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ABSTRACT

Bounds for the roots of polynomials and matrices are sought. A number of such bounds are found for the case where the polynomial has only positive roots. A lower bound for the magnitude of the least root of an arbitrary real polynomial is obtained. By considering matrices in tri-diagonal form, the bounds are extended to matrices. Finally some bounds are given for the least root of a positive definite matrix, depending only on the matrix elements.

I. INTRODUCTION

When looking for polynomial and matrix roots we frequently wish to find only some of them. Often it is the least root or the first two, sometimes the largest. In many cases a lower bound for the magnitude of the least root will suffice.

The purpose of this report is to develop useful bounds for the roots of polynomials and matrices whose roots are all positive. Since most of the bounds given here are in terms of polynomial coefficients, some review and background material is included in Section II. Section's III and IV are concerned with polynomial bounds, while Section V deals with matrices.

The bounds developed here are easily extended to matrices and polynomials whose roots are negative. Some of the work might also be extended, and indeed is, to yield bounds on the magnitude of arbitrary real roots.

II. REVIEW AND BACKGROUND

In this section we review some facts about polynomials with only positive roots. Let

$$F(\lambda) = \lambda^n + C_1 \lambda^{n-1} + C_2 \lambda^{n-2} + \cdots + C_{n-2} \lambda^2 + C_{n-1} \lambda + C_n = 0$$

be a polynomial all of whose roots are positive. Let $0 < \lambda_n \leq \lambda_{n-1} \leq \lambda_{n-2} \leq \cdots \leq \lambda_2 \leq \lambda_1$ be the n roots of $F(\lambda) = 0$. We may express the coefficients C_i of $F(\lambda)$ in terms of the λ_i as follows

$$C_1 = - \sum_{i=1}^{n} \lambda_i, \quad C_2 = \sum_{\substack{i=1 \\ j>i}}^{n-1} \lambda_i \lambda_j, \quad C_3 = - \sum_{\substack{i=1 \\ j>i \\ k>j}}^{n-2} \lambda_i \lambda_j \lambda_k ,$$

$$\cdots, \quad C_{n-1} = (-1)^{n-1} \prod_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \frac{1}{\lambda_j}, \quad C_n = (-1)^n \prod_{i=1}^{n} \lambda_i .$$

Notice that the signs of the C_i alternate with i. If we multiply $F(\lambda)$ by $(-1)^n$, we get

$$(-1)^n F(\lambda) \equiv f(\lambda) = (-\lambda)^n + a_1(-\lambda)^{n-1} + a_2(-\lambda)^{n-2} + \cdots\cdots$$

$$+ a_{n-2} \lambda^2 - a_{n-1} \lambda + a_n = 0,$$

where the a_i are all positive, and the roots of $F(\lambda)$ and $f(\lambda)$ are the same.

We will consider further the polynomial $f(\gamma)$ whose roots γ_i are the inverses of the roots of $f(\lambda)$. Let

$$0 = f(\gamma) = (-\gamma)^n + b_1(-\gamma)^{n-1} + b_2(-\gamma)^{n-2} + \cdots\cdots + b_{n-2}\gamma^2 - b_{n-1}\gamma + b_n.$$

But:

$$b_n = \prod_{i=1}^{n} \frac{1}{\lambda_i} = \frac{1}{a_n}$$

$$b_{n-1} = \sum_{j=1}^{n} \prod_{i=1}^{n} \frac{\lambda_j}{\lambda_i} = \prod_{i=1}^{n} \frac{1}{\lambda_i} \sum_{j=1}^{n} \lambda_j = \frac{1}{a_n} \cdot a_1 = \frac{a_1}{a_n}$$

$$b_{n-2} = \sum_{\substack{j=1\\k>j}}^{n-1} \prod_{i=1}^{n} \frac{\lambda_j \cdot \lambda_k}{\lambda_i} = \frac{1}{a_n} \sum_{\substack{j=1\\k>j}}^{n-1} \lambda_j \lambda_k = \frac{a_2}{a_n}.$$

$$\vdots$$

$$b_2 = \sum_{\substack{i=1\\j>i}}^{n-1} \frac{1}{\lambda_i \lambda_j} = \frac{1}{a_n} \sum_{\substack{i=1\\j>i}}^{n-1} \frac{a_n}{\lambda_i \lambda_j} = \frac{1}{a_n} \sum_{\substack{i=1\\j>i}}^{n-1} \prod_{k=1}^{n} \frac{\lambda_k}{\lambda_i \lambda_j} = \frac{a_{n-2}}{a_n}$$

$$b_1 = \sum_{i=1}^{n} \frac{1}{\lambda_i} = \frac{1}{a_n} \sum_{i=1}^{n} \frac{a_n}{\lambda_i} = \frac{1}{a_n} \sum_{i=1}^{n} \prod_{j=1}^{n} \frac{\lambda_j}{\lambda_i} = \frac{a_{n-1}}{a_n},$$

so that

$$0 = f(\gamma) = (-\gamma)^n + \frac{a_{n-1}}{a_n}(-\gamma)^{n-1} + \frac{a_{n-2}}{a_n}(-\gamma)^{n-2} + \cdots + \frac{a_2}{a_n}\gamma^2 - \frac{a_1}{a_n}\gamma + \frac{1}{a_n}.$$

We will call $f(\gamma)$ the dual polynomial of $f(\lambda)$.

