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Two flows in non-Kähler geometry

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ABSTRACT OF THE DISSERTATION

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We consider various geometric flows which are well adapted for the study of non-Kähler complex manifolds. We first study solutions to the pluriclosed flow on compact complex surfaces, giving a complete classification of the long time behavior of homogeneous solutions to the flow and constructing non-trivial, non-Kähler expanding soliton solutions to the pluriclosed flow. We also give a simple expression for the evolution of the Lee form on a complex surface, and use this to give simplified proofs of various classification results for fixed points of the flow.

We also consider a functional which is closely related to the Dirichlet energy of maps between two Hermitian manifolds and which has holomorphic maps as global minimizers. We derive it's first and second variations and consider the associated parabolic flow. We provide conditions under which the flow converges to a critical point of the functional and give explicit examples of nice solutions to the flow in Hopf surfaces. We additionally demonstrate so called 'bubbling' criteria for solutions of the flow on surfaces and, using this functional, we give a variational proof that submanifolds of Vaisman manifolds are Vaisman.

Chapter 1

Introduction

1.1 Preamble and Statement of Results

Complex geometry has for many years been dominated by the study of Kähler manifolds, that is Hermitian manifolds where the metric and complex structure satisfy a rather strong integrability condition. The intense interest in Kähler manifolds is driven from physics by supersymmetry requirements coming from field theory, while within geometry Kähler manifolds are of independent interest as they form a class of Riemannian manifolds where many historically significant geometric problems have tractable and beautiful solutions. For example, a classical problem in geometry is to determine whether or not a manifold admits an Einstein metric. This problem has not been solved in general as there are very few known obstructions to the existence of an Einstein Riemannian metric. In contrast, when restricted to Kähler manifolds this problem has a definitive solution determined by Chern classes and notions of bundle stability coming from geometric invariant theory. Such a result is obtainable as Kähler geometry provides is the existence of Kähler potentials and torsion-free Hermitian connections, reducing many problems in this setting to studying a partial differential equation for a single function.

The motivation for this thesis is that the world of complex geometry is not entirely Kähler, and while the classification of compact complex surfaces which are Kähler is essentially completely solved through the work of Kodaira, the classification of non-Kähler surfaces is incomplete. The simplest non-Kähler surface is perhaps the Hopf manifold $S^3 \times S^1$, but other famous examples are provided by the Kodaira-Thurston manifold and the Inoue surfaces; explicit constructions of these will be given in Chapter 2. There is a long standing conjecture concerning the structure of compact complex surfaces with first Betti number $b_1 = 1$ and second Betti number $b_2 > 0$, namely

Conjecture 1.1 (Due to Kato, [11]). *A minimal compact complex surface with first Betti number $b_1 = 1$ and second Betti number $b_2 > 0$ is a small deformation of a Hopf surface blown up at $b_2 > 0$ points.*

Here the word minimal is meant in the sense that the surface has no curves with self intersection -1. Much of the work in this thesis it to study geometric analytic tools which, the author believes, can be brought to bear on this problem.

In trying to solve geometric problems in non-Kähler settings, the primary difficulty arises from the lack of Kähler potentials and torsion-free connections. Without these tools, the complexity of many calculations is magnified by the inclusion of torsion terms, terms which in local Riemannian normal coordinates involve derivatives of the complex structure. Nevertheless, careful consideration of such terms can give good estimates and classification results.

The flow approach to solving any geometric problem is to start with one geometric structure and evolve it over time toward one which, hopefully, better solves the problem. In Riemannian geometry, the Ricci flow $g_t = -2Rc(g)$ is one such flow and is an essential tool for obtaining classification results for Riemannian manifolds satisfying a variety of curvature conditions. The issue with using the Ricci flow in Hermitian geometry is that the Ricci tensor is defined only using the Levi-Civita connection, and so the Ricci flow is ignorant of any background complex geometry. In particular, a metric which is initially Hermitian need not remain Hermitian along the flow, see for example the Ricci flow on a Hopf surface with a product metric.

For this reason, it is natural to consider flows which are constructed out of curvature tensors associated to Hermitian compatible connections. Streets and Tian have studied a family of such flows, and Chapter 2 of this thesis studies solutions to a particularly important member of this family, the pluriclosed flow. Specifically, in Chapter 2 the complete classification of locally homogeneous solutions to pluriclosed flow on compact complex surfaces is given. Particularly notable results here are that Hopf surface solutions converge to metrics which are unique up to homothety and Inoue surface solutions converge in the Gromov-Hausdorff sense to circles whose length depends on the complex structure. These results are obtained by first listing all the possible locally homogeneous compact complex surfaces and providing local moving frames for them which simplify the necessary curvature calculations. The qualitative long-time behavior of the resulting ODE is then determined and an understanding of the fundamental group of these spaces gives the Gromov-Hausdorff limits.

A solution to a given geometric flow will often not exist for all time and even if it does it will become singular in some limit. Blowing up the singularity, i.e. rescaling and re-centering near the singular points, often gives specialized blowup or soliton solutions to the flow. This is the subject of the latter half of Chapter 2, where non-trivial expanding soliton solutions to pluriclosed flow are constructed by taking blowdown limits of the homogeneous solutions. This is done because a good knowledge of soliton solutions to any flow is but one step toward using the flow to obtain classification results.

We also provide a very simple evolution equation for the torsion Lee form θ along the pluriclosed flow on a surface, and use it to give an elementary and self contained proof that non-Kähler fixed points of the flow on complex surfaces are Hopf surfaces, bypassing the current known proof which requires an understanding of low dimension Einstein-Weyl structures.

Another commonly studied flow is the harmonic map heat flow $f_t = \tau(f)$ for maps $f : M \to N$ between Riemannian manifolds. This flow arises from gradient descent applied to the Dirichlet energy functional $E = \frac{1}{2}\int_M |Df|^2 dV$. The subject of harmonic maps and the harmonic map heat flow is at this point classical, with many famous results concerning their existence, classification, and singular structure. For example, returning to what can be done with Kähler geometry there is Siu's [21] strong rigidity result for Kähler manifolds of strongly negative curvature. This is a celebrated theorem which exploits the work of Eells and Sampson on the harmonic map heat flow. Similarly, Siu-Yau's proof of the Frankel conjecture [22] uses Kahler geometry to give curvature conditions under which a harmonic S^2 is holomorphic.

Given that much information can be inferred from a complex manifold by studying it's complex submanifolds, it is natural to study functionals of maps $f : M \to N$ between Hermitian manifolds which are small when f is holomorphic. If J_M and J_N are the corresponding complex structures, the functional $E_+ = \frac{1}{4}\int_M |Df + J_N \circ Df \circ J_M|^2 dV$ exactly serves this purpose, and it's critical points are called $\bar{\partial}$-harmonic maps. Chapter 3 is devoted to studying this functional, it's critical points, and it's associated parabolic flow for maps between almost Hermitian manifolds. We start by computing the first and second variations of this functional, noting that the Euler-Lagrange equation differs from the harmonic map equation by two terms, both of which are first order in derivatives of f with one linear and the other quadratic. As holomorphic maps are automatically stable critical points of this functional, we derive a positivity result for a certain elliptic operator on vector fields which holds on any almost-Hermitian manifold. We then apply this positivity result to derive an eigenvalue bound for almost-Kähler manifolds with positive Ricci curvature.

We then study the parabolic flow associated to E_+, giving conditions for the long time existence and convergence of the flow. In particular we show the long time existence of the flow for any almost-Kähler target manifold N with strictly negative sectional curvature. The central difficulty in obtaining compactness or convergence results for E_+ is that it is non-coercive, and so the Dirichlet energy E may blow up along a sequence which is minimizing E_+. As an attempt to circumvent this, we give a family of coercive functionals which contain E_+ as a limiting case, and demonstrate convergence for every member of this family. We also give some additional conditions where the energy E is finite along a solution the flow.

A solution to a geometric flow which is becoming singular is said to develop bubbles if near the singularity the solution is 'pinching off' a non-trivial solution, obtained by localizing and rescaling around the singular point. Bubbling phenomena are often encountered when studying conformally invariant functionals like the Dirichlet energy or Yang-Mills functional. Since the functional E_+ is conformally invariant if the source is a complex curve, bubbling phenomena does indeed occur and we show that on Riemann surfaces if a finite time singularity does occur then a non-trivial $\bar{\partial}$-harmonic S^2 bubbles off.

We end the discussion of the functional E_+ by applying the variational structure of this functional to giving an elementary proof that an immersed complex curve in a Vaisman manifold is a torus.

1.2 Notation and Hermitian Manifolds

In what follows if some geometric structure is fixed along a flow, we will assume that structure is smooth; we will not focus on the weakest assumptions under which the results of this thesis hold, but someone who is analytically minded will notice that we rarely take more than two derivatives of anything. We begin by establishing some notation that is common in Hermitian geometry but for which every author follows slightly different conventions. A standard reference for this material is the book [12].

Definition 1.1. *An almost-complex manifold we mean a smooth manifold M together with a smooth section J, the almost-complex structure, of the bundle $T^*M \otimes TM$ such that $J^2 = -Id$. If M is provided with a Riemannian metric, g, we say that (M, g, J) is almost-Hermitian if $g(JX, JY) = g(X, Y)$ for all tangent vectors $X, Y \in TM$. If (M, g, J) is an almost-Hermitian manifold it's fundamental 2-form ω, also known as it's Kähler form, is $\omega(X, Y) = g(JX, Y)$. If $\alpha \in T^*M$ is a one-form the action of J on α will be given by $J\alpha = -\alpha \circ J$. If J restricts to an immersed submanifold N then we say that N is a complex submanifold of M.*

We note that Hermitian metrics are readily available for any almost-Complex structure. If we view the metric and Kähler form as providing isomorphisms $g, \omega : TM \to T^*M, g(X)(Y) = g(X, Y)$, then in this notation we would have $\omega = gJ$ and the Hermitian condition on g is equivalent to saying that $gJ = Jg$. Next, the complexified bundle of one-forms $\mathbb{C} \otimes T^*M$ admits a decomposition according to the $\pm\sqrt{-1}$-eigenspaces of J. Note that we will often use the same notation for T^*M and it's complexification.

Definition 1.2. *Elements of the $-\sqrt{-1}$-eigenspace, $\Lambda^{1,0}$, of the action of J on T^*M are called* $1,0$*-forms. Elements of the $\sqrt{-1}$-eigenspace, $\Lambda^{0,1}$, of the action of J on T^*M are called* $0,1$*-forms. This decomposition extends to k-forms via*

$$\Lambda^k T^*M = \sum_{i=0}^{k} \Lambda^i(\Lambda^{1,0}) \wedge \Lambda^{k-i}(\Lambda^{0,1}) = \sum_{i=0}^{k} \Lambda^{i,j}.$$

The previous conventions laid out, we demonstrate with a simple example why this notation is convenient. In $\mathbb{C} = \mathbb{R}^2$ let ∂_x and ∂_y denote two tangent vectors with $J\partial_x = \partial_y$. With dx and dy the dual forms, in this notation we have $Jdx = -dx \circ J = dy$. Therefore, with $dz = dx + \sqrt{-1}dy$, we have $Jdz = dy - \sqrt{-1}dx = -\sqrt{-1}dz$, and so dz spans the $(1, 0)$-forms while $d\bar{z}$ spans the $(0, 1)$-forms.

Definition 1.3. *The operators ∂ and $\bar{\partial}$ are defined as follows. Let $\pi^{i,j}$ denote the projection of k-forms, $k = i + j$, onto $\Lambda^{i,j}$. Then, viewing d as mapping smooth sections of Λ^k to Λ^{k+1}, let $\partial = \pi^{i+1,j} \circ d$ and $\bar{\partial} = \pi^{i,j+1} \circ d$. We say the complex structure J is integrable if $\bar{\partial}^2 = 0$ and in this case (M, J) is said to be a complex manifold. Complex manifolds of real dimension two are called complex curves, while complex manifolds of real dimension four are called complex surfaces.*

If we decompose the exterior derivative $d = \partial + \bar{\partial} + N$ into the three pieces corresponding to the above definition, then the action of N on differential forms is tensorial, i.e. $N(f\alpha) = fN(\alpha)$. N is called the Nijenhuis tensor and can be expressed in terms of first order derivatives of J at a point. The integrability of J is equivalent to $N = 0$ and unless otherwise noted we will assume the complex structure is integrable.

On a complex manifold we would have $d = \bar{\partial} + \partial$, motivating us to naively think that the exterior derivative d is two times the real part of ∂. The following definition then concerns the imaginary part.

Definition 1.4. *Let (M, J) be a complex manifold. The operator d^c, mapping smooth k-forms to smooth $k+1$-forms, is given by*

$$d^c = \sqrt{-1}(\bar{\partial} - \partial).$$

Note that d^c is a real operator, meaning it maps real k-forms to real $k+1$-forms. This is made precise in the following lemma, where d^c can be expressed as a composition of other real operators. Also we caution the reader that elsewhere in the literature the notation for d^c often differs from the one here by a sign or a factor of 2.

Lemma 1.1. *Let J act on differential k-forms in the obvious way, i.e. $J(\alpha \wedge \beta \wedge \ldots) = J\alpha \wedge J\beta \wedge \ldots$. Then $d^c = JdJ^{-1}$.*

It is also important to note the following lemma relating the composition of d and d^c.

Lemma 1.2. *Let (M, J) be a complex manifold. Then $\partial\bar{\partial} = -\bar{\partial}\partial$ and $-d^c d = dd^c = 2\sqrt{-1}\partial\bar{\partial}$.*

1.2.1 Metric Integrability Conditions

In this section ∇ will denote the Levi-Civita connection of an almost-Hermitian manifold. In Hermitian geometry, a metric integrability condition is a vanishing assumption about one or more covariant derivatives of the complex structure, or combinations thereof. The most basic first order condition one could then impose, $\nabla J = 0$, is the strongest.

Definition 1.5. *An almost-Hermitian manifold (M, g, J) is said to be Kähler if $\nabla J = 0$.*

Note that a Kähler manifold is automatically a complex manifold as the Nijenhuis tensor depends on first derivatives of J and would therefore vanish at a point in Riemannian normal coordinates. In addition to this, a complex manifold for which $d\omega = 0$ satisfies $\nabla J = 0$, as ∇J can be expressed in terms of N and $d\omega$, see [12] Chapter 3. The following lemma provides many examples of Kähler manifolds.

Lemma 1.3. *Any complex submanifold of a Kähler manifold is a Kähler manifold. In particular any complex submanifold of $\mathbb{C}P^n$, and so every complex variety, is a Kähler manifold.*

Through Hodge theory considerations, every compact Kähler manifold has the property that it's odd Betti numbers are even. Using this, one can check if a compact Hermitian manifold admits any Kähler metrics. For example it is easy to see from this that the Hopf surface $S^3 \times S^1$ admits no Kähler metrics at all but has a natural complex structure coming from the quotient $S^3 \times S^1 \simeq (\mathbb{C}^2 \setminus \{0\})/\Gamma$ by the action $(z, w) \mapsto (2z, 2w)$.

There are numerous other, less restrictive integrability conditions which are encountered in the literature. We will list the more commonly found ones, together with examples of such a manifold in each case.

Definition 1.6. *An almost-Hermitian manifold* (M^{2n}, g, J) *is...*

1. *Almost-Kähler if* $d\omega = 0$ *but* J *is not integrable. Gromov [7] has shown that any symplectic manifold has a compatible almost-Hermitian metric, and so symplectic manifolds provide a large class of examples in this case.*

2. *Nearly-Kähler if the component* $d\omega^{2,1+1,2} = 0$ *vanishes.* S^6 *with it's standard complex structure is one example.*

3. *Balanced if* $d^*\omega = 0$*, equivalently* $d\omega^{n-1} = 0$*. Twistor spaces often have balanced metrics but no Kähler metrics, for constructions, see for example [10].*

4. *Gauduchon if* $dd^c\omega^{n-1} = 0$ *and* J *is integrable. By a result of Gauduchon [5], every compact Hermitian manifold is conformal to a Gauduchon metric.*

5. *Pluriclosed if* $dd^c\omega = 0$*. By Gauduchon's result, any compact complex surface is pluriclosed. Chapter 2 is devoted to the study of a flow of pluriclosed metrics and we construct many examples in this case.*

There is an additional integrability condition that is of interest for conformal geometry and the classification of compact complex surfaces.

Definition 1.7. *A Hermitian manifold* (M, g, J) *is locally conformally Kähler if* $d\omega = \theta \wedge \omega$ *for a closed one-form* θ *called the Lee form. We say that* M *is Vaisman if the stronger condition* $\nabla\theta = 0$*,* $\theta \neq 0$ *holds.*

In the previous definition if we have a locally defined function for which $dq = \theta$, then the metric $\tilde{\omega} = e^{-q}\omega$ is Kähler, motivating the name for this condition. Locally conformally

Kähler manifolds are common among non-Kähler manifolds. For example, the Hopf surfaces $S^3 \times S^1$ admit Vaisman metrics and any blow up of a locally conformally Kähler manifold is locally conformally Kähler. A good reference for this an related results is the book [2]. In particular, if Kato's conjecture on surfaces is true, then all compact complex manifolds with $b_1 = 1$ and $b_2 > 0$ have locally conformally Kähler metrics. A good reference for these and other results is the survey article [19]. For later purposes it is important to note the following proposition, see [2] Chapter 1.

Proposition 1.1. *Let (M, g, J) be a Vaisman manifold with Lee form θ and let $\eta = J\theta$. Let $X = \theta^\sharp$ be the dual Lee vector field and $Y = JX$. Then X is holomorphic and Killing.*

Proof. From the Koszul formula, for any vector fields U, V, W

$$2g(\nabla_U V, W) = (L_V g)(U, W) + dV^\flat(U, W),$$

applying this to $V = X$ and using the Vaisman condition gives $L_X g = 0$. Now on a Hermitian manifold where $d\omega = \theta \wedge \omega$ the covariant derivative of J is

$$2(\nabla_U J)V = -\eta(V)U - \theta(V)JU + g(U, V)Y + \omega(U, V)X$$

and therefore

$$\begin{aligned} 2(\nabla_X J)V &= -\eta(V)X - \theta(V)Y + g(X, V)Y + \omega(X, V)X \\ &= -\omega(X, V)X - g(X, V)Y + g(X, V)Y + \omega(X, V)X \\ &= 0. \end{aligned}$$

Since $\nabla X = 0$, we have $[X, U] = L_X U = \nabla_X U$, so

$$(L_X J)U = [X, JU] - J[X, U] = \nabla_X(JU) - J\nabla_X U = (\nabla_X J)U = 0.$$

$\square$

Vaisman metrics are therefore quite special as they split isometrically. Moreover, the geometry in directions perpendicular to the Lee vector field is Sasakian [2].

1.2.2 Commutator Identities in non-Kähler Geometry

Any Hermitian manifold (M^{2n}, g, J) comes equipped with a map on the algebra of forms $L : \Lambda^* \to \Lambda^*$, given by

$$L\alpha = \omega \wedge \alpha,$$

which is nothing more than wedging a k-form with ω. It's adjoint L^* is of course defined by $\langle X, L^*Y \rangle = \langle LX, Y \rangle$, where we use the canonical inner product on forms where, for $i < j < k < \ldots$ and so on, $\sigma^i \wedge \sigma^j \wedge \sigma^k \wedge \ldots$ is an orthonormal basis whenever the σ^i form an orthonormal basis of one-forms. In particular note that $L^*\beta = \langle \beta, \omega \rangle$ for any $\beta \in \Lambda^2$. The following theorem relates the commutator of the operators L, d, and d^c to their adjoints and the form $d\omega$.

