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# Stability conditions on Kuznetsov components of Gushel-Mukai threefolds and Serre functor

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by

Ethan Robinett

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June 2022

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My research interests lie in the field of algebraic geometry. Specifically, I study existence questions involving Bridgeland stability conditions and properties of the corresponding stability manifolds.

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## Abstract

Stability conditions on Kuznetsov components of Gushel-Mukai threefolds and Serre functor

Ethan Robinett

This dissertation focuses on the construction of Serre-invariant Bridgeland stability conditions on Kuznetsov components of Gushel-Mukai threefolds. In particular, for a Gushel-Mukai threefold $X$ with Kuznetsov component $\mathrm{Ku}(X)$ and Serre functor $S_{\mathrm{Ku}(X)}$, we find a family of stability conditions $\sigma(s,q)$ on $\mathrm{Ku}(X)$ such that $S_{\mathrm{Ku}(X)} \cdot \sigma(s,q) = \sigma(s,q) \cdot \tilde{g}$ for some $\tilde{g}$ residing in the universal cover of $\mathrm{GL}_2^+(\mathbb{R})$. This leads to an explicit construction of Bridgeland stability conditions on Kuznetsov components of special Gushel-Mukai fourfolds, which previously was not known.

# Contents

# 1 Introduction

## 1.1 Slope Stability

The idea of a stability condition on a triangulated category, as introduced by Bridgeland in [Bri07], is rooted in the classical idea of slope-stability for smooth projective curves over $\mathbb{C}$. The following theorem of Grothendieck is arguably the starting point for this line of inquiry:

**Theorem 1.1.** *Let $E$ be a vector bundle on $\mathbb{P}^1$. Then there are integers $n_i, k_i$ such that $E \cong \bigoplus_i \mathcal{O}(n_i)^{\oplus k_i}$.*

For both curves of positive genus and varieties of higher dimension, this theorem fails. However, we can retain some of the principal characteristics by replacing the direct sum decomposition above with a certain filtration and introducing a notion of *(semi)stability*, subsequently replacing the summands above with (semi)stable factors.

If we restrict the present discussion to smooth projective curves $C$, then making these replacements is not very difficult. Fixing a coherent sheaf $E \in \mathrm{Coh}(C)$, we define the rank of $E$, denoted $\mathrm{rk}(E)$, to be the rank of $E$ at the generic point of $C$. Equivalently, there is a short exact sequence:

$$0 \to T_E \to E \to F_E \to 0$$

where $T_E$ is a torsion sheaf and $F_E$ is locally free. We have $\mathrm{rk}(E) = \mathrm{rk}(F_E)$. Denoting the Euler characteristic $\chi(C, -)$, we set the degree of $E$ to be $\deg(E) = \chi(C, E) - \mathrm{rk}(E)\chi(C, \mathcal{O}_C)$. The *slope* of $E$ is then:

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)},$$

where division by 0 is interpreted as $+\infty$. Note that the $\frac{0}{0}$ case cannot occur: if $\mathrm{rk}(E) = 0$, then $E$ is torsion, so $\deg(E) = \chi(C, E) > 0$.

We say $E$ is *(semi)stable* if for any proper, nonzero subsheaf $0 \to F \to E$, we have $\mu(F) < (\leqslant) \mu(E)$. The significance of these definitions lies in the following theorem:

**Theorem 1.2.** *Let $E$ be a nonzero coherent sheaf on $C$. Then there is a unique filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$ of coherent sheaves such that $A_i = E_i/E_{i-1}$ is semi-stable and $\mu(A_1) > \mu(A_2) > \cdots > \mu(A_n) = \mu(E)$.*

The filtration in Theorem 1.2 is called the *Harder-Narasimhan* (HN) filtration of $E$. When $C = \mathbb{P}^1$, Theorem 1.1 gives this filtration directly: the stable quotients are line bundles, and we may build the filtration from the direct sum decomposition.

Besides the content of Theorem 1.2, there are many senses in which slope stability is both convenient and very well-behaved on curves. For instance, stable sheaves satisfy a form of Schur's Lemma. To be precise, if $E, F \in \mathrm{Coh}(C)$ are stable with $\mu(E) \geqslant \mu(F)$ and $\mathrm{Hom}(E, F) \neq 0$, then $E \cong F$ and $\mathrm{Hom}(E, F) \cong \mathrm{Hom}(E, E) \cong \mathbb{C}$. Additionally, semi-stable coherent sheaves admit Jordan-Hölder filtrations: if $E \in \mathrm{Coh}(C)$ is semi-stable, then there is a filtration:

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

of coherent sheaves with stable quotients $A_i = E_i/E_{i-1}$ such that $\mu(A_i) = \mu(A_j)$ for any pair $1 \leqslant i, j \leqslant n$.

It is natural, given this good behavior, to attempt to implement the idea of slope stability in higher dimensions, though the careful reader will notice that certain definitions will need drastic modification in order to accomplish this. Fix a smooth projective variety $X$ with $\dim X = n$ and an ample divisor class $H$ on $X$. The most direct way to obtain an analogous notion of stability of coherent sheaves on $X$ is to define the slope function:

$$\mu_H(E) = \frac{H^{n-1} \cdot \mathrm{ch}_1(E)}{H^n \cdot \mathrm{ch}_0(E)}$$

where $\cdot$ denotes the intersection product and $\mathrm{ch}_i(E)$ denotes the codimension-$i$ part of the exponential Chern character of $E$ (formally, the computations defining the numerator and denominator of $\mu_H$ take place in the graded Chow ring of $X$). Note that, when $X$ is a curve, we recover the formerly defined slope function.

There are immediate problems with this method of evaluating the slope of coherent sheaves. In particular, for $E \in \mathrm{Coh}(X)$ with $\mathrm{ch}_0(E) = 0$, it is not true in general that $H^{n-1} \cdot \mathrm{ch}_1(E) > 0$, in contrast to the case for curves. In fact, if $X$ is a surface and $E$ is a nonzero torsion sheaf supported in dimension 0, then $\mathrm{ch}_1(E) = \mathrm{ch}_0(E) = 0$. Nevertheless, this naive generalization of slope stability will be useful in the sequel.

## 1.2 Bridgeland Stability

Bridgeland stability was introduced by Bridgeland in [Bri07]. In contrast to classical slope stability, where (semi)stability is defined directly on objects in the abelian category $\mathrm{Coh}(X)$, Bridgeland stability allows for notions of (semi)stability to be defined in arbitrary triangulated categories, with a view towards the bounded derived category of coherent sheaves on $X$ (henceforth denoted $D^bX$). For the basics on triangulated categories and the derived category, the reader is referred to [Huy06].

Let $\mathcal{T}$ be a triangulated category. Following Bridgeland in [Bri07], we will always assume $\mathcal{T}$ is essentially small, that is, the object class of $\mathcal{T}$ is a set. Also, we will assume that $\mathcal{T}$ is linear and of finite type over $\mathbb{C}$, meaning that the $\mathrm{Hom}$-sets of $\mathcal{T}$ are $\mathbb{C}$-vector spaces of finite dimension. We begin with the following fundamental definition:

**Definition 1.3.** A *heart of a bounded t-structure* is a full subcategory $\mathcal{A} \subseteq \mathcal{T}$ such that:

1. For any $E, F \in \mathcal{A}$ and $k < 0$, we have $\mathrm{Hom}(E, F[k]) = 0$.

2. For any $E \in \mathcal{T}$, there is a filtration:

$$0 = E_0 \xrightarrow{\phi_1} E_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_m} E_m = E$$

   such that for each $i$, $\mathrm{Cone}(\phi_i) \cong A_i[k_i]$ for some $A_i \in \mathcal{A}$ and $k_1 > k_2 > \cdots > k_m$.

T-structures in general were introduced by Beilinson, Bernstein and Deligne in [BBD82] as a means by which to view the numerous abelian subcategories embedded in a given triangulated category $\mathcal{T}$. If $\mathcal{F} \subseteq \mathcal{T}$ is a subcategory and we define the right orthogonal:

$$\mathcal{F}^{\perp} = \{E \in \mathcal{T} : \mathrm{Hom}(F, E) = 0 \text{ for all } F \in \mathcal{F}\},$$

then $\mathcal{F}$ is a *t-structure* on $\mathcal{T}$ if $\mathcal{F}$ is full, satisfies $\mathcal{F}[1] \subseteq \mathcal{F}$ and every $E \in \mathcal{T}$ fits into a triangle $A \to E \to B$ with $A \in \mathcal{F}$ and $B \in \mathcal{F}^{\perp}$. $\mathcal{F}$ is said to be *bounded* if every $E \in \mathcal{T}$ has $E \in \mathcal{F}[i] \cap \mathcal{F}^{\perp}[j]$ for some $i, j \in \mathbb{Z}$.

Every bounded t-structure $\mathcal{F}$ admits a heart, given by the intersection $\mathcal{F} \cap \mathcal{F}^{\perp}[1]$, which can be verified to satisfy the conditions of Definition 1.3. Conversely, given a heart $\mathcal{A} \subseteq \mathcal{T}$, there is a unique bounded t-structure on $\mathcal{T}$ determined by $\mathcal{A}$. A heart of a bounded t-structure is an abelian subcategory of $\mathcal{T}$, and triangles with objects in $\mathcal{A}$ are actually short exact sequences [BBD82].

The idea of a heart of a bounded t-structure is one of the two major components of Bridgeland stability. The other component is the notion of a (weak) stability function, which we define now:

**Definition 1.4.** Let $\mathcal{A}$ be an abelian category. A *weak stability function* on $\mathcal{A}$ is a homomorphism of groups:

$$\begin{aligned} Z \colon K(\mathcal{A}) &\to \mathbb{C} \\ E &\mapsto \Re Z(E) + i\Im Z(E) \end{aligned}$$

where $K(\mathcal{A})$ denotes the Grothendieck group of $\mathcal{A}$, such that for all $0 \neq E \in \mathcal{A}$, we have $\Im Z(E) \geqslant 0$ and $\Im Z(E) = 0$ implies $\Re Z(E) \leqslant 0$. We say that $Z$ is a *stability function* if, in addition, when $\Im Z(E) = 0$ we have $\Re Z(E) < 0$.

Stability conditions are defined with respect to a fixed, finite-rank lattice $\Lambda$. We also fix a surjective group homomorphism $v : K(\mathcal{A}) \to \Lambda$.

**Definition 1.5.** A *weak stability condition* on $\mathcal{T}$ with respect to $\Lambda$ is a pair $\sigma = (\mathcal{A}, Z)$, where $\mathcal{A}$ is a heart of a bounded t-structure and $Z \colon \Lambda \to \mathbb{C}$ is a group homomorphism, such that:

1. The composition $K(\mathcal{A}) \xrightarrow{v} \Lambda \xrightarrow{Z} \mathbb{C}$ is a weak stability function on $\mathcal{A}$. We will omit the function $v$ and write $Z(E) = Z(v(E))$ for brevity. Given such a $Z$, we may define the slope of any $E \in \mathcal{A}$ as:

$$\mu_\sigma(E) = \begin{cases} -\frac{\Re Z(E)}{\Im Z(E)} & \Im Z(E) \neq 0 \\ +\infty & \text{otherwise.} \end{cases}$$

   We also obtain a notion of semistability (stability): we say $0 \neq E \in \mathcal{A}$ is $\sigma$-semistable ($\sigma$-stable) if for any nonzero, proper subobject $F \hookrightarrow E$, we have $\mu_\sigma(F) \leqslant \mu_\sigma(E)$ ($\mu_\sigma(F) < \mu_\sigma(E/F)$).

2. Any $E \in \mathcal{A}$ admits a Harder-Narasimhan filtration with $\sigma$-semistable factors. Explicitly, this means that given $E \in \mathcal{A}$, there is a filtration:

$$0 = E_0 \xrightarrow{\phi_1} E_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_m} E_m = E$$

   such that $E_i/E_{i-1}$ is $\sigma$-semistable, with $\mu_\sigma(E_1/E_0) > \cdots > \mu_\sigma(E_m/E_{m-1})$.

3. (Support Property) There is a quadratic form $Q$ on $\Lambda \otimes \mathbb{R}$ such that $Q|_{\ker Z}$ is negative-definite, and $Q(E) \geqslant 0$ for all $\sigma$-semistable $E \in \mathcal{A}$.

**Definition 1.6.** A weak stability condition $\sigma = (\mathcal{A}, Z)$ on $\mathcal{T}$ with respect to $\Lambda$ is called a *stability condition* if $Z$ is a stability function.

The support property admits an equivalent formulation avoiding the quadratic form $Q$ above. Fix a norm $||\cdot||$ on $\Lambda \otimes \mathbb{R}$. Then the support property is equivalent to the claim that:

$$\inf \left\{ \frac{|Z(v(E))|}{||v(E)||} : 0 \neq E \in \mathcal{A} \text{ is semistable} \right\} > 0.$$

We remark here that slope stability defines a weak stability condition, so this is indeed a generalization. The heart is the canonical subcategory $\mathrm{Coh}(X) \hookrightarrow D^bX$, and fixing an ample divisor class $H$ on $X$, the stability function is $Z(v(E)) = -H^{n-1} \cdot \mathrm{ch}_1(E) + i(H^n \cdot \mathrm{ch}_0(E))$. The pair $(\mathrm{Coh}(X), Z)$ forms a stability condition on $X$ if and only if $X$ is a curve [Tod09].

Fix a (weak) stability condition $\sigma = (\mathcal{A}, Z)$ on $\mathcal{T}$. Given a semistable object $E \in \mathcal{A}$ with $Z(E) \neq 0$, we define the *phase* of $E$ as

$$\phi(E) = \frac{1}{\pi}\arg(Z(E)).$$

If $Z(E) = 0$, we set $\phi(E) = 1$, and for any shift $E[n]$, we define $\phi(E[n]) = \phi(E) + n$. The notion of phase of a semistable object $E \in \mathcal{A}$ naturally gives rise to a *slicing* of $\mathcal{T}$.

**Definition 1.7.** A *slicing* $\mathcal{P}$ of $\mathcal{T}$ is a collection of full additive subcategories $\mathcal{P}(\phi)$ of $\mathcal{T}$ for each $\phi \in \mathbb{R}$ such that:

1. For $\phi \in (0, 1]$, $\mathcal{P}(\phi)$ is the subcategory of all $\sigma$-semistable objects of phase $\phi$, together with the zero object.

2. For $\phi \in (0, 1]$ and $n \in \mathbb{Z}$, $\mathcal{P}(\phi + n) = \mathcal{P}(\phi)[n]$.

Given a (weak) stability condition $\sigma$ with slicing $\mathcal{P}$, the heart $\mathcal{A}$ is recovered via $\mathcal{A} = \mathcal{P}((0, 1])$, and conversely, a slicing $\mathcal{P}$ arises from $\sigma$ immediately by definition. In the next, we will use the notations $(\mathcal{A}, Z)$ and $(\mathcal{P}, Z)$ for a (weak) stability condition interchangeably.

We write $\mathrm{Stab}_\Lambda(\mathcal{T})$ to denote the set of stability conditions on $\mathcal{T}$. The space $\mathrm{Stab}_\Lambda(\mathcal{T})$ can be given a metrizable topology in a natural way, and Bridgeland [Bri07] proved that with this topology, the map $\mathrm{Stab}_\Lambda(\mathcal{T}) \to \mathrm{Hom}(\Lambda, \mathbb{C})$ given by $(\mathcal{A}, Z) \mapsto Z$ is a local homeomorphism, hence $\mathrm{Stab}(\mathcal{T})$ is a complex manifold of dimension $\mathrm{rk}(\Lambda)$.

The manifold $\mathrm{Stab}_\Lambda(\mathcal{T})$ admits two natural group actions, one from the universal cover of $\mathrm{GL}_2^+(\mathbb{R})$ (denoted $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$) and one from the group $\mathrm{Aut}_\Lambda(\mathcal{T})$ of exact autoequivalences which are compatible with $v$. For the former of these, given some $\tilde{g} = (g, M) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ with $M \in \mathrm{GL}_2^+(\mathbb{R})$ and $g : \mathbb{R} \to \mathbb{R}$ increasing with $g(\phi + 1) = g(\phi) + 1$, the action on a stability condition $\sigma = (\mathcal{P}, Z)$ is given by $\sigma \cdot \tilde{g} = (\mathcal{P}', M^{-1} \circ Z)$, where $\mathcal{P}'(\phi) = \mathcal{P}(g(\phi))$. For the $\mathrm{Aut}_\Lambda(\mathcal{T})$-action, given some $\Phi \in \mathrm{Aut}_\Lambda(\mathcal{T})$, we have $\Phi \cdot \sigma = (\Phi(\mathcal{P}), Z \circ \Phi_*^{-1})$, where $\Phi_*$ is the induced automorphism on $K(\mathcal{T})$.

