)+(a-d)(c-d-1))u+(a-d)(a-b+1)}| \\
&= |\frac{(2c(a-b+1)-(a-d)(c-d-1))u+(a-d)(a-b+1)+2b(c-d-1)}{(d-c-1)u+b-a+1}| \\
&= |\frac{(a+d+2)((d-c-1)u+b-a+1)}{(d-c-1)u+b-a+1}| \\
&= |a+d+2|.
\end{aligned}$$

2.5.4 CS invariants of torus bundles

Any irreducible representation of $\pi_1(M)$ to $\mathrm{SL}(2,\mathbb{C})$ can be conjugated to one into $\mathrm{SU}(2)$ (see Equation 2.10), and Kirk and Klassen computed its CS invariant in [20]. Here we use methods in [21] to compute the CS invariant of both irreducible and reducible but indecomposable ones, the latter of which can not be conjugated to $\mathrm{SU}(2)$.

Let T_i ($i=A,B$) be two copies of the torus, and I be the interval $[0,1]$. Then M is obtained by gluing the two $T_i\times I$ such that $T_B\times\{0\}$ is glued to $T_A\times\{1\}$ via the identity map and $T_B\times\{1\}$ is glued to $T_A\times\{0\}$ via the map $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$. Let (μ_i,λ_i) be a positive basis of $H_1(T_i)$ so that, under the embedding $T_i\times I\hookrightarrow M$, μ_i and λ_i are sent to x and y, respectively. For $\kappa = 0, 1$, denote by μ_i^κ the element of $H_1(T_i \times \{\kappa\})$ that corresponds to μ_i in $H_1(T_i \times I)$, and by λ_i^κ in a similar way. Then (μ_i^1, λ_i^1) is a positive basis for $H_1(T_i \times \{1\})$ and $(-\mu_i^0, \lambda_i^0)$ is a positive basis for $H_1(T_i \times \{0\})$. These basis are identified as follows,

$$(\mu_B^0, \lambda_B^0) = (\mu_A^1, \lambda_A^1), \qquad (\mu_B^1, \lambda_B^1) = (\mu_A^0, \lambda_A^0) \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Set $N = |a + d + 2|$. For an irreducible representation ρ in Equation 2.10 where $\alpha = e^{\frac{2\pi i\,k}{N}}$ and $\beta = e^{\frac{2\pi i\,l}{N}}$, we have

$$\begin{aligned} c_{T_i \times I}(\rho) &= [\frac{k}{N}, \frac{l}{N}, \frac{k}{N}, \frac{l}{N}; 1]_{(\mu_i^1, \lambda_i^1), (\mu_i^0, \lambda_i^0)} \\ &= [\frac{k}{N}, \frac{l}{N}, -\frac{k}{N}, \frac{l}{N}; 1]_{(\mu_i^1, \lambda_i^1), (-\mu_i^0, \lambda_i^0)} \end{aligned}$$

Hence,

$$\begin{aligned} & c_{T_A \times I}(\rho) \\ &= [\frac{k}{N}, \frac{l}{N}, \frac{k}{N}, \frac{l}{N}; 1]_{(\mu_A^1, \lambda_A^1), (\mu_A^0, \lambda_A^0)} \\ &= [\frac{k}{N}, \frac{l}{N}, \frac{ak + cl}{N}, \frac{bk + dl}{N}; 1]_{(\mu_A^1, \lambda_A^1), (\mu_B^1, \lambda_B^1)} \\ &= [\frac{k}{N}, \frac{l}{N}, -\frac{k}{N}, \frac{bk + dl}{N}; \exp(2\pi i(-\nu)\frac{bk + dl}{N})],\ (\nu := \frac{(a+1)k + cl}{N}) \\ &= [\frac{k}{N}, \frac{l}{N}, -\frac{k}{N}, -\frac{l}{N}; \exp(2\pi i(-\nu)\frac{bk + dl}{N} + 2\pi i(-\mu)\frac{k}{N})],\ (\mu := \frac{bk + (d+1)l}{N}) \\ &= [\frac{k}{N}, \frac{l}{N}, -\frac{k}{N}, \frac{l}{N}; \exp(2\pi i f)]_{(\mu_A^1, \lambda_A^1), (-\mu_B^1, \lambda_B^1)} \end{aligned}$$

where,

$$f = \nu\frac{bk + dl}{N} + \mu\frac{k}{N} = \frac{k\mu - l\nu}{N} + \mu\nu.$$

Note that, by Equation 2.13, μ and ν are both integers. Also,

$$\begin{aligned} c_{T_B\times I}(\rho) &= [\frac{k}{N}, \frac{l}{N}, -\frac{k}{N}, \frac{l}{N}; 1]_{(\mu_B^1,\lambda_B^1),(-\mu_B^0,\lambda_B^0)} \\ &= [\frac{k}{N}, \frac{l}{N}, -\frac{k}{N}, \frac{l}{N}; 1]_{(\mu_B^1,\lambda_B^1),(-\mu_A^1,\lambda_A^1)} \end{aligned}$$

By taking the pairing on $c_{T_A\times I}(\rho)$ and $c_{T_B\times I}(\rho)$, we obtain that,

$$\mathrm{CS}(\rho) \;=\; f \;=\; \frac{k\mu - l\nu}{N}. \tag{2.27}$$

For reducible representations $\rho_{\epsilon_x,\epsilon_y}$ in Equation 2.14 depending on the values of ϵ_x and ϵ_y (see Section **??**), the computation of the CS invariant proceeds in the exactly the same way as for irreducible representations by making the substitution,

$$\frac{k}{N} \to \frac{\epsilon_x}{2}, \quad \frac{l}{N} \to \frac{\epsilon_y}{2}.$$

Consequently, by setting

$$\nu = \frac{(a+1)\epsilon_x + c\epsilon_y}{2}, \quad \mu = \frac{b\epsilon_x + (d+1)\epsilon_y}{2},$$

we obtain that,

$$\begin{aligned} \mathrm{CS}(\rho_{\epsilon_x,\epsilon_y}) &= \frac{\epsilon_x\mu - \epsilon_y\nu}{2} = \frac{\epsilon_x\mu + \epsilon_y\nu}{2} \\ &= \frac{(a+d+2)\epsilon_x\epsilon_y + b\epsilon_x + c\epsilon_y}{4} \end{aligned} \tag{2.28}$$

It can be checked that $\mathrm{CS}(\rho_{\epsilon_x,\epsilon_y}) \in \frac{1}{2}\mathbb{Z}$.

Chapter 3

Aspects about algebra

3.1 Modular tensor category

We recall some basics about modular tensor category. We choose $\mathbb{C}$ as base field and denote the set of isomorphism classes of objects of category $\mathcal{C}$ by $\mathcal{O}(\mathcal{C})$.

3.1.1 Monoidal category

Definition 3.1.1. A *monoidal category* is a quintuple $(\mathcal{C}, \otimes, a, 1, l_X, r_X)$ where $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a bifunctor, $a_{:}(- \otimes -) \otimes - \longrightarrow - \otimes (- \otimes -)$ is a natural isomorphism called associativity isomorphism, $1 \in \mathcal{C}$ is an object, $l_X : 1 \otimes X \longrightarrow X$ and $r_X : X \otimes 1 \longrightarrow X$ are natural isomorphisms, subject to the two axioms as shown in Fig. 3.1, 3.2.

$$\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W,X,Y} \otimes id"'] \arrow[dr, "a_{W \otimes X,Y,Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W,X \otimes Y,Z}"'] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y \otimes Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "id \otimes a_{X,Y,Z}"'] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

Figure 3.1. Pentagon axiom

$$\begin{tikzcd}
& X \otimes (1 \otimes Y) \arrow[d, "\mathrm{id}_X \otimes l_Y"] \\
(X \otimes 1) \otimes Y \arrow[ur, "a_{X,1,Y}"] \arrow[r, "r_X \otimes \mathrm{id}_Y"'] & X \otimes Y
\end{tikzcd}$$

Figure 3.2. Triangle axiom

3.1.2 Fusion category

Definition 3.1.2. An object $X^* \in \mathcal{C}$ is said to be a *left dual* of X if there exist morphisms $\mathrm{ev}_X : X^* \otimes X \longrightarrow 1$ and $\mathrm{coev}_X : 1 \longrightarrow X \otimes X^*$, called the evaluation and coevaluation, such that the following compositions are identity morphisms.

$$X \overset{\mathrm{coev}_X \otimes \mathrm{id}_X}{\longrightarrow} (X \otimes X^*) \otimes X \overset{a_{X,X^*,X}}{\longrightarrow} X \otimes (X^* \otimes X) \overset{\mathrm{id}_X \otimes \mathrm{ev}_X}{\longrightarrow} X \tag{3.1}$$

$$X^* \overset{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X}{\longrightarrow} X^* \otimes (X \otimes X^*) \overset{a^{-1}_{X^*,X,X^*}}{\longrightarrow} (X^* \otimes X) \otimes X^* \overset{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}}{\longrightarrow} X^* \tag{3.2}$$

A *right dual* ${}^*X$ of X can be defined similarly.

The left or right dual of an object is unique up to a unique isomorphism.

Definition 3.1.3. An object in a monoidal category is called *rigid* if it has left and right duals. A monoidal category $\mathcal{C}$ is called *rigid* if every object of $\mathcal{C}$ is rigid.

Definition 3.1.4. A *fusion category* is a finite, semisimple, rigid, $\mathbb{C}$-linear monoidal category with a simple tensor unit.

Let $\mathcal{C}$ be a fusion category. We denote the set of isomorphism classes of simple objects of $\mathcal{C}$ by $L(\mathcal{C}) = \{X_0 = 1, \cdots, X_{n-1}\}$. We have the fusion rules given by

$$X_i \otimes X_j \cong \sum_k N_{i,j}^k X_k \tag{3.3}$$

where $N_{i,j}^k = \dim \operatorname{Hom}(X_i \otimes X_j, X_k)$ are called the fusion coefficients. For any $X_i \in L(\mathcal{C})$, the fusion matrix N_i is given by $(N_i)_{k,j} = N_{i,j}^k$. The largest positive eigenvalue of N_i is called the Frobenius-Perron dimension (or FP-dimension) of X_i and is denoted by $\mathrm{FPdim}(X_i)$ (cf. [14]). A simple object $X \in L(\mathcal{C})$ is called invertible if $\mathrm{FPdim}(X) = 1$. A fusion category $\mathcal{C}$ is pointed if any $X \in \mathcal{C}$ is invertible.

3.1.3 Spherical category

Definition 3.1.5. A *pivotal sturcture* on a fusion category $\mathcal{C}$ is a natural isomorphism $a_X : X \longrightarrow X^{**}$ for every $X \in \mathcal{C}$.

Given a_X defined as above and any $f : X \longrightarrow X$, define *left and right quantum traces*

$$\mathrm{Tr}^L(f) : 1 \xrightarrow{coev_{X^*}} X^* \otimes X^{**} \xrightarrow{id \otimes a_X^{-1}} X^* \otimes X \xrightarrow{ev_X} 1 \tag{3.4}$$

$$\mathrm{Tr}^R(f) : 1 \xrightarrow{coev_X} X \otimes X^* \xrightarrow{a \otimes id_{X^*}} X^{**} \otimes X^* \xrightarrow{ev_{X^*}} 1 \tag{3.5}$$

A pivotal structure is *spherical* if $\mathrm{Tr}^L(f) = \mathrm{Tr}^R(f)$ for any $f \in \mathrm{End}(X)$. A *spherical category* $\mathcal{C}$ is a fusion category with a spherical structure. For any $X \in L(\mathcal{C})$, define the *quantum dimension* of X by

$$\dim_a(X) = \mathrm{Tr}(\mathrm{id}_X) \in \mathrm{End}(1). \tag{3.6}$$

Without making confusion, we omit the subscript a. Define the *global dimension* D of $\mathcal{C}$

$$\begin{tikzcd}
& X \otimes (Y \otimes Z) \arrow[rr, "c_{X,Y\otimes Z}"] & & (Y \otimes Z) \otimes X \arrow[dr, "a_{Y,Z,X}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "a_{X,Y,Z}"] \arrow[dr, "c_{X,Y} \otimes id"'] & & & & Y \otimes (Z \otimes X) \\
& (Y \otimes X) \otimes Z \arrow[rr, "a_{Y,X,Z}"] & & Y \otimes (X \otimes Z) \arrow[ur, "id \otimes c_{X,Z}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (X \otimes Y) \otimes Z \arrow[rr, "c_{X\otimes Y,Z}"] & & Z \otimes (X \otimes Y) \arrow[dr, "a^{-1}_{Z,X,Y}"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "a^{-1}_{X,Y,Z}"] \arrow[dr, "id \otimes c_{Y,Z}"'] & & & & (Z \otimes X) \otimes Y \\
& X \otimes (Z \otimes Y) \arrow[rr, "a^{-1}_{X,Z,Y}"] & & (X \otimes Z) \otimes Y \arrow[ur, "c_{X,Z} \otimes id"'] &
\end{tikzcd}$$

Figure 3.3. Hexagonal axiom

by

$$D^2 = \sum_{i=1}^{n} \dim(X_i)^2, \tag{3.7}$$

where $X_i \in L(\mathcal{C})$.

