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Benford's Law and the FSD Distribution of Economic Behavioral Micro Data

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Abstract

In this paper, we focus on the first significant digit (FSD) distribution of European micro income data and use information theoretic-entropy based methods to investigate the degree to which Benford's FSD law is consistent with the nature of these economic behavioral systems. We demonstrate that Benford's law is not an empirical phenomenon that occurs only in important distributions in physical statistics, but that it also arises in self-organizing dynamic economic behavioral systems. The empirical likelihood member of the minimum divergence-entropy family, is used to recover country based income FSD probability density functions and to demonstrate the implications of using a Benford prior reference distribution in economic behavioral system information recovery.

Keywords: Benford's law; Information theoretic methods; Micro income data; Empirical likelihood criterion; Minimum divergence distance measures; Cross entropy

JEL Classification: C1; C10; C24

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1. Introduction

In recent journal articles, Shao and Ma[1,2] investigated three widely used distributions in physical statistics and found that the Boltzman-Gibbs, Fermi-Derac and Bose-Einstein distributions either conform or fluctuate around the Benford[3] first significant digit(FSD) distribution. Since Benford's law is an empirical phenomenon that occurs in a range of data sets, this raises the question as to whether or not the same thing might be true in terms of the very important income distribution in behavioral economics. To pursue this question, in this paper we use time-dated samples of European micro income data to investigate whether in behavioral economics, the FSD income probability density function-distribution conforms to Benford's FSD law. *The answer to this question is important because country based income data-distributions contains information on how the market is functioning, the allocation and distribution system is performing, and in terms of dynamics, how the economic system has changed and is changing over time.*

1.1 Benford's Law

In 1881 astronomer and mathematician Simon Newcomb[4] conjectured that in natural data sets, the first digits did not occur with equal frequency. Instead Newcomb suggested that the occurrence of numbers is such that all mantissa of their logarithms are equally probable. This led him to suggest the following expression for the empirical distribution of first digits, $P(d) = \log_{10}\left(\frac{1+d}{d}\right)$, with the following probability of the digits P(d = 1, 2, ---, 9) =(0.301, 0.176, 0.125, 0.097, 0.790, 0.670. 0.058, 0.051, 0,046).

Fifty-seven years later physicist Frank Benford[3], empirically demonstrated that a large number of seemingly unrelated data sets provided a good fit to the FSD exponential distribution, and gave the FSD exponential distribution law status.

Since then, others have published studies showing that "Benford's Law" not only applies to a surprisingly large number of natural-behavioral data sets, but also has the nice properties of being scale and base invariant (see Varian[5] and Miller[6[. Overviews of the history and theoretical explanations include Raimi[7], Diaconis[8], Hill[9], Berger and Hill[10] Miller and Nigrini[11] and Judge and Scheckter[12]. Even when FSD data sets deviate from the Benford pattern, the lower digits are favored and decline monotonically. Given the nature of the distribution, it has been suggested that Benford's law is a special case of the power law and thus a way of generalizing FSD distributions (see for example, Pietronero, et al., [13]. Furthermore, as noted above, Shao and Ma[1,2] demonstrate that in physical statistics, the Boltmann-Gibbs and Fermi-Derac distributions with respect to the temperature of the system, fluctuate around the Benford distribution and that the Bose-Einstein distribution exactly conforms to it.

In statistical physics and behavioral economics one might naively expect that outcomes of admissible microstates of physical and behavioral systems are equally probable over long periods of time. Alternatively, Benford's law suggests for these and many other real world situations, the occurrence of nonzero digits are not uniformly distributed, but instead favor the smaller digits in a scale and base invariant exponential way. Given this difference in the digit distribution, in this paper we consider the question: "Does Benford's exponential first significant digit (FSD) law reflect a fundamental principle behind the complex and nondeterministic nature of large scale behavioral systems such as country based income probability density function-distributions?"

In the sections ahead, we seek an answer to this question by using the FSD of country based household micro income data and entropy-based information theoretic methods to recover the corresponding exponential Benford's distribution. In doing so, we demonstrate the degree to which the corresponding income FSD are consistent with Benford's law. After introducing the information theoretic conceptual framework and the micro samples of income data, we provide behavioral economics empirical examples using micro income data over a range of several years and European countries. The paper concludes with a summary and the implications of our results.

2. The Conceptual Framework

In seeking a new way to analyze the question posed at the end of Section 1, we recognize that economic income-social systems do not evolve in a deterministic or a random way, but tend to adapt behavior in line with an optimizing principle. While prior research has shown that Benford's law most commonly holds in large naturally occurring numerical datasets, the presence of Benford's law in samples of data from economic behavioral systems is an open question. As we seek a new way to think about Benford's FSD distributional result in large complex and dynamic micro-income systems, we use information theory as a recovery method and entropy as the systems optimizing criterion-status measure.

2.1 Problem Formulation and Solution

In the introduction, we discussed the Newcomb-Benford approach to determining the seemingly general exponential distribution of FSD. Pre analysis knowledge suggests that the FSD distribution of a sequence of positive real numbers from scale-independent multiplicative data should vary with the phenomena in question. In this context information theoretic methods offer a natural way to establish a data based link that captures the varying monotonically decreasing nature of the FSD.