The first issue in the construction of stability conditions is the production of a suitable heart of a bounded t-structure. For instance, the canonical choice of $\mathrm{Coh}(X)$ cannot be the heart of a stability condition with respect to the numerical Grothendieck group $\Lambda = \mathcal{N}(X)$ of $X$, unless $X$ is a curve [Tod09]. However, if we have a (weak) stability condition $(\mathcal{A}, Z)$, it is sometimes possible to produce a new heart by *tilting* the old one. We discuss this procedure now.

**Definition 1.8.** Let $\mathcal{A}$ be an abelian category. A *torsion pair* is a pair of two full, additive subcategories $(\mathcal{F}, \mathcal{T})$ of $\mathcal{A}$ such that:

1. For any $T \in \mathcal{T}, F \in \mathcal{F}$, we have $\mathrm{Hom}(T, F) = 0$.
2. Given any $E \in \mathcal{A}$, there are $T \in \mathcal{T}, F \in \mathcal{F}$ and a short exact sequence:

$$0 \to T \to E \to F \to 0$$

The importance of this notion comes from the following theorem:

**Theorem 1.9** ([HRS96]). *Let $\mathcal{A} \subset \mathrm{D^b}(X)$ be a heart of a bounded t-structure and let $(\mathcal{F}, \mathcal{T})$ be a torsion pair in $\mathcal{A}$. Then the extension-closure $\langle \mathcal{F}[1], \mathcal{T} \rangle$ is also a heart of a bounded t-structure in $\mathrm{D^b}(X)$.*

Given a (weak) stability condition $\sigma = (\mathcal{A}, Z)$ on $\mathrm{D^b}(X)$, one may produce a new heart according to the theorem above by choosing any $\mu \in \mathbb{R}$ and considering the following torsion pair:

$$\mathcal{F}_\sigma^\mu = \langle E \in \mathcal{A} : E \text{ is semistable with } \mu_\sigma(E) \leqslant \mu \rangle$$
$$\mathcal{T}_\sigma^\mu = \langle E \in \mathcal{A} : E \text{ is semistable with } \mu_\sigma(E) > \mu \rangle.$$

We say that the new heart $\langle \mathcal{F}_\sigma^\mu[1], \mathcal{T}_\sigma^\mu \rangle$ is constructed by tilting the (weak) stability condition $\sigma$ at the slope $\mu$. This construction is ubiquitous in what follows.

**Example 1.10.** ([BLMS17, Example 2.8]) Let $X$ be a smooth projective variety of dimension $n$ with an ample class $H$. We have that the group morphism

$$Z_H \colon \Lambda \cong \mathbb{Z}^2 \to \mathbb{C}; \quad (H^n \operatorname{rk}(E), H^{n-1} \operatorname{ch}_1(E)) \mapsto -H^{n-1} \operatorname{ch}_1(E) + H^n \operatorname{rk}(E)\sqrt{-1}$$

defines a weak stability function on $\mathrm{Coh}(X)$. Moreover, the pair $\sigma_H = (\mathrm{Coh}(X), Z_H)$ is a weak stability condition on $\mathrm{D^b}(X)$ with respect to $\Lambda$, known as *slope stability.* The slope with respect to $\sigma_H$ is denoted by $\mu_H$. Furthermore, if $n = 1$, then $\sigma_H$ is a stability condition on $\mathrm{D^b}(X)$.

We remark that slope semistable coherent sheaves satisfy the classical Bogomolov–Gieseker inequality: for every $\mu_H$-semistable $E \in \mathrm{Coh}(X)$ we have the inequality

$$(H^{n-1}\,\mathrm{ch}_1(E))^2 - 2H^n\,\mathrm{rk}(E)H^{n-2}\,\mathrm{ch}_2(E) \geqslant 0 \tag{1}$$

### 1.3 Existence of Bridgeland Stability Conditions

In general, the construction of Bridgeland stability conditions and the study of the stability manifold is very difficult, even when restricted to $D^bX$. The purpose of this section is to give a brief overview of the current state of affairs in this direction.

A complete description of the stability manifold is known only when $X$ is a curve. The stability manifold of $\mathbb{P}^1$ is isomorphic to $\mathbb{C}^2$ [Oka06b]. For curves of positive genus, the action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ on the stability manifold is transitive, and the stability manifold itself is therefore isomorphic to $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ [Bri07, Mac07].

There are some descriptions of connected components of $\mathrm{Stab}(X)$ for some varieties $X$. In [Bri08] and [BB], a connected component of $\mathrm{Stab}(X)$ for $X$ a K3 or abelian surface is described. There are also analogous descriptions for twisted K3 or abelian surfaces [HMS]. In [Oka06a], the stability manifold of the cotangent bundle of $\mathbb{P}^1$ is shown to be connected.

Stability conditions are known to exist for various classes of varieties, including surfaces [AB13], abelian threefolds and some finite resolutions of abelian threefolds [BMS16], Fano threefolds [BMT14] and quintic threefolds [Li19a]. Stability conditions are also known to exist on Gushel-Mukai varieties [PPZ19] and on certain product varieties [Liu19].

Lastly, the content of this paper focuses mainly on stability conditions on Kuznetsov components of certain varieties, a notion that will be properly defined later. It is known that stability conditions exist on Kuznetsov components of all Gushel-Mukai varieties [PPZ19]. In [BLMS17], existence is proven for Fano threefolds and cubic fourfolds as a result of a more general method of inducing stability on the Kuznetsov component via weak stability on the derived category.

## 1.4 Gushel-Mukai Varieties and Kuznetsov Components

A Gushel–Mukai (GM) variety of dimension $n$, for $2 \leqslant n \leqslant 6$, is a smooth intersection

$$\mathrm{Cone}(\mathrm{Gr}(2,5)) \cap Q,$$

where $\mathrm{Cone}(\mathrm{Gr}(2,5))$ is the projective cone over the Plücker-embedded Grassmanian $\mathrm{Gr}(2,5) \hookrightarrow \mathbb{P}^9$ and $Q$ is a quadric hypersurface in some $\mathbb{P}(W) \cong \mathbb{P}^{n+4} \hookrightarrow \mathbb{P}^{10}$. Gushel [Gus82] and Mukai [Muk89] showed that for $n \geqslant 3$, GM varieties are precisely the Fano varieties of Picard number 1, degree 10 and coindex 3, while if $n = 2$, GM surfaces are Brill–Noether general polarized K3 surfaces. If the vertex of the cone $\mathrm{Cone}(\mathrm{Gr}(2,5))$ is not in the linear section $\mathbb{P}(W)$, then $X$ is an *ordinary* GM variety, otherwise $X$ is a *special* GM variety.

GM varieties are intriguing for many reasons. There is a bijective correspondence between triples $(V, W, A)$, where $V$ is a 6-dimensional vector space, $W \hookrightarrow V$ is a hyperplane and $A \hookrightarrow \bigwedge^3 V$ is a Lagrangian subspace (with the symplectic form given by wedge product), and the so-called *GM data sets* [Deb20]. A GM data set of dimension $n$ is a tuple $(W_{n+5}, V_6, V_5, q)$, with $V_6$ a 6-dimensional vector space, $V_5 \hookrightarrow V_6$ a hyperplane, $W_{n+5} \hookrightarrow \bigwedge^2 V_5$ a subspace of dimension $n+5$ and $q : V_6 \to \mathrm{Sym}^2 W_{n+5}^\vee$ a linear map with $q(v)(w,w) = v \wedge w \wedge w$ for all $v \in V_5$ and $w \in W_{n+5}$. Note that given a GM data set as above, for each $v \in V_6$ we may define $Q(v) \hookrightarrow \mathbb{P}(W_{n+5})$ to be the hypersurface cut out by $q(v)$. The intersection:

$$X = \bigcap_{v \in V_6} Q(v)$$

defines a GM variety when it is smooth. When $3 \leqslant n \leqslant 5$, the aforementioned correspondence can be refined further to yield a bijection between smooth, ordinary GM varieties and the the same tuples $(V, W, A)$ above with $A$ containing no decomposable vectors and $\dim(A \cap \wedge^3 V_5) = 5 - n$.

Beyond the correspondence with Lagrangian data sets, GM varieties exhibit interesting rationality properties. All GM varieties in dimensions 5 and 6 are rational, and GM varieties are unirational in dimensions 3 and 4 [Deb20], but general 3 and 4 dimensional GM varieties are not expected to be rational. GM varieties of dimension $n \geqslant 3$ with either isomorphic or dual associated Lagrangians $A$ are birational [DK]. The situation regarding the rationality of GM fourfolds is analogous to the corresponding question for cubic fourfolds: many rational examples are known and it is expected that a very general GM fourfold is irrational, but not a single example of a provably irrational GM fourfold is currently known.

Finally, and most pertinently to our discussion, the derived category of a GM variety has particularly nice properties. Kuznetsov and Perry [KP18] proved that the bounded derived category $\mathrm{D^b}(X)$ of a GM variety $X$ of dimension $n \geqslant 3$ admits a semiorthogonal decomposition of the form

$$\mathrm{D^b}(X) = \langle \mathrm{Ku}(X), \mathcal{O}_X, \mathcal{U}_X^\vee, ..., \mathcal{O}_X((n-3)H), \mathcal{U}_X^\vee((n-3)H) \rangle, \quad (2)$$

where $\mathcal{U}_X$ is the pullback to $X$ of the rank 2 tautological subbundle on the Grassmannian, $H \subset X$ is a hyperplane class and $\mathrm{Ku}(X) := \langle \mathcal{O}_X, \mathcal{U}_X^\vee, ..., \mathcal{O}_X((n-3)H), \mathcal{U}_X^\vee((n-3)H) \rangle^\perp$ is the *Kuznetsov component*. For $n = 2$, set $\mathrm{Ku}(X) := \mathrm{D^b}(X)$.

Since $\mathrm{Ku}(X)$ is an admissible subcategory of $\mathrm{D^b}(X)$, it admits a Serre functor, which we denote by $S_{\mathrm{Ku}(X)}$. By [KP18, Proposition 2.6] (which makes use of [Kuz19, Corollaries 3. 7, 3.8]) the Serre functor of $\mathrm{Ku}(X)$ has the following property:

- if $n$ is even, then $S_{\mathrm{Ku}(X)} \cong [2]$;
- if $n$ is odd, then $S_{\mathrm{Ku}(X)} \cong \sigma[2]$ for a nontrivial involutive autoequivalence $\sigma$ of $\mathrm{Ku}(X)$.

Moreover, computing the Hochschild homology [KP18, Proposition 2.9] one sees that if $n$ is even, then $\mathrm{Ku}(X)$ is a noncommutative K3 surface, while for $n$ odd $\mathrm{Ku}(X)$ is a noncommutative Enriques surface.

Let $X$ be a GM threefold. Since $\omega_X \cong \mathcal{O}_X(-H)$, by Serre duality we can write the semiorthogonal decomposition (2) as

$$\mathrm{D^b}(X) = \langle \mathrm{Ku}(X), \mathcal{O}_X, \mathcal{U}_X^\vee \rangle = \langle \mathcal{U}_X^\vee(-H), \mathrm{Ku}(X), \mathcal{O}_X \rangle.$$

Since $\mathcal{U}_X^\vee(-H) \cong \mathcal{U}_X$, we obtain the alternative semiorthogonal decomposition

$$\mathrm{D^b}(X) = \langle \mathbb{L}_{\mathcal{U}_X}(\mathrm{Ku}(X)), \mathcal{U}_X, \mathcal{O}_X \rangle$$

which is the one used in [BLMS17] for the construction of stability conditions. Note that $\mathrm{Ku}(X)$ and $\mathbb{L}_{\mathcal{U}_X}(\mathrm{Ku}(X))$ are equivalent by [Kuz04, Proposition 3.8], [Bon89]. In order to be compatible with [BLMS17], we set

$$\mathrm{Ku}(X) := \langle \mathcal{U}_X, \mathcal{O}_X \rangle^\perp \tag{3}$$

sitting in

$$\mathrm{D^b}(X) = \langle \mathrm{Ku}(X), \mathcal{U}_X, \mathcal{O}_X \rangle$$

(in fact, in the rest of this paper we will need to be precise on which Kuznetsov component we are working on, see Section 2.1). By [Kuz09, Proposition 3.9], the numerical Grothendieck group $\mathcal{N}(\mathrm{Ku}(X))$ of $\mathrm{Ku}(X)$ satisfies $\mathcal{N}(\mathrm{Ku}(X)) \cong \mathbb{Z}^{\oplus 2}$ and a basis is

$$\begin{aligned} b_1 &= 1 - \frac{3}{10}H^2 + \frac{1}{20}H^3 \\ b_2 &= H - \frac{3}{5}H^2 + \frac{1}{60}H^3. \end{aligned} \tag{4}$$

The Todd class of $X$ is

$$\mathrm{td}(X) = 1 + \frac{1}{2}H + \frac{17}{60}H^2 + \frac{1}{10}H^3.$$

### 1.5 Stability conditions on $\mathrm{Ku}(X)$

The existence of stability conditions on $\mathrm{Ku}(X)$ of a GM variety is known. More precisely, if $X$ has dimension 2, this follows from Bridgeland's work [Bri08]. By the duality conjecture [KP19, Theorem 1.6] if $X$ has dimension 6 or 5, the problem reduces to the same question in dimension 4 and 3, respectively: if $X$ is a GM fourfold, this is proved in [PPZ19], while the case of GM threefolds is solved by Bayer, Lahoz, Macrì and Stellari in [BLMS17].

In this section we focus on GM threefolds and we review the construction of stability conditions on $\mathrm{Ku}(X)$ defined in (3) given in [BLMS17].

Stability conditions on $\mathrm{Ku}(X)$ are induced from double-tilted slope stability on $\mathrm{D^b}(X)$. First for $\alpha > 0$ and $\beta \in \mathbb{R}$, consider the weak stability conditions on $\mathrm{D^b}(X)$ of the form

$$\sigma_{\alpha,\beta} = (\mathrm{Coh}^{\beta}(X), Z_{\alpha,\beta})$$

with respect to the rank-3 lattice $\Lambda$ generated by vectors $(H^3 \operatorname{rk}(E), H \operatorname{ch}_1(E), H^2 \operatorname{ch}_2(E))$ for $E \in \mathrm{D^b}(X)$. Here, $\mathrm{Coh}^{\beta}(X)$ is the heart of a bounded t-structure obtained by tilting $\mathrm{Coh}(X)$ with respect to slope stability at slope $\mu = \beta$ (see Example 1.10), and the central charge $Z_{\alpha,\beta}$ is

$$Z_{\alpha,\beta}(E) = \frac{1}{2}\alpha^2 H^3 \cdot \mathrm{ch}_0^{\beta}(E) - H \cdot \mathrm{ch}_2^{\beta}(E) + iH^2 \cdot \mathrm{ch}_1^{\beta}(E), \tag{5}$$

where $\mathrm{ch}_i^{\beta}(E)$ is the $i$-th component of the twisted Chern character $\mathrm{ch}^{\beta}(-) = e^{-\beta} \cdot \mathrm{ch}(-)$ (see [BLMS17, Proposition 2.12]). For $E \in \mathrm{Coh}^{\beta}(X)$ the slope of $E$ defined by $\sigma_{\alpha,\beta}$ is

$$\mu_{\alpha,\beta}(E) = \begin{cases} \frac{\Re Z_{\alpha,\beta}(E)}{\Im Z_{\alpha,\beta}(E)} & \text{if } \Im Z_{\alpha,\beta}(E) > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Note that $\sigma_{\alpha,\beta}$-semistable objects satisfy the inequality (1) which can be taken as the quadratic form satisfying the support property.

Second for $\mu \in \mathbb{R}$, denote by $\mathrm{Coh}^{\mu}_{\alpha,\beta}(X)$ the heart obtained by tilting $\mathrm{Coh}^{\beta}(X)$ with respect to $\sigma_{\alpha,\beta}$ at slope $\mu_{\alpha,\beta} = \mu$. Fix $u \in \mathbb{C}$ such that $u$ is the unit vector in the upper half plane with $\mu = -\frac{\mathrm{Re}(u)}{\mathrm{Im}(u)}$. By [BLMS17, Proposition 2.15] we have that

$$\sigma^{\mu}_{\alpha,\beta} = (\mathrm{Coh}^{\mu}_{\alpha,\beta}(X), Z^{\mu}_{\alpha,\beta}) \tag{6}$$

is a weak stability condition on $\mathrm{D^b}(X)$ with respect to $\Lambda$, where $Z^{\mu}_{\alpha,\beta} = \frac{1}{u} Z_{\alpha,\beta}$.