Theorem 1.1 ([1], page 306)**.** *Let (M^{2n}, g, J) be a Hermitian manifold and let L, L^* be given as above. Let $\kappa = [L^*, \partial\omega]$ be the zero-th order operator on the algebra of forms given by the commutator of L^* with the operation of taking a (left) wedge product of a form with $\bar{\partial}\omega$ and note that $\kappa : \Lambda^{i,j} \to \Lambda^{i+1,j}$. Then*

1. $[\bar{\partial}^*, L] = \sqrt{-1}(\partial + \kappa)$
2. $[\partial^*, L] = -\sqrt{-1}(\bar{\partial} + \bar{\kappa})$
3. $[L^*, \bar{\partial}] = -\sqrt{-1}(\partial^* + \kappa^*)$

4. $[L^*, \partial] = \sqrt{-1}(\bar{\partial}^* + \bar{\kappa}^*)$

By adding together combinations of these identities we obtain similar statements for the real operators d and d^c.

Corollary 1.1. *Let $\mu = \kappa + \bar{\kappa} = [L^*, d\omega]$ and $\mu^c = [L^*, d^c\omega]$. Then*

1. $[d^*, L] = -(d^c + \mu^c)$
2. $[L^*, d^c] = d^* + \mu^*$

1.2.3 Simplifications for the Torsion on Almost-Complex Surfaces

If (M^4, g, J) is an almost-Hermitian surface then it is easy to see that the map $L : \Lambda^1 \to \Lambda^3$ is an isometry. In this case, one always has $d\omega = \theta \wedge \omega$ for a uniquely defined one-form θ which is called the Lee form of ω. We record a few fundamental lemmas which will be used in a few places throughout the thesis.

Lemma 1.4. *Let $* : \Lambda^k \to \Lambda^{4-k}$ denote the Hodge star operator induced by g and the orientation determined by J.*

- *If α is a one-form, then $*\alpha = J\alpha \wedge \omega$.*
- *If β is a two-form, then $*\beta = \langle \beta, \omega \rangle \omega - J\beta$.*

Lemma 1.5. *Let (M^4, g, J) be an almost-Hermitian surface. If β is a two-form then $L\beta = \beta \wedge \omega = \langle \omega, \beta \rangle dV$. In particular, since $d\theta \wedge \omega = d^2\omega = 0$ we conclude $\langle d\theta, \omega \rangle = 0$ and $*d\theta = -Jd\theta$.*

Proof. Since these are tensorial identities we need only pick some frame at a point and compute these to see if they are equal. Any oriented orthonormal basis σ^i of T^*M such that

$J\sigma^i = \sigma^{i+1}$ for i odd will work. Then, for example $*\sigma^1 = \sigma^{234} = J\sigma^1 \wedge (\sigma^{12} + \sigma^{34}) = J\sigma^1 \wedge \omega$, where $\sigma^{ijk\dots} = \sigma^i \wedge \sigma^j \wedge \sigma^k \wedge \dots$. □

As mentioned before, covariant derivatives of the complex structure can be compactly expressed in terms of the Lee form [12]

Lemma 1.6. *Let (M^4, g, J) be a complex surface. Let θ be its Lee form and $\eta = J\theta$, let X denote the Lee vector field and $Y = JX$, and finally let ∇ denote the Levi-Civita connection of g.*

$$2(\nabla_U J)V = -\eta(V)U - \theta(V)JU + g(U, V)Y + \omega(U, V)X.$$

Integrability conditions of Hermitian metrics on complex surfaces can then be simplified by expressing them in terms of the Lee form. By unwinding the results of the previous subsections we get the following.

Lemma 1.7. *A complex surface (M^4, g, J) is almost-Kähler if, and only if, $\theta = 0$. It is Gauduchon or pluriclosed if, and only if, $d^*\theta = 0$.*

Chapter 2

Pluriclosed Flow

2.1 Flows of Hermitian Metrics

Let (M^{2n}, g, J) be a Hermitian manifold with complex structure J and compatible Riemannian metric g. The fundamental or Kähler 2-form associated to the metric is $\omega = g(J\cdot, \cdot)$ and we remind the reader that the metric is *pluriclosed* if

$$\partial\overline{\partial}\omega = 0.$$

In [23, 24], Streets and Tian introduce a parabolic flow of Hermitian metrics which, when the metric is pluriclosed, is equivalent to the following

$$\frac{\partial \omega}{\partial t} = \partial\partial^*\omega + \overline{\partial\partial^*}\omega + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\det g. \tag{2.1}$$

In this chapter we present the long time behavior of (2.1) on all compact Hermitian surfaces which are locally homogeneous. These include hyperelliptic, Hopf, Inoue, Kodaira, and non-Kähler properly elliptic surfaces. We will also construct expanding soliton solutions of the flow on the universal covers of these spaces. Our theorem to this end will be:

Theorem 2.1. *Let $g(\cdot)$ be a locally homogeneous solution of pluriclosed flow on a compact complex surface which exists on a maximal time interval $[0, T)$. If $T < \infty$ then the complex surface is rational or ruled. If $T = \infty$ and the manifold is a Hopf surface, the evolving metric converges exponentially fast to a canonical form unique up to homothety. Otherwise, there is a blowdown limit*

$$\tilde{g}_\infty(t) = \lim_{s\to\infty} s^{-1}\tilde{g}(st)$$

of the induced metric on the universal cover which is an expanding soliton in the sense that $\tilde{g}(t) = t\tilde{g}(1)$ up to automorphism.

This is analogous to the construction of expanding Ricci solitons in [16]. We also compute the time-rescaled Gromov-Hausdorff limits of our solutions and observe collapse to points, circles, and curves of high genus.

Theorem 2.2. *Let $g(\cdot)$ be a locally homogeneous solution of pluriclosed flow on a compact complex surface (M, J) which exists on the interval $[0, \infty)$ and suppose that (M, J) is not a Hopf surface. Let $\hat{g}(t) = \frac{g(t)}{t}$.*

1. If the surface is a torus, hyperelliptic, or Kodaira surface, then the family $(M, \hat{g}(t))$ converges as $t \to \infty$ to a point in the Gromov-Hausdorff sense.

2. If the surface is an Inoue surface, then the family $(M, \hat{g}(t))$ converges as $t \to \infty$ to a circle in the Gromov-Hausdorff sense and moreover the length of this circle depends only on the complex structure of the surface.

3. If the surface is a properly elliptic surface where the genus of the base curve is at least 2, then the family $(M, \hat{g}(t))$ converges as $t \to \infty$ to the base curve with a metric of constant curvature.

4. If the surface is of general type, then the family $(M, \hat{g}(t))$ converges as $t \to \infty$ to a product *of Kahler-Einstein metrics on M.*

Remark: We note that homogeneous solutions of pluriclosed flow on Inoue and non-Kähler properly elliptic surfaces have similar asymptotics and Gromov-Hausdorff limits as the example solutions to Chern-Ricci flow on these surfaces considered in [27], and our arguments for the Gromov-Hausdorff limits in these cases are the same as in [27].

The organization of the chapter is as follows. In Section 2 we will provide background discussion and calculations for the homogeneous Hermitian geometries considered throughout the chapter. In Section 3 we will analyze the long time behavior of the flow for each of the geometries in Section 2 and prove Theorem 2.2. In Section 4 we complete the proof of Theorem 2.1 by performing blowdown limits on the solutions of Section 3. We then study the evolution of the Lee form under pluriclosed flow on surfaces, giving a simple evolution equation for it and giving an elementary proof that compact fixed points of the flow are locally conformally Kähler. Additionally, we construct a large class of non-compact and non-complete fixed points by conformally modifying Ricci flat Kähler manifolds. We conclude the chapter with a discussion of future work which would build off the results of this chapter.

2.2 Homogeneous and non-Kähler Hermitian Geometry

2.2.1 Hermitian Connections and Ricci Curvature Forms

Let (M^{2n}, g, J) be a Hermitian manifold with fundamental 2-form ω. With a condition on the torsion, Gauduchon [6] has shown that there is a canonical 1-dimensional family of Hermitian connections ∇^τ on M. When $\tau = 1$ we obtain the Chern connection $\nabla^c = \nabla^1$. It is defined by

$$g(\nabla^c_X Y, Z) = g(\nabla^g_X Y, Z) - \frac{1}{2} d\omega(JX, Y, Z)$$

where ∇^g is the Levi-Civita connection of g. In local holomorphic coordinates the Chern connection ∇^c has coefficients

$$\Gamma^k_{ij} = g^{\bar{l}k} g_{j\bar{l},i}$$

where

$$g_{j\bar{l},i} = \frac{\partial}{\partial z^i} g_{j\bar{l}}.$$

If $R^c(X,Y)Z = ([\nabla^c_X, \nabla^c_Y] - \nabla^c_{[X,Y]})Z$ is the $(3,1)$ curvature tensor of the Chern connection, its Ricci form is defined by

$$\rho^c_\omega = \frac{1}{2} \sum_i R^c(X, Y, Je_i, e_i).$$

Here e_i is an orthonormal basis of the (real) tangent space. In local holomorphic coordinates the Ricci form of the Chern connection is given by

$$\rho^c_\omega = -\frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \det g.$$

In a more invariant form, if ω and ω_0 are two Hermitian metrics, their Chern-Ricci forms are related by

$$\rho^c_\omega = \rho^c_{\omega_0} - \sqrt{-1} \partial\bar{\partial} \log \frac{\omega^n}{\omega_0^n},$$

where $\omega^n = \omega \wedge \omega \wedge \ldots \wedge \omega$. This formula is useful for computing with invariant metrics. For example, it is immediate that $\rho^c_\omega = \rho^c_{\omega_0}$ whenever ω and ω_0 have proportionally constant volume forms, so the Chern-Ricci form of a left invariant metric is independent of the metric used to compute it.

The Bismut connection is defined by

$$g(\nabla^b_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2} d\omega(JX, JY, JZ)$$

and is the connection corresponding to $\tau = -1$ in the family of Gauduchon. As with the Chern connection, there is an associated Bismut-Ricci form

$$\rho^b_\omega = \frac{1}{2} \sum_i R^b(X, Y, Je_i, e_i).$$

The Bismut-Ricci and Chern-Ricci forms are related by

$$\rho^b_\omega = \rho^c_\omega - dd^*\omega$$

and so we obtain

Lemma 2.1. *Pluriclosed flow is given by*

$$\frac{d\omega}{dt} = \partial\partial^*\omega + \overline{\partial\partial}^*\omega + \frac{\sqrt{-1}}{2}\partial\overline{\partial} \log\det g = -(\rho^b_\omega)^{(1,1)}.$$

2.2.2 Homogeneous Hermitian Geometries

In this section we will give a short summary of homogeneous Hermitian geometry in complex dimension two. We will begin with a few notions from homogeneous Riemannian geometry.

Definition 2.1. *A Riemannian manifold (M^n, g) is* locally homogeneous *if for any two* *points $x, y \in M$ there exists an isometry $\theta : U \to V$ between open neighborhoods $x \in U$ and $y \in V$ such that $\theta(x) = y$. (M, g) is* globally homogeneous *if the locally defined isometries θ are defined on all of M.*

Given that the universal cover of any complete, locally homogeneous Riemannian manifold is globally homogeneous, in order to classify complete, locally homogeneous Riemannian manifolds it is enough to classify the simply connected, complete, homogeneous Riemannian manifolds together with their co-compact lattices. With a globally homogeneous Riemannian manifold the isometry group $Iso(M, g)$ acts transitively on M. Then the isotropy groups

$$G_x = \{\theta \in Iso(M, g) \mid \theta(x) = x\}$$

are all conjugate, isomorphic to closed subgroups of $O(n)$, and M is diffeomorphic to the quotient G/G_x. A first step toward the classification of such spaces, then, is to find closed subgroups of $O(n)$ of dimension k and embeddings of these subgroups into unimodular Lie groups of dimension $n + k$. That the same manifold M can arise from different quotients in this way is addressed with the following definition.

Definition 2.2. *A (minimal)* model geometry *is*

1. A complete, simply connected, homogeneous Riemannian manifold (M, g) where the metric g is the pullback of a metric on some compact manifold whose universal cover is M, together with

2. A closed subgroup G of $Iso(M, g)$ acting transitively on M such that G is minimal with this property.

A list of the four dimensional model geometries can be found in [9] in the context of homogeneous Ricci flows and is summarized below. Note that we have omitted the groups G.

Proposition 2.1. *The four-dimensional model geometries are either*

1. Products of two-dimensional model geometries $S^2, \mathbb{R}^2$ *and* H^2*,*

2. $S^4, \mathbb{C}P^2, H^3 \times \mathbb{R}, \mathbb{C}H^2, H^4$*, or*

3. A simply connected 4-dimensional Lie group which has a co-compact lattice.

The metrics in the first and second cases are products of canonical, well known Einstein metrics, while the metrics on each of the Lie groups are left invariant.

Here H^n is hyperbolic space of dimension n, $\mathbb{C}P^n$ is complex projective space with a Fubini-Study metric, and $\mathbb{C}H^n$ is complex hyperbolic space: an open ball in $\mathbb{C}^n$ with the Bergman metric.

Definition 2.3. *A complex structure* J *is* compatible *with a model geometry* (M, g, G) *if* G *acts by holomorphic isometries. If* J *is compatible we say that* (M, g, J) *is* homogeneous Hermitian*.*

Wall [30] classified the (integrable) complex structures up to isomorphism and conjugation which are compatible with a given 4-dimensional model geometry.

Proposition 2.2. *The 4-dimensional model geometries which admit a compatible integrable complex structure are either*

1. Products of the two-dimensional model geometries $S^2 = \mathbb{C}P^1$*,* $\mathbb{R}^2 = \mathbb{C}$*, and* $H^2 = \mathbb{C}H^1$*, with canonical product complex structures,*

2. $\mathbb{C}P^2$ *and* $\mathbb{C}H^2$*, with canonical complex structures, or*

3. A 4-dimensional Lie group with a left invariant integrable complex structure

Proof. We will give a brief sketch; for details see [30]. Let J be a compatible integrable complex structure on a four-dimensional model geometry (M, g, G). First, note that J must commute with the action of the isotropy group on the tangent space at each point. For the model geometries S^4, $H^3 \times \mathbb{R}$, and H^4 where the isotropy group contains a copy of $SO(3)$, there can be no such commuting J. The remaining cases are either Lie groups, $\mathbb{C}P^2$, $\mathbb{C}H^2$, or products of two-dimensional models. The standard complex structures on $\mathbb{C}P^2$, $\mathbb{C}H^2$, and the product cases are compatible with the model geometry structure and are the unique compatible complex structures up to automorphism and conjugation. The Lie group cases are treated as follows.

A compatible complex structure on a Lie group G is determined by its action on the Lie algebra $\mathfrak{g}$ of left invariant vector fields. Note that J is integrable if, and only if, the $\sqrt{-1}$-eigenspace of the complexification $J^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ is a Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let e_i be a basis of $\mathfrak{g}$ and let c_{ij}^k be the structure constants of $\mathfrak{g}$ with respect to such a basis, so that

$$[e_i, e_j] = c_{ij}^k e_k.$$

Because there are only two 2-dimensional complex Lie algebras up to isomorphism, if J is integrable there is a basis Z_1, Z_2 of the $\sqrt{-1}$-eigenspace of $J^{\mathbb{C}}$ where either $[Z_1, Z_2] = 0$ or $[Z_1, Z_2] = Z_2$. Writing $Z_i = a_i^j e_j$ for some $a_i^j \in \mathbb{C}$, the integrability condition becomes

$$a_1^i a_2^j c_{ij}^k = \epsilon a_2^k \tag{2.2}$$

where $\epsilon = 0$ or 1 depending on whether or not the $\sqrt{-1}$-eigenspace is abelian. Once a solution a_i^j to the above equations is known, if we let

$$X_1 = (\Re a_1^j) e_j, X_2 = -(\Im a_1^j) e_j, X_3 = (\Re a_2^j) e_j, X_4 = -(\Im a_2^j) e_j,$$

then an integrable complex structure is given by $JX_1 = X_2$ and $JX_3 = X_4$. As there are possibly many solutions to the above system, automorphisms of $\mathfrak{g}$ substantially reduce the number of cases that one needs to consider. Doing this on a case by case basis for each of the Lie algebras in Proposition (2.1) completes the proof and gives the list of complex structures. $\square$

We now list the complex structures and corresponding groups of the previous proposition. In each case we will use a left invariant basis Z_i of $T^{1,0}M$ and give the Lie brackets with respect to this basis; the complex structure is $J^{\mathbb{C}} Z_i = \sqrt{-1} Z_i$. When there is more than one compatible complex structure we list a family of Lie brackets with respect to such a basis.

1. On $\mathbb{R}^4$ there is a unique compatible complex structure. Quotients by a cocompact lattice in this case give complex tori.

2. On $\tilde{E}(2) \times \mathbb{R}$ there is a unique compatible complex structure. Here $\tilde{E}(2)$ is the universal covering group of the rigid motions of the Euclidean plane. The non-vanishing Lie brackets with respect to a $T^{1,0}$ frame are

   $$[Z_1, Z_2] = Z_1 \qquad [Z_1, \overline{Z_2}] = -Z_1$$

   and compact quotients in this case are hyperelliptic surfaces.

3. On $\mathbb{R} \times SU_2$ there is a one parameter family of compatible complex structures (parameterized by $\alpha \in \mathbb{R}$) with brackets

   $$[Z_1, Z_2] = Z_2 \qquad [Z_1, \overline{Z_2}] = -\overline{Z_2}$$

   $$[Z_2, \overline{Z_2}] = (\sqrt{-1}\alpha - 1) Z_1 + (\sqrt{-1}\alpha + 1) \overline{Z_1}.$$

   Here the compact quotients give Hopf surfaces.

4. On $\tilde{SL}_2(\mathbb{R}) \times \mathbb{R}$ there is a one parameter family of compatible complex structures with brackets

   $$[Z_1, Z_2] = \sqrt{-1} Z_1 \qquad [Z_1, \overline{Z_2}] = \sqrt{-1} Z_1$$

   $$[Z_1, \overline{Z_1}] = (\sqrt{-1} - \alpha) Z_2 + (\sqrt{-1} + \alpha) \overline{Z_2}.$$

Quotients by a cocompact lattice in this case give non-Kähler, properly elliptic surfaces.

5. On $Nil^3 \times \mathbb{R}$ there is a unique compatible complex structure whose brackets are

$$[Z_1, \overline{Z_1}] = \sqrt{-1}(Z_2 + \overline{Z_2}).$$

The quotients in this case form Kodaira surfaces.

6. There is a semi-direct product $Nil^3 \rtimes \mathbb{R}$ with two compatible complex structures given by

$$[Z_1, Z_2] = \epsilon Z_1 \qquad [Z_1, \overline{Z_2}] = -\epsilon Z_1,$$

$$[Z_1, \overline{Z_1}] = -\epsilon\sqrt{-1}(Z_2 + \overline{Z_2}).$$

where $\epsilon = \pm 1$. Compact quotients in this case are Kodaira surfaces.

7. There is a family of solvable Lie groups with complex structures

$$[Z_1, Z_2] = \lambda Z_1 \qquad [Z_1, \overline{Z_2}] = -\lambda Z_1,$$

$$[Z_2, \overline{Z_2}] = 2a\sqrt{-1}(Z_2 + \overline{Z_2}),$$

where $\lambda = -b + \sqrt{-1}a$ is a complex number. These do not always have a cocompact lattice; when such a lattice exists the quotient forms an Inoue surface.

8. The solvable Lie group Sol_1^4 has two compatible complex structures. The first is given by

$$[Z_1, Z_2] = -Z_2 \qquad [Z_1, \overline{Z_2}] = -Z_2,$$

$$[Z_1, \overline{Z_1}] = \overline{Z_1} - Z_1,$$

and we will call Sol_1^4 with this complex structure Sol_1. The second is given by

$$[Z_1, Z_2] = -Z_2 \qquad [Z_1, \overline{Z_2}] = -Z_2,$$

$$[Z_1, \overline{Z_1}] = \overline{Z_1} - Z_1 + Z_2 - \overline{Z_2},$$

which we will call Sol_1'. The quotients in each these cases form Inoue surfaces.

2.2.3 The Bismut-Ricci Form of a Left Invariant Metric

We next record a basic lemma concerning the left invariant Hermitian metrics that we will consider on the previous Lie groups.