III. POLYNOMIALS, LOWER BOUNDS

First we will consider polynomials with only positive roots:

$$f(\lambda) = (-\lambda)^n + a_1(-\lambda)^{n-1} + a_2(-\lambda)^{n-2} + \cdots + a_{n-2}\lambda^2 - a_{n-1}\lambda + a_n.$$

Let the roots of $f(\lambda) = 0$ be $0 < \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 \leq \lambda_1$.

<u>Lemma III.1</u> $\lambda_n \geq \dfrac{a_n}{a_{n-1}}$

Proof. We have from algebra that

$$a_n = \prod_{i=1}^{n} \lambda_i$$

and

$$a_{n-1} = \sum_{j=1}^{n} \frac{\prod_{i=1}^{n} \lambda_i}{\lambda_j} = \prod_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \frac{1}{\lambda_j} .$$

Then

$$\frac{a_n}{a_{n-1}} = \frac{\prod_{i=1}^{n} \lambda_i}{\prod_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \frac{1}{\lambda_j}} = \frac{1}{\sum_{j=1}^{n} \frac{1}{\lambda_j}} \leqslant \frac{1}{\frac{1}{\lambda_n}} = \lambda_n . \qquad (3.1)$$

Q. E. D.

Lemma III.2 $\quad a_{n-2}\lambda_n^2 - \frac{n}{2} a_{n-1} \lambda_n + \frac{n}{2} a_n \leqslant 0$

Proof. From algebra we have that

$$a_{n-2} = \prod_{i=1}^{n} \lambda_i \sum_{\substack{j=1 \\ k>j}}^{n-1} \frac{1}{\lambda_j \lambda_k} .$$

$$\frac{a_{n-2}}{a_n} = \sum_{\substack{j=1 \\ k>j}}^{n-1} \frac{1}{\lambda_j \lambda_k} = \frac{1}{\lambda_n} \sum_{j=1}^{n-1} \frac{1}{\lambda_j} + \frac{1}{\lambda_{n-1}} \sum_{j=1}^{n-2} \frac{1}{\lambda_j} + \cdots + \frac{1}{\lambda_2 \lambda_1}$$

$$\leqslant \frac{1}{\lambda_n} \left[\sum_{j=1}^{n-1} \frac{1}{\lambda_j} + \sum_{j=1}^{n-2} \frac{1}{\lambda_j} + \cdots \frac{1}{\lambda_1} \right]. \qquad (3.2)$$

Since $1/\lambda_1 \leqslant 1/\lambda_{n-1}$

we have

$$\sum_{j=1}^{n-2} \frac{1}{\lambda_j} + \frac{1}{\lambda_1} \leqslant \sum_{j=1}^{n-1} \frac{1}{\lambda_j} .$$

If n is even and $k \leqslant n-2/2$,

$$\sum_{j=1}^{n-1-k} \frac{1}{\lambda_j} + \sum_{j=1}^{k} \frac{1}{\lambda_j} \leqslant \sum_{j=1}^{n-1} \frac{1}{\lambda_j} . \qquad (3.3)$$

Putting Eq. (3.3) in Eq. (3.2), we have

$$\text{Eq. } 3.2 \leqslant \frac{1}{\lambda_n} \frac{n}{2} \sum_{j=1}^{n-1} \frac{1}{\lambda_j} . \qquad (3.4)$$

Equation (3.3) also holds when n is odd and $k \leqslant (n-3)/2$. If n is odd, after combining terms in the manner indicated in Eq. (3.3) there remains the term

$$\sum_{j=1}^{\frac{n-3}{2}} \frac{1}{\lambda_j} .$$

But

$$\sum_{j=1}^{\frac{n-3}{2}} \frac{1}{\lambda_j} \leqslant \frac{1}{2} \sum_{j=1}^{n-1} \frac{1}{\lambda_j} .$$

So for n odd

$$\text{Eq. (3.2)} \leqslant \frac{1}{\lambda_n} \left[1 + \frac{n-3}{2} + \frac{1}{2} \right] \sum_{j=1}^{n-1} \frac{1}{\lambda_j} = \frac{1}{\lambda_n} \frac{n}{2} \sum_{j=1}^{n-1} \frac{1}{\lambda_j} .$$

Equation (3.4) has been shown to be true for n even or odd. Continuing, we have

$$\frac{a_{n-2}}{a_n} \leqslant \frac{1}{\lambda_n} \frac{n}{2} \sum_{j=1}^{n-1} \frac{1}{\lambda_j} = \frac{1}{\lambda_n} \frac{n}{2} \left[\sum_{j=1}^{n} \frac{1}{\lambda_j} - \frac{1}{\lambda_n} \right] .$$

From Eq. (3.1)

$$\sum_{j=1}^{n} \frac{1}{\lambda_j} = \frac{a_{n-1}}{a_n} .$$

Then

$$\frac{a_{n-2}}{a_n} \leqslant \frac{1}{\lambda_n} \frac{n}{2} \left[\frac{a_{n-1}}{a_n} - \frac{1}{\lambda_n} \right] ,$$

or

$$a_{n-2} \lambda_n^2 - \frac{n}{2} a_{n-1} \lambda_n + \frac{n}{2} a_n \leqslant 0. \qquad (3.5)$$

Q. E. D.

We will show that the parabola

$$p_1(\lambda) = a_{n-2} \lambda^2 - \frac{n}{2} a_{n-1} \lambda + \frac{n}{2} a_n = 0 \qquad (3.6)$$

has two positive real roots. Consider the discriminant

$$\left[\left(\frac{n}{2}\, a_{n-1}\right)^2 - 4\, a_{n-2}\, \frac{n}{2}\, a_n\right]^{1/2} = \frac{n}{2}\left[a_{n-1}^2 - \frac{8}{n}\, a_{n-2}\, a_n\right]^{1/2}, \qquad (3.7)$$

where

$$a_{n-1}^2 = \prod_{j=1}^{n} \lambda_j^2 \left(\sum_{i=1}^{n} \frac{1}{\lambda_i}\right)^2, \quad a_{n-2}\, a_n = \prod_{j=1}^{n} \lambda_j^2 \left(\sum_{\substack{i=1\\k>i}}^{n-1} \frac{1}{\lambda_i\, \lambda_k}\right).$$