Now we recall the following criterion from [BLMS17], which is useful for determining when weak stability conditions defined on $\mathrm{D^b}(X)$, like those above, restrict to stability conditions on the orthogonal complement of a subcategory determined by an exceptional collection. In the following, $\mathcal{T}$ is a triangulated category with Serre functor $S$, $E_0, \cdots, E_m$ are exceptional objects in $\mathcal{T}$ and $\mathcal{D} = \langle E_0, \cdots, E_m \rangle$, giving a semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{D}^{\perp}, \mathcal{D} \rangle$.

**Proposition 1.11** ([BLMS17], Proposition 5.1)**.** *Let $\sigma = (\mathcal{A}, Z)$ be a weak stability condition on $\mathcal{T}$. Assume that:*

1. $E_i \in \mathcal{A}$
2. $S(E_i) \in \mathcal{A}[1]$
3. $Z(E_i) \neq 0$ *for all $i$.*

*If for all $0 \neq E \in \mathcal{A} \cap \mathcal{D}^{\perp} = \mathcal{A}_1$ we have $Z(E) \neq 0$, then the pair $(\mathcal{A}_1, Z|_{\mathcal{A}_1})$ defines a stability condition on $\mathcal{D}^{\perp}$.*

The criterion above was applied in [BLMS17] to show the following existence result.

**Theorem 1.12** ([BLMS17], Theorem 6.9)**.** *Let $X$ be a GM threefold. Then the weak stability conditions $\sigma^{\mu}_{\alpha,\beta}$ defined in* (6) *induce stability conditions on* $\mathrm{Ku}(X)$ *so long as $\alpha > 0$ is sufficiently close to 0, $\beta > -1$ is sufficiently close to $-1$ and $\mu_{\alpha,\beta}(\mathcal{O}_X(-1)[1]) < \mu < \mu_{\alpha,\beta}(\mathcal{U}_X)$.*

## 1.6 Serre Invariance

In what follows, $X$ is a GM threefold. We let $\sigma(\alpha, \beta)$ denote the stability conditions induced on $\mathrm{Ku}(X)$ described in Theorem 1.12 above. We endeavor to study the action of the Serre functor $S_{\mathrm{Ku}(X)}$ of $\mathrm{Ku}(X)$ on the stability conditions $\sigma(\alpha, \beta)$. Recall that there is a right action of the universal cover $\widetilde{\mathrm{GL}}^+_2(\mathbb{R})$ of the group $\mathrm{GL}^+_2(\mathbb{R})$ of real $2 \times 2$ matrices with positive determinant on the stability manifold. We show that $S_{\mathrm{Ku}(X)}$ preserves the orbit of the stability conditions $\sigma(\alpha,\beta)$ with respect to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action.

**Theorem 1.13** (Theorem 2.6, Corollary 2.17)**.** *Let $X$ be Gushel–Mukai threefold. Let $\sigma$ be a stability condition on the Kuznetsov component $\mathrm{Ku}(X)$ which is in the same orbit of $\sigma(\alpha,\beta)$ with respect to the action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$. Then there exists $\tilde{g} \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that*

$$S_{\mathrm{Ku}(X)} \cdot \sigma = \sigma \cdot \tilde{g}.$$

Theorem 1.13 shows that the stability conditions constructed in [BLMS17] are Serre-invariant in the sense of Definition 3.1.

As an application, we construct stability conditions on the Kuznetsov component of a special GM fourfold. Recall that a special GM fourfold $X$ is a double cover of a linear section of the Grassmannian $\mathrm{Gr}(2,5)$ ramified over an ordinary GM threefold $Z$. By [KP17, Corollary 1.3] there is an exact equivalence

$$\mathrm{Ku}(Z)^{\mathbb{Z}/2\mathbb{Z}} \simeq \mathrm{Ku}(X),$$

where $\mathrm{Ku}(Z)^{\mathbb{Z}/2\mathbb{Z}}$ denotes the category of $\mathbb{Z}/2\mathbb{Z}$-equivariant objects of $\mathrm{Ku}(Z)$ and the $\mathbb{Z}/2\mathbb{Z}$-action on $\mathrm{Ku}(Z)$ is given by $S_{\mathrm{Ku}(Z)}[-2]$.

**Theorem 1.14** (Corollary 3.3, Remark 3.4)**.** *Let $X$ be a special GM fourfold and $Z$ be its associated ordinary GM threefold. Serre-invariant stability conditions on $\mathrm{Ku}(Z)$ induce stability conditions on the equivariant category $\mathrm{Ku}(Z)^{\mathbb{Z}/2\mathbb{Z}}$. In particular, they define stability conditions on $\mathrm{Ku}(X)$.*

In Corollary 3.5 we show that there is a unique $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit of Serre-invariant stability conditions on $\mathrm{Ku}(X)$ of a GM threefold $X$.

Serre invariance can be a very useful property of stability conditions, often allowing for heightened insight into moduli of stable objects. In [PY20, Proposition 5.7] it is shown that the stability conditions induced on the Kuznetsov component of a Fano threefold of Picard rank 1 and index 2 (e.g. a cubic threefold) with the method in [BLMS17] are Serre-invariant. Using this result, the authors further proved that non-empty moduli spaces of stable objects with respect to these stability conditions are smooth. They also gave another proof of the categorical Torelli Theorem for cubic threefolds in [PY20, Theorem 5.17], following the strategy in [BMMS12, Theorem 1.1] where this result was proved for the first time (see also [BBF$^+$20] for a different approach). See, for instance, [JLLZ21, LZ21, FP21] for further recent applications of Serre invariance in the study of moduli spaces of stable objects.

On the other hand, not all triangulated subcategories of the bounded derived category of a smooth projective variety admit Serre-invariant stability conditions. In the recent paper [KP18] the authors show that the Kuznetsov component (called residual category) of almost all Fano complete intersections of codimension $\geqslant 2$ does not admit Serre-invariant stability conditions.

In [FP21, Theorem 1.1] a criterion is proved which ensures that a fractional Calabi–Yau category of dimension $\leqslant 2$ admits a unique Serre-invariant stability condition, up to the action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$. In Corollary 3.5 we show this criterion applies to the Kuznetsov component of a GM threefold. Note that this result was already known by [JLLZ21, Theorem 4.25]. In particular, all known stability conditions on $\mathrm{Ku}(X)$ for $X$ a Fano threefold of Picard rank 1, index 2 or index 1 and even genus $\geqslant 6$ are Serre-invariant.

The next interesting question is to investigate whether the property of Serre-invariance characterize the stability conditions on $\mathrm{Ku}(X)$, providing a complete description of the stability manifold as in the case of curves [Mac07].

Stability conditions on the Kuznetsov component of a GM fourfold have been constructed in [PPZ19]. However, this existence is not shown through an explicit construction for special GM fourfolds, where it follows from the proof of the duality conjecture for GM varieties in [KP19]. The stability conditions constructed in Theorem 1.14 are, to the authors' knowledge, the first explicit ones defined on special GM fourfolds. In the work in preparation [PPZon], Theorems 1.13 and 1.14 are useful to study properties (like non-emptyness) of moduli spaces of stable objects in $\mathrm{Ku}(X)$ of an ordinary GM threefold $X$, together with the results in

[PPZ19] on moduli spaces on the associated special GM fourfold.

## 2 Action of the Serre functor on stability conditions on $\mathrm{Ku}(X)$

This section is devoted to the proof of Theorem 1.13. In Section 2.1 we induce stability conditions on the Kuznetsov component of $X$ from (a tilt of the) tilt stability conditions lying over Li's boundary, defined in [Li19b]. This allows to enlarge the region where there are induced stability conditions on $\mathrm{Ku}(X)$ with the method of [BLMS17] and will be useful in Section 2.4. In Section 2.2 we outline the proof of Theorem 1.13, which will be carried out in Sections 2.3, 2.4, 2.5.

### 2.1 Stability conditions over Li's boundary

Let $X$ be a GM threefold. Note that we have the following semiorthogonal decompositions:

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{Ku}(X)_1, \mathcal{U}_X, \mathcal{O}_X \rangle, \tag{7}$$

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{Ku}(X)_2, \mathcal{O}_X, \mathcal{U}_X^\vee \rangle, \tag{8}$$

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{Ku}(X)_3, \mathcal{U}_X^\vee, \mathcal{O}_X(H) \rangle \tag{9}$$

Here $\mathrm{Ku}(X)_1 := \mathrm{Ku}(X)$ as in (3). For instance, we can obtain (9) tensoring (7) by $\mathcal{O}_X(H)$ and setting $\mathrm{Ku}(X)_3 := \mathrm{Ku}(X)_1(H)$. Analogously, by Serre duality we have

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{Ku}(X)_3, \mathcal{U}_X^\vee, \mathcal{O}_X(H) \rangle = \langle \mathcal{O}_X, \mathrm{Ku}(X)_3, \mathcal{U}_X^\vee \rangle = \langle \mathbb{L}_{\mathcal{O}_X}(\mathrm{Ku}(X)_3), \mathcal{O}_X, \mathcal{U}_X^\vee \rangle$$

so we get (8) setting $\mathrm{Ku}(X)_2 := \mathbb{L}_{\mathcal{O}_X}(\mathrm{Ku}(X)_3)$. Note also that $\mathrm{Ku}(X)_1$, $\mathrm{Ku}(X)_2$, $\mathrm{Ku}(X)_3$ are equivalent to each others by [Kuz04, Proposition 3.8], [Bon89].

As in [LZ19a, Section 1], [Li19b], we consider the following reparametrization of the tilt stability condition $\sigma_{\alpha,\beta}$: for $q > 0, s \in \mathbb{R}$ and $E \in \mathrm{D}^{\mathrm{b}}(X)$ we define

$$Z_{s,q}(E) = -(H \cdot \mathrm{ch}_2(E) - q\,\mathrm{rk}(E)H^3) + \sqrt{-1}(H^2 \cdot \mathrm{ch}_1(E) - s\,\mathrm{rk}(E)H^3).$$

For $E \in \mathrm{Coh}^s(X)$ we have the associated slope function

$$\mu_{s,q}(E) = \frac{H \cdot \mathrm{ch}_2(E) - q\,\mathrm{rk}(E)H^3}{H^2 \cdot \mathrm{ch}_1(E) - s\,\mathrm{rk}(E)H^3}.$$

Then for $\alpha > 0, \beta \in \mathbb{R}$, setting $s = \beta$, $q = \frac{\alpha^2+\beta^2}{2}$, it follows that

$$\mu_{\alpha,\beta} = \mu_{s,q} - s$$

and for $q > \frac{1}{2}s^2$ the pair $\sigma_{s,q} = (\mathrm{Coh}^s(X), Z_{s,q})$ defines a weak stability condition on $\mathrm{D^b}(X)$.

For $E \in \mathrm{D^b}(X)$ we consider the reduced character

$$\widetilde{v}_H(E) := [H^3\,\mathrm{rk}(E) : H^2 \cdot \mathrm{ch}_1(E) : H \cdot \mathrm{ch}_2(E)]$$

which defines a point in a projective plane $\mathbb{P}^2_{\mathbb{R}}$ when $\widetilde{v}_H(E) \neq \mathbf{0}$. If $\mathrm{rk}(E) \neq 0$, we consider the affine coordinates

$$\left(s(E) := \frac{H^2 \cdot \mathrm{ch}_1(E)}{H^3\,\mathrm{rk}(E)},\, q(E) := \frac{H \cdot \mathrm{ch}_2(E)}{H^3\,\mathrm{rk}(E)}\right) \in \mathbb{A}^2_{\mathbb{R}}.$$

Note that since the inequality (1) holds for $\sigma_{s,q}$-semistable objects, we have that points below the parabola $q = \frac{1}{2}s^2$ correspond to $\sigma_{s,q}$-semistable objects. Furthermore, the slope of a $\sigma_{s,q}$-semistable objects $E \in \mathrm{Coh}^s(X)$ is the gradient of the line connecting $(s,q)$ and $(s(E), q(E))$ (see Figure (1)).

By [Li19b, Theorem 0.3] slope stable coherent sheaves on $X$ satisfy a stronger Bogomolov inequality. More precisely, in the affine plane $\mathbb{A}^2_{\mathbb{R}}$ we consider the open region

$$R_{\frac{3}{20}} \tag{10}$$

defined in [Li19b, Definition 3.1] as the set of points above the curve $s^2 - 2q = \frac{3}{20}$ and above the tangent lines to the curve $s^2 - 2q = 0$ at $\widetilde{v}_H(\mathcal{O}_X(kH))$ for all $k \in \mathbb{Z}$ (see Figure (2)).

As a consequence, we obtain the following refined result:

**Proposition 2.1** ([BLMS17], Proposition 2.12, [Li19b], Theorem 0.3)**.** *For* $(s,q) \in R_{\frac{3}{20}}$*, the*

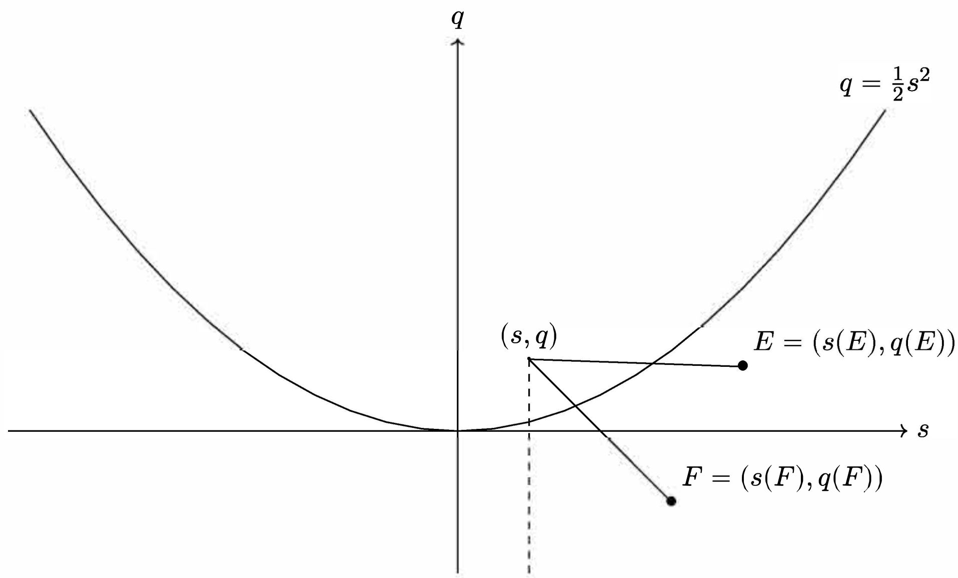


Figure 1: If $E$, $F$ are $\sigma_{s,q}$-semistable, their $\mu_{s,q}$-slope is the gradient of the line connecting the point $(s,q)$ with $(s(E), q(E))$ and $(s(F), q(F))$, respectively. We may also compare the $\mu_{s,q}$-slope of $E$ and $F$, using the picture: $E$ has larger slope than $F$ is and only if the line connecting $E$ to $(s,q)$ is above the line connecting $F$ with $(s,q)$ (see [LZ19b, Lemma 2]).

*pair* $\sigma_{s,q} = (\mathrm{Coh}^s(X), Z_{s,q})$ *defines a weak stability condition on* $\mathrm{D^b}(X)$ *with respect to the lattice* $\Lambda^2_H \cong \mathbb{Z}^{\oplus 3}$ *generated by the reduced Chern character.*

Now using the same strategy as in [BLMS17] we can induce stability conditions on the Kuznetsov components (7),(8),(9) from $\sigma_{s,q}$ for certain values of $(s,q) \in R_{\frac{3}{20}}$. As done in (6), we need to tilt a second time. For $\mu \in \mathbb{R}$, we denote by $\mathrm{Coh}^{\mu}_{s,q}(X)$ the heart obtained by tilting $\mathrm{Coh}^s(X)$ with respect to $\sigma_{s,q}$ at $\mu$. By [BLMS17, Proposition 2.15], which applies in the same way to the reparametrized tilt stability conditions, implies that $\sigma^{\mu}_{s,q} = (\mathrm{Coh}^{\mu}_{s,q}(X), Z^{\mu}_{s,q})$ is a weak stability condition on $\mathrm{D^b}(X)$.

