Title
Scattering Resonances for Convex Obstacles

Author
Jin, Long

Publication Date
2015

Peer reviewed|Thesis/dissertation

Scattering Resonances for Convex Obstacles

by

Long Jin

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Maciej R. Zworski, Chair
Professor Daniel I. Tataru
Professor Robert G. Littlejohn

Spring 2015

Scattering Resonances for Convex Obstacles

Abstract

Scattering Resonances for Convex Obstacles

by

Long Jin

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Maciej R. Zworski, Chair

In the setting of obstacle scattering in Euclidean spaces, the poles of meromorphic continuation of the resolvent of the Laplacian on the exterior region are called the resonances or scattering poles. Each resonance corresponds to a resonant wave. The real part of a resonance corresponds to the frequency of the wave, while the imaginary part corresponds to the decay rate of the wave. Consequently understanding the distribution of the resonances is important in understanding the long time behavior of the solution to wave equations in the exterior domain.

We study the distribution of resonances in the case of a strictly convex obstacle with smooth boundary. In particular, under general boundary conditions, we prove the existence of the cubic resonance free regions near the real axis. Moreover, if the obstacle is close to a sphere, in the sense that it satisfies certain pinched curvature conditions, we prove that the resonances close to the real axis are separated into cubic bands and in each band, the counting function of resonances satisfies a Weyl law. We also generalize these results to totally convex obstacles in more general asymptotic Euclidean metrics.

To my parents.

Contents

List of Figures

Acknowledgments

First and foremost, I would like to thank my advisor, Maciej Zworski, for all the encouragement and advice in years. He also suggested these projects and taught me the techniques to approach the problems. It would be impossible for me to finish the thesis without him. This research is also partially supported by the National Science Foundation grant DMS-1201417 and I would like to express my gratitude.

I am very grateful for Steve Zelditch, Semyon Dyatlov and Kiril Datchev for the valuable mentoring and teaching during the last several years of graduate school. I would also like to thank Michael Christ, Daniel Tataru, Craig Evans, Denis Auroux, Ian Agol, John Lott, Nicolas Burq, Plamen Stefanov, Antonio Sa Barreto, Christopher Sogge, Jeffery Galkowski, Boaz Haberman, Benjamin Harrop-Griffiths and Alexis Drouot for many helpful discussions and teaching me different aspects in mathematics. I am also thankful to anonymous referees of my papers for suggesting many improvements.

I would also like to thank many friends who made the graduate school happy years for me. Thanks to Baoping Liu, Yi Liu, An Huang, Shuchao Bi, Yuhao Huang, Junjie Zhou, Te Zhang, Qingchun Ren, Weihua Liu, Luming Wang, Jue Chen, Mingxiao Lee, Xin Jin, Penghui Li, Qiao Zhou, Tao Su, Kun Chen, Chen Shen, Bo Lin, Ze Xu, Zhengyi Zhou, Yindi Qin, Shenghan Zhang.

Finally, I would like to thank my family and friends back home, especially my parents for their constant patience and encouragement.

Chapter 1

Introduction

1.1 Statement of results

In this thesis, we study the distribution of resonances for convex strictly obstacles $\mathcal{O}$ in the Euclidean space $\mathbb{R}^n$ or more generally, in a Riemmanian manifold X which is diffeomorphic to $\mathbb{R}^n$ and equipped with an analytic, non-trapping, asymptotically Euclidean metric g. We also consider general boundary conditions including the Dirichlet boundary condition $u|_{\partial\mathcal{O}} = 0$, the Neumann boundary condition $\partial_\nu u|_{\partial\mathcal{O}} = 0$ and the Robin boundary condition $\partial_\nu u + \eta u|_{\partial\mathcal{O}} = 0$, where $\eta \in C^\infty(\partial\mathcal{O})$.

1.1.1 Notations

Let Q be the second fundamental form of $\partial\mathcal{O}$ and $S\partial\mathcal{O}$ be the sphere bundle of $\partial\mathcal{O}$. We suppose that $\partial\mathcal{O}$ is strictly convex, in the sense that $\min_{S\partial\mathcal{O}} Q > 0$. We shall write

$$\kappa = 2^{-1/3}\cos(\pi/6)\min_{S\partial\mathcal{O}} Q^{2/3}, \quad K = 2^{-1/3}\cos(\pi/6)\max_{S\partial\mathcal{O}} Q^{2/3}.$$

Let Ai be the Airy function,

$$0 < \zeta_1 < \zeta_2 < \cdots < \zeta_j < \zeta_{j+1} < \cdots$$

be the negative of the zeroes of Ai and

$$0 < \zeta_1' < \zeta_2' < \cdots < \zeta_j' < \zeta_{j+1}' < \cdots$$

be the negatives of the zeroes of Ai$'$.

We shall write $\mathrm{Res}(P)$ to be the set of resonances of the resonances of $P = -\Delta$ on the exterior region $\mathbb{R}^n \setminus \mathcal{O}$. For $\lambda \in \mathrm{Res}(P)$, we write $m_P(\lambda)$ to be the multiplicity of the resonance λ.

1.1.2 Euclidean case: resonance-free region

There is a cubic resonance-free region near the real axis. More precisely, there exists a constant $C > 0$ such that there are no resonances in the region

$$C \leqslant \operatorname{Re}\lambda, \quad 0 \leqslant -\operatorname{Im}\lambda \leqslant \kappa\zeta_1'(\operatorname{Re}\lambda)^{1/3} - C \tag{1.1.1}$$

in the case of Neumann or Robin boundary condition and with ζ_1' replaced by ζ_1 in the case of the Dirichlet boundary condition.

1.1.3 Euclidean case: band structure

If the obstacle is close to the sphere, then the resonances close to the real axis are separated into cubic bands by cubic resonance-free strips. More precisely, in the case of Neumann or Robin boundary condition, suppose we have the following pinched curvature condition

$$\frac{\max_{S\partial\mathcal{O}} Q}{\min_{S\partial\mathcal{O}} Q} < \left(\frac{\zeta_{j_0+1}'}{\zeta_{j_0}'}\right)^{3/2}$$

for some $j_0 \geqslant 1$, then there exists a constant $C > 0$ such that for all $0 \leqslant j \leqslant j_0$, there are no resonances in the regions

$$C \leqslant \operatorname{Re}\lambda, \quad K\zeta_j'(\operatorname{Re}\lambda)^{1/3} + C \leqslant -\operatorname{Im}\lambda \leqslant \kappa\zeta_{j+1}'(\operatorname{Re}\lambda)^{1/3} - C. \tag{1.1.2}$$

Moreover, we have a Weyl law for the counting function for the number of the resonances in each band,

$$\begin{aligned}\sum\{m_P(\lambda) : \lambda \in \operatorname{Res}(P), |\lambda| \leqslant r, \kappa\zeta_j'(\operatorname{Re}\lambda)^{1/3} - C < -\operatorname{Im}\lambda < K\zeta_j'(\operatorname{Re}\lambda)^{1/3} + C\} \\ = (1 + o(1))(2\pi)^{1-n}\operatorname{vol}(B^{n-1}(0,1))\operatorname{vol}(\partial\mathcal{O})r^{n-1},\end{aligned} \tag{1.1.3}$$

where $B^{n-1}(0,1)$ is the unit ball in $\mathbb{R}^{n-1}$.

Again, the same statements hold in the case of Dirichlet boundary condition if we replace ζ_j' by ζ_j.

1.1.4 Asymptotically Euclidean case

We also extend the results above the case of totally convex obstacles on analytic asymptotically Euclidean case. We need to make assumptions such that the dynamics of the geodesics are similar to the Euclidean case. For the detailed assumption, we refer to section 7.1.1.

1.2 History and previous results

The study of the distribution of resonances for convex bodies dates back to the work of Watson [39] on electromagnetic scattering by the earth almost a hundred years ago. In

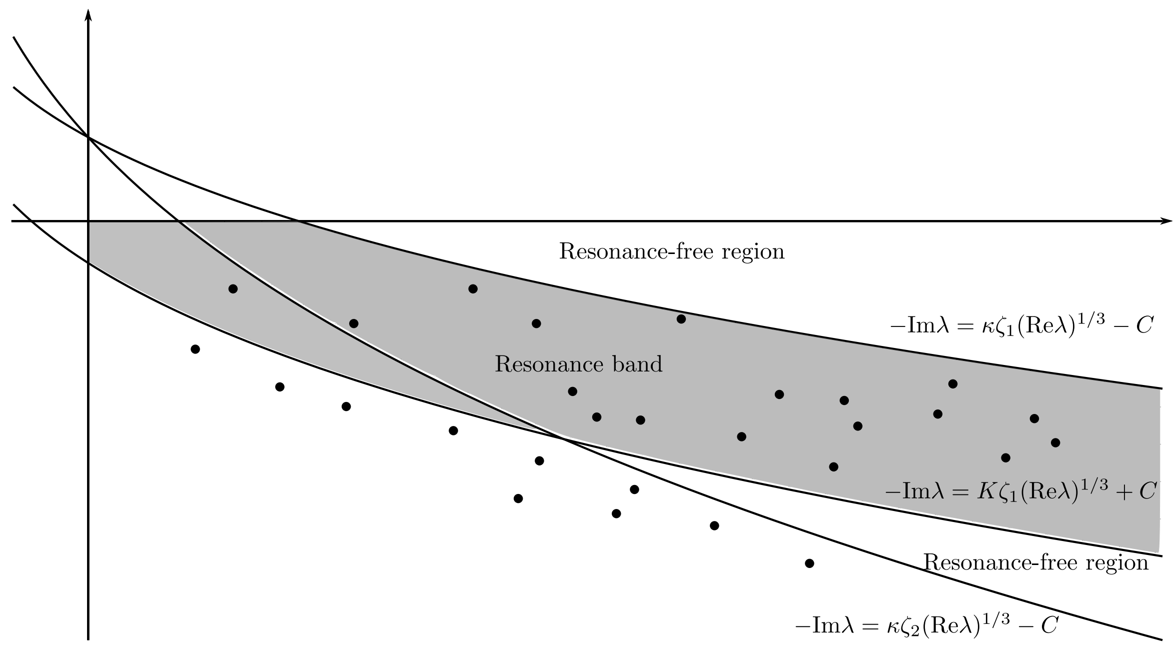

Figure 1.1: Resonance bands and resonance free regions.

that case, for the Dirichlet problems, the resonances are given in terms of the zeroes of Hankel functions. The study of these zeroes has been conducted by Watson [39], Olver [24], Nussenzverig [23] and others. See Stefanov [36] for a modern account and references. For more general convex obstacles, the resonances have been studied by Babich-Grigoreva [2], Bardos-Lebeau-Rauch [4], Filippov-Zayaev [41], Hargé-Lebeau [9], Lascar-Lascar [15] and Sjöstrand-Zworski [27, 35, 28, 30].

1.2.1 Resonance free region

A resonance free region is a gap near the real axis in which there are no resonances. The work of Lax-Phillips [16] and Vainberg [37] connects the presence of such regions to the propagation of singularities for the wave equations and hence to the geometry of the obstacle. For example, if the obstacle is smooth and non-trapping, in the sense that all the reflecting rays escape to infinity, then there are no resonances in the region

$$\mathrm{Im}\,\lambda > -M \log|\lambda| + C_M$$

for any $M > 0$.

When the boundary is *real analytic*, and the obstacle is *nontrapping*, the work of Lebeau [18] on propagation of Gevrey-3 singularities implies that the resonance free region is cubic, see Popov [25] and Bardos-Lebeau-Rauch [4]. This result is sharp as was shown already in [4] where the analysis of Gevrey-3 singularities of the wave trace gave a string of resonances near a cubic curve, see also [28], [15].

A remarkable discovery was made by Hargé-Lebeau [9] who showed that for *smooth* strictly convex obstacles and the Dirichlet boundary condition, the resonance free region is also cubic. In [35], Sjöstrand-Zworski gave a more direct proof of the cubic resonance free region. The case of general Robin boundary condition is given in [13].

1.2.2 Number of resonances

A global optimal upper bound for the counting functions of resonances was proved in Melrose [21] in odd dimension and Vodev [38] in even dimension

$$\sum\{m_P(\lambda) : |\lambda| \leqslant r\} \leqslant Cr^n + C.$$

Sjöstrand-Zworski [31] proved the same result in the general black-box setting. However, an optimal lower bound for the counting functions is still unknown. Lax-Phillips [17] gave a general lower bound for resonances on the imaginary axis. In fact, they proved that for star-shaped obstacles with smooth boundary in odd dimensions,

$$c_n(r_I(\mathcal{O}))^{n-1}r^{n-1} \leqslant \sum\{m_P(\lambda) : C \leqslant |\operatorname{Im}\lambda| \leqslant r, \operatorname{Re}\lambda = 0\} \leqslant c_n(r_O(\mathcal{O}))^{n-1}r^{n-1}.$$

Here r_I and r_O are the inscribed radius and superscribed radius of the obstacle, respectively. Away from the imaginary axis, Sjöstrand-Zworski [34] proved a weaker lower bound

$$\sum\{m_P(\lambda) : C \leqslant |\lambda| \leqslant r, \operatorname{Re}\lambda \neq 0\} \geqslant r^{n-1-\epsilon}/C_\epsilon, \quad \epsilon > 0$$

for the Dirichlet boundary condition when $n = 4k + 1$, and for the Neumann boundary condition when $n = 4k - 1$, see also [33]. On the other hand, optimal lower bounds for counting functions have been obtained for obstacles with trapping trajectories for the broken geodesic flow. In general, the recent work of Christiansen [6] and [5] show that the order of the counting function is n for generic star-shaped obstacles in odd dimensions and all obstacles in even dimensions. For more results on the distribution of resonances in other settings, we refer to the surveys for resonances [20] and [42] and the book [8].

1.3 Open problems

Although the study for the resonances for convex obstacles has a long history, there are still many open problems. Here we list a few of them that are related to the current research.

1.3.1 Sharp constants

The constant $S = \kappa\zeta_1'$ (or $\kappa\zeta_1$ in the Dirichlet case) in (1.1.1) is optimal if the obstacle $\mathcal{O}$ is a ball. This can be seen from Weyl's law on counting function in the band (1.1.3). However, in general, a better constant S is given in [4] and [28] for analytic obstacles under Dirichlet boundary condition and in [15] for Gevrey-s ($s < 3$) obstacles:

$$S = 2^{-1/3}\cos(\pi/6)\zeta_1 \lim_{T\to\infty} \frac{1}{T}\int_0^T Q(\dot{\gamma}(t))^{2/3} dt$$

where $\gamma = \gamma(t)$ varies in the set of geodesics on the boundary $\partial\mathcal{O}$ with unit speed. This reflects the heuristic picture of the formation of resonances by waves creeping along the geodesics on the boundary and losing energy at a rate relating to the curvature. It is also shown in [4] that this constant is optimal under certain assumptions on the geodesics on the boundary. The optimal constant S in other cases is still unknown.

1.3.2 Sharp error terms

The error term in Weyl's law (1.1.3) is only a very weak one and very likely not optimal in general. For the ball, the error term is $O(r^{n-2})$. The general sharp error terms are unknown. However, it seems the best one could get from our treatment for the problem is $O(r^{n-5/3})$.

1.3.3 Other problems

As discussed previously, the general optimal lower bounds of size r^n on the counting number of resonances are still unknown. Another difficult problem is to study whether there is a Weyl's law for the resonances on the imaginary axis in odd dimensions. In Lax-Phillips [17], the proof of upper bound and lower bound is based on comparison principles. It is unclear what kind of geometric properties of the obstacles govern the behavior of pure imaginary resonances and give such a Weyl's law.

1.4 Outline

In chapter 2, we first review the definition of scattering resonances in the general black box setting and other equivalent characterizations of resonances in the special case of obstacle scattering in Euclidean spaces. Then we review some basic facts about Airy functions Ai and Airy differential operators $D_t^2 + t$ which are crucial in our analysis. Finally we study the distribution of resonances when the obstacle is a sphere using the method of separation of variables, where we mostly follow [36].

In chapter 3, we review necessary background from semiclassical microlocal analysis, including two important tools: the Fourier-Bros-Iagolnitzer(FBI) transform and the second microlocalization with respect to a hypersurface.

In chapter 4, we review the method of complex scaling which originates from the work of [1], [3] on continuous spectrum of Schrödinger operator. We follow the framework developed by Sjöstrand and Zworski [31], [27], [35]. This method characterizes the resonances as eigenvalues of a non-selfadjoint operator and thus allows us to apply microlocal techniques for partial differential operators.

In chapter 5, we prove the presence of the cubic resonance-free region in the Euclidean case. The strategy is similar to the proof of Dirichlet case given by [35]. Near the obstacle, we use the tool of FBI transform on a smooth manifold developed in [40] and reduce the study the lower bound of an Airy differential operator. However, to deal with general Robin boundary conditions, we have to start with such an Airy differential operator with no boundary conditions, thus non-selfadjoint. The argument is based on viewing this as a perturbation of the Neumann boundary conditions and the lower bounds involve Neumann boundary terms. Away from the obstacle, the scaled operator is elliptic and thus a better lower bound is available. Combining these lower bounds we get a global lower bounds on the complex scaled operator which implies the cubic resonance free region. This chapter is based on [13].

In chapter 6, we prove the band structure under pinched curvature conditions. Our approach is based on a modification of the work [30] in which a Grushin problem is built to further reduce the problem to an operator on the boundary of the obstacle, often referred as the effective Hamiltonian operator, whose symbol is in certain exotic classes arise from second microlocalization. The key observation is that in the semiclassical setting, the Robin boundary operator is a perturbation of the Neumann boundary operator. We build the complex scaled differential operator and the boundary operator together into the Grushin problem and obtain the effective Hamiltonian. Then the symbol properties of this operator will connect the geometry of the boundary to the distribution of the resonances. In particular, we obtain the band structure and Weyl's law for counting functions. This chapter is based on [14].

Finally in chapter 7, we extend the results to the case of totally convex obstacles in analytic asymptotically Euclidean manifolds. We first describe the dynamical assumptions needed for the manifolds and the obstacles. Then we construct modified complex scaling contours and associated escape functions which are used to construct microlocally weighted spaces that the scaled operators act on. Next we prove a lower bound on the scaled operators and deduce the cubic resonance free region. Finally we construct a global Grushin problem as before and prove the result on resonance bands.

Chapter 2

Preliminaries

In this chapter, we review some basic results in the field of obstacle scattering. First, we give the definition and some equivalent characterization of scattering resonances. Then we review some basic properties of the Airy function and the Airy differential operators. Finally we study the special example of scattering by the unit sphere in the Euclidean space.

2.1 Resonances in obstacle scattering

In this section, we start with the general setup that $\mathcal{O}$ is a smooth obstacle in $\mathbb{R}^n$, i.e. an open bounded set with smooth boundary $\partial\mathcal{O}$. The scattering problem is to understand the Laplace operator $P = -\Delta|_{\mathbb{R}^n\setminus\mathcal{O}}$ on the exterior domain realized with certain boundary conditions on $\partial\mathcal{O}$. We restrict ourselves to the Dirichlet boundary condition

$$\mathcal{D}(P) = \{u \in H^2(\mathbb{R}^n \setminus \mathcal{O}) : u|_{\partial\mathcal{O}} = 0\}$$

or the Neumann/Robin boundary condition

$$\mathcal{D}(P) = \{u \in H^2(\mathbb{R}^n \setminus \mathcal{O}) : \partial_\nu u + \eta u|_{\partial\mathcal{O}} = 0\} \tag{2.1.1}$$

where ν is the exterior normal derivatives on $\partial\mathcal{O}$ and η is a real-valued smooth function on $\partial\mathcal{O}$. When $\eta = 0$, the boundary condition is the Neumann boundary condition. Most of our analysis will be focused on the case of the Neumann/Robin boundary condition, while the case of Dirichlet boundary condition could be easily dealed in the same way.

2.1.1 Definition of resonances

By the general spectral theory, P is a self-adjoint operator on $L^2(\mathbb{R}^n \setminus \mathcal{O})$ with continuous spectrum $[0,\infty)$, (possibly with a finite number of eigenvalues in the case of Robin boundary condition). Therefore the resolvent

$$R(\lambda) = (P - \lambda^2)^{-1} : L^2(\mathbb{R}^n \setminus \mathcal{O}) \to \mathcal{D}(P) \subset H^2(\mathbb{R}^n \setminus \mathcal{O}) \tag{2.1.2}$$

is holomorphic (or at least meromorphic with finite number of poles on the positive imaginary axis) in the upper half plane $\{\lambda \in \mathbb{C} : \operatorname{Im} \zeta > 0\}$. We are interested in the meromorphic continuation of the resolvent (2.1.2) across the continuous spectrum, to an operator

$$R(\lambda) = R_P(\lambda) : L^2_{\text{comp}}(\mathbb{R}^n \setminus \mathcal{O}) \to \mathcal{D}_{\text{loc}} \subset H^2_{\text{loc}}(\mathbb{R}^n \setminus \mathcal{O})$$

where L^2_{comp} is the space of compact supported L^2 functions, H^2_{loc} is the space of locally H^2 functions and $\mathcal{D}_{\text{loc}}$ is the space of locally H^2 functions satisfies the given boundary conditions at $\partial\mathcal{O}$.

Equivalently, we can consider the Green function, $G(\lambda, x, y)$ defined as the kernel of the resolvent (2.1.2):

$$u(x) = \int_{\mathbb{R}^n \setminus \mathcal{O}} G(\lambda, x, y) f(y) dy, \quad f \in C^\infty_{\text{comp}}(\mathbb{R}^n \setminus \mathcal{O}),$$

if u is the solution to the boundary value problem

$$\begin{aligned} (-\Delta - \lambda^2) u(x) = f(x), &\quad x \in \mathbb{R}^n \setminus \mathcal{O}; \\ \partial_\nu u(x) + \eta(x) u(x) = 0, &\quad x \in \partial\mathcal{O}. \end{aligned}$$

The Green function has a meromorphic continuation across the real line $\{\lambda : \operatorname{Im} \lambda = 0\}$ to the whole complex plane $\mathbb{C}$ when n is odd; and to the logarithmic plane Λ defined as the logarithmic covering space of the punctured complex plane $\mathbb{C} \setminus \{0\}$ when n is even. This fact is a special case in the black box framework of more general scattering problem.

Definition 2.1.1. *The poles of this meromorphic continuation of the Green function $G(\lambda, x, y)$, or equivalently, the resolvent $R(\lambda)$ are called the* resonances *or* scattering poles *of the operator P. The multiplicity of a pole λ_0 is defined as*

$$m(\lambda_0) = m_P(\lambda_0) = \operatorname{rank} \oint_{|\lambda - \lambda_0| = \varepsilon} R_P(\lambda) 2\lambda d\lambda, \quad 0 < \epsilon \ll 1. \tag{2.1.3}$$

2.1.2 Black box scattering in $\mathbb{R}^n$

Now we briefly review the black box frame work. Let $\mathcal{H}$ be a complex separable Hilbert space with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0))$$

where $R_0 > 0$ is fixed and $B(x, R)$ is the ball of radius R centered at x. The orthogonal projections are denoted by

$$\begin{aligned} &u \mapsto \mathbf{1}_{B(0,R_0)} u = u|_{B(0,R_0)} \in \mathcal{H}_{R_0} \\ &u \mapsto \mathbf{1}_{\mathbb{R}^n \setminus B(0,R_0)} u = u|_{\mathbb{R}^n \setminus B(0,R_0)} \in L^2(\mathbb{R}^n \setminus B(0, R_0)). \end{aligned}$$

For any bounded continuous function χ such that $\chi = c_0$ is a constant on $B(0, R_0)$, we can define $\chi u \in \mathcal{H}$ for $u \in \mathcal{H}$ to be

$$\chi u := c_0(u|_{B(0,R_0)}) + (\chi|_{\mathbb{R}^n \setminus B(0,R_0)})(u|_{\mathbb{R}^n \setminus B(0,R_0)}).$$

To study the scattering problem, we also have a smaller space of compactly supported elements in $\mathcal{H}$

$$\mathcal{H}_{\mathrm{comp}} = \{u \in \mathcal{H} : u|_{\mathbb{R}^n \setminus B(0,R_0)} \in L^2_{\mathrm{comp}}(\mathbb{R}^n \setminus B(0, R_0))\},$$

and a larger space of elements that locally in $\mathcal{H}$:

$$\mathcal{H}_{\mathrm{loc}} = \mathcal{H}_{R_0} \oplus L^2_{\mathrm{loc}}(\mathbb{R}^n \setminus B(0, R_0)).$$

We consider an unbounded selfadjoint operator $P : \mathcal{H} \to \mathcal{H}$ with domain $\mathcal{D} \subset \mathcal{H}$ and assume that

$$\mathbf{1}_{\mathbb{R}^n \setminus B(0,R_0)} \mathcal{D} \subset H^2(\mathbb{R}^n \setminus B(0, R_0))$$

and if $u \in H^2(\mathbb{R}^n \setminus B(0, R_0))$ with $u = 0$ near $\partial B(0, R_0)$, then $u \in \mathcal{D}$. We also assume that for $u \in \mathcal{D}$,

$$\mathbf{1}_{\mathbb{R}^n \setminus B(0,R_0)}(Pu) = -\Delta(u|_{\mathbb{R}^n \setminus B(0,R_0)}).$$

Later on, we shall also consider a long range perturbation in the semiclassical setting so that the assumption above is replaced by

$$\mathbf{1}_{\mathbb{R}^n \setminus B(0,R_0)}(Pu) = Q(u|_{\mathbb{R}^n \setminus B(0,R_0)}).$$

Here Q is a formally self-adjoint second order semiclassical differential operator

$$Qu = \sum_{|\alpha| \leqslant 2} a_\alpha(x; h)(hD_x)^\alpha u,$$

where the coefficients a_α are bounded in C_b^∞ when h varies and the principal terms $a_\alpha(x; h) = a_\alpha(x), |\alpha| = 2$ are independent of h. Moreover, we assume that there exists a constant $C > 0$ such that for all $\xi \in \mathbb{R}^n$,

$$\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geqslant \frac{1}{C}|\xi|^2,$$

and as $|x| \to \infty$,

$$\sum_{|\alpha| \leqslant 2} a_\alpha(x; h)\xi^\alpha \to \xi^2$$

uniformly in h.