For $n = 4$:

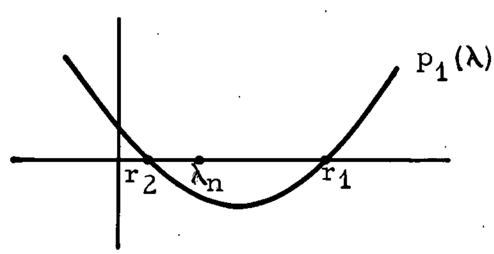
$$\left(\sum_{i=1}^{4} \frac{1}{\lambda_i}\right)^2 - 2 \sum_{\substack{i=1\\k>i}}^{3} \frac{1}{\lambda_i\, \lambda_k} = \sum_{i=1}^{4} \frac{1}{\lambda_i^2} > 0.$$

For $n > 4$ we have $8/n < 2$, and Eq. (3.7) $\geqslant \sum_{i=1}^{n} 1/\lambda_i^2 > 0$.

For $n = 2$, we have $a_2 = \lambda_1\, \lambda_2$, $a_1 = \lambda_1 + \lambda_2$, $a_0 = 1$, and

$$(\lambda_1 + \lambda_2)^2 - 4\, \lambda_1\, \lambda_2 = \lambda_1^2\, \lambda_2^2 + 2\, \lambda_1\, \lambda_2 - 4\, \lambda_1\, \lambda_2 = (\lambda_1 - \lambda_2)^2 \geqslant 0.$$

For $n = 3$ Eq. (3.7) also holds, but the result does not warrant the space necessary to demonstrate it. That the roots are positive follows from "Descartes' Rules" and the above proof that they are real. $p_1(\lambda)$ is convex upwards (slope increasing with λ). From Eq. (3.5), we have $p_1(\lambda_n) \leqslant 0$. Consequently if $r_2 \leqslant r_1$ are the roots of $p_1(\lambda) = 0$, then $0 < r_2 \leqslant \lambda_n \leqslant r_1$.



For the sake of keeping all the results concerning lower bounds in the same section, the following result of the next section is presented without proof.

If $p_2(\lambda) = 0 = a_2\, \lambda^2 - (n/2)\, a_1\, \lambda + n/2$ has roots $s_1 \geqslant s_2 > 0$, and λ_1 is the largest root of $f(\lambda) = 0$, then

$$\frac{1}{s_1} \leqslant \lambda_1 \leqslant \frac{1}{s_2}\,.$$

Lemma III.3 $\quad 0 < \frac{a_n}{a_{n-1}} < r_2 \leq \lambda_n.$

Proof. From Lemma III.1, we have $a_n/a_{n-1} < \lambda_n$. We will show that $p_1\left(a_n/a_{n-1}\right) > 0$, from which the Lemma follows.

$$p_1\left(\frac{a_n}{a_{n-1}}\right) = a_{n-2}\left(\frac{a_n}{a_{n-1}}\right)^2 - \frac{n}{2}\, a_{n-1}\, \frac{a_n}{a_{n-1}} + \frac{n}{2}\, a_n$$

$$= a_{n-2}\left(\frac{a_n}{a_{n-1}}\right)^2 \geq 0.$$

Q. E. D.

Lemmas III.1 through III.3 have been concerned with polynomials with positive roots. Lemma III.4 considers arbitrary polynomials with real roots and gives a lower bound for the root with least absolute value.

Consider the polynomial with real roots.

$$0 = f(\lambda) = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_{n-2}\lambda^2 + b_{n-1}\lambda + b_n.$$

Let the roots of $f(\lambda) = 0$ be $0 < |\lambda_n| \leq |\lambda_{n-1}| \leq \cdots \leq |\lambda_2| \leq |\lambda_1|$.

Lemma III.4

$$|\lambda_n| \geq \frac{|b_n|}{\left(b_{n-1}^2 - 2\, b_n\, b_{n-2}\right)^{1/2}}.$$

Proof. $\quad 0 = f(-\lambda) = (-\lambda)^n + b_1(-\lambda)^{n-1} + \cdots + b_{n-2}\lambda^2 - b_{n-1}\lambda + b_n.$

$$0 = f(\lambda)\cdot f(-\lambda) = (-1)^n\lambda^{2n} + \cdots - (b_{n-1}^2 - 2\, b_n\, b_{n-2})\,\lambda^2 + b_n^2$$

is an even valued polynomial in λ. Let $\gamma = \lambda^2$. Then

$$0 = f(\lambda)\cdot f(-\lambda) = f(\gamma) = (-\gamma)^n + \cdots - (b_{n-1}^2 - 2\, b_n\, b_{n-2})\,\gamma + b_n^2.$$

$0 = f(\gamma)$ is a polynomial with real positive roots, so that $(b_{n-1}^2 - 2\, b_n\, b_{n-2}) \geq 0$. In fact, $\gamma_i = \lambda_i^2$. We may use either Lemmas III.1 or Lemma III.2 to bound γ_n. From Lemma III.1

$$\gamma_n \geq \frac{b_n^2}{(b_{n-1}^2 - 2\, b_n\, b_{n-2})}$$

or

$$|\lambda_n| \geq \frac{|b_n|}{\left(b_{n-1}^2 - 2\, b_n\, b_{n-2}\right)^{1/2}}$$

Q. E. D.

Suppose we return to considering $f(\lambda) = 0$ with only positive roots. Then $b_n = a_n > 0$ and $b_{n-2} = a_{n-2} > 0$.