For $i = 1, 2, 3$, we set

$$\mathcal{A}(s,q) := \mathrm{Coh}^{\mu}_{s,q}(X) \cap \mathrm{Ku}(X)_i$$

and

$$Z(s,q) := Z^{\mu}_{s,q}|_{\mathrm{Ku}(X)_i}.$$

We also note that the exceptional bundles in the semiortoghonal decompositions (7), (8), (9)

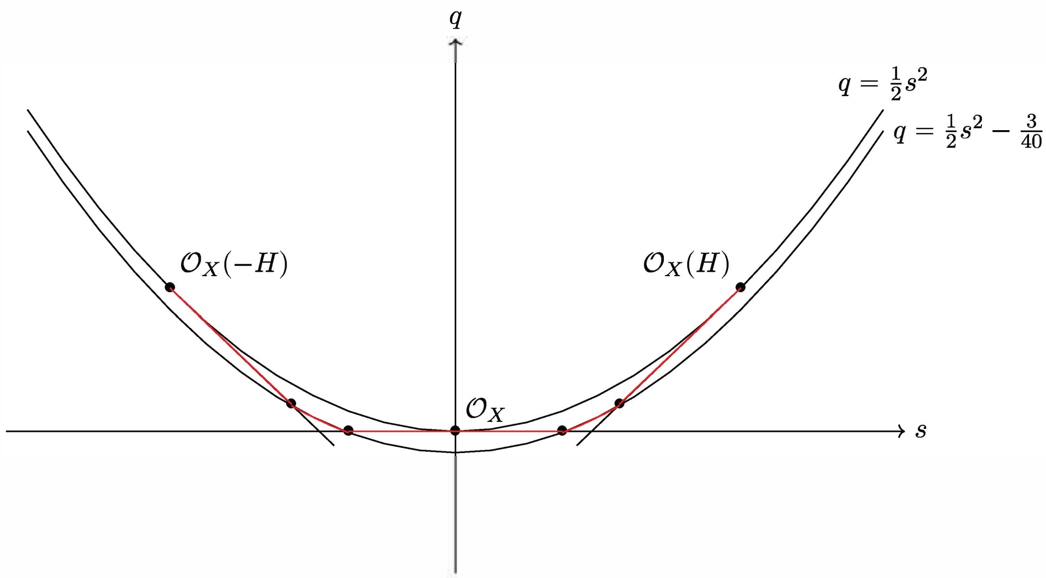


Figure 2: We represent the boundary of the region $R_{\frac{3}{20}}$ among $\widetilde{v}_H(\mathcal{O}_X(-H))$ and $\widetilde{v}_H(\mathcal{O}_X(H))$ in red.

are on the boundary of $R_{\frac{3}{20}}$ as

$$\mathrm{ch}_{\leqslant 2}(\mathcal{O}_X(kH)) = (1, kH, \frac{k^2}{2}H^2), \quad \mathrm{ch}_{\leqslant 2}(\mathcal{U}_X^\vee) = (2, H, \frac{1}{10}H^2),$$

$$\mathrm{ch}_{\leqslant 2}(\mathcal{U}_X) = (2, -H, \frac{1}{10}H^2), \quad \mathrm{ch}_{\leqslant 2}(\mathcal{U}_X(-H)) = (2, -3H, \frac{21}{10}H^2).$$

**Proposition 2.2.** *Let $(s, q)$ be points in the region $R_{\frac{3}{20}}$.*

1. *If $(s, q)$ is below the segment connecting $\widetilde{v}_H(\mathcal{O}_X(-H))$ and $\widetilde{v}_H(\mathcal{U}_X)$, then the pair $\sigma(s, q) = (\mathcal{A}(s, q), Z(s, q))$ defines a Bridgeland stability condition on* $\mathrm{Ku}(X)$ *with respect to $\Lambda_H^2$ for $\mu \in \mathbb{R}$ satisfying $\mu_{s,q}(\mathcal{O}_X(-H)[1]) \leqslant \mu < \mu_{s,q}(\mathcal{U}_X)$.*

2. *If $(s, q)$ is below the segment connecting $\widetilde{v}_H(\mathcal{O}_X)$ and $\widetilde{v}_H(\mathcal{U}_X)$, then the pair $\sigma(s, q) = (\mathcal{A}(s, q), Z(s, q))$ defines a Bridgeland stability condition on* $\mathrm{Ku}(X)_2$ *with respect to $\Lambda_H^2$ for $\mu \in \mathbb{R}$ satisfying $\mu_{s,q}(\mathcal{U}_X[1]) \leqslant \mu < \mu_{s,q}(\mathcal{O}_X)$.*

3. *If $(s, q)$ is below the segment connecting $\widetilde{v}_H(\mathcal{O}_X)$ and $\widetilde{v}_H(\mathcal{U}_X^\vee)$, then the pair $\sigma(s, q) = (\mathcal{A}(s, q), Z(s, q))$ defines a Bridgeland stability condition on* $\mathrm{Ku}(X)_3$ *with respect to $\Lambda_H^2$ for $\mu \in \mathbb{R}$ satisfying $\mu_{s,q}(\mathcal{O}_X[1]) \leqslant \mu < \mu_{s,q}(\mathcal{U}_X^\vee)$.*

In Figure 3 we represent the regions where there are induced stability conditions as in

Proposition 2.2.

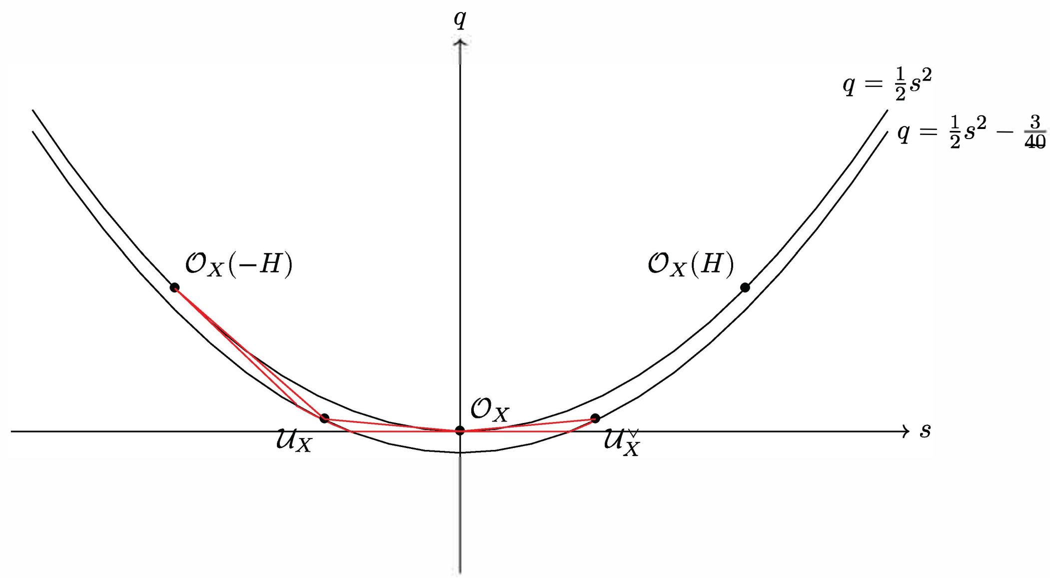


Figure 3: We represent in red the boundary of the regions defined in Proposition 2.2.

*Proof.* This is a refinement of [BLMS17, Theorem 6.8], where the statement is proved in the case of $\mathrm{Ku}(X)_1$ for $(s,q)$ above the parabola $q - \frac{1}{2}s^2 = 0$ and $\mu$ as in item 1.

We study the case of $\mathrm{Ku}(X)_1$, the others can be treated analogously. Note that $\mathcal{U}_X$, $\mathcal{O}_X$, $\mathcal{U}_X(-H)$, $\mathcal{O}_X(-H)$ are slope stable sheaves with slope $-\frac{1}{2}$, 0, $-\frac{3}{2}$, $-1$, respectively. Thus $\mathcal{U}_X, \mathcal{O}_X, \mathcal{U}_X(-H)[1], \mathcal{O}_X(-H)[1]$ belong to $\mathrm{Coh}^s(X)$ for $-1 \leqslant s < -\frac{1}{2}$. Since these objects are on the boundary of $R_{\frac{3}{20}}$, by [BMS16, Corollary 3.11] we have that $\mathcal{U}_X, \mathcal{O}_X, \mathcal{U}_X(-H)[1]$, $\mathcal{O}_X(-H)[1]$ are $\sigma_{s,q}$-stable in $\mathrm{Coh}^s(X)$. For $(s,q)$ as in the assumptions of item 1, by a direct computation or comparing the slopes using the picture, we see that

$$\mu_{s,q}(\mathcal{U}_X[1]) < \mu_{s,q}(\mathcal{O}_X(-H)[1]) < \mu_{s,q}(\mathcal{U}_X) < \mu_{s,q}(\mathcal{O}_X(H)).$$

Thus for $\mu$ as in the statement, we have $\mathcal{U}_X$, $\mathcal{O}_X$, $\mathcal{U}_X(-H)[2]$, $\mathcal{O}_X(-H)[2]$ in $\mathrm{Coh}^{\mu}_{s,q}(X)$. Finally, by [BLMS17, Lemma 2.16] objects in $\mathrm{D^b}(X)$ with vanishing central charge $Z^{\mu}_{s,q}$ are torsion sheaves supported on points, which do not belong to $\mathrm{Ku}(X)_1$. Then Proposition 1.11 implies the statement. □

Note that we omit $\mu$ from the notation of the induced stability condition. In fact, $\sigma(s,q)$

does not depend on $\mu$, up to the action of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, as we show in the next lemma.

**Lemma 2.3.** *Fix $i=1,2,3$. Let $(s,q)$ be a point in $R_{\frac{3}{20}}$, $\mu > \mu' \in \mathbb{R}$ satisfying the conditions in item (i) of Proposition 2.2. Then the stability condition induced from $\sigma^{\mu}_{s,q}$ is the same as the one induced from $\sigma^{\mu'}_{s,q}$, up to the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action.*

*Proof.* Denote by $\sigma(s,q,\mu)$ and $\sigma(s,q,\mu')$ the induced stability conditions on $\mathrm{Ku}(X)_i$ corresponding to the choice of $\mu$ and $\mu'$, respectively. We claim that

$$\mathrm{Coh}^{\mu}_{s,q}(X) \subset \langle \mathrm{Coh}^{\mu'}_{s,q}(X), \mathrm{Coh}^{\mu'}_{s,q}(X)[1]\rangle.$$

Indeed, consider $F \in \mathrm{Coh}^s(X)$ semistable with $\mu_{s,q}(F) > \mu$, which is an object in $\mathrm{Coh}^{\mu}_{s,q}(X)$. Then $\mu_{s,q}(F) > \mu'$, so $F \in \mathrm{Coh}^{\mu'}_{s,q}(X)$. Otherwise, consider $F \in \mathrm{Coh}^s(X)$ semistable with $\mu_{s,q}(F) \leqslant \mu$, so $F[1] \in \mathrm{Coh}^{\mu}_{s,q}(X)$. If $\mu_{s,q}(F) \leqslant \mu'$, then $F[1] \in \mathrm{Coh}^{\mu'}_{s,q}(X)$, while if $\mu_{s,q}(F) > \mu'$, then $F[1] \in \mathrm{Coh}^{\mu'}_{s,q}(X)[1]$. By the definition of $\mathrm{Coh}^{\mu}_{s,q}(X)$, we deduce the claim.

As a consequence, we have the same relation between the restrictions of the hearts on $\mathrm{Ku}(X)_i$ by [BLMS17, Lemma 4.3], i.e.

$$\mathcal{A}(s,q,\mu) \subset \langle \mathcal{A}(s,q,\mu'), \mathcal{A}(s,q,\mu')[1]\rangle.$$

By definition $Z^{\mu}_{s,q} = \frac{1}{u} Z_{s,q}$ and $Z^{\mu'}_{s,q} = \frac{1}{u'} Z_{s,q}$, for unit vectors $u, u'$ in the upper half plane. Recall the generators $b_1$ and $b_2$ of $\mathcal{N}(\mathrm{Ku}(X)_1)$ defined in (4). Since $\mathrm{Ku}(X)_3 = \mathrm{Ku}(X)_1(H)$ and $\mathrm{Ku}(X)_2 = \mathbb{L}_{\mathcal{O}_X}(\mathrm{Ku}(X)_3)$, we have that

$$\begin{aligned} d_1 &:= b_1(H) = (1, H, \frac{1}{5}H^2, -\frac{5}{6}), \\ d_2 &:= b_2(H) = (0, H, \frac{2}{5}H^2, -\frac{5}{6}) \end{aligned} \tag{11}$$

form a basis of $\mathcal{N}(\mathrm{Ku}(X)_3)$, and

$$\begin{aligned} c_1 &:= \mathbb{L}_{\mathcal{O}_X}(d_1) = (-3, H, \frac{1}{5}H^2, -\frac{5}{6}), \\ c_2 &:= \mathbb{L}_{\mathcal{O}_X}(d_2) = (-4, H, \frac{2}{5}H^2, -\frac{5}{6}) \end{aligned} \tag{12}$$

for $\mathcal{N}(\mathrm{Ku}(X)_2)$. An easy computation shows that multiplying by $1/u$ and $1/u'$ does not change the orientation of the basis $Z_{s,q}(b_1), Z_{s,q}(b_2)$ of $\mathbb{C}$. Thus the basis $Z^{\mu}_{s,q}(b_1)$, $Z^{\mu}_{s,q}(b_2)$ and $Z^{\mu'}_{s,q}(b_1)$, $Z^{\mu'}_{s,q}(b_2)$ have the same orientation. Analogous comments hold for $d_1, d_2$ and $c_1, c_2$.

Note that $Z^{\mu}_{s,q} = \frac{u'}{u} Z^{\mu'}_{s,q}$, thus setting $M := \frac{u}{u'}$, we have $Z(s,q,\mu) = M^{-1}Z(s,q,\mu')$ and there exists a cover $\widetilde{g} = (g, M) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ such that $\sigma(s,q,\mu') \cdot \widetilde{g} = (\mathcal{A}', M^{-1}Z(s,q,\mu') = Z(s,q,\mu))$, where

$$\mathcal{A}' \subset \langle \mathcal{A}(s,q,\mu'), \mathcal{A}(s,q,\mu')[1] \rangle.$$

It follows that the stability conditions $\sigma(s,q,\mu)$ and $\sigma(s,q,\mu')\cdot\widetilde{g}$ have the same central charge and their hearts are tilt of the same heart $\mathcal{A}(s,q,\mu')$. [BMS16, Lemma 8.11] implies that they are the same stability condition. □

We end this section by showing that the induced stability conditions on each $\mathrm{Ku}(X)_i$ are in the same orbit with respect to the action of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$.

**Proposition 2.4.** *Fix $i = 1, 2, 3$. The stability conditions induced in item (i) of Proposition 2.2 on $\mathrm{Ku}(X)_i$ are in the same orbit with respect to the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$- action.*

*Proof.* We explain the proof for $i = 1$, the other cases are analogous. Let $(s,q)$, $(s',q')$ as in Proposition 2.2(1). It is not restrictive to assume $s' \geqslant s$ and $q \geqslant q'$. By Lemma 2.5 below, we only need to show that the central charges of $\sigma(s,q)$ and $\sigma(s',q')$ are in the same orbits with respect to the action of $\mathrm{GL}^+(2,\mathbb{R})$. Note that for every $(s,q)$ as in Proposition 2.2(1), we can choose to tilt at $\mu = -\frac{9}{10}$. Indeed, since $(s,q)$ is below the line $q = -\frac{9}{10}s - \frac{2}{5}$ passing through $\widetilde{v}_H(\mathcal{O}_X(-H))$ and $\widetilde{v}_H(\mathcal{U}_X)$, it satisfies the inequalities

$$\begin{cases} \mu_{s,q}(\mathcal{O}_X(-H)[1]) = \frac{\frac{1}{2}-q}{-1-s} \geqslant -\frac{9}{10} \\ \mu_{s,q}(\mathcal{U}_X) = \frac{\frac{1}{10}-2q}{-1-2s} > -\frac{9}{10}. \end{cases}$$

By Lemma 2.3, the stability condition $\sigma(s,q)$ does not depend on the choice of $\mu$, so we can assume $\mu = -\frac{9}{10}$. In particular, $u = \frac{1}{\sqrt{181}}(9 + 10\sqrt{-1})$.