We equip $\mathcal{D}$ with the h-dependent Hilbert space norm by

$$\|u\|^2_{\mathcal{D}} = \|u\|^2_{\mathcal{H}} + \|P(h)u\|^2_{\mathcal{H}}$$

and define the spaces

$$\mathcal{D}_{\text{comp}} = \mathcal{D} \cap \mathcal{H}_{\text{comp}}$$

and

$$\mathcal{D}_{\text{loc}} = \{u \in \mathcal{H}_{\text{loc}} : \chi \in C^\infty, \chi = 1 \text{ on } B(0, R_0) \Rightarrow \chi u \in \mathcal{D}\}.$$

We make the final assumption that

$$\mathbf{1}_{B(0,R_0)}(P(h) + i)^{-1} \text{ is compact.}$$

Then we have (see [31])

Proposition 2.1.2. *If $Q = -\Delta$, then the spectrum of P in $(-\infty, 0)$ is discrete and $(P - \lambda^2)^{-1} : \mathcal{H} \to \mathcal{D}$ is meromorphic for $\operatorname{Im}\lambda > 0$ and has a meromorphic continuation to a family of operator*

$$R(\lambda) : \mathcal{H}_{\text{comp}} \to \mathcal{D}_{\text{loc}}$$

for $\lambda \in \mathbb{C}$ if n is odd; $\lambda \in \Lambda$ if n is even.

Now just as before, we can define the *resonances* of P as the poles of the meromorphic continuation for $R(\lambda)$ and define the multiplicities of resonances as in (2.1.3).

In the case of long range perturbation, under certain analyticity assumptions that will be specified in section 7.1.1, we have a meromorphic continuation of the resolvent to a sector near the real axis, and the poles are called the *resonances* of P.

2.1.3 Other characterizations for resonances

Although the definition of the resonances is simple, it is hard to study the distribution of resonances directly using the definition with meromorphic continuation. In practice, we often use other characterizations for resonances. From both the theoretic and computational view, the most useful one is given by the method of complex scaling which we shall review in the next section. In this part, we give several other equivalent characterizations in the Euclidean case. For details, we refer to [8].

Outgoing solutions and resonance states

Let $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$ be the resolvent of the free Laplacian on the Euclidean space, first defined for $\operatorname{Im}\lambda > 0$, then analytically continued to $\lambda \in \mathbb{C}$ if n is odd; $\lambda \in \Lambda$ if n is even. Then a solution to

$$(P - \lambda^2)u = f \in \mathcal{H}_{\text{comp}}$$

is called outgoing if there exists $g \in L^2_{\text{comp}}(\mathbb{R}^n)$ such that $u = R_0(\lambda)g$ on $\mathbb{R}^n \setminus B(0, R)$ for some large $R > 0$. Then λ is a resonance for P if and only if $(P - \lambda^2)u = 0$ has a nonzero outgoing solution. In this case, we call the outgoing solutions the resonance states.

Resonance expansion for solutions to wave equations

Under certain assumptions, the solution to the wave equation

$$(\partial_t^2 + P)u = 0, \;\; u|_{t=0} = f_0, \;\; \partial_t u|_{t=0} = f_1$$

has an asymptotic expansion on compact sets

$$u(t,x) \sim \sum_{\lambda_j \in \mathrm{Res}(P)} \sum_{k=0}^{m_P(\lambda_j)-1} e^{-it\lambda_j} t^k w_{\lambda_j,k}(x). \tag{2.1.4}$$

For example, if there are only finitely many resonances in $\{\operatorname{Im}\lambda > -A\}$ for some $A > 0$, then we can truncate the sum in (2.1.4) to a finite sum over all resonances with imaginary part greater than $-A$ and get an error term which is of size e^{-tA} in any reasonable local norms.

Scattering matrices

For any $\lambda \in \mathbb{R} \setminus \{0\}$ and any $g \in C^\infty(\mathbb{S}^{n-1})$, we can find a unique $f \in C^\infty(\mathbb{S}^{n-1})$ and $v \in \mathcal{D}_{\mathrm{loc}}$ such that $(P - \lambda^2)v = 0$ and v satisfies the asymptotic formula

$$v(r\theta) = r^{-\frac{n-1}{2}}(e^{i\lambda r} f(\theta) + e^{-i\lambda r} g(\theta)) + O(r^{-\frac{n+1}{2}}).$$

The the scattering matrix $S(\lambda)$ is defined to be unitary operator $S(\lambda) : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1})$ which maps $g(\theta)$ to $i^{n-1} f(-\theta)$. Roughly speaking, the scattering matrix maps the income pieces of the solution to the outgoing pieces of the solution. The resonances are exactly the poles of the meromorphic continuation of $S(\lambda)$.

Lax-Phillips semigroups

Lax-Phillips scattering theory [16] gives another characterization of resonances from the view of wave equations in odd dimensions. Let $U(t)$ and $U_0(t)$ be the unitary group of solution operators to the perturbed and unperturbed wave equations, respectively. Let D_+ and D_- be the spaces of initial values for which the solution to the unperturbed wave equations are outgoing and incoming, respectively. Let Π be the projection onto the orthogonal complement to $D_+ \oplus D_-$, then the Lax-Phillips semigroup is defined as

$$Z(t) = \Pi U(t) \Pi.$$

Then the infinitesimal generator of the Lax-Phillips semigroup has discrete spectrum and the eigenvalues coincide with the resonances (up to multiplication by i). For the details of Lax-Phillips semigroups for black-box operators, see [34].

Boundary layer potentials and Dirichle-to-Neumann operators

In the case of obstacle scattering, one can also use boundary layer potentials and Dirichlet-to-Neumann operators to describe the resonances. The formulations depends on the boundary conditions. We omit the details and refer to [21].

2.2 Airy functions and Airy differential operators

In this section, we review some basic properties of the Airy function Ai and the Airy differential operators $A = D_t^2 + t$. They play crucial roles in our analysis near the convex obstacle.

In the real variables, the Airy function Ai is given by the integral formula

$$\mathrm{Ai}(t) = \frac{1}{2\pi} \int_{\mathrm{Im}\,\sigma=\delta>0} e^{i(\sigma^3/3)+i\sigma t} d\sigma, \tag{2.2.1}$$

or equivalently,

$$\mathrm{Ai}(t) = \frac{1}{\pi} \int_0^\infty \cos(\frac{s^3}{3} + st) ds.$$

In the complex variable, Ai is an entire function given by the same integral formula (2.2.1). From the integral formula, it is clear that Ai satisfies the Airy differential equation (also known as the Stokes equation)

$$Ay = -y'' + ty = 0 \tag{2.2.2}$$

in the real domain as well as the complex domain. Its initial data is given by

$$\mathrm{Ai}(0) = \frac{1}{3^{2/3}\Gamma(\frac{2}{3})} = 0.355\ldots, \quad \mathrm{Ai}'(0) = -\frac{1}{3^{1/3}\Gamma(\frac{1}{3})} = -0.258\ldots.$$

2.2.1 Asymptotic properties of the Airy function

The Airy function and its derivative have different asymptotic behaviors in different directions in the complex plane. For example, in the positive real direction, they decay superexponentially:

$$\begin{aligned} \mathrm{Ai}(t) &= (2\sqrt{\pi})^{-1} t^{-1/4} e^{-\frac{2}{3}t^{3/2}} (1 + \mathcal{O}(t^{-3/2})), \\ \mathrm{Ai}'(t) &= -(2\sqrt{\pi})^{-1} t^{1/4} e^{-\frac{2}{3}t^{3/2}} (1 + \mathcal{O}(t^{-3/2})), \end{aligned} \tag{2.2.3}$$

as $t \to \infty$; while in the negative real direction, they oscillate faster and faster,

$$\begin{aligned} \mathrm{Ai}(-t) &= \pi^{-1/2} t^{-1/4} \left(\sin(\frac{2}{3}t^{3/2} + \frac{\pi}{4}) + \mathcal{O}(t^{-3/2}) \right), \\ \mathrm{Ai}'(-t) &= -\pi^{-1/2} t^{1/4} \left(\cos(\frac{2}{3}t^{3/2} + \frac{\pi}{4}) + \mathcal{O}(t^{-3/2}) \right), \end{aligned} \tag{2.2.4}$$

as $t \to \infty$. Moreover, the formulas (2.2.3) actually hold away from the negative real axis,

$$\begin{aligned} \mathrm{Ai}(z) &= (2\sqrt{\pi})^{-1} e^{-\zeta} z^{-1/4}(1+\mathcal{O}(|\zeta|^{-1})), \\ \mathrm{Ai}'(z) &= -(2\sqrt{\pi})^{-1} e^{-\zeta} z^{1/4}(1+\mathcal{O}(|\zeta|^{-1})), \end{aligned} \tag{2.2.5}$$

uniformly for $0 \leqslant |\arg z| \leqslant \pi - \delta$, where $\delta > 0$ is fixed. Here $\zeta = \frac{2}{3} z^{3/2}$ and we choose the branch such that if z is real and positive, then so is ζ. Another asymptotic formula similar to (2.2.4) holds uniformly in the sectors $0 \leqslant |\arg z - \pi| \leqslant \frac{2}{3}\pi - \delta$ for fixed $\delta > 0$. In fact, we can extend the formulas above to full asymptotic expansions.

All the zeroes of Ai and Ai$'$ are real, negative and simple. We write $0 < \zeta_1 < \zeta_2 < \cdots$ and $0 < \zeta_1' < \zeta_2' < \cdots$ to be the negatives of the zeroes of Ai and Ai$'$, respectively. They appear alternatively in the sense that for any j, $\zeta_j' < \zeta_j < \zeta_{j+1}'$. The first several zeroes are given by

$$\begin{aligned} \zeta_1 &= 2.338\ldots, \quad \zeta_1' = 1.018\ldots; \\ \zeta_2 &= 4.807\ldots, \quad \zeta_2' = 3.248\ldots; \\ \zeta_3 &= 5.520\ldots, \quad \zeta_3' = 4.820\ldots. \end{aligned}$$

By Sturm's comparison theorem, the distances between the zeroes get closer as $j \to \infty$, i.e.

$$\zeta_{j+1} - \zeta_j \searrow 0 \text{ and } \zeta_{j+1}' - \zeta_j' \searrow 0, \quad j \to \infty.$$

As a consequence, as $j \to \infty$, $\zeta_{j+1}/\zeta_j \searrow 1$, $\zeta_{j+1}'/\zeta_j' \searrow 1$. The first several ratios are given by

$$\begin{aligned} \zeta_2/\zeta_1 &= 1.748\ldots, \quad \zeta_2'/\zeta_1' = 3.188\ldots; \\ \zeta_3/\zeta_2 &= 1.350\ldots, \quad \zeta_3'/\zeta_2' = 1.483\ldots. \end{aligned}$$

In fact, we have more precise asymptotic formula for the zeroes:

$$\begin{aligned} \zeta_j &= (\frac{3}{8}(4j-1)\pi)^{2/3}(1+\mathcal{O}(j^{-2})) \\ \zeta_j' &= (\frac{3}{8}(4j-3)\pi)^{2/3}(1+\mathcal{O}(j^{-2})). \end{aligned} \tag{2.2.6}$$

The other data at the zeroes also have the asymptotic formula

$$\begin{aligned} \mathrm{Ai}'(-\zeta_j) &= (-1)^{j-1}\pi^{-1/2}(\frac{3}{8}(4j-1)\pi)^{1/6}(1+\mathcal{O}(j^{-2})) \\ \mathrm{Ai}(-\zeta_j') &= (-1)^{j-1}\pi^{-1/2}(\frac{3}{8}(4j-3)\pi)^{-1/6}(1+\mathcal{O}(j^{-2})). \end{aligned}$$

2.2.2 Airy differential operators

Now we turn to the Airy differential operator $A = D_t^2 + t$, in particular, we only consider A on the positive half-line $(0,\infty)$. With the natural domain $\mathcal{D}(A) = C_0^\infty[0,\infty) \subset L^2 = L^2(0,\infty)$, it is clearly a symmetric positive operator. It has deficiency indices $(1, 1)$ and thus by von Neumann's theory of self-adjoint extensions, it has a one-parameter family of self-adjoint extensions. Each self-adjoint extension can be realized by fixing a boundary condition at 0. For our purpose, we only consider two special cases: the Dirichlet boundary condition $u(0) = 0$ and the Neumann boundary condition $u'(0) = 0$. We shall write A_0 and A_1 to be the Dirichlet and Neumann realization of the Airy operator A, respectively. Their domain are given by

$$\mathcal{D}(A_0) = \{u \in L^2 : D_t^2 u, tu \in L^2, u(0) = 0\},$$
$$\mathcal{D}(A_1) = \{u \in L^2 : D_t^2 u, tu \in L^2, u'(0) = 0\}.$$

Both A_0 and A_1 has discrete spectrum and all the eigenfunctions are given by translations of the Airy function

$$(D_t^2 + t) \operatorname{Ai}(t - \zeta_j) = \zeta_j \operatorname{Ai}(t - \zeta_j),$$
$$(D_t^2 + t) \operatorname{Ai}(t - \zeta_j') = \zeta_j' \operatorname{Ai}(t - \zeta_j').$$

In particular, by the spectral theorem, for any $u \in C_0^\infty[0, \infty)$, if $u(0) = 0$, then

$$\langle (D_t^2 + t)u, u\rangle \geqslant \zeta_1 \|u\|^2; \tag{2.2.7}$$

if $D_t u(0) = 0$, then

$$\langle (D_t^2 + t)u, u\rangle \geqslant \zeta_1' \|u\|^2. \tag{2.2.8}$$

We also need the normalized eigenfunctions

$$e_j(\cdot) = c_j^{-1} \operatorname{Ai}(\cdot - \zeta_j), \qquad c_j = \| \operatorname{Ai} \|_{L^2(-\zeta_j, \infty)};$$
$$e_j'(\cdot) = c_j'^{-1} \operatorname{Ai}(\cdot - \zeta_j'), \qquad c_j' = \| \operatorname{Ai}' \|_{L^2(-\zeta_j', \infty)}.$$

To compute the norms, we notice that for any $z \neq w$, integration by parts gives

$$\begin{aligned}
& z\langle \operatorname{Ai}(t - z), \operatorname{Ai}(t - w)\rangle_{L^2} \\
= & \langle (D_t^2 + t) \operatorname{Ai}(t - z), \operatorname{Ai}(t - w)\rangle_{L^2} \\
= & \operatorname{Ai}'(-z)\overline{\operatorname{Ai}(-w)} - \operatorname{Ai}(-z)\overline{\operatorname{Ai}'(-w)} + \langle \operatorname{Ai}(t - z), (D_t^2 + t) \operatorname{Ai}(t - w)\rangle_{L^2} \\
= & \operatorname{Ai}'(-z)\overline{\operatorname{Ai}(-w)} - \operatorname{Ai}(-z)\overline{\operatorname{Ai}'(-w)} + \bar{w}\langle \operatorname{Ai}(t - z), \operatorname{Ai}(t - w)\rangle_{L^2}.
\end{aligned}$$

Therefore

$$\langle \operatorname{Ai}(t - z), \operatorname{Ai}(t - w)\rangle_{L^2} = \frac{\operatorname{Ai}'(-z)\overline{\operatorname{Ai}(-w)} - \operatorname{Ai}(-z)\overline{\operatorname{Ai}'(-w)}}{z - \bar{w}}.$$

Take $z = \zeta_j$, we have

$$\langle \operatorname{Ai}(t - \zeta_j), \operatorname{Ai}(t - w)\rangle_{L^2} = \frac{\operatorname{Ai}'(-\zeta_j)\overline{\operatorname{Ai}(-w)}}{\zeta_j - \bar{w}}.$$

Let $w \to \zeta_j$, using $\mathrm{Ai}(-\zeta_j)=0$, we have

$$c_j^2 = \|\,\mathrm{Ai}\,\|^2_{L^2(-\zeta_j,\infty)} = \|\,\mathrm{Ai}(\cdot-\zeta_j)\|^2_{L^2} = |\,\mathrm{Ai}'(-\zeta_j)|^2.$$

Similarly, if we first take $z=\zeta_j'$, then let $w\to\zeta_j'$, and use $\mathrm{Ai}'(-\zeta_j')=0$ as well as the Airy equation (2.2.2), we obtain

$$c_j'^2 = \|\,\mathrm{Ai}\,\|^2_{L^2(-\zeta_j',\infty)} = \|\,\mathrm{Ai}(\cdot-\zeta_j')\|^2_{L^2} = -\,\mathrm{Ai}(-\zeta_j')\,\mathrm{Ai}''(-\zeta_j') = \zeta_j'|\,\mathrm{Ai}(-\zeta_j')|^2.$$

Finally, we can obtain the other initial data for the normalized eigenfunctions

$$\begin{aligned}\partial_t e_j(0) &= |\,\mathrm{Ai}'(-\zeta_j)|^{-1}\,\mathrm{Ai}'(-\zeta_j) = (-1)^{j-1};\\ e_j'(0) &= \zeta_j'^{-1/2}|\,\mathrm{Ai}(-\zeta_j')|^{-1}\,\mathrm{Ai}(-\zeta_j') = (-1)^{j-1}\zeta_j'^{-1/2}.\end{aligned} \tag{2.2.9}$$

2.3 Scattering by a sphere

In this section, we consider the special case of scattering by the unit sphere, i.e. the obstacle $\mathcal{O}=B(0,1)$ is the unit ball in $\mathbb{R}^n$. For simplicity, we assume that n is odd, so that the resonances are in the complex plane $\mathbb{C}$.

Our main tool will be the method of separation of variables, and to apply such method, we restrict ourselves to either Dirichlet boundary condition $u|_{\partial\mathcal{O}}=0$ or Neumann/Robin boundary condition $\partial_\nu u+\eta u|_{\partial\mathcal{O}}=0$ with η a real constant.

In the polar coordinates $x\mapsto(r,\omega)=(|x|,x/|x|)\in\mathbb{R}^+\times\mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$, the Laplace operator has the following form

$$\Delta = \frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{dr}+\frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}}$$

where $\Delta_{\mathbb{S}^{n-1}}$ is the Laplacian on the unit sphere.

The eigenfunctions of $\Delta_{S^{n-1}}$ are called the spherical harmonic functions. We shall choose an orthonormal basis $\{Y_l^m : l=0,1,\ldots;m=1,\ldots,m(l)\}$ of $L^2(\mathbb{S}^{n-1})$ consisting of spherical harmonics

$$-\Delta_{S^{n-1}}Y_l^m(\omega) = l(l+n-2)Y_l^m(\omega).$$

Here the multiplicity of the eigenvalue $\mu_l=l(l+n-2)$ is given by

$$m(l) = \frac{2l+n-2}{n-2}\binom{l+n-3}{n-3}.$$

If we write the solution to the Helmholtz equation $(-\Delta-\lambda^2)u=0$ in the form

$$u=\sum_{l,m} g_l^m(r)Y_l^m(\omega),$$

then $g = g_l^m$ solves the equation

$$r^2 \frac{d^2 g}{dr^2} + (n-1) r \frac{dg}{dr} + [\lambda^2 r^2 - l(l+n-2)] g = 0.$$

Write $g(r) = (\lambda r)^{1-\frac{n}{2}} H(\lambda r)$, then H solves the Bessel equation

$$z^2 H''(z) + z H'(z) + (z^2 - \nu^2) H(z) = 0, \quad \nu = l + \frac{n}{2} - 1 \tag{2.3.1}$$

For u to be outgoing, i.e. the Sommerfeld radiation condition

$$\lim_{r \to \infty} r^{(n-1)/2} (\partial_r - i\lambda) u = 0$$

holds, H is a multiple of the Hankel function $H_\nu^{(1)}(z)$, also known as the Bessel function of the third kind, which is the unique solution to (2.3.1) satisfies the asymptotic properties

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i(z - \nu\pi/2 - \pi/4)},$$

as $z \to \infty$ in the sector $-2\pi + \delta \leqslant \arg z \leqslant \pi - \delta$ where $\delta > 0$ is a fixed small constant. Therefore the corresponding outgoing solution is

$$u(x) = \sum_{l,m} c_{lm} (\lambda r)^{1-n/2} H_\nu^{(1)}(\lambda r) Y_l^m(\omega).$$

An explicit definition of the Hankel function $H_\nu^{(1)}(z)$ is given by

$$H_\nu^{(1)}(z) = J_\nu(z) + i Y_\nu(z),$$

where $J_\nu(z)$ is the Bessel function of the first kind given by

$$J_\nu(z) = (\frac{z}{2})^\nu \sum_{k=0}^\infty (-1)^k \frac{(z^2/4)^k}{k! \Gamma(\nu + k + 1)},$$

and $Y_\nu(z)$ is the Weber function, also known as the Bessel function of the second kind,

$$Y_\nu(z) = \frac{J_\nu(z) \cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}.$$

An integral formula is valid when $-\pi/2 < \arg z < \pi$,

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \frac{e^{i(z - \nu\pi/2 - \pi/4)}}{\Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{-s} s^{\nu - 1/2} \left(1 - \frac{s}{2iz}\right)^{\nu - 1/2} ds.$$

We introduce the following variables: $\xi = \frac{2}{3}w^{3/2}$,

$$\rho(z) = \frac{2}{3}\zeta^{3/2} = \log\frac{1+\sqrt{1-z^2}}{z} - \sqrt{1-z^2}$$

for $|\arg z| < \pi$ and if z is real then ζ is real. Then the Hankel function $H_\nu^{(1)}$ has the following asymptotic expansions

$$\begin{aligned} H_\nu^{(1)}(\nu z) \sim 2e^{-\pi i/3}\left(\frac{4\zeta}{1-z^2}\right)^{\frac{1}{4}} \Bigg[& \mathrm{Ai}(e^{2\pi i/3}\nu^{\frac{2}{3}}\zeta)\sum_{k=0}^{\infty}\frac{A_k(\zeta)}{\nu^{2k+\frac{1}{3}}} \\ & + e^{2\pi i/3}\,\mathrm{Ai}'(e^{2\pi i/3}\nu^{\frac{2}{3}}\zeta)\sum_{k=0}^{\infty}\frac{B_k(\zeta)}{\nu^{2k+\frac{5}{3}}}\Bigg] \end{aligned} \tag{2.3.2}$$

and

$$\begin{aligned} H_\nu^{(1)\prime}(\nu z) \sim \frac{4e^{-2\pi i/3}}{z}\left(\frac{1-z^2}{4\zeta}\right)^{\frac{1}{4}} \Bigg[& e^{-2\pi i/3}\,\mathrm{Ai}(e^{2\pi i/3}\nu^{\frac{2}{3}}\zeta)\sum_{k=0}^{\infty}\frac{C_k(\zeta)}{\nu^{2k+\frac{4}{3}}} \\ & + \mathrm{Ai}'(e^{2\pi i/3}\nu^{\frac{2}{3}}\zeta)\sum_{k=0}^{\infty}\frac{D_k(\zeta)}{\nu^{2k+\frac{2}{3}}}\Bigg] \end{aligned}$$

uniformly in $|\arg z| \leqslant \pi - \delta$, for any fixed $\delta > 0$. All the coefficients A_k, B_k, C_k and D_k are real smooth functions of $\zeta \in \mathbb{R}$.