Theorem 3.1.6 (P. Etingof, D. Nikshych, and V. Ostrik 05)**.** Let $\mathcal{C}$ be a spherical category, then $\dim(X)^2 > 0$ for any $X \in L(\mathcal{C})$.

Accoring to the above theorem, D^2 is always a positive number. Generally we choose the positive value for D.

3.1.4 Premodular tensor category

Definition 3.1.7. A *braiding* on $\mathcal{C}$ is a natural isomorphism $c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$ for any $X, Y \in \mathcal{C}$ satisfying the hexagonal axiom as shown in Fig. 3.3.

Definition 3.1.8. A *twist* on $\mathcal{C}$ is an element $\theta \in \mathrm{Aut}(\mathrm{id}_{\mathcal{C}})$ such that

$$\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y)c_{X,Y}c_{Y,X} \tag{3.8}$$

θ is called a *ribbon structure* if $\theta_X^* = \theta_{X^*}$. A *premodular tensor category* is a braided fusion category with a ribbon category.

Theorem 3.1.9 (C. Vafa 88, G.Andersen and G.Moore 88, B. Bakalov and A. Kirillov Jr. 01)**.** Let $\mathcal{C}$ be a premodular category. Twist θ_X is a root of unity for any $X \in L(\mathcal{C})$.

Definition 3.1.10. (1) The T*-matrix* of $\mathcal{C}$ is a n by n diagonal matrix $\mathrm{diag}\{\theta_{X_i}\}$ for $X_i \in L(\mathcal{C})$.

(2) The S*-matrix* of $\mathcal{C}$ is a n by n matrix with each entry S_{X_i,X_j} defined by

$$S_{X_iX_j} = \mathrm{Tr}(c_{X_j,X_i}c_{X_i,X_j}) \tag{3.9}$$

for any $X_i, X_j \in L(\mathcal{C})$.

S, T-matrix are called the *modular data* of $\mathcal{C}$.

Definition 3.1.11. A *modular tensor category* is a premodular category with a nondegenerate S-matrix.

Theorem 3.1.12 (P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik 15...)**.** Let $\mathcal{C}$ be a modular category. S and T-matrix form a projective representation of $\mathrm{SL}(2,\mathbb{Z})$.

3.1.5 Unitary category

Definition 3.1.13. A *Hermitian ribbon category* $\mathcal{C}$ is a ribbon category with a conjugation $^-$: $\mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(Y,X)$ for any $X, Y \in \mathcal{C}$, such that

(1) $\bar{\bar{f}} = f$, $\overline{f\otimes g} = \bar{f}\otimes\bar{g}$, $\overline{fg} = \bar{g}\bar{f}$.

(2) $\overline{c_{X,Y}} = c^{-1}_{X,Y}$.

(3) $\overline{\theta_X} = \theta_X^{-1}$, $\overline{\mathrm{coev}_X} = \mathrm{ev}_X c_{X,X^*}(\theta_X \otimes \mathrm{id}_{X^*})$, $\overline{\mathrm{ev}_X} = (\mathrm{id}_{X^*} \otimes \theta_X^{-1}) c^{-1}_{X^*,X} \mathrm{coev}_X$.

$\mathcal{C}$ is *unitary* if the Hermitian form $(f,g) = \mathrm{Tr}(f\bar{g})$ is positive definite on $\mathrm{Hom}(X,Y)$ for any $X, Y \in \mathcal{C}$.

On a unitary ribbon category, the quantum dimension of every object is a positive real number.

3.2 Examples

We recall some algebraic examples that will be used in the program.

3.2.1 Temperley-Lieb-Jones category

Let A be an indeterminant over $\mathbb{C}$, and $d = -A^2 - A^{-2}$. We will call A the Kauffman variable, and d the loop variable. Let $\mathbb{F} = \mathbb{C}[A, A^{-1}]$. Let $I = [0,1]$ be the unit interval, and $R = I \times I$ be the square in the plane. The generic Temperley-Lieb-Jones category $\mathrm{TLJ}(A)$ is defined as follows. An object of $\mathrm{TLJ}(A)$ is the unit interval with a finite set of points in the interior of I, allowing the empty set, with each point colored by a natural number. Given $X, Y \in \mathrm{TLJ}(A)$, morphisms in $\mathrm{Hom}(X,Y)$ are formal $\mathbb{F}$-linear combinations of uni-trivalent graphs connecting X, Y with admissible compatible colorings, modulo d-isotopic relation. $\mathrm{TLJ}(A)$ has a tensor product from horizontal juxtaposition of formal diagrams. The empty object is a tensor unit. Every object is self-dual. The involution $X \longrightarrow \bar{X}$ is the duality. For more details, please refer to [33].

$\mathrm{TLJ}(A)$ is not a premodular tensor category. But by setting A to be some roots of unity, we will get (pre)modular tensor categories. The following theorem is known to experts.

Theorem 3.2.1.

(1) If $A \in \mathbb{C}/\{0\}$ such that the loop value d is not a root of any Chebyshev polynomial, then the structure of $\mathrm{TLJ}(A)$ is the same as generic $\mathrm{TLJ}(A)$.

(2) If A is a primitive $4r$-th root of unity, then $\mathrm{TLJ}(A)$ is a modular tensor category.

(3) If r is odd, and A is a primitive $2r$-th root of unity, then $\mathrm{TLJ}(A)$ is a premodular tensor category with $S-$matrix $S = S_{\text{even}} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ where S_{even} is the submatrix of S indexed by even labels. Furthermore, S_{even} is nondengenerate.

3.2.2 $\mathrm{SU}(2)_k$ anyon model

For each integer $r \geq 2$, there is a unitary MTC, usually denoted by $\mathrm{SU}(2)_{r-2}$ [4], which is closely related to the Temperley-Lieb-Jones categories. Here $r-2$ is called the level of the MTC. It has the same label set as $\mathrm{TLJ}(e^{\frac{2\pi i}{4r}})$, say $L = \{0, 1, ..., r-2\}$, but differs from it in modular data by some signs. Explicitly, setting $A = e^{\frac{2\pi i}{4r}}$, the modular data for $\mathrm{SU}(2)_{r-2}$ is given as follows where $i, j \in L$,

$$\theta_j \;=\; A^{j(j+2)} \;=\; e^{\frac{2\pi i\, j(j+2)}{4r}}, \tag{3.10}$$

$$S_{ij} \;=\; [(i+1)(j+1)]_A \;=\; \frac{\sin\frac{(i+1)(j+1)\pi}{r}}{\sin\frac{\pi}{r}}. \tag{3.11}$$

In particular, its quantum dimensions are all positive (since it is unitary),

$$d_j \;=\; [j+1]_A \;=\; \frac{\sin\frac{(j+1)\pi}{r}}{\sin\frac{\pi}{r}}, \tag{3.12}$$

and the total dimension is

$$D = \sqrt{\frac{r}{2}} \frac{1}{\sin \frac{\pi}{r}}. \tag{3.13}$$

Note that $d_j = |d_j(A)|$ and $D = D(A)$, where $d_j(A)$ and $D(A)$ are the quantum dimension of j and total dimension of $\mathrm{TLJ}(A)$, respectively.

3.2.3 Pointed category

Let $\mathcal{C}$ be a premodular tensor category. We say an object $X \in \mathcal{C}$ is invertible if $\dim(X) = 1$. We call $\mathcal{C}$ a pointed category if every object in $\mathcal{C}$ is invertible. Every pointed category is equivalent to Vec_G^ω, which is the category of finite dimensional vector spaces graded by a finite group G with the associativity given by the 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$ [13].

Let G be a finite abelian group, $q : G \to \mathbb{C}^\times$ be a quadratic form, and $\chi : G \to \mathbb{C}^\times$ be a character such that $\chi^2 = 1$. As shown in [12], there exists a pointed premodular category $\mathcal{C}(G, q, \chi)$ with the following properties:

- the simple objects of $\mathcal{C}(G, q, \chi)$ are parametrized by G, and the monoidal product is given by the group product;
- $S_{gh} = b(g, h)\chi(g)\chi(h)$, where b is the bicharacter $b(g, h) := \frac{q(gh)}{q(g)q(h)}$; and
- $T_g = q(g)\chi(q)$.

Moreover, every pointed premodular category is equivalent to some $\mathcal{C}(G, q, \chi)$. When χ is trivial, we simply denote it as $\mathcal{C}(G, q)$.

3.2.4 Equivariantization

Let G be a finite group, and $\mathcal{C}$ be a fusion category. We recall some notions about equivariantization. For more details, please refer to [5, 12, 24].

Definition 3.2.2. Let $\underline{G}$ be the fusion category whose objects are elements of G and morphisms are identities. The tensor product is given by the multiplication of G.

Define the action of G on $\mathcal{C}$ by the tensor functor $T : \underline{G} \to \mathrm{Aut}_{\otimes}(\mathcal{C})$; $g \mapsto T_g$. For any $g, h \in G$ let $\gamma_{g,h}$ be the isomorphism $T_g \circ T_h \simeq T_{gh}$ that defines the tensor structure on the functor T.

Definition 3.2.3. Define the G-equivariantization of $\mathcal{C}$ as follows.

- An object of $\mathcal{C}^G$ is a pair (X, u), where $X \in \mathcal{C}$ and $u = \left\{u_g : T_g(X) \xrightarrow{\sim} X \mid g \in G\right\}$, such that $u_{gh} \circ \gamma_{g,h} = u_g \circ T_g(u_h)$ for all $g, h \in G$. (X, u) is called G-equivariant object.

- The morphisms between equivariant objects are morphisms in $\mathcal{C}$ commuting with $u_g, g \in G$.

- The tensor product given by $(X, u) \otimes (Y, w) := (X \otimes Y, u \otimes w)$, where $(u \otimes w)_g := (u_g \otimes w_g) \circ (\mu^g_{X,Y})^{-1}$ and $\mu^g_{X,Y} : T_g(X) \otimes T_g(Y) \to T_g(X \otimes Y)$ is the tensor structure for T.

$\mathcal{C}^G$ is a fusion category.

Chapter 4

From 3-manifolds to modular data

4.1 Program to construct modular data

The modular data of an MTC or a premodular category consist of the modular S- and T- matrices. Given a 3-manifold M with certain conditions, [8] contains an algorithm for choosing the T-matrix and the first row of the S-matrix, i.e. all quantum dimensions. The next step for the full S-matrix is a trial-and-error algorithm based on finding the right loop operators for each simple object. When all the loop operators are given, then the modular data can be computed. There are no general algorithms to define loop operators, but in the cases of SFSs and Sol manifolds, we find the relevant loop operators completely.

4.1.1 From adjoint-acyclic non-Abelian characters to simple object types

Each premodular category has a label set—the isomorphism classes of the simple objects, and a label is an isomorphism class of simple objects, so we will refer to a label also as a simple object type. In physics, an anyon model is a unitary MTC and a label is called an anyon type or a topological charge.

A candidate label from a 3-manifold M and $\mathrm{SL}(2,\mathbb{C})$ is morally an irreducible representation of the fundamental group $\pi_1(M)$ to $\mathrm{SL}(2,\mathbb{C})$. But the precise definition is more subtle and based on our examples later, we make the following definition. In particular, we discover that reducible but indecomposable representations cannot be discarded and play important roles in the construction of premodular categories from torus bundles over the circle. Our definition is specific for representations to $\mathrm{SL}(2,\mathbb{C})$ and we expect an appropriate generalization is needed for other Lie groups such as $\mathrm{SL}(n,\mathbb{C}), n \geq 3$.

Definition 4.1.1. Let $\chi \in \chi(M)$ be a non-Abelian $\mathrm{SL}(2,\mathbb{C})$-character of a 3-manifold M.

- A non-Abelian character χ is *adjoint-acyclic* if each non-Abelian representation $\rho : \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$ with character χ is adjoint-acyclic, namely, the chain complex associated with the universal cover $\tilde{M}$ twisted by $\mathrm{Adj} \circ \rho$ is acyclic (see Definition 2.3.2), and furthermore, the adjoint Reidemeister torsion of all such non-Abelian representations with character χ are the same.
- A candidate label is an *adjoint-acyclic non-Abelian character.*
- A candidate label set $L(M)$ from a 3-manifolds M is a finite set of adjoint-acyclic non-Abelian characters in $\chi(M)$ with a pre-chosen character such that the difference of the CS invariant of each character $L(M)$ with that of the pre-chosen character is a rational number.

  The pre-chosen character is the candidate tensor unit.

Note that by definition, the adjoint Reidemeister torsion is well-defined for adjoint-acyclic non-Abelian characters. The CS invariant only depends on characters, and is hence also well-defined for such characters.

