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NO SOLITARY WAVES IN 2-D GRAVITY AND CAPILLARY WAVES IN DEEP WATER

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Abstract. A fundamental question in the study of water waves is the existence and stability of solitary waves. Solitary waves have been proved to exist and have been studied in many interesting situations, and often arise from the balance of different forces/factors influencing the fluid dynamics, e.g. gravity, surface tension or the fluid bottom. However, the existence of solitary waves has remained an open problem in two of the simplest cases, namely for either pure gravity waves or pure capillary waves in infinite depth.

In this article we settle both of these questions in two space dimensions. Precisely, we consider incompressible, irrotational, infinite depth water wave equation, either with gravity and without surface tension, or without gravity but with surface tension. In both of these cases we prove that there are no solitary waves.

1. Introduction

The water wave equations describe the motion of the free surface of an inviscid incompressible fluid. Under the additional assumption of irrotationality, Zakharov [40] observed that this motion can be described purely by the evolution of the fluid surface.

Solitary waves are waves which move with constant velocity while maintaining a fixed asymptotically flat profile. The question of existence of solitary waves is a classical problem, going back to Russell's 1844 experimental observation of such waves. Equally interesting is the periodic setting, where one instead seeks periodic traveling waves.

Water waves come in many flavors depending on a number of physical parameters, most notably gravity, surface tension (capillarity) and fluid depth. Solitary waves have been discovered in many of these settings in both two and higher dimensions, beginning with the results in finite depth in [18], [7] and [4], followed by the bifurcation result in [31]. An extensive literature exists now on this subject, including both results for gravity and for gravity/capillary waves, see for instance [3, 12, 13, 15, 19–21, 32] and the review in [19]. This famously includes for instance the Stokes waves of greatest height, which have an angular crest; these were conjectured in [34] and proved to exist in [36], [2]. Within this family of problems, one of the most difficult cases turned out to be the case of deep water.

For water waves in deep water, solitary waves have been recently proved to exist provided that both gravity and surface tension are present, see [9, 10, 22, 27], following numerical work in [28, 29]. However, in the seemingly simpler cases where exactly one of these forces is active, the problem has remained largely open, and only partial nonexistence results are known [16, 24, 35].

The goal of the present work is to settle both of these problems in the two dimensional setting. We show that with either gravity and no surface tension, or with surface tension but no gravity, no solitary waves exist in infinite depth. It remains an open problem to prove a similar result in higher dimension.

The difficulty in both of these problems is that the water wave equations are not only fully nonlinear but also nonlocal. The equations we consider do have a lot of structure, as well as a scaling symmetry. However, taking advantage of this structure in the classical Eulerian formulation seems untractable at this time, particularly for large data solutions. Instead, in the present paper we use the holomorphic (conformal) formulation of the equations, which is specific to two dimensional problems. In this setting we eventually arrive at a simpler, symmetric formulation of the solitary wave equations, though still nonlinear and nonlocal. Our final argument is vaguely reminiscent of proofs of the absence of embedded resonances for elliptic operators, with an added twist which is due to the nonlocality.

1.1. **The Eulerian equations.** We consider the incompressible, infinite depth water wave equation in two space dimensions, either with gravity but no surface tension, or without gravity but with surface tension. This is governed by the incompressible Euler's equations with boundary conditions on the water surface.

We first describe the equations. We denote the water domain at time t by $\Omega(t)$, and the water surface at time t by $\Gamma(t)$. We think of $\Gamma(t)$ as being either asymptotically flat at infinity or periodic. The fluid velocity is denoted by u and the pressure is p. Then u solves the Euler's equations inside $\Omega(t)$,

$$\begin{cases} u_t + u\cdot\nabla u = -\nabla p - gj \\ \text{div } u = 0 \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

while on the boundary we have the dynamic boundary condition

$$p = -2\sigma\mathbf{H} \quad \text{on } \Gamma(t), \tag{1.2}$$

and the kinematic boundary condition

$$\partial_t + u\cdot\nabla \text{ is tangent to } \bigcup \Gamma(t). \tag{1.3}$$

Here g represents the gravity, $\mathbf{H}$ is the mean curvature of the boundary and σ represents the surface tension.

Under the additional assumption that the flow is irrotational, we can write u in terms of a velocity potential ϕ as $u = \nabla\phi$, where ϕ is harmonic within the fluid domain, with appropriate decay at infinity. Thus ϕ is determined by its trace on the free boundary $\Gamma(t)$. Denote by η the height of the water surface as a function of the horizontal coordinate. Following Zakharov [40], we introduce $\psi = \psi(t,x) \in \mathbb{R}$ to be the trace of the velocity potential ϕ on the boundary, $\psi(t,x) = \phi(t,x,\eta(t,x))$. Then the fluid dynamics can be expressed in terms of a one-dimensional evolution of the pairs of variables (η,ψ), namely

$$\begin{cases} \partial_t\eta - G(\eta)\psi = 0 \\ \partial_t\psi + g\eta - \sigma\mathbf{H}(\eta) + \dfrac{1}{2}|\nabla\psi|^2 - \dfrac{1}{2}\dfrac{(\nabla\eta\cdot\nabla\psi + G(\eta)\psi)^2}{1+|\nabla\eta|^2} = 0. \end{cases} \tag{1.4}$$

Here G represents the Dirichlet to Neumann map associated to the fluid domain. This is Zakharov's Eulerian formulation of the gravity/capillary water wave equations. One limitation of this formulation is that it assumes that the fluid surface is a graph.

It is known since Zakharov [40] that the water-wave system is Hamiltonian, where the Hamiltonian (conserved energy) is given by

$$\mathcal{H}(\eta,\psi) = \frac{g}{2}\int_{\mathbb{R}} \eta^2\, dx + \sigma\int_{\mathbb{R}} \left(\sqrt{1+\eta_x^2} - 1\right) dx + \frac{1}{2}\int_{\mathbb{R}}\int_{-\infty}^{\eta} |\nabla_{x,y}\phi|^2\, dx\, dy.$$

The momentum is another conserved quantity which arises by Noether's theorem as a consequence of the fact that the problem is invariant with respect to horizontal translations (see Benjamin and Olver [8] for a complete study of the invariants and symmetries of the water-wave equations):

$$\mathcal{M} = \int_{\mathbb{R}} \int_{-\infty}^{\eta(t,x)} \phi_x(t,x,y)\, dy\, dx.$$

Solitary waves are solutions to the water wave equations which have a constant profile which moves with constant speed,

$$(\eta, \psi)(x,t) = (\eta, \psi)(x - ct),$$

and which decay at infinity. Here c represents the horizontal speed of propagation for the free surface. The horizontal fluid speed, however, is in general not equal to c.