Now consider the central charges $Z^{\mu}_{s,q}$ and $Z^{\mu}_{s',q'}$. Since multiplying by $\frac{1}{u}$ does not change the orientation, we reduce to compare the orientations of $Z_{s,q}$ and $Z_{s',q'}$ on the basis $b_1$, $b_2$. We have

$$Z_{s,q}(b_1) = 10(q + \frac{3}{10}) + 10i(-s), \quad Z_{s,q}(b_2) = 10(\frac{3}{5} + i).$$

Then

$$\begin{vmatrix} q + \frac{3}{10} & \frac{3}{5} \\ -s & 1 \end{vmatrix} = q + \frac{3}{5}s + \frac{3}{10} > \frac{1}{2}s^2 + \frac{3}{5}s + \frac{9}{40} > 0,$$

since $(s,q)$ is above the parabola $q = \frac{1}{2}s^2 - \frac{3}{40}$. In particular, there exists $N \in \mathrm{GL}^+(2,\mathbb{R})$ such that $Z_{s',q'} = N^{-1} \cdot Z_{s,q}$. We write $N = \frac{1}{\det(N^{-1})}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where

$$a = \frac{10q + 3 + 6s'}{10q + 3 + 6s}, \quad b = \frac{6(q' - q)}{10q + 3 + 6s}$$

$$c = \frac{10(s' - s)}{10q + 3 + 6s}, \quad d = \frac{10q' + 3 + 6s}{10q + 3 + 6s}.$$

As a consequence, we have $Z^{\mu}_{s',q'} = M^{-1}Z^{\mu}_{s,q}$, where $M = \frac{1}{u}N^{-1}u$ and there exists $(g, M) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ such that $g(0,1) \subset (0,2)$. This implies $\sigma(s,q) \cdot (g, M) = (\mathcal{A}', Z^{\mu}_{s',q'}|_{\mathcal{N}(\mathrm{Ku}(X))_1})$ with

$$\mathcal{A}' \subset \langle \mathcal{A}(s,q), \mathcal{A}(s,q)[1] \rangle.$$

Thus the stability conditions $\sigma(s',q')$ and $\sigma(s,q) \cdot (g, M)$ have the same central charge and their hearts are tilt of $\mathcal{A}(s,q)$. We conclude that they are the same stability condition by [BMS16, Lemma 8.11]. □

**Lemma 2.5.** *Fix $i = 1,2,3$. Let $(s,q)$, $(s',q')$ as in Proposition 2.2(i). If $s < s'$, then $\mathcal{A}(s',q') \subset \langle \mathcal{A}(s,q), \mathcal{A}(s,q)[1] \rangle$, while if $s = s'$, then $\mathcal{A}(s',q') = \mathcal{A}(s,q)$.*

*Proof.* The argument is similar to the one used in the proof of [PY20, Lemma 3.8]. Consider the case $i = 1$, the other are analogous. By Lemma 2.3 we can fix $\mu = -\frac{9}{10}$. We denote by $\mathcal{P}_{s,q}$ the slicing defined by $\sigma_{s,q}$. We claim that $\mathrm{Coh}^{\mu}_{s,q}(X) = \mathcal{P}_{s,q}(\phi_u, \phi_u + 1]$, where $\phi_u = \arg(u^{-1})$. Indeed, assume $E \in \mathrm{Coh}^{\mu}_{s,q}(X)$ is $\sigma^{\mu}_{s,q}$-semistable. Then there is a triangle $A[1] \to E \to B$, where $A \in \mathrm{Coh}^s(X)$ (resp. $B \in \mathrm{Coh}^s(X)$) and its $\sigma_{s,q}$-semistable factors have slope $\mu_{s,q} \leqslant \mu$ (resp. $> \mu$). If $Z_{s,q}(B) \neq 0$, then $A[1]$ has larger slope than $B$ with respect to $\sigma^{\mu}_{s,q}$. This would contradict the semistability of $E$, unless either $E = B$, or $E = A[1]$. If

$E = B$, the $\sigma_{s,q}$-semistable factors of $E$ would have phase in the interval $(\phi_u, 1]$ by definition of $B$. Actually, this also shows $E$ is $\sigma_{s,q}$-semistable, as a destabilizing sequence of $E$ with respect to $\sigma_{s,q}$ would destabilize $E$ with respect to $\sigma^{\mu}_{s,q}$. A similar observation shows that $A[1] \in \mathcal{P}_{s,q}(1, \phi_u + 1]$. It remains to consider the case when $Z_{s,q}(B) = 0$, i.e. $B$ is a torsion sheaf supported on points. Then $B$ is $\sigma_{s,q}$-semistable of phase 1. Since $A[1] \in \mathcal{P}_{s,q}(1, \phi_u + 1]$, we conclude that $E \in \mathcal{P}_{s,q}(\phi_u, \phi_u + 1]$. This shows $\mathrm{Coh}^{\mu}_{s,q}(X) \subset \mathcal{P}_{s,q}(\phi_u, \phi_u + 1]$. Since those are hearts of bounded t-structures, we deduce that they are equal.

Now if $s' > s$, it is easy to see that $\mathrm{Coh}^{s'}(X)$ is a tilt of $\mathrm{Coh}^{s}(X)$. Equivalently, $\mathcal{P}_{s',q'}(0,1] \subset \mathcal{P}_{s,q}(0,2]$. The action by multiplication with $u^{-1}$ preserves the distance of the slicings $\mathcal{P}_{s,q}$ and $\mathcal{P}_{s',q'}$, thus

$$\mathrm{Coh}^{\mu}_{s',q'}(X) = \mathcal{P}_{s',q'}(\phi_u, \phi_u + 1] \subset \mathcal{P}_{s,q}(\phi_u, \phi_u + 2] = \langle \mathrm{Coh}^{\mu}_{s,q}(X), \mathrm{Coh}^{\mu}_{s,q}(X)[1] \rangle.$$

Consider $\mathcal{A}(s', q') = \mathrm{Ku}(X) \cap \mathrm{Coh}^{\mu}_{s',q'}(X)$. Since the cohomology with respect to the restricted heart of an objects $E \in \mathrm{Ku}(X)$ is the same as the cohomology in $\mathrm{Coh}^{\mu}_{s,q}(X)$ by [BLMS17, Lemma 4.3], we deduce that

$$\mathcal{A}(s', q') \subset \langle \mathcal{A}(s, q), \mathcal{A}(s, q)[1] \rangle.$$

If $s' = s$, we get $\mathrm{Coh}^{\mu}_{s',q'}(X) = \mathrm{Coh}^{\mu}_{s,q}(X)$, which implies $\mathcal{A}(s', q') = \mathcal{A}(s, q)$. ☐

**Notation:** In the next, we will use the subscript $s, q$ (resp. $\alpha, \beta$) when we refer to the parametrized tilt stability condition (resp. to the classical tilt stability). If we work in the region above the parabola $q - \frac{1}{2}s^2 = 0$, we will prefer to use the classical tilt stability condition depending on $\alpha$ and $\beta$, and we will make use of the tilt stability below this parabola and above Li's boundary only where it is necessary.

We will denote by $\mathrm{Coh}^{s}(X)_{\mu_{s,q} > \mu}$ (resp. $\mathrm{Coh}^{s}(X)_{\mu_{s,q} \leqslant \mu}$) the subcategory of $\mathrm{Coh}^{s}(X)$ generated by $\mu_{s,q}$-semistable objects with slope $\mu_{s,q} > \mu$ (resp. $\leqslant \mu$), and analogous notation with the subscript $\alpha, \beta$.

## 2.2 Proof of Theorem 1.13

Consider $\mathrm{Ku}(X)_3$ defined in (9). By [Bon89], since $S_X(-) = - \otimes \mathcal{O}_X(-H)[3]$, the Serre functor $S_{\mathrm{Ku}(X)_3}$ on $\mathrm{Ku}(X)_3$ satisfies

$$S_{\mathrm{Ku}(X)_3}^{-1}(-) = \mathbb{L}_{\mathcal{U}_X^\vee} \circ \mathbb{L}_{\mathcal{O}_X(H)} \circ (- \otimes \mathcal{O}_X(H))[-3] = (- \otimes \mathcal{O}_X(H)) \circ \mathbb{L}_{\mathcal{U}_X} \circ \mathbb{L}_{\mathcal{O}_X}[-3]. \quad (13)$$

The goal of the next sections is to prove Theorem 1.13, which can be stated more precisely as follows.

**Theorem 2.6.** *Let $\sigma(s_3, q_3)$ be a stability condition on $\mathrm{Ku}(X)_3$ as induced in Proposition 2.2(3). Then there exists $\tilde{g} \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that*

$$S_{\mathrm{Ku}(X)_3}^{-1} \cdot \sigma(s_3, q_3) = \sigma(s_3, q_3) \cdot \tilde{g}.$$

Here we outline the strategy of the proof. The idea is to decompose $S_{\mathrm{Ku}(X)_3}^{-1}$ as in (13) and study the action of $\mathbb{L}_{\mathcal{O}_X}$ on $\sigma(s_3, q_3)$ and then of $\mathbb{L}_{\mathcal{U}_X}$ on $\mathbb{L}_{\mathcal{O}_X} \cdot \sigma(s_3, q_3)$. In fact, $\mathbb{L}_{\mathcal{O}_X}$ (resp. $\mathbb{L}_{\mathcal{U}_X^\vee}$) induces an equivalence between $\mathrm{Ku}(X)_3$ and $\mathrm{Ku}(X)_2$ (resp. $\mathrm{Ku}(X)_2$ and $\mathrm{Ku}(X)_1$), so $\mathbb{L}_{\mathcal{O}_X} \cdot \sigma(s_3, q_3)$ and $\mathbb{L}_{\mathcal{U}_X} \cdot \mathbb{L}_{\mathcal{O}_X} \cdot \sigma(s_3, q_3)$ are stability conditions on $\mathrm{Ku}(X)_2$ and $\mathrm{Ku}(X)_1$, respectively.

First, in Section 2.3 we consider special values of $s_3$ and $q_3$ very close to 0. Here it is not necessary to work with the reparametrized tilt stability conditions, so we use the notation with $\alpha$ and $\beta$. In particular we consider the stability condition $\sigma(\alpha, \epsilon)$, for $\epsilon > 0$ very small and $0 < \alpha < \epsilon$. In Lemma 2.9 we show that the heart $\mathbb{L}_{\mathcal{O}_X}(\mathcal{A}(\alpha, \epsilon))$ on $\mathrm{Ku}(X)_2$ is a tilting of $\mathcal{A}(\alpha, -\epsilon)$. The basic idea is that when moving from $\epsilon$ to $-\epsilon$, the only problematic object in $\mathrm{Coh}^0_{\alpha,\epsilon}(X)$ is $\mathcal{O}_X[2]$, which belongs to $\mathrm{Coh}^0_{\alpha,-\epsilon}(X)[2]$. Then we show in Proposition 2.10 that the stability condition $\mathbb{L}_{\mathcal{O}_X} \cdot \sigma(\alpha, \epsilon)$ on $\mathrm{Ku}(X)_2$ is the same as $\sigma(\alpha, -\epsilon)$ up to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action. This implies the same statement for every stability condition $\sigma(s_3, q_3)$ on $\mathrm{Ku}(X)_3$ (see Corollary 2.11).

Next, in Section 2.4 we follow the same argument for the stability conditions $\sigma(s_2, q_2)$ on $\mathrm{Ku}(X)_2$ and the left mutation $\mathbb{L}_{\mathcal{U}_X}$. Here we need to work with the stability conditions over

Li's boundary, as we need to consider $s_2$ very close to $-\frac{1}{2}$ and $q_2$ close to $\frac{1}{20}$. Analogously, we show in Lemma 2.14 that the heart $\mathbb{L}_{\mathcal{U}_X}(\mathcal{A}(-\frac{1}{2}+\epsilon, q_2))$ on $\mathrm{Ku}(X)_1$ is a tilt of $\mathcal{A}(-\frac{1}{2}-\epsilon, q_2)$. This allows to show in Corollary 2.16 that $\mathbb{L}_{\mathcal{U}_X} \cdot \sigma(s_2, q_2)$ on $\mathrm{Ku}(X)_1$ is in the same orbit with respect to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action of the induced stability conditions $\sigma(s_1, q_1)$ on $\mathrm{Ku}(X)_1$.

Finally, we simply observe that acting via $(-)\otimes\mathcal{O}_X(H)$ on a stability condition $\sigma(s_1, q_1)$ on $\mathrm{Ku}(X)_1$, we get $\sigma(s_1+1, q_1')$, namely a stability condition on $\mathrm{Ku}(X)_3$ in the same orbit of $\sigma(s_3, q_3)$.

### 2.3 Stability conditions on $\mathrm{Ku}(X)_3$ and action of $\mathbb{L}_{\mathcal{O}_X}$

In this section we study the action of $\mathbb{L}_{\mathcal{O}_X}$ on the stability conditions $\sigma(s_3, q_3)$ on $\mathrm{Ku}(X)_3$ defined in Proposition 2.2(3). The main result is Corollary 2.11.

We start by considering $(s_3, q_3)$ close to $(0,0)$. For this reason, we can simply work with the usual parametrization of the tilt stability $\sigma_{\alpha,\beta}$ and $\beta = \epsilon > 0$ very small.

**Lemma 2.7.** *There exist $\epsilon > 0$, $0 < \alpha < \epsilon$ such that*

$$\mathrm{Coh}^0_{\alpha,\epsilon}(X) \subset \langle \mathrm{Coh}^0_{\alpha,-\epsilon}(X), \mathrm{Coh}^0_{\alpha,-\epsilon}(X)[1], \mathcal{O}_X[2]\rangle.$$

*Proof.* **Step 1:** We show that, up to taking $\epsilon \to 0$, every object in $\mathrm{Coh}^\epsilon(X)$ is an extension of objects in $\mathrm{Coh}^{-\epsilon}(X)$ and objects of the form $\mathcal{G}[1]$, where $\mathcal{G}$ is a slope stable coherent sheaf with $\mu_H(\mathcal{G}) = 0$ and $\mu^+_{\alpha,-\epsilon}(\mathcal{G}) \leqslant \mu_{\alpha,-\epsilon}(\mathcal{O}_X)$.

Take $E \in \mathrm{Coh}^\epsilon(X)$. By definition $E$ is an extension of the form

$$\mathcal{H}^{-1}(E)[1] \to E \to \mathcal{H}^0(E),$$

where $\mathcal{H}^0(E)$ (resp. $\mathcal{H}^{-1}(E)$) is in $\mathrm{Coh}(X)$ and its slope semistable factors have slope $\mu_H > \epsilon$ (resp. $\leqslant \epsilon$). Clearly, $\mathcal{H}^0(E) \in \mathrm{Coh}^{-\epsilon}(X)$. Consider $\mathcal{H}^{-1}(E)$ and denote by $\mathcal{G}_1, \dots, \mathcal{G}_k$ its slope semistable factors. Note that for $\epsilon \to 0$, we have $\mu_H(\mathcal{G}_i) \to 0$. Thus up to taking $\epsilon \to 0$, we can reduce to treat the case that $\mu_H(\mathcal{G}_i) \leqslant 0$ for every $i = 1, \dots, k$, and if $\mu_H(\mathcal{G}_i) < 0$ for some $i$, we have $\mu_H(\mathcal{G}_i) \leqslant -\epsilon$. In this latter case, $\mathcal{G}_i[1] \in \mathrm{Coh}^{-\epsilon}(X)$.

On the other hand, consider $\mathcal{G}_i$ such that $\mu_H(\mathcal{G}_i) = 0$. Denote by $A_1^i, \dots, A_m^i$ its Harder–Narasimhan factors with respect to $\sigma_{\alpha,-\epsilon}$. Note that

$$\mathrm{ch}_1^{-\epsilon}(\mathcal{G}_i) = \sum_{j=1}^{m} \mathrm{ch}_1^{-\epsilon}(A_j^i)$$

with $\mathrm{ch}_1^{-\epsilon}(A_j^i)H^2 \geqslant 0$. Since $\mathrm{ch}_1^{-\epsilon}(\mathcal{G}_i)H^2 = \mathrm{ch}_1(\mathcal{G}_i)H^2 + \epsilon\,\mathrm{rk}(\mathcal{G}_i)H^3 = \epsilon\,\mathrm{rk}(\mathcal{G}_i)H^3 \to 0$ for $\epsilon \to 0$, it follows that $\mathrm{ch}_1(A_j^i)H^2 + \epsilon\,\mathrm{rk}(A_j^i)H^3 \to 0$, which implies $\mathrm{ch}_1(A_j^i)H^2 \to 0$ for $\epsilon \to 0$. Note that

$$\mu_{\alpha,-\epsilon}(A_j^i) \leqslant \mu_{\alpha,-\epsilon}(\mathcal{O}_X).$$

Indeed, writing $\mathrm{ch}_2(A_j^i) = \frac{c_j^i}{2}H^2$ for $c_j^i \in \mathbb{Z}$, we see that, when $\mathrm{rk}(A_j^i) \neq 0$, the above inequality is equivalent to $\frac{c_j^i}{\mathrm{rk}(A_j^i)} \leqslant 0$. The latter inequality holds by (1). In the case some $A_j^i$ has rank 0, then there would be a torsion sheaf supported in codimension $\geqslant 2$ with a morphism to $\mathcal{G}_i$, in contradiction with the slope stability of $\mathcal{G}_i$. We conclude that $\mu^+_{\alpha,-\epsilon}(\mathcal{G}_i) \leqslant \mu_{\alpha,-\epsilon}(\mathcal{O}_X)$.