In this paper, our candidate label set is in general maximal in the sense it consists of all the adjoint-acyclic non-Abelian characters of the given three manifold. It is also true that the CS invariants of all the candidate labels including the candidate tensor unit are all rational in our examples. We are not aware of any example of a candidate label set for which not all CS invariants are rational numbers.

4.1.2 Vacuum choices, loop operators, and modular data

Each simple object x of a premodular category $\mathcal{C}$ has a quantum dimension d_x and a topological twist θ_x. The set $Td(\mathcal{C}) := \cup_{i\in L(\mathcal{C})}\{d_{x_i}, \theta_{x_i}\}$ will be called the twist-dimension set of $\mathcal{C}$, where $L(\mathcal{C})$ is the label set of $\mathcal{C}$ and $\{x_i, i \in L(\mathcal{C})\}$ form a complete representative set of simple objects of $\mathcal{C}$. A candidate label set of a three manifold M will lead to a candidate twist-dimension set in the following.

The choice of a tensor unit or vacuum from a collection of adjoint-acyclic non-Abelian characters is not unique in general and it is known that different choices could produce different premodular categories. Once a vacuum is chosen, then the adjoint Reidemeister torsion of each character is scaled to the absolute value of normalized quantum dimension and the difference of the CS invariant of the character with that of the vacuum is the conformal weight of the simple object up to a sign[1].

Given a 3-manifold M and a Lie group G, a central representation of $\pi_1(M)$ is a homomorphism from $\pi_1(M)$ to the center $Z(G)$ of G. For $G = \mathrm{SL}(2,\mathbb{C})$, a central representation of $\pi_1(M)$ is simply a homomorphism from $\pi_1(M)$ to $\mathbb{Z}_2$. The group of central representations can be identified with $H^1(M,\mathbb{Z}_2)$. A central representation $\sigma \in H^1(M,\mathbb{Z}_2)$ of $\pi_1(M)$ naturally acts on $R(M)$ by tensoring $\rho \in R(M)$, i.e. by sending ρ to

[1]The sign and hence the negative sign in front of CS invariant below is not important and the choice is made to be the same as in [8].

$\rho \otimes \sigma$. Moreover, this action induces an action of central representations on the character variety $\chi(M)$.

Definition 4.1.2. 1. Given a candidate label set $L(M)$ from a 3-manifold M, a central representation σ is *bosonic* with respect to $L(M)$ if the action of σ keeps $L(M)$ invariant and preserves the CS invariant of every candidate label. If the action of σ changes the CS invariants of all candidate labels in $L(M)$ by either 0 or $\frac{1}{2}$, then χ is called *fermionic* if it is not bosonic.

2. Two candidate labels are centrally related if they are in the same orbit under the action of $H^1(M, \mathbb{Z}_2)$ and they have the same CS and torsion invariant.

Given a candidate label set $L(M)$ of M that is invariant under the action of $H^1(M, \mathbb{Z}_2)$, the candidate symmetric center $s(M)$ consists of all characters in $L(M)$ that are centrally related to the candidate tensor unit. Let $G_0(M)$ be the maximal subgroup of $H^1(M, \mathbb{Z}_2)$ such that $G_0(M)$ maps the candidate tensor unit onto $s(M)$. The action of $G_0(M)$ separates $L(M)$ into orbits $\{O_0, \cdots, O_m\}$, where each subset O_i of $L(M)$ consists of candidate labels that are centrally related to each other, and O_0 is the subset for the candidate vacuum.

We often represent a candidate label (a character) by arbitrarily choosing a representative (a representation of $\pi_1(M)$) for it.

Definition 4.1.3. A candidate label set $L(M) = \{\rho_\alpha\}$ of a three manifold M with ρ_0 the candidate vacuum is *admissible* if the following two equations hold with the notations as above:

$$\sum_{\rho_\alpha \in L(M)} \frac{1}{2\mathrm{Tor}(\rho_\alpha)} = 1, \tag{4.1}$$

$$\left| \sum_\alpha \frac{\exp(-2\pi i\,\mathrm{CS}(\rho_\alpha))}{2\mathrm{Tor}(\rho_\alpha)} \right| = \frac{1}{s_L} \frac{\sqrt{|s(M)|}}{\sqrt{2\mathrm{Tor}(\rho_0)}}, \tag{4.2}$$

where $s_L = 1$ if all central representations in $G_o(M)$ are bosonic and $s_L = \sqrt{2}$ if there is a fermionic one.

The conditions above are derived from the conjecture that the Mueger center of the potential premodular category is a collection of Abelian anyons parameterized by the subset O_0. In the condensed category, each subset O_i will be identified into a single composite object which has the same quantum dimension as that of any simple object in O_i and which splits into a number of simple objects of the same quantum dimension. The resulting condensed category is either modular or super-modular depending on if there is a fermion in the candidate Mueger center. In a particular case when M is a $\mathbb{Z}_2$ homology sphere, that is, $H^1(M, \mathbb{Z}_2) = 0$, Equation 4.2 reduces to,

$$\left| \sum_{\alpha} \frac{\exp(-2\pi i \, \mathrm{CS}(\rho_\alpha))}{2\mathrm{Tor}(\rho_\alpha)} \right| = \frac{1}{\sqrt{2\mathrm{Tor}(\rho_0)}}. \tag{4.3}$$

Given an admissible candidate label set $L(M)$ with the chosen candidate tensor unit ρ_0, then the candidate twist-dimension set is constructed as follows:

$$\theta_\alpha = e^{-2\pi i (\mathrm{CS}(\rho_\alpha) - \mathrm{CS}(\rho_0))}, \tag{4.4}$$

$$D^2 = 2\mathrm{Tor}(\rho_0) \tag{4.5}$$

$$d_\alpha^2 = \frac{D^2}{2\mathrm{Tor}(\rho_\alpha)}, \tag{4.6}$$

where D^2 is the total dimension squared of the candidate premodular category.

Next, we discuss the construction of the S-matrix.

Definition 4.1.4. Given a three manifold M, a primitive loop operator of M is a pair (a, R), where a is a conjugacy class of the fundamental group $\pi_1(M)$ of X and R a finite dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{C})$.

Given an $\mathrm{SL}(2,\mathbb{C})$-representation ρ of $\pi_1(M)$ and a primitive loop operator (a,R), then the weight of the loop operator (a,R) with respect to ρ is $W_\rho(a,R) := \mathrm{Tr}_R(\rho(a))$. Denote by Sym^j the unique $(j+1)$-dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{C})$. Then $W_\rho(a,\mathrm{Sym}^j)$ can be computed from the Chebyshev polynomial $\Delta_j(t)$ defined recursively by,

$$\Delta_{j+2}(t) = t\Delta_{j+1}(t) - \Delta_j(t), \quad \Delta_0(t) = 1, \Delta_1(t) = t. \tag{4.7}$$

Explicitly,

$$W_\rho(a,\mathrm{Sym}^j) = \Delta_j(t), \quad t = W_\rho(a,\mathrm{Sym}^1) = \mathrm{Tr}(\rho(a)). \tag{4.8}$$

From the above two equations, it follows that $W_\rho(a,\mathrm{Sym}^j)$ only depends on the character χ induced by ρ. It is direct to check that,

$$\Delta_j(2\cos\theta) = \sin((j+1)\theta)/\sin\theta, \quad \Delta_j(-t) = (-1)^j\Delta_n(t). \tag{4.9}$$

A fundamental assumption in constructing the S-matrix is that each candidate label ρ_α should correspond to a finite collection of primitive loop operators:

$$\rho_\alpha \mapsto \{(a_\alpha^\kappa, R_\alpha^\kappa)\}_\kappa. \tag{4.10}$$

Obtaining the above correspondence involves a guess-and-trial process as follows. With a guess in hand and a choice $\epsilon = \pm 1$, we define the W-symbols

$$W_\beta(\alpha) \;:=\; \prod_\kappa W_{\epsilon\,\rho_\beta}(a_\alpha^\kappa, R_\alpha^\kappa) \;=\; \prod_\kappa \mathrm{Tr}_{R_\alpha^\kappa}(\epsilon\,\rho_\beta(a_\alpha^\kappa)), \quad \rho_\alpha, \rho_\beta \in L(X). \tag{4.11}$$

The W-symbols and the un-normalized S-matrix $\tilde{S} = D\,S$ are related by,

$$W_\beta(\alpha) \;=\; \frac{\tilde{S}_{\alpha\beta}}{\tilde{S}_{0\beta}} \quad \text{or} \quad \tilde{S}_{\alpha\beta} \;=\; W_\beta(\alpha)W_0(\beta), \tag{4.12}$$

where 0 denotes the tensor unit ρ_0. In particular, the quantum dimension

$$d_\alpha = W_0(\alpha) \tag{4.13}$$

Hence, we can try to guess a correspondence between candidate labels and loop operators so that the quantum dimension computed by Equation 4.13 matches (in absolute value) with that computed by Equation 4.6.

We expect that the resulting modular data corresponds to a MTC if and only if $H^1(M,\mathbb{Z}_2) = 0$. Note that, this is purely a topological condition, independent of the choice of loop operators. Hence, if $H^1(M,\mathbb{Z}_2) = 0$, we can also validate a choice of the loop operators by checking whether the resulting S and T matrices define a representation to $\mathrm{SL}(2,\mathbb{Z})$.

4.2 Modular data from Seifert fiber spaces

Depending on different choices of the characters for a unit, we found that modular data from three components Seifert fiber spaces can be realized by Temperley-Lieb-Jones categories and $\mathrm{SU}(2)_k$ MTCs, respectively.

4.2.1 Realization of Temperley-Lieb-Jones categories

We will show that the modular data constructed from 3-component SFSs for some choice of unit are related to the Temperley-Lieb-Jones categories at root of unit. So let us collect some basic facts about those. For references, see for instance [33].

Let A be a complex number such that $A^4 \neq 1$. For an integer n, define the quantum integer $[n]_A = \frac{A^{2n}-A^{-2n}}{A^2-A^{-2}}$. So $[0]_A = 0,\ [1]_A = 1,\ [2]_A = A^2 + A^{-2}$. For each A, usually called the Kauffman variable, such that A^4 is a primitive r-th root of unity for some integer $r \geq 2$, there is an associated premodular category, called the Temperley-Lieb-Jones category and denoted by $\mathrm{TLJ}(A)$. The category has the label set (simple objects) $[0 \cdots r-2]$ where the label 0 is the unit object. For $i, j \in [0 \cdots r-2]$, the quantum dimension is

$$d_j(A) = (-1)^j [j+1]_A = (-1)^j \frac{A^{2j+2} - A^{-2j-2}}{A^2 - A^{-2}},$$

the twist is

$$\theta_j(A) = (-A)^{j(j+2)},$$

and the (un-normalized) S-matrix is

$$S_{ij}(A) = (-1)^{i+j} [(i+1)(j+1)]_A.$$

The total dimension can be computed directly,

$$D(A) = \frac{\sqrt{2r}}{|A^2 - A^{-2}|}.$$

Denote by $\mathrm{TLJ}(A)_0$ (resp. $\mathrm{TLJ}(A)_0$) the subcategory linearly spanned by even (resp. odd) labels. We call $\mathrm{TLJ}(A)_0$ and $\mathrm{TLJ}(A)_1$ the even and odd subcategory of $\mathrm{TLJ}(A)$, respectively. The even and odd subcategory has the same dimension, both equal to $\frac{D(A)}{\sqrt{2}}$.

It is well known that if A is a primitive $4r$-th root of unity, then $\mathrm{TLJ}(A)$ is non-degenerate. If r is odd and A is a primitive $2r$-th root of unity, then $\mathrm{TLJ}(A)$ is degenerate, but the even subcategory $\mathrm{TLJ}(A)_0$ is non-degenerate.

Now we consider the construction of modular data. As before, set

$$M = \{0; (o, 0); (p_1, q_1), (p_2, q_2), (p_3, q_3)\}$$

. Here each pair (p_k, q_k) are co-prime. Choose integers s_k and r_k such that $p_k s_k - q_k r_k = 1$. If q_k is odd, set $c_k = p_k q_k s_k - r_k$. Otherwise, set $c_k = p_k q_k s_k - r_k(p_k - 1)^2$. Let $A_k = -\exp(\frac{2\pi i}{4p_k} c_k)$. Note that while c_k depends on the choice of s_k and r_k, A_k does not. Moreover, A_k is a primitive $4p_k$-th root of unity if q_k is odd, a primitive $2p_k$-th root of unity if $q_k = 0 \mod 4$, and a primitive p_k-th root of unity if $q_k = 2 \mod 4$. In the latter two cases, p_k clearly must be odd. Hence, in all cases, A_k^4 is a primitive p_k-th root of unity.