1.2. **Scale invariance and critical regularity.** We now discuss natural regularity assumptions for solitary waves in the Eulerian setting. The energy space for solutions (η, ψ), provided by the Hamiltonian, is different for gravity versus capillary waves,

$$E^g := L^2 \times \dot{H}^{\frac{1}{2}}, \qquad E^c := \dot{H}^1 \times \dot{H}^{\frac{1}{2}}.$$

We will not a-priori assume that a solitary wave has to have finite energy, though a-posteriori this will be established as an intermediate step within the proof. However, in order to study the existence of solitary waves, it is natural to consider solutions which have at least critical regularity. This is best understood from the scaling properties of the equation, which also differ depending on whether we consider gravity or capillary waves.

For gravity waves, the equations admit the scaling law

$$(\eta(x,t), \psi(x,t)) \to (\lambda^{-1}\eta(\lambda x, \lambda^{\frac{1}{2}} t), \lambda^{-\frac{3}{2}} \lambda x, \lambda^{\frac{1}{2}} t).$$

Thus one might be led to consider (η, ψ) in the space $\dot{H}^{\frac{3}{2}} \times \dot{H}^2$. However, this is not accurate due to the fact that the water wave system, viewed at the linearized level, is degenerate hyperbolic and in non diagonal form. Because of this, one needs to work instead with *Alihnac's good variable*, or in other words the diagonal variable, which in this context is best understood in terms of the differentiated variables, namely $(\eta_x, \nabla \phi_{|y=\eta})$. A scale invariant space for these variables is for instance

$$E^g_{crit} := \dot{H}^{\frac{1}{2}} \times \dot{H}^1.$$

Here we could also add the L^∞ norm for η_x, which would provide the Lipschitz property for the free surface.

For capillary waves, the equations admit the scaling law

$$(\eta(x,t), \psi(x,t)) \to (\lambda^{-1}\eta(\lambda x, \lambda^{\frac{3}{2}} t), \lambda^{-\frac{1}{2}} \lambda x, \lambda^{\frac{3}{2}} t).$$

Here the spaces are the same whether we look at $(\eta_x, \nabla \phi_{|y=\eta})$ or (η_x, ψ_x), namely

$$E^g_{crit} := \dot{H}^{\frac{1}{2}} \times L^2.$$

Our a-priori assumptions on the solitary waves will be slightly stronger than the above critical norms, but still at critical scaling; this is in part due to the need to define the conformal map in a way that does not looses the critical regularity. In particular the $\dot{H}^{\frac{1}{2}}$

norm for η_x is not good enough, as it does not yield a Lipschitz bound on Γ. Instead, we shall use a stronger bound, namely the norm in the homogeneous Besov space $\dot{B}^{\frac{1}{2}}_{2,1}$, where the dyadic pieces of η_x are still measured in $\dot{H}^{\frac{1}{2}}$ but summed in l^1. Precisely, this is defined in terms of a standard dyadic Littlewood-Paley decomposition

$$1 = \sum_{k\in\mathbb{Z}} P_k,$$

where the projectors P_k select functions with frequencies $\approx 2^k$, as

$$\|u\|_{\dot{B}^{\frac{1}{2}}_{2,1}} = \sum_{k\in\mathbb{Z}} 2^{\frac{k}{2}} \|P_k u\|_{L^2}.$$

1.3. **Solitary waves and the non-existence result.** Now we are in a position to provide the Eulerian formulation of our main results. We begin with the gravity waves:

Theorem 1. *The two dimensional gravity wave equation in deep water admits no solitary waves (η, ψ) with critical regularity $\eta_x \in \dot{B}^{\frac{1}{2}}_{2,1}, \nabla\phi_{|y=\eta} \in \dot{H}^1$.*

Here it is not essential to specify also the regularity of the velocity $\nabla\phi$ on the free surface, as for solitary waves this is immediately seen to be a consequence the regularity of η.

In earlier work it was shown[1] in [16] that there are no solitary waves with either positive elevation ($\eta \geq 0$) or negative elevation ($\eta \leq 0$) in both two and three dimensions, and more recently, that there are no two dimensional solitary waves with at least $|x|^{-1-\epsilon}$ decay at infinity, see [24, 35]. The expansion at infinity is discussed in greater detail in [37].

Compared with these earlier works, here we impose no sign condition on η, as well as no decay condition on η; indeed, our only "decay" assumption is $\eta_x \in \dot{B}^{\frac{1}{2}}_{2,1}$. In view of the embedding $\dot{B}^{\frac{1}{2}}_{2,1} \subset L^\infty$ this does require η to be Lipschitz continuous and thus of at most linear growth. However, it allows for functions with almost linear growth at infinity, which is the natural critical assumption, and far from any actual decay condition.

Here it is important to work at low regularity, as the solitary wave equations are not necessarily elliptic, and in particular may allow angular crests, e.g. as in the Stokes waves which have $2\pi/3$ angles. It is known that such crests can locally occur only at points of maximal elevation, and that the crest points are stationary point for the flow, where the fluid speed and the solitary wave speed coincide. Away from these stationary points for the flow, the equations for the solitary waves are seen to be elliptic, and the solitary waves must be C^∞.

In this work we do not impose any condition to guarantee the absence of stationary points, though the above Besov regularity property does preclude angular crests. However, while proving the result we give a stronger form of it in conformal coordinates, see Theorem 3, which also allows for angular crests. Then, at the end of Section 3, we show that this form excludes crests with angles $\theta \in (\pi/2, \pi)$, and in particular it excludes Stokes types crests, which are the only ones consistent with the solitary wave equations.

Next we continue with our result for capillary waves, where we use similar a-priori bounds. Here there are no prior results in this direction, as far as we are aware.

[1] This is both for the two and the three dimensional problem.