Lemma 2.2. *Let (M^4, J) be a 4-dimensional Lie group with a left invariant complex structure. With a basis Z_i of left invariant $T^{1,0}M$ vector fields, any Hermitian metric g is determined by complex valued functions $x = g(Z_1, \overline{Z_1})$, $y = g(Z_2, \overline{Z_2})$, and $z = g(Z_1, \overline{Z_2})$ satisfying $x, y > 0$ and $xy - |z|^2 > 0$. If ζ^i is the dual basis to the Z_i, the Kähler form $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$ is given by*

$$\omega = \sqrt{-1}(x\zeta^{1\overline{1}} + y\zeta^{2\overline{2}} + z\zeta^{1\overline{2}} + \overline{z}\zeta^{2\overline{1}}),$$

where $\zeta^{i\overline{j}} = \zeta^i \wedge \zeta^{\overline{j}}$ is the wedge product. The metric g is left invariant if and only if x, y, and z are constant on M.

A formula for the Ricci form associated to each of the canonical connections ∇^τ for a left invariant Hermitian metric was computed in [29]; the following proposition concerns the specific case of the Bismut connection.

Proposition 2.3 ([29], Proposition 4.1)**.** *Let (M^{2n}, g, J) be a Lie group with a left invariant*

Hermitian structure. Then the Bismut-Ricci form can be written as $\rho^b = d\eta$, where

$$\eta = \eta_i \zeta^i + \eta_{\bar{i}} \zeta^{\bar{i}}$$

$$\eta_i = \sqrt{-1} c_{ij}^j - \sqrt{-1} g^{\bar{j}k} c_{k\bar{j}}^{\bar{l}} g_{i\bar{l}}$$

and $c_{ij}^k, c_{\bar{i}\bar{j}}^{\bar{k}}$ are the structure constants of the Lie algebra with respect to the $Z_i, \overline{Z_i}$

2.3 Pluriclosed Flow on the Model Geometries

We are now ready to compute the pluriclosed flow equations

$$\frac{d\omega}{dt} = -(\rho_\omega^b)^{(1,1)}$$

for the homogeneous Hermitian metrics of the previous section and to prove Theorem 2.2. Recall that our metrics have the form

$$\omega = \sqrt{-1}(x\zeta^{1\bar{1}} + y\zeta^{2\bar{2}} + z\zeta^{1\bar{2}} + \bar{z}\zeta^{2\bar{1}})$$

with respect to the above $T^{1,0}M$ frames.

2.3.1 Hyperelliptic Surfaces

Lemma 2.3. *Let ω be a left invariant Hermitian metric on $\tilde{E}(2) \times \mathbb{R}$ and let ζ^i be a $(T^{1,0})^*M$ frame satisfying*

$$d\zeta^1 = -\zeta^{12} + \zeta^{1\bar{2}} \qquad d\zeta^2 = 0.$$

Then

$$\eta_1 = \sqrt{-1}\frac{zx}{xy - |z|^2},$$

$$\rho^b = \sqrt{-1}\frac{zx}{xy - |z|^2}(-\zeta^{12} + \zeta^{1\bar{2}}) + conjugates.$$

Corollary 2.1. *Pluriclosed flow for a left invariant metric on $\tilde{E}(2) \times \mathbb{R}$ satisfies*

$$x' = y' = 0$$

$$z' = -\frac{xz}{xy - |z|^2}.$$

Where x', y', z' denote time derivatives of x, y, z. In particular, $|z| = O(e^{-Ct})$ for some positive constant C depending on the initial condition.

Corollary 2.2. *Under pluriclosed flow a homogeneous Hermitian metric g on a hyperelliptic surface converges exponentially fast in the C^∞ topology to a flat Kähler metric. Under the family of metrics $\frac{g(t)}{t}$, a hyperelliptic surface converges to a point in the Gromov-Hausdorff sense.*

Remark: Notice that ω is a flat Kähler metric if and only if $z = 0$. This is therefore an example of pluriclosed flow taking non-Kähler initial data to a Kähler metric.

Proof. A direct calculation gives the Bismut-Ricci form and the resulting pluriclosed ODE. From here, we compute

$$(|z|^2)' = -2\frac{x_0|z|^2}{x_0 y_0 - |z|^2} \leq -2\frac{|z|^2}{y_0},$$

and so

$$|z| \leq |z_0| e^{-\frac{1}{y_0}t}.$$

$\square$

2.3.2 Hopf Surfaces

Lemma 2.4. *Let ω be a left invariant Hermitian metric on $\mathbb{R} \times SU_2$. With respect to a frame satisfying*

$$d\zeta^1 = (1 - \sqrt{-1}\alpha)\zeta^{2\overline{2}} \qquad d\zeta^2 = -\zeta^{12} - \zeta^{2\overline{1}}$$

we have

$$\eta_1 = \frac{\alpha x^2 + \sqrt{-1}(xy - x^2 - 2|z|^2)}{xy - |z|^2}$$

$$\eta_2 = \frac{\alpha x\overline{z} - \sqrt{-1}\overline{z}(x+y)}{xy - |z|^2},$$

and so

$$\rho^b = (1 - \sqrt{-1}\alpha)\frac{\alpha x^2 + \sqrt{-1}(xy - x^2 - 2|z|^2)}{xy - |z|^2}\zeta^{2\overline{2}} +$$

$$\frac{-\alpha x\overline{z} + \sqrt{-1}\overline{z}(x+y)}{xy - |z|^2}(\zeta^{12} + \zeta^{2\overline{1}}) + conjugates.$$

Corollary 2.3. *Pluriclosed flow for a left invariant metric on $\mathbb{R} \times SU_2$ is given by*

$$x' = 0$$

$$y' = 2\frac{x((\alpha^2+1)x - y) + 2|z|^2}{xy - |z|^2}$$

$$z' = \frac{\alpha\sqrt{-1}xz - z(x+y)}{xy - |z|^2}$$

in particular, $|z| = O(e^{-\frac{1}{x_0}t})$ and $y \to (1+\alpha^2)x_0$.

Corollary 2.4. *Under pluriclosed flow, a locally homogeneous Hermitian metric on a Hopf surface converges in the C^∞ topology to a metric which is independent of the initial condition and is unique up to homothety.*

Proof. We compute

$$(|z|^2)' = -2\frac{|z|^2(x_0+y)}{x_0 y - |z|^2}$$

and so

$$(|z|^2)' \leq -2\frac{|z|^2}{x_0}.$$

Thus

$$|z|^2 \leq |z_0|^2 e^{-\frac{2}{x_0}t}.$$

Note that y is increasing whenever $y < (1+\alpha^2)x_0$ and if y is nondecreasing then

$$y \leq (\alpha^2+1)x_0 + 2\frac{|z_0|^2}{x_0}e^{-\frac{2}{x_0}t}.$$

Therefore $y \to (1+\alpha^2)x_0$. □

Remark: A homogeneous Hopf surface is a compact complex surface whose universal cover is $\mathbb{C}^2\backslash\{0\}$ and which has a finite index subgroup of the fundamental group generated by the map $\theta(z,w) = (az, bw)$, where $a, b \in \mathbb{C}^*$ and $|a| = |b| < 1$. One can identify $\mathbb{R} \times SU_2$ with $\mathbb{C}^2\backslash\{0\}$ so that the induced complex structure is left invariant and the map θ is given by left multiplication by the element $(|a|, id) \in \mathbb{R} \times SU_2$. The parameter α then encodes information on the angles $\arg a$ and $\arg b$ of a and b. Because $|a| = |b|$, the metric

$$\omega = \sqrt{-1}\frac{1}{r^2}\partial\bar{\partial}r^2$$

descends to the quotient Hopf surface. Up to homothety this can be the only non-Kähler fixed point of the flow on compact complex surfaces. This result was proved initially by Ivanov and Gauduchon [4] through their study of Hermitian-Einstein-Weyl manifolds. We will give a different proof of this result later in this chapter which is much simpler and requires only basic knowledge of Hermitian geometry.

2.3.3 Non-Kähler, Properly Elliptic Surfaces

Lemma 2.5. *Let ω be a left invariant Hermitian metric on $\tilde{SL_2}(\mathbb{R}) \times \mathbb{R}$. With the frame given by*

$$d\zeta^1 = -\sqrt{-1}(\zeta^{12} + \zeta^{1\bar{2}}) \qquad d\zeta^2 = (\alpha - \sqrt{-1})\zeta^{1\bar{1}},$$

we compute

$$\eta_1 = \frac{z(y-x)}{xy - |z|^2} + \sqrt{-1}\frac{-\alpha yz}{xy - |z|^2}$$

$$\eta_2 = \frac{xy + y^2 - 2|z|^2}{xy - |z|^2} + \sqrt{-1}\frac{-\alpha y^2}{xy - |z|^2},$$

and therefore

$$\rho^b = (\frac{-\alpha yz + \sqrt{-1}z(x-y)}{xy - |z|^2})(\zeta^{12} + \zeta^{1\bar{2}}) +$$

$$(\alpha - \sqrt{-1})(\frac{xy + y^2 - 2|z|^2 - \sqrt{-1}\alpha y^2}{xy - |z|^2})\zeta^{1\bar{1}} + conjugates.$$

Corollary 2.5. *Pluriclosed flow of a left invariant Hermitian metric on $\tilde{SL_2}(\mathbb{R})\times\mathbb{R}$ is given by*

$$x' = 2(1 + \frac{(1+\alpha^2)y^2 - |z|^2}{xy - |z|^2}) \qquad y' = 0$$

$$z' = \frac{-\sqrt{-1}\alpha yz + z(y-x)}{xy - |z|^2}.$$

In particular, $z = O(e^{-Ct})$ for some positive constant C depending on the initial condition and $x \sim 2t$.

Corollary 2.6. *If $\omega(t)$ is a locally homogeneous solution to pluriclosed flow on a non-Kähler properly elliptic surface, then under the family of metrics $\frac{\omega(t)}{t}$ the surface converges to the base curve in the Gromov-Hausdorff sense.*

Proof. We compute that $xy - |z|^2$ is increasing since

$$2xy^2 + 2(1+\alpha^2)y^3 - 4|z|^2y + 2|z|^2(x-y) > 0$$

whenever $xy - |z|^2 > 0$. Then note that $x' \geq 4$ whenever $x \leq y$, so for any $\delta > 0$

$$(|z|^2)' \leq |z|^2 2(\delta - \frac{1}{y_0})$$

for sufficiently large t. Thus

$$|z| \leq Ce^{(\delta - \frac{1}{y_0})t}$$

for some constant C. $x \sim 2t$ is then immediate.

If $\pi : \Gamma\backslash\tilde{SL_2}(\mathbb{R}) \times \mathbb{R} \to \Sigma$ is the projection of a non-Kähler properly elliptic surface to the base curve Σ, the fibers are the leaves of the distribution spanned by the real and imaginary parts of Z_2. Moreover, there is a unique metric ω_Σ on Σ such that $\pi^*\omega_\Sigma = 2\sqrt{-1}\zeta^{1\bar{1}}$. Now, if $f : \Sigma \to \Gamma\backslash \tilde{SL_2}(\mathbb{R}) \times \mathbb{R}$ is any function (not necessarily continuous) such that $\pi \circ f = id$ then, for any $\epsilon > 0$, π and f are ϵ-Gromov-Hausdorff approximations with respect to the metrics $\frac{\omega(t)}{t}$ and ω_Σ as long as t is sufficiently large. $\square$

2.3.4 Kodaira Surfaces

Lemma 2.6. *Let ω be a left invariant metric on $Nil^3 \times \mathbb{R}$. With a frame satisfying*

$$d\zeta^1 = 0 \qquad d\zeta^2 = -\sqrt{-1}\zeta^{1\bar{1}},$$

we compute

$$\eta_2 = \frac{y^2}{xy - |z|^2},$$

and so

$$\rho^b = -\sqrt{-1}\frac{y^2}{xy - |z|^2}\zeta^{1\bar{1}} + conjugate.$$

Corollary 2.7. *Pluriclosed flow of a left invariant metric on $Nil^3 \times \mathbb{R}$ is given by*

$$x' = 2\frac{y^2}{xy - |z|^2}$$

$$y' = z' = 0.$$

In particular

$$\frac{1}{2}x^2 y_0 - x|z_0|^2 = 2y_0^2 t + \frac{1}{2}x_0^2 y_0 - x_0|z_0|^2$$

for all t.

Corollary 2.8. *Let $g(t)$ be a homogeneous solution of pluriclosed flow on a primary Kodaira surface. Then under the metrics $\frac{g(t)}{t}$ the surface converges to a point in the Gromov-Hausdorff sense.*

Proof. This is a direct computation. $\square$

Lemma 2.7. *Let ω be a left invariant Hermitian metric on $Nil \rtimes \mathbb{R}$. With a frame satisfying*

$$d\zeta^1 = \epsilon(-\zeta^{12} + \zeta^{1\bar{2}}) \qquad d\zeta^2 = \epsilon\sqrt{-1}\zeta^{1\bar{1}},$$

we compute

$$\eta_1 = \epsilon(-\frac{yz}{xy - |z|^2} + \sqrt{-1}\frac{xz}{xy - |z|^2}),$$

$$\eta_2 = \epsilon(-\frac{y^2}{xy - |z|^2} + \sqrt{-1}\frac{2|z|^2 - xy}{xy - |z|^2}),$$

and therefore

$$\rho^b = \frac{-yz + \sqrt{-1}xz}{xy - |z|^2}(-\zeta^{12} + \zeta^{1\bar{2}}) +$$

$$\frac{xy - 2|z|^2 - \sqrt{-1}y^2}{xy - |z|^2}\zeta^{1\bar{1}} + conjugates.$$

Corollary 2.9. *Pluriclosed flow of a left invariant metric on $Nil \rtimes \mathbb{R}$ is given by*

$$x' = 2\frac{y^2}{xy - |z|^2} \qquad y' = 0$$

$$z' = -\frac{(x + \sqrt{-1}y)z}{xy - |z|^2}.$$

In particular $x \sim 2\sqrt{y_0 t}$ and $|z| = O(e^{-Ct})$ for some positive constant C depending on the *initial data.*

Corollary 2.10. *Let $g(t)$ be a homogeneous solution of pluriclosed flow on a secondary Kodaira surface. Then under the metrics $\frac{g(t)}{t}$ the surface converges to a point in the Gromov-Hausdorff sense.*

Proof. Similar to the previous cases, we compute

$$(|z|^2)' \leq -\frac{|z|^2}{y_0},$$

and so $|z| = O(e^{-Ct})$. Then note that

$$\frac{2y_0}{x} \leq x' \leq \frac{2y_0^2}{xy_0 - |z_0|^2}.$$

□

2.3.5 Inoue Surfaces

Lemma 2.8. *Let ω be a left invariant Hermitian metric on the solvable Lie group with frame satisfying*

$$d\zeta^1 = -\lambda\zeta^{12} + \lambda\zeta^{1\bar{2}} \qquad d\zeta^2 = -2a\sqrt{-1}\zeta^{2\bar{2}}.$$

Then we compute

$$\eta_1 = \frac{2azx + \sqrt{-1}\bar{\lambda}zx}{xy - |z|^2},$$

$$\eta_2 = \frac{\sqrt{-1}(\lambda+\bar{\lambda})|z|^2 + (2a - \sqrt{-1}\lambda)xy}{xy - |z|^2}.$$

Therefore the Bismut-Ricci form is

$$\rho^b = \frac{2azx + \sqrt{-1}\bar{\lambda}zx}{xy - |z|^2}(-\lambda\zeta^{12} + \lambda\zeta^{1\bar{2}}) +$$

$$(\frac{2a(\lambda + \bar{\lambda})|z|^2 + (-4a^2\sqrt{-1} - 2a\lambda)xy}{xy - |z|^2})(\zeta^{2\bar{2}}) + conjugates.$$

Corollary 2.11. *Pluriclosed flow of a left invariant metric on the solvable family is given by*

$$x' = 0,$$

$$y' = 12a^2(1 + \frac{|z|^2}{xy - |z|^2}),$$

$$z' = -(3a^2 + b^2 + 2ab\sqrt{-1})\frac{xz}{xy - |z|^2}.$$

In particular, $y \sim 12a^2 t$ *and* $|z|$ *is bounded.*

Proof. We compute

$$(|z|^2)' = -(3a^2 + b^2)\frac{x_0|z|^2}{x_0 y - |z|^2},$$

which shows that $|z|$ is bounded. It is then immediate that $y(t)/t \to 12a^2$. □

Corollary 2.12. *Let* $\omega(t)$ *be a locally homogeneous solution to pluriclosed flow on an Inoue surface of type* S_A*. Then under the family of metrics* $\frac{g(t)}{t}$*,* S_A *converges to a circle of length* $\sqrt{6}|a|$ *in the Gromov-Hausdorff sense.*

Proof. We recall the construction of an Inoue surface of type S_A. Choose a matrix $A \in SL_3(\mathbb{Z})$ with eigenvalues α, $\bar{\alpha}$, and $c = |\alpha|^{-2}$, where $\alpha \neq \bar{\alpha}$ and $|\alpha| \neq 1$. Write $\alpha = |\alpha|e^{i\theta}$.

Consider the solvable Lie group G of matrices of the form

$$\begin{pmatrix} \alpha^s & 0 & w \\ 0 & c^s & r \\ 0 & 0 & 1 \end{pmatrix},$$

where $r, s \in \mathbb{R}$, $w \in \mathbb{C}$. This group has Lie brackets given by $[X_4, X_1] = aX_1 + bX_2$, $[X_4, X_2] = -bX_1 + aX_2$, and $[X_4, X_3] = -2aX_3$, where $a = \log|\alpha|$ and $b = \theta$. There is a natural identification of G with $\mathbb{C} \times \mathbb{C}H^1$ given by sending an element as before to $(w, r + c^s\sqrt{-1})$. Under such an identification, left multiplication is a biholomorphism. Let $(a_1, a_2, a_3)^T$ be an eigenvector of α and $(c_1, c_2, c_3)^T$ be a real eigenvector of c. Consider the lattice Γ generated by the elements

$$g_0 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad g_i = \begin{pmatrix} 1 & 0 & a_i \\ 0 & 1 & c_i \\ 0 & 0 & 1 \end{pmatrix}.$$

As shown by Inoue [8], the quotient $\Gamma \backslash G = S_A$ forms a compact complex surface of class VII_0 with the property that it has no complex curves. Moreover, S_A is a T^3 bundle over S^1, where the projection $\pi : S_A \to S^1$ is given by mapping (the equivalence classes of) $(w, r + c^s\sqrt{-1})$ to c^s; this projection is induced by the natural projection of $\mathbb{C} \times \mathbb{C}H^1$ to the imaginary axis of the second factor.

Consider the closed curve $\gamma : S^1 \to S_A$ given by $\gamma(s) = (0, c^s\sqrt{-1})$. Note that this is well defined as a map of equivalence classes and $\pi \circ \gamma = id$. Let $\omega(t)$ be a left invariant solution to pluriclosed flow on G. With respect to $\omega(t)$, γ has length $L_\omega(\gamma) = \sqrt{\frac{y(t)}{2}}$. Therefore, with respect to the metrics $\frac{\omega(t)}{t}$, the length of γ approaches $\sqrt{6}|a|$ as $t \to \infty$. Because there are no curves in S_A, the real and imaginary parts of Z_1 form an integrable distribution whose leaves are dense in each T^3 fiber. Now, because the length of Z_1 with respect to $\omega(t)$ is fixed in time, for any $\epsilon > 0$ the diameter of each T^3 fiber with respect to the metric $\frac{\omega(t)}{t}$ is less than ϵ for t sufficiently large. Therefore γ and π are ϵ-Gromov-Hausdorff approximations between the circle of length $\sqrt{6}|a|$ and S_A with the metric $\frac{\omega(t)}{t}$ for t sufficiently large. $\square$

Lemma 2.9. *Let ω be a left invariant Hermitian metric on Sol_1 with frame given by*

$$d\zeta^1 = \zeta^{1\bar{1}} \qquad d\zeta^2 = \zeta^{12} + \zeta^{1\bar{2}}.$$

Then

$$\eta_1 = -\sqrt{-1}\frac{2xy - |z|^2 - z^2}{xy - |z|^2},$$

$$\eta_2 = -\sqrt{-1}\frac{y(\bar{z} - z)}{xy - |z|^2},$$

and so

$$\rho^b_{1\bar{1}} = -\sqrt{-1}\frac{4xy - (z + \bar{z})^2}{xy - |z|^2},$$

$$\rho^b_{2\bar{2}} = 0,$$

$$\rho^b_{1\bar{2}} = -\sqrt{-1}\frac{y(\bar{z} - z)}{xy - |z|^2}$$

Corollary 2.13. *Pluriclosed flow of a left invariant metric on Sol_1 is given by*

$$x' = \frac{4xy - (z + \bar{z})^2}{xy - |z|^2}$$

$$y' = 0$$

$$z' = \frac{y(\bar{z} - z)}{xy - |z|^2}.$$

In particular, $x \sim 4t$ and z is bounded.