First, we consider the Dirichlet boundary condition $u|_{\mathbb{S}^{n-1}} = 0$, which implies that for some l, m, $H_\nu^{(1)}(\lambda) = 0$. Therefore the resonances are given by the zeroes of the Hankel function $H_\nu^{(1)}$. Recall that $-\zeta_k$ is the k-th zero of Ai, using the first term in the asymptotic expansion (2.3.2), approximate resonances can be given by

$$\tilde{\lambda}_{\nu k}^{(0)} = \nu\zeta^{-1}(e^{\pi i/3}\nu^{-2/3}\zeta_k) = \nu\rho^{-1}\left(\frac{i}{\nu}\left(\frac{2}{3}\zeta_k^{3/2}\right)\right).$$

For Neumann/Robin boundary condition $\frac{\partial u}{\partial \nu} + \eta u|_{\mathbb{S}^{n-1}} = 0$, the resonances are given by the zeroes of

$$g_\eta(z) = zH_\nu^{(1)\prime}(z) + (1 - \frac{n}{2} + \eta)H_\nu^{(1)}(z)$$

Similarly, we can use the zeroes of Ai' to give approximate resonances

$$\tilde{\lambda}_{\nu k}^{(1)} = \nu\zeta^{-1}(e^{\pi i/3}\nu^{-2/3}\zeta_k'^{3/2}) = \nu\rho^{-1}\left(\frac{i}{\nu}\left(\frac{2}{3}\zeta_k'^{3/2}\right)\right).$$

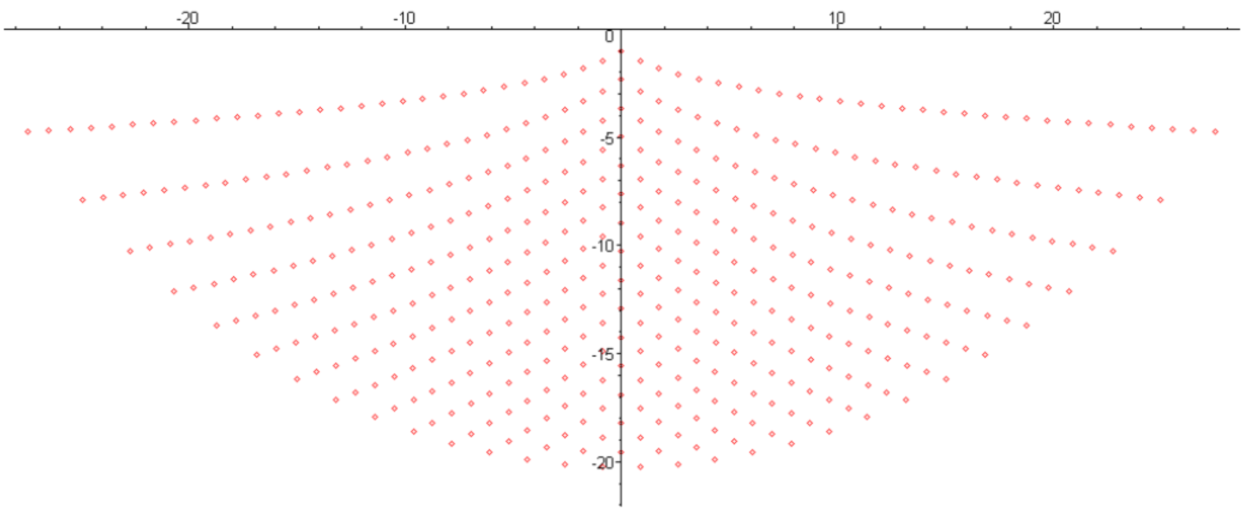

Figure 2.1: The Resonances for S^2 with Dirichlet boundary condition, from Stefanov [36].

Chapter 3

Semiclassical microlocal analysis

In this chapter, we review some basic results from semiclassical microlocal analysis. The standard references are [7],[43] and [19].

3.1 Symbol classes and quantization

3.1.1 Semiclassical pseudodifferential operators on $\mathbb{R}^{2n}$

Let $m = m(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ be an order function in the sense that

$$|\partial^\alpha m| \leqslant C_\alpha m, \quad \forall \alpha \in \mathbb{N}^{2n}.$$

The symbol class $S(m)$ is the collections of all smooth functions $a = a(x,\xi;h)$ such that

$$|\partial^\alpha a| \leqslant C_\alpha m, \quad \forall \alpha \in \mathbb{N}^{2n}.$$

We also consider the symbol class $S_\delta^{m,k}$ $(m,k \in \mathbb{Z}, 0 \leqslant \delta < 1/2)$ of all smooth functions $a = a(x,\xi,h)$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \leqslant C_{\alpha\beta} h^{-m-\delta(|\alpha|+|\beta|)} \langle\xi\rangle^{k-(1-\delta)|\beta|+\delta|\alpha|}, \quad \forall \alpha,\beta \in \mathbb{N}^n.$$

The semiclassical principal symbol of $a \in S_\delta^{m,k}$ is the equivalence class of a in $S_\delta^{m,k}/S_\delta^{m-1+\delta,k-1}$. We also write

$$S^{-\infty,-\infty} = \bigcap_{m,k} S_\delta^{m,k}.$$

The Weyl quantization of a symbol $a \in S(m)$ or $S_\delta^{m,k}$ to be the operator $\mathrm{Op}_h^w(a) = a^w(x,hD) \in \Psi(m)$ or $\Psi_\delta^{m,k}$ defined as

$$(\mathrm{Op}_h^w(a)u)(x) = \frac{1}{(2\pi h)^n} \iint e^{i(x-y)\cdot\xi/h} a(\frac{x+y}{2},\xi)u(y)dyd\xi. \tag{3.1.1}$$

Let $H_h^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ be the semiclassical Sobolev space of order s with the norm

$$\|u\|_{H_h^s(\mathbb{R}^n)} = \|\langle hD\rangle^s u\|_{L^2(\mathbb{R}^n)}, \quad \langle hD\rangle = (1+(hD)^2)^{\frac{1}{2}}.$$

If $a \in S_\delta^{m,k}$, then $h^k \operatorname{Op}_h^w(a)$ is uniformly bounded $H_h^s(\mathbb{R}^n) \to H_h^{s-m}(\mathbb{R}^n)$ for any s.

Moreover, for any $A \in \Psi_\delta^{m,k}$, $B \in \Psi_\delta^{m',k'}$, the composition $AB \in \Psi_\delta^{m+m',k+k'}$ and the commutator $[A,B] \in \Psi_\delta^{m+m'-1,k+k'-1}$. If $a = \sigma(A)$ and $b = \sigma(B)$ are the principal symbol of A, B, respectively, then $\sigma(AB) = ab$ and $\sigma([A,B]) = ih^{-1}\{a,b\}$ where $\{\cdot,\cdot\}$ is the Poisson bracket.

For $A = \operatorname{Op}_h(a) \in S_\delta^{m,k}$ we say that $(x_0,\xi_0) \in \mathbb{R}^{2n}$ is not in the semiclassical wave front set $\operatorname{WF}_h(A)$ if $|\partial^\alpha a| = O(h^\infty)$ near (x_0,ξ_0) for any α. We have if $\operatorname{WF}_h(A) = \emptyset$, then for any N, $\|A\|_{H_h^{-N} \to H_h^N} = O(h^\infty)$.

We say that $a \in S_\delta^{m,k}(\mathbb{R}^n)$ is elliptic on $U \subset \mathbb{R}^{2n}$ if for $(x,\xi) \in U$,

$$|a(x,\xi)| \geqslant C^{-1}h^{-k}\langle\xi\rangle^m$$

and $A \in \Psi_\delta^{m,k}$ is elliptic if its principal symbol $a = \sigma(A)$ is elliptic. If A is elliptic on U and $\operatorname{WF}_h(B) \subset U$, then

$$\|Bu\|_{H_h^s} \leqslant Ch^k \|ABu\|_{H_h^{s+m}}.$$

Finally we recall the sharp Gårding's inequality which states that if $A =\in \Psi_0^{0,0}(\mathbb{R}^n)$ has principal symbol $a \geqslant 0$ on $U \subset \mathbb{R}^{2n}$, then for any $B \in \Psi_0^{0,0}$ with $\operatorname{WF}_h(B) \subset U$, then

$$\langle ABu, Bu\rangle_{L^2(\mathbb{R}^n)} \geqslant -Ch\|Bu\|^2_{L^2(\mathbb{R}^{2n})}.$$

3.1.2 Semiclassical pseudodifferential operators on a compact manifold

Let X be a compact smooth manifold, we can choose a finite cover $X_1,\dots,X_p$ of X where $X_1,\dots,X_p$ are coordinate charts with local coordinates $x_1,\dots,x_n$. Then there exists a partition of unity $\chi_j \in C_0^\infty(X_j)$, $\sum_{j=1}^p \chi_j = 1$. We define the semiclassical Sobolev space $H_h^s(X)$ to be the space of all $u \in \mathcal{D}'(X)$ such that

$$\|u\|^2_{H_h^s(X)} = \sum_{j=1}^p \|\chi_j\langle hD\rangle^s \chi_j u\|^2_{L^2(X_j)} < \infty.$$

For different choice of the coordinate charts and partition of unity, the norms are equivalent uniformly for $h > 0$. Also, another equivalent norm can be given by

$$\|u\|_{H_h^s(X)} = \|(I - h^2\Delta)^{\frac{s}{2}} u\|_{L^2(X)},$$

where Δ is the Laplacian operator with respect to some Riemannian metric. From this norm, we see that $H_h^s(X)$ is a Hilbert space with inner product

$$\langle u, v\rangle_{H_h^s(X)} = \langle (I-h^2\Delta)^{\frac{s}{2}} u, (I-h^2\Delta)^{\frac{s}{2}} v\rangle_{L^2(X)}.$$

We can also generalize the symbol class $S_\delta^{m,k}$ to a compact manifold X:

$$S_\delta^{m,k}(T^*X) = \{a \in C^\infty(T^*X \times (0,1]) : |\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \leqslant C_{\alpha\beta} h^{-m-\delta(|\alpha|+|\beta|)} \langle \xi \rangle^{k-(1-\delta)|\beta|+\delta|\alpha|}\}.$$

For any $a \in S_\delta^{m,k}(T^*X)$, we can quantize a to an operator $\mathrm{Op}_h(a) : C^\infty(X) \to C^\infty(X)$ by using a partition of unity and (3.1.1) in local coordinate patches.

A linear operator $A : C^\infty(X) \to C^\infty(X)$ is said to be in $\Psi_\delta^{m,k}(X)$ if

(i) for each coordinate patch (U,γ) where $\gamma : U \to V \subset \mathbb{R}^{2n}$ is a diffeomorphism, there exists a symbol $a \in S_\delta^{m,k}$ such that

$$\varphi A(\psi u) = \varphi \gamma^* \mathrm{Op}_h(a)[(\gamma^{-1})^*(\psi u)],$$

for any $\varphi, \psi \in C_c^\infty(U_\gamma)$ and $u \in C^\infty(X)$;

(ii) for any $\chi_1, \chi_2 \in C^\infty(X)$ with disjoint support and any N,

$$\|\chi_1 A \chi_2\|_{H^{-N}(X) \to H^N(X)} = O(h^\infty).$$

We have a quantization map

$$\mathrm{Op}_h : S_\delta^{m,k}(T^*X) \to \Psi_\delta^{m,k}(X)$$

and the principal symbol map

$$\sigma : \Psi_\delta^{m,k}(X) \to S_\delta^{m,k}(T^*X)/S_\delta^{m-1+\delta,k-1}(T^*X).$$

The semiclassical wavefront set $\mathrm{WF}_h(A)$ of $A \in \Psi_\delta^{m,k}(X)$ is defined in local coordinates as in the case of $\mathbb{R}^n$, since it is invariant under change of variables. All the results for semiclassical wavefront set extend to the case of a compact manifold.

Finally, we remark that the notion of ellipticity and the sharp Gårding inequality can be extended to the case of a compact manifold without difficulty.

3.1.3 Trace operator on the boundary

Now we go back to the setting that $\mathcal{O}$ is a convex obstacle in $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{O}$. We consider the trace operator $\mathrm{Tr} : C^\infty(\mathbb{R}^n \setminus \mathcal{O}) \to C^\infty(\partial\mathcal{O}), u \mapsto u|_{\partial\mathcal{O}}$. In the normal geodesic coordinates given in the previous section, it is equivalent to the operator $\mathrm{Tr} : C^\infty(X \times [0,\infty)) \to C^\infty(X), \mathrm{Tr}\, u(y) = u(y,0)$.

Proposition 3.1.1. *For $u \in C_0^\infty(X \times [0,\infty))$, we have $\|\mathrm{Tr}\, u\|_{H_h^1(X)}^2 \leqslant Ch^{-1}\|u\|_{H_h^2(X \times [0,\infty))}^2$.*

Proof. Since $u \in C_0^\infty(X \times [0,\infty))$, we know there exists $L > 0$ such that u is supported in $X \times [0,L]$. Therefore

$$\begin{aligned}\|\mathrm{Tr}\, u\|_{H_h^1(X)}^2 &= -h^{-1}\int_0^\infty hD_t\|u(\cdot,t)\|_{H_h^1(X)}^2 dt \leqslant 2h^{-1}\int_0^\infty |\langle hD_t u(\cdot,t), u(\cdot,t)\rangle_{H_h^1(X)}| dt \\ &\leqslant h^{-1}\int_0^\infty [\|hD_t u(\cdot,t)\|_{H_h^1(X)}^2 + \|u(\cdot,t)\|_{H_h^1(X)}^2] dt \leqslant Ch^{-1}\|u\|_{H_h^2(X \times [0,\infty))}^2.\end{aligned}$$

∎

Remark 3.1.2. *A more careful analysis will give* $\mathrm{Tr} = O_s(h^{-\frac{1}{2}}) : H_h^s(\Omega) \to H_h^{s-\frac{1}{2}}(\partial\Omega)$ *when* $s > \frac{1}{2}$, $\Omega \subset\subset \mathbb{R}^n$ *an open set with smooth boundary. We do not need this strong version in our argument.*

3.2 Second microlocalization with respect to a hypersurface

In this part, we review some facts about second microlocalization with respect to a hypersurface. For details, see [30].

We always assume that X is a n-dimensional compact smooth manifold and $\Sigma \subset T^*X$ is a smooth compact hypersurface. In our application, $X = \partial\mathcal{O}$ will be the boundary of the obstacle and $\Sigma = \Sigma_w = \{(x', \xi') \in T^*\partial\mathcal{O} : R(x', \xi') = w\}$ will be the glancing hypersurface. Here $R(x', \xi') = |\xi'|_{x'}$ is the first fundamental form and w is in a compact subset of $\mathbb{R}$.

We shall also fix a distance function $d(\Sigma, \cdot)$ on T^*X as the absolute value of a defining function of Σ. In particular, $d(\Sigma, \cdot)$ vanishes only on Σ and behaves like $\langle\xi\rangle$ near the infinity in T^*X.

For any $0 \leqslant \delta < 1$ we define a class of symbols associated to Σ: $a \in S^{m,k_1,k_2}_{\Sigma,\delta}(T^*X)$ if

$$\begin{aligned} &\text{near } \Sigma : V_1 \cdots V_{l_1} W_1 \cdots W_{l_2} a = O(h^{-m-\delta l_1} \langle h^{-\delta} d(\Sigma, \cdot)\rangle^{k_1}), \\ &\text{where } V_1, \dots, V_{l_1} \text{ are vector fields tangent to } \Sigma, \\ &\ \text{and } W_1, \dots, W_{l_2} \text{ are any vector fields;} \\ &\text{away from } \Sigma : \partial_x^\alpha \partial_\xi^\beta a(x, \xi; h) = O(h^{-m-\delta k_1} \langle\xi\rangle^{k_2 - |\beta|}). \end{aligned} \tag{3.2.1}$$

To define the corresponding class of operators $\Psi^{m,k_1,k_2}_{\Sigma,\delta}$, we start locally by assuming Σ is of the normal form $\Sigma_0 = \{\xi_1 = 0\}$. Then near $\xi_1 = 0$, we can write $a = a(x, \xi, \lambda; h)$ with $\lambda = h^{-\delta}\xi_1$. Then (3.2.1) becomes

$$\partial_x^\alpha \partial_\xi^\beta \partial_\lambda^l a(x, \xi, \lambda, h) = O(h^{-m})\langle\lambda\rangle^{k-l},$$

which we shall write $a = \tilde{O}(h^{-m}\langle\lambda\rangle^k)$. Then we can define

$$\widetilde{\mathrm{Op}}_h(a)u(x) = \frac{1}{(2\pi h)^n} \int e^{i(x-y)\cdot\xi/h} a(x, \xi, h^{-\delta}\xi_1, h)u(y)dyd\xi.$$

Then as in the standard semiclassical calculus, we have the composition formula: for $a = \tilde{O}(h^{-m_1}\langle\lambda\rangle^{k_1})$ and $b = \tilde{O}(h^{-m_2}\langle\lambda\rangle^{k_2})$,

$$\widetilde{\mathrm{Op}}_h(a) \circ \widetilde{\mathrm{Op}}_h(b) = \widetilde{\mathrm{Op}}_h(a \#_h b) \mod \Psi^{-\infty,-\infty}(X),$$

where

$$a\#_h b(x, \xi, \lambda; h) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} (h\partial_{\xi'})^{\alpha'} (h\partial_{\xi_1} + h^{1-\delta}\partial_\lambda)^{\alpha_1} a D_x^\alpha b \in \tilde{O}(h^{-m_1-m_2}\langle\lambda\rangle^{k_1+k_2}).$$

We also have a version of Beals's characterization of pseudodifferential operators: Let $A = A_h : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ and put $x' = (x_2, \dots, x_n)$. Then $A = \tilde{\mathrm{O}}\mathrm{p}_h(a)$ for some $a = \tilde{O}(h^{-m}\langle\lambda\rangle^k)$ if and only if for all $N, p, q \geqslant 0$ and every sequence $l_j(x', \xi'), j = 1, \dots, N$ of linear forms on $\mathbb{R}^{2(n-1)}$ there exists $C > 0$ such that

$$\| \mathrm{ad}_{l_1(x', hD_{x'})} \circ \cdots \circ \mathrm{ad}_{l_N(x', hD_{x'})} \circ (\mathrm{ad}_{h^{1-\delta}D_{x_1}})^p \circ (\mathrm{ad}_{x_1})^q Au\|_{(q-\min(k,0))} \\ \leqslant Ch^{N+(1-\delta)(p+q)}\|u\|_{(\max(k,0))},$$

where $\|u\|_{(p)} = \|u\|_{L^2} + \|(h^{1-\delta}D_{x_1})^p u\|_{L^2}$.

The global definition of the class $\Psi^{m,k_1,k_2}_{\Sigma,\delta}(X)$ relies on the invariance of $\widetilde{\mathrm{Op}}_h(\tilde{O}(\langle\lambda\rangle^m))$ under conjugation by h-Fourier integral operators whose associated canonical relation fixes $\{\xi_1 = 0\}$. See [30, Section 4.2]. Now we define $A \in \Psi^{m,k_1,k_2}_{\Sigma,\delta}(X)$ if and only if
(1) for any $m_0 \in \Sigma$ and any h-Fourier integral operator $U : C^\infty(X) \to C^\infty(\mathbb{R}^n)$ elliptic near $((0,0), m_0)$ whose corresponding canonical transformation κ satisfies $\kappa(m_0) = (0,0)$, $\kappa(\Sigma \cap V) \subset \{\xi_1 = 0\}$ for some neighborhood V of m_0, we have $UAU^{-1} = \widetilde{\mathrm{Op}}_h(\tilde{O}(h^{-m}\langle\lambda\rangle^{k_1})$, microlocally near $(0,0)$;
(2) for any m_0 outside any fixed neighborhood of Σ, $A \in \Psi^{m+\delta k_1,k_2}(X)$ microlocally near m_0 in both classical and semiclassical sense.

In particular, we have a quantization map

$$\mathrm{Op}_{\Sigma,h} : S^{m,k_1,k_2}_{\Sigma,\delta}(T^*X) \to \Psi^{m,k_1,k_2}_{\Sigma,\delta}(X),$$

and the principal symbol map

$$\sigma_{\Sigma,h} : \Psi^{m,k_1,k_2}_{\Sigma,\delta}(X) \to S^{m,k_1,k_2}_{\Sigma,\delta}(T^*X)/S^{m-1+\delta,k_1-1,k_2-1}_{\Sigma,\delta}(T^*X).$$

For $a \in S^{m,k_1,-\infty}_{\Sigma,\delta}$ we introduce a notion of essential support. We say for an h-dependent family of sets $V_h \subset T^*X$,

$$\operatorname{esssupp} a \cap V_h = \emptyset$$

if and only if there exists $\chi \geqslant 0$, $\chi \in S^{0,0,-\infty}(T^*X)$, such that

$$\chi|_{V_h} \geqslant 1, \chi a \in S^{-\infty,-\infty}(T^*X).$$

As in the standard case, if $a, b \in S^{m,k,-\infty}_{\Sigma,\delta}(T^*X)$ satisfies $\mathrm{Op}_{\Sigma,h}(a) = \mathrm{Op}_{\Sigma,h}(b)$, then $\operatorname{esssupp} a = \operatorname{esssupp} b$. Therefore we can define for $A \in \Psi^{m,k,-\infty}_{\Sigma,\delta}(X)$ the semiclassical wave front set as $\mathrm{WF}_h(A) = \operatorname{esssupp} a$ if $A = \mathrm{Op}_{\Sigma,h}(a)$.