If some q_k's are even, we re-arrange the elements of $\chi^{\mathrm{nab}}(M)$ as follows. For (p, q) co-prime, $j \in [0 \cdots p-2]$, let

$$n_{p,q}(j) = \begin{cases} \frac{p-1-j}{2}, & q \text{ even and } j \text{ even} \\ \frac{j+1}{2}, & \text{otherwise} \end{cases}$$

Then from Equation 2.8, $\chi^{\mathrm{nab}}(M)$ can also be written as

$$\begin{aligned} &\left\{(n_{p_1,q_1}(j_1), n_{p_2,q_2}(j_2), n_{p_3,q_3}(j_3), \frac{1}{2}) \mid j_k \in [0 \cdots p_k - 2]^e, k = 1, 2, 3\right\} \\ \sqcup &\left\{(n_{p_1,q_1}(j_1), n_{p_2,q_2}(j_2), n_{p_3,q_3}(j_3), 0) \;\; \mid j_k \in [0 \cdots p_k - 2]^o, k = 1, 2, 3\right\} \end{aligned} \tag{4.14}$$

Thus, the elements of $\chi^{\mathrm{nab}}(M)$ are indexed by $\vec{j} \in \prod_{k=1}^{3}[0 \cdots p_k - 2]^e \sqcup \prod_{k=1}^{3}[0 \cdots p_k - 2]^o$. Given such a $\vec{j} = (j_1, j_2, j_3)$, denote a corresponding representation by $\rho_{\vec{j}}$. (The choice of a representative is irrelevant.)

Proposition 2.5.1 shows that all non-Abelian characters of M are adjoint acyclic and

Proposition 2.5.2 shows that the CS invariants of non-Abelian characters are all rational. We choose the candidate label set $L(M)$ to be $\chi^{\text{nab}}(M)$.

We propose the correspondence between $L(M)$ and loop operators by the following map,

$$\rho_{\vec{j}} \mapsto \left\{(x_k^{c_k}, \text{Sym}^{j_k}) \mid k = 1, 2, 3\right\}. \tag{4.15}$$

Moreover, we designate $\rho_{\vec{0}} = \rho_{(0,0,0)}$ as the unit object, which of course corresponds to the loop operator

$$\mathbf{1} = \rho_{\vec{0}} \mapsto \left\{(x_k^{c_k}, \text{Sym}^0) \mid k = 1, 2, 3\right\}. \tag{4.16}$$

The following two lemmas are direct consequences of Proposition 2.5.2 and Proposition 2.5.1, respectively.

Lemma 4.2.1. Let $M,\ c_k,\ A_k$ be given as above. For each $\vec{j} = (j_1, j_2, j_3) \in \prod_{k=1}^3 [0 \cdots p_k - 2]^e \ \sqcup\ \prod_{k=1}^3 [0 \cdots p_k - 2]^o$ with $\rho_{\vec{j}}$ a corresponding representation, then

$$\text{CS}(\rho_{\vec{j}}) = \sum_{k=1}^{3} \frac{-c_k}{4p_k}(j_k + 1)^2. \tag{4.17}$$

As a consequence,

$$e^{-2\pi i \text{CS}(\rho_{\vec{j}})} = \prod_{k=1}^{3} (-A_k)^{(j_k+1)^2} = (-A_1A_2A_3) \prod_{k=1}^{3} \theta_{j_k}(A_k). \tag{4.18}$$

Lemma 4.2.2. Let $M,\ c_k,\ A_k$ be given as above and let $D = D(A_1)D(A_2)D(A_3)/2$. For each $\vec{j} = (j_1, j_2, j_3) \in \prod_{k=1}^3 [0 \cdots p_k - 2]^e \ \sqcup\ \prod_{k=1}^3 [0 \cdots p_k - 2]^o$ with $\rho_{\vec{j}}$ a corresponding representation, then

$$\mathrm{Tor}(\rho_{\vec{j}}) = \prod_{k=1}^{3} \frac{p_k}{4\sin^2(\frac{\pi r_k(j_k+1)}{p_k})}, \tag{4.19}$$

and hence,

$$\left(2\mathrm{Tor}(\rho_{\vec{j}})\right)^{-\frac{1}{2}} = 2\prod_{k=1}^{3}\left|\frac{d_{j_k}(A_k)}{D(A_k)}\right| = \frac{|\prod_{k=1}^{3} d_{j_k}(A_k)|}{D}. \tag{4.20}$$

The main result of the section is the following theorem.

Theorem 4.2.3. Let $M = \{0;(p_1,q_1),(p_2,q_2),(p_3,q_3)\}$ and $\{A_k\}_{k=1,2,3}$ be given as above. With the operators and tensor unit given in Equations 4.15 and 4.16, respectively, the modular data constructed from M matches that of the following premodular category,

$$\mathcal{C} := \left(\ \boxtimes_{k=1}^{3}\mathrm{TLJ}(A_k)_0\ \right)\bigoplus\left(\ \boxtimes_{k=1}^{3}\mathrm{TLJ}(A_k)_1\ \right)$$

Proof: Since A_k^4 is a primitive p_k-th root of unity, the label set for $\mathcal{C}$ is clearly $L := \prod_{k=1}^{3}[0\cdots p_k-2]^e \ \sqcup\ \prod_{k=1}^{3}[0\cdots p_k-2]^o$, the same index set for $L(M)$. The modular data of $\mathcal{C}$ can be easily expressed in terms of that of the individual $\mathrm{TLJ}(A_k)$. For $\vec{i},\vec{j}\in L$,

$$d_{\vec{j}} = \prod_{k=1}^{3} d_{j_k}(A_k), \quad \theta_{\vec{j}} = \prod_{k=1}^{3}\theta_{j_k}(A_k), \quad \tilde{S}_{\vec{i}\,\vec{j}} = \prod_{k=1}^{3}\tilde{S}_{i_kj_k}(A_k).$$

Also, the total dimension of $\mathcal{C}$ is $D = D(A_1)D(A_2)D(A_3)/2$.

Lemma 4.2.1 shows that, up to a global phase, the Chern-Simons invariant gives the twist,

$$e^{-2\pi i\mathrm{CS}(\rho_{\vec{j}})} = \theta_{\vec{j}},$$

and Lemma 4.2.2 shows that the torsion matches the absolute value of the normalized quantum dimension,

$$\big(2\mathrm{Tor}(\rho_{\vec{j}})\big)^{-\frac{1}{2}} = \frac{d_{\vec{j}}}{D}.$$

Lastly, We check the S-matrix computed from local operators. Given $\vec{i} = (i_1, i_2, i_3)$, $\vec{j} = (j_1, j_2, j_3) \in L$, we have (choosing $\epsilon = -1$)

$$W_{\vec{i}}(\vec{j}) = \prod_{k=1}^{3} \mathrm{Tr}_{\mathrm{Sym}^{j_k}}(-\rho_{\vec{i}}(x_k^{c_k})).$$

Note that,

$$\mathrm{Tr}\big(\rho_{\vec{i}}(x_k^{c_k})\big) \;=\; 2\cos\frac{2n_{p_k,q_k}(i_k)\pi c_k}{p_k} \;=\; 2\cos\frac{(i_k+1)\pi c_k}{p_k},$$

where the second equality holds irrelevant of the parity of q_k. Combining the previous two equations, we get

$$W_{\vec{i}}(\vec{j}) \;=\; \prod_{k=1}^{3}\Delta_{j_k}(-2\cos\frac{(i_k+1)\pi c_k}{p_k}) \;=\; \prod_{k=1}^{3}(-1)^{j_k}\frac{\sin\frac{(i_k+1)(j_k+1)\pi c_k}{p_k}}{\sin\frac{(i_k+1)\pi c_k}{p_k}},$$

where $\Delta_{j_k}(\cdot)$ is the Chebyshev polynomial (see Equation 4.9). Therefore, the $(\vec{j},\vec{i})$-entry of the potential un-normalized S matrix is,

$$\begin{aligned} W_{\vec{i}}(\vec{j})W_{\vec{0}}(\vec{i}) \;&=\; \prod_{k=1}^{3}(-1)^{i_k+j_k}\frac{\sin\frac{(i_k+1)(j_k+1)\pi c_k}{p_k}}{\sin\frac{\pi c_k}{p_k}} \\ &=\; \prod_{k=1}^{3}\tilde{S}(A_k)_{j_k i_k}, \end{aligned}$$

which is precisely $\tilde{S}_{\vec{j}\,\vec{i}}$ of $\mathcal{C}$.

The premodular category produced in the previous theorem may not be modular in general, and it depends crucially on the topology of the three manifold. For a three-component SFS M, it is a $\mathbb{Z}_2$ homology sphere, i.e., $H^1(M, \mathbb{Z}_2) = 0$, if and only if

$$p_1p_2p_3(\frac{q_1}{p_1} + \frac{q_2}{p_2} + \frac{q_3}{p_3}) \in 2\mathbb{Z} + 1$$

Lemma 4.2.4. Assume that r is odd. Suppose that

$$T(p, j, l, *) \;=\; \sum_{m \in [p]^*} \left(e^{(j+l)mr\frac{\pi}{p}i} - e^{(j-l)mr\frac{\pi}{p}i} - e^{(-j+l)mr\frac{\pi}{p}i} + e^{(-j-l)mr\frac{\pi}{p}i}\right)$$

where $* = 1, 0$, and $[p]^*$ denotes the set of odd integers from 1 to $p-1$ if $*$ is 1 and the set of even integers in the same range otherwise.

When p is odd, $j \neq l$, $j + l$ is odd,

$$T(p, j, l, *) = \begin{cases} 0 & j + l \neq p \\ (-1)^* p & j + l = p \end{cases}$$

When p is odd, $j \neq l$, $j + l$ is even,

$$T(p, j, l, *) = 0$$

When p is odd, $j = l$,

$$T(p, j, l, *) = -p$$

When p is even, $j \neq l$, $j + l$ is odd,

$$T(p, j, l, *) = 0$$

When p is even, $j \neq l$, $j+l$ is even,

$$T(p,j,l,*) = \begin{cases} 0 & j+l \neq p \\ (-1)^* p & j+l = p \end{cases}$$

When p is even, $j = l$,

$$T(p,j,l,0) = \begin{cases} -p & j+l \neq p \\ 0 & j+l = p \end{cases}$$

$$T(p,j,l,1) = \begin{cases} -p & j+l \neq p \\ -2p & j+l = p \end{cases}$$

Proof: We prove the lemma by direct computation.

When p is odd, $j \neq l$, $j+l$ is odd,

$$\begin{aligned}
T(p,j,l,1) &= \sum_{m=1, m \text{ odd}}^{p-2} (e^{(j+l)mr\frac{\pi}{p}i} - e^{(j-l)mr\frac{\pi}{p}i} + e^{(j-l)(p-m)r\frac{\pi}{p}i} - e^{(j+l)(p-m)r\frac{\pi}{p}i}) \\
&= \sum_{m=1, m \text{ odd}}^{p-2} (e^{(j+l)mr\frac{\pi}{p}i} - e^{(j-l)mr\frac{\pi}{p}i}) + \sum_{m=2, \text{ even}}^{p-1} (e^{(j-l)mr\frac{\pi}{p}i} - e^{(j+l)mr\frac{\pi}{p}i}) \\
&= -\sum_{m=1}^{p-1} (-e^{(j+l)r\frac{\pi}{p}i})^m + \sum_{m=1}^{p-1} (-e^{(j-l)r\frac{\pi}{p}i})^m \\
&= \begin{cases} 0 & j+l \neq p \\ -p & j+l = p \end{cases} \\
&= -T(p,j,l,0)
\end{aligned}$$

Similarly, we get other cases.

Proposition 4.2.5. Given a three-component SFS M, the premodular category $\mathcal{C}_M$ produced in Theorem 4.2.3 is modular if and only if M is a $\mathbb{Z}_2$ homology sphere.

Proof: Since the structure from the above subsection respects the change of parametrization of Seifert fiber space, it suffices to verify the following 5 cases for $(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$.

$$(\frac{odd}{odd}, \frac{odd}{odd}, \frac{odd}{odd}), (\frac{odd}{odd}, \frac{odd}{odd}, \frac{even}{odd}), (\frac{odd}{odd}, \frac{even}{odd}, \frac{even}{odd}),$$

$$(\frac{even}{odd}, \frac{even}{odd}, \frac{even}{odd}), (\frac{odd}{odd}, \frac{odd}{odd}, \frac{odd}{even})$$

The first two cases correspond to $\mathbb{Z}_2$-homology sphere. In the following, we will explicitly calculate S^2, which directly implies the proposition.

When q_1, q_2, q_3 are odd, $j_1 = j_2 = j_3 \mod 2$, $l_1 = l_2 = l_3 \mod 2$.