Theorem 2. *The two dimensional capillary wave equations in deep water admits no solitary waves (η, ψ) with critical regularity $\eta_x \in \dot{B}^{\frac{1}{2}}_{2,1}$, $\nabla\phi_{|y=\eta} \in L^2$.*

Our approach for both of these problems relies on the use of the holomorphic formulation of the equations, which uses conformal coordinates within the fluid domain. This approach[2] has been initially used precisely in the study of periodic traveling waves in work of Babenko [5,6], and later for solitary waves in [11,24,31] and many other works. It has also been implemented in the study of the dynamical problem for gravity waves in work of Wu [38], Dyachenko-Kuznetsov-Spector-Zakharov [17], and more recently by the authors and Hunter [23], as well as for capillary waves by the authors in [26]. The equivalence between the solitary wave equation and the Babenko equation has also been explored in [24, 33].

In addition to the Babenko equations [6], an alternate method is to use a different conformal transformation, called a hodograph transform, see e.g. [27] and references therein. This last method applies only for the study of steady flows, and not for the dynamical problem.

In the present work we derive a set of equations for steady gravity waves in holomorphic coordinates, which are easily seen to imply the Babenko equations for gravity waves; however, the converse is far less obvious. The equivalence of the Eulerian and the conformal formulation of the solitary wave equations has been extensively studied, see [33] and the references therein.

Similarly, we also produce a related set of equations for capillary waves. This is done in the next section, where we also recall the holomorphic form of the water wave evolution.

In the last two sections we prove the solitary wave nonexistence result, including also a statement of the main results which applies directly in holomorphic coordinates. This is a stronger statement, as in particular it does not require the free fluid surface to be a graph, and also precludes angular crests in the case of gravity waves.

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2. Water waves in holomorphic coordinates and the Babenko equations

2.1. **Holomorphic coordinates.** One main difficulty in the study of the above equations is the presence of the Dirichlet to Neumann operator $\mathcal{D}$, which depends on the free boundary. This is one of the motivations for our choice to use a different framework to study these equations, namely the holomorphic coordinates. In the context of the dynamical problem these were first introduced by Ovsiannikov [30], and further developed by Wu [39] and Dyachenko-Kuznetsov-Spector-Zakharov [17], and were heavily exploited by the authors in earlier work [23], [25] in the context of two dimensional gravity water waves and [26] for two dimensional capillary waves.

The holomorphic coordinates are defined via a conformal map $Z : \mathbb{H} \to \Omega(t)$, where $\mathbb{H}$ is the lower half plane, $\mathbb{H} := \{\alpha - i\beta \ : \ \beta > 0\}$. Here Z maps the real line into the free boundary $\Gamma(t)$. Thus the real coordinate α parametrizes the free boundary, which is denoted

[2]Babenko does not phrase his approach explicitly in terms of conformal coordinates, but the outcome is nevertheless the same.

by $Z(t,\alpha)$. Requiring Z to satisfy the condition

$$\lim_{\alpha\to\pm\infty} Z_\alpha = 1$$

uniquely determines it modulo horizontal translations. If in addition the fluid surface is assumed to be asymptotically flat then one can remove this remaining degree of freedom in the choice of the parametrization by imposing a stronger boundary condition at infinity,

$$\lim_{\alpha\to\pm\infty} Z(t,\alpha) - \alpha = 0.$$

In order to describe the velocity potential ϕ we use the function $Q = \phi + iq$ where q represents the harmonic conjugate of ϕ (the stream function). Both functions Z and Q admit Lipschitz holomorphic extensions into the lower half plane, which implies that their Fourier transforms are supported in $(-\infty, 0]$. By a slight abuse of terminology we call such functions holomorphic functions. They can be described by the relation $Pf = f$, where P represents the projector operator to negative frequencies. For later use we recall that P can be represented as

$$P := \frac{1}{2}\left(I - iH\right),$$

where H is the classical Hilbert transform on the real line.

The water wave equations will define a flow in the class of holomorphic functions for the pair of variables (W, Q) where $W := Z - \alpha$.

For a full derivation of the holomorphic form of gravity/capillary water waves we refer the reader to the paper [23] for gravity waves, and [26] for capillary waves. The equations have the form

$$\begin{cases} W_t + F(1+W_\alpha) = 0 \\ Q_t + FQ_\alpha - igW + P\left[\dfrac{|Q_\alpha|^2}{J}\right] - \sigma P\left[\dfrac{-i}{2+W_\alpha+\bar{W}_\alpha}\dfrac{d}{d\alpha}\left(\dfrac{W_\alpha - \bar{W}_\alpha}{|1+W_\alpha|}\right)\right] = 0, \end{cases} \tag{2.1}$$

where

$$F := P\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right], \qquad J := |1+W_\alpha|^2. \tag{2.2}$$

Simplifying we obtain the fully nonlinear system

$$\begin{cases} W_t + F(1+W_\alpha) = 0 \\ Q_t + FQ_\alpha - igW + P\left[\dfrac{|Q_\alpha|^2}{J}\right] + i\sigma P\left[\dfrac{W_{\alpha\alpha}}{J^{1/2}(1+W_\alpha)} - \dfrac{\bar{W}_{\alpha\alpha}}{J^{1/2}(1+\bar{W}_\alpha)}\right] = 0. \end{cases} \tag{2.3}$$

This is a still Hamiltonian system, where the Hamiltonian has a simple form when expressed in holomorphic coordinates:

$$\mathcal{H}(W,Q) = \int \Im(Q\bar{Q}_\alpha) + 2\sigma\left(J^{\frac{1}{2}} - 1 - \Re W_\alpha\right) + g\left(|W|^2 - \frac{1}{2}(\bar{W}^2 W_\alpha + W^2 \bar{W}_\alpha)\right) d\alpha.$$

The invariance with respect to horizontal translations leads by Noether's theorem to the conservation of momentum,

$$\mathcal{M}(W,Q) = i\int \bar{Q}W_\alpha - Q\bar{W}_\alpha \, d\alpha.$$

One downside of working in holomorphic coordinates is that here the symplectic form is not as simple as in the Eulerian case. Fortunately this is not needed in the present paper.