Now we generalize the symbol class to an arbitrary order function m and vector valued operators from a Banach space $\mathcal{B}$ to another Banach space $\mathcal{H}$. We assume that $m = m(x, \xi, \lambda; h)$ is an order function with respect to the metric $g = dx^2 + d\xi^2/\langle\xi\rangle + d\lambda^2/\langle\lambda\rangle$ in the sense that

$$|g_{(x,\xi,\lambda)}(y, \eta, \mu)| \leqslant c \Rightarrow C^{-1}m(x, \xi, \lambda) \leqslant m(x + y, \xi + \eta, \lambda + \mu) \leqslant Cm(x, \xi, \lambda).$$

(See [12] for instance.) We also assume that $\mathcal{B}$ and $\mathcal{H}$ are equipped with $(x, \xi, \lambda; h)$-dependent norms $\|\cdot\|_{m_\mathcal{B}}$, $\|\cdot\|_{m_\mathcal{H}}$ which are equivalent to some fixed norm (may not uniformly in h), respectively. In addition, we assume that the norms are continuous with respect to the metric g, uniformly with respect to h. Then we say that $a \in S_{\Sigma,\delta}(T^*X, m, \mathcal{L}(\mathcal{B},\mathcal{H}))$ if

$$\|a(x,\xi;h)u\|_{m_{\mathcal{H}}(x,\xi,\lambda;h)} \leqslant Cm(x,\xi,\lambda;h)\|u\|_{m_{\mathcal{B}}(x,\xi,\lambda;h)}, \lambda = h^{-\delta}d(\Sigma,\cdot), \text{ for all } u \in \mathcal{B},$$

and if this statement is stable under applications of vector fields in the sense of (3.2.1), namely,

$$\begin{aligned}
&\text{near } \Sigma : V_1 \cdots V_{l_1} W_1 \cdots W_{l_2} a = O_{\mathcal{L}(\mathcal{B},\mathcal{H})}(mh^{-\delta l_1}),\\
&\text{where } V_1,\ldots,V_{l_1} \text{ are vector fields tangent to } \Sigma,\\
&\text{ and } W_1,\ldots,W_{l_2} \text{ are any vector fields;}\\
&\text{away from } \Sigma : \partial_x^\alpha \partial_\xi^\beta a(x,\xi;h) = O_{\mathcal{L}(\mathcal{B},\mathcal{H})}(m\langle\xi\rangle^{-|\beta|}).
\end{aligned}$$

Then we can obtain a class of operators $\Psi_{\Sigma,\delta}(X; m, \mathcal{L}(\mathcal{B},\mathcal{H}))$ and the corresponding principal symbol map

$$\begin{aligned}
\sigma_{\Sigma,h} : &\Psi_{\Sigma,\delta}(X; m, \mathcal{L}(\mathcal{B},\mathcal{H}))\\
&\to S_{\Sigma,\delta}(T^*X; m, \mathcal{L}(\mathcal{B},\mathcal{H}))/S_{\Sigma,\delta}(T^*X; m\langle h^{-\delta}d(\Sigma,\cdot)\rangle^{-1}, \mathcal{L}(\mathcal{B},\mathcal{H})).
\end{aligned}$$

3.3 The Fourier-Bros-Iagolnitzer(FBI) transform

3.3.1 FBI transform on $\mathbb{R}^n$

Let $\phi = \phi(x,y)$ be a holomorphic quadratic form on $\mathbb{C}^n \times \mathbb{C}^n$. Then we can find $n \times n$ matrices A, B, D with complex entries such that $A^T = A$, $D^T = D$ and

$$\phi(x,y) = \frac{1}{2}\langle Ax, x\rangle + \langle Bx, y\rangle + \frac{1}{2}\langle Dy, y\rangle.$$

Here $\langle\cdot,\cdot\rangle$ is the complex bilinear product on $\mathbb{C}^n$ given by

$$\langle x, y\rangle = \sum_{j=1}^n x_j y_j.$$

We assume that

$$\det \frac{\partial^2\phi}{\partial x \partial y} = \det B \neq 0$$

and

$$\operatorname{Im} \frac{\partial^2\phi}{\partial y^2} = \operatorname{Im} D > 0.$$

We also let

$$\Phi(x) = \sup_{y\in\mathbb{R}^n} -\operatorname{Im}\phi(x,y).$$

The supreme is attained at a unique point $y = y(x) \in \mathbb{R}^n$. Now we define the Fourier-Bros-Iagolnitzer transform associated to ϕ to be the operator T defined on $\mathscr{S}'(\mathbb{R}^n)$ by

$$Tu = T_\phi u(x;h) = c_\phi h^{-3n/4} \int_{\mathbb{R}^n} e^{i\phi(x,y)/h} u(y)dy,$$

where

$$c_\phi = 2^{-n/2}\pi^{-3n/4}(\det \operatorname{Im} \partial^2_{xx}\phi)^{-1/4}|\det \partial^2_{xz}\phi|.$$

The image Tu is an entire function and satisfies

$$|Tu(x;h)| \leqslant Ch^{-N}\langle x\rangle^N e^{\Phi(x)/h}$$

for some N depends on u. Moreover, if $u \in L^\infty$,

$$|Tu| \leqslant Ch^{-n/4}\|u\|_{L^\infty} e^{\Phi(x)/h};$$

if $u \in \mathscr{S}(\mathbb{R}^n)$, then for every N, there exists a constant C_N depends on u,

$$|Tu| \leqslant C_N h^{-n/4}\langle x\rangle^{-N} e^{\Phi(x)/h}.$$

We can view T as a semiclassical Fourier integral operator, then the associated linear canonical transformation is given by

$$\kappa_T : \mathbb{C}^{2n} \ni (y, -\phi'_y(x,y)) \mapsto (x, \phi'_x(x,y)) \in \mathbb{C}^{2n}.$$

Then we have

$$\kappa_T(\mathbb{R}^{2n}) = \Lambda_\Phi := \{(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}(x)); x \in \mathbb{C}^n\}.$$

We consider the complex symplectic form

$$\sigma = \sum_{j=1}^n d\eta_j \wedge dy_j$$

on $\mathbb{C}^{2n}$. A submanifold of $\mathbb{C}^{2n}$ is called an IR-manifold if it is Lagrangian for $\operatorname{Im}\sigma$ and symplectic for $\operatorname{Re}\sigma$. An example of IR-manifold is $\mathbb{R}^{2n}$. Since κ_T is canonical for σ, we see Λ_Φ is also an IR-manifold. This means that Φ is strictly plurisubharmonic:

$$\frac{\partial^2\Phi}{\partial\bar{x}\partial x} > 0.$$

To consider the mapping properties of T, we consider the following weighted space $L^2_\Phi(\mathbb{C}^n) = L^2(\mathbb{C}^n, e^{-2\Phi(x)/h}L(dx))$ with Hilbert norm

$$\|u\|^2_{L^2_\Phi} = \int_{\mathbb{C}^n} |u|^2 e^{-2\Phi(x)/h} L(dx).$$

We also consider the space of holomorphic functions in L^2_Φ,

$$H_\Phi(\mathbb{C}^n) = A(\mathbb{C}^n) \cap L^2_\Phi(\mathbb{C}^n).$$

Then T is a unitary operator from $L^2(\mathbb{R}^n) \to H_\Phi(\mathbb{C}^n)$ with formal adjoint

$$T^*v(y) = \bar{c}_\varphi h^{-3n/4} \int_{\mathbb{C}^n} e^{-i\phi^*(x,y)/h} v(x) e^{-2\Phi(x)/h} L(dx)$$

where $\phi^*(x,y) = \overline{\phi(\bar{x},\bar{y})}$ is holomorphic. So $T^*T = I$ and $\Pi = TT^*$ is the orthogonal projection from L^2_Φ onto H_Φ.

Now for the purpose of applying the method of complex scaling, we consider a slight different contour and consider an FBI transform $T : L^2(\Gamma_0) \to H_{\Phi_0}(\mathbb{C}^n)$ where the contour Γ_0 is given by $\operatorname{Im} x = \epsilon_0 \operatorname{Re} x$ where $\epsilon_0 > 0$ is a small fixed number:

$$Tu(x;h) = c_\phi h^{-3n/4} \int_{\Gamma_0} e^{i\phi(x,y)/h} u(x) dx.$$

Then the associated canonical transformation κ_T maps $T^*\Gamma_0 \subset \mathbb{C}^{2n}$ given by

$$T^*\Gamma_0 = \{(1+i\epsilon_0)x, (1-i\epsilon_0)\xi) : (x,\xi) \in \mathbb{R}^{2n}\}$$

to an IR-manifold

$$\Lambda_{\Phi_0} = \{(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x)) : x \in \mathbb{C}^n\}.$$

Here

$$\Phi_0(x) = \sup_{y \in \Gamma_0} -\operatorname{Im} \phi(x,y) = \Phi(x) + O(\epsilon_0 |x|^2)$$

is again a strictly plurisubharmonic quadratic function. All the previous properties also hold for this FBI transform.

Example 3.3.1. *A standard example is given by*

$$\phi(x,y) = \frac{i}{2}(x-y)^2,$$

then

$$\Phi(x) = \frac{1}{2}(\operatorname{Im} x)^2$$

and we get the Bargmann transform

$$Tu(z;h) = 2^{-n/2}(\pi h)^{3n/4} \int_{\mathbb{R}}^{n} e^{-(z-y)^2/2h} u(y) dy$$

We can compute the associated canonical transformation

$$\kappa_T(x,\xi) = (x - i\xi, \xi)$$

and thus it is easy to check that

$$\kappa_T(\mathbb{R}^{2n}) = \Lambda_\Phi = \{(x, -\operatorname{Im} x) : x \in \mathbb{C}^n\}.$$

We also have the FBI transform $T : L^2(\Gamma_0) \to H_{\Phi_0}(\mathbb{C}^n)$ *associated to the contour* Γ_0 *and we have*

$$\Phi_0(x) = \sup_{y\in\Gamma_0} -\operatorname{Im}\phi(x,y) = \frac{1}{2(1-\epsilon_0^2)}(\epsilon_0 \operatorname{Re} x - \operatorname{Im} x)^2.$$

We can also check that

$$\kappa_T(T^*\Gamma_0) = \Lambda_{\Phi_0} = \{(x, \frac{1-i\epsilon_0}{1-\epsilon_0^2}(\epsilon_0 \operatorname{Re} x - \operatorname{Im} x)) : x \in \mathbb{C}^n\}.$$

If we identify $z = x - i\xi \in \mathbb{C}^n$ *with* $(x,\xi) \in T^*\mathbb{R}^n$*, then this FBI transform on* $L^2(\mathbb{R}^n)$ *is related to the classical form of FBI transform (see [19])*

$$\tilde{T}_h u = \tilde{T}u(x,\xi;h) = 2^{-n/2}(\pi h)^{-3n/4}\int_{\mathbb{R}^n} e^{-(x-y)^2/2h + i(x-y)\cdot\xi/h} u(y)dy$$

by

$$\tilde{T}u(x,\xi;h) = e^{-\xi^2/2h} Tu(z;h).$$

The classical FBI transform $\tilde{T}_h$ *is an isometry from* $L^2(\mathbb{R}^n)$ *into* $L^2(\mathbb{R}^{2n})$ *with image*

$$\tilde{T}_h(L^2(\mathbb{R}^n)) = L^2(\mathbb{R}^{2n}) \cap e^{-\xi^2/2h} A(\mathbb{C}^n_{x-i\xi})$$

where $A(\mathbb{C}^n)$ *is the space of entire functions in* $\mathbb{C}^n$*. The adjoint* $\tilde{T}_h^* : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^n)$ *is given by*

$$\tilde{T}_h^* v(y) = 2^{-n/2}(\pi h)^{-3n/4}\int_{\mathbb{R}^{2n}} e^{-(x-y)^2/2h - i(x-y)\cdot\xi/h} v(x,\xi)dxd\xi.$$

3.3.2 Pseudodifferential operators with holomorphic symbols

Now we consider the actions of pseudodifferential operators. We shall only discuss the case for FBI transform on $\mathbb{R}^n$, the situation for Γ_0 is exactly the same.

Let m be an order function, $S(\Lambda_\Phi, m)$ be the space of all smooth functions a on Λ_Φ such that

$$\partial^\alpha a = O_\alpha(m), \quad \forall \alpha \in \mathbb{N}^{2n}.$$

Here we identify Λ_Φ linearly with $\mathbb{R}^{2n}$. Then for $u = O(\langle x\rangle^{-\infty})e^{\Phi(x)/h}$, we put

$$\operatorname{Op}_h(a)u(x) = \frac{1}{(2\pi h)^n}\iint_{\Gamma(x)} e^{i(x-y)\cdot\theta/h} a(\frac{x+y}{2}, \theta)u(y)dyd\theta,$$

where the contour $\Gamma(x)$ is given by

$$\theta = \frac{2}{i}\frac{\partial\Phi}{\partial x}(\frac{x+y}{2}).$$

Let $\tilde{m}$ be a second order function on Λ_Φ, and we view both m and $\tilde{m}$ to be functions on $\mathbb{C}^n_x$ by the natural identification $\pi_x : \Lambda_\Phi \ni (x, \xi) \mapsto x \in \mathbb{C}^n$. We consider the spaces

$$L^2_{\Phi,\tilde{m}} = L^2(\mathbb{C}^n, \tilde{m}^2 e^{-2\Phi/h} L(dx)), \quad H_{\Phi,\tilde{m}} = A(\mathbb{C}^n) \cap L^2_{\Phi,\tilde{m}}.$$

Then for $a \in S(\Lambda_\Phi, m)$, $\mathrm{Op}_h(a)$ extends to a bounded operator from $H_{\Phi,\tilde{m}}$ to $H_{\Phi,\tilde{m}/m}$ with norm uniformly bounded in h. Moreover, if $b \in S(\mathbb{R}^{2n}, m \circ \kappa_T)$ be such that

$$a \circ \kappa_T = b,$$

then

$$\mathrm{Op}_h(a) \circ T = T \circ \mathrm{Op}_h(b).$$

Now we assume that $a(x, \xi)$ is holomorphic in a neighborhood $\Lambda_\Phi + W$ of Λ where W is an open neighborhood of $0 \in \mathbb{C}^n$. Let $a \in S(m)$ in $\Lambda_\Phi + W$, where m is an order function first defined on Λ_Φ, then extended to $\Lambda_\Phi + W$ by $m(x, \xi) = m(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}(x))$.

We can perturb the contour in the definition of pseudodifferential operators by

$$\mathrm{Op}_h(a)u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma_c(x)} e^{i(x-y)\cdot\theta/h} a(\frac{x+y}{2}, \theta) u(y) dy d\theta,$$

where the contour $\Gamma_c(x)$ is given by

$$\Gamma_c(x) : \theta = \frac{2}{i}\frac{\partial \Phi}{\partial x}(\frac{x+y}{2}, \theta) + i\frac{c}{2}(\bar{x} - \bar{y}),$$

so that for c small, $\Gamma_c(x) \subset \Lambda_\Phi + W$.

Consider $\tilde{\Phi}(x) = \Phi(x) + f(x)$ where $f \in C^{1,1}(\mathbb{C}^n)$ is supported in some fixed compact set. Moreover, we assume

$$\|f\|_{C^{1,1}} = \|\nabla f\|_{\mathrm{Lip}} + \sup_{\mathbb{C}^n} |f|$$

is small. We can define the Hilbert spaces $H_{\tilde{\Phi},\tilde{m}}$ which coincides with $H_{\Phi,\tilde{m}}$ as spaces, with equivalent norms for fixed h, but not uniformly as $h \to 0$. The associated Lipschitz manifold

$$\Lambda_{\tilde{\Phi}} : \xi = \frac{2}{i}\frac{\partial \tilde{\Phi}}{\partial x}(x)$$

coincides with Λ_Φ outside a compact set and is close to Λ_Φ in the sense of Lipschitz graphs. We can replace the contour in the definition again by

$$\tilde{\Gamma}_c(x) : \theta = \frac{2}{i}\frac{\partial \tilde{\Phi}}{\partial x}(\frac{x+y}{2}) + ic\frac{\bar{x} - \bar{y}}{\langle x - y \rangle}.$$

Then $\mathrm{Op}_h(a)$ is a bounded operator from $H_{\tilde{\Phi},\tilde{m}}$ to $H_{\tilde{\Phi},\tilde{m}/m}$ with norm uniformly bounded in h. Moreover, we can approximate such operators by multiplication operators

Theorem 1. *Let $a \in S(m)$ be a holomorphic symbol as above and $\tilde{\Phi}$ a small perturbation of Φ, then for $u \in H_{\tilde{\Phi},\tilde{m}}$,*

$$\mathrm{Op}_h(a)u(x) = a(x,\xi(x))u(x) + \sum_{j=1}^{n} \frac{\partial a}{\partial \xi_j}(x,\xi(x))(hD_{x_j} - \xi_j(x))u(x) + Ru(x),$$

where $\xi(x) = \frac{2}{i}\frac{\partial\tilde{\Phi}}{\partial x}$ and $R = O(h) : H_{\tilde{\Phi},\tilde{m}} \to L^2_{\tilde{\Phi},\tilde{m}/m}$.

As a corollary, we get results on scalar products

Proposition 3.3.2. *Let $a \in S(m_a)$ and $b \in S(m_b)$ be holomorphic symbols as above, $\psi(x)$ locally Lipschitz on $\mathbb{C}^n$ such that for some order function m_ψ,*

$$\partial^\alpha \psi(x) = O(m_\psi(x)), \quad |\alpha| \leqslant 1.$$

(1) If m_1 and m_2 are order functions such that $m_\psi m_a \leqslant m_1 m_2$, then for $u \in H_{\tilde{\Phi},m_1}$, $v \in H_{\tilde{\Phi},m_2}$,

$$(\psi \,\mathrm{Op}_h(a)u, v)_{L^2_{\tilde{\Phi}}} = \int_{\mathbb{C}^n} \psi(x)a(x,\xi(x))u(x)\overline{v(x)}e^{-2\tilde{\Phi}(x)/h}L(dx) + O(h)\|u\|_{H_{\tilde{\Phi},m_1}}\|v\|_{H_{\tilde{\Phi},m_2}}.$$

(2) If m_1 and m_2 are order functions such that $m_\psi m_a m_b \leqslant m_1 m_2$, then for $u \in H_{\tilde{\Phi},m_1}$, $v \in H_{\tilde{\Phi},m_2}$,

$$\begin{aligned}(\psi \,\mathrm{Op}_h(a)u, \mathrm{Op}_h(b)v)_{L^2_{\tilde{\Phi}}} = &\int_{\mathbb{C}^n} \psi(x)a(x,\xi(x))u(x)\overline{b(x,\xi(x))v(x)}e^{-2\tilde{\Phi}(x)/h}L(dx) \\ &+ O(h)\|u\|_{H_{\tilde{\Phi},m_1}}\|v\|_{H_{\tilde{\Phi},m_2}}.\end{aligned}$$

(3) In particular, we have for $u \in H_{\tilde{\Phi},m_a}$,

$$\|\mathrm{Op}_h(a)u\|^2_{H_{\tilde{\Phi}}} = \|a(x,\xi(x))u(x)\|^2_{L^2_{\tilde{\Phi}}} + O(h)\|u\|^2_{H_{\tilde{\Phi},m_a}}.$$

3.3.3 FBI transform on a compact smooth manifold

In this section, we consider a form of FBI transform on a compact smooth manifold, which is motivated by the classical form of FBI transform. We follow the presentation in [40].

Now let (X,g) be a compact Riemannian manifold. We write y to be a point on X, dy to be the volume form on X; (x,ξ) a point on T^*X (where $x \in X, \xi \in T^*_x X$) and $dxd\xi$ to be the canonical volume form on T^*X.

Let $\Delta \subset T^*X \times X$ be the diagonal set

$$\Delta = \{(x,\xi,y) \in T^*X \times X : x = y\}.$$

An admissible phase function $\varphi(x, \xi, y)$ is a smooth function on $T^*X \times X$ satisfying the following conditions:

(1) φ is an elliptic polyhomogeneous symbol of order one in ξ;
(2) $\operatorname{Im}\varphi \geqslant 0$;
(3) $d_y\varphi|_\Delta = -\xi dy$;
(4) $d_y^2 \operatorname{Im}\varphi|_\Delta \sim \langle\xi\rangle$;
(5) $\varphi|_\Delta = 0$.

Therefore near the diagonal Δ, $\varphi = \xi \cdot (x-y) + \langle Q(x,\xi,y)(x-y), (x-y)\rangle$ where Q is a symmetric matrix-valued symbol of degree 1 in ξ with $\operatorname{Im} Q|_\Delta \sim \langle\xi\rangle I$.

The symbol class $h^m S^k_{phg}(T^*X \times X)$ is defined to be the collection of all smooth functions

$$a = a(x,\xi,y;h) \sim h^m(a_k(x,\xi,y) + h a_{k-1}(x,\xi,y) + \cdots)$$

where $a_j(x,\xi,y)$ are polyhomogeneous symbols of degree j in ξ and the asymptotic expansion means that

$$|a - h^m(a_k + \cdots + h^j a_{k-j})| \leqslant C_j h^{m+j+1}|\xi|^{k-j-1}, |\xi| > 1.$$

The symbol class $h^m S^k_{phg}(T^*X)$ is defined similarly, without the y-components.

A symbol $a \in h^m S^k_{phg}$ is called elliptic if the principal part $|a_k| \sim \langle\xi\rangle$ uniformly with respect to other variables. The quantization of a is defined as the operator

$$\operatorname{Op}(a)u(y) = \frac{1}{(2\pi h)^n}\int e^{i\xi \exp_x^{-1}(y)/h} a(x,\xi,y;h)u(x)\chi(y,x)dxd\xi,$$

where exp is the exponential map with respect to g on X. We shall write $h^m\Psi^k(X)$ to be the algebra of all operators $\operatorname{Op}(a) + R, a \in h^m S^k, R = O(h^\infty) : C^{-\infty}(X) \to C^\infty(X)$.

Then following [40], an FBI transform on X is an operator $T_h : C^\infty(X) \to C^\infty(T^*X)$ given by

$$T_h u(x,\xi) = \int_X e^{i\varphi(x,\xi,y)/h} a(x,\xi,y;h)\chi(x,\xi,y)u(y)dy. \tag{3.3.1}$$

Here $\varphi(x,\xi,y)$ is an admissible phase function; $a(x,\xi,y;h) \in h^{-3n/4}S^{n/4}$ is an elliptic polyhomogeneous symbol; $\chi(x,\xi,y)$ is a cut-off function to a small neighborhood of the diagonal $\Delta = \{(x,\xi,y) \in T^*X \times X : x = y\}$ such that $\operatorname{Im}\varphi \leqslant -C^{-1}d(x,y)^2$ on the support of χ.

The following properties of the FBI transform were proved in [40]:
(1) $T_h : L^2(X) \to L^2(T^*X)$ is bounded for $h < h_0$;
(2) We can choose a suitable phase φ and elliptic symbol a such that

$$\|T_h u\|_{L^2(T^*X)} = (1 + O(h^\infty))\|u\|_{L^2(X)}, \tag{3.3.2}$$

i.e. T_h is an isometry modulo h^∞.

From now on, we shall always use such kind of FBI transforms. Furthermore, we know from [40]:

Lemma 3.3.3. *Let $P = \mathrm{Op}(p) \in h^k\Psi^m(X)$, then $T_h^* p T_h - P \in h^{k+1}\Psi^{m-1}$.*

We can apply this to prove the following

Proposition 3.3.4. *(1) For any $u \in C^\infty(X)$,*

$$\|\langle\xi\rangle T_h u\|_{L^2(T^*X)} \leqslant C\|u\|_{H_h^1(X)}. \tag{3.3.3}$$

(2) If $A(x, hD_x)$ is a second-order differential operator on X, then for any $u \in C^\infty(X)$,

$$\|A(x,\xi)Tu\|^2_{L^2(T^*X)} = \|A(x,hD_x)u\|^2_{L^2(X)} + O(h)\|u\|^2_{H_h^2}. \tag{3.3.4}$$

Proof. (1)

$$\begin{aligned}\|\langle\xi\rangle T_h u\|^2 &= \langle\langle\xi\rangle T_h u, \langle\xi\rangle T_h u\rangle = \langle T_h^*\langle\xi\rangle^2 T_h u, u\rangle \\ &= \langle(I-\Delta)u, u\rangle + \langle Ru, u\rangle = \|u\|_{H_h^1(X)} + \langle Ru, u\rangle,\end{aligned}$$

where $R \in h\Psi^1$. So $\langle Ru, u\rangle = O(h)\|u\|^2_{H_h^{1/2}(X)}$.

(2) Notice that by symbol calculus

$$(\bar{A}A)(x,hD) = A(x,hD)^*A(x,hD) \mod h\Psi^3$$

we have the following

$$\begin{aligned}\|A(x,\xi)Tu\|^2_{L^2(T^*X)} &= \langle A(x,\xi)Tu, A(x,\xi)Tu\rangle = \langle T^*|A(x,\xi)|^2 Tu, u\rangle \\ &= \langle A(x,hD)^*A(x,hD)u, u\rangle + \langle Ru, u\rangle = \|A(x,hD)u\|^2 + \langle Ru, u\rangle,\end{aligned}$$

where $R \in h\Psi^3$. So $\langle Ru, u\rangle = O(h)\|u\|^2_{H_h^{3/2}(X)}$. □

Remark 3.3.5. *We also notice that all the discussion above work for functions with value in a Hilbert space $\mathscr{H}$. In our case, we shall choose $X = \partial\mathcal{O}$ and $\mathscr{H} = L^2([0,\infty))$.*

Chapter 4

The method of complex scaling

In this section, we review the method of complex scaling to characterize the resonances in scattering by convex obstacles following a series of works of Sjöstrand-Zworski [31, 27, 35]. The advantage of this method is that it gives a characterization of resonances as eigenvalues of a non-selfadjoint differential operator to which we can apply the microlocal techniques for partial differential equations.

4.1 Complex scaling in the black box setting

A smooth manifold $\Gamma \subset \mathbb{C}^n$ is called totally real if

$$T_x\Gamma \cap i(T_x\Gamma) = 0, \quad \forall x \in \Gamma$$

where i denotes the natural action on tangent vectors induced by the multiplication by the imaginary unit. It is clear that if Γ is totally real, then $\dim \Gamma \leqslant n$. A natural example is given by $\Gamma = \mathbb{R}^n$. Since totally real manifolds are mapped to totally real manifolds under holomorphic diffeomorphisms, we can extend the notion of totally real manifolds to submanifolds of complex manifolds.

Let $\Omega \subset \mathbb{C}^n$ be open and $\Gamma \subset \Omega$ be a totally real smooth manifold of dimension n. For a differential operator

$$P(x, D_x) = \sum_{|\alpha| \leqslant m} a_\alpha(x) D_x^\alpha$$

with coefficients holomorphic in Ω, then we can define a differential operator $P_\Gamma : C^\infty(\Gamma) \to C^\infty(\Gamma)$ with smooth coefficients. We identify $T^*\Gamma$ with a submanifold of $\mathbb{C}^n \times \mathbb{C}^n$ via

$$T^*\Gamma \ni (x, d\phi(x)) \mapsto (x, \partial_x \tilde{\phi}(x)) \in \Gamma \times \mathbb{C}^n,$$

where $\phi \in C^\infty(\Gamma; \mathbb{R})$ and $\tilde{\phi}$ is an almost analytic extension of ϕ which can be defined since Γ is totally real. Then the principal symbol p_Γ of the differential operator P_Γ is given by

$$p_\Gamma = p|_{T^*\Gamma}.$$

The key result is the following deformation result which is standard and proved in [31].

Proposition 4.1.1. *Let $\omega \subset \mathbb{R}^n$ be an open set, $f : [0,1] \times \omega \ni (t,y) \mapsto f(t,y) \in \mathbb{C}^n$ be a smooth proper map such that*

(1) $\det(\partial_y f(t,y)) \neq 0$ for all (t,y);

(2) $f(t,\cdot)$ is injective;

(3) $f(t,y) = f(0,y)$ for $y \in \omega \setminus K$ where K is a compact subset of ω.