To use information theoretic methods to recover the FSD distribution from a sequence of positive real numbers, we assume for the discrete random variable d_i (for $i = 1, 2, \ldots, 9$), that at each trial, one of nine digits is observed with probability p_i. Suppose after n trials, we have first-moment information in the form of the average value of the FSD:

$$\sum_{j=1}^{9} d_j\, p_j = \bar{d}. \qquad (2.1)$$

Based on $sample\ information, \sum_{j=1}^{9} d_j\, p_j = \bar{d}, \sum_{j=1}^{9} p_j = 1$, and $0 \leq p_j \leq 1'$, the nine digit FSD ill-posed inverse recovery problem cannot be solved for a unique solution. In such a situation it seems useful to have an approach that permits the investigator to use sample based information recovery methods without having to choose a parametric family of probability densities on which to base the FSD probability density function.

2.2 An Information Theoretic Approach

One way to solve this ill-posed inverse problem for the unknown p_j without making a large number of assumptions or introducing additional information is to formulate it as an extremum-optimization problem. In this context a solution is achieved by minimizing the divergence between the two sets of probabilities and an optimizing goodness-of-fit criterion, subject to data-moment constraints. One attractive set of divergence measures is the Cressie-Read (CR) power divergence family of statistics (Cressie and Read[14], Read and Cressie[15], and Judge and Mittelhammer, [16,17]):

$$I(\boldsymbol{p},\boldsymbol{q},\gamma)=\frac{1}{\gamma(1+\gamma)}\sum_{j=1}^{N}\left(p_j\left[\left(\frac{p_j}{q_j}\right)^{\gamma}-1\right]\right), \tag{2.2}$$

where γ is an arbitrary unspecified parameter. All well known entropy divergences belong to the class of CR functions. In the context of recovering the unknown sample information FSD distribution, we make use of the CR criterion (3.2) and seek a solution to the following extremum problem:

$$\widehat{\boldsymbol{p}}=\arg\min_{\boldsymbol{p}}\left[I(\boldsymbol{p},\boldsymbol{q},\gamma)\mid\sum_{j=1}^{N}p_j\,d_j=\bar{d},\sum_{j=1}^{N}p_j=1,p_j\geq 0\right]. \tag{2.3}$$

When $\gamma\rightarrow -1$ and $I(\boldsymbol{p},\boldsymbol{q},\gamma)$ converges to an estimation criterion equivalent to the empirical likelihood (EL) criterion $\sum_{j=1}^{N}\ln\,(p_j)$. As γ varies, power law like behavior is efficiently described and the resulting estimators that minimize power divergence exhibit qualitatively different sampling behavior. Over defined ranges of the divergence measures, the CR and entropy families are equivalent.

2.3 CR (gamma->-1) Mean Related FSD Distribution

In terms of the information-theoretic variants of the CR $I(\boldsymbol{p},\boldsymbol{q},\gamma)$ we demonstrate for the Benford recovery problem the case of the CR $\gamma\rightarrow -1$ criterion, $with$ a uniform reference distribution $\boldsymbol{q}$ ($q_j=1/9,\forall j$). First moment information $\bar{d}$ is used as a basis for recovering discrete FSD probability distributions. As noted above, under the criterion CR

$\gamma \to -1$, the CR $I(\boldsymbol{p},\boldsymbol{q},\gamma)$ converges to the empirical likelihood criterion metric $9^{-1}\sum_{j=1}^{9}(\,ln p_j)$ and the extremum likelihood function

$$\max_{\boldsymbol{p}}\left[9^{-1}\sum_{j=1}^{9}\ln\, p_j \mid \sum_{j=1}^{9} p_j\, d_j = \bar{d}, \sum_{j=1}^{9} p_j \quad = 1\right]. \tag{2.4}$$

The corresponding Lagrange function is

$$L(\boldsymbol{p},\eta,\lambda) \equiv 9^{-1}\textstyle\sum_{j=1}^{9}\ln\, p_j - \eta\left(\sum_{j=1}^{9} p_j - 1\right) - \lambda\left(\sum_{j=1}^{9} p_j\, d_j - \bar{d}\right) \tag{2.5}$$

with the solution

$$\widehat{p_j}(\bar{d},\lambda) = \left[9^{-1}\left(1+\hat{\lambda}\left(d_j - \bar{d}\right)\right)\right]^{-1}, \tag{2.6}$$

for the jth FSD outcome. As the mean of the significant first digits varies a family of probability density functions-distributions result. For mean FSD values less than 5, the resulting estimated FSD distribution reflect the monotonic decreasing FSD probabilities exhibited by the Benford distribution. As the FSD mean approaches the Benford mean 3.44, the CR-EL and FSD distributions are approximately equal. If we use the CR in the limit $\gamma \to -1$ criterion and a Benford reference distribution $I(\boldsymbol{p},\boldsymbol{q}_B,\gamma) = \sum_{j=1}^{9}(\ln p_j/q_{jB})$, then with the first moment condtion of 3.44, the Benford FSD distribution is exactly reproduced.