Since W and Q only appear in differentiated form in the equations above, differentiating with respect to α yields a self contained system for (W_α, Q_α). This system is best expressed using *Alihnac's good variable*, which in this case has the form

$$(\mathbf{W}, R) = \left(W_\alpha, \frac{Q_\alpha}{1+W_\alpha} \right).$$

Here $1+\mathbf{W}$ describes the slope of the free surface and R is simply the velocity vector in complex notation, both in the holomorphic parametrization. The system for $(\mathbf{W}, R)$ has the form

$$\begin{cases} \mathbf{W}_t + b\mathbf{W}_\alpha + \dfrac{(1+\mathbf{W})R_\alpha}{1+\bar{\mathbf{W}}} = (1+\mathbf{W})M \\ R_t + bR_\alpha + \dfrac{i(g+a)}{1+\mathbf{W}} + i\sigma \dfrac{1}{1+\mathbf{W}} P\left[\dfrac{\mathbf{W}_\alpha}{J^{1/2}(1+\mathbf{W})} \right]_\alpha = i\sigma P\left[\dfrac{\bar{\mathbf{W}}_\alpha}{J^{1/2}(1+\bar{\mathbf{W}})} \right]_\alpha, \end{cases} \tag{2.4}$$

where the real advection velocity b is given by

$$b := 2\Re P\left[\frac{Q_\alpha}{J} \right]. \tag{2.5}$$

The real *frequency-shift* a is given by

$$a := i\left(\bar{P}\left[\bar{R}R_\alpha \right] - P\left[R\bar{R}_\alpha \right] \right), \tag{2.6}$$

and the auxiliary function M is given by

$$M := \frac{R_\alpha}{1+\bar{\mathbf{W}}} + \frac{\bar{R}_\alpha}{1+\mathbf{W}} - b_\alpha = \bar{P}[\bar{R}Y_\alpha - R_\alpha \bar{Y}] + P[R\bar{Y}_\alpha - \bar{R}_\alpha Y]. \tag{2.7}$$

Here we used the notation $Y := \dfrac{\mathbf{W}}{1+\mathbf{W}}$. Primarily, the equations (2.3) will be used in the sequel. However the above discussion motivates the fact that all the results here can also be phrased in terms of the differentiated variables.

2.2. **The regularity of W_α and R.** Here we transfer the a-priori regularity of the Eulerian variables to the holomorphic variables (W, Q).

Lemma 2.1. *a) Assume that $\eta_x \in \dot{B}^{\frac{1}{2}}_{2,1}$. Then $W_\alpha \in \dot{B}^{\frac{1}{2}}_{2,1}$ and the following inequality holds*

$$1 + \Re W_\alpha \geq \delta > 0. \tag{2.8}$$

b) Assume in addition that the trace $\nabla\phi|_{y=\eta}$ of the velocity on the free surface is in L^2, respectively $\dot{H}^1$. Then R also has regularity L^2, respectively $\dot{H}^1$.

Proof. To obtain the conformal map Z and thus W_α it suffices to construct the harmonic function β inside the fluid domain $\Omega(t)$. Then α is the harmonic conjugate of β, normalized so that it vanishes on the top. Here β is a negative function, which solves the Dirichlet problem

$$
\begin{cases} -\Delta\beta = 0 & \text{in } \Omega(t) \\ \beta = 0 & \text{in } \Gamma(t) \\ \lim\limits_{(x,y)\to\infty} \dfrac{\beta - y}{|x|+|y|} = 0. \end{cases} \tag{2.9}
$$

STEP 1. We first prove the uniform bound (2.8). A direct computation using the inverse function theorem shows that

$$
1 + \Re W_\alpha = \frac{\partial_y \beta}{|\nabla_{(x,y)}\beta|^2},
$$

where due to the homogeneous boundary condition we have

$$
|\beta_x| \lesssim |\beta_y|.
$$

Thus we need to show that $|\beta_y| \lesssim 1$ on the top, or equivalently that $|\nabla\beta| \lesssim 1$, or equivalently that

$$
|\beta(x,y)| \approx |y - \eta|. \tag{2.10}
$$

Our starting point for this is the authors' result in [1], Proposition 3.3, which asserts that if η_x is small in $\dot{B}^{\frac{1}{2}}_{2,1}$ then $\mathbf{W}$ is small in the same space. This further implies that $\mathbf{W}$ has a small uniform bound, and thus $|1 + \mathbf{W} \approx 1$. For β this shows that (2.10) holds.

For our proof here we will combine this small data result with the maximum principle in two steps. The maximum principle shows that if η_1, η_2 are two elevation functions so that $\eta_1 \leq \eta_2$ then for the corresponding holomorphic coordinates β_1, β_2 we have $|\beta_1| \leq |\beta_2|$.

We first regularize η at a small frequency scale $\lambda \ll 1$, namely $\eta_{\leq\lambda}$. If λ is small enough then $\eta_{\leq\lambda}$ is small in $\dot{B}^{\frac{1}{2}}_{2,1}$, so the result in [1] applies. Thus, the corresponding function β_λ satisfies

$$
\beta_\lambda(x,y) \approx |y - \eta_{\leq\lambda}|.
$$

Here β_λ is the harmonic function solving (2.9), where the defining function for Γ is $\eta_{\leq\lambda}$.

On the other hand, by Bernstein's inequality, we have

$$
|\eta - \eta_{\leq\lambda}| \lesssim 1.
$$

Comparing $\eta < \eta_{\leq\lambda} + C$, by the maximum principle it follows that

$$
|\beta(x,y) \leq |\beta_\lambda(x, y - C) + C|,
$$

and thus

$$
|\beta(x,y)| \leq |y - \eta(x)| + 2C.
$$

This provides the desired bound (2.10) at distance $\gtrsim 1$ from Γ, and localizes the problem to the unit spatial scale.

For the next step of the proof of (2.10), we fix $x_0 \in \mathbf{R}$ and seek to estimate β near x_0. After a rotation (which preserves the $B^{\frac{1}{2}}_{2,1}$ regularity), we can assume without any restriction in generality that $\eta'(x_0) = 0$. Then, by localizing η near x_0, we can find a function η_1 which is small in $B^{\frac{1}{2}}_{2,1}$ so that

$$
\eta_1(x) = \eta(x_0) + (x - x_0)^2, \qquad |x - x_0| \ll 1.
$$

Then comparing the corresponding functions β and β_1 in a set

$$\{|x - x_0| \leq \epsilon, \quad \eta(x) - \epsilon \leq y \leq \eta(x)\},$$

by the maximum principle it follows that in this set we have

$$|\beta| \leq C|\beta_1|.$$

In particular at $x = x_0$ we have

$$|\beta(x_0, y)| \leq |\beta_1(x_0, y)| \lesssim |\eta_1(x_0) - y| \lesssim |\eta(x_0) - y|,$$

as needed.