We write $\Gamma_t = f(\{t\} \times \omega)$. Let $P(x, D_x)$ be a partial differential operator with holomorphic coefficients defined in a neighborhood of $f([0,1]\times\omega)$ such that $P|_{\Gamma_t}$ is elliptic for $0 \leqslant t \leqslant 1$. If $u_0 \in \mathscr{D}'(\Gamma_0)$ and $P_{\Gamma_0} u_0$ extends to a holomorphic function in a neighborhood of $f([0,1] \times \omega)$, then u_0 extends to a holomorphic function in a neighborhood of $f([0,1] \times \omega)$.

Now we construct the complex scaled operator for the black box operator. To do this we need an additional assumption on the analyticity of the coefficients:

> There exists a constant $C > 0$ such that the coefficients $a_\alpha(x;h)$ extend holomorphically in x to $\{x \in \mathbb{C}^n; |\operatorname{Im} x| < C^{-1}\langle \operatorname{Re} x\rangle\}$, and the relevant estimates remain valid.

Now we construct the contour Γ_θ for the complex scaled operator. For given $\epsilon_0 > 0$, $R_1 > R_0$, we can construct a smooth function $f : [0,\pi] \times [0,\infty) \ni (\theta, t) \mapsto f_\theta(t) \in \mathbb{C}$, injective for every θ such that

(1) $f_\theta(t) = t$ for $0 \leqslant t \leqslant R_1$;

(2) $0 \leqslant \arg f_\theta(t) \leqslant \theta, \partial_t f_\theta \neq 0$;

(3) $\arg f_\theta(t) \leqslant \arg \partial_t f_\theta(t) \leqslant \arg f_\theta(t) + \epsilon_0$,

(4) $f_\theta(t) = e^{i\theta} t$, for $t \geqslant T_0$ where T_0 only depends on ϵ_0 and R_1.

Consider the map

$$\kappa_\theta : \mathbb{R}^n \ni x = r\omega \mapsto f_\theta(r)\omega \in \mathbb{C}^n$$

where $(r = |x|, \omega = x/|x|) \in (0,\infty) \times \mathbb{S}^{n-1}$ is the standard polar coordinates on $\mathbb{R}^n$. The image $\Gamma_\theta = \kappa_\theta(\mathbb{R}^n)$ is a totally real manifold of dimension n which coincides with $\mathbb{R}^n$ along $B(0, R_1)$.

For $0 < \theta < \theta_0$, where θ_0 is small, so that the coefficients of Q has analytic extension to all Γ_θ, we define the complex scaled Hilbert space to be

$$\mathcal{H}_\theta = \mathcal{H}_{R_0} \oplus L^2(\Gamma_\theta \setminus B(0,R_0)).$$

Via the map κ_θ, we can identify $\mathcal{H}_\theta$ with $\mathcal{H}$ and define $\mathcal{D}_\theta = \mathcal{D}$. Let $\chi \in C_0^\infty(B(0,R_1))$ be equal to 1 on a neighborhood of $\overline{B(0,R_0)}$, then we can define the complex scaled operator $P_\theta : \mathcal{H} \to \mathcal{H}$ with domain $\mathcal{D}$ by

$$P_\theta u = P(\chi u) + Q_{\Gamma_\theta}(1-\chi)u.$$

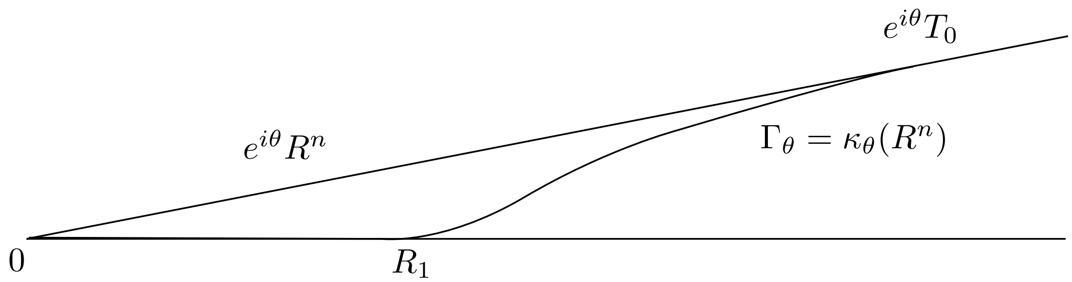

Figure 4.1: The contour for complex scaling on $\mathbb{R}^n$.

We start with the Laplacian $-\Delta_{\Gamma_\theta}$. Again, via the map κ_θ, we see that in polar coordinates

$$-\Delta_{\Gamma_\theta} = \left(\frac{1}{f'_\theta(r)} D_r\right)^2 - \frac{n-1}{f_\theta(r) f'_\theta(r)} i D_r + (f_\theta(r))^{-2} \Delta_{\mathbb{S}^{n-1}}$$

with principal symbol

$$p_{0,\theta} = \frac{r^{*2}}{f'_\theta(t)^2} + \frac{\omega^{*2}}{f_\theta(t)^2},$$

where (r^*, ω^*) are the dual variable of (r, ω). If $\epsilon > 0$ is small, then $-\Delta_{\Gamma_\theta}$ is elliptic. Its principal symbol $p_{0,\theta}$ takes values in a sector of angle $2\epsilon_0$ for any fixed r, and globally in

$$-2(\theta + \epsilon_0) \leqslant \arg p_{0,\theta} \leqslant 0;$$

near infinity it satisfies $\arg p_{0,\theta} = -2\theta$.

Now if we choose R_1 large enough, then in $\mathbb{R}^n \setminus B(0, R_0)$, $h^{-2} P_\theta$ is an elliptic differential operator with classical principal symbol takes values in a sector of angle $\leqslant 3\epsilon_0$ over any fixed point in Γ_θ, and globally in the sector

$$-2\theta - 3\epsilon_0 \leqslant \arg z \leqslant \epsilon_0.$$

When $x \in \Gamma_\theta \to \infty$, all the coefficients of $P_\theta - e^{-2i\theta}(-h^2 \Delta_{\mathbb{R}^n})$ tend to zero uniformly in h.

From [26], we have

Proposition 4.1.2. *If $z \in \mathbb{C} \setminus e^{-2i\theta}[0, \infty)$, then $P_\theta - z : \mathcal{D} \to \mathcal{H}$ is a Fredholm operator of index 0. In particular, such a z belongs to $\sigma(P_\theta)$ if and only if*

$$\ker(P_\theta - z) \neq 0,$$

i.e z is an eigenvalue of P_θ. Moreover, if $0 \leqslant \theta_1 < \theta_2 \leqslant \theta_0$, and $z \in \mathbb{C} \setminus e^{-2i[\theta_1,\theta_2]}[0,\infty)$, then

$$\dim\ker(P_{\theta_2} - z) = \dim\ker(P_{\theta_1} - z).$$

Therefore, we have a discrete subset $\mathrm{Res}(P,\theta_0)$ of the sector $e^{-2i[0,\theta_0)}[0,\infty)$ which agrees with $\sigma(P_\theta)$ in $e^{-2i[0,\theta)}[0,\infty)$ for any $\theta \in [0,\theta_0]$ counting multiplicity. Moreover, this is exactly the set of resonances of P_θ.

Proposition 4.1.3. *The resolvent $R(z)$ of the black box Hamiltonian P, first defined for $\operatorname{Im} z > 0$, can be extended meromorphically across $(0,+\infty)$ to the sector $e^{-i[0,2\theta_0)}(0,+\infty)$ with poles in $\mathrm{Res}(P,\theta_0)$ counting multiplicity.*

4.2 Complex scaling all the way to the obstacle

From now on, we let $\mathcal{O}$ be a strictly convex, bounded open set in $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{O}$, then $d(x) = \mathrm{dist}(x,\mathcal{O})$ is a smooth convex function in $\mathbb{R}^n \setminus \mathcal{O}$. Moreover, $d''(x) \geqslant 0$ and $\dim\ker d''(x) = 1$, generated by $x - y(x)$ where $y(x)$ is the closest point to x on $\partial\mathcal{O}$, so that $d(x) = |x - y(x)|$. At $y(x)$, the exterior unit normal vector of $\partial\mathcal{O}$ is

$$\nu(y(x)) = \nabla d(y(x)) = \frac{x - y(x)}{|x - y(x)|}.$$

We introduce the following normal geodesic coordinates on the exterior domain $\mathbb{R}^n \setminus \mathcal{O}$:

$$x = (x', x_n) \mapsto x' + x_n \nu(x'), \quad x' = y(x) \in \partial\mathcal{O}, \quad x_n = d(x, \partial\mathcal{O}),$$

where $\nu(x') = \nu(y(x))$ is the exterior unit normal vector to $\mathcal{O}$ at x'. Then in this coordinates, we have the following expression for the Laplacian operator in the exterior domain

$$-\Delta_{\mathbb{R}^n\setminus\mathcal{O}} = D_{x_n}^2 + R(x', D_{x'}) - 2x_n Q(x_n, x', D_{x'}) + G(x_n, x')D_{x_n},$$

where $R(x', D_{x'})$, $Q(x_n, x', D_{x'})$ are second order operators on $\partial\mathcal{O}$:

$$R(x', D_{x'}) = -\Delta_{\partial\mathcal{O}} = (\det(g^{ij}))^{1/2} \sum_{i,j=1}^{n-1} D_{y_i} (\det(g_{ij}))^{1/2} g^{ij} D_{y_j}$$

is the Laplacian with respect to the induced metric $g = (g_{ij})$ on $\partial\mathcal{O}$ and $Q(x', D_{x'}) = Q(0, x', D_{x'})$ is of the form

$$\det(g^{ij})^{1/2} \sum_{i,j=1}^{n-1} D_{y'_j} (\det(g_{ij}))^{1/2} a_{ij} D_{y'_i}$$

in any local coordinates such that the principal symbol of Q is the second fundamental form of $\partial\mathcal{O}$ lifted by the duality to $T^*\partial\mathcal{O}$:

$$Q(x', \xi') = \sum_{i,j=1}^{n-1} a_{ij}(x')\xi_i\xi_j.$$

Thus the principal curvatures of $\partial\mathcal{O}$ are the eigenvalues of the quadratic form $Q(x', \xi')$ with respect to the quadratic form $R(x', \xi')$.

Now following we introduce the family of complex scaling contours $(0 < \theta < \theta_0)$

$$\Gamma_\theta = \{z = x + i\theta f'(x) : x \in \mathbb{R}^n \setminus \mathcal{O}\} \subset \mathbb{R}^n \setminus \mathcal{O} + i\mathbb{R}^n$$

where $f : \mathbb{R}^n \setminus \mathcal{O} \to \mathbb{R}$ is a smooth function such that for x near $\mathcal{O}$, $f(x) = \frac{1}{2}d(x)^2$, so $f'(x) = d(x)d'(x)$.

If we parametrize Γ_θ by $x \in \mathbb{R}^n \setminus \mathcal{O}$, then we can compute the principal symbol of $-\Delta|_{\Gamma_\theta}$:

$$p_\theta(x, \xi) = \langle (1 + i\theta f''(x))^{-1})\xi, (1 + i\theta f''(x))^{-1}\xi\rangle = a_\theta - ib_\theta$$

where

$$\begin{aligned} a_\theta &= \langle (1 - (\theta f''(x))^2\tilde{\xi}, \tilde{\xi}\rangle, \\ b_\theta &= 2\theta\langle f''(x)\tilde{\xi}, \tilde{\xi}\rangle. \\ \tilde{\xi} &= (1 + (\theta f''(x))^2)^{-1}\xi. \end{aligned}$$

In normal geodesic coordinates, we can write the contours Γ_θ as the image of

$$U \times [0, \infty) \ni (x', x_n) \mapsto x' + (1 + i\theta)x_n\nu(x') \in \mathbb{C}^n,$$

locally for x_n small. In the global version, by rescaling $t = |(1 + i\theta)|x_n$, the contours Γ_θ are the image of

$$\partial\mathcal{O} \times [0, \infty) \ni (x', t) \mapsto x' + g_\theta(t)\nu(x') \in \mathbb{C}^n,$$

where $g_\theta : [0, \infty) \to \mathbb{C}$ is a smooth injective map such that

$$|g'_\theta| = 1, \quad g(0) = 0, \quad g(t) = t\frac{1 + i\theta}{|1 + i\theta|}$$

for t near 0;

$$g(t) = t\frac{1 + i\varphi(\theta)}{|1 + i\varphi(\theta)|}$$

outside a small neighborhood of 0 and

$$\begin{aligned} \arg(1 + i\varphi(\theta)) \leqslant \arg g(t) \leqslant \arg(1 + i\theta), \\ \frac{1}{2}\arg(1 + i\varphi(\theta)) \leqslant \arg g'(t) \leqslant \arg(1 + i\theta). \end{aligned}$$

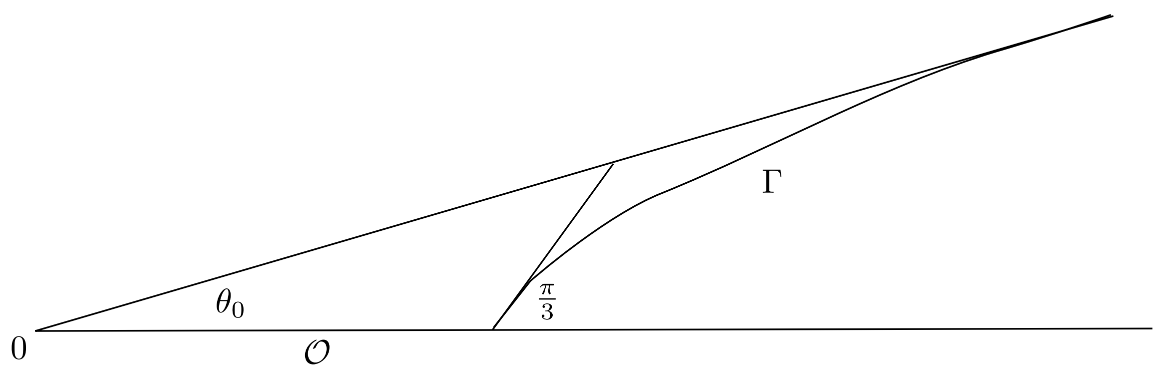

Figure 4.2: The contour for complex scaling all the way to the obstacle.

We shall choose $\varphi(\theta)$ small enough such that Γ_θ satisfy the conditions given above. In particular, we shall work on the contour $\Gamma = \Gamma_\theta$ for $\theta = \theta_1$ such that $g = g_{\theta_1}(t)$ equals to $te^{\pi i/3}$ for t near 0 (later on we shall use $t \in [0, L^{-1}]$ for L large enough).

Now we introduce the semiclassical parameter h, then on this contour Γ,

$$\begin{aligned}-h^2\Delta|_\Gamma = \frac{1}{g'(t)^2}(hD_t)^2 + R(x', hD_y) - 2g(t)Q(y, hD_y)\\ + O(t^2(hD_y)^2) + O(h)hD_{y,t} + O(h^2),\end{aligned}$$

which is elliptic in both semiclassical sense and the usual sense.

For t small such that $g(t) = te^{\pi i/3}$,

$$-h^2\Delta|_\Gamma = e^{-2\pi i/3}((hD_{x_n})^2 + 2x_nQ(x_n, x', hD_{x'}; h)) + R(x', hD_{x'}, h) + hF(x_n, x')hD_{x_n}.$$

Let p be the principal symbol of $-h^2\Delta|_\Gamma$. We notice that as in [27], if $g = g_{\theta_1}, \varphi = \varphi(\theta_1)$ are chosen suitably, then p lies in the lower half plane and for every $\delta > 0$, there exists $\epsilon > 0$ such that

$$t \geqslant \delta \Rightarrow \epsilon \leqslant -\arg p \leqslant \pi - \epsilon.$$

Moreover, we can extend Γ to a totally real submanifold $\tilde{\Gamma}$ in $\mathbb{C}^n$ such that the classical symbol of $-\Delta|_{\tilde{\Gamma}}$ takes values in

$$\arg(1 + i\theta) \leqslant -\arg z \leqslant \frac{2}{3}\pi + \epsilon.$$

Now we consider the boundary condition. In the normal geodesic coordinate $(x', x_n) \in \partial\mathcal{O} \times [0, \infty)$, the Robin boundary condition becomes

$$\partial_{x_n} u + \eta u|_{x_n=0} = 0$$

Therefore in the scaled operator, we shall choose the boundary condition

$$e^{-\pi i/3}\partial_{x_n}u + \eta u|_{x_n=0} = 0$$

or

$$\partial_\nu u + ku|_{\partial\Gamma} = 0 \tag{4.2.1}$$

where $k : \partial\Gamma \to \mathbb{C}$ is a smooth function.

We shall define the scaled operator $P = P_\Gamma = -\Delta|_\Gamma : \mathcal{D}_\eta(\Gamma) \to L^2(\Gamma)$ where

$$\mathcal{D}(\Gamma) = \{u \in H^2(\Gamma) | \partial_\nu u + ku|_{\partial\Gamma} = 0\}.$$

Again, we regard P as an operator on $\mathbb{R}^n \setminus \mathcal{O}$ by the parametrization of Γ given above.

Proposition 4.2.1. *The spectrum of $-\Delta_\Gamma$ is discrete in $-2\theta_0 < \arg z < 0$ and the resonances of $-\Delta_{\mathbb{R}^n\setminus\mathcal{O}}$ in the sector $-\theta_0 < \arg\zeta < 0$ are the same as the square root of the eigenvalues of $-\Delta_\Gamma$ with corresponding boundary condition in $-2\theta_0 < \arg z < 0$. Moreover, they have the same multiplicities:*

$$m_{\mathcal{O}}(\zeta) = m(z) := \operatorname{tr} \frac{1}{2\pi i} \oint_{|\tilde{z}-z|=\epsilon} (-\Delta_\Gamma - \tilde{z})^{-1} d\tilde{z}$$

where $z = \zeta^2$, $0 < \epsilon \ll 1$ so that there are no other eigenvalues of $-\Delta_\Gamma$ in $|\tilde{z} - z| \leqslant \epsilon$.

To prove the proposition, we only need the following lemma from [27] and combine it with proposition 4.1.3.

Lemma 4.2.2. *Let $u \in C^\infty(\mathbb{R}^n \setminus \mathcal{O})$ satisfy $(-\Delta - \lambda^2)^{k_0}u = 0, \partial^\alpha u|_{\partial\Omega} = \bar{u}_\alpha \in C^\infty(\partial\Omega)$ in a neighborhood of x_0. Then there exists a complex neighborhood W of x_0 such that (1) u extends holomorphically to a function U in a complex open neighborhood of $W \cap \bigcup_{|\theta|\leqslant\theta_0} \Gamma_\theta^0$;*
(2) $u_\theta = U|_{\Gamma_\theta}$ is smooth up to $\partial\Gamma_\theta = \partial\mathcal{O}$;
(3) $(-\Delta|_{\Gamma_\theta} - \lambda^2)^{k_0}u_\theta = 0, \partial^\alpha u_\theta|_{\partial\Gamma_\theta} = \bar{u}_\alpha$ in $\Gamma_\theta \cap W$.
Moreover, we may replace $\mathbb{R}^n \setminus \mathcal{O}$ by any fixed Γ_η with $|\eta| < \theta_0$.

Part (1) and the first equation in (3) follows from Lemma 4.1.1 by choosing intermediate contours between Γ_θ and $\mathbb{R}^n \setminus \mathcal{O}$ away from the boundary. The difficulty lies in justification of the boundary condition for which we need to estimate the norm of u_θ. To do this, we first review the strong uniqueness property of the scaled operator and its corollary.

Proposition 4.2.3. *Assume P is an m-th order differential operator with holomorphic coefficients, Γ a totally real submanifold of $\mathbb{C}^n$ of maximal dimension such that $P|_\Gamma$ is elliptic. Then if $u \in \mathscr{D}'(\Gamma)$ satisfies $P_\Gamma u = 0$ on Γ and $u = 0$ in a neighborhood of some $x_0 \in \Gamma$, then $u \equiv 0$.*

Corollary 4.2.4. *Let P, Γ be as in the proposition above, $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset \Gamma$ are open sets, then there exists some constant $C > 0$ such that for all $u \in H^m(\Omega_3)$,*

$$\|u\|_{H^m(\Omega_2)} \leqslant C(\|Pu\|_{H^0(\Omega_3)} + \|u\|_{H^0(\Omega_3 \setminus \Omega_1)}). \tag{4.2.2}$$

Also if $P, \Gamma, \Omega_1, \Omega_2, \Omega_3$ depends continuously on some parameters varying in some compact set, then we have the estimate for some constant C independent of the parameters.

This corollary shows that if $Pu = 0$, then the part of u in $\Omega_3 \setminus \Omega_1$ controls the whole part of u in Ω_2. We shall apply this property to a family of intermediate contours between Γ_θ and $\mathbb{R}^n \setminus \mathcal{O}$ and use the part in $\mathbb{R}^n \setminus \mathcal{O}$ to control the part in Γ_θ.

Now we describe our family of intermediate contours. To do this, we first blow up a neighborhood of x_0 by introducing the following change of variables:

$$x \mapsto \tilde{x}, x = y + \epsilon\tilde{x}$$

where $y \in \partial\mathcal{O}$ is some boundary point near x_0, $\epsilon > 0$ is a parameter which we will let tend to 0. We shall choose $\tilde{x}_0$ such that $|\tilde{x}_0| = 1$ is close to the normal direction to boundary through y and focus on the region $B(\tilde{x}_0, 1)$ in the new coordinates.

The intermediate contours are constructed as follows: Let $\Gamma_{\theta,y,\epsilon}$ be the image of Γ_θ in the complexified $\tilde{x}$-space. Since Γ_θ is parameterized by $z = x + i\theta f'(x)$, we have the following parametrization of $\Gamma_{\theta,y,\epsilon}$:

$$\tilde{z} = \tilde{x} + i\theta\epsilon^{-1} f'(y + \epsilon\tilde{x}) = \tilde{x} + i\theta\partial_{\tilde{x}} f_{\epsilon,y}(\tilde{x})$$

where $f_{\epsilon,y}(\tilde{x}) = \epsilon^{-2} f(y + \epsilon\tilde{x})$. The derivatives of $f_{\epsilon,y}$ can be estimated as follows:

$$\partial_{\tilde{x}} f_{\epsilon,y}(\tilde{x}) = \begin{cases} O(|\tilde{x}|^{2-|\alpha|}) & \text{if} \quad |\alpha| \leqslant 2 \\ O(\epsilon^{|\alpha|-2}) & \text{if} \quad |\alpha| \geqslant 2 \end{cases}.$$

We choose a cut-off function $\chi \in C_0^\infty(B(\tilde{x}_0, \frac{1}{2})), 0 \leqslant \chi \leqslant 1$ and $\chi \equiv 1$ on $B(\tilde{x}_0, \frac{1}{4})$. Out intermediate contours will be the image of

$$\tilde{x} \mapsto \tilde{z} = \tilde{x} + i\theta\partial_{\tilde{x}}(\chi f_{\epsilon,y}(\tilde{x}))$$

and we will write $\Omega_0, \Omega_1, \Omega_2, \Omega_3$ to be the images of the balls $B(\tilde{x}_0, \frac{1}{4})$, $B(\tilde{x}_0, \frac{1}{2})$, $B(\tilde{x}_0, \frac{5}{8})$, $B(\tilde{x}_0, \frac{3}{4})$, respectively. See Figure 2(where we omit Ω_2).

By the strong uniqueness property of $(-\Delta_{\tilde{z}} - \epsilon^2\lambda^2)^{k_0}$ and (4.2.2), we have the following estimates uniformly with respect to y in a neighborhood of x_0 and ϵ small,

$$\|v\|_{H^{2k_0}(\Omega_2)} \leqslant C(\|(-\Delta_{\tilde{z}} - \epsilon^2\lambda^2)^{k_0}\|_{H^0(\Omega_3)} + \|v\|_{H^0(\Omega_3 \setminus \Omega_1)}).$$

We shall only use the following weak version:

$$\|v\|_{L^2(\Omega_1)} \leqslant C(\|(-\Delta_{\tilde{z}} - \epsilon^2\lambda^2)^{k_0}\|_{L^2(\Omega_3)} + \|v\|_{L^2(\Omega_3 \setminus \Omega_1)}). \tag{4.2.3}$$

Since in the $\tilde{x}$ coordinates, $\tilde{u}(\tilde{x}) = u(x)$ satisfies the equation $(-\Delta_{\tilde{x}} - \epsilon^2\lambda^2)^{k_0}\tilde{u} = 0$. By Lemma 4.1.1, $\tilde{u}$ extends to a holomorphic solution $\tilde{U}$ of $(-\Delta_{\tilde{z}} - \epsilon^2\lambda^2)^{k_0}\tilde{U} = 0$ over a neighborhood of a family of intermediate contours between $B(\tilde{x}_0, \frac{3}{4})$ and $\Gamma^{\circ}_{\theta,y,\epsilon}$ including $\Omega_2 \subset \tilde{\Gamma}_{\theta,y,\epsilon}$. Back to the x-coordinates, we get the holomorphic extension U of u in $W \cap \Gamma^{\circ}_{\theta}$ if we let y varies near x_0 and ϵ goes to 0.