Proof. We compute

$$(|z|^2)' = \frac{y(z-\bar{z})^2}{xy-|z|^2} \leq 0,$$

which shows that $|z|$ is bounded. Then, noting that $x' \geq 4$, we conclude the result. $\square$

Lemma 2.10. *Let ω be a left invariant Hermitian metric on Sol_1' with frame satisfying*

$$d\zeta^1 = \zeta^{1\bar{1}} \qquad d\zeta^2 = -\zeta^{1\bar{1}} + \zeta^{1\bar{2}} + \zeta^{12}.$$

Then

$$\eta_1 = -\sqrt{-1}\frac{2xy - yz - |z|^2 - z^2}{xy-|z|^2},$$

$$\eta_2 = -\sqrt{-1}\frac{y(\bar{z}-z) - y^2}{xy-|z|^2},$$

and so

$$\rho^b_{1\bar{1}} = -\sqrt{-1}\frac{4xy - y(z+\bar{z}) - (z+\bar{z})^2 + 2y^2}{xy-|z|^2},$$

$$\rho^b_{2\bar{2}} = 0$$

$$\rho^b_{1\bar{2}} = -\sqrt{-1}\frac{y(\bar{z}+z) - y^2}{xy-|z|^2}$$

Corollary 2.14. *Pluriclosed flow of a left invariant metric on Sol_1' is given by*

$$x' = \frac{4xy - y(z+\bar{z}) - (z+\bar{z})^2 + 2y^2}{xy-|z|^2} \qquad y' = 0,$$

$$z' = \frac{y(\bar{z}-z) - y^2}{xy-|z|^2}.$$

In particular, $x \sim 4t$ and $|z| = O(\log t)$.

Proof. First, note that the imaginary part of z is bounded and it suffices to assume that z is real. For simplicity, rescale the initial condition so that $y_0 = 1$. Then x and z satisfy the system

$$x' = 4 + 2\frac{1-z}{x-z^2} \qquad z' = -\frac{1}{x-z^2}.$$

This implies that

$$0 > z' \geq -\frac{1}{4t + x_0 - z_0^2},$$

which gives $|z| = O(\log t)$. ☐

Corollary 2.15. *Let $\omega(t)$ be a locally homogeneous solution to pluriclosed flow on an Inoue surface of type S^+. Then under the family of metrics $\frac{\omega(t)}{t}$ the surface S^+ converges to a circle of length $\sqrt{2}|\log \lambda|$ in the Gromov-Hausdorff sense, where λ depends on the construction of S^+. For an Inoue surface of type S^- the surface converges to a circle of length $2\sqrt{2}|\log \lambda|$.*

Proof. We recall the construction of an Inoue surface of type S^+. Let $N \in SL_2(\mathbb{Z})$ have positive eigenvalues $\lambda \neq 1, \lambda^{-1}$ and corresponding eigenvectors $(a_1, a_2)^T, (b_1, b_2)^T$. Choose integers j, k, l with $l \neq 0$ and a complex number κ. Let Sol_1^4 be the group of matrices of the form

$$\begin{pmatrix} 1 & u & v \\ 0 & q & r \\ 0 & 0 & 1 \end{pmatrix}$$

where $q, r, u, v \in \mathbb{R}$ and $q > 0$. For $m \in \mathbb{R}$ this group has transitive actions on $\mathbb{C}H^1 \times \mathbb{C}$ with trivial stabilizers so that an element as before maps $(\sqrt{-1}, 0)$ to

$$(r + \sqrt{-1}q, v + \sqrt{-1}(u + m \log q),$$

with Sol_1 corresponding to the $m = 0$ and Sol_1' corresponding to the $m \neq 0$ cases. By taking $m = \frac{\Im(\kappa)}{\log \lambda}$ we can obtain a cocompact lattice Γ generated by

$$g_0 = \begin{pmatrix} 1 & 0 & \Re(\kappa) \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad g_i = \begin{pmatrix} 1 & b_i & c_i \\ 0 & 1 & a_i \\ 0 & 0 & 1 \end{pmatrix} \qquad g_3 = \begin{pmatrix} 1 & 0 & \frac{b_1 a_2 - b_2 a_1}{l} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where c_i are determined by the previously chosen data. The quotient by this lattice gives an Inoue surface of type S^+. An Inoue surface of type S^- is formed by a similar quotient where $m = 0$ and λ is replaced with λ^2. As shown by Inoue, these are compact complex surfaces of class VII_0 with no curves and, similarly to the type S_A case, these are bundles over S^1 such that the real and imaginary parts of Z_2 span an integrable distribution whose leaves are dense in each fiber. The curve $\gamma : [0, 1] \to S$ given by $\gamma(t) = (\sqrt{-1}\lambda^t, 0)$ provides the relevant Gromov-Hausdorff approximation for t sufficiently large and we observe that the fibers are shrinking with respect to the metrics $\frac{\omega(t)}{t}$ as in the case of Inoue surfaces of type S_A. □

We can now complete the proof of Theorem 2.2.

Proof. Let $\omega(t)$ be a locally homogeneous solution to pluriclosed flow on a compact complex surface which exists on $[0, \infty)$. If ω_0 is Kähler then it is a product of Kähler-Einstein metrics with non-positive scalar curvatures. Under the rescaled metrics $\frac{\omega(t)}{t}$ the surface either converges to a product of Kähler-Einstein metrics with negative scalar curvature, as in the case where the universal covering metric is $\mathbb{C}H^2$ or $\mathbb{C}H^1 \times \mathbb{C}H^1$, collapses to a curve of genus $g \geq 2$, as in the case $\mathbb{C}H^1 \times \mathbb{C}$, or collapses to a point, as in the case of a flat metric on $\mathbb{C}^2$. If ω_0 is non-Kähler, note that it must be a left invariant Hermitian metric on one of the Lie groups considered before. Given that it is not a solution on the Hopf surface, the claimed Gromov-Hausdorff convergence follows from the case by case analysis considered throughout this section. $\square$

2.4 Blowdown Limits of Homogeneous Solutions

Let $(M, g(t))$ be a 1-parameter family of Riemannian manifolds for $t \in (0,\infty)$. Suppose that a blowdown limit $g_\infty(t) = \lim_{s\to\infty} s^{-1}g(st)$ exists in the sense that there is a 1-parameter family of diffeomorphisms θ_s of M such that $\theta_{s_1} \circ \theta_{s_2} = \theta_{s_1 s_2}$ and $g_\infty(t) = \lim_{s\to\infty} \theta_s^* s^{-1} g(st)$ uniformly on compact subsets of $M \times (0,\infty)$. Fix some $a > 0$. Then

$$g_\infty(at) = \lim_{s\to\infty} \theta_s^* s^{-1} g(sat) = a\theta_{a^{-1}}^* \lim_{\tilde{s}\to\infty} \theta_{\tilde{s}}^* \tilde{s}^{-1} g(\tilde{s}t) = a\theta_{a^{-1}}^* g_\infty(t).$$

This implies that

$$g_\infty(t) = tg_\infty(1)$$

up to diffeomorphisms of M. Therefore, if $g_\infty(t)$ satisfies some geometric flow then it is an expanding soliton solution of that flow. This construction has been used in [16] to give expanding soliton solutions to Ricci flow by performing blowdown limits of type III homogeneous Ricci flows.

In this section we will construct expanding soliton solutions to pluriclosed flow by applying blowdown limits to the homogeneous solutions of the previous section. We will write X_1 and $-X_2$ for the real and imaginary parts of Z_1 respectively, and similarly for X_3, $-X_4$, and Z_2. With respect to the dual basis σ^i to the X_i we see that

$$\zeta^1 = \frac{1}{2}(\sigma^1 + \sqrt{-1}\sigma^2) \qquad \zeta^2 = \frac{1}{2}(\sigma^3 + \sqrt{-1}\sigma^4),$$

and so our metrics have the form

$$\omega = \frac{1}{2}(x\sigma^{12} + y\sigma^{34} - \Im(z)(\sigma^{13} + \sigma^{24}) + \Re(z)(\sigma^{14} - \sigma^{23})).$$

2.4.1 The Hyperelliptic Case

Proposition 2.4. *Let $\omega(t)$ be a left invariant solution of pluriclosed flow on $\tilde{E}(2) \times \mathbb{R}$. There is a blowdown limit $\omega_\infty(t) = \lim_{s\to\infty} s^{-1}\omega(st)$ of the form*

$$\omega_\infty(t) = \frac{1}{2}(x_0\sigma^{12} + y_0\sigma^{34})$$

which is an expanding soliton solution. It is isometric to the flat metric on $\mathbb{C}^2$.

Proof. Recall that $x' = y' = 0$, and $|z| = O(e^{-Ct})$ for some positive constant C depending on the initial conditions. Define diffeomorphisms $\psi_s : \mathbb{R}^4 \to \tilde{E}(2) \times \mathbb{R}$ by

$$\psi_s(q, r, u, v) = \alpha_s(q, r)\beta_s(u, v),$$

where

$$\alpha_s(q, r) = \exp(\sqrt{s}(qX_1 + rX_2))$$

and

$$\beta_s(u, v) = \exp(\sqrt{s}(uX_3 + vX_4)).$$

Write $\theta_s = \psi_s \circ \psi_1^{-1}$. We see that

$$s^{-1}\theta_s^*(\omega(st)) = \frac{1}{2}(x_0\sigma^{12} + y_0\sigma^{34}) + O(e^{-Cst})$$

and so there is a blowdown limit

$$\omega_\infty(t) = \frac{1}{2}(x_0\sigma^{12} + y_0\sigma^{34}).$$

□

2.4.2 The Non-Kähler, Properly Elliptic Case

Recall that our $T^{1,0}$ frame has Lie brackets

$$[Z_1, Z_2] = \sqrt{-1}Z_2 \qquad [Z_1, \overline{Z_2}] = \sqrt{-1}Z_1$$

$$[Z_1, \overline{Z_1}] = (\sqrt{-1} - \alpha)Z_2 + (\sqrt{-1} + \alpha)\overline{Z_2}.$$

With respect to the basis X_i described above, the Lie brackets are

$$[X_1, X_2] = X_3 - \alpha X_4 \qquad [X_3, X_2] = X_1 \qquad [X_1, X_3] = X_2$$

and the complex structure is given by $JX_1 = X_2$ and $JX_3 = X_4$.

Lemma 2.11. *Any element of $\tilde{SL}_2(\mathbb{R}) \times \mathbb{R}$ can be written uniquely as*

$$\exp(qX_1 + rX_2)\exp(uX_3 + vX_4).$$

Proof. This follows from [16], Lemma 3.34, where we note that X_4 is central and our X_1, X_2, and X_3 correspond, respectively, to X_3, X_2, and X_1 of that Lemma. □

Proposition 2.5. *Let $\omega(t)$ be a left invariant solution to pluriclosed flow on $\tilde{SL}_2(\mathbb{R}) \times \mathbb{R}$. Then there is a blowdown limit $\omega_\infty(t)$ given by a product metric*

$$\omega_\infty(t) = \omega_{\mathbb{C}H^1}(t) \oplus \omega_{\mathbb{C}}$$

on $\mathbb{C}H^1 \times \mathbb{C}$ which is an expanding soliton solution.

Proof. The argument is the same as in [16] for the blowdown limit for a homogeneous Ricci flow on $\tilde{SL}_2(\mathbb{R})$. Recall that $x \sim 2t$, $y' = 0$, and $|z| = O(e^{-Ct})$ for some positive constant C. Consider the family of diffeomorphisms $\psi_s : \mathbb{R}^4 \to \tilde{S}L_2(\mathbb{R}) \times \mathbb{R}$ given by

$$\psi_s(q, r, u, v) = \exp(qX_1 + rX_2)\exp(s^{\frac{1}{2}}(uX_3 + vX_4)),$$

write $A(q, r) = \exp(qX_1 + rX_2)$ and $B_s(u, v) = \exp(s^{\frac{1}{2}}(uX_3 + vX_4))$ and let

$$h^{-1}dh = B_s^{-1}A^{-1}dAB_s + s^{\frac{1}{2}}(duX_3 + dvX_4)$$

be the Maurer-Cartan form. We compute

$$s^{-1}\psi_s^*\omega(st) \sim \frac{1}{2}(2t((B_s^{-1}A^{-1}dAB_s)_1 \wedge (B_s^{-1}A^{-1}dAB_s)_2) + y_0(du \wedge dv)).$$

The proof is concluded by noting that conjugation by B_s gives a rotation in the (q, r)-plane, so there is a blowdown limit

$$\omega_\infty(t) = \frac{1}{2}(2t(A^{-1}dA)_1 \wedge (A^{-1}dA)_2 + y_0 du \wedge dv),$$

which is now a product metric solution on $\mathbb{C}H^1 \times \mathbb{C}$. □

2.4.3 The Kodaira Surface Cases

Proposition 2.6. *Let $\omega(\cdot)$ be a left invariant solution to pluriclosed flow on $Nil \times \mathbb{R}$. Then there is a blowdown limit*

$$\omega_\infty(t) = \frac{1}{2}(2\sqrt{y_0 t}\sigma^{12} + y_0\sigma^{34})$$

of $\omega(\cdot)$ which is an expanding soliton solution.

Proof. Recall that $x \sim 2\sqrt{y_0 t}$ and $y' = z' = 0$. Define diffeomorphisms $\psi_s : \mathbb{R}^4 \to Nil \times \mathbb{R}$ by

$$\psi_s(q, r, u, v) = \left(\begin{pmatrix} 1 & s^{\frac{1}{4}} r & s^{\frac{1}{2}} u \\ 0 & 1 & s^{\frac{1}{4}} q \\ 0 & 0 & 1 \end{pmatrix}, s^{\frac{1}{2}} v \right),$$

and let $\theta_s = \psi_s \circ \psi_1^{-1}$. We see that

$$s^{-1} \theta_s^* \omega(st) = \frac{1}{2}(2\sqrt{y_0 t}\sigma^{12} + y_0 \sigma^{34}) + O(s^{-\frac{1}{4}}).$$

Therefore the blowdown limit is given by

$$\omega_\infty(t) = \frac{1}{2}(2\sqrt{y_0 t}\sigma^{12} + y_0 \sigma^{34})$$

□

Proposition 2.7. *Let $\omega(t)$ be a left invariant solution to pluriclosed flow on $Nil \rtimes \mathbb{R}$. Then there is a blowdown limit*

$$\omega_\infty(t) = \frac{1}{2}(2\sqrt{y_0 t}\sigma^{12} + y_0 \sigma^{34})$$

given by the expanding soliton solution on $Nil \times \mathbb{R}$

Proof. Recall that $x \sim 2\sqrt{y_0 t}$, $y' = 0$, and $|z| = O(e^{-Ct})$ for some positive constant C. Note that these groups $Nil \rtimes \mathbb{R}$ and $Nil \times \mathbb{R}$ are diffeomorphic. For the same coordinates as in the previous case, use the same diffeomorphisms θ_s to obtain the desired blowdown limit. □

2.4.4 The Inoue Surface Cases

Let $\omega(t)$ be a left invariant solution to pluriclosed flow on the solvable family. Recall that $x' = 0$, $y \sim 12a^2 t$, and $|z| = O(1)$.

Proposition 2.8. *There is a blowdown limit*

$$\omega_\infty(t) = \frac{1}{2}(x_0\sigma^{12} + 12a^2 t\sigma^{34})$$

Proof. Define diffeomorphisms $\psi_s : \mathbb{R}^4 \to G$ by

$$\psi_s(q,r,u,v) = \begin{pmatrix} e^{av}\cos(bv) & -e^{av}\sin(bv) & 0 & s^{\frac{1}{2}}q \\ e^{av}\sin(bv) & e^{av}\cos(bv) & 0 & s^{\frac{1}{2}}r \\ 0 & 0 & e^{-2av} & u \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and let $\theta_s = \psi_s \circ \psi_1^{-1}$. Then

$$s^{-1}\theta_s^* g(st) = \frac{1}{2}(x_0\sigma^{12} + 12a^2 t\sigma^{34}) + O(s^{-\frac{1}{2}})$$

□

Proposition 2.9. *Let $\omega(t)$ be a left invariant solution to pluriclosed flow on Sol_1. There is a blowdown limit*

$$\omega_\infty(t) = \frac{1}{2}(4t\sigma^{12} + y_0\sigma^{34})$$

Proof. Recall that $x \sim 4t$, $y' = 0$, and z is bounded. Define diffeomorphisms $\psi_s : \mathbb{R}^3 \times \mathbb{R}^+ \to$

Sol_1 by

$$\psi_s(u,v,r,q) = \begin{pmatrix} 1 & s^{\frac{1}{2}}u & s^{\frac{1}{2}}v \\ 0 & q & r \\ 0 & 0 & 1 \end{pmatrix}$$

and let θ_s be as before. We see that

$$s^{-1}\theta_s^* g(st) = \frac{1}{2}(4t\sigma^{12} + y_0\sigma^{34}) + O(s^{-\frac{1}{2}})$$

□

Proposition 2.10. *Let $\omega(t)$ be a left invariant solution to pluriclosed flow on Sol_1'. There is a blowdown limit*

$$\omega_\infty(t) = \frac{1}{2}(4t\sigma^{12} + y_0\sigma^{34})$$

given by the expanding soliton solution with respect to the other complex structure Sol_1.

Proof. Recall that $x \sim 4t$, $y' = 0$, and $|z| = O(\log t)$. Let σ^i be one forms dual to the X_i associated to the complex structure Sol_1. The one forms $\tilde{\sigma}^i$, dual to the $\tilde{X}_i$ associated to the complex structure Sol_1', are given by

$$\tilde{\sigma}^i = \sigma^i \qquad (i = 1,2,3) \qquad \tilde{\sigma}^4 = \sigma^4 - \sigma^2.$$

If

$$\omega = \frac{1}{2}(x\tilde{\sigma}^{12} + y\tilde{\sigma}^{34} - \Im(z)(\tilde{\sigma}^{13} + \tilde{\sigma}^{24}) + \Re(z)(\tilde{\sigma}^{14} - \tilde{\sigma}^{23})$$

is a left invariant Hermitian metric with respect to the complex structure Sol_1', we see that

$$\omega = \frac{1}{2}(x\sigma^{12} + y(\sigma^{34} - \sigma^{32}) - \Im(z)(\sigma^{13} + \sigma^{24}) + \Re(z)(\sigma^{14} - \sigma^{12} - \sigma^{23})).$$

Therefore, using the same diffeomorphisms θ_s as in the previous proposition, we find a blowdown limit

$$\omega_\infty(t) = \frac{1}{2}(4t\sigma^{12} + y_0\sigma^{34})$$

□

We are now able to prove Theorem 2.1

Proof. Let $\omega(t)$ be a locally homogeneous solution to pluriclosed flow on a compact complex surface M which exists on the maximal time interval $[0, T)$. If T is finite then M must be either $\mathbb{C}P^2$ or a product of $\mathbb{C}P^1$ with a curve. Suppose then that $T = \infty$. If $\omega(t)$ is Kähler then the induced metric on the universal cover of M is one of $\mathbb{C}H^2$, $\mathbb{C}H^1 \times \mathbb{C}H^1$, $\mathbb{C}H^1 \times \mathbb{C}$, or $\mathbb{C}^2$. In each case, the induced metric is a product of Kähler-Einstein metrics and it is easy to obtain the required diffeomorphisms to show the existence of a blowdown limit. If $\omega(t)$ is non-Kähler, it is either a Hopf surface or one of the metrics considered in this section. It was shown that the pluriclosed flow of a homogeneous metric on a Hopf surface converges to a canonical metric up to homothety, and we have already constructed blowdown limits for the remaining cases. □

2.5 Fixed Points and Locally Conformal Kähler Geometry

Let $(M^4, g(t), J, \omega(t))$ be a solution to Pluriclosed flow on a complex surface. The Lee form of ω is as always defined by $d\omega = \theta \wedge \omega$. In this section we will compute the evolution of θ under the pluriclosed flow and derive classification results for the fixed points of the flow from it.