STEP 2. Here we transfer the Sobolev regularity from η_x to W_α, and from $\nabla\phi|_{y=\eta}$ to R. On the top we have the relations

$$\Im Z(\alpha) = \eta(x), \qquad R(\alpha) = \phi_x + i\phi_y,$$

where the relation between the two parametrizations is given by

$$x = \Re Z(\alpha).$$

By (2.8) this is a bi-Lipschitz map, so we have the straightforward Sobolev norm equivalence

$$\|R\|_{L^2_\alpha} \approx \|\nabla\phi|_{y=\eta}\|_{L^2_x}, \qquad \|R\|_{\dot{H}^1_\alpha} \approx \|\nabla\phi|_{y=\eta}\|_{\dot{H}^1_x}.$$

Interpolating between L^2 and $\dot{H}^1$, we also obtain the Besov norm equivalence

$$\|\eta_x\|_{\dot{B}^{\frac{1}{2}}_{2,1,x}} \approx \|\eta_x\|_{\dot{B}^{\frac{1}{2}}_{2,1,\alpha}}$$

We still need to transfer the last bound to W_α. For that we compute

$$\eta_x = \frac{\Im Z_\alpha}{\Re Z_\alpha}.$$

This implies that

$$\frac{\eta_x}{1+\eta_x^2} = -\Im\left(\frac{1}{Z_\alpha}\right) = \Im\left(\frac{W_\alpha}{1+W_\alpha}\right) \tag{2.11}$$

Now we use the algebra property of the Besov space $\dot{B}^{\frac{1}{2}}_{2,1}$, as well as the fact that standard Moser estimates hold for this space. For later reference we state the result in the following

Lemma 2.2. *a) The space $\dot{B}^{\frac{1}{2}}_{2,1}$ is an algebra.*

b) Let $W \in \dot{B}^{\frac{1}{2}}_{2,1}$, and G a smooth function with $G(0) = 0$. Then we have the Moser estimate

$$\|G(W)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \lesssim C(\|W\|_{L^\infty})\|W\|_{\dot{B}^{\frac{1}{2}}_{2,1}}. \tag{2.12}$$

This is a standard result and the proof is omitted. Such a property is proved for instance in $\dot{H}^{\frac{1}{2}} \cap L^\infty$ in [23]; the Besov case is completely similar.

This property implies that

$$\left\|\frac{\eta_x}{1+\eta_x^2}\right\|_{\dot{B}^{\frac{1}{2}}_{2,1,\alpha}} \lesssim \|\eta_x\|_{\dot{B}^{\frac{1}{2}}_{2,1,\alpha}} C(\|\eta_x\|_{L^\infty})$$

Thus we obtain

$$\Im\left(\frac{W_\alpha}{1+W_\alpha}\right) \in \dot{B}^{\frac{1}{2}}_{2,1,\alpha}$$

But the function $\dfrac{W_\alpha}{1+W_\alpha}$ is bounded holomorphic in the lower half-plane with decay at infinity, so we have the usual relation between its real and imaginary part,

$$\Re\frac{W_\alpha}{1+W_\alpha} = H\Im\frac{W_\alpha}{1+W_\alpha}$$

Thus we obtain

$$\frac{W_\alpha}{1+W_\alpha} \in \dot{B}^{\frac{1}{2}}_{2,1,\alpha}$$

Here we know that $1+W_\alpha$ is bounded away from zero and thus W_α is bounded. This, again by the Moser property, yields

$$W_\alpha \in \dot{B}^{\frac{1}{2}}_{2,1,\alpha}$$

as needed.

□

2.3. **Solitary waves and the Babenko equations.** Solitons are solutions for (2.1) of the form $(Q(\alpha-ct), W(\alpha-ct))$. Here the sign of c is not important due to the time reversal symmetry $(W(t,\alpha), Q(t,\alpha)) \to (W(-t,\alpha), -Q(-t,\alpha))$. Substituting in the equations (2.3) we obtain the system

$$\begin{cases} -cW_\alpha + F(1+W_\alpha) = 0 \\ -cQ_\alpha + FQ_\alpha - igW + P\left[\dfrac{|Q_\alpha|^2}{J}\right] - \sigma P\left[\dfrac{-i}{2+W_\alpha+\bar{W}_\alpha}\dfrac{d}{d\alpha}\left(\dfrac{W_\alpha - \bar{W}_\alpha}{|1+W_\alpha|}\right)\right] = 0. \end{cases} \tag{2.13}$$

Before pursuing this venue, we make a brief parenthesis to review the Hamiltonian formalism and the corresponding more classical derivation of the Babenko equations. While we will not rely on this in the present paper, this is nevertheless an instructive exercise. We recall that the Hamiltonian and the horizontal momentum are given by

$$\mathcal{H}(W,Q) = \int \Im(Q\bar{Q}_\alpha) + 2\sigma\left(J^{\frac{1}{2}} - 1 - \Re W_\alpha\right) + g\left(|W|^2 - \frac{1}{2}(\bar{W}^2 W_\alpha + W^2\bar{W}_\alpha)\right) d\alpha,$$

respectively

$$\mathcal{M}(W,Q) = i\int \bar{Q}W_\alpha - Q\bar{W}_\alpha \, d\alpha.$$

Since $\mathcal{M}$ is the generator of the group of horizontal translations with respect to the same symplectic form, it is natural to expect that solitary waves must formally solve the system

$$D\mathcal{H}(W,Q) = cD\mathcal{M}(W,Q). \tag{2.14}$$

This can be interpreted as saying that solitary waves are critical points for the Hamiltonian on level sets of the momentum. In this interpretation the velocity c plays the role of the Lagrange multiplier.