Substitute $v = \tilde{U}_{\Omega_3}$ in (4.2.3), noticing that $\Omega_3 \setminus \Omega_1 \subset \mathbb{R}^n_{\tilde{x}}$, so $\tilde{U} = \tilde{u}$ on $\Omega_3 \setminus \Omega_1$, we have

$$\|\tilde{U}\|_{L^2(\Omega_1)} \leqslant C\|\tilde{u}\|_{L^2(\Omega_3\setminus\Omega_1)}.$$

Back to x-coordinates, we will have similar estimate for U and u with the same constant C uniformly for all y near x_0 and $\epsilon > 0$. In particular, this shows that $u_\theta = U|_{\Gamma_\theta}$ is well-defined in L^2 near x_0. Since u_θ satisfies the non-characteristic equation $(-\Delta_z - \lambda^2)u_\theta = 0$, if we identify Γ_θ with $\mathbb{R}^n \setminus \mathcal{O}$ and use the normal geodesic coordinate (x', x_n) (again, only locally near x_0), then $u_\theta \in C([0, \epsilon_0); \mathscr{D}'(\mathbb{R}^{n-1}))$. In particular, it has a boundary value $u_\theta(x', 0) \in \mathscr{D}'(\mathbb{R}^{n-1})$. (For the proof of this, see e.g. [11].)

Now it only remains to show that $u_\theta(x', 0)$ coincides with the original boundary value $\bar{u}(x')$. To do this, we substitute $v = \tilde{U} - \tilde{u}(0)$ in (4.2.3), noticing that $\tilde{u}(0) = \bar{u}(y)$,

$$(-\Delta_{\tilde{z}} - \epsilon^2\lambda^2)^{k_0}(\tilde{U} - \tilde{u}(0)) = -(-\epsilon^2\lambda^2)^{k_0}\tilde{u}(0),$$

we have

$$\|\tilde{U} - \tilde{u}(0)\|_{L^2(\Omega_1)} \leqslant C((\epsilon^2\lambda)^{k_0}|\tilde{u}(0)| \operatorname{vol}(\Omega_3)^{\frac{1}{2}} + \|\tilde{u} - \tilde{u}(0)\|_{L^2(\Omega_3\setminus\Omega_1)}).$$

Since $\tilde{u}$ is smooth, the last term tends to 0 when $\epsilon \to 0$. Therefore

$$\operatorname{vol}(\Omega_0)^{-1}\|\tilde{U} - \tilde{u}(0)\|^2_{L^2(\Omega_1)} = o(1), \epsilon \to 0.$$

Back to x-space and use the normal geodesic coordinates (x', x_n), we have

$$\epsilon^{-n}\|u_\theta(x) - \bar{u}(x')\|^2_{B((x',\epsilon),\frac{\epsilon}{4})} = o(1), \epsilon \to 0 \tag{4.2.4}$$

uniformly in x'. Now let $\chi_n \in C_0^\infty(\mathbb{R}), \chi'(x') \in C_0^\infty(\mathbb{R}^{n-1})$ be cut-off functions with support close to 1 and 0, respectively. Also let $\int \chi_n(x_n)dx_n = \int \chi'(x')dx' = 1$. Then for any test function $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$,

$$\begin{aligned}\langle u_\theta(x', 0) - \bar{u}(x'), \varphi(x')\rangle &= \lim_{\epsilon\to 0}(u_\theta(x) - \bar{u}(x'))\epsilon^{-1}\chi_n(\epsilon^{-1}x_n)\varphi(x')dx \\ &= \lim_{\epsilon\to 0}\int \epsilon^{-n}\int(u_\theta(x) - \bar{u}(x'))\chi_n(\epsilon^{-1}x_n)\chi'(\epsilon^{-1}(x' - y'))\varphi(x')dxdy' = 0,\end{aligned}$$

since $\epsilon^{-n}\int(u_\theta(x) - \bar{u}(x'))\chi_n(\epsilon^{-1}x_n)\chi'(\epsilon^{-1}(x' - y'))\varphi(x')dx$ has a uniformly compact support with respect to y' and tends to zero uniformly in y' by Cauchy-Schwarz inequality and (4.2.4).

To get the desired global deformation with suitable boundary condition, we only need to glue all the local deformation together using the strong uniqueness property.

For higher order derivatives, we can repeat the argument for every $\partial^\alpha u$ which satisfies the differential equation $(-\Delta - \lambda^2)^{k_0}(\partial^\alpha u) = 0$ to get the holomorphic extension of $\partial^\alpha u$. The strong uniqueness property shows that this is exactly the derivative of the holomorphic extension of u.

Chapter 5

Euclidean case: Resonance-free region

In this chapter, we prove the following theorem on resonance-free region in the Euclidean case under general Robin boundary conditions.

Theorem 2. *Let $\mathcal{O}$ be a bounded, strictly convex open set with a smooth boundary $\partial\mathcal{O}$. Let $P = P^{(\eta)}$ be the Laplacian $-\Delta_{\mathbb{R}^n\backslash\mathcal{O}}$ on the exterior region with Robin boundary condition $\partial_\nu u + \eta u|_{\partial\mathcal{O}} = 0$. Then there are no resonances of P in the cubic region*

$$\operatorname{Im}\zeta > -S|\zeta|^{1/3} + C, \tag{5.0.1}$$

for some $C > 0$, depending on η and $\mathcal{O}$. The constant S is given by

$$S = \kappa\zeta_1' = 2^{-1/3}\cos(\pi/6)\,\zeta_1' \left(\min_{y\in\partial\mathcal{O},i=1,\dots,n-1} K_i(y)\right)^{2/3} \tag{5.0.2}$$

with $K_i(y)$ the principal curvatures of $\partial\mathcal{O}$ at y, $-\zeta_1'$ the first zero of the derivative of the Airy function.

5.1 Estimates for model Airy operators

In this section, we give the lower bounds for the ordinary differential operator

$$P = e^{-2\pi i/3}((hD_t)^2 + t) + \mathcal{O}(h)hD_t + \mathcal{O}(h + t^2) \tag{5.1.1}$$

defined on $[0,\infty)$ with general conditions at $t = 0$.

5.1.1 The Dirichlet and Neumann realization

We start by considering the semiclassical version of the Airy differential operator $(hD_t)^2 + t$. By changing the variables $t = h^{2/3}s$, we have

$$(hD_t)^2 + t = h^{2/3}(D_s^2 + s).$$

For $u = u(t)$, we define $v(s) = h^{1/3}u(h^{2/3}s)$, then

$$u(t) = h^{-1/3}v(h^{-2/3}t), \quad \|u\|_{L^2_t} = \|v\|_{L^2_s}$$

and

$$((hD_t)^2 + t)u(t) = h^{1/3}(D_s^2 + s)v(s).$$

Therefore

$$\langle((hD_t)^2 + t)u, u\rangle_{L^2_t} = h^{2/3}\langle(D_s^2 + s)v, v\rangle_{L^2_s}.$$

Applying the estimates (2.2.7) and (2.2.8), we have for $u \in C_0^\infty[0, \infty)$, if $u(0) = 0$, then

$$\langle((hD_t)^2 + t)u, u\rangle \geqslant \zeta_1 h^{2/3}\|u\|^2 \tag{5.1.2}$$

if $D_t u(0) = 0$, then

$$\langle((hD_t)^2 + t)u, u\rangle \geqslant \zeta_1' h^{2/3}\|u\|^2 \tag{5.1.3}$$

We also have the following useful identity: for $u \in C_0^\infty([0, \infty)), u(0) = 0$ or $D_t u(0) = 0$,

$$\langle((hD_t)^2 + t)u, u\rangle = \langle(hD_t)^2 u, u\rangle + \langle tu, u\rangle = \|hD_t u\|^2 + \|t^{1/2}u\|^2,$$

and consequently

$$\langle((hD_t)^2 + t)u, u\rangle \geqslant \|hD_t u\|^2; \tag{5.1.4}$$

$$\langle((hD_t)^2 + t)u, u\rangle \geqslant \|t^{1/2}u\|^2. \tag{5.1.5}$$

Now we could estimate $\|((hD_t)^2 + t)u\|$ by the Cauchy-Schwartz inequality

$$\|((hD_t)^2 + t)u\|\|u\| \geqslant \langle((hD_t)^2 + t)u, u\rangle.$$

If $u(0) = 0$, then by (5.1.2),

$$\|((hD_t)^2 + t)u\| \geqslant \zeta_1 h^{2/3}\|u\|, \tag{5.1.6}$$

and by (5.1.4)

$$\|((hD_t)^2 + t)u\|^2 \geqslant \zeta_1 h^{2/3}\langle((hD_t)^2 + t)u, u\rangle \geqslant \zeta_1 h^{2/3}\|hD_t u\|^2,$$

thus

$$\|((hD_t)^2 + t)u\| \geqslant \sqrt{\zeta_1} h^{1/3}\|hD_t u\| \tag{5.1.7}$$

Similarly, if $D_t u(0) = 0$, by (5.1.3) and (5.1.4), we have

$$\|((hD_t)^2 + t)u\| \geqslant \zeta_1' h^{2/3}\|u\|; \tag{5.1.8}$$

$$\|((hD_t)^2 + t)u\| \geqslant \sqrt{\zeta_1'} h^{1/3}\|hD_t u\|. \tag{5.1.9}$$

Another way to estimate $\|((hD_t)^2+t)u\|$ is to use the following identity: If $u(0)=0$ or $D_tu(0)=0$, since $\langle u, hD_tu\rangle$ is real

$$\begin{aligned}\|((hD_t)^2+t)u\|^2 &= \|(hD_t)^2u\|^2+\|tu\|^2+2\operatorname{Re}\langle tu,(hD_t)^2u\rangle\\ &= \|(hD_t)^2u\|^2+\|tu\|^2+2\operatorname{Re}\langle hD_t(tu),hD_tu\rangle\\ &= \|(hD_t)^2u\|^2+\|tu\|^2+2\operatorname{Re}\langle thD_tu,hD_tu\rangle+h\operatorname{Re}\frac{2}{i}\langle u,hD_tu\rangle\\ &= \|(hD_t)^2u\|^2+\|tu\|^2+2\|t^{1/2}hD_tu\|^2.\end{aligned} \tag{5.1.10}$$

This gives us the following estimates

$$\|((hD_t)^2+t)u\|^2 \geqslant \|(hD_t)^2u\|^2+\|tu\|^2 \geqslant \|(hD_t)^2u\|^2. \tag{5.1.11}$$

5.1.2 General condition

Now we remove the Dirichlet or Neumann condition at $t=0$ and try to get a lower bound of $\langle((hD_t)^2+t)u,u\rangle$. In this case, $(hD_t)^2+t$ is no longer a self-adjoint operator, but the semiclassical setting allows us to view it as a perturbation of the Neumann realization. We shall start with the following basic estimate:

$$\begin{aligned}\|hD_tu\|^2 = \langle hD_tu,hD_tu\rangle &= \langle(hD_t)^2u,u\rangle - ih^2D_tu(0)\bar u(0)\\ &\leqslant \langle((hD_t)^2+t)u,u\rangle - ih^2D_tu(0)\bar u(0).\end{aligned}$$

Since the right hand side is real, we have

$$\begin{aligned}\|hD_tu\|^2 &\leqslant \operatorname{Re}\langle((hD_t)^2+t)u,u\rangle - \operatorname{Re}(ih^2D_tu(0)\bar u(0))\\ &\leqslant \operatorname{Re}\langle((hD_t)^2+t)u,u\rangle + h^2|D_tu(0)||u(0)|\\ &\leqslant \operatorname{Re}\langle((hD_t)^2+t)u,u\rangle + \mathcal{O}(h^2)|D_tu(0)|^2+\mathcal{O}(h^2)|u(0)|^2,\end{aligned}$$

or

$$\operatorname{Re}\langle((hD_t)^2+t)u,u\rangle \geqslant \|hD_tu\|^2-\mathcal{O}(h^2)|D_tu(0)|^2-\mathcal{O}(h^2)|u(0)|^2. \tag{5.1.12}$$

which is the analogue of (5.1.4) for general u. Next we try to get an analogue of (5.1.2) and (5.1.3) with bounary terms.

Lemma 5.1.1. *Suppose $u\in C_0^\infty([0,\infty))$, then we have the following estimate:*

$$\operatorname{Re}\langle((hD_t)^2+t)u,u\rangle \geqslant \zeta_1' h^{2/3}(1-\mathcal{O}(h^{2/3}))\|u\|^2-\mathcal{O}(h^2)|D_tu(0)|^2, \tag{5.1.13}$$

$$|\operatorname{Im}\langle((hD_t)^2+t)u,u\rangle| \leqslant \mathcal{O}(h^{2/3})\operatorname{Re}\langle((hD_t)^2+t)u,u\rangle+\mathcal{O}(h^2)|D_tu(0)|^2. \tag{5.1.14}$$

Proof. Write $D_tu(0)=a$ for simplicity. First, we use the scaling $t=h^{2/3}s$, $v(s)=h^{1/3}u(h^{2/3}s)$ as before. We have

$$\langle((hD_t)^2+t)u,u\rangle_{L^2_t} = h^{2/3}\langle(D_s^2+s)v,v\rangle_{L^2_s},$$

and more importantly, $D_s v(0) = hD_t u(0) = ha$. Now let $v(s) = w(s) + has\chi(s)$ where $\chi \in C_0^\infty([0,\infty))$ is a fixed function such that $\chi \equiv 1$ near 0. Then we get the decomposition

$$\begin{aligned}&\langle (D_s^2+s)v, v\rangle = \\ &\quad \langle (D_s^2+s)w, w\rangle + ha\langle (D_s^2+s)s\chi, w\rangle + h\bar{a}\langle (D_s^2+s)w, s\chi\rangle + h^2|a|^2\langle (D_s^2+s)s\chi, s\chi\rangle.\end{aligned}$$

Since w satisfies the Neumann condition $D_s w(0) = 0$, we know the first term is real. We can integrate by parts to rewrite the third term as $-h\bar{a}\langle D_s w, D_s(s\chi)\rangle + h\bar{a}\langle w, s^2\chi\rangle$. Therefore for the real part, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\operatorname{Re}\langle (D_s^2+s)v, v\rangle &\geqslant \langle (D_s^2+s)w, w\rangle - \mathcal{O}(h)|a|\|D_s w\| - \mathcal{O}(h)|a|\|w\| - \mathcal{O}(h^2)|a|^2 \\ &\geqslant \langle (D_s^2+s)w, w\rangle - \mathcal{O}(h^{4/3})|a|^2 - \mathcal{O}(h^{2/3})(\|w\|^2 + \|D_s w\|^2).\end{aligned}$$

From

$$\begin{aligned}\|D_s w\|^2 = \langle D_s w, D_s w\rangle = \langle D_s^2 w, w\rangle &\leqslant \langle (D_s^2+s)w, w\rangle \\ \|w\|^2 \leqslant \zeta_1'^{-1}\langle (D_s^2+s)w, w\rangle & \qquad \text{(by (2.2.7))},\end{aligned}$$

we have

$$\operatorname{Re}\langle (D_s^2+s)v, v\rangle \geqslant (1-\mathcal{O}(h^{2/3}))\langle (D_s^2+s)w, w\rangle - \mathcal{O}(h^{4/3})|a|^2. \tag{5.1.15}$$

By (2.2.7) and

$$\begin{aligned}\|w\|^2 &\geqslant (\|v\| - \|has\chi\|)^2 = \|v\|^2 - \mathcal{O}(h)|a|\|v\| + \mathcal{O}(h^2)|a|^2 \\ &\geqslant (1-\mathcal{O}(h^{2/3}))\|v\|^2 - \mathcal{O}(h^{4/3})|a|^2,\end{aligned}$$

we get

$$\begin{aligned}\operatorname{Re}\langle (D_s^2+s)v, v\rangle &\geqslant \zeta_1'(1-\mathcal{O}(h^{2/3}))\|w\|^2 - \mathcal{O}(h^{4/3})|a|^2 \\ &\geqslant \zeta_1'(1-\mathcal{O}(h^{2/3}))\|v\|^2 - \mathcal{O}(h^{4/3})|a|^2.\end{aligned}$$

For the imaginary part, we have

$$\begin{aligned}|\operatorname{Im}\langle (D_s^2+s)v, v\rangle| &\leqslant \mathcal{O}(h^{4/3})|a|^2 + \mathcal{O}(h^{2/3})(\|w\|^2 + \|D_s w\|^2) \\ &\leqslant \mathcal{O}(h^{4/3})|a|^2 + \mathcal{O}(h^{2/3})\langle (D_s^2+s)w, w\rangle.\end{aligned}$$

Using (5.1.15) again, we have

$$|\operatorname{Im}\langle (D_s^2+s)v, v\rangle| \leqslant \mathcal{O}(h^{4/3})|a|^2 + \mathcal{O}(h^{2/3})\operatorname{Re}\langle (D_s^2+s)v, v\rangle.$$

Scaling back from v to u, we get the desired estimates (5.1.13) and (5.1.14). □

Finally, we need an analogue of (5.1.11). In the equation (5.1.10), we show that

$$\|((hD_t)^2+t)u\|^2 = \|(hD_t)^2u\|^2 + \|tu\|^2 + 2\|t^{1/2}hD_tu\|^2 + h\operatorname{Re}\frac{2}{i}\langle u, hD_tu\rangle.$$

Although the last term is no longer zero, we can calculate it through integration by parts. Since

$$\langle u, hD_tu\rangle = \langle hD_tu, u\rangle - hi|u(0)|^2,$$

we have

$$\operatorname{Im}\langle u, hD_tu\rangle = -\frac{h}{2}|u(0)|^2.$$

Therefore

$$\operatorname{Re}\frac{2}{i}\langle u, hD_tu\rangle = -h|u(0)|^2,$$

and we get

$$\|((hD_t)^2+t)u\|^2 \geqslant \|(hD_t)^2u\|^2 - h^2|u(0)|^2 \tag{5.1.16}$$

5.1.3 Restriction to a small interval

Now we restrict the support of u to a small fixed interval and get a better estimate.

Lemma 5.1.2. *If $L>0$ is sufficiently large, $0<h<h_0(L)$, then the following estimates holds uniformly for $u\in C_0^\infty([0,L^{-1}])$:*

$$\operatorname{Re}\langle((hD_t)^2+t)u,u\rangle \geqslant (\zeta_1'h^{2/3}-\mathcal{O}(hL))\|u\|^2 - \mathcal{O}(h^2)|D_tu(0)|^2 + \frac{L}{2}\|tu\|^2. \tag{5.1.17}$$

Proof. We choose $\chi_0,\chi_1\in C^\infty(\mathbb{R})$ such that

$$\begin{aligned}\chi_0^2+\chi_1^2 = 1, 0\leqslant \chi_j &\leqslant 1\\ \chi_0\equiv 1 &\text{ on } (-\infty, h^{1/2}]\\ \chi_1\equiv 1 &\text{ on } [2h^{1/2},\infty)\\ \partial^\alpha\chi_j &= \mathcal{O}_\alpha(h^{-\frac{\alpha}{2}}), \alpha=0,1,2.\end{aligned}$$

Then we can deduce that

$$\chi_0[\chi_0,(hD_t)^2] + \chi_1[\chi_1,(hD_t)^2] = -[\chi_0(hD_t)^2(\chi_0) + \chi_1(hD_t)^2(\chi_1)] = \mathcal{O}(h).$$

We notice that

$$\begin{aligned}\langle((hD_t)^2+t)u,u\rangle &= \langle\chi_0((hD_t)^2+t)u,\chi_0u\rangle + \langle\chi_1((hD_t)^2+t)u,\chi_1u\rangle\\ &= \langle((hD_t)^2+t)\chi_0u,\chi_0u\rangle + \langle((hD_t)^2+t)\chi_1u,\chi_1u\rangle\\ &\quad - \langle(\chi_0[\chi_0,(hD_t)^2] + \chi_1[\chi_1,(hD_t)^2])u,u\rangle.\end{aligned}$$

Since $\chi_1 u(0) = 0$, also $\chi_0 u$ and u have the same condition at $t = 0$, we can apply the estimates (5.1.5) and (5.1.13) to get

$$\mathrm{Re}\langle((hD_t)^2+t)u,u\rangle \geqslant \zeta_1' h^{2/3}(1-\mathcal{O}(h^{2/3}))\|\chi_0 u\|^2+\|t^{1/2}\chi_1 u\|^2-\mathcal{O}(h^2)|D_t u(0)|^2-\mathcal{O}(h)\|u\|^2.$$

From the construction of χ_0, χ_1, it is easy to see

$$\|u\|^2 = \|\chi_0 u\|^2 + \|\chi_1 u\|^2, \quad \|tu\|^2 = \|t\chi_0 u\|^2 + \|t\chi_1 u\|^2.$$

Therefore we only need to prove for some $c_0 = \mathcal{O}(L)$, we have

$$c_0 h\|\chi_0 u\|^2 \geqslant \frac{L}{2}\|t\chi_0 u\|^2 \tag{5.1.18}$$

and

$$\|t^{1/2}\chi_1 u\|^2 \geqslant (\zeta_1' h^{2/3} - c_0 h)\|\chi_1 u\|^2 + \frac{L}{2}\|t\chi_1 u\|^2. \tag{5.1.19}$$

Since χ_0 is supported on $[-\infty, 2h^{1/2}]$, we only need to choose $c_0 = 2L$ to get (5.1.18). To prove (5.1.19), we need to show that for $t \in [h^{1/2}, L^{-1}]$,

$$t \geqslant \zeta_1' h^{2/3} - c_0 h + \frac{L}{2}t^2$$

or equivalently,

$$\frac{L}{2}(t-L^{-1})^2 \leqslant \frac{1}{2L} + 2hL - \zeta_1' h^{2/3}.$$

The left hand side achieves its maximum at $t = h^{1/2}$, so we only need

$$h^{1/2} \geqslant \zeta_1' h^{2/3} - c_0 h + \frac{L}{2}h$$

which can be achieved by choosing $h < h_0(L)$ small enough. This finishes the proof. □

5.1.4 Airy operator with lower order terms

Now we shall include the lower order terms in (5.1.1) and prove the main result of this section.