Proposition 2.11. *If $\omega(t)$ is a solution to pluriclosed flow $\omega' = -\rho^{1,1}$ then the Lee form θ evolves according to*

$$\begin{aligned}\theta' \wedge \omega &= \theta \wedge \rho^{1,1} + \frac{1}{2} dd^c\theta \\ \theta' &= \frac{1}{2}\Delta_d \theta + L^*(\theta \wedge \rho^{1,1})\end{aligned}$$

Proof. We have

$$\omega' = -\rho^{1,1}.$$

Applying this to $d\omega = \theta \wedge \omega$ gives

$$\theta' \wedge \omega - \theta \wedge \rho^{1,1} = -d\rho^{1,1}.$$

Now recall that the Bismut-Ricci form ρ is related to the Chern-Ricci form ρ^c via

$$\rho = \rho^c + dJ\theta.$$

In particular,

$$\begin{aligned}\rho^{1,1} &= \rho^c + \frac{dJ\theta + JdJ\theta}{2},\\ d\rho^{1,1} &= \frac{dJdJ\theta}{2} = -\frac{dd^c\theta}{2},\end{aligned}$$

using the fact that ρ^c is a closed $(1,1)$-form. This proves the first equality.

The second equality is obtained by applying L^* to the first. From the commutator identities of Chapter 1 we have $[L^*, d^c] = d^* + \mu^*$, and therefore

$$\begin{aligned}L^*(dd^c\theta) &= -L^*d^cd\theta\\ &= -d^cL^*d\theta - d^*d\theta - \mu^*d\theta\\ &= -d^*d\theta\\ &= \Delta_d\theta.\end{aligned}$$

The fourth equality comes from $\Delta_d = -(dd^* + d^*d)$ and the fact that $d^*\theta = 0$ because the metric is pluriclosed, while the second and third equalities follow from

$$\begin{aligned}\mu &= [L^*, d\omega] = L^*\theta L - \theta LL^* = L^*\theta L - L\theta L^*,\\ \mu^* &= L^*\theta^*L - L\theta^*L^*\\ \mu^*d\theta &= 0\end{aligned}$$

□

A remarkable feature about the Bismut-Ricci form is that it behaves very well with respect to conformal transformations of a Kähler metric. We will exploit this property to give an elementary proof of the fact that pluriclosed, Bismut-Ricci flat metrics are Vaisman.

Proposition 2.12. *Suppose that ω_0 is a Kähler metric on the complex manifold M^{2n}. Let*

u be some smooth function and consider the conformally equivalent Hermitian metric $\omega = e^{2u}\omega_0$. Then

$$\rho^b_\omega = \rho^c_{\omega_0} + (2-n)dd^c u.$$

In particular, on a locally conformally Kähler surface (M^4, ω)

$$\rho^b_\omega = \rho^c_{\omega_0},$$

where ω_0 is the locally defined Kähler metric conformal to ω.

Proof. Let ω_0 and ω be as above. Remember the two formulas for the Bismut and Chern Ricci forms:

$$\rho^b_\omega = \rho^c_\omega + dJ\theta,$$
$$\rho^c_\omega = \rho^c_{\omega_0} - \sqrt{-1}\partial\bar{\partial}\log(\frac{\omega^n}{\omega_0^n}).$$

Note that the Lee form of ω is $2du$ and that $dJdu = dd^c u$. Additionally

$$\log(\frac{\omega^n}{\omega_0^n}) = 2nu.$$

This combined with the fact that $2\sqrt{-1}\partial\bar{\partial} = dd^c$ gives the result. $\square$

The previous proposition demonstrates that without assumptions of completeness or pluriclosedness, there are many Bismut-Ricci flat surfaces.

Corollary 2.16. *Let $\Omega \subseteq \mathbb{C}^2$ be any domain with the standard flat metric ω_0 and let e^{2u} be any positive function on Ω. Then $e^{2u}\omega_0$ is Bismut-Ricci flat. It is moreover pluriclosed if e^{2u} is a positive harmonic function.*

Proposition 2.13. *Suppose that a locally conformally Kähler surface* (M^4, g, J) *is Bismut-Ricci flat. Then the Ricci curvature of* g *is*

$$Rc_g = \frac{1}{2}(|\theta|^2 + d^*\theta)g - \frac{1}{2}\theta \otimes \theta - \nabla\theta.$$

Proof. By the previous proposition, if ω is locally conformally Kähler and Bismut-Ricci flat, then the local conformal Kähler metric ω_0 is Ricci flat. Recall next the formula for the conformal change of the Ricci curvature: if $g_0 = e^{-2u}g$ on a Riemannian manifold M^m then

$$Rc_{g_0} = Rc_g + (m-2)(Hess(u) + du \otimes du) - (\Delta u + (m-2)|du|^2)g,$$

the derivatives on the right hand side computed with respect to g. Under our hypothesis we have local conformally Ricci flat metrics g_0 and $2du = \theta$ is a globally defined one-form. Therefore

$$0 = Rc_g + \nabla\theta + \frac{1}{2}\theta \otimes \theta + \frac{1}{2}(d^*\theta - |\theta|^2)g.$$

□

Proposition 2.14. *Under the same assumptions as the previous proposition,* $\nabla\theta = 0$*, so* (M^4, g, J) *is in fact Vaisman with non-negative Ricci curvature.*

Proof. From the previous proposition we have

$$Rc_g = \frac{1}{2}(|\theta|^2 + d^*\theta)g - \frac{1}{2}\theta \otimes \theta - \nabla\theta.$$

Note that if θ is pluriclosed and locally conformally Kähler, then $\nabla\theta$ is traceless and symmetric. We then take the inner product of the above identity with $\nabla\theta$ and integrate over

M, obtaining

$$\int_M \langle Rc, \nabla\theta\rangle + \frac{1}{2}\langle \theta\otimes\theta, \nabla\theta\rangle = -\int_M |\nabla\theta|^2.$$

We conclude the proof by showing that both terms in the left hand side are divergences. For the first, we integrate by parts and apply the second Bianchi identity to get

$$\int_M \langle Rc, \nabla\theta\rangle = -\frac{1}{2}\int_M \langle ds, \theta\rangle = -\frac{1}{2}\int_M div(s\theta^\sharp),$$

using that $d^*\theta = 0$.

For the second, we have

$$\int_M \langle \theta\otimes\theta, \nabla\theta\rangle = 2\int_M \langle (d^*\theta)\theta, \theta\rangle = 0,$$

using the fact that $\nabla\theta$ is symmetric. $\square$

We can use the previous calculations to give elementary proofs of classification results for fixed points of the flow among compact surfaces.

Definition 2.4. *A pluriclosed metric is static for pluriclosed flow if $-\rho^{1,1} = \lambda\omega$ for some constant λ. It is respectively expanding, steady, or shrinking if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$.*

Proposition 2.15. *Suppose that ω is a pluriclosed metric on a compact complex surface (M^4, J) such that $-\rho^{1,1} = \lambda\omega$. Then*

$$\int_M \lambda|\theta|^2 + \frac{1}{2}|d\theta|^2 dV = 0.$$

In particular,

- *If $\lambda > 0$ we must have $\theta = 0$ and so ω is Kähler-Einstein.*

- *If $\lambda = 0$ then M is locally conformally Kähler and Bismut-Ricci flat, and is therefore Vaisman, splits locally as a product of a positively curved space form with a line, and must therefore be a diagonal Hopf surface, a quotient of $S^3 \times S^1$ with a product metric.*

Proof. With $-\rho^{1,1} = \lambda\omega$ we can apply the previous propositions to conclude

$$\frac{1}{2}\Delta_d \theta = \lambda\theta.$$

Integrating this with θ gives us the desired identity. If $\lambda > 0$ then we must have $\theta = 0$, while if $\lambda = 0$ we conclude that $d\theta = 0$ and by the previous propositions (M^4, g, J) is a Vaisman manifold with Ricci curvature

$$Rc = \frac{1}{2}(|\theta|^2 g - \theta \otimes \theta).$$

Note that $|\theta|^2 \neq 0$ is a constant over M. If V, W are parallel to $\theta^\sharp$ then $Rc(V, W) = 0$. If they are perpendicular to $\theta^\sharp$ we have $Rc(V, W) = \frac{|\theta|^2}{2} g(V, W)$ so the Ricci curvature is a positive constant in these directions. We conclude that $g = \frac{\theta}{|\theta|} \otimes \frac{\theta}{|\theta|} + \frac{2}{|\theta|^2} Rc$ splits isometrically as a product of a positively curved three dimensional space form. Therefore M must be a Hopf surface.

□

2.6 Speculative Remarks

We have seen the long time behavior of pluriclosed flow on a wide variety non-Kähler complex surfaces and have observed that information of the complex structure of such a surface is contained in the asymptotic behavior of the flow. This is a small step toward using the flow to study all non-Kähler complex surfaces, and the philosophy is that one would like to take a result which holds for the (Kähler-)Ricci flow and prove an analogous result in this pluriclosed setting. We mention two directions of future work which would build off the results in this chapter.

One might be interested in the behavior of the family of flows

$$\frac{d\omega}{dt} = -(\rho_\omega^\tau)^{(1,1)},$$

for different values of τ. Here ρ_ω^τ is the Ricci-form associated to the connection ∇^τ in the canonical family of Gauduchon [6] mentioned briefly before. The case $\tau = -1$ gives the Bismut connection and corresponds to the pluriclosed flow considered in this chapter. The case $\tau = 1$ gives the Chern connection and the flow corresponds to the Chern-Ricci flow considered by Tosatti and Weinkove and, similar to what we have done here, homogeneous and soliton solutions to Chern-Ricci flow on Lie groups have been studied in [13]. We are interested in the bifurcation theory for this family of flows on homogeneous complex surfaces. For example, the Chern-Ricci flow of a homogeneous metric on the Hopf surface must have a finite time singularity, but we have seen that pluriclosed flow always converges to a canonical metric in this case. Additionally, as shown in [13], any left invariant Hermitian structure on a nilpotent Lie group is fixed under Chern-Ricci flow, while we have seen non-trivial solutions to the pluriclosed flow on $Nil \times \mathbb{R}$. Therefore there are values of τ which induce qualitative changes in the behavior of solutions to this family of flows. The corresponding connections ∇^τ may be canonical for the complex surface in some sense.

We have given a self contained proof which classifies the fixed points of pluriclosed flow on complex surfaces and have shown how to construct a large class of non-compact fixed points. The proof of these facts involves the identity $2L^*d\rho^{1,1} = -\Delta_d\theta$ and exploits the conformal relationship between the Bismut and Chern-Ricci forms. The next step in this direction is to classify and construct non-trivial soliton solutions to pluriclosed flow. This is equivalent to finding pluriclosed metrics where

$$-\rho^{1,1} = \lambda\omega + \mathcal{L}_V\omega$$

for some non-zero holomorphic vector field V. The work in [25] shows that any compact expanding solitons, where $\lambda > 0$, are Kähler-Einstein. In contrast to this result we constructed non-compact, non-Kähler expanding solitons earlier in this chapter. Constructing non-Kähler steady and shrinking pluriclosed solitons is still an open problem and the identities we obtained in this chapter for classifying the fixed points of the flow were initially sought after in an attempt to construct such solitons and study their relationship to locally conformally Kähler, Vaisman, and Sasakian geometry.

Chapter 3

$\bar{\partial}$-Harmonic Maps

Given two Riemannian manifolds (M, g) and (N, h), a map $f : M \to N$ is said to be harmonic if it is a critical point of the energy functional

$$E(f) = \frac{1}{2}\int_M |Df|^2 dV$$

with respect to compactly supported test variations of f, where $|Df|^2$ is the norm squared of the derivative $Df : TM \to TN$ as a section of $T^*M \otimes f^{-1}TN$. The Euler-Lagrange equation associated to E is called the tension of f and has the local coordinate expression

$$(tr_g \nabla Df)^i = \tau^i(f) = g^{\alpha\beta}(f^i_{\alpha\beta} - f^i_\gamma \Gamma^\gamma_{\alpha\beta} + f^j_\alpha f^k_\beta \Gamma^i_{jk}).$$

Here we use the convention that coordinates on M are denoted with the Greek indices α, β, ... and so on while Roman indices i, j, ... will denote coordinates on the target N. Harmonic maps are well studied objects in geometric analysis, with many results obtained toward their existence, regularity, and compactness; see the survey in the book [15]. We remind the reader specifically of the celebrated existence result of Eells and Sampson.

Theorem 3.1. *Suppose that (M, g) and (N, h) are closed Riemannian manifolds where the sectional curvatures of h are non-positive. Then any smooth $f_0 : M \to N$ is homotopic to a harmonic map.*

This result, proved with gradient descent methods, has inspired many of the heat flows which are topics of current research in differential geometry. In this chapter, inspired by the result of Eells-Sampson, we study a functional of maps between almost Hermitian manifolds and its associated parabolic flow.

Let $f : M \to N$ be a differentiable map between the almost Hermitian manifolds (M, J_M, g) and (N, J_N, h), and let its derivative be $Df : TM \to TN$. In the case where the complex structures are integrable, f is holomorphic in local complex coordinates if, and only if, $J_N Df = Df J_M$. In situations where complex coordinates do not exist, a map satisfying $J_N Df = Df J_M$ is said to be complex or pseudo-holomorphic. Note that the derivative $Df : TM \to TN$ has an orthogonal decomposition

$$Df = \frac{1}{2}(Df + J_N Df J_M) + \frac{1}{2}(Df - J_N Df J_M)$$

where the first component of this decomposition vanishes if, and only if, f is pseudo-holomorphic.

Holomorphic maps are important objects in complex and symplectic geometry, as their moduli often contain information on the global structure of the source and target manifolds. For example, Gromov invariants count families of holomorphic maps from curves into a fixed symplectic manifold [7], while the classification of compact complex surfaces with $b_1 = 1$ would be complete if certain homology classes in these surfaces had holomorphic representatives [18].

Counting such curves motivates the search for holomorphic maps in a given homotopy class, and a perhaps naive attempt towards finding these maps is to apply gradient descent methods to functionals which are small when evaluated at holomorphic maps. For this reason, it is natural to define the pseudo-holomorphic energy of a map $f : (M, g, J_M) \to (N, h, J_N)$ to be

$$E_+(f) = \frac{1}{4}\int_M |Df + J_N Df J_M|^2 dV_g.$$

This energy was first studied by Lichnerowicz in [14], where it is called $E''(f)$. In that paper the relationship between E_+ and the harmonic map energy E is studied for maps between almost Kähler manifolds.

The first result of this section is to give explicit formulas for the first and second variations of E_+, exact expressions of which are not found in the literature, and note that incorrect forms of this variation have appeared in various preprints.

Proposition 3.1. *Let $f : M \times (-\epsilon, \epsilon) \to N$ be a smooth one parameter family of maps between the almost Hermitian manifolds (M, g, J_M) and (N, h, J_N) with compactly supported variation field $\partial_t f = v$. Then*

$$\frac{d}{dt}E_+(f) = -\int_M \langle \tau + A, v\rangle dV,$$

where τ is the tension of f and $2A = \langle d^*\omega_M, f^\times \omega_N\rangle + \langle \omega_M, f^\times d\omega_N\rangle$, which has the local coordinate expression

$$2A^i = (d^*\omega)_\alpha f^j_\beta \omega_{lj} g^{\alpha\beta} h^{li} + \omega_{\alpha\beta} f^j_\gamma f^k_\delta (d\omega)_{ljk} g^{\alpha\gamma} g^{\beta\delta} h^{li},$$

where $\omega_M(X, Y) = g(JX, Y)$.

Maps f satisfying $\tau_+(f) = \tau(f) + A(f) = 0$ will be called $\bar{\partial}$-harmonic. We pause to make some small observations about this expression. First, one immediately gets the result of Lichnerowicz in [14] on the equivalence between E_+ and E critical maps if the source is balanced and the target is almost Kähler; we will give explicit, non-Kähler examples which show that E_+ critical maps are distinct from E critical maps in general. The second observation is that, independent of any integrability assumptions on source or target, the Euler-Lagrange equation of E_+ is elliptic and semi-linear. We then compute the second variation of E_+ at a C^2 critical point.

Proposition 3.2. *Let $f : M \to N$ be a C^2 critical point of E_+ with respect to all compactly supported variations $\partial_t f = v$, so that $\tau_+(f) = \tau(f) + A(f) = 0$. Then the second variation of E_+ at f is given by*

$$\frac{d^2}{dt^2}|_{t=0} E_+(f) = \int_M \langle Lv, v\rangle dV,$$

where L is the operator on sections of $f^{-1}TN$ given by

$$\begin{aligned}-Lv = \Delta v + tr_g R(v, Df)Df + \frac{1}{2}\{&\langle d^*\omega_M, \omega_N(\cdot, \nabla v)\rangle^\sharp \\ &+ \langle d^*\omega_M, (\nabla_v \omega_N)(\cdot, Df)\rangle^\sharp + \langle \omega_M, (d\omega_N)(\cdot, \nabla v, Df)\rangle^\sharp \\ &+ \langle \omega_M, (d\omega_N)(\cdot, Df, \nabla v)\rangle^\sharp \\ &+ \langle \omega_M, (\nabla_v d\omega_N)(\cdot, Df, Df)\rangle^\sharp\}.\end{aligned}$$

A simple corollary of the previous proposition is:

Corollary 3.1. *Suppose that M is closed and $f : M \to N$ is pseudo-holomorphic. Then the operator $L : C^\infty(f^{-1}TN) \to C^\infty(f^{-1}TN)$ in the previous proposition is non-negative. In particular, if (M, g, J) is a closed almost Hermitian manifold the operator $L : C^\infty(TM) \to$*

$C^\infty(TM)$, given by

$$-Lv = \Delta v + Rc(v) + \frac{1}{2}\{\langle d^*\omega, \omega(\cdot, \nabla v) + (\nabla_v \omega)(\cdot, id)\rangle^\sharp$$
$$+ \langle \omega, d\omega(\cdot, \nabla v, id)\rangle^\sharp + \langle \omega, d\omega(\cdot, id, \nabla v)\rangle^\sharp$$
$$+ \langle \omega, (\nabla_v d\omega)(\cdot, id, id)\rangle^\sharp\},$$

is non-negative. Moreover, if $v \in C^\infty(TM)$ is a pseudo-holomorphic vector field then $Lv = 0$.

The non-negativity of L and its relationship to holomorphic vector fields is known in the Kähler setting. For example, by applying the non-negativity of L to the gradient of an eigenfunction of the Laplacian, a version of the Obata-Lichnerowicz theorem for Kähler manifolds is proved in [28]. We can then give a similar lowest eigenvalue bound in the almost-Kähler setting.

Corollary 3.2. *If (M, g, J) is a compact, almost Kähler manifold with Ricci curvature $Rc \geq \alpha > 0$ then the first eigenvalue of the Laplacian satsfies $\lambda_1 \geq 2\alpha$.*

Our next proposition concerns the well posedness of the gradient flow associated to E_+ and shows that it has the same obstructions to long time existence as the harmonic map heat flow.