Lemma III.5. If the roots of $f(\lambda) = 0$ are all positive, then

$$\lambda_n \geq \frac{a_n}{\left(a_{n-1}^2 - 2\, a_n\, a_{n-2}\right)^{1/2}} \geq \frac{a_n}{a_{n-1}} > 0 .$$

Proof. $a_n > 0$, $a_{n-1} > 0$, $a_{n-2} > 0$ imply that

$$\frac{a_n}{\left(a_{n-1}^2 - 2\, a_n\, a_{n-2}\right)^{1/2}} \geq \frac{a_n}{\left(a_{n-1}^2\right)^{1/2}} = \frac{a_n}{a_{n-1}} .$$

Q. E. D.

We will now give a technique for finding a lower bound for any of the roots of a polynomial with real positive roots.

Lemma III.6 If the roots of $f(\lambda) = 0$ are all positive then,

$$\lambda_i \geq \frac{a_i}{(n + 1 - i)\, a_{i-1}} \qquad i = 1, 2, \cdots n,$$

where n is the order of the polynomial.

Proof. We know that the roots of $f'(\lambda) = 0$ separate the roots of $f(\lambda) = 0$. Let λ'_{n-1} be the minimum root of $f'(\lambda) = 0$. $\lambda_n \leq \lambda'_{n-1} \leq \lambda_{n-1}$:

$$f'(\lambda) = - n(-\lambda)^{n-1} - (n-1)a_1\, (-\lambda)^{n-2} - \cdots + 2\, a_{n-2}\, \lambda - a_{n-1}.$$

From Lemma III.1 we have for $f'(\lambda) = 0$ that $\lambda'_{n-1} \geq a_{n-1}/2\, a_{n-2}$, so that $\lambda_{n-1} \geq a_{n-1}/2\, a_{n-2}$. In the same manner the roots of $f''(\lambda) = 0$ separate the roots of $f'(\lambda) = 0$. So $\lambda'_{n-1} \leq \lambda''_{n-2} \leq \lambda'_{n-2} \leq \lambda_{n-2}$. We can show in the same way as above that

$$\lambda''_{n-2} \geq \frac{2\, a_{n-2}}{3.2\, a_{n-3}} = \frac{a_{n-2}}{3\, a_{n-3}} .$$

So

$$\lambda_{n-2} \geq \lambda''_{n-2} \geq \frac{a_{n-2}}{3\, a_{n-3}} .$$

Continuing in the same manner, we find for $i = 1, 2, \cdots n$ that

$$\lambda_i \geq \frac{a_i}{(n + 1 - i)\, a_{i-1}} .$$

Q. E. D.

It does not follow that $\dfrac{a_i}{(n + 1 - i)\, a_{i-1}} < \dfrac{a_{i-1}}{(n + 1 - i + 1)a_{i-2}}$,

so that a lower bound obtained for λ_i is frequently a lower bound for λ_{i-1}. A better lower bound for λ_i could be obtained by using the root-separating relationship used in Lemma III.6 and the bounds found in Lemma's III.3 and III.5.

Lemma III.7 If the roots of $f(\lambda) = 0$ are all positive, then for $i = 2, 3, \cdots n$

$$\lambda_i \geq \frac{a_i}{\left((n-i+1)^2 a_{i-1}^2 - (n-i+1)(n-i+2) a_i a_{i-2}\right)^{1/2}} .$$

Proof. Let $c_i^{(j)}$ be the coefficients of the polynomial $f^{(j)}(\lambda) = d^j f(\lambda)/d\lambda^j$, where $j = 0, 1, 2, \cdots, n-2$ and $i = 1, 2, \cdots, n-j$. In terms of the coefficients a_i of $f(\lambda)$

$$c_i^{(j)} = \frac{a_i (n-i)!}{(n-i-j)!} .$$

Let the roots of $f^{(j)}(\lambda) = 0$ be $\lambda_i^{(j)}$ with $\lambda_i^{(j)} \geq \lambda_{i+1}^{(j)}$. As shown in Lemma III.6, $\lambda_{n-i} \geq \lambda_{n-i}^{(j)}$. From Lemma III.5

$$\lambda_{n-j} \geq \lambda_{n-j}^{(j)} \geq \frac{c_{n-j}^{(j)}}{\left(\left[c_{n-j-1}^{(j)}\right]^2 - 2 c_{n-j}^{(j)} c_{n-j-2}^{(j)}\right)^{1/2}} .$$

Let $i = n-j$; then we have

$$\lambda_i \geq \frac{c_i^{(n-i)}}{\left\{\left[c_{i-1}^{(n-i)}\right]^2 - 2 c_i^{(n-i)} c_{i-2}^{(n-i)}\right\}^{1/2}}$$

$$= \frac{a_i (n-i)!}{\left[a_{i-1}^2 (n-i+1)!^2 - 2 a_i (n-i)! \, a_{i-2} (n-i+2)!/2!\right]^{1/2}}$$

$$= \frac{a_i}{\left[(n-i+1)^2 a_{i-1}^2 - (n-i+2)(n-i+1) a_i a_{i-2}\right]^{1/2}} ,$$