Proposition 5.1.3. *Let P be in the form* (5.1.1) *and $\omega_0 \in \mathbb{C}$ with $\arg\omega_0 \in (-\pi/6, 5\pi/6)$. If $L > 0$ is sufficiently large, $h > 0$ sufficiently small depending on L and ω_0, then for $u \in C_0^\infty([0, L^{-1}))$,*

$$\begin{aligned}\|(P-\omega_0)u\|^2 \geqslant{}& (|e^{2\pi i/3}\omega_0 - \zeta_1' h^{2/3}|^2 - \mathcal{O}(hL))\|u\|^2 \\ &+ C^{-1}\|(hD_t)^2 u\|^2 - \mathcal{O}(h^2)|D_t u(0)|^2 - \mathcal{O}(h^2)|u(0)|^2.\end{aligned} \tag{5.1.20}$$

Proof. We begin with the following identity

$$\begin{aligned}\|(P-\omega_0)u\|^2 &= \|(e^{-2\pi i/3}((hD_t)^2+t)-\omega_0)u\|^2 + \|[\mathcal{O}(h)hD_t+\mathcal{O}(h+t^2)]u\|^2 \\ &\quad + 2\operatorname{Re}\langle (e^{-2\pi i/3}((hD_t)^2+t)-\omega_0)u, [\mathcal{O}(h)hD_t+\mathcal{O}(h+t^2)]u\rangle \\ &\geqslant \|(e^{-2\pi i/3}((hD_t)^2+t)-\omega_0)u\|^2 - [\mathcal{O}(h)\langle((hD_t)^2+t)u, hD_tu\rangle \\ &\quad + \mathcal{O}(h)\langle u, hD_tu\rangle + \langle((hD_t)^2+t)u, \mathcal{O}(h+t^2)u\rangle + \langle u, \mathcal{O}(h+t^2)u\rangle].\end{aligned}$$

The lower order terms are estimated as follows,

$$\begin{aligned}\mathcal{O}(h)\langle((hD_t)^2+t)u, hD_tu\rangle &\leqslant \mathcal{O}(h)\|((hD_t)^2+t)u\|^2 + \mathcal{O}(h)\|hD_tu\|^2 \\ &\leqslant \mathcal{O}(h)\|((hD_t)^2+t)u\|^2 + \mathcal{O}(h)\operatorname{Re}\langle((hD_t)^2+t)u,u\rangle \\ &\quad + \mathcal{O}(h^3)|D_tu(0)|^2 + \mathcal{O}(h^3)|u(0)|^2 \quad \text{(by (5.1.12))}; \\ \mathcal{O}(h)\langle u, hD_tu\rangle &\leqslant \mathcal{O}(h)\|u\|^2 + \mathcal{O}(h)\|hD_tu\|^2 \\ &\leqslant \mathcal{O}(h)\|u\|^2 + \mathcal{O}(h)\operatorname{Re}\langle((hD_t)^2+t)u,u\rangle \\ &\quad + \mathcal{O}(h^3)|D_tu(0)|^2 + \mathcal{O}(h^3)|u(0)|^2 \quad \text{(by (5.1.12))}; \\ \langle((hD_t)^2+t)u, \mathcal{O}(h+t^2)u\rangle &\leqslant \mathcal{O}(h)\langle((hD_t)^2+t)u,u\rangle + \mathcal{O}(1)\langle((hD_t)^2+t)u, t^2u\rangle \\ &\leqslant \mathcal{O}(h)\|((hD_t)^2+t)u\|^2 + \mathcal{O}(h)\|u\|^2 \\ &\quad + \tfrac{1}{2}\|((hD_t)^2+t)u\|^2 + \mathcal{O}(1)\|t^2u\|^2, \\ \langle u, \mathcal{O}(h+t^2)u\rangle &\leqslant \mathcal{O}(h)\|u\|^2 + \mathcal{O}(1)\|tu\|^2.\end{aligned}$$

For the leading terms, we use the following identities

$$\begin{aligned}\|(e^{-2\pi i/3}((hD_t)^2+t)-\omega_0)u\|^2 &= \|((hD_t)^2+t)u\|^2 + |\omega_0|^2\|u\|^2 \\ &\quad - 2\operatorname{Re}\langle e^{-2\pi i/3}((hD_t)^2+t)u, \omega_0u\rangle,\end{aligned}$$

and

$$\begin{aligned}-2\operatorname{Re}\langle e^{-2\pi i/3}((hD_t)^2+t)u, \omega_0u\rangle &= 2\operatorname{Re}[e^{\pi i/3}\bar{\omega}_0\langle((hD_t)^2+t)u,u\rangle] \\ &= \operatorname{Re}(2e^{\pi i/3}\bar{\omega}_0)\cdot\operatorname{Re}\langle((hD_t)^2+t)u,u\rangle \\ &\quad - \operatorname{Im}(2e^{\pi i/3}\bar{\omega_0})\cdot\operatorname{Im}\langle((hD_t)^2+t)u,u\rangle.\end{aligned}$$

By (5.1.14), the second term is bounded below by

$$-2|\omega_0||\operatorname{Im}\langle((hD_t)^2+t)u,u\rangle| \geqslant -\mathcal{O}(h^{2/3})\operatorname{Re}\langle((hD_t)^2+t)u,u\rangle - \mathcal{O}(h^2)|D_tu(0)|^2.$$

Therefore

$$\begin{aligned}&-2\operatorname{Re}\langle e^{-2\pi i/3}((hD_t)^2+t)u, \omega_0u\rangle \\ &\quad\geqslant (2\cos((\pi/3)-\arg\omega_0) - \mathcal{O}(h^{2/3}))|\omega_0|\operatorname{Re}\langle((hD_t)^2+t)u,u\rangle - \mathcal{O}(h^2)|D_tu(0)|^2.\end{aligned}$$

Now combining all the terms together, we get the following estimate

$$\begin{aligned}\|(P-\omega_0)u\|^2 \geqslant &(2\cos((\pi/3)-\arg\omega_0)-\mathcal{O}(h^{2/3}))|\omega_0|\operatorname{Re}\langle((hD_t)^2+t)u,u\rangle\\ &+(|\omega_0|^2-\mathcal{O}(h))\|u\|^2+(\tfrac{1}{2}-\mathcal{O}(h))\|((hD_t)^2+t)u\|^2\\ &-\mathcal{O}(1)\|tu\|^2-\mathcal{O}(1)\|t^2u\|^2-\mathcal{O}(h^2)|D_tu(0)|^2-\mathcal{O}(h^3)|u(0)|^2.\end{aligned}$$

Since $|(\pi/3)-\arg\omega_0|<\pi/2, \cos((\pi/3)-\arg\omega_0)>0$, when h is small enough, we can apply (5.1.17) to the first term and use

$$|\omega_0|^2-2\operatorname{Re}(e^{-2\pi i/3}\bar{\omega_0})\zeta_1'h^{2/3}=|e^{2\pi i/3}\omega_0-\zeta_1'h^{2/3}|^2-\mathcal{O}(h)$$

to get

$$\begin{aligned}\|(P-\omega_0)u\|^2\geqslant &\left((|e^{2\pi i/3}\omega_0-\zeta_1'h^{2/3}|^2-\mathcal{O}(hL)\right)\|u\|^2+((1/2)-\mathcal{O}(h))\,\|((hD_t)^2+t)u\|^2\\ &+(|\omega_0|\cos((\pi/3)-\arg\omega_0)-\mathcal{O}(h))L-\mathcal{O}(1))\,\|tu\|^2\\ &-\mathcal{O}(1)\|t^2u\|^2-\mathcal{O}(h^2)|D_tu(0)|^2-\mathcal{O}(h^3)|u(0)|^2.\end{aligned}$$

Since u is supported in $[0,L^{-1}]$, we have $\|t^2u\|^2\leqslant L^{-2}\|tu\|^2$. So if $L>L_0(\omega_0)$ large enough, $h<h_0(L,\omega_0)$ small enough, we get

$$\begin{aligned}\|(P-\omega_0)u\|^2\geqslant &(|e^{2\pi i/3}\omega_0-\zeta_1'h^{2/3}|^2-\mathcal{O}(hL))\|u\|^2+((1/2)-\mathcal{O}(h))\,\|((hD_t)^2+t)u\|^2\\ &+C^{-1}L\|tu\|^2-\mathcal{O}(h^2)|D_tu(0)|^2-\mathcal{O}(h^3)|u(0)|^2.\end{aligned}$$

Applying (5.1.16), we conclude the proof of (5.1.20). □

Remark 5.1.4. *If we replaced $(hD_t)^2+t$ by $(hD_t)^2+Qt$, then the estimates (with $\zeta_1'h^{2/3}$ replaced by $\zeta_1'Q^{2/3}h^{2/3}$) remain uniform for*

$$Q\in[C^{-1},C],\quad |\omega_0|\in[C^{-1},C]\ \textit{and}\ \ \arg(\omega_0)\in[-(\pi/6)+\delta,(5\pi/6)-\delta]$$

if $\delta,C>0$ are fixed.

Remark 5.1.5. *For the Dirichlet and Neumann realization, we can get the same estimate without the last two lower order terms $-\mathcal{O}(h^2)|D_tu(0)|^2-\mathcal{O}(h^2)|u(0)|^2$. For Dirichlet realization, we can also improve ζ_1' to ζ_1.*

5.2 Lower bounds on the scaled operator

In section 4.2, we defined the scaled operator $P=-\Delta|_\Gamma$ and find the explicit formula in normal coordinates with respect to the boundary: $\Omega\ni x\mapsto(y,t)\in\partial\mathcal{O}\times(0,\infty)$. If we freeze $(y,\eta)\in T^*\partial\mathcal{O}$, then the (semiclassical) symbol of h^2P is given by

$$P(y,t;\eta,hD_t)=e^{-2\pi i/3}((hD_t)^2+2tQ(y,\eta))+R(y,\eta)+\mathcal{O}(t^2+h)\langle\eta\rangle^2+\mathcal{O}(h)hD_t. \quad (5.2.1)$$

In this section, we first estimate $P(y,t;\eta,hD_t)-\omega_0$ and then through the FBI transform T_h defined in (3.3.1) to get a lower bound on $h^2P-\omega_0$.

5.2.1 Estimate in the glancing region

When $\operatorname{Re}\omega_0 - R(y,\eta)$ is small, the principal part of $P(y,t;\eta,hD_t)-\omega_0$ is given by the Airy-type operator $e^{-2\pi i/3}((hD_t)^2+2tQ(y,\eta))$. We shall apply our previous results to get the following estimate. We notice that such (y,η) lies in a compact subset of $T^*\partial\mathcal{O}$.

Lemma 5.2.1. *Let $\omega_0\in\mathbb{C}$ with $\operatorname{Re}\omega_0>0, \operatorname{Im}\omega_0=r_0>0$. Suppose $|\operatorname{Re}\omega_0 - R(x',\xi')|<c$ where c is sufficiently small, L is large enough and $0<h<h_0(L)$. Then For any $v\in C_0^\infty([0,L^{-1}))$, we have*

$$\begin{aligned}\|(P(y,t,\eta,hD_t)-\omega_0)v\|^2 \geqslant{}& |r_0+2S(\operatorname{Re}\omega_0)^{2/3}h^{2/3}-\mathcal{O}(h)|^2\|v\|^2\\ &+C^{-1}\|(hD_t)^2v\|^2-\mathcal{O}(h^2)|D_tv(0)|^2-\mathcal{O}(h^2)|v(0)|^2\end{aligned}\tag{5.2.2}$$

where $S=\kappa\zeta_1'$ is given by (5.0.2).

Proof. Let $c=r_0\tan(\pi/6)$, then since $|\operatorname{Re}\omega_0-R(y,\eta)|<c$, we have

$$\arg(\omega_0-R(y,\eta))\in[\pi/3,2\pi/3].$$

It follows immediately from (5.1.20) by replacing ω_0 with $\omega_0-R(y,\eta)$ that

$$\begin{aligned}\|(P(y,t,\eta,hD_t)-\omega_0)v\|^2 \geqslant{}& (|\omega_0-R(y,\eta)-e^{-2\pi i/3}\zeta_1'(2Q(y,\eta))^{2/3}h^{2/3}|^2-\mathcal{O}(hL))\|v\|^2\\ &+C^{-1}\|(hD_t)^2v\|^2-\mathcal{O}(h^2)|D_tv(0)|^2-\mathcal{O}(h^2)|v(0)|^2.\end{aligned}$$

The uniformity of the constants follows from the ellipticity of Q and R.

Now we need to find a uniform lower bound for

$$\begin{aligned}&|\omega_0-R(y,\eta)-e^{-2\pi i/3}\zeta_1'(2Q(y,\eta))^{2/3}h^{2/3}|^2\\ ={}&|\operatorname{Re}\omega_0-R(y,\eta)+\sin\left(\frac{\pi}{6}\right)\zeta_1'(2Q(y,\eta))^{2/3}h^{2/3}|^2\\ &+|r_0+\cos\left(\frac{\pi}{6}\right)\zeta_1'(2Q(y,\eta))^{2/3}h^{2/3}|^2\end{aligned}\tag{5.2.3}$$

over (y,η) such that $|\operatorname{Re}\omega_0-R(y,\eta)|<c$. The minimum is obtained at $R(y,\eta)=\operatorname{Re}\omega_0+\mathcal{O}(h^{2/3})$ and the minimum of $\zeta_1'(2Q(y,\eta))^{2/3}$ under such constraint. Since the principal curvatures are the eigenvalues of the quadratic form $Q(y,\eta)$ with respect to the quadratic form $R(y,\eta)$, we have

$$Q(y,\eta)\geqslant\left(\min_{y\in\partial\mathcal{O},j=1,\dots,n-1}K_j(y)\right)R(y,\eta).$$

Thus

$$(5.2.3)\geqslant|r_0+2S(\operatorname{Re}\omega_0)^{2/3}h^{2/3}|^2+\mathcal{O}(h^{4/3})$$

which completes the proof of (5.2.2). $\square$

5.2.2 Estimate away from the glancing region

When $|\operatorname{Re}\omega_0 - R(y,\eta)|$ is bounded from below, $2tQ(y,\eta)$ is dominated by $R(y,\eta)-\omega_0$ for small t. In this case, we can give a better estimate for P from $e^{-2\pi i/3}(hD_t)^2 + R(y,\eta)-\omega_0$.

Lemma 5.2.2. *Suppose that $\omega_0\in\mathbb{C}$ with $\operatorname{Re}\omega_0>0, \operatorname{Im}\omega_0 = r_0>0$, $|\operatorname{Re}\omega_0 - R(y,\eta)|>c$, then for L large enough, h sufficiently small, and $v\in C_0^\infty([0,L^{-1}))$,*

$$\begin{aligned}\|(P(y,t,\eta,hD_t)-\omega_0)v\|^2 \geqslant (r_0+C^{-1})^2\|v\|^2 + C^{-1}(\|(hD_t)^2v\|^2 + \langle\eta\rangle^4\|v\|^2)\\ - \mathcal{O}(h^2)\langle\eta\rangle^2|v(0)||D_tv(0)|.\end{aligned}\tag{5.2.4}$$

Proof. Since

$$[P(y,t,\eta,hD_t)-\omega_0]v = [e^{-2\pi i/3}(hD_t)^2 + R(y,\eta)-\omega_0]v + [\mathcal{O}(t^2+t+h)\langle\eta\rangle^2 + \mathcal{O}(h)hD_t]v,$$

we have

$$\begin{aligned}\|(P(y,t,\eta,hD_t)-\omega_0)v\|^2 \geqslant (\|[e^{-2\pi i/3}(hD_t)^2 + R(y,\eta)-\omega_0]v\|\\ - \|[\mathcal{O}(t^2+t+h)\langle\eta\rangle^2 + \mathcal{O}(h)hD_t]v\|)^2.\end{aligned}\tag{5.2.5}$$

Now

$$\begin{aligned}\|[e^{-2\pi i/3}(hD_t)^2 + R(y,\eta)-\omega_0]v\|^2 = \|(hD_t)^2v\|^2 + |R(y,\eta)-\omega_0|^2\|v\|^2\\ + 2\operatorname{Re}[e^{-2\pi i/3}(R(y,\eta)-\bar\omega_0)\langle(hD_t)^2v,v\rangle],\end{aligned}\tag{5.2.6}$$

where

$$\begin{aligned}\operatorname{Re}[e^{-2\pi i/3}(R(y,\eta)-\bar\omega_0)\langle(hD_t)^2v,v\rangle] = \operatorname{Re}[e^{-2\pi i/3}(R(y,\eta)-\bar\omega_0)]\operatorname{Re}\langle(hD_t)^2v,v\rangle\\ - \operatorname{Im}[e^{-2\pi i/3}(R(y,\eta)-\bar\omega_0)]\operatorname{Im}\langle(hD_t)^2v,v\rangle.\end{aligned}\tag{5.2.7}$$

Notice that

$$\langle(hD_t)^2v,v\rangle = \|hD_tv\|^2 + ih^2D_tv(0)\overline{v(0)}.$$

Therefore

$$\operatorname{Re}\langle(hD_t)^2v,v\rangle = \|hD_tv\|^2 + \operatorname{Re}(ih^2D_tv(0)\overline{v(0)}) \geqslant -h^2|D_tv(0)||v(0)|;\tag{5.2.8}$$

$$|\operatorname{Im}\langle(hD_t)^2v,v\rangle| = |\operatorname{Im}(ih^2D_tv(0)\overline{v(0)})| \leqslant h^2|D_tv(0)||v(0)|.\tag{5.2.9}$$

We can compute that

$$\operatorname{Re}[e^{-2\pi i/3}(R(y,\eta)-\bar\omega_0)] = -\frac{1}{2}(R(y,\eta)-\operatorname{Re}\omega_0) + \frac{\sqrt{3}}{2}r_0.$$

In the identity

$$\begin{aligned}\operatorname{Re}[e^{-2\pi i/3}(R(y,\eta)-\bar\omega_0)]\operatorname{Re}\langle(hD_t)^2v,v\rangle = &-\frac{1}{2}(R(y,\eta)-\operatorname{Re}\omega_0)\operatorname{Re}\langle(hD_t)^2v,v\rangle\\ &+\frac{\sqrt{3}}{2}r_0\operatorname{Re}\langle(hD_t)^2v,v\rangle,\end{aligned}$$

we apply (5.2.8) to the second term and

$$|(R(y,\eta)-\operatorname{Re}\omega_0)\operatorname{Re}\langle (hD_t)^2v,v\rangle| \leqslant \frac{1}{2}(\|(hD_t)^2v\|^2+|(R(y,\eta)-\operatorname{Re}\omega_0)|^2\|v\|^2)$$

to the first term, and get

$$\begin{aligned}\operatorname{Re}[e^{-2\pi i/3}(R(y,\eta)-\bar{\omega}_0)]\operatorname{Re}\langle (hD_t)^2v,v\rangle \geqslant &-\frac{1}{4}(\|hD_tv\|^2+|R(y,\eta)-\operatorname{Re}\omega_0|^2\|v\|^2)\\ &-h^2\frac{\sqrt{3}}{2}r_0|D_tv(0)||v(0)|.\end{aligned} \tag{5.2.10}$$

By (5.2.9), we also have

$$-\operatorname{Im}[e^{-2\pi i/3}(R(y,\eta)-\bar{\omega}_0)]\operatorname{Im}\langle (hD_t)^2v,v\rangle \geqslant -h^2|R(y,\eta)-\omega_0||D_tv(0)||v(0)|. \tag{5.2.11}$$

Combining (5.2.6),(5.2.7),(5.2.10),(5.2.11) together, we have

$$\begin{aligned}\|[e^{-2\pi i/3}(hD_t)^2+R(y,\eta)-\omega_0]v\|^2 \geqslant &\frac{1}{2}\|(hD_t)^2v\|^2+(r_0^2+\frac{1}{2}|R(y,\eta)-\operatorname{Re}\omega_0|^2)\|v\|^2\\ &-h^2(|R(y,\eta)-\omega_0|+\sqrt{3}r_0)|D_tv(0)||v(0)|.\end{aligned} \tag{5.2.12}$$

Now we estimate the remainder terms, since $|R(y,\eta)-\operatorname{Re}\omega_0|>c$ and $R(y,\eta)$ is a quadratic form in η, we have

$$C^{-1}\langle\eta\rangle^2 \leqslant |R(y,\eta)-\operatorname{Re}\omega_0| \leqslant C\langle\eta\rangle^2 \tag{5.2.13}$$

for some constant $C>0$ independent of y and η. Therefore if L is large enough and h is small enough, we have for $v\in C_0^\infty([0,L^{-1}])$,

$$\|[\mathcal{O}(t^2+t+h)\langle\eta\rangle^2+\mathcal{O}(h)hD_t]v\|^2 \leqslant C^{-2}|R(y,\eta)-\operatorname{Re}\omega_0|^2\|v\|^2+\mathcal{O}(h^2)\|hD_tv\|^2.$$

We can apply

$$\langle (hD_t)^2v,v\rangle = \|hD_tv\|^2+ih^2D_tv(0)\overline{v(0)}$$

again to get

$$\|hD_tv\|^2 \leqslant |\langle (hD_t)^2v,v\rangle|+h^2|D_tv(0)||v(0)| \leqslant \frac{1}{2}(\|(hD_t)^2v\|^2+\|v\|^2)+h^2|D_tv(0)||v(0)|.$$

Therefore

$$\begin{aligned}\|[\mathcal{O}(t^2+t+h)\langle\eta\rangle^2+\mathcal{O}(h)hD_t]v\|^2 \leqslant &C^{-2}(|R(y,\eta)-\operatorname{Re}\omega_0|^2\|v\|^2\\ &+\|(hD_t)^2v\|^2)+\mathcal{O}(h^4)|D_tv(0)||v(0)|.\end{aligned} \tag{5.2.14}$$

It is easy to prove the following elementary inequality:

$$(\sqrt{a-h^2b}-\sqrt{C^{-2}a+h^4b})^2 \geqslant (1-2C^{-1})a-2h^2b \tag{5.2.15}$$

for C large and h small, independent of $a, b > 0$.

Applying (5.2.15) with

$$a = \frac{1}{2}\|(hD_t)^2 v\|^2 + (r_0^2 + \frac{1}{2}|R(y,\eta) - \operatorname{Re}\omega_0|^2)\|v\|^2, \quad b = (|R(y,\eta) - \omega_0| + \frac{\sqrt{3}}{2} r_0)|D_t v(0)||v(0)|,$$

by (5.2.5),(5.2.12) and (5.2.14), we have

$$\begin{aligned}\|(P(y,t,\eta,hD_t) - \omega_0)v\|^2 \geqslant (1 - 2C^{-1})[\frac{1}{2}\|(hD_t)^2 v\|^2 + (r_0^2 + \frac{1}{2}|R(y,\eta) - \operatorname{Re}\omega_0|^2)\|v\|^2] \\ -\mathcal{O}(h^2)(|R(y,\eta) - \omega_0| + \sqrt{3}r_0)|D_t v(0)||v(0)|.\end{aligned}$$

Now by our assumption, $|R(y,\eta) - \operatorname{Re}\omega_0| > c = r_0\tan(\pi/6)$, and (5.2.13), we finish the proof of the lemma. □

5.2.3 Lower bounds for the scaled operator near the boundary

We first consider an estimate valid for functions supported in a sufficiently small neighborhood of the boundary.

Proposition 5.2.3. *Suppose that* $u \in C^\infty(\mathbb{R}^n \setminus \mathcal{O})$ *satisfies* $\operatorname{supp}(u) \subset \partial\mathcal{O} \times [0, L^{-1})$, *and the Robin boundary condition* $\partial_\nu u + ku|_{\partial\mathcal{O}} = 0$ *for* $k \in C^\infty(\partial\mathcal{O}, \mathbb{C})$. *Then*

$$\|(h^2P - \omega_0)u\|^2 \geqslant |r_0 + S(\operatorname{Re}\omega_0)^{2/3}h^{2/3} - \mathcal{O}(h)|^2\|u\|^2. \tag{5.2.16}$$

Proof. Recall that T is the FBI transform defined in (3.3.1), by (3.3.2), we have

$$\|u\|^2_{L^2(\partial\mathcal{O}\times[0,L^{-1}))} = \|Tu\|^2_{L^2(T^*\partial\mathcal{O}\times[0,L^{-1}))} + \mathcal{O}(h)\|u\|^2_{L^2}.$$

By (3.3.4), we have

$$\|u\|^2_{H_h^2(\partial\mathcal{O}\times[0,L^{-1}))} \sim \|\langle\eta\rangle^2 Tu\|^2_{L^2(T^*\partial\mathcal{O}\times[0,L^{-1}))} + \|(hD_t)^2 Tu\|^2_{L^2(T^*\partial\mathcal{O}\times[0,L^{-1}))},$$

also

$$\|(h^2P - \omega_0)u\|^2_{L^2(\partial\mathcal{O}\times[0,L^{-1}))} = \|(P(y,t,\eta,hD_t) - \omega_0)Tu\|^2_{L^2(T^*\partial\mathcal{O}\times[0,L^{-1}))} + \mathcal{O}(h)\|u\|^2_{H_h^2}.$$

Now (5.2.2) shows that if $|R(y,\eta) - \operatorname{Re}\omega_0| < c$,

$$\begin{aligned}\int_0^\infty |(P(y,t,\eta,hD_t) - \omega_0)Tu|^2 dt \geqslant (r_0 + 2S(\operatorname{Re}\omega_0)^{2/3}h^{2/3} - \mathcal{O}(h))^2 \int_0^\infty |Tu|^2 dt \\ +C^{-1}\int_0^\infty |(hD_t)^2 Tu|^2 dt - \mathcal{O}(h^2)|D_t Tu(0)|^2 - \mathcal{O}(h^2)|Tu(0)|^2;\end{aligned}$$

and (5.2.4) shows that if $|R(y,\eta)-\operatorname{Re}\omega_0|>c$,

$$\int_0^\infty |(P(y,t,\eta,hD_t)-\omega_0)Tu|^2dt \geqslant (r_0+C^{-1})^2\int_0^\infty |Tu|^2dt + C^{-1}\int_0^\infty |(hD_t)^2Tu|^2dt$$
$$+C^{-1}\langle\eta\rangle^4\int_0^\infty |Tu|^2dt - \mathcal{O}(h^2)\langle\eta\rangle^2|D_tTu(0)||Tu(0)|.$$

If we integrate $\int_0^\infty |(P(y,t,\eta,hD_t)-\omega_0)Tu|^2dt$ in $(y,\eta)\in T^*\partial\mathcal{O}$, we get

$$\begin{aligned}
&\|(P(y,t,\eta,hD_t)-\omega_0)Tu\|^2_{L^2(T^*\partial\mathcal{O}\times[0,L^{-1}))}\\
&\quad\geqslant (r_0+2S(\operatorname{Re}\omega_0)^{2/3}h^{2/3}-\mathcal{O}(h))^2\|Tu\|^2_{L^2(T^*\partial\mathcal{O}\times[0,L^{-1}))}\\
&\quad+C^{-1}\left(\int_{T^*\partial\mathcal{O}}\int_0^\infty |(hD_t)^2Tu|^2dt+\int_{|R(y,\eta)-\operatorname{Re}\omega_0|>c}\int_0^\infty \langle\eta\rangle^4|Tu|^2dt\right)\\
&\quad-\mathcal{O}(h^2)\|\langle\eta\rangle D_tTu(y,\eta,0)\|^2_{L^2(T^*\partial\mathcal{O})}-\mathcal{O}(h^2)\|\langle\eta\rangle Tu(y,\eta,0)\|^2_{L^2(T^*\partial\mathcal{O})}.
\end{aligned}\tag{5.2.17}$$

Here $D_tTu(y,\eta,0)=T(D_tu(\cdot,0))(y,\eta)=T(-ku(\cdot,0))$, so by (3.3.3),

$$\begin{aligned}
\|\langle\eta\rangle D_tTu(y,\eta,0)\|^2_{L^2(T^*\partial\mathcal{O})} =&\|\langle\eta\rangle T(k(\cdot)u(\cdot,0))\|^2_{L^2(T^*\partial\mathcal{O})}\\
\leqslant& C\|k(y)u(y,0)\|^2_{H^1_h(\partial\mathcal{O})}\leqslant C\|k\|^2_{H^1(\partial\mathcal{O})}\|u(y,0)\|^2_{H^1_h(\partial\mathcal{O})};\\
\|\langle\eta\rangle Tu(\cdot,0)\|^2_{L^2(T^*\partial\mathcal{O})} \leqslant& C\|u(y,0)\|^2_{H^1_h(\partial\mathcal{O})}.
\end{aligned}$$

Now we can apply Proposition 3.1.1 to the last two terms in (5.2.17) to show that they are bounded by $\mathcal{O}(h)\|u\|_{H^2_h}$.