Proposition 3.3. *Let $f_0 : M \to N$ be a smooth map between the closed almost Hermitian manifolds (M, g, J_M) and (N, h, J_N). Then there is a maximal T such that the initial value problem*

$$\partial_t f = \tau_+(f)$$
$$f|_{t=0} = f_0$$

has a unique smooth solution on $M \times [0,T)$. Moreover, if $T < \infty$ then

$$\lim_{t \to T} |Df|_{C^0} = \infty.$$

Equipped with this proposition, we prove the following long time existence theorem for solutions to the $\bar{\partial}$-harmonic map heat flow with negatively curved, almost Kähler targets.

Theorem 3.2. *Let $f_0 : M \to N$ be a smooth map between the closed almost Hermitian manifolds (M, g, J_M) and (N, h, J_N). Suppose that ω_h is almost Kähler and the sectional curvature of h is negative. Then the solution to the $\bar{\partial}$-harmonic map heat flow with initial condition f_0 has a unique smooth solution on $M \times [0, \infty)$.*

In some cases, for example when J_N is integrable, the negative curvature assumption in this result can be weakened to non-positive curvature. We will also discuss the difficulties in improving this result to the general case where $d\omega_N \neq 0$; stronger conditions on the curvature in relation to the complex structure are needed for our proof to go through without substantial modification.

Convergence of the harmonic map heat flow at infinite time requires a parabolic Harnack inequality to prove a bound on $|Df|^2$ on all of $M \times [0, \infty)$ given a uniform bound on $|Df|_{L^2}$. In the case of the harmonic map energy this bound is free because the flow is precisely given by following the negative L^2-gradient of E. In the context of the previous theorem, we do not have such a bound because the pseudo-holomorphic energy E_+ is in general non-coercive; we do not expect the energy to be bounded at infinite time. In the presence of such a bound, convergence to a $\bar{\partial}$-harmonic map at infinite time follows. We attempt to get around this difficulty by considering a family of energies E_a which are coercive if $|a| < 1$ and which contain $E_+ = E_1$ as a limiting case. This family is given by linear interpolation $E_a = (1-a)E + aE_+$ between the harmonic map energy $E_0 = E$ and the pseudo-holomorphic energy $E_1 = E_+$. Because $(1 - |a|)E \leq E_a$, we obtain the following existence result for these modified functionals.

Theorem 3.3. *Let $f : M \to N$ be a smooth map between the closed, almost Hermitian manifolds (M, g, J_M) and (N, h, J_N). Suppose that the sectional curvatures of N are negative and that ω_h is almost Kähler. Then for all $|a| < 1$ the parabolic flow corresponding to the functional E_a, beginning with f, exists for all time and has subsequential convergence to a critical point of E_a.*

An additional case of interest is when $M = \Sigma$ is a compact Riemann surface and N is an arbitrary compact almost Hermitian manifold, not necessarily almost Kähler. This is the critical dimension for the functionals we consider, and just like the harmonic map energy, E_+ is conformally invariant in this case. Understanding $\bar{\partial}$-harmonic maps in this case is particularly important given the use of J-holomorphic curves in almost Hermitian geometry. We consider examples of the flow in this case as well as prove the existence of $\bar{\partial}$-harmonic bubbles at a finite time singularity.

Specifically, we will work through an interesting example of the flow restricted to a family of harmonic tori $f : T^2 \to S^3 \times S^1$ inside a Hopf surface. This family is parameterized by orthonormal pairs in $\mathbb{R}^4$ and we show that the flow both preserves this family, restricting to an ODE, and converges to a holomorphic or anti-holomorphic map at infinite time.

3.1 Background and Variation Computations

In this section we will give a brief overview of some calculations useful in the context of harmonic maps, establish the notation which is used throughout this section, and derive the first and second variations of the anti-holomorphic energy E^+.

In what follows (M, g, J_M) and (N, h, J_N) are almost Hermitian manifolds, assumed complete and without boundary. When writing some object in local coordinates we reserve Greek indices $\alpha, \beta, \ldots$ for coordinates on M and Roman indices $i, j, \ldots$ for coordinates on N. We will also use the standard summation convention unless stated otherwise, and will often abbreviate coordinate derivatives of functions with subscripts: $\partial_\alpha u = u_\alpha$.

Let $f : M \times I \to N$ be a smooth one-parameter family of maps defined on some open interval I. For each $t \in I$, let $Df : M \to N$ be the derivative of the map $f(\cdot, t)$, viewed as a section of $T^*M \otimes f(\cdot, t)^{-1}TN$, with $f^{-1}E$ denoting the pullback bundle. In local coordinates we have $Df = f^i_\alpha \partial_i \otimes d^\alpha$. Note that Df can also be viewed as a section of $T^*(M \times I) \otimes f^{-1}TN$ by pre-composing the (full) derivative of the family $f_* : T(M \times I) \to TN$ with the canonical projection to TM.

The manifolds M, $M \times I$, and N have Levi-Civita connections which induce connections on the various tensor and pullback bundles that we will consider. The symbol ∇ will denote the full covariant derivative operator with respect to spacial variables only, meaning if s is a section of some bundle over $M \times I$ we pre-compose the full covariant derivative of s with the projection to TM, so that $(\nabla s)^i_\alpha = s^i_\alpha + s^j f^k_\alpha \Gamma^i_{kj}$ in coordinates. We reserve ∇_t for the covariant derivative in the direction of ∂_t. Let $v = f^i_t \partial_i$ denote the variational vector field of f, which in various notations can be written

$$v = \frac{\partial f}{\partial t} = f_*(\partial_t).$$

By unwinding the notation we have:

Lemma 3.1. *Let f be a smooth one-parameter family of maps with variational vector field v. Then*

$$\nabla_t Df = \nabla v.$$

We next recall the derivation of the tension tensor τ as the Euler-Lagrange equation of the energy functional. Given a map $f : M \to N$, let its energy density be $e(f) = \frac{1}{2}|Df|^2$, where in coordinates

$$|Df|^2 = g^{\alpha\beta} f^i_\alpha f^j_\beta h_{ij}.$$

The energy of f is then the integral $E(f) = \int_M e(f) dV_g$ of the energy density.

Proposition 3.4. *Let f be a smooth one parameter family of maps whose variational vector field v has compact support. Then*

$$\frac{d}{dt}E(f) = \int_M \langle \nabla v, Df \rangle dV_g = -\int_M \langle v, \tau \rangle dV_g,$$

where $\tau = tr_g \nabla Df$ is the tension tensor of f, given in local coordinates by

$$\tau^i = g^{\alpha\beta}(f^i_{\alpha\beta} - f^i_\gamma \Gamma^\gamma_{\alpha\beta} + f^j_\alpha f^k_\beta \Gamma^i_{jk}).$$

Proof. The first equality is immediate from the previous lemma by differentiating under the integral, while the second follows from the divergence theorem applied to the vector field $X = \langle v, Df \rangle^\sharp$. □

The next proposition concerns the second variation of the energy.

Proposition 3.5. *Let f be a smooth one parameter family of maps whose variational vector field v has compact support. Then*

$$\frac{d^2}{dt^2}E(f) = -\int_M \langle \nabla_t v, \tau \rangle + \langle v, \Delta v + tr_g R^N(v, Df)Df \rangle dV_g,$$

where $\Delta v = tr_g \nabla^2 v$.

Proof. The variation of τ is given by

$$\begin{aligned}\nabla_t \tau = \nabla_t tr_g \nabla Df &= tr_g \nabla_t \nabla Df \\ &= tr_g \nabla \nabla_t Df + tr_g R^N(v, Df)Df \\ &= tr_g \nabla^2 v + tr_g R^N(v, Df)Df,\end{aligned}$$

where $R^N(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ is the curvature tensor of N. The result follows from differentiation under the integral of the previous proposition, noting that $\partial_t \langle v, \tau \rangle = \langle \nabla_t v, \tau \rangle + \langle v, \nabla_t \tau \rangle$. $\square$

Next we look at what can be done when the (almost) complex structures are taken into account. We first focus on the orthogonal decomposition of Df and the norm of its various pieces.

Lemma 3.2. *Let $f : M \to N$ be a differentiable map between the almost Hermitian manifolds (M, g, J_M) and (N, h, J_N). Then the derivative has an orthogonal decomposition*

$$Df = \frac{1}{2}(Df + J_N Df J_M) + \frac{1}{2}(Df - J_N Df J_M)$$

Proof. We compute

$$\langle Df + J_N Df J_M, Df - J_N Df J_M \rangle = |Df|^2 - |J_N Df J_M|^2 = |Df|^2 - |Df|^2 = 0.$$

$\square$

Lemma 3.3. *With the same assumptions of the previous lemma, we have*

$$\frac{1}{4}|Df + J_N Df J_M|^2 = \frac{1}{2}|Df|^2 + \frac{1}{2}\langle Df, J_N Df J_M \rangle$$

and

$$\langle Df, J_N Df J_M \rangle = -\langle \omega_M, f^*\omega_N \rangle,$$

where the inner product on two-forms has the normalization $\langle a, b\rangle = a_{\alpha\beta} b_{\gamma\delta} g^{\alpha\gamma} g^{\beta\delta}$, *i.e. is the inner product as real tensors. In particular, if* M *is a Riemann surface we have*

$$-\frac{1}{2}\langle \omega_M, f^*\omega_N \rangle dV_g = -f^*\omega_N$$

Proof. The first equality comes from expanding the inner product, while the second is done in coordinates:

$$\begin{aligned} \langle Df, J_N Df J_M \rangle = f^i_\alpha J^k_j f^j_\beta J^\beta_\gamma g^{\alpha\gamma} h_{ik} &= -J^\beta_\gamma g^{\alpha\gamma} (f^*\omega_N)_{\alpha\beta} \\ &= -(\omega_M)_{\gamma\delta} g^{\gamma\alpha} g^{\delta\beta} (f^*\omega_N)_{\alpha\beta}. \end{aligned}$$

The proof is completed by noting that on a Riemann surface $|\omega_M|^2 = 2$ and $dV_g = \omega_M$, so that $\frac{1}{\sqrt{2}}\omega_M$ is an orthonormal basis of Ω^2 at each point and $\langle \omega_M, f^*\omega_N \rangle dV_g = 2f^*\omega_N$. $\square$

The remark about the normalization of the inner product of forms is necessary because it is not the convention used in the rest of geometry. For example, if M is $2m$ dimensional we have $|\omega_M|^2 = 2m$, while other authors would say $|\omega_M|^2 = m$. We note that the volume form is still given by $dV_g = \frac{1}{m!}\omega_M^m$ and that our normalization does not change the adjoint d^* of the exterior derivative.

Our next proposition concerns how the pullback $f^*\omega$ of a two-form varies with a variation of f and is necessary for computing the first variation of E^+. If f is a one parameter group of diffeomorphisms of M, the following is nothing more than the familiar Cartan formula $\mathcal{L}_X\omega = d\iota_X\omega + \iota_X d\omega$, but for arbitrary smooth families of maps $f : M \to N$ the generalization we give is perhaps unknown to the reader.

Proposition 3.6. *Let $f : M \times I \to N$ be a smooth one parameter family of maps with $\partial_t f = v$. Fix a differential two-form ω on N, and let $f^*\omega$ be the pullback of ω with respect to the map $f(t) : M \to N$. Then as a one parameter family of forms on M we have*

$$\frac{d}{dt} f^*\omega = df^*\iota_v\omega + f^*\iota_v d\omega.$$

Proof. We compute directly in coordinates

$$(\frac{d}{dt} f^*\omega)_{\alpha\beta} = v^i_\alpha f^j_\beta \omega_{ij} + f^i_\alpha v^j_\beta \omega_{ij} + f^i_\alpha f^j_\beta \omega_{ij,k} v^k.$$

On one hand we have

$$(d\omega)_{ijk} = \omega_{ij,k} + \omega_{ki,j} + \omega_{jk,i},$$

$$(\iota_v d\omega)_{ij} = v^k(\omega_{ij,k} + \omega_{ki,j} + \omega_{jk,i}),$$

and finally

$$(f^*\iota_v d\omega)_{\alpha\beta} = f^i_\alpha f^j_\beta v^k(\omega_{ij,k} + \omega_{ki,j} + \omega_{jk,i}).$$

While on the other hand,

$$(\iota_v\omega)_i = v^j\omega_{ji},$$

$$(f^*\iota_v\omega)_\alpha = f^i_\alpha v^j\omega_{ji},$$

and

$$(df^*\iota_v\omega)_{\alpha\beta} = f^j_\beta v^i_\alpha \omega_{ij} - f^j_\beta v^k \omega_{jk,i} f^i_\alpha + f^i_\alpha v^j_\beta \omega_{ij} - f^i_\alpha v^k \omega_{ki,j} f^j_\beta.$$

Adding these gives the result. $\square$

Now, given a smooth of map $f : M \to N$ between almost Hermitian manifolds, the anti-holomorphic energy E_+ and holomorphic energy E_- of f decompose

$$E_\pm(f) = \frac{1}{4}\int_M |Df \pm J_N Df J_M|^2 dV_g = E(f) \pm K(f)$$

into a sum of the standard energy $E(f) = \frac{1}{2}\int_M |Df|^2 dV_g$ and an additional term

$$K(f) = -\frac{1}{2}\int_M \langle \omega_M, f^*\omega_N\rangle dV_g.$$

Some obvious relations between these functionals are:

$$E = \frac{1}{2}(E_+ + E_-)$$
$$K = \frac{1}{2}(E_+ - E_-).$$

These belong to a family E_a of energies which will be considered in a later section. These are given by

$$E_a = aE_+ + (1-a)E = E + aK.$$

We next give the first variation of K. Remarkably, its Euler-Lagrange equation depends only on first derivatives of f.

Proposition 3.7. *Let $K = K(f) = -\frac{1}{2}\int_M \langle \omega_M, f^*\omega_N\rangle dV_g$ be the difference $E_+ - E$ between the Dirichlet energy $E(f) = \frac{1}{2}\int_M |Df|^2 dV_g$ and the anti-holomorphic energy $E_+(f) =$*

$\frac{1}{4}\int_M |Df + J_N Df J_M|^2 dV_g$. *Let* $v = \partial_t f$ *be a variation of* f *with compact support. Then*

$$\frac{d}{dt}K(f) = -\int_M \langle v, A\rangle dV_g$$

where A *is given by*

$$2A = \langle d^*\omega_M, f^\times \omega_N\rangle^\sharp + \langle \omega_M, f^\times d\omega_N\rangle^\sharp,$$

$f^\times\omega$ *denotes the pullback of a form* ω *on all indices except for the first, and* $\sharp : T^* \to T$ *is the musical isomorphism given by the metric.*

Proof. Given the generalized Cartan formula of the previous proposition we differentiate under the integral to obtain:

$$\begin{aligned}\frac{d}{dt}K(f) &= -\frac{1}{2}\int_M \langle \omega_M, df^*\iota_v\omega_N + f^*\iota_v d\omega_N\rangle dV_g \\ &= -\frac{1}{2}\int_M \langle d^*\omega_M, f^*\iota_v\omega_N\rangle + \langle \omega_M, f^*\iota_v d\omega_N\rangle dV_g.\end{aligned}$$

□

As a corollary we obtain the aforementioned result of Lichnerowicz in [14] for harmonic maps between almost Kähler manifolds.

Corollary 3.3. *If* $d\omega_N = 0$ *and* $d^*\omega_M = 0$ *then* K *is a smooth homotopy invariant of* f*. Therefore, under these assumptions, the critical points of* E_+ *coincide with the critical points of* E*, i.e. harmonic maps. In particular, a holomorphic map between closed almost Kähler manifolds is harmonic and minimizes the energy in its homotopy class.*

We are also equipped to give a complete proof of the first variation formula for E_+.

Proof. Noting that $E^+ = E + K$, we combine the first variation of E with the first variation of K to get the result. $\square$

We next compute the second variation of E_+ and prove Proposition 3.2 and Corollaries 3.1 and 3.2.

Proof. From the first variation we have

$$\frac{\partial}{\partial t}E_+ = -\int_M \langle v, \tau_+\rangle dV_g.$$

We then compute

$$\begin{aligned}\nabla_t\tau_+ = \nabla_t\tau + \nabla_t A &= \nabla_t tr_g \nabla Df + \nabla_t A \\ &= tr_g\nabla\nabla_t Df + tr_g R(v, Df)Df + \nabla_t A \\ &= \Delta v + tr_g R(v, Df)Df + \nabla_t A,\end{aligned}$$

therefore

$$\frac{\partial^2}{\partial t^2}|_{t=0}E_+ = -\int_M \langle v, \Delta v + tr_g R(v, Df)Df + \nabla_t A\rangle dV_g.$$

Finally,

$$\begin{aligned}\nabla_t A = \frac{1}{2}\{&\langle d^*\omega_M, \omega_N(\cdot, \nabla v)\rangle^\sharp \\ &+ \langle d^*\omega_M, (\nabla_v\omega_N)(\cdot, Df)\rangle^\sharp + \langle \omega_M, (d\omega_N)(\cdot, \nabla v, Df)\rangle^\sharp \\ &+ \langle \omega_M, (d\omega_N)(\cdot, Df, \nabla v)\rangle^\sharp \\ &+ \langle \omega_M, (\nabla_v d\omega_N)(\cdot, Df, Df)\rangle^\sharp\}\end{aligned}$$

accounts for the remaining terms. $\square$

We clarify what is meant by the various inner products in the $\nabla_t A$ term of the previous proposition by expressing some of them in local coordinates. For example,

$$\langle \omega_M, (\nabla_v d\omega_N)(\cdot, Df, Df)\rangle_i = \omega_{\alpha\beta}(\nabla_v d\omega_N)_{ijk} f^j_\gamma f^k_\delta g^{\alpha\gamma} g^{\beta\delta}$$

and

$$\langle \omega_M, (d\omega_N)(\cdot, Df, \nabla v)\rangle_i = \omega_{\alpha\beta} f^j_\gamma f^l_\delta (d\omega)_{ijk} g^{\alpha\gamma} g^{\beta\delta} (\nabla_l v)^k.$$

Proof of Corollary 3.1. Consider maps $f : M \to N$. Note that $E_+(f) \geq 0$ and $E_+(f) = 0$ if, and only if, f is (pseudo) holomorphic. Therefore holomorphic maps are stable critical points of E_+ as they are global minimizers of this functional.

Now, the identity map $id : M \to M$ is always holomorphic and is therefore a stable E_+ critical point, hence the operator L is non-negative. If v is a holomorphic vector field then it generates a one parameter family of holomorphic maps beginning at the identity and so $Lv = 0$. □

Proof of Corollary 3.2. Let $f : M \to \mathbb{R}$ satisfying $\Delta f = -\lambda f$ be an eigenfunction. Recall the Bochner formula $\Delta \nabla f = \nabla \Delta f + Rc(\nabla f)$. If (M, g, J) is almost Kähler then by the previous corollary the operator $L = -\Delta - Rc$ on smooth vector fields is non-negative. We then compute

$$L\nabla f = -\Delta \nabla f - Rc(\nabla f) = -\nabla \Delta f - 2Rc(\nabla f) = \lambda \nabla f - 2Rc(\nabla f).$$

Thus, since $Rc \geq \alpha$, we have

$$0 \leq \int_M \langle L\nabla f, \nabla f\rangle dV \leq (\lambda - 2\alpha) \int_M |\nabla f|^2 dV.$$

$\square$

3.2 The Gradient Flow of E_+

In this section we will study the negative gradient flow corresponding to E_+. Consider the initial value problem

$$\begin{aligned} \frac{\partial}{\partial t} f &= \tau_+(f) \\ f|_{t=0} &= f_0. \end{aligned}$$

A solution to this problem is said to solve the $\bar{\partial}$-harmonic map heat flow with initial condition f_0. We've seen that $\tau_+ = \tau + A$, while A consists of two terms, one linear and one quadratic in Df. Therefore the linearized operator of τ_+ has the same principal symbol as that of τ. Applying a dose of semi-linear parabolic existence and regularity theory we therefore obtain

Proposition 3.8. *Let M be a closed, smooth, almost Hermitian manifold and let $f_0 : M \to N$ be a smooth map from M to the smooth, almost Hermitian manifold N. Then there is a $T > 0$ such that the initial value problem*

$$\begin{aligned} \frac{\partial}{\partial t} f &= \tau_+(f) \\ f|_{t=0} &= f_0 \end{aligned}$$

has a unique smooth solution on $[0, T) \times M$.