Up to a scalar,

$$S_{(j_1,j_2,j_3),(l_1,l_2,l_3)} = (-1)^{j_1+l_1} \prod_{k=1}^{3} \sin j_k l_k r_k \frac{\pi}{p_k}$$

$$\begin{aligned}
&(S^2)_{(j_1,j_2,j_3),(l_1,l_2,l_3)} \\
&= \sum_{(m_1,m_2,m_3)} (-1)^{j_1+m_1+m_1+l_1} \prod_{k=1}^{3} \sin j_k m_k r_k \frac{\pi}{p_k} \sin m_k l_k r_k \frac{\pi}{p_k} \\
&= (-1)^{j_1+l_1} \sum_{(m_1,m_2,m_3)} \prod_{k=1}^{3} -\frac{1}{4}(e^{(j_k+l_k)m_k r_k \frac{\pi}{p_k} i} - e^{(j_k-l_k)m_k r_k \frac{\pi}{p_k} i} - e^{(-j_k+l_k)m_k r_k \frac{\pi}{p_k} i} \\
&\quad + e^{(-j_k-l_k)m_k r_k \frac{\pi}{p_k} i}) \\
&= (-1)^{j_1+l_1} (\sum_{(m_1,m_2,m_3), m_i \text{ odd}} + \sum_{(m_1,m_2,m_3), m_i \text{ even}}) ... \\
&= (-1)^{j_1+l_1} (\prod_{k=1}^{3} T(p_k, j_k, l_k, 1) + \prod_{k=1}^{3} T(p_k, j_k, l_k, 0))
\end{aligned}$$

When p_1, p_2, p_3 are odd,

$$(S^2)_{(j_1,j_2,j_3),(l_1,l_2,l_3)} = \begin{cases} 0 & (j_1,j_2,j_3) \neq (l_1,l_2,l_3) \\ \frac{p_1p_2p_3}{32} & (j_1,j_2,j_3) = (l_1,l_2,l_3) \end{cases}$$

When p_1, p_2 are odd, p_3 is even,

$$(S^2)_{(j_1,j_2,j_3),(l_1,l_2,l_3)} = \begin{cases} 0 & (j_1,j_2,j_3) \neq (l_1,l_2,l_3) \\ \frac{p_1p_2p_3}{32} & (j_1,j_2,j_3) = (l_1,l_2,l_3) \end{cases}$$

Thus $S^2 = cI$ for the above two cases.

When p_1 is odd, p_2, p_3 are even,

$$(S^2)_{(1,1,1),(l_1,l_2,l_3)} = \begin{cases} \frac{p_1p_2p_3}{32} & (l_1,l_2,l_3) = (1,1,1),(1,p_2-1,p_3-1) \\ 0 & \text{otherwise} \end{cases}$$

$$(S^2)_{(1,p_2-1,p_3-1),(l_1,l_2,l_3)} = \begin{cases} \frac{p_1p_2p_3}{32} & (l_1,l_2,l_3) = (1,1,1),(1,p_2-1,p_3-1) \\ 0 & \text{otherwise} \end{cases}$$

$$(S^2)_{(1,1,1)} = (S^2)_{(1,p_2-1,p_3-1)}$$

When p_1, p_2, p_3 are even,

$$(S^2)_{(1,1,1),(l_1,l_2,l_3)} = \begin{cases} \frac{p_1p_2p_3}{32} & (l_1,l_2,l_3) = (1,1,1),(1,p_2-1,p_3-1) \\ 0 & \text{otherwise} \end{cases}$$

$$(S^2)_{(1,p_2-1,p_3-1),(l_1,l_2,l_3)} = \begin{cases} \frac{p_1p_2p_3}{32} & (l_1,l_2,l_3) = (1,1,1),(1,p_2-1,p_3-1) \\ 0 & \text{otherwise} \end{cases}$$

$$(S^2)_{(1,1,1)} = (S^2)_{(1,p_2-1,p_3-1)}$$

S^2 is degenerate for above two cases.

When q_1, q_2 are odd, q_3 is even, $j_1 = j_2 \mod 2$, $l_1 = l_2 \mod 2$, $j_3 = 0 \mod 2$, $l_3 = 0 \mod 2$.

$$(S^2)_{(j_1,j_2,j_3),(l_1,l_2,l_3)} = \prod_{k=1}^{2} T(p_k, j_k, l_k, 1) T(p_3, j_3, l_3, 0) + \prod_{k=1}^{3} T(p_k, j_k, l_k, 0)$$

When p_1, p_2, p_3 are odd,

$$(S^2)_{(1,1,2),(l_1,l_2,l_3)} = \begin{cases} -\dfrac{p_1p_2p_3}{32} & (l_1,l_2,l_3) = (1,1,2), (p_1-1, p_2-1, 2) \\ 0 & \text{otherwise} \end{cases}$$

$$(S^2)_{(p_1-1,p_2-1,2),(l_1,l_2,l_3)} = \begin{cases} -\dfrac{p_1p_2p_3}{32} & (l_1,l_2,l_3) = (1,1,2), (p_1-1, p_2-1, 2) \\ 0 & \text{otherwise} \end{cases}$$

S^2 is degenerate.

It is worth noting even if every TLJ(A_k) appearing in the construction of $\mathcal{C}_M$ in Theorem 4.2.3 is not modular, $\mathcal{C}_M$ could still be modular. For instance, for the SFS $M_0 = (0; (o,0); (5,1), (3,2), (5,4))$, the corresponding Kauffman variables are $A_1 = -e^{\frac{i\pi}{10}}$, $A_2 = -e^{\frac{i\pi}{3}}$, $A_3 = -e^{\frac{2i\pi}{5}}$. It is direct to see that TLJ(A_1) is modular, but TLJ(A_2) and TLJ(A_3) are not. However, M_0 is a $\mathbb{Z}_2$ homology sphere, by Proposition 4.2.5, $\mathcal{C}_{M_0}$ is modular, a rank-8 MTC.

4.2.2 Realization of $\mathrm{SU}(2)_k$ MTCs

Here we study a special class of SFSs with three components, namely, $M(r) := \{0; (o,0); (3,1), (3,1), (r,1)\}$. We show explicitly that different choice of characters as the unit object may lead to different theories. In fact, it will be proved that from $M(r)$ we can construct either the MTC $\mathrm{SU}(2)_{r-2}$ or $\mathrm{TLJ}(e^{\frac{2\pi i}{4r}})$.

For each integer $r \geq 2$, there is a unitary MTC, usually denoted by $\mathrm{SU}(2)_{r-2}$ [4], which is closely related to the Temperley-Lieb-Jones categories. Here $r-2$ is called the level of the MTC. It has the same label set as $\mathrm{TLJ}(e^{\frac{2\pi i}{4r}})$, but differs from it in modular data by some signs. Explicitly, setting $A = e^{\frac{2\pi i}{4r}}$, the modular data for $\mathrm{SU}(2)_{r-2}$ is given as follows,

$$\begin{aligned} \theta_j &= A^{j(j+2)} = e^{\frac{2\pi i\, j(j+2)}{4r}}, \\ \tilde{S}_{ij} &= [(i+1)(j+1)]_A = \frac{\sin\frac{(i+1)(j+1)\pi}{r}}{\sin\frac{\pi}{r}}. \end{aligned}$$

In particular, its quantum dimensions are all positive (since it is unitary),

$$d_j = [j+1]_A = \frac{\sin\frac{(j+1)\pi}{r}}{\sin\frac{\pi}{r}},$$

and the total dimension is

$$D = \sqrt{\frac{r}{2}}\frac{1}{\sin\frac{\pi}{r}}.$$

Note that $d_j = |d_j(A)|$ and $D = D(A)$, where $d_j(A)$ and $D(A)$ are the quantum dimension of j and total dimension of $\mathrm{TLJ}(A)$, respectively.

We will use notations from Section 2.2. The non-Abelian characters of $M(r)$ is given by

$$\begin{aligned} \chi^{\mathrm{nab}}(M(r)) = &\left\{\left(\frac{1}{2},\frac{1}{2},\frac{j+1}{2},\frac{1}{2}\right) \mid (0,0,j) \in \{0\}\times\{0\}\times[0\cdots r-2]^e\right\} \\ &\sqcup\left\{\left(1,1,\frac{j+1}{2},0\right) \mid (1,1,j) \in \{1\}\times\{1\}\times[0\cdots r-2]^o\right\}. \end{aligned} \tag{4.21}$$

Thus, each $j \in [0\cdots r-2]$ corresponds to a non-Abelian character indexed by (j mod $2, j$ mod $2, j$). We denote the corresponding representation by ρ_j (instead of using the triple as the subscript). The eigenvalues of $\rho_j(x_3)$ are $e^{\pm\frac{(j+1)\pi i}{r}}$. The eigenvalues of $\rho_j(x_1)$ and those of $\rho_j(x_2)$ are both $e^{\pm\frac{a_j\pi i}{3}}$, where $a_j = 1$ if j even and $a_j = 2$ otherwise.

Also, it is direct to see that $c_1 = c_2 = c_3 = 1$, and $A_1 = A_2 = -e^{\frac{\pi i}{6}}$, $A_3 = -e^{\frac{2\pi i}{4r}}$.

In Section **??**, we chose the candidate label set $L(M(r))$ to be $\chi^{\mathrm{nab}}(M(r))$, and defined the following map from $\chi^{\mathrm{nab}}(M(r))$ to local operators,

$$\rho_j = \mapsto \left\{(x_1, \mathrm{Sym}^{j \bmod 2}), (x_2, \mathrm{Sym}^{j \bmod 2}), (x_3, \mathrm{Sym}^j)\right\}. \tag{4.22}$$

It can be checked directly that for $i, j \in [0 \cdots r-2]$, $\mathrm{Tr}(\rho_i(x_1)) = \mathrm{Tr}(\rho_i(x_2)) = \pm 1$, and it follows that,

$$\begin{aligned} W_i(j) &= \mathrm{Tr}_{\mathrm{Sym}^{j \bmod 2}}(-\rho_i(x_1))\,\mathrm{Tr}_{\mathrm{Sym}^{j \bmod 2}}(-\rho_i(x_2))\,\mathrm{Tr}_{\mathrm{Sym}^j}(-\rho_i(x_3)) \\ &= \mathrm{Tr}_{\mathrm{Sym}^j}(-\rho_i(x_3)). \end{aligned}$$

Hence, we may as well choose a simplified map to local operators,

$$\rho_j \mapsto \{(x_3, \mathrm{Sym}^j)\}. \tag{4.23}$$

The unit object was chosen to be ρ_0 which corresponds to the local operator (x_3, Sym^0). By Theorem 4.2.3, the modular data match that of the premodular category,

$$\mathcal{C}_{M(r)} = \left(\ \boxtimes_{k=1}^3 \mathrm{TLJ}(A_k)_0\ \right) \bigoplus \left(\ \boxtimes_{k=1}^3 \mathrm{TLJ}(A_k)_1\ \right). \tag{4.24}$$

Note that $\mathrm{TLJ}(A_1) = \mathrm{TLJ}(-e^{\frac{\pi i}{6}})$ has label set $\{0, 1\}$, the twists $\theta_0 = 1$, $\theta_1 = i$, and un-normalized S-matrix,

$$\tilde{S} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.$$

This means that $\mathcal{C}_{M(r)}$ has the same twists for even labels and S-matrix as $\mathrm{TLJ}(A_3)$. The twists for odd labels differ by a minus sign between the two theories. Let $A(r) = -A_3 = e^{\frac{2\pi i}{4r}}$. Note that a change of the Kauffman variable from A to $-A$ does not change the S-matrix. It follows that $\mathcal{C}_{M(r)}$ and $\mathrm{TLJ}(A(r))$ has the same modular data. In fact, they are isomorphic.

Therefore, by using the local operator correspondence in Equation 4.23 and letting ρ_0 be the unit object, we recover the MTC $\mathrm{TLJ}(A(r))$.

Now we examine an alternative choice of the unit object. Since $M(r)$ is a $\mathbb{Z}_2$ homology sphere, a potential unit object ρ_{α_0} can be determined by the equation,

$$\left| \sum_{\rho \in \chi^{\mathrm{nab}}(M(r))} \frac{\exp(-2\pi i \mathrm{CS}(\rho))}{2\mathrm{Tor}(\rho)} \right| = (2\mathrm{Tor}(\rho_{\alpha_0}))^{-\frac{1}{2}}. \tag{4.25}$$

Such a ρ_{α_0} would have quantum dimension in absolute value equal to 1 in any MTC produced by $M(r)$. Since we already know that we can produce $\mathrm{TLJ}(A(r))$ from $M(r)$ and the only non-unit object in $\mathrm{TLJ}(A(r))$ whose quantum dimension is 1 in absolute value is ρ_{r-2}, we can choose ρ_{r-2} as the unit object in a new theory.

In this case, we reverse the previous order of the simple objects. Denote by $\tilde{\rho}_j := \rho_{r-2-j}$, $j \in [0 \cdots r-2]$. Set $\tilde{\rho}_0 = \rho_{r-2}$ as the unit object. The correspondence between characters and local operators is now defined as,

$$\tilde{\rho}_j \mapsto (x_3, \mathrm{Sym}^j). \tag{4.26}$$

We claim that with above choice of unit object and local operators, the modular data produced from $M(r)$ matches that of $\mathrm{SU}(2)_{r-2}$ where $\tilde{\rho}_j$ corresponds to j in the label set of $\mathrm{SU}(2)_{r-2}$. See the above section for a collection of facts about $\mathrm{SU}(2)_{r-2}$.