We now use the above relation to formally derive the Babenko's equations. We first compute some simple variational derivatives.

$$\frac{d\mathcal{H}}{d\bar{Q}} = iQ_\alpha, \qquad \frac{d\mathcal{M}}{d\bar{Q}} = iW_\alpha.$$

Then from the first component of (2.14) we obtain

$$Q_\alpha = cW_\alpha. \tag{2.15}$$

Next we have

$$\frac{d\mathcal{M}}{d\bar{W}} = iQ_\alpha.$$

Thus we are left with a single equation for W, namely

$$\frac{d\mathcal{H}}{d\bar{W}} = ic^2 W_\alpha.$$

We have

$$\frac{d\mathcal{H}}{d\bar{W}} = -\sigma\partial_\alpha\left(\frac{1+W_\alpha}{|1+W_\alpha|}\right) + g(W - \bar{W}W_\alpha + WW_\alpha).$$

This leads us to the following formulation of the Babenko equations:

$$P\left[-\sigma\partial_\alpha\left(\frac{1+W_\alpha}{|1+W_\alpha|}\right) + g(W - \bar{W}W_\alpha + WW_\alpha)\right] = ic^2 W_\alpha. \tag{2.16}$$

We remark that this is not exactly the standard formulation, which is done at the level of the function $\Im W$ which represents the elevation in the conformal parametrization. In this article we prefer instead to work with the holomorphic variable W_α. Critically this allows us to take advantage of the algebra structure for this class of functions.

We specialize the above equations to gravity, respectively capillary waves.

(i) For gravity waves, the Babenko equations have the form

$$gW - gP[\bar{W}W_\alpha - WW_\alpha] = ic^2 W_\alpha \tag{2.17}$$

These admit a variational interpretation, namely as critical points for the reduced Hamiltonian (i.e. potential energy)

$$\mathcal{H}_0(W) = \int g\left(|W|^2 - \frac{1}{2}(\bar{W}^2 W_\alpha + W^2\bar{W}_\alpha)\right) d\alpha,$$

on level sets of the reduced momentum

$$\mathcal{M}_0(W) = i\int \bar{W}W_\alpha - W\bar{W}_\alpha \, d\alpha.$$

(ii) For capillary waves, the Babenko equations are

$$-P\partial_\alpha\left(\frac{1+W_\alpha}{|1+W_\alpha|}\right) = ic^2 W_\alpha \tag{2.18}$$

As above, these can be seen as the equations for critical points for the reduced Hamiltonian (i.e. potential energy)

$$\mathcal{H}_0(W) = \int 2\sigma\left(J^{\frac{1}{2}} - 1 - \Re W_\alpha\right) d\alpha,$$

on level sets of the reduced momentum $\mathcal{M}_0$.

Rigorously deriving these equations in the low regularity setting via the Hamiltonian formalism requires inverting the symplectic map, which is not so simple in the holomorphic coordinate system. Also, we are not assuming enough low frequency regularity in order to guarantee that either the Hamiltonian or the momentum are finite. Because of this we will derive these equations directly from (2.13). We begin with the first equation in (2.13), which we rewrite as

$$F = P\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right] = \frac{cW_\alpha}{1+W_\alpha}.$$

The expression inside the projection is imaginary, thus by taking the imaginary part of both sides, leads to the following equality

$$\frac{1}{2}\frac{Q_\alpha - \bar{Q}_\alpha}{J} = \frac{1}{2}\frac{c\left(W_\alpha - \bar{W}_\alpha\right)}{J}.$$

Hence, $\Im Q_\alpha = c\Im W_\alpha$; therefore

$$Q_\alpha = cW_\alpha. \tag{2.19}$$

To obtain the counterpart of the second Babenko equation (2.15) we start from the second equation in (2.13), we substitute the expression of F from the first equation, use (2.19), and equate the real part of both sides of the resulting equation. This leads to the following relation

$$-\frac{c^2}{2}(W_\alpha+\bar{W}_\alpha)+\frac{c^2}{2}\left(\frac{W_\alpha^2}{1+W_\alpha}+\frac{\bar{W}_\alpha^2}{1+\bar{W}_\alpha}\right)+\frac{c^2}{2}\frac{|W_\alpha|^2}{J}-ig\frac{W-\bar{W}}{2}+\frac{i\sigma}{1+W_\alpha}\partial_\alpha\left(\frac{1+W_\alpha}{|1+W_\alpha|}\right)=0.$$

Further computations give

$$-ig\frac{W-\bar{W}}{2}+\frac{i\sigma}{1+W_\alpha}\partial_\alpha\left(\frac{1+W_\alpha}{|1+W_\alpha|}\right)=\frac{c^2}{2}\frac{W_\alpha+\bar{W}_\alpha+W_\alpha\bar{W}_\alpha}{J}.$$

We now specialize to gravity, respectively capillary waves.

(i) For gravity waves we have

$$-ig(W-\bar{W}) = c^2\frac{W_\alpha+\bar{W}_\alpha+W_\alpha\bar{W}_\alpha}{|1+W_\alpha|^2}. \tag{2.20}$$

We multiply (2.20) by $(1+W_\alpha)$ to obtain

$$-ig(W-\bar{W})(1+W_\alpha) = c^2\left[W_\alpha + \frac{\bar{W}_\alpha}{1+\bar{W}_\alpha}\right]. \tag{2.21}$$

The right hand side of (2.21) can be written as a holomorphic plus an antiholomorphic part, but the left hand side cannot be decomposed in a similar fashion. We therefore need to use the projection P and multiply by i to arrive at (2.17).

(ii) For capillary water waves we have a self-contained equation for $\mathbf{W} = W_\alpha$,

$$i\frac{\sigma}{1+\mathbf{W}}\partial_\alpha\left(\frac{1+\mathbf{W}}{|1+\mathbf{W}|}\right) = \frac{c^2}{2}\frac{\mathbf{W}+\bar{\mathbf{W}}+\mathbf{W}\bar{\mathbf{W}}}{J}. \tag{2.22}$$

As before, we multiply (2.22) by $(1+\mathbf{W})$ to obtain

$$i\sigma\partial_\alpha\left(\frac{1+\mathbf{W}}{|1+\mathbf{W}|}\right) = c^2\left[\mathbf{W} + \frac{\bar{\mathbf{W}}}{1+\bar{\mathbf{W}}}\right], \tag{2.23}$$

and take the projection of the resulting equation to arrive at (2.18).