We next obtain some basic apriori estimates for the $\bar{\partial}$-harmonic map heat flow which will give sufficient conditions to conclude long time existence.

Proposition 3.9. *Let f be a solution to the τ_+-flow. Then*

$$\nabla_t \tau_+ = \Delta \tau_+ + tr_g R(\tau_+, Df)Df + \nabla_t A,$$

and

$$\nabla_t Df = \Delta Df - Q + \nabla A.$$

Therefore

$$(\partial_t - \Delta)\frac{1}{2}|\tau_+|^2 = -|\nabla \tau_+|^2 + \langle tr_g R^N(\tau_+, Df)Df, \tau_+\rangle + \langle \nabla_t A, \tau_+\rangle,$$

and

$$(\partial_t - \Delta)\frac{1}{2}|Df|^2 = -|\nabla Df|^2 - \langle Q, Df\rangle + \langle \nabla A, Df\rangle,$$

Proof. We compute

$$\begin{aligned}
\nabla_t \tau_+ &= \nabla_t \tau + \nabla_t A \\
&= tr_g \nabla_t \nabla Df + \nabla_t A \\
&= tr_g \nabla^2 \tau_+ + tr_g R^N(\tau_+, Df)Df + \nabla_t A \\
&= \Delta \tau_+ + tr_g R^N(\tau_+, Df)Df + \nabla_t A,
\end{aligned}$$

using the commutator formula $\nabla_t \nabla - \nabla \nabla_t = R^N(\dot{f}, Df)$. This proves the first claim. The second claim follows from a similar calculation

$$\begin{aligned}
\nabla_t Df = \nabla \tau_+ &= \nabla \tau + \nabla A \\
&= \Delta Df - Q + \nabla A,
\end{aligned}$$

where

$$Q(\cdot) = -R^N(Df\cdot, Df_\alpha)Df_\alpha + Df(Rc^M\cdot)$$

and we have used the Bochner formula

$$\Delta Df = \nabla\tau + Q.$$

The final claims follow from

$$\begin{aligned}
\partial_t \frac{1}{2}|\tau_+|^2 &= \langle \nabla_t \tau_+, \tau_+ \rangle \\
&= \langle \Delta\tau_+ + tr_g R^N(\tau_+, Df)Df + \nabla_t A, \tau_+ \rangle \\
&= \Delta \frac{1}{2}|\tau_+|^2 - |\nabla\tau_+|^2 + \langle tr_g R^N(\tau_+, Df)Df + \nabla_t A, \tau_+ \rangle,
\end{aligned}$$

with a similar calculation for $\partial_t \frac{1}{2}|Df|^2$. □

Proposition 3.10. *If the solution to the $\bar{\partial}$-harmonic heat flow between two compact, almost Hermitian manifolds exists on a maximal time interval $[0, T)$, with $T < \infty$, then* $\limsup_{t\uparrow T} |Df|_\infty = \infty$.

Proof. This is a standard obstruction in semi-linear parabolic systems. If there is some positive constant C such that $|Df| < C$ on $[0, T)$ then we conclude convergence to a smooth map at time T. Using this map as the new condition for the flow, we extend the solution smoothly past T and contradict maximality. □

We can now prove a long time existence result for $\bar{\partial}$-harmonic map heat flow, a corollary of which is Theorem 1.5.

Proposition 3.11. *Suppose that M is compact and the target N is almost Kahler with sectional curvature bounded above by a negative constant and with bounded Nijenhaus tensor.*

Then the $\bar{\partial}$-harmonic map heat flow beginning at any smooth $f_0 : M \to N$ exists smoothly for all time.

Proof. Let f be the corresponding solution to the flow. It amounts to proving that $|Df|$ is bounded on any interval of the form $[0, T)$ where $T < \infty$. Recall from 3.9 that

$$(\partial_t - \Delta)\frac{1}{2}|Df|^2 = -|\nabla Df|^2 - \langle Q, Df\rangle + \langle \nabla A, Df\rangle.$$

Since the target is almost Kähler we have $2A^\flat = \langle d^*\omega_M, f^\times \omega_N\rangle$. We then compute $\langle \nabla A, Df\rangle$, which has the form

$$2\langle \nabla A, Df\rangle = \nabla d^*\omega_M * Df \wedge Df + d^*\omega_M * \nabla Df \wedge Df + d^*\omega_M * \nabla^h \omega_N * (Df \wedge Df) * Df,$$

where we have used schematic notation. To clarify, if A and B are some tensors then $A * B$ denotes a tensor constructed from taking contractions, either with the metrics g, h or the forms ω_M, ω_N, of $A \otimes B$ and $B \otimes A$. We have also used the notation that, if A and B are in $T^*M \otimes E$ for some vector bundle E then $A \wedge B(X, Y) = A(X) \wedge B(Y)$. In particular, since M is compact and $\nabla^h \omega_N$ is bounded, there is some constant C_1 independent of f such that

$$\langle \nabla A, Df\rangle \leq C_1(|Df|^2 + |\nabla Df||Df| + |Df \wedge Df||Df|).$$

Now, if the sectional curvatures of N are bounded above by K and the Ricci curvature of M is bounded below by R, then the curvature term $\langle Q, Df\rangle$ satisfies

$$-\langle Q, Df\rangle \leq K|Df \wedge Df|^2 - R|Df|^2.$$

An application of Young's inequality $ab \leq \frac{1}{2}(\epsilon a^2 + \frac{1}{\epsilon}b^2)$ then gives

$$-|\nabla Df|^2 - \langle Q, Df\rangle + \langle \nabla A, Df\rangle \leq C_1(1 + \frac{1}{2\epsilon_1} + \frac{1}{2\epsilon_2})|Df|^2 - R|Df|^2 + (\frac{C_1\epsilon_1}{2} - 1)|\nabla Df|^2 + (\frac{C_1\epsilon_2}{2} + K)|Df \wedge Df|^2.$$

Because K is negative, by choosing ϵ_1 and ϵ_2 small enough the last two terms of this expression are negative. Therefore there is some constant C such that

$$(\partial_t - \Delta)\,|Df|^2 \leq C|Df|^2,$$

and so $|Df|$ is bounded on any finite length time interval by the maximum principle. $\square$

We note that if the complex structure of N is integrable, so that N is genuinely Kähler, then there is no $\nabla^h \omega_N$ term in the above estimates. This would mean we can weaken the negative sectional curvature assumption to just non-positive curvature. With a more careful analysis several of the assumptions in this theorem can be weakened using standard methods; certainly regularity in f_0 or compactness/boundedness assumptions can be weakened.

A quick look at the form of $\langle \nabla A, Df\rangle$ should indicate to the reader why obtaining a long time existence result in the general non-Kähler case using a basic maximum principle type argument as above is a little delicate. Without the almost Kähler assumption on N there are three additional terms which have the schematic form

$$\nabla\omega_M * d\omega_N * Df \wedge Df \wedge Df + d\omega_N * \nabla Df \wedge Df \wedge Df + \nabla^h d\omega_N * Df \wedge Df \wedge Df * Df,$$

each of which gives "bad ODE terms" when trying to apply parabolic maximum principles. It is not unreasonable to think that there are some curvature conditions on N of the form $Rm^N(X, Y, Y, X) + K|X \wedge Y|^2 + P(X, Y, Y, X) \leq 0$, $K > 0$, which guarantees the long time existence of this flow. Here P would be some algebraic curvature tensor depending on the torsion and it's covariant derivatives.

3.3 Bubbling

In this section we will consider the $\bar{\partial}$-harmonic map heat flow for maps $f : \Sigma \to N$ between a compact Riemann surface Σ and a compact, almost Hermitian manifold N. In this setting the pseudoholomorphic energy E_+ is conformally invariant, as is readily seen from the fact that we now have

$$E_+(f) = \int_\Sigma \frac{1}{2}|Df|^2 dV - \int_\Sigma f^*\omega_N.$$

The most important observation we can make is that in this form the functional is exactly amenable to the result of Riviere [20] on the regularity of two variable conformally invariant elliptic systems.

Theorem 3.4 (Theorem 1.1 of [20])**.** *Let B be a ball in $\mathbb{R}^2$ and let $u \in W^{1,2}(B, \mathbb{R}^n)$ be a weak solution to the system*

$$\Delta u^i = \Omega^i_j(\nabla u^j),$$

where $\Omega \in L^2(B, \mathfrak{so}(n) \otimes \wedge^1\mathbb{R}^2)$. Then u is locally Hölder continuous in B.

We derive a number of corollaries in applying this result to $\bar{\partial}$-harmonic maps of surfaces. Analogues of these are well known in the theory of harmonic maps and other conformally invariant elliptic systems.

Corollary 3.4. *Let $N \subseteq \mathbb{R}^n$ be a smooth, compact, almost Hermitian manifold. Let $u \in W^{1,2}(B, N)$ be a weakly $\bar{\partial}$-harmonic map. Then u is smooth. In particular, if $u : B/\{0\} \to N$ is smooth and $\bar{\partial}$-harmonic with finite energy, then u is smooth in B.*

Proof. As observed in Theorem 1.2 of [20], any conformally invariant quadratic energy functional in two-dimensions has Euler-Lagrange equation in the form for which the previous theorem applies. Specifically, any functional of the form

$$\frac{1}{2}\int_\Sigma |Tu|^2 dV + \int_\Sigma u^*\omega,$$

where ω is any C^1 section of $\wedge^2 T^*N$ has Euler-Lagrange equation in the form required by the theorem. The E_+ energy is exactly of this form, so any weakly $\bar\partial$-harmonic map $u : B \to N$ for which $E(u) < \infty$ is Hölder continuous. Smoothness of u then follows from the smoothness of N, the Hölder continuity of u, and higher regularity theory of elliptic systems. □

Note that in the previous corollary the assumption of finite energy is essential, as the map $z \mapsto z^{-1}$ is clearly $\bar\partial$-harmonic with E_+ finite but is not smooth in a ball centered at the origin.

Corollary 3.5. *If $u : \mathbb{R}^2 \to N$ is $\bar\partial$-harmonic and has finite energy, then u extends to a smooth $\bar\partial$-harmonic map $\tilde{u} : S^2 \to N$.*

Proof. Consider the map given by composing u with stereographic projection from S^2 to $\mathbb{R}^2$. The previous proposition then implies that there is a smooth extension of this map to the point at infinity. □

Corollary 3.6. *Suppose that a solution u to the $\bar\partial$-harmonic map heat flow from a compact Riemann surface Σ to a compact, almost Hermitian manifold N exists on a maximal time interval $[0, T)$, where $T < \infty$, and there is a uniform energy bound on the solution. Then there exists a point $p \in \Sigma$, a sequence of times $t_i \nearrow T$, and a sequence of $r_i \searrow 0$ such that the family of maps $u_i(x) = u(exp_p(r_i x), t_i)$ converges to a limiting map $u_\infty : \mathbb{R}^2 \to N$ in $H^{2,2}_{loc}$ to a non-constant, smooth harmonic map with finite energy.*

Proof. As shown in Proposition 3.3 we know there is a sequence of times $t_i \nearrow T$ and points

$p_i \to p \in \Sigma$ such that $\lim_{i\to\infty} |Df|(p_i, t_i) = \infty$ and $|Df|(p_i, t_i) = \sup_{p\in\Sigma, t\leq t_i} |Df|(p, t)$. Let $r_i^{-1} = |Df|(p_i, t_i)$ and consider a geodesic ball centered at p of some small radius ρ. For $x \in B_{r_i^{-1}\rho}(0) \subset T\Sigma_p$ let

$$u_i(x, t) = u(exp_p(xr_i), t_0 + r_i^2 t).$$

Note that u_i solves the $\bar{\partial}$-harmonic map heat flow with respect to the metric $g_i = exp_p(r_i\cdot)^* g$. Since this metric is converging in C^2_{loc} to the flat metric on $\mathbb{R}^2$ we can extract a subsequence $u_i \to u_\infty$ converging locally in $C^2(\mathbb{R}^2 \times (-\infty, 0], N)$ where $u_\infty : \mathbb{R}^2 \times (-\infty, 0] \to N$ is a non-trivial solution to the $\bar{\partial}$-harmonic map heat flow with finite energy and constant E_+-energy, in particular it is a $\bar{\partial}$-harmonic map with finite energy. □

Corollary 3.7. *With the assumptions of the previous corollary, there must exist a $\bar{\partial}$-harmonic sphere in N. In particular, if N does not admit a non-trivial $\bar{\partial}$-harmonic S^2, then any solution to the $\bar{\partial}$-harmonic map heat flow with a uniform energy bound from any compact Riemann surface Σ to N exists smoothly for all time.*

Proof. By the previous corollary, if a finite time singularity occurs then there is a non-trivial $\bar{\partial}$-harmonic map $u_\infty : \mathbb{R}^2 \to N$ with finite energy. This u then extends by Corollary 4.3 to a $\bar{\partial}$-harmonic map of S^2 into N. □

3.4 Uniform Energy Bounds

The Dirichlet energy E of a solution to the $\bar{\partial}$-harmonic map heat flow may not be bounded along a solution to the flow, but such a uniform bound is necessary to conclude the existence of weak solutions or obtain convergence results for the flow. We know of no examples where such a uniform energy bound fails to hold along the flow, but we cannot rule it out in general.

The non-coerciveness of E_+ can be demonstrated with a simple example. Let $\omega_0 = \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$ be a flat metric on $\mathbb{C}$ and let ω_Σ be a metric on some compact Riemann surface Σ. Consider the metric $\omega = e^{2u(z)}\omega_\Sigma + \omega_0$ on the product $\Sigma \times \mathbb{C}$, where u is some smooth real valued function on $\mathbb{C}$ which is unbounded above. For each $z \in \mathbb{C}$ consider the inclusion $f_z(p) = (p, z)$. Then $E_+(f_z) = 0$ for each z while $E(f_z) = e^{2u(z)} Area(\omega_\Sigma)$. Note that this example is locally conformally Kähler with Lee form $\theta = 2du$. In particular, if u is unbounded we can make it as large as we want by following the Lee vector field of the metric. This motivates the following investigation.

If $f : \Sigma \to N$ is a map from a complex curve to a locally conformally Kähler manifold, we note that the energy takes the simple form $E_+ = E - K = E - \int_\Sigma f^*\omega_N$, and so if we are looking to construct bad sequences of maps where where E_+ is bounded but E becomes large, we necessarily have to find ways of increasing $K = \int_\Sigma f^*\omega_N$. Now with $d\omega = \theta \wedge \omega$, the first variation of K through $\frac{df}{dt} = v$ is given by

$$\frac{d}{dt}K = \int_\Sigma f^*(\theta(v)\omega - \theta \wedge \omega v).$$

In particular, variations through v which are tangent to $J\theta^\sharp$ do not increase K. In the previous example, $J\theta^\sharp = J\nabla u$ is directed tangentially to the level sets of u, and flowing in these directions obviously does not increase the energy. If some initial map f_0 is holomorphic then the form $f_0^*(\theta(v)\omega - \theta \wedge \omega v)$ is maximized when v is parallel to $\theta^\sharp$. In other words, to rapidly increase K one should follow the Lee vector field.

If the target is almost Kähler we at least have such a bound on any finite length time interval.

Proposition 3.12. *If N is almost Kähler then a solution to $\bar{\partial}$-heat flow has bounded energy on time intervals of finite length.*

Proof. Along a C^2 solution to the flow

$$\frac{d}{dt}E = -\int |\tau|^2 - \int_M \langle \tau, A\rangle$$

Since N is almost Kähler, we have that A is some tensor which is linear in Df and contracted with only $d^*\omega_M$ and ω_N. Therefore there are some positive constants C_1 and C_2 such that $-\langle \tau, A\rangle \leq C_1|\tau||Df| \leq \frac{1}{2}(|\tau|^2 + C_2|Df|^2)$. Therefore

$$\frac{d}{dt}E \leq C_3 E$$

for some constant C_3. Therefore $E(t) \leq E(0)e^{C_3 t}$. $\square$

If our manifolds are uniformly equivalent to balanced and almost Kähler manifolds, then remarkably an energy bound does hold. This result stands in direct comparison to the well known a priori energy bounds for pseudoholomorphic curves tamed by a symplectic structure.

Proposition 3.13. *Suppose that (M, g, J_M) is compact and uniformly equivalent to a J_M-compatible balanced metric g_0, so that $d^*\omega_{g_0} = 0$. Suppose also (N, h, J_N) is uniformly equivalent to a J_N-compatible almost Kähler metric h_0, non necessarily complete. Suppose further that f_t is a smooth one-parameter family of maps such that $E_+(f_t)$, computed with respect to g and h, is uniformly bounded. Then there is a uniform bound on $E(f_t)$ computed with respect to g and h.*

Proof. Let $E^{gh}(f_t)$ and $E_+^{gh}(f_t)$ denote the energy and pseudoholomorphic energy of f_t computed with respect to the metrics g and h, and let $K^{gh}(f_t)$ be the difference between these. Note that $K^{g_0h_0}(f_0)$ is a smooth homotopy invariant of f_0, and so $K^{g_0h_0}(f_t)$ is constant. By the uniform equivalence of the metrics, we have

$$\begin{aligned}E^{gh}(f_t) \leq CE^{g_0h_0}(f_t) &= C(E_+^{g_0h_0}(f_t) - K^{g_0h_0}(f_t)) \\ &= C(E_+^{g_0h_0}(f_t) - K^{g_0h_0}(f_0))\end{aligned}$$

for some constant $C > 0$ witnessing the equivalence of the metrics. But again by the uniform equivalence of the metrics

$$E_+^{g_0h_0}(f_t) \leq CE_+^{gh}(f_t).$$

and so $E^{gh}(f_t)$ is uniformly bounded. □

The following corollary is immediate from the previous proposition, and gives a rough picture of what is occurring when we fail to have a uniform energy bound along a solution to the $\bar{\partial}$-harmonic map heat flow.

Corollary 3.8. *Let $f : M \times [0,T) \to N$, $0 < T \leq \infty$ be a smooth solution to the $\bar{\partial}$-harmonic map heat flow between a compact, balanced, almost Hermitian manifold M and a (not necessarily complete) almost Hermitian manifold N. If there is not a uniform energy bound $E(f_t) \leq C$ for all $t \in [0,T)$, then for each $t_0 \in [0,T)$ no neighborhood of $f(M \times [t_0,T))$ in N can admit a uniformly equivalent Kähler metric.*

Coercive bounds for E_+ with source manifold a complex curve can be obtained in special settings from a smallness condition on the energy of the initial map. Such an estimate was obtained by Toda [26]. We state the main estimate in a form which is directly applicable to our flow.

Theorem 3.5. *Suppose that Σ is a closed Riemann surface and N is a compact almost-Hermitian manifold with Kähler form ω. Then there exists an absolute constant C depending* *only on dimensions such that, if*

$$|d\tilde{\omega}|^2 E(u_0) < C,$$

then this bound is preserved along the $\bar{\partial}$-harmonic map heat flow beginning near this u_0 and there is a minimizing $\bar{\partial}$-harmonic map in the free homotopy class of u_0. Here we embed $N \to \mathbb{R}^l$ and $\tilde{\omega}$ is some two-form on $\mathbb{R}^l$ which extends the one on N.

What this theorem essentially boils down to is showing a coercivity estimate for E_+ under the hypotheses given. Such an estimate is obtained as follows: for a given u in $H^{1,2}(\Sigma; N)$, the map $\omega \mapsto \int_\Sigma u^*\omega$ is a closed integral two-current. Once extended to $\mathbb{R}^l$ we can invoke the Almgren, Federer-Fleming [3] isoperimetric inequality for two-currents to conclude

$$\int_\Sigma u^*\omega \leq C|d\omega|E(u)^{3/2},$$

with C an absolute constant. We therefore conclude a bound of the form

$$E_+ \geq E - C|d\omega|E^{\frac{3}{2}},$$

which assuming that $E < \delta$ sufficiently small gives a bound of the form

$$E_+ \geq C_2 E,$$

for some constant C_2 depending on δ and C. Choosing δ appropriately ensures that this bound is preserved along the flow.