$i = 2, 3, \cdots, n.$

Q. E. D.

Lemma III.8 If the roots of $f(\lambda) = 0$ are all positive, then for $i = 2, 3, \cdots n$

$$\lambda_i \geq \frac{i}{(n-i+2)} \left\{ \frac{a_{i-1} - \left[a_{i-1}^2 - \frac{4}{i} \frac{(n-i+2)}{(n-i+1)} a_{i-2} a_i\right]^{1/2}}{2 a_{i-2}} \right\} .$$

Proof. Let $c_i^{(j)}$ be as defined in Lemma III.7. Also as in Lemma III.7, $\lambda_{n-i} \geq \lambda_{n-i}^{(j)} \geq \lambda_{n-i+1}$. As in Lemma III.2, the least root of $f^{(j)}(\lambda) = 0$, $\lambda_{n-j}^{(j)}$ satisfies the polynomial inequality

$$c_{n-j-2}^{(j)} \lambda_{n-j}^{(j)\,2} - \left(\frac{n-j}{2}\right) c_{n-j-1}^{(j)} \lambda_{n-j}^{(j)} + \frac{(n-j)}{2} c_{n-j}^{(j)} \leq 0.$$

Substituting for $c_i^{(j)} = a_i \frac{(n-i)!}{(n-i-j)!}$ we have,

$$0 \geq a_{n-j-2} \frac{(j+2)!}{2!} \lambda_{n-j}^{(j)^2} - \left(\frac{n-j}{2}\right) a_{n-j-1} \frac{(j+1)!}{1!} \lambda_{n-j}^{(j)} + \left(\frac{n-j}{2}\right) a_{n-j} \frac{j!}{0!}.$$

Let $i = n-j$ and multiply both sides of the inequality by $2!/(j+2)!$:

$$0 \geq a_{i-2} \lambda_i^{(j)^2} - \frac{i}{2} \frac{2}{(j+2)!} \frac{(J+1)!}{1!} a_{i-1} \lambda_i^{(j)} + \frac{i}{2} \frac{2}{(j+2)!} \frac{j!}{0!} a_i$$

$$= a_{i-2} \lambda_i^{(j)^2} - \frac{i}{(n-i-2)} a_{i-1} \lambda_i^{(j)} + \frac{i}{(n-i-2)(n-i-1)} a_i .$$

Let the roots of $p_i(\lambda) = 0 = a_{i-2} \lambda^2 - \frac{i}{(n-i+2)} a_{i-1} \lambda + \frac{i}{(n-i+2)(n-i+1)} a_i$ be $r_{i,1} \geq r_{i,2}$. As in Lemma III, we have

$$r_{i,1} \geq \lambda_i^{(j)} > r_{i,2} .$$

Then

$$\lambda_i \geq \lambda_i^{(j)} > r_{i,2} \qquad i = 2, 3, \cdots, n.$$

Q.E.D.

IV. COMPLETE BOUNDS FOR POLYNOMIAL ROOTS

In this section we will derive upper bounds for the roots of real polynomials. We consider the dual polynomial whose roots are the inverses of the polynomial in Section III. The results of Section III for the dual polynomial will give upper bounds for the polynomial roots. The upper and lower bounds are combined to give complete bounds.

Again, $f(\lambda)$ is the polynomial

$$f(\lambda) = (-\lambda)^n + a_1(-\lambda)^{n-1} + a_2(-\lambda)^{n-2} + \cdots + a_{n-2}\lambda^2 - a_{n-1}\lambda + a_n ,$$

and the roots of $f(\lambda) = 0$ are

$$0 < \lambda_n \leq \lambda_{n-1} \leq \lambda_{n-2} \leq \cdots \leq \lambda_2 \leq \lambda_1 .$$

It is easy to show that <u>Lemma III.1</u> is the dual of the bound

Lemma IV.1: $\lambda_1 \leq \sum_{i=1}^{n} \lambda_1 = a_1 .$

From Section II the roots of the dual polynomial are the roots of

$$0 = f(\gamma) = (-\gamma)^n + \frac{a_{n-1}}{a_n}(-\gamma)^{n-1} + \cdots + \frac{a_2}{a_n}\gamma^2 - \frac{a_1}{a_n}\gamma + \frac{1}{a_n} .$$

Since

$$\frac{1}{\lambda_n} = \gamma_1 \leq \sum_{i=1}^{n} \gamma_i = \frac{a_{n-1}}{a_n} ,$$

which is Lemma III.1.

If we apply Lemma III.2 to the dual polynomial we obtain

Lemma IV.2:

$$\frac{a_2}{a_n}\gamma_n^2 - \frac{n}{2}\frac{a_1}{a_n}\gamma_n + \frac{n}{2}\frac{1}{a_n} \leq 0.$$

The polynomial

$$p_1(\gamma) = a_2\gamma^2 - \frac{n}{2}a_1\gamma + \frac{n}{2} = 0$$

with roots $s_2 \leq s_1$ bound γ_n with $0 < s_2 \leq \gamma_n \leq s_1$ or $s_2 \leq 1/\lambda_1 \leq s_1$, giving

$$\frac{1}{s_1} \leq \lambda_1 \leq \frac{1}{s_2} .$$

Lemma IV.3: $\lambda_1 \leq \frac{1}{s_2} \leq a_1$.

Proof. Apply Lemma III.3 to the dual polynomial

$$0 < \frac{\frac{1}{a_n}}{\frac{a_1}{a_n}} < s_2 \leq \gamma_n$$

or

$$\lambda_1 = \frac{1}{\gamma_n} \leq \frac{1}{s_2} < a_1 .$$

Q. E. D.

In Lemma IV.4 we consider a polynomial with arbitrary real roots.