2.4 Discussion

The relationship between the CR-EL minimum divergence entropy measure and the Benford FSD distribution provide a basis for recovering information regarding the unknown FSD probability density function from samples of micro income behavioral data. In seeking an optimizing criterion with the income behavioral data, we follow Wissner-Gross and Freer[18] and recognize the connection between adaptive intelligent behavior, causal entropy maximization and self-organized equilibrium seeking behavior. As noted early in Section 2, the connection between causal adaptive behavior and entropy maximization, based on a causal generalization of entropic forces, suggests that behavioral systems do not evolve in a deterministic or a random way, but tend to adapt behavior in line with an optimizing principle. As we think about the connection between Benford's FSD distribution and information recovery in the causal adaptive behavior of large complex and dynamic micro income-economic systems, entropy emerges as the systems status measure and a basis for gauging performance. Given the entropy adaptive behavior connection we now turn to an empirical example of European income data and the resulting information theoretic FSD income behavioral distributions.

3. The Eurostat Micro Income Data

Eurostat is a Directorate-General of the European Commission. Its main responsibilities are to provide statistical information to the institutions of the European Union (EU). Considering data availability and country characteristics, we use income household data for the following 13 countries listed in Table 3.1.

Table 3. 1

COUNTRIES AND ABBREVIATIONS AND NUMBER OF OBSERVATIONS

Abbreviation	Country	Number Observations
AT	Austria	58,669
BE	Belgium	59,053
CZ	Czech republic	77,726
DK	Denmark	57,735
EL	Greece	45,245
ES	Spain	102,211
FR	France	77,053
IE	Ireland	51,895
IT	Italy	138,943
PL	Poland	124,487
PT	Portugal	37,395
SK	Slovakia	47,166
UK	United Kingdom	83,797
Total in 13 Country Sample		961,375
Total Negative		804
Total Missing		59
Percent Data Used From Raw Data		99.94%

Source: EUROSTAT.

Among all income related variables in the Eurostat's micro survey database, we use the variable "Total household gross income" to measure the income level. It measures in Euros without an inflation factor the sum of gross personal income components for all household members. The total household gross income variable is a comprehensive and well-defined variable for our study and is consistent with income measures used in previous studies. In Table 3.2 we specify which countries have data available for each of the years from 2004 to 2013.

TABLE 3.2

AVAILABLE YEARLY INCOME DATA SAMPLES BY COUNTRY

Years	Countries
2004	AT,BE,DK,IE
2005	AT,BE,CZ,DK,IE,PL,SK,UK
2006	AT,BE,CZ,DK,ES,HU,IE,PL,SK,UK
2007	AT,BE,CZ,DK,ES,FR,GR,IE,IT,PL
2008	AT,BE,CZ,DK,EL,ES,FR,IE,IT,,PL,PT,SK,UK
2009	AT,BE,CZ,DK,EL,ES,FR,IE,IT,PL,PT,SK,UK
2010	AT,BE,CZ,DK,EL,ES,FR,IE,IT,PL,PT,SK,UK
2011	AT,BE,CZ,DK,EL,ES,FR,IE,IT,PL,PT,SK,UK
2012	AT,BE,CZ,DK,EL,ES,FR,IE,IT,PL,PT,SK,UK
2013	AT,BE,CZ,DK,EL,ES,FR,IE,IT,PL,PT,SK,UK

Source: EUROSTAT.

The household level micro income data are in Euros for all of the 13 countries. Summary statistics of the household income data by country and by year are reported in Villas-Boas, et al[17]. There are a total number of 961,375 data observations by country-year and the data are quite complete and clean. After removing negative and missing entries, we keep 99.94% of the original household sample data. We use the complete sample of income data for each country to obtain the FSD distributions by year.

4. Information Theoretic European Income FSD distributions

In this section we compare Benford and the information theoretic FSD income distributions for thirteen European countries. We focus on the analysis of the FSD samples of the micro income data from 13 European countries that range over the years 2004 to 2013. In the analysis of the samples of FSD income data, we make use of the information theoretic methods of Section 2 as a basis for summarizing the country based samples of income data in the form of FSD probability density functions so that they can be compared to the Benford FSD distribution.

4.1 Entropy FSD Income Measure

In order to provide the information that is needed to group and compare the FSD income distributions of the 13 European countries, we make use of the entropy measure $E = \gamma \rightarrow -1 =. 9^{-1} \sum_{j=1}^{9} \ln\ p_j$. Using this entropy criterion-measure we seek a FSD probability density function solution for each country for the combined years 2009-2013.This entropy measure is defned as a measure of the uncertainty-diversity-information contained in the FSD income probability density functions. As a benchmark the entropy measure for the Benford probability density function distribution is approximately 2.0. Making use of this country based FSD entropy measure for the combined sample of years 2009-2013, we are able to produce ranking of the 13 European countries as shown in Figure 4.1 below.