**Step 2:** We improve the computation in Step 1, by showing the objects of the form $\mathcal{G}[1]$ are extensions of copies of $\mathcal{O}_X[1]$ and shifts by 1 of $\sigma_{\alpha,-\epsilon}$-semistable objects with slope $\mu_{\alpha,-\epsilon} \leqslant 0$. Note that $\mu_{\alpha,-\epsilon}(\mathcal{O}_X) = \frac{\epsilon^2-\alpha^2}{2\epsilon} > 0$ and converges to 0 for $\alpha \to \epsilon$. Thus if $\mu^+_{\alpha,-\epsilon}(\mathcal{G}_i) < \mu_{\alpha,-\epsilon}(\mathcal{O}_X)$, up to choosing $\epsilon$ small and $\alpha$ close to $\epsilon$, we have $\mu_{\alpha,-\epsilon}(A_j^i) \leqslant 0$. Otherwise, assume $\mu^+_{\alpha,-\epsilon}(\mathcal{G}_i) = \mu_{\alpha,-\epsilon}(\mathcal{O}_X)$. Then we can assume that $\mu_{\alpha,-\epsilon}(A_1^i) = \mu_{\alpha,-\epsilon}(\mathcal{O}_X)$ and $\mu_{\alpha,-\epsilon}(A_j^i) \leqslant 0$ for $j = 2, \dots, m$. Since $\mathrm{ch}_1(A_1^i) = 0$, we have that $\Delta_H(A_1^i) = 0$. Then $\mathrm{ch}_{\leqslant 2}(A_1^i) = (r, 0, 0)$ for $r > 0$. Up to replacing $A_1^i$ with a stable factor, we can assume $A_1^i$ is stable. By [BMS16, Corollary 3.11(c)] we have that $A_1^i$ is a slope semistable torsion free sheaf. Moreover, $\mathrm{ch}_3(A_1^j) = e \leqslant 0$, since by [Li19b, Theorem 0.1], Conjecture 4.1 of [BMS16] holds.

We claim that $A_1^i \cong \mathcal{O}_X$. Indeed, note that $\chi(A_1^i, \mathcal{O}_X) = r - e > 0$. Thus $\hom(A_1^i, \mathcal{O}_X) + \hom(A_1^i, \mathcal{O}_X[2]) > 0$. By Serre duality, we have

$$\mathrm{Hom}(A_1^i, \mathcal{O}_X[2]) = \mathrm{Hom}(\mathcal{O}_X(H), A_1^i[1]) = 0,$$

where the last equality follows from the fact that $\mathcal{O}_X(H)$ and $A_1^i[1]$ are $\sigma_{\alpha,\epsilon}$-stable in $\mathrm{Coh}^{\epsilon}(X)$ with slopes $\mu_{\alpha,\epsilon}(\mathcal{O}_X(H)) > 0 > \mu_{\alpha,\epsilon}(\mathcal{O}_X[1]) = \mu_{\alpha,\epsilon}(A_1^i[1])$. Thus there exists a non-zero morphisms $f\colon A_1^i \to \mathcal{O}_X$. Using the slope (semi)stability of $\mathcal{O}_X$ ($A_1^i$), one can show that if $r = 1$, then $f$ is an isomorphism. If $r > 1$, then $\mathrm{ch}_{\leqslant 2}(\ker f) = (r-1, 0, 0)$. By induction, we lead back to the case $r = 1$ and we get $A_1^i \cong \mathcal{O}_X$. This implies the claim in Step 2.

**Step 3:** We show $\mathrm{Coh}^0_{\alpha,\epsilon}(X) \subset \langle \mathrm{Coh}^0_{\alpha,-\epsilon}(X), \mathrm{Coh}^0_{\alpha,-\epsilon}(X)[1], \mathcal{O}_X[2]\rangle$ for some $0 < \alpha < \epsilon$ and $\epsilon$ small enough.

Consider $F \in \mathrm{Coh}^0_{\alpha,\epsilon}(X)$. By definition $F$ is an extension of the form

$$A[1] \to F \to B$$

where $B$ (resp. $A$) belongs to $\mathrm{Coh}^{\epsilon}(X)_{\mu_{\alpha,\epsilon}>0}$ (resp. $\mathrm{Coh}^{\epsilon}(X)_{\mu_{\alpha,\epsilon}\leqslant 0}$). In the next, we show that

$$B \in \langle \mathrm{Coh}^{-\epsilon}(X)_{\mu_{\alpha,-\epsilon}>0}, \mathrm{Coh}^{-\epsilon}(X)[1]\rangle \tag{14}$$

and

$$A[1] \in \langle \mathrm{Coh}^{-\epsilon}(X)[1], \mathrm{Coh}^0_{\alpha,-\epsilon}(X)[1], \mathcal{O}_X[2]\rangle, \tag{15}$$

which imply the statement.

Note that $B \subset \langle \mathrm{Coh}^{-\epsilon}(X), \mathrm{Coh}^{-\epsilon}(X)[1]\rangle$ by Step 1. By [LZ19b, Lemma 3], if $B_i$ is a $\sigma_{\alpha,-\epsilon}$-semistable factor of $B$, then its slope satisfies

$$\mu_{\alpha,-\epsilon}(B^-) \leqslant \mu_{\alpha,-\epsilon}(B_i) \leqslant \mu_{\alpha,-\epsilon}(B^+),$$

where $B^+$, $B^-$ are the intersection points of the parabola $q = \frac{1}{2}s^2$ with the line connecting the point $(-\epsilon, \frac{\alpha^2+\epsilon^2}{2})$. For $\epsilon \to 0$, we have $\mu_{\alpha,-\epsilon}(B^-) \to \mu_{\alpha,-\epsilon}(B)$ (see Figure 4). Also note that for $\epsilon \to 0$, we have

$$\mu_{\alpha,-\epsilon}(B) \to \mu_{\alpha,\epsilon}(B) > 0.$$

Thus up to taking $\epsilon$ small enough, we can assume that $\mu_{\alpha,-\epsilon}(B)$, and thus $\mu_{\alpha,-\epsilon}(B^-)$, remains positive. This proves (14).

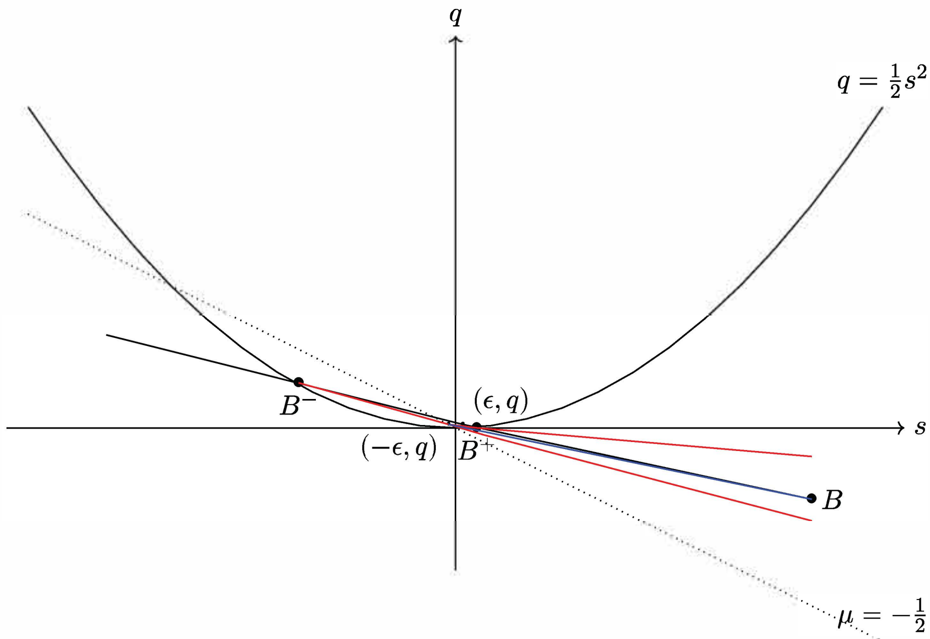


Figure 4: The tilt-stability $\sigma_{\alpha,\epsilon}$ (resp. $\sigma_{\alpha,-\epsilon}$) corresponds to the point $(\epsilon, q := \frac{\alpha^2+\epsilon^2}{2})$ (resp. $(-\epsilon, \frac{\alpha^2+\epsilon^2}{2})$) in the affine plane. Tilting at $\mu_{\alpha,\epsilon} = 0$ is equivalent to tilting at $\mu_{s,q} = -\frac{1}{2}$. The line connecting $B^{\pm}$ with $(-\epsilon, q)$ is represented in red, while the line in blue connects $B$ with $(-\epsilon, q)$. When $\epsilon$ approaches 0 the slopes of $B^+$ and $B^-$ converge to the slope of $B$ with respect to $(-\epsilon, q)$.

By Step 2, we have that $A[1]$ is an extension of objects in $\mathrm{Coh}^{-\epsilon}(X)[1]$, copies of $\mathcal{O}_X[2]$ and objects of the form $G[2]$, where $G \in \mathrm{Coh}^{-\epsilon}(X)$ is semistable with $\mu_{\alpha,-\epsilon}(G) \leqslant 0$. Then $G[1] \in \mathrm{Coh}^0_{\alpha,-\epsilon}(X)$. This implies (15) and ends the proof of the lemma. ☐

**Lemma 2.8.** *There exists $\epsilon > 0$, $0 < \alpha < \epsilon$ such that for $F \in \mathrm{Coh}^0_{\alpha,\epsilon}(X)$ we have that $\mathbb{L}_{\mathcal{O}_X}(F)$ is in $\mathrm{Coh}^0_{\alpha,\epsilon}(X)$.*

*Proof.* As in [BLMS17, Lemma 5.9] we have the five terms exact sequence

$$0 \to \mathcal{H}^{-1}(\mathbb{L}_{\mathcal{O}_X}(F)) \to \mathcal{O}_X[2]^{\oplus k_0} \to F \to \mathcal{H}^0(\mathbb{L}_{\mathcal{O}_X}(F)) \to \mathcal{O}_X[2]^{\oplus k_1} \to 0,$$

where $\mathcal{H}^i(\mathbb{L}_{\mathcal{O}_X}(F))$ denotes the cohomology in $\mathrm{Coh}^0_{\alpha,\epsilon}(X)$ and $k_0, k_1$ are integers. Note that

$$\mu^0_{\alpha,\epsilon}(\mathcal{O}_X[2]) = \frac{2\epsilon}{\epsilon^2 - \alpha^2} \to +\infty \text{ if } \alpha \to \epsilon.$$

Thus up to replacing $\alpha$, we can assume that $\mathcal{O}_X[2]$ is the stable factor of $F$ with larger slope.

It follows that $\mathcal{H}^{-1}(\mathbb{L}_{\mathcal{O}_X}(F))=0$, thus $\mathbb{L}_{\mathcal{O}_X}(F)\in \mathrm{Coh}^0_{\alpha,\epsilon}(X)$. □

**Lemma 2.9.** *For $\epsilon>0$ very small and $0<\alpha<\epsilon$ we have*

$$\mathbb{L}_{\mathcal{O}_X}(\mathcal{A}(\alpha,\epsilon))\subset\langle\mathcal{A}(\alpha,-\epsilon),\mathcal{A}(\alpha,-\epsilon)[1]\rangle.$$

*Proof.* Consider $F\in\mathcal{A}(\alpha,\epsilon)$ and its left mutation $\mathbb{L}_{\mathcal{O}_X}(F)$. By Lemma 2.8 we can find $\alpha$ and $\epsilon$ such that $\mathbb{L}_{\mathcal{O}_X}(F)\in\mathrm{Coh}^0_{\alpha,\epsilon}(X)$. By Lemma 2.7, we have

$$\mathbb{L}_{\mathcal{O}_X}(F)\in\langle\mathrm{Coh}^0_{\alpha,-\epsilon}(X),\mathrm{Coh}^0_{\alpha,-\epsilon}(X)[1],\mathcal{O}_X[2]\rangle$$

for some $\epsilon,\alpha$. Note that $\mathcal{A}(\alpha,-\epsilon)=\mathrm{Coh}^0_{\alpha,-\epsilon}(X)\cap\mathrm{Ku}(X)_2$ is a heart of a stability condition on $\mathrm{Ku}(X)_2$ by Proposition 2.2(2). Since $\mathbb{L}_{\mathcal{O}_X}(F)$ is in $\mathrm{Ku}(X)_2$ and by [BLMS17, Lemma 4.3] its cohomology in $\mathrm{Coh}^0_{\alpha,-\epsilon}(X)$ belongs to $\mathrm{Ku}(X)_2$ as well, we deduce that $\mathbb{L}_{\mathcal{O}_X}(F)\in\langle\mathcal{A}(\alpha,-\epsilon),\mathcal{A}(\alpha,-\epsilon)[1]\rangle$. Finally, we observe that the statement does not depend on $\alpha<\epsilon$, by Lemma 2.5. □

**Proposition 2.10.** *There exists $\widetilde{g}\in\widetilde{\mathrm{GL}}^+_2(\mathbb{R})$ such that*

$$\mathbb{L}_{\mathcal{O}_X}\cdot\sigma(\alpha,\epsilon)=\sigma(\alpha,-\epsilon)\cdot\widetilde{g}.$$

*Proof.* Recall that the stability condition $\mathbb{L}_{\mathcal{O}_X}\cdot\sigma(\alpha,\epsilon)$ has heart $\mathbb{L}_{\mathcal{O}_X}(\mathcal{A}(\alpha,\epsilon))$ and stability function $Z':=Z(\alpha,\epsilon)\circ(\mathbb{L}_{\mathcal{O}_X})_*^{-1}$. As done for instance in Proposition 2.4, we can check that there exists $\widetilde{g}\in\widetilde{\mathrm{GL}}^+_2(\mathbb{R})$ such that $\sigma(\alpha,-\epsilon)\cdot\widetilde{g}=\sigma'$, where $\sigma'=(\mathcal{A}',Z')$ and $\mathcal{A}'$ is a tilt of $\mathcal{A}(\alpha,-\epsilon)$, up to shifting. More precisely, one first needs to check there exists $M\in\mathrm{GL}^+_2(\mathbb{R})$ such that $Z'=M^{-1}\cdot Z(\alpha,-\epsilon)$, or equivalently,

$$Z(\alpha,\epsilon)=M^{-1}\cdot Z(\alpha,-\epsilon)\cdot(\mathbb{L}_{\mathcal{O}_X})_*. \tag{16}$$

In order to do this, recall the basis $d_1:=b_1(H)$, $d_2:=b_2(H)$ of $\mathcal{N}(\mathrm{Ku}(X)_3)$ given in (11).