Lemma IV.4: $|\lambda_1| \leq (b_1^2 - 2\,b_2)^{1/2}$.

Proof. Apply Lemma III.4 to the dual polynomial.

Q. E. D.

Lemma IV.5: If the roots of $f(\lambda) = 0$ are all positive, then

$$\lambda_1 \leq (a_1^2 - 2\,a_2)^{1/2} \leq a_1 .$$

Proof. From Lemma IV.4 and inspection.

Lemma IV.6: If the roots of $f(\lambda) = 0$ are all positive, then

$$j \frac{a_j}{a_{j-1}} \geq \lambda_j \qquad j = 1, 2, \cdots, n.$$

Proof. Lemma III.6 gives the inequality

$$\lambda_i \geq \frac{a_i}{(n+1-i)\, a_{i-1}}$$

for the roots of $f(\lambda) = 0$. To apply this to the dual $f(\lambda) = 0$, notice that we must replace a_i by a_{n-i} and a_{i-1} by a_{n-i+1}. Then

$$\frac{1}{\lambda_{n-i+1}} = \gamma_i \geq \frac{a_{n-i}}{(n+1-i)\, a_{n-i+1}} .$$

Let $n - i + 1 = j$; then we have

$$\frac{1}{\lambda_j} \geq \frac{a_{j-1}}{j\, a_j}.$$

Q. E. D.

Theorem 1: If the roots of $f(\lambda) = 0$ are all positive, then

$$\frac{a_i}{(n-i+1)\, a_{i-1}} \leq \lambda_i \leq \frac{i\, a_i}{a_{i-1}} .$$

Proof. Combine Lemmas III.6 and IV.6.

Q. E. D.

Theorem 2: If the roots of $f(\lambda) = 0$ are all positive, then for $i = 2, 3, \cdots, n-1$, we have

$$\frac{(i^2 a_i^2 - i(i+1) a_{i-1}\, a_{i+1})^{1/2}}{a_{i-1}} \geq \lambda_i$$

$$\geq \frac{a_i}{\left[(n-i+1)^2 a_{i-1}^2 - (n-i+1)(n-i+2)\, a_i\, a_{i-2}\right]^{1/2}} \tag{4.1}$$

for $i = 1$

$$(a_1^2 - 2\, a_2)^{1/2} \geq \lambda_1 \geq \frac{a_1}{n} \tag{4.2}$$

for $i = n$

$$\frac{n\, a_n}{a_{n-1}} \geq \lambda_n \geq \frac{a_n}{(a_{n-1}^2 - 2\, a_n\, a_{n-2})^{1/2}} . \tag{4.3}$$

Proof: The right-hand side of the inequality (4.1) is Lemma III.7. Inequalities (4.2) and (4.3) are obtained from Eqs. (4.1) and Theorem 1. The left-hand side of Eq. (4.1) remains to be proven. From the dual polynomial and Lemma III.7 we have

$$\gamma_i \geqslant \frac{b_i}{\left[(n-i+1)^2 b_{i-1}^2 - (n-i+1)(n-i+2) b_i b_{i-2}\right]^{1/2}},$$

where $b_i = a_{n-i}/a_n$, $\gamma_i = 1/\lambda_{n-i+1}$, and $i = 2, 3, \cdots, n$. Making the substitutions and putting $j = n - i + 1$, we have

$$\frac{1}{\lambda_j} \geqslant \frac{a_{j-1}}{\left[j^2 a_j^2 - j(j+1) a_{j-1} a_{j+1}\right]^{1/2}}, \quad j = 1, 2, \cdots, n-1.$$

Now, replacing j by i and inverting, we have the left-hand side of Eq. (4.1).

Q. E. D.

Theorem 3: If the roots of $f(\lambda) = 0$ are all positive, then we have,

for $i = 2, 3, \cdots, n-1$,

$$\left\{\frac{(i+1)}{(n-i+1)} \frac{2 a_{i+1}}{a_i - \left[a_i^2 - \frac{4(i+1) a_{i-1} a_{i+1}}{i(n-i+1)}\right]^{1/2}}\right\} \geqslant \lambda_i$$

$$\geqslant \frac{i}{(n-i+2)} \left\{\frac{a_{i-1} - \left[a_{i-1}^2 - \frac{4(n-i+2)}{i(n-i+1)} a_{i-2} a_i\right]^{1/2}}{2 a_{i-2}}\right\}; \tag{4.4}$$

for $i = 1$,

$$\frac{4 a_2}{n a_1 - (n^2 a_1^2 - 8 n a_2)^{1/2}} \geqslant \lambda_1 \geqslant \frac{4 a_2}{n a_1 + (n^2 a_1^2 - 8 n a_2)^{1/2}}; \tag{4.5}$$

and for $i = n$,

$$\frac{n a_{n-1} + (n^2 a_{n-1}^2 - 8 n a_{n-2} a_n)^{1/2}}{4 a_{n-2}} \geqslant \lambda_n$$

$$\geqslant \frac{n a_{n-1} - (n^2 a_{n-1}^2 - 8 n a_{n-2} a_n)^{1/2}}{4 a_{n-2}}. \tag{4.6}$$

Proof. The right-hand sides of the inequalities (4.4) and (4.6) come from Lemma III.8. The left-hand side of (4.6) comes from the comments preceding Lemma III.3.