In both cases, we remark that the above equations provide additional[3] information compared to the Babenko type equations (2.17) and (2.18). We will take advantage of this in the sequel.

3. Gravity waves

Here we prove the desired nonexistence statement for solitary waves in the case of the gravity waves, expressed in holomorphic coordinates. Our starting point is the equation (2.20) for W, which we recall here:

$$2g\Im W = c^2\frac{W_\alpha + \bar{W}_\alpha + |W_\alpha|^2}{|1+W_\alpha|^2}. \tag{3.1}$$

For this equation we consider solutions W whose derivative W_α has regularity as follows:

$$W_\alpha \in \dot{B}^{\frac{1}{2}}_{2,1}, \qquad |1+W_\alpha| \geq \delta > 0. \tag{3.2}$$

Then our main nonexistence result in holomorphic coordinates is

Theorem 3. *The above equation has no nontrivial holomorphic solutions W which have regularity* (3.2)*.*

Proof. We first directly dispense with the case $c=0$. For $c \neq 0$, on the other hand, we can improve the low frequency regularity of W. We begin with (3.2) and the Moser estimates in $\dot{B}^{\frac{1}{2}}_{2,1}$ to conclude that $\Im W \in \dot{B}^{\frac{1}{2}}_{2,1}$, and thus $W \in \dot{B}^{\frac{1}{2}}_{2,1} \subset L^\infty$ (after possibly subtracting an appropriate constant). Interpolating this with (3.2) we obtain $W_\alpha \in L^2$. Now we reiterate the same steps to succesively conclude that $\Im W \in L^2$, then $W \in L^2$. Since $W_\alpha \in L^2$, we also conclude that W has limit zero at infinity.

Multiplying the equation (3.1) by $(1+W_\alpha)$ gives

$$2g(1+W_\alpha)\Im W = c^2\left(W_\alpha + \frac{\bar{W}_\alpha}{1+\bar{W}_\alpha}\right),$$

which after projection yields the Babenko's equation

$$-igW + 2gP[\Im W W_\alpha] = c^2 W_\alpha.$$

We rewrite this equation in the form

$$gW = P[iVW_\alpha], \qquad V = c^2 + ig(W-\bar{W}). \tag{3.3}$$

Then it remains to show that this problem admits no H^1 solutions.

[3] Of course, the two different sets of equations are ultimately equivalent, but proving this requires extra work.

The key remark is that the coefficient V of W_α is purely real, and also bounded. If there were no projection, we could choose a bounded increasing function χ, multiply the equation by $2\chi\bar{W}_\alpha$, take the real part and integrate by parts to get

$$0 = 2\Re\int \chi \bar{W}_\alpha W \, d\alpha = -\int \chi' |W|^2 \, d\alpha,$$

which yields $W = 0$. In our case, we need to be more careful. We set χ to be a smooth bump function which increases from 0 to 1, and for $r > 0$ we define

$$\chi_r := \chi(\alpha/r).$$

Then, multiplying as above by $\chi\bar{W}_\alpha$ and integrating by parts on the left, we have the following equality

$$g\int \chi_r' |W|^2 \, d\alpha = -2\Re \int \chi_r \bar{W}_\alpha P(iVW_\alpha)\, d\alpha = -2\Re\int \overline{[P,\chi_r]W_\alpha} \cdot (iVW_\alpha)\, d\alpha,$$

where we have used the fact that P is L^2 self-adjoint. We rewrite this in the form

$$g\int \chi'(\alpha/r)' |W|^2 \, d\alpha = -2r\Re\int \overline{[P,\chi_r]W_\alpha} \cdot (iVW_\alpha)\, d\alpha.$$

Here we have two favourable features. First, we have a commutator which yields the r^{-1} gain via the classical Coifman-Meyer estimate [14],

$$\|[P,\chi_r]W_\alpha\|_{L^2} \lesssim \|\chi_r'\|_{L^\infty}\|W\|_{L^2} \lesssim r^{-1}\|W\|_{L^2}. \tag{3.4}$$

This is almost enough but not quite. Secondly only the frequencies in W which are less than r^{-1} affect the commutator. Because of this we claim that that the bound (3.4) admits a qualitative improvement in the limit as $r \to \infty$:

Lemma 3.1. *Let $W \in L^2$. Then*

$$\lim_{r\to\infty} r\|[P,\chi_r]W_\alpha\|_{L^2} = 0.$$

To conclude the proof of the theorem, we let $r \to \infty$ in the last integral relation. By Lebesque dominated convergence theorem, we obtain

$$\chi'(0)\int |W|^2 \, d\alpha \leq \lim_{r\to\infty}\int r\chi_r' |W|^2 \, d\alpha = 0,$$

which implies $W = 0$ and indeed concludes the proof of the theorem.

□

It remains to prove the Lemma.

Proof of Lemma 3.1. As mentioned above, the idea is to use the fact that primarily only the frequencies $\lesssim r^{-1}$ of W contribute to the commutator. Precisely, given a frequency threshold $\lambda > 0$ we separate W into a low and a high frequency part,

$$W = W_{<\lambda} + W_{\geq\lambda}.$$

For the contribution of the low frequency part we use directly (3.4),

$$\|[P,\chi_r]W_{\alpha,<\lambda}\|_{L^2} \lesssim r^{-1}\|W_{<\lambda}\|_{L^2}.$$

For the high frequency part we write

$$[P, \chi_r] W_{\geq \lambda, \alpha} = \left[P, (\chi_r)_{\geq \frac{\lambda}{4}} \right] W_{\geq \lambda, \alpha},$$

where we took the $\lambda/4$ truncation in χ_r to make sure that the excluded frequencies in χ_r are strictly smaller than the frequencies in W and thus do not contribute to the commutator. Here we have rapid decay as r approaches infinity,

$$\left| \partial_\alpha (\chi_r)_{\geq \frac{\lambda}{4}} \right| \lesssim r^{-1} (\lambda r)^{-N},$$

where N is any fixed large positive integer.