Firstly, by Lemma 4.2.1, up to an irrelevant phase factor,

$$\mathrm{CS}(\rho_j) = -\frac{j(j+2)}{4r} + \frac{1-(-1)^j}{4} \mod 1. \tag{4.27}$$

Then rewriting above equation in terms of $\tilde{\rho}_j$, we get, again up to an irrelevant factor,

$$\mathrm{CS}(\tilde{\rho}_j) = -\frac{j(j+2)}{4r} \mod 1. \tag{4.28}$$

Thus,

$$e^{-2\pi i \mathrm{CS}(\tilde{\rho}_j)} = e^{\frac{2\pi i\, j(j+2)}{4r}} \tag{4.29}$$

is the twist θ_j of $\mathrm{SU}(2)_{r-2}$.

Next, we check the S-matrix.

$$W_0(j) = \mathrm{Tr}_{\mathrm{Sym}^j}(-\tilde{\rho}_0(x_3)) = \Delta_j(2\cos\frac{\pi}{r}) = \frac{\sin\frac{(j+1)\pi}{r}}{\sin\frac{\pi}{r}}, \tag{4.30}$$

and the (j,i)-entry of the potential S-matrix is,

$$\begin{aligned} W_i(j)W_0(i) &= \mathrm{Tr}_{\mathrm{Sym}^j}(-\tilde{\rho}_i(x_3))\, W_0(i) && (4.31)\\ &= \Delta_j(2\cos\frac{(i+1)\pi}{r})\Delta_i(2\cos\frac{\pi}{r}) && (4.32)\\ &= \frac{\sin\frac{(i+1)(j+1)\pi}{r}}{\sin\frac{\pi}{r}}, && (4.33) \end{aligned}$$

which is S_{ji} of $\mathrm{SU}(2)_{r-2}$.

Lastly, by Lemma 4.2.2,

$$\big(2\mathrm{Tor}(\tilde{\rho}_j)\big)^{-\frac{1}{2}} = \big(2\mathrm{Tor}(\rho_{r-2-j})\big)^{-\frac{1}{2}} = \frac{|d_{r-2-j}(A_3)|}{D(A_3)}, \tag{4.34}$$

where we used the fact that in $\mathrm{TLJ}(A_1) = \mathrm{TLJ}(A_2)$, the two simple objects have quantum dimensions ± 1 and thus the dimension of the category is $D(A_1) = \sqrt{2}$. Also note that $A_3 = -e^{\frac{2\pi i}{4r}}$, then $|d_{r-2-j}(A_3)| = |d_j(A_3)|$ and $D(A_3)$ are equal to the quantum dimension d_j and the total dimension D, respectively, in $\mathrm{SU}(2)_{r-2}$. Hence, the torsion invariant computes the normalized quantum dimension,

$$\big(2\mathrm{Tor}(\tilde{\rho}_j)\big)^{-\frac{1}{2}} = \frac{d_j}{D}. \tag{4.35}$$

To summarize, for the SFS $M(r)$, two choices of the unit object together with appropriate definition of loop operators produce the MTCs $\mathrm{TLJ}(e^{\frac{2\pi i}{4r}})$ and $\mathrm{SU}(2)_{r-2}$, with the former non-unitary and the latter unitary.

4.2.3 Graded product of graded premodular categories

In the above subsection, we have seen that the premoduar category resulting from three-component SFSs is formed from three Temperley-Lieb-Jones categories, by taking the Deligne product of the even sectors, that of the odd sectors, and suming them up. Here we generalize the operation.

Definition 4.2.6. Let $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ and $\mathcal{D} = \oplus_{g\in G}\mathcal{D}_g$ be two G-graded premodular tensor categories for some finite group G (which must be Abelian). The graded product of $\mathcal{C}$ and $\mathcal{D}$ is again a G-graded premodular category $\mathcal{C} \boxtimes_{gr} \mathcal{D} = \oplus_{g\in G}(\mathcal{C} \boxtimes_{gr} \mathcal{D})_g$ such that $(\mathcal{C} \boxtimes_{gr} \mathcal{D})_g := \mathcal{C}_g \boxtimes \mathcal{D}_g$.

The monoidal and braiding structure on $\mathcal{C} \boxtimes_{gr} \mathcal{D}$ is defined in the obvious way which make it into a premodular category. Another way to see this is that $\mathcal{C} \boxtimes_{gr} \mathcal{D}$ is a full subcategory of the premodular category $\mathcal{C} \boxtimes \mathcal{D}$ and is closed under tensor product and braiding. The graded product operation $\boxtimes_{gr}$ is associative up to canonical equivalence.

For a Kauffman variable A, $\mathrm{TLJ}(A)$ is a $\mathbb{Z}_2$-graded premodular category with $\mathrm{TLJ}(A)_0$ spanned by even labels and $\mathrm{TLJ}(A)_1$ odd labels. Hence, Theorem 4.2.3 states that, for a three-component SFS $M = \{0; (o, 0); (p_1, q_1), (p_2, q_2), (p_3, q_3)\}$ with $A_k, k = 1, 2, 3$, the premodular category resulting from M is $\mathcal{C}_M = \mathrm{TLJ}(A_1) \boxtimes_{gr} \mathrm{TLJ}(A_2) \boxtimes_{gr} \mathrm{TLJ}(A_3)$.

The graded product operation provides method to construct new premodular categories from old ones. A very interesting question is when the graded product of two pre-modular categories is modular. For instance, take $A_1 = -e^{\frac{i\pi}{6}},\ A_2 = -e^{-\frac{i\pi}{5}}$. Here A_1 is a primitive 12-th root of unity and A_2 a primitive 5-th root of unity. Hence $\mathrm{TLJ}(A_1)$ is modular of rank 2 and $\mathrm{TLJ}(A_2)$ is none modular of rank 4. Their S-matrices are given by,

$$\tilde{S}(A_1) = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \quad \tilde{S}(A_2) = \begin{pmatrix} 1 & \varphi & \varphi & 1 \\ \varphi & -1 & -1 & \varphi \\ \varphi & -1 & -1 & \varphi \\ 1 & \varphi & \varphi & 1 \end{pmatrix}, \tag{4.36}$$

where $\varphi = \frac{1}{2}(1-\sqrt{5})$. Then the S-matrix of $\mathrm{TLJ}(A_1) \boxtimes_{gr} \mathrm{TLJ}(A_2)$ with its simple objects ordered as $\{0 \boxtimes 0, 0 \boxtimes 2, 1 \boxtimes 1, 1 \boxtimes 3\}$ is,

$$\tilde{S} = \begin{pmatrix} 1 & \varphi & -\varphi & -1 \\ \varphi & -1 & 1 & -\varphi \\ -\varphi & 1 & 1 & -\varphi \\ -1 & -\varphi & -\varphi & -1 \end{pmatrix}, \tag{4.37}$$

which can be checked straightforwardly to be non-degenerate. Thus $\mathrm{TLJ}(A_1) \boxtimes_{gr} \mathrm{TLJ}(A_2)$ is modular.

We leave the question of when the graded product of two arbitrary graded (and more generally multiple) premodular categories is modular as a future direction. In the rest of this section, we focus on the case where the group is $\mathbb{Z}_2$ and study a special class of $\mathbb{Z}_2$-graded modular categories, namely $\mathrm{SU}(2)_k$. For basic facts, see Section 4.2.2.

Let $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$ be a $\mathbb{Z}_2$-graded MTC. Denote by I the label set of $\mathcal{C}$ and partition $I = I_0 \sqcup I_1$ where I_α consists of objects of I that are in the $\mathcal{C}_\alpha$ sector. To avoid confusion, when there is more than one MTC present, we write $I(\mathcal{C})$, $\tilde{S}(\mathcal{C})$, etc.

Proposition 4.2.7. Let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathbb{Z}_2$-graded MTCs. Then $\mathcal{C} \boxtimes_{gr} \mathcal{D}$ is a proper (i.e., degenerate) premodular category if and only if there exist $i \in I(\mathcal{C})$, $j \in I(\mathcal{D})$, scalars $c_0(\mathcal{C})$, $c_1(\mathcal{C})$, $c_0(\mathcal{D})$, and $c_1(\mathcal{D})$, such that,

1. i and j belong to sectors of the same parity;

2. the following equations concerning S-entries hold:

$$\tilde{S}(\mathcal{C})_{ik} = \begin{cases} c_0(\mathcal{C}) d_k(\mathcal{C}) & k \in I_0(\mathcal{C}) \\ c_1(\mathcal{C}) d_k(\mathcal{C}) & k \in I_1(\mathcal{C}) \end{cases} \qquad \tilde{S}(\mathcal{D})_{jk} = \begin{cases} c_0(\mathcal{D}) d_k(\mathcal{D}) & k \in I_0(\mathcal{D}) \\ c_1(\mathcal{D}) d_k(\mathcal{D}) & k \in I_1(\mathcal{D}) \end{cases}$$

3. $c_0(\mathcal{C})/c_1(\mathcal{C}) = c_1(\mathcal{D})/c_0(\mathcal{D}) \neq 1$.

Proof: The main idea is to show that the conditions presented in the statement of the proposition are equivalent to the property that in the S-matrix of $\mathcal{C} \boxtimes_{gr} \mathcal{D}$, the row corresponding to the object $i \boxtimes j$ is proportional to the first row (i.e., the row corresponding to the unit object).

Remark 4.2.8. In the above proposition, the conditions $c_0(\mathcal{C})/c_1(\mathcal{C}) \neq 1$ and $c_1(\mathcal{D})/c_0(\mathcal{D}) \neq 1$ are used to eliminate the trivial case where i and j are both the unit object. When neither of i nor j is the unit object, those conditions automatically hold since otherwise the S-matrix of $\mathcal{C}$ or $\mathcal{D}$ would be degenerate. Also, note that if either $\mathcal{C}_0$ or $\mathcal{D}_0$ is non-degenerate, then i and j must be in the sector of odd parity.

For $m \geq 0$, $\mathrm{SU}(2)_m$ is a $\mathbb{Z}_2$-graded MTC with $(\mathrm{SU}(2)_m)_0$ spanned by even labels and $(\mathrm{SU}(2)_m)_1$ by odd labels.

Theorem 4.2.9. For $m, n \geq 0$, $\mathrm{SU}(2)_m \boxtimes_{gr} \mathrm{SU}(2)_n$ is an MTC if and only if the pair (m, n) have different parity. In particular, $\mathrm{SU}(2)_m \boxtimes_{gr} \mathrm{SU}(2)_m$ is always degenerate.
Proof: In $\mathrm{SU}(2)_m$, the un-normalized S-matrix is given by,

$$\tilde{S}_{ab} = \frac{\sin \frac{(a+1)(b+1)\pi}{m+2}}{\sin \frac{\pi}{m+2}}.$$

Hence, $\tilde{S}_{mb} = (-1)^b \tilde{S}_{0b} = (-1)^b d_b$. For (m, n) with the same parity, with the notation from the statement of Proposition 4.2.7, we choose $i = m, j = n$. Then the relevant constants are $c_0(\mathrm{SU}(2)_m) = c_0(\mathrm{SU}(2)_n) = 1$, $c_1(\mathrm{SU}(2)_m) = c_1(\mathrm{SU}(2)_n) = -1$ which satisfies the conditions stated in that proposition, and hence $\mathrm{SU}(2)_m \boxtimes_{gr} \mathrm{SU}(2)_n$ is degenerate. For the converse direction, it can be seen that the only non-unit simple object in $\mathrm{SU}(2)_m$ for which $c_0(\mathrm{SU}(2)_m)$ and $c_1(\mathrm{SU}(2)_m)$ exist is the object m. Therefore, if (m, n) have different parity, the only pair of indexes for (i, j) is (m, n) which contradicts the first condition of Proposition 4.2.7. This implies that $\mathrm{SU}(2)_m \boxtimes_{gr} \mathrm{SU}(2)_n$ is non-degenerate.