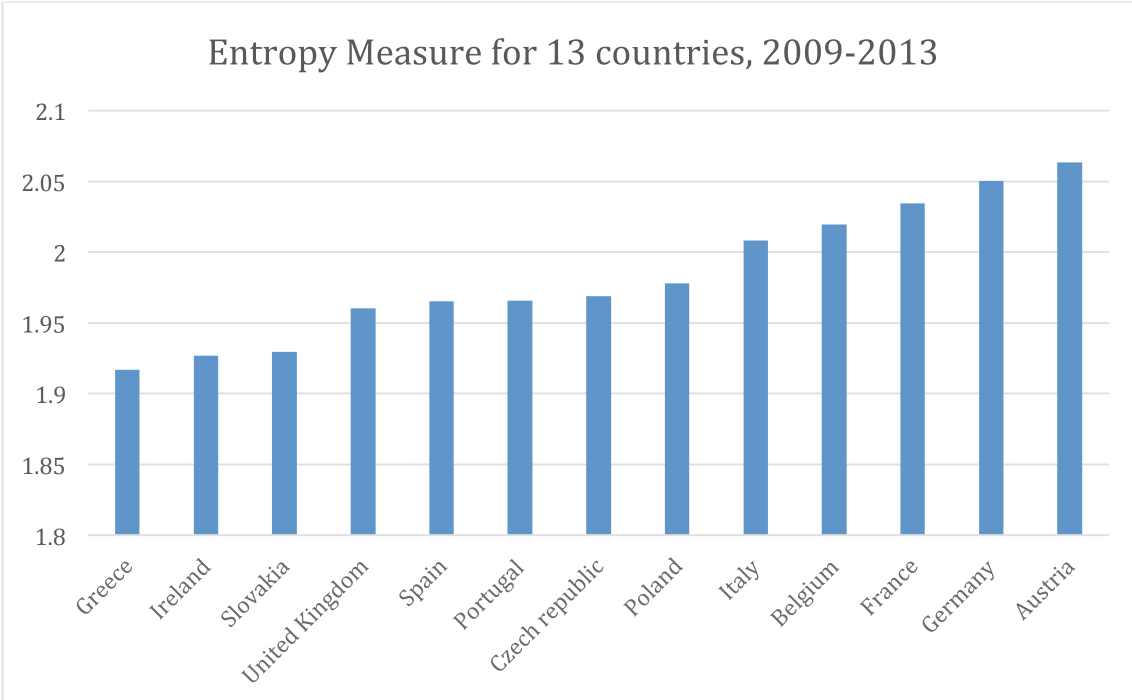

Figure 4.1. Entropy Measure of the FSDs for 13 European Countries

The FSD entropy measures for all the European countries are closely associated with the Benford FSD entropy measure-distribution of 1.9938. Additionally, the higher the entropy measure, the more uniformly distributed the FSD income distribution. For example, Austria's high entropy measure of 2.06 indicates an FSD income density function that is not only more uniform than the Benford FSD distribution, but also the most uniform of the 13 European countries studied. On the other hand, the low entropy measure of 1.91 for Greece indicates a large departure from the Benford distribution, and the least uniform FSD income probability density function-greatest income inequality of the European countries studied. It is of interest to note that the three low entropy countries Greece, Ireland and Slovakia, are countries that recently have been facing economic-financial problems.

4.2 Income FSD Probability Density functions for Central and Eastern European Countries

In Figure 4.2, we jointly display for the combined 2009-2013 data, the FSD probability density functions for AT- Austria, DE-Germany, FR-France, BE-Belgium and

IT-Italy, PL-Poland, CZ-Czech Republic, PT-Portugal, ES-Spain and the UK-United Kingdom, SK- Slovakia, IE-Ireland and EL-Greece.

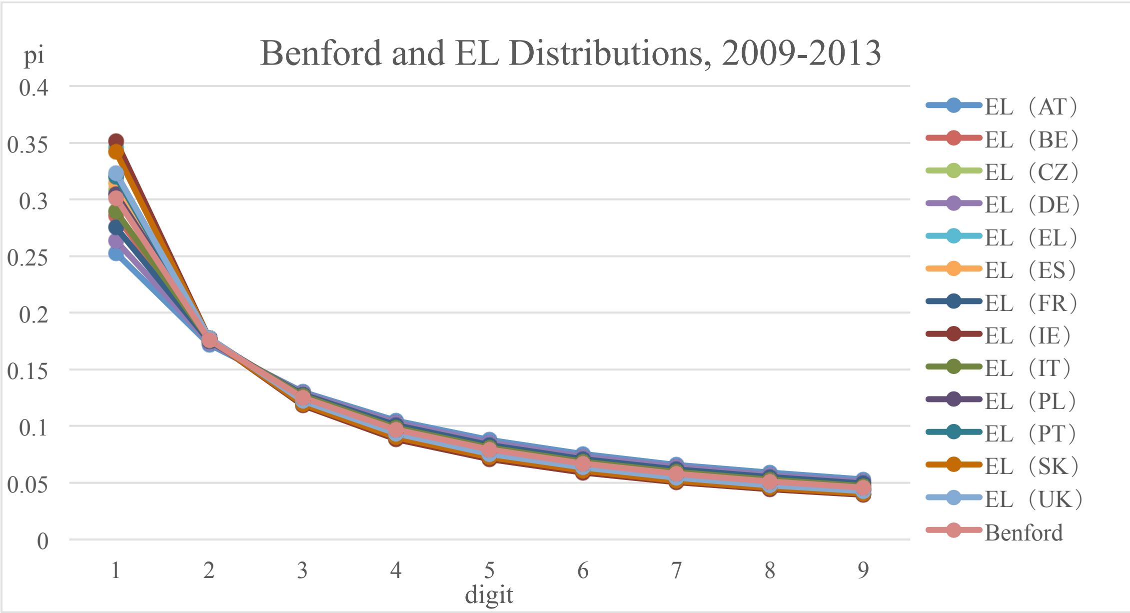

Figure 4.2 Income FSD Probability Density Functions for Central and Eastern European countries

Like the FSD distributions noted earlier in physical statistics, and the entropy measures in Figure 4.1, all the 13 country FSD density functions, 1) closely fluctuate around the Benford distribution, 2) have the Benford exponential distribution shape and 3) share similar features relative to their FSD probability density-income distribution functions and entropy measure. Also the first moment information-mean of the distributions of the European countries, closely fluctuate around the first moment Benford mean 3.4402.