Then

$$Z(\alpha,\epsilon)(d_1) = (1-\epsilon) + \sqrt{-1}(\frac{1}{5} - \epsilon + \frac{\epsilon^2}{2} - \frac{\alpha^2}{2}),$$
$$Z(\alpha,\epsilon)(d_2) = 1 + \sqrt{-1}(\frac{2}{5} - \epsilon),$$
$$Z(\alpha,-\epsilon)(c_1) = (1-3\epsilon) + \sqrt{-1}(\frac{1}{5} + \epsilon - \frac{3}{2}\epsilon^2 + \frac{3}{2}\alpha^2),$$
$$Z(\alpha,-\epsilon)(c_2) = (1-4\epsilon) + \sqrt{-1}(\frac{2}{5} + \epsilon - 2\epsilon^2 + 2\alpha^2).$$

The determinant of the two matrices having on the columns the components with respect to the standard basis of $\mathbb{C} = \mathbb{R}^2$ of $Z(\alpha,\epsilon)(d_1)$, $Z(\alpha,\epsilon)(d_2)$, and $Z(\alpha,-\epsilon)(c_1)$, $Z(\alpha,-\epsilon)(c_2)$ respectively, have determinant

$$\frac{\alpha^2}{2} + \frac{\epsilon^2}{2} - \frac{2}{5}\epsilon + \frac{1}{5} > 0.$$

Thus there exists $M \in \mathrm{GL}_2^+(\mathbb{R})$ satisfying (16). More explicitly, setting $M^{-1} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, condition (16) translates into

$$\begin{cases} Z(\alpha,\epsilon)(d_1) = M^{-1} \cdot Z(\alpha,-\epsilon) \cdot (\mathbb{L}_{\mathcal{O}_X})_*(d_1), \\ Z(\alpha,\epsilon)(d_2) = M^{-1} \cdot Z(\alpha,-\epsilon) \cdot (\mathbb{L}_{\mathcal{O}_X})_*(d_2) \end{cases}$$

which is equivalent to solve the linear system

$$\begin{cases} (1-3\epsilon)x_1 + (\frac{1}{5} + \epsilon - \frac{3}{2}\epsilon^2 + \frac{3}{2}\alpha^2)x_2 = 1-\epsilon \\ (1-3\epsilon)x_3 + (\frac{1}{5} + \epsilon - \frac{3}{2}\epsilon^2 + \frac{3}{2}\alpha^2)x_4 = \frac{1}{5} - \epsilon + \frac{\epsilon^2}{2} - \frac{\alpha^2}{2} \\ (1-4\epsilon)x_1 + (\frac{2}{5} + \epsilon - 2\epsilon^2 + 2\alpha^2)x_2 = 1 \\ (1-4\epsilon)x_3 + (\frac{2}{5} + \epsilon - 2\epsilon^2 + 2\alpha^2)x_4 = \frac{2}{5} - \epsilon. \end{cases}$$

Using a computer, we can find the solution of the above linear system and check the existence of a cover $\tilde{g} = (g, M) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ with the desired properties.

Since by Lemma 2.9, the heart $\mathbb{L}_{\mathcal{O}_X}(\mathcal{A}(\alpha,\epsilon))$ is a tilt of $\mathcal{A}(\alpha,-\epsilon)$, by [BMS16, Lemma

8.11] we conclude $\sigma' = \mathbb{L}_{\mathcal{O}_X} \cdot \sigma(\alpha, -\epsilon)$, as we wanted. $\square$

**Corollary 2.11.** *For $i = 2, 3$, if $\sigma(s_i, q_i)$ is a stability condition on $\mathrm{Ku}(X)_i$, then there exists $\widetilde{g} \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that*

$$\mathbb{L}_{\mathcal{O}_X} \cdot \sigma(s_3, q_3) = \sigma(s_2, q_2) \cdot \widetilde{g}.$$

*Proof.* By Proposition 2.4, we have that $\sigma(s_3, q_3)$ (resp. $\sigma(s_2, q_2)$) is in the same orbit of $\sigma(\alpha, \epsilon)$ (resp. $\sigma(\alpha, -\epsilon)$) with respect to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action. Since the action of $\mathbb{L}_{\mathcal{O}_X}$ commutes with the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action, by Proposition 2.10, we deduce the claim. $\square$

## 2.4 Stability conditions on $\mathrm{Ku}(X)_2$ and action of $\mathbb{L}_{\mathcal{U}_X}$

Our goal is now to investigate the action of the left mutation $\mathbb{L}_{\mathcal{U}_X}$ on a stability condition $\sigma(s_2, q_2)$ on $\mathrm{Ku}(X)_2$ as in Proposition 2.2(2). The main statement is Corollary 2.16.

We would like to apply the same technique as in the previous section. In particular, we need to consider a stability condition corresponding to $(s_2, q_2)$ very close to the point $(-\frac{1}{2}, \frac{1}{20}) \in \partial R_{\frac{3}{20}}$. Thus we have to work with the stability conditions induced from (a tilt of) the tilt stability conditions $\sigma_{s,q}$ below the parabola $q - \frac{1}{2}s^2 = 0$.

We start by fixing $s = -\frac{1}{2} + \epsilon$ for $\epsilon > 0$ very small such that $s < -\sqrt{\frac{3}{20}}$. For simplicity, we can assume $\epsilon < \frac{1}{10}$. Consider $q > 0$ such that the point $(s, q)$ satisfies the conditions in Proposition 2.2(2). Explicitly, we have that

$$q \in (\frac{1}{20} + \frac{1}{2}\epsilon^2 - \frac{1}{2}\epsilon, \frac{1}{20} - \frac{1}{10}\epsilon). \tag{17}$$

On the other side of the vertical wall for $\mathcal{U}_X$, we consider $s' = -\frac{1}{2} - \epsilon$ and $q' = q + \epsilon$. Note that $q'$ varies in

$$(\frac{1}{20} + \frac{1}{2}\epsilon^2 - \frac{1}{2}\epsilon + \epsilon, \frac{1}{20} - \frac{1}{10}\epsilon + \epsilon) = (\frac{1}{20} + \frac{1}{2}\epsilon^2 + \frac{1}{2}\epsilon, \frac{1}{20} + \frac{9}{10}\epsilon).$$

In particular, $(s', q')$ satisfies the conditions in Proposition 2.2(1), so this point induces a stability condition on $\mathrm{Ku}(X)_1$ after suitable tilting.

We fix $\bar{\mu} = -\frac{1}{10}$. Note that

$$\mu_{s,q}(\mathcal{O}_X) = \frac{-q}{\frac{1}{2} - \epsilon} > -\frac{1}{10} \Longleftrightarrow q < \frac{1}{20} - \frac{1}{10}\epsilon$$

and

$$\mu_{s,q}(\mathcal{U}_X[1]) = \frac{\frac{1}{10} - 2q}{-2\epsilon} < -\frac{1}{10} \Longleftrightarrow q < \frac{1}{20} - \frac{1}{10}\epsilon$$

which holds by (17).

On the other side, fix $\bar{\mu}' = -\frac{9}{10}$. The above computation shows

$$\mu_{s',q'}(\mathcal{O}_X(-H)[1]) < \bar{\mu}' < \mu_{s',q'}(\mathcal{U}_X).$$

**Lemma 2.12.** *For $\bar{\mu} = -\frac{1}{10}$, $\bar{\mu}' = -\frac{9}{10}$, there exists $\epsilon > 0$, $q$ satisfying (17) such that*

$$\mathrm{Coh}^{\bar{\mu}}_{-\frac{1}{2}+\epsilon,q}(X) \subset \langle \mathrm{Coh}^{\bar{\mu}'}_{-\frac{1}{2}-\epsilon,q+\epsilon}(X), \mathrm{Coh}^{\bar{\mu}'}_{-\frac{1}{2}-\epsilon,q+\epsilon}(X)[1], \mathcal{U}_X[2]\rangle.$$

*Proof.* Fix $s = -\frac{1}{2} + \epsilon$, $s' = -\frac{1}{2} - \epsilon$, $q$ satisfying (17) and $q' = q + \epsilon$.

**Step 1:** Following the same argument as in Step 1 of the proof of Lemma 2.7, we have that, up to taking $\epsilon \to 0$, every object in $\mathrm{Coh}^s(X)$ is an extension of objects in $\mathrm{Coh}^{s'}(X)$ and objects of the form $\mathcal{G}[1]$, where $\mathcal{G}$ is a slope stable coherent sheaf with $\mu_H(\mathcal{G}) = -\frac{1}{2}$ and $\mu^+_{s',q'}(\mathcal{G}) \leqslant \mu_{s',q'}(\mathcal{U}_X)$.

**Step 2:** We claim that the objects of the form $\mathcal{G}[1]$ in Step 1 are extensions of copies of $\mathcal{U}_X[1]$ and shifts by 1 of $\sigma_{s',q'}$-semistable objects with slope $\mu_{s',q'} \leqslant \bar{\mu}'$.

Indeed, we have $\mu_{s',q'}(\mathcal{U}_X) = \left(\frac{1}{10} - 2q'\right)\frac{1}{2\epsilon} > -\frac{9}{10}$ which converges to $-\frac{9}{10}$ for $q' \to \frac{1}{20} + \frac{9}{10}\epsilon$. Thus we can argue as in Step 2 of Lemma 2.7 and it remains to show that if $A$ is a stable object in $\mathrm{Coh}^{s'}(X)$ with $\mu_{s',q'}(A) = \mu_{s',q'}(\mathcal{U}_X)$ and $\frac{\mathrm{ch}_1(A)}{\mathrm{rk}(A)} = -\frac{1}{2}$, then $A \cong \mathcal{U}_X$. In order to prove this, note that under these assumptions, the point $\left(\frac{\mathrm{ch}_1(A)}{\mathrm{rk}(A)}, \frac{\mathrm{ch}_2(A)}{\mathrm{rk}(A)}\right)$ belongs to the boundary of $R_{\frac{3}{20}}$. Thus by [Li19b, Proposition 3.2] $\mathrm{rk}(A)$ is either 1 or 2. We exclude the case $\mathrm{rk}(A) = 1$, as the numerical Grothendieck group of $X$ does not contain the class of such object by [Kuz09, Proposition 3.9]. Moreover, $A$ is tilt stable everywhere. In particular, it is a slope semistable torsion-free sheaf. Since $\mathrm{Hom}(A, \mathcal{O}_X[2]) = \mathrm{Hom}(\mathcal{O}_X(H), A[-1]) = 0$ by stability, [LM16, Theorem 3.14] implies that $A$ is a vector bundle with $\mathrm{ch}(A) = \mathrm{ch}(\mathcal{U}_X)$. It follows that $A \cong \mathcal{U}_X$.

**Step 3:** We end by showing $\mathrm{Coh}^{\bar{\mu}}_{s,q}(X) \subset \langle \mathrm{Coh}^{\bar{\mu}'}_{s',q'}(X), \mathrm{Coh}^{\bar{\mu}'}_{s',q'}(X)[1], \mathcal{U}_X[2] \rangle$ for some $q'$ and $\epsilon$ small enough.

Consider $F \in \mathrm{Coh}^{\bar{\mu}}_{s,q}(X)$. By definition $F$ is an extension of the form

$$A[1] \to F \to B$$

where $B$ (resp. $A$) belongs to $\mathrm{Coh}^s(X)_{\mu_{s,q} > \bar{\mu}}$ (resp. $\mathrm{Coh}^s(X)_{\mu_{s,q} \leqslant \bar{\mu}}$).

Note that $B \subset \langle \mathrm{Coh}^{s'}(X), \mathrm{Coh}^{s'}(X)[1] \rangle$ by Step 1. For $\epsilon \to 0$, we have $q \to \frac{1}{20}$ and as in Step 3 of Lemma 2.7, we have

$$\mu^-_{s',q'}(B) \to \mu_{s,q}(B) > \bar{\mu} = -\frac{1}{10} > \bar{\mu}' = -\frac{9}{10}.$$

Thus up to taking a smaller $\epsilon$, we can assume that $\mu^-_{s',q'}(B) > \bar{\mu}'$. This implies

$$B \in \langle \mathrm{Coh}^{\bar{\mu}'}_{s',q'}(X), \mathrm{Coh}^{\bar{\mu}'}_{s',q'}(X)[1] \rangle.$$

By Step 2, we have that $A[1]$ is an extension of objects in $\mathrm{Coh}^{s'}(X)[1]$, copies of $\mathcal{U}_X[2]$ and objects of the form $G[2]$, where $G \in \mathrm{Coh}^{s'}(X)$ is semistable with $\mu_{s',q'}(G) \leqslant \bar{\mu}'$. Then $G[1] \in \mathrm{Coh}^{\bar{\mu}'}_{s',q'}(X)$. This implies the proof of the lemma. □

**Lemma 2.13.** *For $\bar{\mu} = -\frac{1}{10}$, there exists $\epsilon > 0$, $q$ satisfying* (17) *such that for $F \in \mathrm{Coh}^{\bar{\mu}}_{-\frac{1}{2}+\epsilon,q}(X)$ we have that $\mathbb{L}_{\mathcal{U}_X}(F)$ is in $\mathrm{Coh}^{\bar{\mu}}_{-\frac{1}{2}+\epsilon,q}(X)$.*

*Proof.* Note that $\bar{\mu} = -\frac{\mathrm{Re} u}{\mathrm{Im} u}$ for $u = \frac{1}{\sqrt{11}}(1 + 10\sqrt{-1})$. In particular, by definition

$$Z^{\bar{\mu}}_{s,q}(-) = \frac{1}{u} Z_{s,q}(-) = \left(\frac{\sqrt{11}}{101} - \frac{10\sqrt{11}}{101}\sqrt{-1}\right) Z_{s,q}(-).$$

Set $s = -\frac{1}{2} + \epsilon$. Then $\mathrm{Im} Z^{\bar{\mu}}_{s,q}(\mathcal{U}_X[2]) = \frac{\sqrt{11}}{101}(-2\epsilon + 1 - 20q)$ which converges to 0 for $\epsilon \to 0$, as $q \to \frac{1}{20}$. Thus $\mu^{\bar{\mu}}_{s,q}(\mathcal{U}_X[2]) \to +\infty$ for $\epsilon \to 0$. The same argument of Lemma 2.8 implies the statement. $\square$

The next results follow from Lemma 2.12 and Lemma 2.13, arguing as in Lemma 2.9, Proposition 2.10, Corollary 2.11.

**Lemma 2.14.** *For $\epsilon > 0$ very small and $q$ satisfying* (17) *we have*

$$\mathbb{L}_{\mathcal{O}_X}(\mathcal{A}(-\frac{1}{2} + \epsilon, q)) \subset \langle \mathcal{A}(-\frac{1}{2} - \epsilon, q + \epsilon), \mathcal{A}(-\frac{1}{2} - \epsilon, q + \epsilon)[1] \rangle.$$

**Proposition 2.15.** *There exists $\tilde{g} \in \widetilde{\mathrm{GL}}^+_2(\mathbb{R})$ such that*

$$\mathbb{L}_{\mathcal{U}_X} \cdot \sigma(-\frac{1}{2} + \epsilon, q) = \sigma(-\frac{1}{2} - \epsilon, q + \epsilon) \cdot \tilde{g}.$$

**Corollary 2.16.** *For $i = 1, 2$, if $\sigma(s_i, q_i)$ is a stability condition on $\mathrm{Ku}(X)_i$, then there exists $\tilde{g} \in \widetilde{\mathrm{GL}}^+_2(\mathbb{R})$ such that*

$$\mathbb{L}_{\mathcal{U}_X} \cdot \sigma(s_2, q_2) = \sigma(s_1, q_1) \cdot \tilde{g}.$$

## 2.5 End of the proof

We are now ready to complete the proof of our main result.

*Proof of Theorem 2.6.* Let $\sigma(s_3, q_3)$ be a stability condition on $\mathrm{Ku}(X)_3$ as induced in Proposition 2.2(3). Consider a stability condition $\sigma(s_1, q_1)$ on $\mathrm{Ku}(X)_1$ as in Proposition 2.2(1) which is above the parabola $q - \frac{1}{2}s^2 = 0$. By Corollary 2.11 and Corollary 2.16 there exists $\tilde{g} \in \widetilde{\mathrm{GL}}^+_2(\mathbb{R})$ such that

$$\mathbb{L}_{\mathcal{U}_X} \cdot \mathbb{L}_{\mathcal{O}_X} \cdot \sigma(s_3, q_3) = \sigma(s_1, q_1) \cdot \tilde{g}.$$

Note that if $F \in \mathrm{Ku}(X)_1$ is $\sigma_{s_1,q_1}$-semistable, then $F(H) \in \mathrm{Ku}(X)_3$ is $\sigma_{s_1+1,q'_1}$-semistable for $q'_1 = \frac{1}{2} + s_1 + q_1$ (see for instance [LZ19b, Proof of Lemma 2.3]). Moreover, $(s_1 + 1, q'_1)$

satisfies the conditions in Proposition 2.2(3). This implies

$$\mathcal{A}(s_1, q_1)(H) \subset \langle \mathcal{A}(s_1+1, q_1'), \mathcal{A}(s_1+1, q_1') \rangle.$$

Arguing as in Proposition 2.10, it follows that

$$(- \otimes \mathcal{O}_X(H)) \cdot \sigma(s_1, q_1) = \sigma(s_1+1, q_1') \cdot \tilde{f} \tag{18}$$

for $\tilde{f} \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$. Since the action by equivalences and by $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ on the stability manifold commute, by (13) this implies

$$S^{-1}_{\mathrm{Ku}(X)_3} \cdot \sigma(s_3, q_3) = \sigma(s_1+1, q_1') \cdot \tilde{h}$$

for $\tilde{h} \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$. By Proposition 2.4 we have that $\sigma(s_3, q_3)$ and $\sigma(s_1+1, q_1')$ are in the same orbit with respect to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action, which implies the claim. □

As a consequence, we obtain the same result for the Serre functors of $\mathrm{Ku}(X)_2$ and $\mathrm{Ku}(X)_1$.