Example 4.2.10. By Theorem 4.2.9, $\mathrm{SU}(2)_2 \boxtimes_{gr} \mathrm{SU}(2)_3$ is an MTC of rank 6. Its un-normalized S-matrix and T-matrix are given by,

$$\tilde{S} = \begin{pmatrix} 1 & \frac{1}{2}\left(1+\sqrt{5}\right) & 1 & \frac{1}{2}\left(1+\sqrt{5}\right) & \frac{1+\sqrt{5}}{\sqrt{2}} & \sqrt{2} \\ \frac{1}{2}\left(1+\sqrt{5}\right) & -1 & \frac{1}{2}\left(1+\sqrt{5}\right) & -1 & -\sqrt{2} & \frac{1+\sqrt{5}}{\sqrt{2}} \\ 1 & \frac{1}{2}\left(1+\sqrt{5}\right) & 1 & \frac{1}{2}\left(1+\sqrt{5}\right) & -\frac{1+\sqrt{5}}{\sqrt{2}} & -\sqrt{2} \\ \frac{1}{2}\left(1+\sqrt{5}\right) & -1 & \frac{1}{2}\left(1+\sqrt{5}\right) & -1 & \sqrt{2} & -\frac{1+\sqrt{5}}{\sqrt{2}} \\ \frac{1+\sqrt{5}}{\sqrt{2}} & -\sqrt{2} & -\frac{1+\sqrt{5}}{\sqrt{2}} & \sqrt{2} & 0 & 0 \\ \sqrt{2} & \frac{1+\sqrt{5}}{\sqrt{2}} & -\sqrt{2} & -\frac{1+\sqrt{5}}{\sqrt{2}} & 0 & 0 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\frac{4i\pi}{5}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{\frac{4i\pi}{5}} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{27i\pi}{40}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -ie^{\frac{3i\pi}{8}} \end{pmatrix}$$

Since $\mathrm{SU}(2)_2 \boxtimes_{gr} \mathrm{SU}(2)_3$ contains the even part of $\mathrm{SU}(2)_3$ as a subcategory which is itself an MTC (Fibonacci), $\mathrm{SU}(2)_2 \boxtimes_{gr} \mathrm{SU}(2)_3$ must split. In fact, $\mathrm{SU}(2)_2 \boxtimes_{gr} \mathrm{SU}(2)_3 \simeq \mathrm{Fib} \boxtimes \mathrm{TLJ}(-ie^{\frac{\pi i}{8}})$.

4.3 Modular data from Sol torus bundles

We will show that the modular data constructed from torus bundles with Sol geometry can be realized by the $\mathbb{Z}_2$-equivariantization of pointed categories.

4.3.1 Equivariantization of $\mathbb{Z}_2$ symmetry

Let $\mathcal{C}(G, q)$ denote the premodular category associated to a finite Abelian group G and a quadratic form $q : G \longrightarrow \mathbb{C}$ as defined in [12]. In this section, we consider the $\mathbb{Z}_2$-equivariantization $\mathcal{C}(G, q)^{\mathbb{Z}_2}$ of this premodular category, where the action $\underline{\mathbb{Z}_2} \to \mathrm{Aut}_{\otimes}(\mathcal{C}(G, q))$ corresponds to the involution $g \mapsto -g$ in G. Commonly referred to as the "particle-hole symmetry," this action previously appeared in the classification of metaplectic modular categories [1, 7, 6] and equivariantization of Tambara-Yamagami categories [17]. It is clear that this action preserves the braiding as well since any quadratic form is invariant under inversion of its argument, and for any braided pointed fusion category $\mathcal{C}(G, q)$ the braiding is given by the bilinear form associated to q.

Proposition 4.3.1. As a fusion category, $\mathcal{C}(G,q)^{\mathbb{Z}_2}$ has the following simple objects:

Invertible objects: X_b^+, X_b^-, for each $b \in G$ such that $b = -b$.

Two-dimensional objects: $Y_{\{a,-a\}}$ for each $a \in G$ such that $a \neq -a$.

For simplicity, we denote $Y_a := Y_{\{a,-a\}}$, and hence $Y_a = Y_{-a}$.

The fusion rules of $\mathcal{C}(G,q)^{\mathbb{Z}_2}$ are given by

$$X_b^\epsilon \otimes X_{b'}^{\epsilon'} \cong X_{b+b'}^{\epsilon\epsilon'},$$

$$X_b^\epsilon \otimes Y_a \cong Y_{a+b},$$

$$Y_a \otimes Y_{a'} \cong \begin{cases} X_0^+ \oplus X_0^- \oplus Y_{2a}, & \text{if } a = \pm a', \\ Y_{a+a'} \oplus Y_{a-a'}, & \text{if } a \neq \pm a', \end{cases} \quad \text{where } \epsilon, \epsilon' = \pm 1.$$

Proof: We can pick the following representatives for each isomorphism class of simple objects: $X_g^\pm$ is given by $(g, u^\pm)$, where $u_\varepsilon^\pm : g \to g$ is given by $u_\varepsilon^\pm = (\pm 1)^\varepsilon \mathrm{id}_g$ for every $\varepsilon \in \mathbb{Z}_2$. Similarly, for all $g \neq -g$, there is a $\mathbb{Z}_2$-equivariant object Y_g given by $(g \oplus -g, u)$, where $u_0 : g \oplus -g \to g \oplus -g$ is given by

$$\begin{pmatrix} \mathrm{id}_g & 0 \\ 0 & \mathrm{id}_{-g} \end{pmatrix},$$

while $u_1 : -g \oplus g \to g \oplus -g$ is given by

$$\begin{pmatrix} 0 & \mathrm{id}_g \\ \mathrm{id}_{-g} & 0 \end{pmatrix}.$$

To see that these objects are simple, one can easily check that their endomorphism rings are one-dimensional. For example, if $f : Y_g \to Y_g$ is a $\mathbb{Z}_2$-equivariant morphism, then $f = x\,\mathrm{id}_g \oplus y\,\mathrm{id}_{-g}$ and $f \circ u_1 = u_1 \circ T_1(f) = u_1 \circ (x\,\mathrm{id}_{-g} \oplus y\,\mathrm{id}_g) = y\,\mathrm{id}_g \oplus x\,\mathrm{id}_{-g}$. This implies $x = y$.

These simple objects are clearly pairwise non-isomorphic (except $Y_a = Y_{-a}$ as mentioned in the statement of the theorem), and the fusion rules follow from a simple calculation. To see that they form a complete set of representatives, one can compare the sum of the squares of their Frobenius-Perron dimensions with the categorical dimension of $\mathcal{C}(G,q)^{\mathbb{Z}_2}$, which must be twice that of $\mathcal{C}(G,q)$ by [13, Prop. 7.21.15].

Table 4.1 goes into more detail in the special case $G = \mathbb{Z}_r \times \mathbb{Z}_{N/r}$.

$(r, \frac{N}{r})$	$X^{\pm}_{(a,b)}$	$\left\lvert L(\mathcal{C}\left(\mathbb{Z}_r \times \mathbb{Z}_{N/r}, q\right)^{\mathbb{Z}_2}_{\mathrm{pt}})\right\rvert$	$Y_{(a,b)}$	Number of $Y_{(a,b)}$
(o,o)	$(a,b) \in \langle (0,0) \rangle$	2	$a = 1, \cdots, \frac{r-1}{2}$, $b = 1, \cdots, \frac{N/r-1}{2}$	$\frac{N-1}{2}$
(o,e)	$(a,b) \in \langle (0, \frac{N}{2r}) \rangle$	4	$a = 1, \cdots, \frac{r-1}{2}$, $b = 1, \cdots, \frac{N}{2r} - 1$	$\frac{N}{2} - 1$
(e,o)	$(a,b) \in \langle (\frac{r}{2}, 0) \rangle$	4	$a = 1, \cdots, \frac{r}{2} - 1$, $b = 1, \cdots, \frac{N/r-1}{2}$	$\frac{N}{2} - 1$
(e,e)	$(a,b) \in \langle (\frac{r}{2}, 0), (0, \frac{N}{2r}) \rangle$	8	$a = 1, \cdots, \frac{r}{2} - 1$, $b = 1, \cdots, \frac{N}{2r} - 1$	$\frac{N}{2} - 2$

Table 4.1: Simple objects for $\mathcal{C}(\mathbb{Z}_r \times \mathbb{Z}_{N/r}, q)^{\mathbb{Z}_2}$. In the first column, we use 'e' to denote 'even' and 'o' for 'odd'.

4.3.2 S- and T- matrices in a special case

We now specialize to the case that the minimal number of generators for G is at most 2. Fixing a surjective homomorphism $\mathbb{Z} \times \mathbb{Z} \to G$, we further assume the existence of a well-defined quadratic form $q : G \longrightarrow \mathbb{Z}_N$ given by

$$q(x_1, x_2) = c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_2^2 \tag{4.38}$$

for some $c_1, c_2, c_3 \in \mathbb{Z}$ and independent of the choice of representative $(x_1, x_2) \in \mathbb{Z} \times \mathbb{Z}$. We denote the associated bilinear form by λ, where $\lambda : G \times G \longrightarrow \mathbb{Z}_N$ defined by $\lambda(x, y) = \tilde{q}(x + y) - \tilde{q}(x) - \tilde{q}(y)$, where $x = (x_1, x_2), y = (y_1, y_2) \in G$. Thus λ can be expressed explicitly as

$$\lambda(x, y) = 2c_1x_1y_1 + c_2(x_1y_2 + x_2y_1) + 2c_3x_2y_2. \tag{4.39}$$

In this case, we consider the pointed premodular category $\mathcal{C}(G, q)$ where q is a quadratic form $q : G \to U(1)$ defined by $q = \exp \frac{2\pi i \tilde{q}}{N}$. Let $F : \mathcal{C}(G, q)^{\mathbb{Z}_2} \to \mathcal{C}(G, q)$ be the forgetful functor. We can equip the fusion category $\mathcal{C}(G, q)^{\mathbb{Z}_2}$ defined in the previous section with a premodular structure as follows. We define the braiding $c_{X,Y}$ in $\mathcal{C}(G, q)^{\mathbb{Z}_2}$ by $c_{X,Y} = c_{F(X),F(Y)}$. Similarly, we define θ_X for $X \in \mathcal{C}(G, q)^{\mathbb{Z}_2}$ by $\theta_X = \theta_{F(X)}$.

Combining the twists with the fusion rules described in Proposition 4.3.1, we compute the corresponding S-matrix using the balancing equation:

- $S_{X^{\pm}_{(a,b)}, X^{\pm}_{(a',b')}} = \exp\left(\dfrac{2\pi i}{N}\lambda(a, b, a', b')\right);$
- $S_{X^{\pm}_{(a,b)}, Y_{(a',b')}} = 2\exp\left(\dfrac{2\pi i}{N}\lambda(a, b, a', b')\right);$
- $S_{Y_{(a,b)}, Y_{(a',b')}} = 4\cos\left(\dfrac{2\pi}{N}\lambda\left(a, b, a', b'\right)\right).$

4.3.3 Realization of $\mathbb{Z}_2$-equivariantization

Let M be a torus bundle over $\mathbb{S}^1$ with Sol geometry, i.e., the monodromy map $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ satisfies $|a + d| > 2$. Set $N = |a + d + 2| > 0$.

Recall the Chern-Simons invariants and adjoint Reidemeister torsions in 2.5. In particular, we have

$$\mathrm{Tor}(\rho) = \begin{cases} \dfrac{|a+d+2|}{4}, & \rho \text{ is irreducible} \\ |a+d+2|, & \rho \text{ is reducible} \end{cases} \tag{4.40}$$

and

$$\mathrm{CS}(\rho) = \begin{cases} \dfrac{k\nu - l\mu}{N} & \rho \text{ is irreducible} \\ \dfrac{(a+d+2)\epsilon_x\epsilon_y + b\epsilon_x + c\epsilon_y}{4} & \rho \text{ is reducible} \end{cases} \tag{4.41}$$

We now define a map $q : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_N$ by $q(\mu, \nu) = c\nu^2 + (a-d)\mu\nu - b\mu^2$.

Lemma 4.3.2. The map q induces a quadratic form $q : G \to \mathbb{Z}_N$.

Proof: Since $\mathrm{Ker}(f) = \mathrm{Im}(g) = \left\{ \begin{pmatrix} a+1 & c \\ b & d+1 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} i, j \in \mathbb{Z} \right\}$, it suffices to show that $q(\mu+a+1, \nu+b) = q(\mu, \nu)$ and $q(\mu+c, \nu+d+1) = q(\mu, \nu)$ for general μ and ν. We have

$$\begin{aligned} q(\mu+a+1, \nu+b) - q(\mu,\nu) &= c(\nu+b)^2 + (a-d)(\mu+a+1)(\nu+b) \\ &\quad - b(\mu+a+1)^2 - q(\mu,\nu) \\ &= -b(d-a+2a+2)\mu + (2bc + (a-d)(a+1))\nu \\ &\quad - b(-bc + (d-a)(a+1) + (a+1)^2) \\ &= (-2 + 2ad - ad + a^2 + a - d)\nu \\ &\quad - b(1 - ad + ad - a^2 + d - a + a^2 + 2a + 1) \\ &= (-2 + a(-a-2) + a^2 + a + 2 + a)\nu \\ &= 0, \end{aligned}$$

and

$$
\begin{aligned}
q(\mu+c,\nu+d+1)-q(\mu,\nu) &= c(\nu+(d+1))^2+(a-d)(\mu+c)(\nu+d+1) \\
&\quad - b(\mu+c)^2-\hat{q}(\mu,\nu) \\
&= c(2(d+1)+(a-d))\nu+((a-d)(d+1)-2bc)\mu \\
&\quad + c((d+1)^2+(a-d)(d+1)-bc) \\
&= (-d^2-d+a-ad+2)\mu+c(d+1+ad+a-bc) \\
&= 0
\end{aligned}
$$

Thus q induces a well defined map $q : G \to \mathbb{Z}_N$. It is routine to check that this map is a quadratic form. □

We define the loop operators for non-Abelian characters by

$$X^{\pm}(\mu,\nu) \mapsto (x^m y^n, \mathrm{Sym}^0)$$

$$Y(\mu,\nu) \mapsto (x^m y^n, \mathrm{Sym}^1)$$

where $m = -b\mu + (a-1)\nu$, $n = (-d+1)\mu + c\nu$, and Sym^j denotes the unique $(j+1)$-dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{C})$. We choose $X^+(0,0)$ to correspond to the monoidal unit object. Each character can be represented by infinitely many representatives $(\mu,\nu) \in \mathbb{Z}\times\mathbb{Z}$, but as the following lemma shows, the S-matrix is independent of this choice.