For clarification and comparative purposes and to provide some additional information for the combined FSD distributions in Figures 4.1 and 4.2, in order to make sure the countries are sufficiently differentiated, we break the European countries into the following three groups: first the Central European countries AT-Austria, DE-Germany, FR-France, BE-Belgium and IT-Italy; second PL-Poland, CZ-Czech Republic, PO-

Portugal, ES-Spain and the UK-United Kingdom; and third SK-Slovakia, IE-Ireland and EL-Greece. As a basis for gauging the goodness of fit when comparing the country income FSD data distributions with the Benford distribution, we make use of the commonly used Chi Square test statistic. If the counts of the digits are statistically independent, then assuming the null hypothesis of Benfords law to hold, the statistic may be compared to the chi square distribution with nine degrees of freedom. The goodness of fit-Chi Square test statistics comparisons of the empirical income FSD distributions and the Benford distribution for the countries in Figure 4.2, are presented in Tables 4.2.1, 4.2.2 and 4.3.3 of sections 4.2.1, 4.2.2 and 4.2.3 that follow. As will be noted in these Tables in the subsections of Section 4.2, all of these countries have income FSD probability density functions that are highly correlated with the Benford FSD distribution and inference wise relative to the Benford FSD distribution, have very low Chi Square values of statistical significance.

4.2 .1 Central European Countries

Figure 4.3 permits us to see visually see how closely the EL income FSD distributions for Austria (AT), Germany (DE), France (FR), Belgium (BE) and Italy (IT) follow the Benford distribution and the chi square goodness of fit test in Table 4.1 confirms we cannot reject the null hypothesis of equality of the country distributions with Benford. .

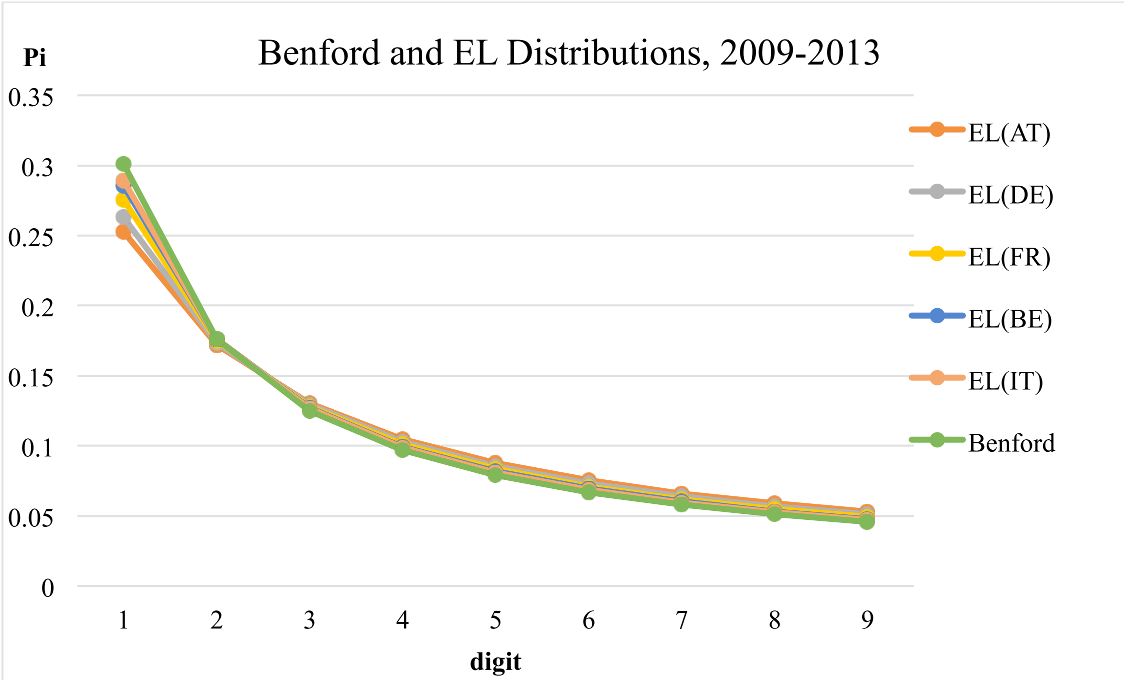

Figure 4.3 Central European Countries

Table 4.1 Chi Square and correlation Values for Figure 4.3

	EL(AT)	EL(DE)	EL(FR)	EL(BE)	EL(IT)
Chi square	0.01411	0.00839	0.00374	0.00130	0.00074
Correlation	0.99518	0.99712	0.99868	0.99950	0.99969

4.2.2 Central and Eastern European Countries

Oce again the visual evidence of distribution compatibility in Figure 4.4 is clear, and inference wise the chi square goodness of fit statistic is once again smaller than the critical value. Thus we cannot reject the null of equality between Benford and the EL distribution of the first significant digits for these countries. Additionally, the correlations between Benford and EL distributions in Table 4.2 are very high for all pair wise comparisons.