**Corollary 2.17.** *For $i = 1, 2, 3$, let $\sigma(s_i, q_i)$ be a stability condition on $\mathrm{Ku}(X)_i$ as induced in Proposition 2.2(i). Then there exists $\tilde{g} \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that*

$$S^{-1}_{\mathrm{Ku}(X)_i} \cdot \sigma(s_i, q_i) = \sigma(s_i, q_i) \cdot \tilde{g}.$$

*Proof.* The case of $i = 3$ is Theorem 2.6. For $i = 1$ it is enough to note that $(- \otimes \mathcal{O}_X(H))$ induces an equivalence between $\mathrm{Ku}(X)_1$ and $\mathrm{Ku}(X)_3$. Using the fact that the Serre functors commute with equivalences and (18), we deduce the statement for $\mathrm{Ku}(X)_1$. If $i = 2$, we apply the same argument since $\mathbb{L}_{\mathcal{O}_X}$ induces an equivalence between $\mathrm{Ku}(X)_3$ and $\mathrm{Ku}(X)_2$ and using Corollary 2.11. □

# 3 Serre-invariant stability conditions

In this section, we drop the superfluous subscript and write $\mathrm{Ku}(X) = \mathrm{Ku}(X)_i$ for any given $i = 1, 2, 3$ to simplify the notation, as the results contained herein hold for all such choices. We introduce the following definition (see [FP21, Definition 3.1]).

**Definition 3.1.** A stability condition $\sigma$ on $\mathrm{Ku}(X)$ is Serre-invariant, or $S_{\mathrm{Ku}(X)}$-invariant, if $S_{\mathrm{Ku}(X)} \cdot \sigma = \sigma \cdot \widetilde{g}$ for some $\widetilde{g} \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$.

In Theorem 2.6, we have established that the stability conditions on $\mathrm{Ku}(X)$ as in Proposition 3.2 are $S_{\mathrm{Ku}(X)}$-invariant. We now aim to explore the implications that this fact has for the existence of Bridgeland stability conditions on special GM fourfolds (Corollary 3.3) and to show that there is a unique orbit with respect to the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action of $S_{\mathrm{Ku}(X)}$-invariant stability conditions.

### 3.1 Stability conditions on special GM fourfolds

We begin by setting up some notation. Let $Y$ be a variety with a line bundle $\mathcal{O}_Y(1)$. We say that $\mathrm{D^b}(Y)$ admits a rectangular Lefschetz decomposition with respect to $\mathcal{O}_Y(1)$ if there is an admissible subcategory $\mathcal{B} \hookrightarrow \mathrm{D^b}(Y)$ such that

$$\mathrm{D^b}(Y) = \langle \mathcal{B}, \mathcal{B}(1), \cdots, \mathcal{B}(m-1) \rangle \tag{19}$$

is a semiorthogonal decomposition for some integer $m$. Given such a decomposition of $\mathrm{D^b}(Y)$, pick $n, d \in \mathbb{N}$ such that $nd \leqslant m$. Suppose we have a degree-$n$ cyclic cover $f : X \to Y$ of $Y$ ramified in a Cartier divisor $Z$ in the linear system corresponding to $\mathcal{O}_Y(nd)$. If $i : Z \hookrightarrow Y$ is the inclusion, then the derived pullbacks $i^*$ and $f^*$ are fully faithful upon restriction to $\mathcal{B}$. We obtain semiorthogonal decompositions

$$\mathrm{D^b}(X) = \langle \mathcal{A}_X, f^*\mathcal{B}, \cdots, f^*\mathcal{B}(m-(n-1)d-1) \rangle, \tag{20}$$

$$\mathrm{D^b}(Z) = \langle \mathcal{A}_Z, i^*\mathcal{B}, \cdots, i^*\mathcal{B}(m-nd-1) \rangle, \tag{21}$$

with $\mathcal{A}_X = \langle f^*\mathcal{B}, \cdots, f^*\mathcal{B}(m-(n-1)d-1) \rangle^{\perp}$ and $\mathcal{A}_Z$ defined similarly. The following theorem of Kuznetsov and Perry relates $\mathcal{A}_X$ and $\mathcal{A}_Z$ in the above scenario.

**Theorem 3.2** ([KP17], Theorem 1.1)**.** *In the setup above, there are fully faithful functors $\Phi_k : \mathcal{A}_Z \to \mathcal{A}_X^{\mu_n}$ for $0 \leqslant k \leqslant n-2$ such that there is a semiorthogonal decomposition:*

$$\mathcal{A}_X^{\mu_n} = \langle \Phi_0(\mathcal{A}_Z), \cdots, \Phi_{n-2}(\mathcal{A}_Z) \rangle. \tag{22}$$

Here, $\mu_n$ is the group of $n^{th}$ roots of unity, acting on $X$ via automorphisms over $Y$ and $\mathcal{A}_X^{\mu_n}$ is the corresponding equivariant category.

If we now assume that $X$ is a special GM fourfold, then the map $X \to \mathrm{Gr}(2,5)$ is a double cover of its image $Y$, ramified over an ordinary GM threefold $Z \hookrightarrow Y$. In the notation of [KP17], we have $n = 2$, $d = 1$, $e = 2$ and $\mathcal{A}_X = \mathrm{Ku}(X)$, $\mathcal{A}_Z = \mathrm{Ku}(Z)$ are the Kuznetsov components of the GM fourfold and threefold. By Theorem 3.2, the map $\Phi_0$ provides an equivalence of categories $\mathrm{Ku}(Z) \cong \mathrm{Ku}(X)^{\mu_2}$. As shown in [KP17, Corollary 1.3, Proposition 7.10], which makes use of [Ela15], we have dually an equivalence

$$\mathrm{Ku}(Z)^{\mathbb{Z}/2\mathbb{Z}} \cong \mathrm{Ku}(X).$$

The action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{Ku}(Z)$ is induced by the rotation functor $\mathbb{L}_{i^*\mathcal{B}}(- \otimes \mathcal{O}_X(H))[-1]$, where $i^*\mathcal{B} =^{\perp} \mathrm{Ku}(Z)$. Using this equivalence and Theorem 2.6, we have the following result.

**Corollary 3.3.** *Let $X$ be a special GM fourfold and $Z$ be its associated ordinary GM threefold. The stability conditions $\sigma(s,q)$ on $\mathrm{Ku}(Z)$ defined in Proposition 2.2 induce stability conditions on the equivariant category $\mathrm{Ku}(Z)^{\mathbb{Z}/2\mathbb{Z}}$. In particular, they define stability conditions on $\mathrm{Ku}(X)$.*

*Proof.* It is sufficient to prove that $S_{\mathrm{Ku}(Z)}[-2] \cdot \sigma(s,q) = \sigma(s,q)$, since the $\mathbb{Z}/2\mathbb{Z}$-action on $\mathrm{Ku}(Z)$ is induced by $\mathbb{L}_{i^*\mathcal{B}}(- \otimes \mathcal{O}_X(H))[-1] = S^{-1}_{\mathrm{Ku}(Z)}[2]$, or equivalently by $S_{\mathrm{Ku}(Z)}[-2]$. By Theorem 2.6, there is some $\tilde{g} = (M, g) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that $S_{\mathrm{Ku}(Z)}[-2] \cdot \sigma(s,q) = \sigma(s,q) \cdot \tilde{g}$. Applying the involution $S_{\mathrm{Ku}(Z)}[-2]$ to both sides of this equality yields: $\sigma(s,q) = \sigma(s,q) \cdot \tilde{g}^2$. Writing $\sigma(s,q) = (\mathcal{P}, Z)$, at the level of slicings this gives $\mathcal{P}(\phi) = \mathcal{P}(g^2(\phi))$ for any $\phi \in \mathbb{R}$, hence $g : \mathbb{R} \to \mathbb{R}$ is an increasing involution, so we must have $g = \mathrm{id}$. On the other hand, on central charges we have $M^{-2} \circ Z = Z$. The image of $Z$ is not contained in a line, hence $M^{-2}$ agrees with the identity on two linearly independent vectors in $\mathbb{C} \cong \mathbb{R}^2$, thus $M^2 = \mathrm{id}$. There are only three conjugacy classes of $2 \times 2$ matrices over $\mathbb{R}$ squaring to the identity, one of which has negative determinant, hence $M = \pm I$. We cannot have $M = -I$, since $M$ induces the identity on the circle, thus $M = I$ and we deduce that $S_{\mathrm{Ku}(Z)}[-2] \cdot \sigma(s,q) = \sigma(s,q)$ as claimed. □

**Remark 3.4.** Note that the above proof does not use anything specific on the stability conditions $\sigma(s, q)$. In particular, Corollary 3.3 holds more generally for every Serre-invariant stability conditions on $\mathrm{Ku}(Z)$.

### 3.2 Uniqueness

Let $X$ be a GM threefold. The aim of this section is to prove the following result.

**Corollary 3.5.** *If $\sigma_1$, $\sigma_2$ are $S_{\mathrm{Ku}(X)}$-invariant stability conditions, then there exists $\widetilde{g} \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that $\sigma_2 = \sigma_1 \cdot \widetilde{g}$.*

Corollary 3.5 has been recently proved in [JLLZ21, Lemmas 4.27, 4.28, 4.29]. Here we give an alternative proof making use of the following result obtained from [FP21].

**Theorem 3.6** ([FP21], Theorem 3.2, Lemma 3.6)**.** *Let $\mathcal{T}$ be a $\mathbb{C}$-linear triangulated category of finite type whose Serre functor satisfies $S_{\mathcal{T}}^2 = [4]$ and whose numerical Grothendieck group $\mathcal{N}(\mathcal{T})$ has rank 2. Assume further the following conditions hold:*

1. $\ell_{\mathcal{T}} := \max\{\chi(v, v) \colon 0 \neq v \in \mathcal{N}(\mathcal{T})\} < 0$.

2. *There are three objects $Q_1, Q_2, Q_2' \in \mathcal{T}$ such that $Q_2$ and $Q_2'$ have the same class in $\mathcal{N}(\mathcal{T})$, $Q_1$ is not isomorphic to $Q_2$, or $Q_2'[1]$, and*

$$\begin{aligned}
&-\ell_{\mathcal{T}} + 1 \leqslant \hom^1(Q_i, Q_i) < -2\ell_{\mathcal{T}} + 2,\\
&\hom(Q_2, Q_1) \neq 0\\
&\hom(Q_1, Q_2'[1]) \neq 0\\
&\hom(Q_2', Q_2[3]) = 0.
\end{aligned}$$

*Then there exists a unique orbit of $S_{\mathcal{T}}$-invariant stability conditions on $\mathcal{T}$ with respect to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action.*

Let us check the conditions of Theorem 3.6 for the Kuznetsov component $\mathrm{Ku}(X) := \langle \mathcal{U}_X, \mathcal{O}_X \rangle^{\perp}$ of a GM threefold $X$. Recall that $S_{\mathrm{Ku}(X)}^2 = [4]$ and $\mathcal{N}(\mathrm{Ku}(X))$ has rank 2. By [Kuz09] the basis $b_1$, $b_2$ of (4) has intersection form

$$\begin{pmatrix} -2 & -3 \\ -3 & -5 \end{pmatrix}.$$

For $v = \alpha b_1 + \beta b_2 \in \mathcal{N}(\mathrm{Ku}(X))$, we have

$$v^2 = -2\alpha^2 - 6\alpha\beta - 5\beta^2 = -(\alpha + 2\beta)^2 - (\alpha + \beta)^2 \leqslant -1,$$

so $\ell_{\mathrm{Ku}(X)} = -1$. To find the suitable objects $Q_i$, we argue similarly as in [JLLZ21, Lemma 4.26], just using conics instead of lines. Let $C \subset X$ be a smooth conic. Its ideal sheaf $\mathcal{I}_C$ is in $\langle \mathcal{O}_X \rangle^{\perp}$ and the left mutation $\mathbb{L}_{\mathcal{U}_X}(\mathcal{I}_C)$ is in $\mathrm{Ku}(X)$ by definition, sitting in the triangle

$$\mathcal{U}_X \to \mathcal{I}_C \to \mathbb{L}_{\mathcal{U}_X}(\mathcal{I}_C). \tag{23}$$

The latter can be computed using the short exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

and $h^0(\mathcal{U}_X^{\vee}|_C) = 4$, $h^i(\mathcal{U}_X^{\vee}|_C) = 0$ for $i \neq 0$, $h^0(\mathcal{U}_X^{\vee}) = 5$, $h^i(\mathcal{U}_X^{\vee}) = 0$ for $i \neq 0$. Assume further that $C$ is generic on $X$. Consider now a smooth twisted cubic $D' \subset X$ such that $D'$ does not intersect $C$ and its ideal sheaf $\mathcal{I}_{D'} \in \mathrm{Ku}(X)$. Note that the generic twisted cubic $D'$ satisfies these conditions. Finally, pick a twisted cubic $D \subset X$ such that $\mathcal{I}_D \in \mathrm{Ku}(X)$ and $C$ is an irreducible component of $D$. The existence of such $D$ has been proved in [JLLZ21, Lemma 4.29]. Set

$$Q_1 := \mathbb{L}_{\mathcal{U}_X}(\mathcal{I}_C), \quad Q_2 := \mathcal{I}_D, \quad Q_2' := \mathcal{I}_{D'}.$$

Clearly $Q_2$ and $Q_2'$ have the same class in $\mathcal{N}(\mathcal{T})$ and they are not isomorphic to $Q_1$. The following lemma ends the proof of Corollary 3.5.

**Lemma 3.7.** *With the notation above, we have*

$$\hom^1(Q_1, Q_1) = 2, \quad \hom^1(Q_2, Q_2) = \hom^1(Q_2', Q_2') = 3,$$
$$\hom(Q_2, Q_1) \neq 0 \quad \hom(Q_1, Q_2'[1]) \neq 0, \quad \hom(Q_2', Q_2[3]) = 0.$$

*Proof.* Note that $\hom^1(Q_1, Q_1) = \hom^1(\mathcal{I}_C, Q_1)$ as $Q_1 \in \mathrm{Ku}(X)$. By Serre duality, we have $\hom^i(\mathcal{I}_C, \mathcal{U}_X) = \hom^{3-i}(\mathcal{U}_X^\vee, \mathcal{I}_C) = \hom^{2-i}(\mathcal{U}_X^\vee, \mathcal{O}_C) = h^{2-i}(\mathcal{U}_X|_C) = 0$ for every $i$, as $C$ is a generic conic. Thus $\hom^1(\mathcal{I}_C, Q_1) = \hom^1(\mathcal{I}_C, \mathcal{I}_C) = 2$ (see [IP99, Lemma 4.2.1]). With a similar computation as in [Zha20, Proposition 3.8], we get $\hom^1(Q_2, Q_2) = \hom^1(Q_2', Q_2') = 3$. By Serre duality, $\hom^3(Q_2', Q_2) = \hom(Q_2, Q_2'(-H)) = 0$ by slope stability of $Q_2$ and $Q_2'(-H)$. Now note that

$$\hom(Q_2, \mathcal{U}_X) = \hom(\mathcal{O}_D, \mathcal{U}_X[1]) = \hom(\mathcal{U}_X, \mathcal{O}_D(-H)[2]) = h^2(\mathcal{U}_X|_D) = 0$$

since $h^i(\mathcal{U}_X) = 0$ for every $i$ and Serre duality. It follows that the space $\mathrm{Hom}(Q_2, \mathcal{I}_C)$ has an injection in $\mathrm{Hom}(Q_2, Q_1)$. Since $C$ is a component of $D$, the former is not 0 and we get $\hom(Q_2, Q_1) \neq 0$. Finally, we have

$$\hom(Q_1, Q_2'[1]) = \hom(\mathcal{I}_C, \mathcal{I}_{D'}[1]) = \hom(\mathcal{O}_C, \mathcal{I}_{D'}[2]) = \hom(\mathcal{O}_C, \mathcal{O}_X[2]) = h^1(\mathcal{O}_C(-H)) = 1,$$

where in the first and second equality we have used $\mathcal{I}_{D'} \in \mathrm{Ku}(X)$ and in the third the fact that $C \cap D' = \varnothing$. □

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