Therefore using directly the Coifman-Meyer commutator estimate we obtain

$$r \left\| \left[P, (\chi_r)_{\geq \frac{\lambda}{4}} \right] W_{\geq \lambda, \alpha} \right\|_{L^2} \lesssim r^{-1} (\lambda r)^{-N} \| W \|_{L^2} \to 0.$$

Adding the low and high frequency contributions, it follows that

$$\limsup_{r \to \infty} r \| [P, \chi_r] W_\alpha \|_{L^2} \lesssim \| W_{\leq \lambda} \|_{L^2}.$$

Now we let $\lambda \to 0$ to conclude the proof. □

3.1. **Further comments on crested waves.** Here we show how our result in Theorem 3 also precludes the existence of crested solitary waves:

Corollary 3.2. *There are no crested wave solutions W with*

$$\frac{W_\alpha}{1 + W_\alpha} \in \dot{B}^{\frac{1}{2}}_{2,1} \ \textit{and} \ |1 + W_\alpha| \geq \delta > 0,$$

for the equation (3.1) *.*

Here, for simplicity, by crested waves we mean waves which are smooth except for finitely many angular crests. Certainly, there is room to refine this further but we choose no to pursue it here.

We next show how to prove this corollary. We will not use directly the result of Theorem 3, as near a crest W_α cannot be expected to have the regularity (3.2). Instead, we will use the second part of the proof of the theorem, which shows that the Babenko equation (3.3) has no solutions W with $W_\alpha \in L^2$.

To set the stage, observe that the equation (3.1) implies that

$$\Im W \leq h_0 := \frac{c^2}{2g}.$$

Here h_0 is the maximum possible height of a solitary wave with speed c. In the the region where $\Im W < h_0$, the coefficient V of W_α in (3.3) is strictly positive, which makes the equation (3.3) elliptic. This in turn implies that $W_\alpha \in C^\infty$ there. Thus the only way we can have a crest is at a point α_0 of maximum height h_0.

Away from the crests, the first argument in the proof of Theorem 3 combined with the hypothesis of the corollary still yields the regularity $W_\alpha \in L^2$. So we need to investigate what happens near an angular crest.

Suppose that at some point α_0 we have an angular crest of angle $\theta \in (0, \pi)$. Then near α_0 the conformal map Z must have the form

$$Z(z) \approx i h_0 + C (z - \alpha_0)^{\frac{\theta}{\pi}},$$

and

$$Z_\alpha = 1 + W_\alpha \approx C(\alpha - \alpha_0)^{\frac{\theta}{\pi}-1}.$$

Inserting such an ansatz in the equations (3.1) the leading singular part on the left hand side is $(z-\alpha_0)^{\frac{\theta}{\pi}}$, whereas on the right hand side, the similar power is $(z-\alpha_0)^{2(1-\frac{\theta}{\pi})}$. Matching the two, we see that the only admissible angular crest corresponds to the angle $\theta = 2\pi/3$ (i.e. Stokes type waves). But if all the crests have $2\pi/3$ angles then the condition $W_\alpha \in L^2$ is satisfied[4], so the existence of W excluded by the second part of the proof of Theorem 3.

4. Capillary waves

Here we start with the algebraically equivalent equations

$$i\sigma\partial_\alpha \frac{1+\mathbf{W}}{|1+\mathbf{W}|} = c^2\left[\mathbf{W} + \frac{\bar{\mathbf{W}}}{1+\bar{\mathbf{W}}}\right], \tag{4.1}$$

or

$$-\partial_\alpha \frac{\mathbf{W}-\bar{\mathbf{W}}}{|1+\mathbf{W}|} = ic^2\left[\frac{\mathbf{W}(2+\mathbf{W})}{1+\mathbf{W}} + \frac{\bar{\mathbf{W}}(2+\bar{\mathbf{W}})}{1+\bar{\mathbf{W}}}\right]. \tag{4.2}$$

Theorem 4. *The above equation* (4.1) *has no nontrivial solutions W with regularity* (3.2).

Proof. If $c = 0$ then from the second equation we immediately obtain $\Im\mathbf{W} = 0$ which yields $\mathbf{W} = 0$. Suppose now that $c \neq 0$. We first improve the regularity of $\mathbf{W}$ both at low and at high frequencies.

At low frequencies we project in the first equation to obtain

$$i\sigma P\partial_\alpha \frac{1+\mathbf{W}}{|1+\mathbf{W}|} = c^2\mathbf{W}.$$

Next we integrate once,

$$i\sigma P\left(\frac{1+\mathbf{W}}{|1+\mathbf{W}|} - 1\right) = c^2 W.$$

Now we argue as in the case of gravity waves. Since $\mathbf{W} \in \dot{B}^{\frac{1}{2}}_{2,1}$, by Moser estimates this yields n $W \in \dot{B}^{\frac{1}{2}}_{2,1}$. Now by interpolation we get $\mathbf{W} \in L^2$, which after reiteration yields $W \in L^2$ and also $W \in L^\infty$.

At high frequencies we work with the function

$$T := \log(1+\mathbf{W}) \in L^\infty \cap L^2,$$

which is also holomorphic.

From the first equation (4.1) we get $\Im T_\alpha \in L^2$, which shows that $T_\alpha \in L^2$, and eventually $\mathbf{W}_\alpha \in L^2$. Differentiating and repeating the argument it follows that $\mathbf{W} \in H^\infty$.

Combining the low and high frequency information, it now suffices to prove that the equation (4.1) has no solutions $\mathbf{W} \in H^\infty$ with $|1+\mathbf{W}| \geq \delta > 0$. For this we switch to polar coordinates, denoting

$$T = \log(1+\mathbf{W}) =: U + iV \in H^\infty.$$

Substituting this into (4.1) we have

$$-\sigma V_\alpha e^{iV} = c^2(e^{U+iV} - e^{-U+iV}),$$

[4]More generally, we have $W_\alpha \in L^2$ iff all crest angles are in the range $\theta \in (\frac{\pi}{2}, \pi)$.

or equivalently

$$-\sigma V_\alpha = 2c^2 \sinh U.$$

We rewrite this in the form

$$HU_\alpha = 2c^2 \sinh U.$$

Then we repeat the argument for gravity waves, multiplying by $\chi_r U_\alpha$ and integrating by parts, using the same commutator bound as in Lemma 3.1 (which holds equally for the Hilbert transform). This yields $U = 0$, and then $V = const$. Then we must have $\mathbf{W} = const$ and further $\mathbf{W} = 0$ since $\mathbf{W} \in L^2$.

□

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