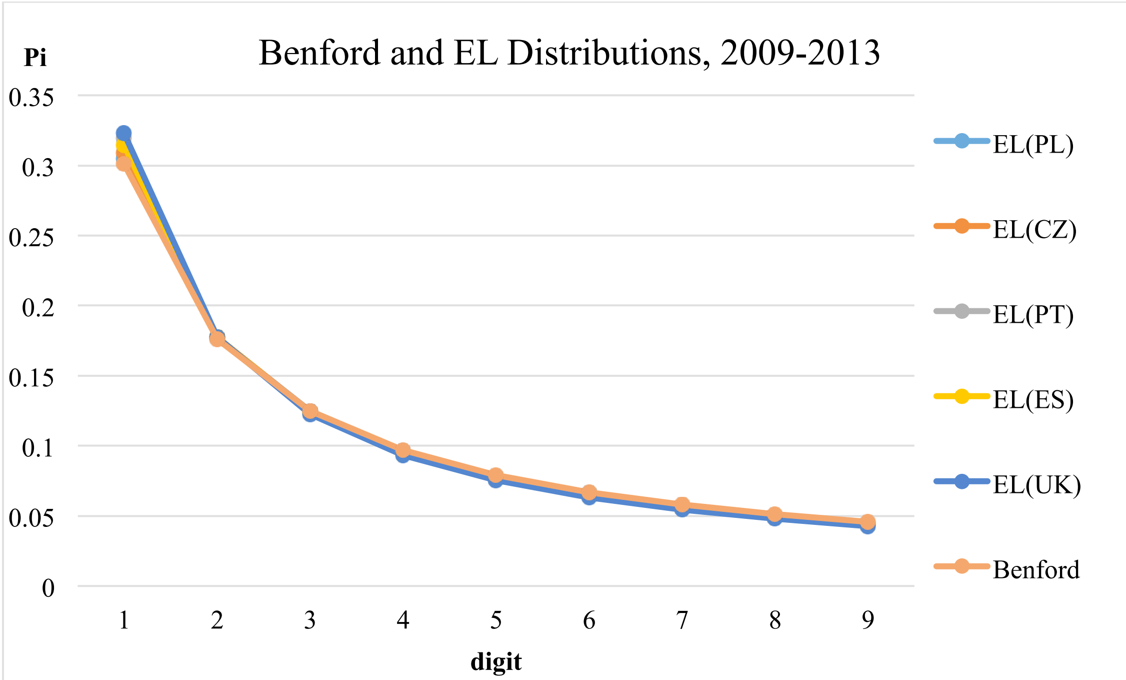

Figure 4.4 Central and Eastern European Countries

Table 4.2 Chi Square and Correlation Values for Figure 4.4

	EL(PL)	EL(CZ)	EL(PT)	EL(ES)	EL(UK)
Chi square	0.01092	0.00039	0.00202	0.01092	0.00282
Correlation	0.99750	0.99994	0.99958	0.99750	0.99939

4.2.3 Slovakia, Ireland and Greece

Again visually and inference wise the compatibility between the income distributions and the Benford distribution are demonstrated in Figure 4.5 and Table 4.3.

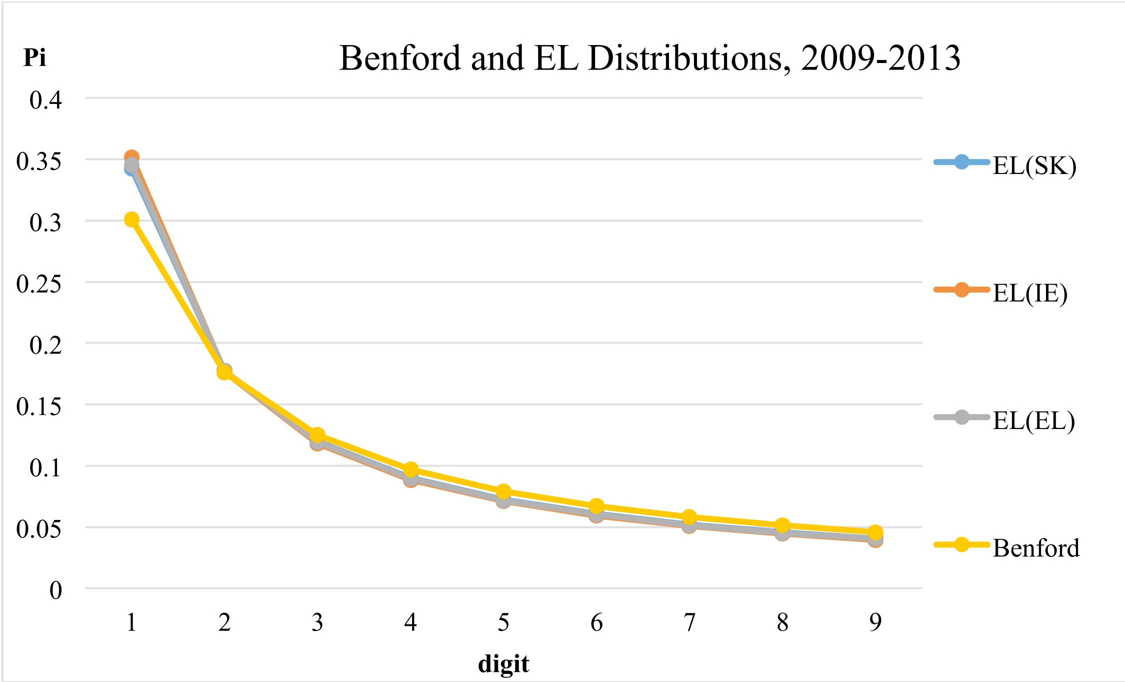

Figure 4.5. Income FSD Density Functions and the Benford Distribution

Table 4.3 Chi Square and Correlation Values for Figure 4.5

	EL(SK)	EL(IE)	EL(EL)
Chi square	0.00936	0.01400	0.01092
Correlation	0.99786	0.99680	0.99750

As an example the time ordered-dynamic characteristics of income FSD distributions, annual comparisons between the empirical likelihood(EL) FSD distributions and the Benford FSD distribution for Germany, are presented in the Appendix A for the years 2005-2013.