To prove the left-hand side of Eq. (4.4) we use the dual polynomial and Lemma III.8:

$$\gamma_i \geqslant \frac{i}{(n-i+2)} \left\{ \frac{b_{i-1} - \left[b_{i-1}^2 - 4\,\frac{(n-i+2)}{i(n-i+1)}\, b_{i-2}\, b_i \right]^{1/2}}{2\, b_{i-2}} \right\},$$

where $\gamma_i = 1/\lambda_{n-i+1}$ and $b_i = a_{n-i}/a_n$. Making the substitution and putting $j = n-i+1$ we have

$$\frac{1}{\lambda_j} \geqslant \frac{n-j+1}{(j+1)} \left\{ \frac{a_j - \left[a_j^2 - \frac{4(j+1)}{(n-j+1)j}\, a_{j-1}\, a_{j+1} \right]^{1/2}}{2\, a_{j+1}} \right\}. \tag{4.7}$$

The left-hand sides of (4.4) and (4.5) follow by inverting (4.7) and replacing j by i. The right-hand side of (4.5) follows from the comments preceding Lemma III.3 applied to the dual polynomial.

Q. E. D.

V. BOUNDS FOR THE EIGENVALUES OF POSITIVE DEFINITE MATRICES

In this section methods are described which allow us to apply to positive definite matrices the inequalities for polynomials derived in Sections III and IV. Some bounds for the smallest eigenvalue not derived from the coefficients of the characteristic polynomial are also discussed.

There are many techniques for finding eigenvalues of matrices which depend on first finding the characteristic polynomial. Several of these methods are discussed by Fadeeva.* In general these methods antedate large-scale computers and are no longer used.

More recent methods** rely on first reducing the given matrix to tri-diagonal form. In this form the characteristic polynomial can be evaluated for different values of λ without explicitly finding its coefficients. This is

*V. N. Fadeeva, Computational Methods of Linear Algebra, Chapter 3 (Dover, New York, 1959).

**See the methods of A. S. Householder and W. Givens in Simultaneous Linear Equations and Determination of Eigenvalues, National Bureau of Standards Applied Mathematics Series 29 (U. S. Government Printing, Office, Washington, 1953).

sufficient for finding the roots of the polynomial. As we are concerned with bounding the roots, the coefficients of the characteristic polynomial are desired, and an algorithm for finding them is developed. This technique may also be used for bounding in absolute value the least eigenvalue of a symmetric matrix as in Lemma III.4.

We will suppose that the given real symmetric matrix has been transformed into the tri-diagonal matrix $A = (a_{ij})$, where

$$a_{ij} \begin{cases} 0 & i > j+1 \\ b_i \neq 0 & i = j+1 \\ a_i \neq 0 & i = j \\ 1 & i = j-1 \\ 0 & i < j-1, \end{cases}$$

and $i, j = 1, 2, \cdots, n$. The determinant $|A - \lambda I|$ is evaluated for any λ by the recursion relation

$$\alpha_i(\lambda) = (a_i - \lambda)\, \alpha_{i-1}(\lambda) - b_{i-1}\, \alpha_{i-2}(\lambda), \tag{5.1}$$

where

$$\alpha_0(\lambda) = 1, \ \alpha_{-1}(\lambda) = 0, \ b_0 = 1, \ \text{and} \ i = 1, 2, \cdots, n, \ \text{and}$$
$$|A - \lambda I| = \alpha_n(\lambda).$$

The roots of $\alpha_n(\lambda) = 0$ are the eigenvalues of A. The coefficients of the polynomial $\alpha_n(\lambda)$ are obtained by an extension of the recursion relation (5.1).

Consider Table I; the second column is the sequence of polynomials $\alpha_i(\lambda)$. In column three we define a new sequence δ_i, $i = 0, 1, \cdots, n$, where δ_i is the constant term in $\alpha_i(\lambda)$. By inspection of $\alpha_{i+1}(\lambda)$ we see that δ_{i+1} can be determined if we know δ_i and δ_{i-1}. In column four we define the sequence γ_i, $i = 0, 1, 2, \cdots, n$, where γ_i is the coefficient of λ in the polynomial $\alpha_i(\lambda)$. By inspection of $\alpha_{i+1}(\lambda)$ we see that γ_{i+1} can be determined if we know γ_i, γ_{i-1}, and δ_{i-1}. In the same manner we can define a new sequence for each power of λ in the polynomial sequence $\alpha_i(\lambda)$. The terms in the new sequence will depend on the two previous terms in that sequence and on a term in the preceding sequence.

The coefficients in $\alpha_n(\lambda)$ can then all be found successively in the manner described above. The computation is very simple and the Table may be generated either columnwise or rowwise.

The above result is summarized as follows:

Table I. Coefficients of some terms in $\alpha_i(\lambda)$.