The assumption that N is an almost Hermitian manifold whose Kähler form can be extended to $\mathbb{R}^l$

3.5 The Perturbed Energies

In this section we will consider the perturbed energies $E_a = E + aK$. These should be viewed as coercive approximations to the energy E_+. Our first lemma is obvious from the previous sections.

Lemma 3.4. *The Euler-Lagrange equation of E_a is $\tau_a = \tau + aA$.*

We also readily have a long time existence result for the flow associated to E_a, as well as an Eells-Sampson type result if $|a| < 1$.

Proof. Let f be the solution to the flow. The long time existence of the flow was established in 3.11, where we note that the change from E_+ to E_a is a trivial modification of the proof. Specifically, along a solution to the flow there is a constant $C_1 > 0$ depending on a and background data such that

$$(\partial_t - \Delta)|Df|^2 \leq C_1 |Df|^2,$$

from which we conclude long time existence. We then must prove convergence at infinite time, which we do so in essentially the same way as for the Eells-Sampson result.

Since $(1 - |a|)E \leq E_a$, we have a uniform energy bound $E < C$ along a solution to the flow when $|a| < 1$. Recall Moser's Harnack inequality for subsolutions to the heat equation [17]: if g is non-negative and there is some positive constant C so that $(\partial_t - \Delta)g \leq Cg$ in a parabolic cylinder $P_R(x_0, t_0) = \{(x,t) | d(x, x_0) \leq R, t_0 - R^2 \leq t \leq t_0\}$ centered at (x_0, t_0), then there exists another constant $C' > 0$ such that

$$g(x_0, t_0) \leq C' R^{-(n+2)} \int_{P_R(x_0,t_0)} g dV.$$

We then apply this inequality to $g = |Df|^2$, giving, for some constant C depending only on background data and a,

$$
\begin{aligned}
|Df|^2(x,t) &\leq CR^{-(n+2)} \int_{P_R(x_0,t_0)} |Df|^2 \\
&\leq CR^{-(n+2)} \int_{t-R^2}^{t} E(f(s))ds \\
&\leq \frac{C}{1-|a|} R^{-(n+2)} \int_{t-R^2}^{t} E_a(f(s))ds \\
&\leq \frac{C}{1-|a|} R^{-n} E_a(f_0).
\end{aligned}
$$

We therefore have a uniform bound on $|Df|$ on all of $[0,\infty)$. By the higher regularity theory for second-order parabolic equations we conclude the existence of constants $C(f_0, M, N, k, a)$ such that $\sup_{M\times[0,\infty)} |\nabla^k Df| \leq C(f_0, M, N, k, a)$. Note also that there is some constant C such that

$$
(\partial_t - \Delta)|\tau_a|^2 \leq C|\tau_a|^2.
$$

Again by Moser's Harnack inequality we conclude

$$
\begin{aligned}
|\tau_a|^2(x,t) &\leq CR^{-(n+2)} \int_{t-R^2}^{t} \int_M |\tau_a|^2 dV ds \\
&= CR^{-(n+2)} \int_{t-R^2}^{t} -\partial_s E_a ds \\
&= CR^{-(n+2)}(E_a(t-R^2) - E_a(t)),
\end{aligned}
$$

and so $|\tau_a|^2 \to 0$ as $t \to \infty$. Taking any sequence of t_i going to ∞, by the previous estimates we can extract a subsequence such that $f(t_i)$ converges in any C^k norm to some f_∞ satisfying $\tau_a(f) = 0$. □

We note that if $|a|$ is sufficiently small we can also invoke Toda's estimate to get a long time existence result for this family of functionals.

3.6 An example of the Flow into a Non-Kähler Surface

Pick some $\alpha > 1$ and let $N = S^3 \times S^1 = \mathbb{C}^2 \setminus \{0\}/\mathbb{Z}$ be the Hopf surface generated by the $\mathbb{Z}$ action given by $(z, w) \mapsto (\alpha z, \alpha w)$ on $\mathbb{C}^2 \setminus \{0\}$. N is a compact complex manifold which cannot admit any Kähler metrics for Hodge theoretic reasons. If ρ denotes the distance to 0 in $\mathbb{C}^2$, a Hermitian metric h on N has corresponding two form $\omega_N(\cdot, \cdot) = h(J_N \cdot, \cdot)$ given by

$$\omega_N = \frac{1}{2\rho^2}\sqrt{-1}\partial\bar{\partial}\rho^2.$$

Notice that this is indeed a metric on N, as it is invariant under scalar multiplication and unitary transformations, and is moreover locally conformal to the standard Euclidean metric on $\mathbb{R}^4$. A similar construction applied to $\mathbb{C}^*$ gives a torus $M = T^2 = \mathbb{C}^*/\mathbb{Z}$, and Kähler metric g with corresponding Kähler form

$$\omega_M = \frac{1}{2\rho^2}\sqrt{-1}\partial\bar{\partial}\rho^2,$$

which is likewise invariant, in fact it is a flat metric on M.

Pick some orthonormal basis e_i of $\mathbb{R}^4$ so that the standard complex structure on $\mathbb{C}^2 \sim \mathbb{R}^4$ takes the form $Je_1 = e_2$ and $Je_3 = e_4$. Take coordinates y^i on $\mathbb{R}^4$ induced by this basis. Do a similar construction for $\mathbb{R}^2 \sim \mathbb{C}$ and call the corresponding coordinates x^i. For any pair (u, v) of orthonormal vectors in $\mathbb{R}^4$, consider the $\mathbb{R}$-linear map $f : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$f(x^1, x^2) = x^1 u + x^2 v.$$

Notice that f is an orthogonal embedding since u and v are orthonormal. Linearity implies that f is equivariant with respect to the $\mathbb{Z}$ action and therefore descends to a map $f : M \to N$ of the torus into the Hopf surface. By the orthogonality, it is evident that $f^*h = g$ and, moreover, f is totally geodesic. This is most easily seen by noting that h can be viewed as a bi-invariant metric with respect to some Lie group structure on N for which $f : M \to N$ is the inclusion of a torus subgroup.

These f are therefore a family of harmonic maps $f : M \to N$ parameterized by orthonormal pairs in $\mathbb{R}^4$. We can compute $K(f)$ for these maps directly. First, note that since $\dim_{\mathbb{C}} M = 1$,

$$K(f) = -\frac{1}{2}\int_M \langle \omega_M, f^*\omega_N \rangle \omega_M = -\int_M f^*\omega_N.$$

In the coordinates y^i

$$\omega_N = \frac{1}{\rho^2}(dy^1 \wedge dy^2 + dy^3 \wedge dy^4),$$

and so

$$\begin{aligned} f^*\omega_N &= \frac{1}{\rho^2}(u^1v^2 - u^2v^1 + u^3v^4 - u^4v^3)(dx^1 \wedge dx^2) \\ &= (u^1v^2 - u^2v^1 + u^3v^4 - u^4v^3)\omega_M, \end{aligned}$$

therefore

$$K(f) = -(u^1v^2 - u^2v^1 + u^3v^4 - u^4v^3)V,$$

where $V = 2\pi \log \alpha$ is the volume of T^2 with respect to g. In particular, it is clear from this example that K is not a homotopy invariant. In fact, the homotopy invariance of K is exactly what allows the pseudoholomorphic energy to distinguish holomorphic maps from harmonic maps.

Proposition 3.14. *Let $\mathcal{F}$ denote the family of all such $f : T^2 \to S^3 \times S^1$ constructed as above. Then this family is preserved by the $\bar{\partial}$-harmonic map heat flow and, for any $f_0 \in \mathcal{F}$, the flow exists for all time and converges subsequentially to a holomorphic or anti-holomorphic map $f_\infty : T^2 \to S^3 \times S^1$.*

Proof. Note that $\langle A(f), Df \rangle = 0$. This is because this depends on $d\omega_N(Df, Df, Df)$ but Df only has rank 2. Therefore, since each $f \in \mathcal{F}$ is harmonic, we have that $\tau_+ = \tau + A = A$ is perpendicular to the image of f at each point. This, together with the fact that f is linear and so A is linear in the coordinates x^i, implies that the evolution equation $\partial_t f = A$ preserves the family. We can then view the flow in this family as given by a smooth vector field on the space of oriented orthonormal 2-frames in $\mathbb{R}^4$, establishing long time existence. Convergence to a $\bar{\partial}$-harmonic map is then immediate from monotonicity of E_+ and compactness of this family.

To see that the limiting map must be holomorphic or anti-holomorphic, note that the energy E_+ is invariant $E_+(U \circ f) = E_+(f)$ under the unitary group of $\mathbb{C}^2$. Since these act transitively on the unit vectors, we can assume the limiting map has $u = e_1$. But then a direct computation shows that, when $u = e_1$, $A(f) = 0$ if, and only if, $v = \pm e_2$. Thus the limiting map is holomorphic or anti-holomorphic. □

This relatively simple example demonstrates that the flow can distinguish a holomorphic map from a harmonic map in non-Kähler settings. In light of Corollary 3.9, if there is a singularity in the flow and a uniform energy bound does not hold, then the image of the flow near the singularity must be quite wild in $S^3 \times S^1$; any proper compact subset of $S^3 \times S^1$ admits a Kähler metric which, due to compactness, is uniformly equivalent to h. Therefore the lack of a uniform energy bound would mean the solution leaves every proper compact subset of the Hopf surface.

3.7 A Variational Approach to Studying Submanifolds of Vaisman manifolds

Let (N, h, J) be a Vaisman manifold with Kähler form $\omega(X, Y) = g(JX, Y)$ and Lee form θ. Let $f : \Sigma \to N$ be a holomorphic immersion of a smooth compact holomorphic curve Σ into N, let $g = f^*h$ be the induced metric on Σ, and let j be the complex structure on Σ. The functional

$$K = \int_\Sigma f^*\omega$$

has appeared a lot in this chapter as the defect between the Dirichlet energy E and the anti-holomorphic energy E_+ of a map. In this final section we demonstrate how the variational structure of this and related functionals can be used to obtain results on the structure of complex submanifolds of a Vaisman manifold. These results have appeared previously, see for example [19], but the proofs we provide require nothing beyond studying the first variation of these functionals.

Proposition 3.15. *Let (M, g, J) be a Vaisman manifold. Then any holomorphic curve in (M, J) is a torus and is tangent to the Lee vector field.*

Proof. Let $X = \theta^\sharp$ be the Lee vector field of ω. Since N is Vaisman, X is parallel and generates a one parameter group of holomorphic isometries of N. Consider the variation of f given by X and let $f : \mathbb{R} \times \Sigma \to N, f : (t, x) \mapsto f_t(x)$ be the corresponding family of maps, with the slight abuse of notation $f_0(x) = f(x)$. Since X is holomorphic and killing, the energies E and E_+ are constant along this family, and therefore

$$\frac{d}{dt}\Big|_{t=0} \int_\Sigma f^*\omega = 0.$$

Now for any one parameter family of maps with variation v, we recall that

$$\frac{d}{dt} f^* \omega = df^* i_v \omega + f^* i_v d\omega,$$

where $i_v \omega = \omega(v, \cdot)$ denotes interior multiplication by v. Therefore, as we are assuming $d\omega = \theta \wedge \omega$

$$\frac{d}{dt} \int_\Sigma f^* \omega = \int_\Sigma f^* (\theta(v)\omega - \theta \wedge \omega v).$$

Substituting $v = \theta^\sharp$, we get

$$\int_\Sigma f^* (|\theta|^2 \omega - \theta \wedge J\theta) = 0.$$

Now consider a Hermitian vector space V with a complex structure J and associated symplectic form ω_V. Fix some 1-form $\theta \in V^*$ with dual vector field X. Let $f : \mathbb{C} \to V$ be a linear isometric holomorphic inclusion of $\mathbb{C}$ into V, so then $f^* \omega_N = \omega_{\mathbb{C}}$. Fix some (real) basis σ^i of orthonormal 1-forms on V^*, so that $J\sigma^i = \sigma^{i+1}$ for i odd and where the vectors dual to σ^1 and σ^2 span the image of f. Let $\tau^1 = f^*\sigma^1$ and $\tau^2 = f^*\sigma^2$, so that $f^*\sigma^i = 0$ for all $i > 2$. With $\theta = \theta_i \sigma^i$, we therefore have

$$f^*(\theta \wedge J\theta) = (\theta_1^2 + \theta_2^2)\omega_{\mathbb{C}},$$

and so

$$f^*(|\theta|^2 \omega - \theta \wedge J\theta) = (\theta_3^2 + \theta_4^2 + \ldots)\omega_{\mathbb{C}}.$$

We conclude that $f^*(|\theta|^2\omega - \theta \wedge J\theta)$ is a non-negative $(1,1)$-form on Σ, and since its integral is 0 it must vanish pointwise. But this implies that $\theta_i = 0$ for all $i > 2$, giving us that X is in the image, i.e. that X is tangent to Σ in N.

Since X is parallel, $|X| > 0$ is constant. Since it is tangent to Σ we conclude that Σ admits a nowhere vanishing tangent vector field and is therefore a torus. $\square$

In fact a similar result holds for any complex submanifold of a Vaisman manifold by considering a variation of the analogous functional in higher dimensions.

Proposition 3.16. *If N^{2n} is a complex submanifold of a Vaisman manifold M then N is tangent to the Lee vector field and is therefore also Vaisman.*

Proof. Consider the functional

$$K : f \mapsto \int_N f^* \omega^n.$$

As before, if f_0 is holomorphic then K is exactly (a universal constant multiple) of the area of N. Let $\frac{df}{dt} = \theta^\sharp$ be the variation of f_0 determined by the Lee vector field. Since the Lee vector field is Killing and holomorphic, K must be constant along this family because the flow preserves the area and holomorphicity of submanifolds. But we can compute the variation of K as

$$0 = \frac{d}{dt} K = n \int_N f^*(|\theta|^2 \omega^n - n\theta \wedge J\theta \wedge \omega^{n-1}).$$

As f is holomorphic, a pointwise calculation shows that

$$f^*(|\theta|^2 \omega^n - n\theta \wedge J\theta \wedge \omega^{n-1}) = n!(\theta_{2n+1}^2 + \theta_{2n+2}^2 + \ldots + \theta_{2m}^2) dV$$

in a local complex frame adapted to N at a point. Therefore, since the integral of this form vanishes, we conclude $\theta_i = 0$ for all $i > 2n$, so the Lee vector field is tangent. $\square$

Bibliography

[1] J.P. Demailly, *Complex analytic and differential geometry*, unpublished, available online, 2012.

[2] Sorin Dragomir and Liviu Ornea, *Locally conformal Kähler geometry*, Progress in Mathematics, vol. 155, Birkhäuser Boston, Inc., Boston, MA, 1998. MR 1481969

[3] Herbert Federer and Wendell H. Fleming, *Normal and integral currents*, Ann. of Math. (2) **72** (1960), 458–520. MR 0123260

[4] P. Gauduchon and S. Ivanov, *Einstein-Hermitian surfaces and Hermitian Einstein-Weyl structures in dimension* 4, Math. Z. **226** (1997), no. 2, 317–326. MR 1477631 (98m:53063)

[5] Paul Gauduchon, *Le théorème de l'excentricité nulle*, C. R. Acad. Sci. Paris Sér. A-B **285** (1977), no. 5, A387–A390. MR 0470920

[6] ______, *Hermitian connections and Dirac operators*, Boll. Un. Mat. Ital. B (7) **11** (1997), no. 2, suppl., 257–288. MR 1456265 (98c:53034)

[7] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), no. 2, 307–347. MR 809718

[8] Masahisa Inoue, *On surfaces of Class* VII_0, Invent. Math. **24** (1974), 269–310. MR 0342734 (49 #7479)

[9] James Isenberg, Martin Jackson, and Peng Lu, *Ricci flow on locally homogeneous closed 4-manifolds*, Comm. Anal. Geom. **14** (2006), no. 2, 345–386. MR 2255014 (2007e:53087)

[10] D. Kaledin and M. Verbitsky, *Non-Hermitian Yang-Mills connections*, Selecta Math. (N.S.) **4** (1998), no. 2, 279–320. MR 1669956

[11] Ma. Kato, *Compact complex manifolds containing "global" spherical shells. I*, Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), Kinokuniya Book Store, Tokyo, 1978, pp. 45–84. MR 578853

[12] Shoshichi Kobayashi and Katsumi Nomizu, *Foundations of differential geometry. Vol. II*, Interscience Tracts in Pure and Applied Mathematics, No. 15 Vol. II, Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1969. MR 0238225

[13] J. Lauret and E. Alejandro Rodriguez Valencia, *On the Chern-Ricci flow and its solitons for Lie groups*, ArXiv e-prints (2013).

[14] André Lichnerowicz, *Applications harmoniques et variétés kähleriennes*, Symposia Mathematica, Vol. III (INDAM, Rome, 1968/69), Academic Press, London, 1968/1969, pp. 341–402. MR 0262993

[15] Fanghua Lin and Changyou Wang, *The analysis of harmonic maps and their heat flows*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008. MR 2431658

[16] John Lott, *On the long-time behavior of type-III Ricci flow solutions*, Math. Ann. **339** (2007), no. 3, 627–666. MR 2336062 (2008i:53093)

[17] Jürgen Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964), 101–134. MR 0159139

[18] Iku Nakamura, *On surfaces of class* VII_0 *with curves. II*, Tohoku Math. J. (2) **42** (1990), no. 4, 475–516. MR 1076173

[19] L. Ornea and M. Verbitsky, *A report on locally conformally Kähler manifolds*, Harmonic maps and differential geometry, Contemp. Math., vol. 542, Amer. Math. Soc., Providence, RI, 2011, pp. 135–149. MR 2796645

[20] Tristan Rivière, *Conservation laws for conformally invariant variational problems*, Invent. Math. **168** (2007), no. 1, 1–22. MR 2285745

[21] Yum Tong Siu, *The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds*, Ann. of Math. (2) **112** (1980), no. 1, 73–111. MR 584075

[22] Yum Tong Siu and Shing Tung Yau, *Compact Kähler manifolds of positive bisectional curvature*, Invent. Math. **59** (1980), no. 2, 189–204. MR 577360

[23] Jeffrey Streets and Gang Tian, *A parabolic flow of pluriclosed metrics*, Int. Math. Res. Not. IMRN (2010), no. 16, 3101–3133. MR 2673720 (2011h:53091)

[24] ______, *Hermitian curvature flow*, J. Eur. Math. Soc. (JEMS) **13** (2011), no. 3, 601–634. MR 2781927 (2012f:53142)

[25] ______, *Regularity results for pluriclosed flow*, Geom. Topol. **17** (2013), no. 4, 2389–2429. MR 3110582

[26] Masahito Toda, *On the existence of H-surfaces into Riemannian manifolds*, Calc. Var. Partial Differential Equations **5** (1997), no. 1, 55–83. MR 1424349

[27] Valentino Tosatti and Ben Weinkove, *The Chern-Ricci flow on complex surfaces*, Compos. Math. **149** (2013), no. 12, 2101–2138. MR 3143707

[28] Hajime Urakawa, *Stability of harmonic maps and eigenvalues of the Laplacian*, Trans. Amer. Math. Soc. **301** (1987), no. 2, 557–589. MR 882704

[29] Luigi Vezzoni, *A note on canonical Ricci forms on* 2*-step nilmanifolds*, Proc. Amer. Math. Soc. **141** (2013), no. 1, 325–333. MR 2988734

[30] C. T. C. Wall, *Geometries and geometric structures in real dimension* 4 *and complex dimension* 2, Geometry and topology (College Park, Md., 1983/84), Lecture Notes in Math., vol. 1167, Springer, Berlin, 1985, pp. 268–292. MR 827276 (87e:57023)