Lemma 4.3.3. Let S^l be the S-matrix constructed from loop operators as above, then

$$S^l_{X^{\pm}(\mu_1,\nu_1),X^{\pm}(\mu_2,\nu_2)} = 1$$

$$S^l_{X^{\pm}(\mu_1,\nu_1),Y(\mu_2,\nu_2)} = 2$$

$$S^l_{Y(\mu_1,\nu_1),Y(\mu_2,\nu_2)} = 4\cos\left(\frac{2\pi}{N}\lambda\left(\mu_1,\nu_1,\mu_2,\nu_2\right)\right)$$

where $\lambda(\mu_1,\nu_1,\mu_2,\nu_2) = q(\mu_1+\mu_2,\nu_1+\nu_2) - q(\mu_1,\nu_1) - q(\mu_2,\nu_2)$ is the bilinear form associated to the quadratic form $q : G \to \mathbb{Z}_N$ defined in Lemma 4.3.2.

Proof: From Equation 4.11, we have the following W-symbols

$$W_{X^{\pm}(\mu_1,\nu_1)}(X^{\pm}(\mu_2,\nu_2)) = W_{Y(\mu_1,\nu_1)}(X^{\pm}(\mu_2,\nu_2)) = 1$$

$$W_{X^{\pm}(\mu_1,\nu_1)}(Y(\mu_2,\nu_2)) = \mathrm{Tr}(X^{\pm}(\mu_1,\nu_1)(x^{m_2}y^{n_2}))$$

$$W_{Y(\mu_1,\nu_1)}(Y(\mu_2,\nu_2)) = \mathrm{Tr}(Y(\mu_1,\nu_1)(x^{m_2}y^{n_2}))$$

Thus,

$$S^l_{X^{\pm}(\mu_1,\nu_1),X^{\pm}(\mu_2,\nu_2)} = W_{X^{\pm}(\mu_2,\nu_2)}(X^{\pm}(\mu_1,\nu_1))W_{X^{+}(0,0)}(X^{\pm}(\mu_2,\nu_2)) = 1$$

$$S^l_{X^{\pm}(\mu_1,\nu_1),Y(\mu_2,\nu_2)} = W_{Y(\mu_2,\nu_2)}(X^{\pm}(\mu_1,\nu_1))W_{X^{+}(0,0)}(Y(\mu_2,\nu_2)) = 2$$

$$S^l_{Y(\mu_1,\nu_1),Y(\mu_2,\nu_2)} = W_{Y(\mu_2,\nu_2)}(Y(\mu_1,\nu_1))W_{X^{+}(0,0)}(Y(\mu_2,\nu_2)) = 2\mathrm{Tr}(Y(\mu_2,\nu_2)(x^{m_1}y^{n_1}))$$

and

$$\begin{aligned}
\mathrm{Tr}(Y(\mu_2,\nu_2)(x^{m_1}y^{n_1})) &= 2\cos\left(2\pi\frac{k_2m_1+l_2n_1}{N}\right)\\
&= 2\cos\left(\frac{2\pi}{N}\begin{pmatrix} m_1 & n_1\end{pmatrix}\begin{pmatrix} k_2\\ l_2\end{pmatrix}\right)\\
&= 2\cos\left(\frac{2\pi}{N}\begin{pmatrix} \mu_1 & \nu_1\end{pmatrix}\begin{pmatrix} -b & -d+1\\ a-1 & c\end{pmatrix}\begin{pmatrix} d+1 & -c\\ -b & a+1\end{pmatrix}\begin{pmatrix} \mu_2\\ \nu_2\end{pmatrix}\right)\\
&= 2\cos\left(\frac{2\pi}{N}\begin{pmatrix} \mu_1 & \nu_1\end{pmatrix}\begin{pmatrix} -2b & a-d\\ a-d & 2c\end{pmatrix}\begin{pmatrix} \mu_2\\ \nu_2\end{pmatrix}\right)\\
&= 2\cos\left(\frac{2\pi}{N}\lambda(\mu_1,\nu_1,\mu_2,\nu_2)\right).
\end{aligned}$$

□

Defining $q: G \to U(1)$ by $q(x) = e^{\frac{2\pi i\hat{q}(x)}{N}}$, we have the premodular category $\mathcal{C}(G,q)$ and its $\mathbb{Z}_2$-equivariantization $\mathcal{C}(G,q)^{\mathbb{Z}_2}$ as described in the above subsection. Our main theorem is the following.

Theorem 4.3.4. The S- and T-matrices constructed from torus bundles with Sol geometry coincide with those of the $\mathbb{Z}_2$-equivariantization $\mathcal{C}(G,q)^{\mathbb{Z}_2}$.

Proof: From Equations 2.13 and 4.41, we have $\mathrm{CS}(\rho) = \frac{-c\nu+(d-a)\mu\nu+b\mu^2}{N} = -\frac{\hat{q}(\mu,\nu)}{N}$. Thus, the T-matrix of $\mathcal{C}(G,q)^{\mathbb{Z}_2}$ as defined in Section 4.3.2 coincides with the one constructed directly from the torus bundle as defined in Equation 4.4.

Let S^e denote the S-matrix from the $\mathbb{Z}_2$-equivariantization $\mathcal{C}(G,q)^{\mathbb{Z}_2}$ as defined in Section 4.3.2, and let S^l denote the S-matrix from the loop operator construction as defined in Lemma 4.3.3. We first consider the following entry:

$$S^e_{X^\pm(\mu_1,\nu_1),X^\pm(\mu_2,\nu_2)} = \frac{q(X(\mu_1+\mu_2,\nu_1+\nu_2))}{q(X(\mu_1,\nu_1))q(X(\mu_2,\nu_2))}$$

When $X(\mu_1,\nu_1) = X(\mu_2,\nu_2)$, according to the group structure of G we have $X(\mu_1+\mu_2,\nu_1+\nu_2) = X(0,0)$. Thus $S^e_{X^\pm(\mu_1,\nu_1),X^\pm(\mu_2,\nu_2)} = 1$. Similarly, if $X(\mu_i,\nu_i) = X(0,0)$ for either i, then clearly $S^e_{X^\pm(\mu_1,\nu_1),X^\pm(\mu_2,\nu_2)} = 1$.

When $X(\mu_1,\nu_1) \neq X(\mu_2,\nu_2)$ and $(\mu_i,\nu_i) \neq (0,0)$ for all i, then the characters $X(\mu_1+\mu_2,\nu_1+\nu_2)$, $X(\mu_1,\nu_1)$, and $X(\mu_2,\nu_2)$ are all distinct. Using the notation of Section **??**, these characters must correspond to the cases $(\epsilon_x,\epsilon_y) \in \{(1,0),(0,1),(1,1)\}$. As mentioned in that section, this can only occur if the parities of $(a,d;b,c)$ are $(o,o;e,e)$. Using the fact that $ad-bc = 1$, one obtains that $N = a+d+2 = 0 \pmod 4$. Thus Equation 4.41 reduces to $\mathrm{CS}(X(\mu,\nu)) = (b\epsilon_x + c\epsilon_y)/4$. By inspection, one finds that applying $q(\mu,\nu) = \exp(-2\pi i \mathrm{CS}(X(\mu,\nu)))$ to the (μ,ν) corresponding to $(\epsilon_x,\epsilon_y) \in \{(1,0),(0,1),(1,1)\}$ yields either the multiset $-1,-1,1$ or $1,1,1$. Thus $S^e_{X^\pm(\mu_1,\nu_1),X^\pm(\mu_2,\nu_2)} = 1$.

Next we consider

$$S^e_{X^\pm(\mu_1,\nu_1),Y(\mu_2,\nu_2)} = 2\frac{q(Y(\mu_1+\mu_2,\nu_1+\nu_2))}{q(X(\mu_1,\nu_1))q(Y(\mu_2,\nu_2))}.$$

Without loss of generality, we only need to consider two cases: (μ_1,ν_1) corresponding to $(k_1 = \frac{N}{2}, l_1 = 0)$ where the parity of $(a,d;b,c)$ is $(o,o;e,o)$, and (μ_1,ν_1) corresponding to $(k_1 = \frac{N}{2}, l_1 = \frac{N}{2})$ for $(o,o;e,e)$ and $(e,e;o,o)$.

When $k_1 = \frac{N}{2}$ and $l_1 = 0$,

$$\begin{aligned}
S^e_{X^\pm(\mu_1,\nu_1),Y(\mu_2,\nu_2)} &= 2\exp\left(2\pi i \frac{(k_2+\frac{N}{2})(\nu_2+\frac{b}{2}) - l_2(\mu_2+\frac{a+1}{2}) - k_2\nu_2 + l_2\mu_2 - \frac{Nb}{4}}{N}\right) \\
&= 2\exp\left(2\pi i \frac{N\nu_2 + k_2 b - l_2(a+1)}{2N}\right) \\
&= 2\exp\left(2\pi i \frac{N\nu_2 + \nu_2 N - l_2(d+1) - l_2(a+1)}{2N}\right) \\
&= 2\exp\left(2\pi i \frac{-l_2(a+d+2)}{2N}\right) \\
&= 2\exp\left(2\pi i \frac{-l_2}{2}\right)
\end{aligned}$$

Since $l_2 = -b\mu_2 + (a+1)\nu_2$ and $b, a+1$ are both even, l_2 is even. Thus $S^e_{X^\pm(\mu_1,\nu_1),Y(\mu_2,\nu_2)} = 2$.

When $k_1 = \frac{N}{2}$ and $l_1 = \frac{N}{2}$,

$$\begin{aligned}
S^e_{X^\pm(\mu_1,\nu_1),Y(\mu_2,\nu_2)} &= 2\exp\Bigg(\frac{2\pi i}{N}((k_2+\frac{N}{2})(\nu_2+\frac{b+d+1}{2}) - (l_2+\frac{N}{2})(\mu_2+\frac{a+c+1}{2}) \\
&\qquad -k_2\nu_2 + l_2\mu_2 - \frac{N(a+c+b+d+2)}{4})\Bigg) \\
&= 2\exp\left(\frac{\pi i}{N}(N(\nu_2-\mu_2) + k_2(b+d+1) - l_2(a+c+1))\right) \\
&= 2\exp\Bigg(\frac{\pi i}{N}(N(\nu_2-\mu_2) + N\nu_2 - (d+1)l_2 \\
&\qquad + k_2(d+1) - N\mu_2 + k_2(a+1) - l_2(a+1))\Bigg) \\
&= 2\exp\left(\frac{\pi i}{N}(k_2-l_2)(a+d+2)\right) \\
&= 2\exp\left(\pi i(k_2-l_2)\right)
\end{aligned}$$

Since $k_2 - l_2 = (b+d+1)\mu_2 - (a+c+1)\nu_2$ and $b+d+1$, $a+c+1$ are both even, $k_2 - l_2$ is even. Thus $S^e_{X^\pm(\mu_1,\nu_1),Y(\mu_2,\nu_2)} = 2$.

Lastly, it follows from their definitions in Lemma 4.3.3 and Section 4.3.2 that

$$S^{e}_{Y(\mu_1,\nu_1),Y(\mu_2,\nu_2)} = S^{l}_{Y(\mu_1,\nu_1),Y(\mu_2,\nu_2)}.$$

Chapter 5

Future Questions

The program is still at its infancy, and there remain many questions which we want to resolve. We hope the insights obtained will lead to an intrinsic understanding of how and why this program works.

(1) The program only provides an algorithm to compute the S- and T-matrices. Other data such as such as the F-symbols and R-symbols, which specify the associators and braidings, respectively [32], are still missing.

(2) Even for the modular data, the computation for the S-matrix follows a trial-and-error procedure. A definite algorithm to construct S-matrix is in demand.

(3) There are also a number of subtleties in choosing the correct set of characters as simple objects, determining the proper unit object, etc.

(4) Connect sum of two 3-manifolds should correspond to Deligne product of the corresponding categories which need to be verified.

(5) The current program concerns closed manifolds whose Chern-Simons invariants are all real, thus hyperbolic manifolds do not fit in with the program to some extent.

The interaction between the program and hyperbolic manifolds is still mysterious.

(6) M-theory also suggests possibility about constructing vertex operator algebra from 4-dimensional manifolds [15], and the category of modules over a vertex operator algebra is a modular tensor category [18]. We are interested in a potential relationship of the two frameworks.

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