i	Polynomials $\alpha_i(\lambda)$	Coefficient of constant term in $\alpha_i(\lambda)$	Coefficient of λ term in $\alpha_i(\lambda)$	Coefficient of λ^2 term in $\alpha_i(\lambda)$
0	$\alpha_0(\lambda) = 1$	$\delta_0 = 1$	$\gamma_0 = 0$	$\epsilon_0 = 0$
1	$\alpha_1(\lambda) = (a_1 - \lambda)\alpha_0$	$\delta_1 = a_1\delta_0$	$\gamma_1 = -1$	$\epsilon_1 = 0$
2	$\alpha_2(\lambda) = (a_2 - \lambda)\alpha_1 - b_1\alpha_0$	$\delta_2 = a_2\delta_1 - b_1\delta_0$	$\gamma_2 = a_2\gamma_1 - b_1\gamma_0 - \delta_1$	$\epsilon_2 = 1 = a_2\epsilon_1 - b_1\epsilon_0 - \gamma_1$
3	$\alpha_3(\lambda) = (a_3 - \lambda)\alpha_2 - b_2\alpha_1$	$\delta_3 = a_3\delta_2 - b_2\delta_1$	$\gamma_3 = a_3\gamma_2 - b_2\gamma_1 - \delta_2$	$\epsilon_3 = a_3\epsilon_2 - b_2\epsilon_1 - \gamma_2$
4	$\alpha_4(\lambda) = (a_4 - \lambda)\alpha_3 - b_3\alpha_2$	$\delta_4 = a_4\delta_3 - b_3\delta_2$	$\gamma_4 = a_4\gamma_3 - b_3\gamma_2 - \delta_3$	$\epsilon_4 = a_4\epsilon_3 - b_3\epsilon_2 - \gamma_3$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
n-1	$\alpha_{n-1}(\lambda) = (a_{n-1} - \lambda)\alpha_{n-2} - b_{n-2}\alpha_{n-3}$	$\delta_{n-1} = a_{n-1}\delta_{n-2} - b_{n-2}\delta_{n-3}$	$\gamma_{n-1} = a_{n-1}\gamma_{n-2} - b_{n-2}\gamma_{n-3} - \delta_{n-2}$	$\epsilon_{n-1} = a_{n-1}\epsilon_{n-2} - b_{n-2}\epsilon_{n-3} - \gamma_{n-2}$
n	$\alpha_n(\lambda) = (a_n - \lambda)\alpha_{n-1} - b_{n-1}\alpha_{n-2}$	$\delta_n = a_n\delta_{n-1} - b_{n-1}\delta_{n-2}$	$\gamma_n = a_n\gamma_{n-1} - b_{n-1}\gamma_{n-2} - \delta_{n-1}$	$\epsilon_n = a_n\epsilon_{n-1} - b_{n-1}\epsilon_{n-2} - \gamma_{n-1}$

Lemma V.1 Let the coefficient of λ^j in the polynomial $\alpha_i(\lambda)$ be

$$\alpha_{ij} = \begin{cases} 0 & i < j \\ (-1)^i & i = j \\ a_i \alpha_{i-1\,j} - b_{i-1}\alpha_{i-2\,j} - \alpha_{i-1\,j-1} & i > j\,, \end{cases}$$

where $j = 0, 1, \cdots, n$ and $i = 0, 1, 2, \cdots, n-1$. In particular α_{nj} is the coefficient of λ^j in the characteristic polynomial $\alpha_n(\lambda)$ of the matrix A.

Two lower bounds for the least eigenvalue of a positive definite matrix will now be given. Each of the bounds has a region in which it is best, although the first which is due to Kato is easier to evaluate and is best in a larger region.

Kato's Lemma V.2.* If $A = (a_{ij})$ is an nxn positive definite matrix, then

$$\lambda_n \geq \frac{|A|}{\left(\frac{\sum_{i=1}^{n} a_{ii}}{n-1}\right)^{n-1}} .$$

Proof. This result follows from the relation between the geometric and arithmetic mean:

$$\prod_{i=1}^{n-1} \lambda_i \leq \left(\frac{\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_{n-1}}{n-1}\right)^{n-1} ,$$

from which

$$\frac{|A|}{\lambda_n} = \frac{\prod_{i=1}^{n} \lambda_i}{\lambda_n} \leq \left(\frac{\sum_{i=1}^{n-1} \lambda_i}{n-1}\right)^{n-1} \leq \left(\frac{\sum_{i=1}^{n} \lambda_i}{n-1}\right)^{n-1} = \left(\frac{\sum_{i=1}^{n} a_{ii}}{n-1}\right)^{n-1} ,$$

or

$$\lambda_n \geq \frac{|A|}{\left(\frac{\sum_{i=1}^{n} a_{ii}}{n-1}\right)^{n-1}} .$$

Q. E. D.

Lemma V.3. If $A = (a_{ij})$ is an nxn positive definite matrix, then

$$\lambda_n > \frac{|A|}{\prod_{j=1}^{n} a_{jj} \sum_{i=1}^{n} \frac{1}{a_{ii}}}$$

*T. Kato, Estimation of Iterated Matrices with Application to the von Neumann Condition, Numerische Mathematik 2, 22 (1960).

Proof. Let $B = (b_{ij})$ be the inverse of A. Then we can write

$$\frac{1}{\lambda_n} < \sum_{i=1}^{n} \frac{1}{\lambda_i} = \sum_{i=1}^{n} b_{ii} = \frac{\sum_{i=1}^{n} A_{ii}}{|A|}, \tag{5.1}$$

where A_{ii} is the cofactor of a_{ii}. By considering the quadratic form of A, one has that A_{ii} is also positive definite. The following inequality exists for positive definite hermitian matrices

$$|A| \leqslant \prod_{j=1}^{n} a_{jj}.^{*}$$

Then we have

$$\sum_{i=1}^{n} A_{ii} \leqslant \prod_{j=1}^{n} a_{jj} \sum_{i=1}^{n} \frac{1}{a_{ii}},$$

which combined with (5.1) gives

$$\frac{1}{\lambda_n} < \frac{\prod_{j=1}^{n} a_{jj} \sum_{i=1}^{n} a_{ii}}{|A|}$$

or

$$\lambda_n > \frac{|A|}{\prod_{j=1}^{n} a_{jj} \sum_{i=1}^{n} a_{ii}}.$$

Q. E. D.

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*See R. Bellman, Introduction to Matrix Analysis (McGraw-Hill Book Publishing Co., New York, 1960), p. 126.