5. Income Density Function FSDs With Benford PRIOR Distribution

In estimating the FSD income probability density functions, to acknowledge their decreasing monotonic nature, instead of a uniform distribution suppose we follow Grendar, et al. [19], and use the Benford distribution, q_B, as the empirical likelihood reference distribution in (2.4). Thus, in the CR formulation, $\gamma \rightarrow -1\ in\ (2.4)$, Benford reference distribution probabilities q_B replace the uniform reference distribution. This leads to the BEL or Benford EL, criterion

$$\lim_{\gamma\to -1} \mathrm{I}(\mathrm{p}, \mathrm{q_B}, \gamma) = \sum_{\mathrm{j}=1}^{9} \mathrm{q_{jB}} \ln\left(\mathrm{p_j}/\mathrm{q_{jB}}\right) = \sum_{\mathrm{j}=1}^{9} \mathrm{q_{jB}} \ln\left(\mathrm{p_j}\right) - \sum_{\mathrm{j}=1}^{9} \mathrm{q_{jB}} \ln\left(\mathrm{q_{jB}}\right). \quad (5,1)$$

where $\sum_{j=1}^{9} q_{jB} \ln\left(q_{jB}\right)$ is an added constant. Using this revised criterion, the data constraint $\sum_{j=1}^{9} d_j\, p_j = \bar{d}$, and the probabilities adding-up condition, results in

$$\widehat{\mathrm{p_{jB}}}\left(\bar{\mathrm{d}}, \hat{\lambda}\right) = \mathrm{q_{jB}}\left(1 + \hat{\lambda}\left(\mathrm{d_j} - \bar{\mathrm{d}}\right)\right)^{-1}, \qquad (5.2)$$

where $\hat{\lambda}$ is such that $\widehat{\mathrm{p_B}}\,(\mathrm{d}, \hat{\lambda})$ satisfies the mean FSD constraint.

As an example of the implications of using Benford as a reference distribution, we use of the Central and Eastern European micro income data discussed in Section 4.2.2.

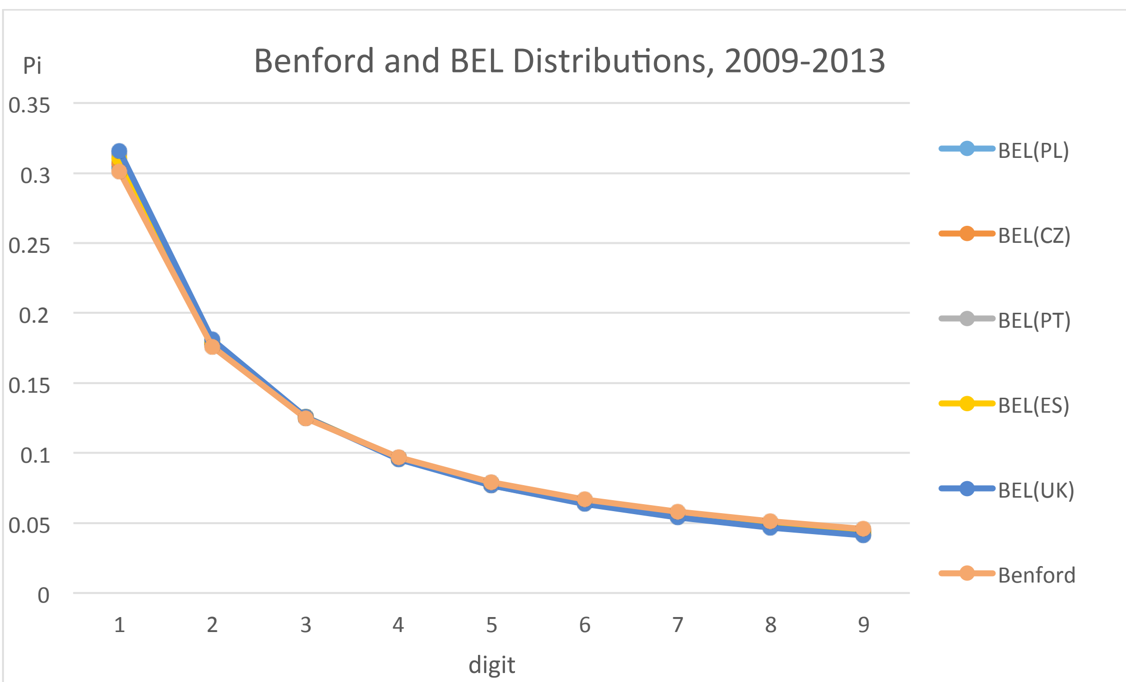

Figure 5.1 Income FSD Distributions For Central and Eastern European Countries with a Benford Reference Prior

Table 5.1 Chi Square and Correlations for Figure 5.1

	BEL(PL)	BEL(CZ)	BEL(PT)	BEL(ES)	BEL(UK)
Chi square	0.00010	0.00034	0.00162	0.00089	0.00223
Correlation	0.99999	0.99999	0.99999	0.99999	0.99999

As indicated in Figure 5.1 and Table 5.1, when a prior reference distribution is used, the FSD income distributions for these countries, almost exactly follow the Benford FSD distribution. Similar results follow for the other eight European countries.

6. Some Concluding Remarks

In this paper we have presented evidence that important economic behavioral systems FSD distributions in are closely linked to the Benford's FSD distribution. Both inference and correlation wise we have noted the close relationship between the income FSD distribution of micro data in Europe and Benford's FSD law. These results that relate to important distributions from both the physical and social-behavioral worlds, add another bit of evidence in the direction that Benfords's law is not an artifact but a natural law. As we have demonstrated in the CR family of entropic functions, in the limit as $\gamma \rightarrow -1$, the empirical likelihood distribution with a Benford as the reference distribution closely follows the Benford distribution. Although not presented, other members of the CR family of entropy distributions also reflect the exponential nature of the distribution of income in behavioral systems, and appear to denote when used with income micro data the universal nature of Benford's law. From a methodological standpoint we have demonstrated in a behavioral systems context, how information theoretic methods may be used to identify income FSD probability density functions and make distributional comparisons with Benford's law. In the CR family of entropic functions, in the limit as $\gamma \rightarrow -1$, the empirical likelihood distribution with a known reference distribution provides a new basis for combining distributions and possibly introducing a dynamic income distribution component.

Appendix: Time Path of Income FSD Probability Density Functions For Germany

As an example of the time ordered-dynamic characteristics of income FSD distributions, annual comparisons between the Empirical Likelihood (EL) FSD distributions and the Benford FSD distribution for Germany are presented in Figure A.1 for the years 2005-2013. As expected the annual income FSDs for Germany fluctuate closely around the Benford FSD distribution. As indicated by the Chi-Square and correlation values in Table A.1, the annual distributions match with Benford is excellent. If we use the Benford distribution as the reference distribution (BEL) in obtaining the estimate of the income distribution, the Benford and BEL distributions are almost identical.

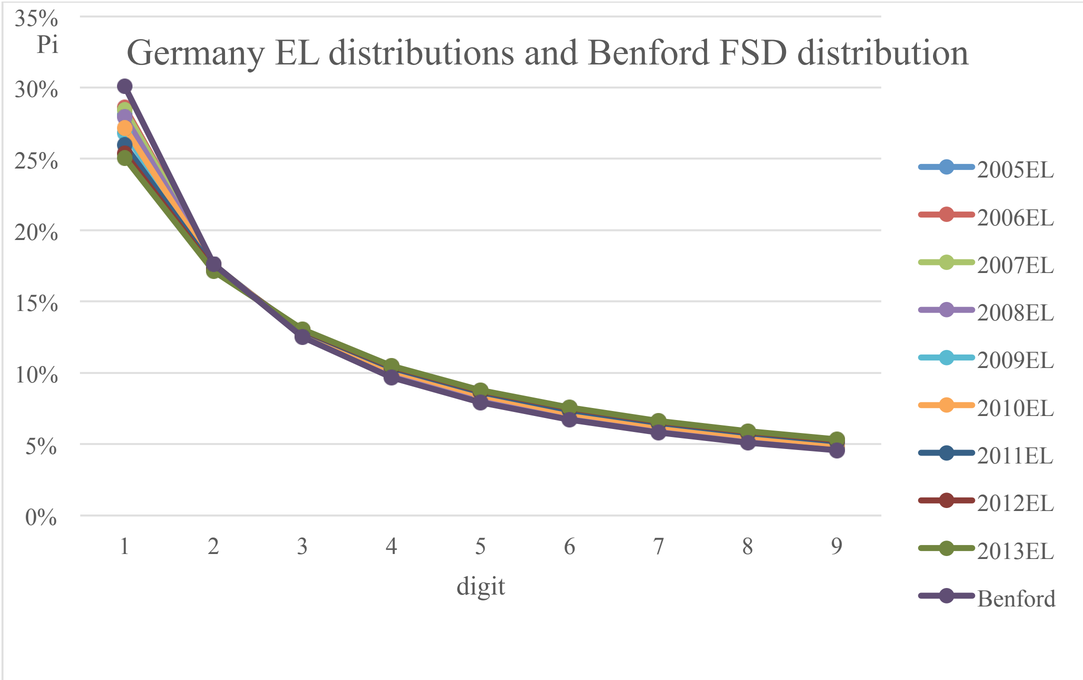

Figure A.1 Annual German EL distributions and the Benford FSD distribution

	2005EL	2006EL	2007EL	2008EL	2009EL	2010EL	2011EL	2012EL	2013EL
Chi square	0.0023	0.0012	0.0016	0.0026	0.0062	0.0049	0.0103	0.0134	0.0154
Correlation	0.9992	0.9995	0.9994	0.9991	0.9978	0.9983	0.9965	0.9954	0.9947

Table A.1 Chi Square values, correlations and significance for yearly EL German FSD distributions and the Benford distribution

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