Notice that if $|R(y,\eta)-\operatorname{Re}\omega_0|<c$, then (y,η) lies in a compact region of $T^*\partial\mathcal{O}$, we have

$$\int_{|R(y,\eta)-\operatorname{Re}\omega_0|<c}\int_0^\infty \langle\eta\rangle^4|Tu|^2dt\leqslant C\|Tu\|^2_{L^2(T^*\partial\mathcal{O}\times[0,L^{-1}))},$$

thus

$$\int_{T^*\partial\mathcal{O}}\int_0^\infty |(hD_t)^2Tu|^2dt+\int_{|R(y,\eta)-\operatorname{Re}\omega_0|>c}\int_0^\infty \langle\eta\rangle^4|Tu|^2dt\geqslant\max\{0,C^{-1}\|u\|^2_{H^2_h}-C\|u\|^2_{L^2}\}.$$

Therefore from (5.2.17), we have

$$\begin{aligned}
\|(h^2P-\omega_0)u\|^2\geqslant&(r_0+2S(\operatorname{Re}\omega_0)^{2/3}h^{2/3}-\mathcal{O}(h))^2\|u\|^2_{L^2}\\
&+C^{-1}\max\{0,C^{-1}\|u\|^2_{H^2_h}-C\|u\|^2_{L^2}\}-\mathcal{O}(h)\|u\|^2_{H^2_h}.
\end{aligned}$$

This concludes the proof of (5.2.16). □

5.2.4 Lower bounds for the scaled operator

The estimate away from the boundary is now combined with elliptic estimates away from the boundary to give the lower bounds for the scaled operator

Proposition 5.2.4. *There exists some $\epsilon > 0$ such that for ω_0 satisfying $\arg\omega_0 \in (\epsilon, \frac{\pi}{2} - \epsilon)$, $\operatorname{Re}\omega_0 \in (1-\epsilon, 1+\epsilon)$, and $u \in C_c^\infty(\mathbb{R}^n \setminus \mathcal{O})$ satisfying the Robin boundary condition, we have*

$$\|(h^2P - \omega_0)u\|^2 \geqslant |r_0 + 2S(\operatorname{Re}\omega_0)^{2/3}h^{2/3} - \mathcal{O}(h)|^2\|u\|^2. \tag{5.2.18}$$

for all sufficiently small $h > 0$.

Proof. We only need to estimate the part away from the boundary and connect it with (5.2.16). Let $\varphi_0, \varphi_1 \in C^\infty(\mathbb{R}^n; [0,1])$ such that $\varphi_0^2 + \varphi_1^2 = 1$, $\operatorname{supp}\varphi_0 \subset \{x : d(x) < L^{-1}\}$, $\varphi_1 = 0$ on $\{x : d(x) > (2L)^{-1}\}$ where $d(x)$ is the distance from x to $\mathcal{O}$. We claim that

$$\|(h^2P - \omega_0)\varphi_1 u\|^2 \geqslant (r_0 + 2S(\operatorname{Re}\omega_0)^{2/3}h^{2/3})^2\|\varphi_1 u\|^2. \tag{5.2.19}$$

In fact, we recall that in Section 4.2, the symbol p of $-\Delta|_\Gamma$ when $d(x) > (2L)^{-1}$ takes its values in $\epsilon < -\arg z < \pi - \epsilon$ for some $\epsilon > 0$. So by the assumption on ω_0,

$$\begin{aligned} \inf|p - \omega_0| &> \operatorname{Im}(e^{i\epsilon}\omega_0) = |\omega_0|\sin(\epsilon + \arg\omega_0) \\ &> |\omega_0|\sin(\arg\omega_0) + 2S(\operatorname{Re}\omega_0)^{2/3}h^{2/3} \\ &= r_0 + 2S(\operatorname{Re}\omega_0)^{2/3}h^{2/3}. \end{aligned}$$

We have the estimate (5.2.19).

Since

$$(h^2P - \omega_0)\varphi_j u = \varphi_j(h^2P - \omega_0)u - [\varphi_j, h^2P]u,$$

we have

$$\begin{aligned} \|(h^2P - \omega_0)\varphi_j u\|^2 &= \|\varphi_j(h^2P - \omega_0)u - [\varphi_j, h^2P]u\|^2 \\ &\leqslant \|\varphi_j(h^2P - \omega_0)u\|^2 + 2\|\varphi_j(h^2P - \omega_0)u\|\|[\varphi_j, h^2P]u\| \\ &\quad + \|[\varphi_j, h^2P]u\|^2, \end{aligned}$$

thus

$$\begin{aligned} \|(h^2P - \omega_0)u\|^2 &= \sum_{j=0,1}\|\varphi_j(h^2P - \omega_0)u\|^2 \\ &\geqslant \sum_{j=0,1}\|(h^2P - \omega_0)\varphi_j u\|^2 - \sum_{j=0,1}\|[\varphi_j, h^2P]u\|^2 \\ &\quad - \sum_{j=0,1}\|\varphi_j(h^2P - \omega_0)u\|\|[\varphi_j, h^2P]u\|. \end{aligned}$$

The commutators can be estimated by

$$\|[\varphi_j, h^2P]u\| = \mathcal{O}(h)(\|hD_x u\|_{L^2(\partial\mathcal{O}\times[(2L)^{-1},L^{-1}])} + \|u\|) \leqslant \mathcal{O}(h)(\|(h^2P-\omega_0)u\| + \|u\|)$$

since $h^2P - \omega_0$ is elliptic when $d(x) > (2L)^{-1}$. Therefore

$$\|(h^2P-\omega_0)u\|^2 \geqslant \sum_{j=0,1} \|(h^2P-\omega_0)\varphi_j u\|^2 - \mathcal{O}(h)(\|(h^2P-\omega_0)u\|^2 + \|u\|^2).$$

Now we can conclude that

$$\begin{aligned}
\|(h^2P-\omega_0)u\|^2 &\geqslant (1-\mathcal{O}(h)) \sum_{j=0,1} \|(h^2P-\omega_0)\varphi_j u\|^2 - \mathcal{O}(h)\|u\|^2 \\
&\geqslant (1-\mathcal{O}(h)) \sum_{j=0,1} |r_0 + 2S(\operatorname{Re}\omega_0)^{2/3}h^{2/3} - \mathcal{O}(h)|^2 \|\varphi_j u\|^2 - \mathcal{O}(h)\|u\|^2 \\
&\geqslant |r_0 + 2S(\operatorname{Re}\omega_0)^{2/3}h^{2/3} - \mathcal{O}(h)|^2\|u\|^2.
\end{aligned}$$

□

5.3 The pole-free region

Now we prove Theorem 2. An equivalent formulation is to say that there are no resonances for $P^{(\eta)}$ in the region

$$\operatorname{Re}\zeta > c_0, 0 < -\operatorname{Im}\zeta < S(\operatorname{Re}\zeta)^{1/3} - c_1 \tag{5.3.1}$$

for some constant $c_0, c_1 > 0$. We suppose that ζ is a resonance of $P^{(\eta)}$ such that

$$0 < -\operatorname{Im}\zeta < S(\operatorname{Re}\zeta)^{1/3} - c_1.$$

Then by proposition 4.2.1, $\lambda = \zeta^2$ is an eigenvalue of P. Let $h = (\operatorname{Re}\zeta)^{-1}$, then $h^2\zeta^2$ is an eigenvalue of h^2P: $h^2Pu = h^2\zeta^2 u$ for some $u \in D(\mathbb{R}^n \setminus \mathcal{O})$. Now we apply proposition 5.2.4 to u and $\omega_0 = \operatorname{Re}(h^2\zeta^2) + ir_0, r_0 > 0$. Since $h\operatorname{Re}\zeta = 1$, we have

$$\operatorname{Re}(h^2\zeta^2) = h^2(\operatorname{Re}\zeta)^2 - h^2(\operatorname{Im}\zeta)^2 = 1 + \mathcal{O}(h^{4/3});$$

and

$$\operatorname{Im}(h^2\zeta^2) = 2h^2(\operatorname{Re}\zeta)(\operatorname{Im}\zeta) = 2h\operatorname{Im}\zeta = \mathcal{O}(h^{4/3}).$$

It is easy to choose some r_0 such that ω_0 satisfies the condition in Proposition 5.2.4, say $r_0 = 1$. Therefore we get

$$|h^2\zeta^2 - \omega_0^2|\|u\|_{L^2}^2 = \|(h^2P-\omega_0)u\|_{L^2}^2 \geqslant |r_0 + 2S(\operatorname{Re}\omega_0)^{2/3}h^{2/3} - \mathcal{O}(h)|^2\|u\|_{L^2}^2,$$

where

$$h^2\zeta^2 - \omega_0 = i(\operatorname{Im}(h^2\zeta^2) - r_0) = i(2h\operatorname{Im}\zeta - r_0);$$

and

$$\operatorname{Re}\omega_0 = \operatorname{Re}(h^2\zeta^2) = 1 - h^2(\operatorname{Im}\zeta)^2.$$

Thus we have

$$|r_0 - 2h\operatorname{Im}\zeta|^2 \geqslant |r_0 + 2S[1 - h^2(\operatorname{Im}\zeta)^2]^{2/3}h^{2/3} - \mathcal{O}(h)|^2.$$

It follows that

$$-\operatorname{Im}\zeta \geqslant S[1 - h^2(\operatorname{Im}\zeta)^2]^{2/3}h^{-1/3} - M.$$

Now by the assumption that $-\operatorname{Im}\zeta < Sh^{-1/3} - c_1$, we can choose c_1 large, say $c_1 \geqslant S + M$, so that we have

$$1 - h^{1/3} \geqslant (1 - h^2(\operatorname{Im}\zeta)^2)^{2/3} = (1 - \mathcal{O}(h^{4/3}))^{2/3}.$$

Let $h \to 0$, we have a contradiction. Therefore we can choose $c_0, c_1 > 0$ large such that there are no poles in (5.3.1).

Chapter 6

Euclidean case: Band structure

In this chapter, we prove the band structure of the resonances near the real axis under pinched curvature conditions.

Theorem 3. *Suppose we have the following pinched curvature condition*

$$\frac{\max_{S\partial\mathcal{O}} Q}{\min_{S\partial\mathcal{O}} Q} < \left(\frac{\zeta'_{j_0+1}}{\zeta'_{j_0}}\right)^{3/2} \tag{6.0.1}$$

for some $j_0 \geqslant 1$. Then there exists a constant $C > 0$ such that for all $0 \leqslant j \leqslant j_0$, there are no resonances in the regions

$$C \leqslant \operatorname{Re}\lambda, \quad K\zeta'_j(\operatorname{Re}\lambda)^{1/3} + C \leqslant -\operatorname{Im}\lambda \leqslant \kappa\zeta'_{j+1}(\operatorname{Re}\lambda)^{1/3} - C. \tag{6.0.2}$$

Theorem 4. *Under the assumption in theorem 3, for some $C > 0$ and all $1 \leqslant j \leqslant j_0$,*

$$\begin{aligned} \sum\{M_{\mathcal{O}}(\lambda) : |\lambda| \leqslant r, \kappa\zeta'_j(\operatorname{Re}\lambda)^{1/3} - C < -\operatorname{Im}\lambda < K\zeta'_j(\operatorname{Re}\lambda)^{1/3} + C\} \\ = (1 + o(1))(2\pi)^{1-n}\operatorname{vol}(B^{n-1}(0,1))\operatorname{vol}(\partial\mathcal{O})r^{n-1}, \end{aligned} \tag{6.0.3}$$

where $B^{n-1}(0,1)$ is the unit ball in $\mathbb{R}^n$.

6.1 More preparations

6.1.1 A simple model

We conclude this section by presenting a simple model motivating our approach to boundary value problems using a Grushin reduction for an operator combining a differential operator and a boundary operator.

We consider the differential operator $P = -\frac{d^2}{dx^2}$ with Neumann boundary condition on the interval $[0, \pi]$. The spectrum of the operator is discrete: $\sigma(P) = \{\lambda_k = k^2 : k = 0, 1, 2, \dots\}$ and each eigenspace is one-dimensional:

$$E_k = \{f \in H^2[0,1] | f'(0) = f'(1) = 0, -f'' = \lambda_k f\} = \mathbb{C}\cos kx.$$

We set up a Grushin problem to capture the first m eigenvalues using a finite matrix. For simplicity, let us consider the case $m = 1$ so that the first eigenvalue is $\lambda_0 = 0$ with unit eigenvector $e_0 = \frac{1}{\pi}$. Put

$$\begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{D} \times \mathbb{C} \to L^2[0, \pi] \times \mathbb{C},$$

where

$$\mathcal{D} = \{u \in H^2[0, \pi] : u'(0) = u'(\pi) = 0\}$$

and

$$Pu = -u'', \quad R_+ u = \langle u, e_0 \rangle = \frac{1}{\pi} \int_0^\pi u dx, \quad R_- u_- = u_- e_0 = \frac{u_-}{\pi}.$$

Then

$$\begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}$$

is equivalent to

$$-u'' - zu + \frac{u_-}{\pi} = v, \quad \frac{1}{\pi} \int_0^\pi u dx = v_+.$$

We can integrate the first equation on $[0, \pi]$ to get

$$-(u'(\pi) - u'(0)) - z \int_0^\pi u dx + u_- = \int_0^\pi v dx$$

and thus

$$u_- = (u'(\pi) - u'(0)) + z \int_0^\pi u dx + \int_0^\pi v dx = \pi z v_+ + \int_0^\pi v dx.$$

It is then not difficult to see that for $z < 1$, we can use this u_- to solve u uniquely. Therefore the Grushin problem is well-posed with inverse

$$\begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} : L^2[0, \pi] \times \mathbb{C} \to \mathcal{D} \times \mathbb{C},$$

which has an explicit expression and we have seen that $E_{-+} = \pi z$ which is invertible if and only if $z \neq \lambda_0 = 0$.

The situation is somewhat similar to our case of obstacle scattering if we regard the left end point $x = 0$ as the boundary, and the right end point $x = \pi$ as infinity. Recall that in the case of obstacle scattering, since the outgoing condition becomes L^2-condition after complex scaling, we get a "boundary condition" at infinity. Now, we consider another Grushin problem for $-\frac{d^2}{dx^2}$, or rather the following operator

$$\begin{pmatrix} -\frac{d^2}{dx^2} - z \\ \gamma_1 \end{pmatrix} : \mathcal{D}' = \{u \in H^2[0, 1] | u'(\pi) = 0\} \to L^2[0, 1] \times \mathbb{C}$$

where $\gamma_1 u = u'(0)$. We use the same R_+ and R_- as above to construct the Grushin problem

$$\begin{pmatrix} -\frac{d^2}{dx^2} - z & R_- \\ \gamma_1 & 0 \\ R_+ & 0 \end{pmatrix} : \mathcal{D}' \times \mathbb{C} \to L^2[0,\pi] \times \mathbb{C} \times \mathbb{C}.$$

Now

$$\begin{pmatrix} -\frac{d^2}{dx^2} - z & R_- \\ \gamma_1 & 0 \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_0 \\ v_+ \end{pmatrix}$$

is equivalent to

$$-u'' - zu + \frac{u_-}{\pi} = v, \quad u'(0) = v_0, \quad \frac{1}{\pi}\int_0^\pi u dx = v_+.$$

Again, integrating the first equation gives

$$-(u'(\pi) - u'(0)) - z\int_0^\pi u dx + u_- = \int_0^\pi v dx$$

and thus

$$u_- = (u'(\pi) - u'(0)) + z\int_0^\pi u dx + \int_0^\pi v dx = -v_0 + \pi z v_+ + \int_0^\pi v dx.$$

Again, using this u_-, it is not difficult to solve u uniquely for $z < 1$. Hence this Grushin problem is also well-posed with inverse

$$\begin{pmatrix} E & K & E_+ \\ E_- & K_- & E_{-+} \end{pmatrix} : L^2[0,\pi] \times \mathbb{C} \times \mathbb{C} \to \mathcal{D}' \times \mathbb{C},$$

which again has an explicit expression. We find that $E_{-+} = \pi z$ coincides with E_{-+} we found in the previous Grushin problem.

Of course in this trivial example we can compute everything explicitly without Grushin reduction. The importance of the Grushin problem is that we can perturb the operator and the invertibility of the perturbed operator is captured by the finite matrix E_{-+} (in our case it is a 1×1 matrix, i.e. a scalar.) This reduces the infinite-dimensional problem to a finite-dimensional one. The second Grushin problem also allows us to perturb the boundary condition at 0 which turns out to be crucial in our setting.

6.1.2 Further reductions to a combined operator

We work in the semiclassical setting and introduce $P(h) := -h^2\Delta_\Gamma$. Near the boundary, we have the expression

$$P(h) = e^{-2\pi i/3}((hD_{x_n})^2 + 2x_n Q(x_n, x', hD_{x'}; h)) + R(x', hD_{x'}; h) + hF(x_n, x')hD_{x_n}. \quad (6.1.1)$$

Also for $w \in W \Subset (0, \infty)$ and $|\operatorname{Im} z| \leqslant C$, $|\operatorname{Re} z| \ll \delta^{-1}$, we let $P - z = h^{-2/3}(P(h) - w) - z$, so near the boundary,

$$\begin{aligned} P - z = e^{-2\pi i/3}&(D_t^2 + 2tQ(h^{2/3}t, x', hD_{x'}; h)) \\ &+ h^{-2/3}(R(x', hD_{x'}; h) - w) + F(h^{2/3}t, x')h^{2/3}D_t - z, \end{aligned} \tag{6.1.2}$$

where $t = h^{-2/3}x_n$.

There are certain difficulties in working with Robin boundary conditions with the domain (2.1.1). In normal geodesic coordinates introduced above, the domain will change as the function η changes and this causes the difficulty in the formulation of the model problem later.

To avoid this issue, notice that in the t-coordinates, the condition (4.2.1) can be rewritten as

$$\partial_t u + h^{2/3}ku|_{t=0} = 0,$$

where $k = e^{\pi i/3}\eta$. The top order term in h corresponds to the Neumann boundary condition. This motivates us to consider the Robin boundary problem with general $\eta \in C^\infty(\partial\mathcal{O})$ as a perturbation of the Neumann boundary problem. To achieve this, we shall combine our differential operator $P - z$ with the boundary operator and consider

$$\begin{pmatrix} P - z \\ \gamma \end{pmatrix} : H^2(\mathbb{R}^n \setminus \mathcal{O}) \to L^2(\mathbb{R}^n \setminus \mathcal{O}) \times H^l(\partial\mathcal{O}) \tag{6.1.3}$$

where for Dirichlet problem, $l = \frac{3}{2}$,

$$\gamma = \gamma_0 : H^2(\mathbb{R}^n \setminus \mathcal{O}) \to H^{3/2}(\partial\mathcal{O}), \quad u \mapsto u|_{\partial\mathcal{O}};$$

and for Neumann or Robin problem ($k = e^{\pi i/3}\gamma$) that we shall focus on, $l = \frac{1}{2}$,

$$\gamma = h^{2/3}(\gamma_1 + k\gamma_0) : H^2(\mathbb{R}^n \setminus \mathcal{O}) \to H^{1/2}(\partial\mathcal{O}), \quad u \mapsto h^{2/3}(\partial_\nu u + ku)|_{\partial\mathcal{O}}. \tag{6.1.4}$$

(γ_1 is defined by equation (6.1.4)). In the coordinates (t, x'), we have $\gamma(u) = u(0, \cdot)$ (Dirichlet) or

$$\gamma(u) = (\partial_t u + h^{2/3}ku)(0, \cdot) \quad \text{(Neumann or Robin)}.$$

Therefore from now on we shall think of $P - z$ as the first component of the combined operator (6.1.3), i.e. the differential operator from $H^2(\mathbb{R}^n \setminus \mathcal{O})$ to $L^2(\mathbb{R}^n \setminus \mathcal{O})$ instead of an operator with a smaller domain (2.1.1). Moreover, to avoid confusion, we shall write $R_P(z)$ to be the resolvent of P with domain (2.1.1), or in other words, $R_P(z)$ is the right inverse of $P - z : H^2 \to L^2$ satisfying $\gamma R_P(z) = 0$. We wish to use our new operator (6.1.3) to give an equivalent description of resonances instead of

$$m(h^{-2}(w + h^{2/3}z)) = \operatorname{tr} \frac{1}{2\pi i} \oint_{|\tilde{z} - z| = \epsilon} R_P(\tilde{z}) d\tilde{z}, \quad 0 < \epsilon \ll 1. \tag{6.1.5}$$

Proposition 6.1.1. *The eigenvalues of P are exactly the poles of*

$$\begin{pmatrix} P-z \\ \gamma \end{pmatrix}^{-1} : L^2(\mathbb{R}^n \setminus \mathcal{O}) \times H^l(\partial\mathcal{O}) \to H^2(\mathbb{R}^n \setminus \mathcal{O}) \tag{6.1.6}$$

as a meromorphic operator-valued function in z. Moreover, they have the same multiplicity:

$$m(h^{-2}(w + h^{2/3}z)) = \operatorname{tr} -\frac{1}{2\pi i} \oint_{|\tilde{z}-z|=\epsilon} \begin{pmatrix} P-\tilde{z} \\ \gamma \end{pmatrix}^{-1} \frac{d}{d\tilde{z}} \begin{pmatrix} P-\tilde{z} \\ \gamma \end{pmatrix} d\tilde{z}, \tag{6.1.7}$$

where $0 < \epsilon \ll 1$ is chosen in a way that there are no other poles for the operator (6.1.6) *in* $|\tilde{z} - z| < \epsilon$.

Proof. Let K be a right inverse of γ:

$$K : L^2(\partial\mathcal{O}) \to H^2(\mathbb{R}^n \setminus \partial\mathcal{O}), \quad \gamma K g = g, \quad \forall g \in H^l(\partial\mathcal{O}). \tag{6.1.8}$$

One possible choice is the so-called Poisson operator, but any choice will be good for us. Then we have

$$\begin{pmatrix} P-z \\ \gamma \end{pmatrix}^{-1} = (R_P(z), K - R_P(z)(P-z)K), \tag{6.1.9}$$

In fact, for any $(v, g) \in L^2(\mathbb{R}^n \setminus \mathcal{O}) \times H^l(\partial\mathcal{O})$, let

$$u = R_P(z)v + (K - R_P(z)(P-z)K)g,$$

by the construction of K, (6.1.8), and the fact that $\gamma R_P(z) = 0$,

$$(P-z)u = v + (P-z)Kg - (P-z)Kg = v, \quad \gamma u = \gamma K g = g.$$

Therefore (6.1.9) gives

$$\begin{pmatrix} P-z \\ \gamma \end{pmatrix}^{-1} \frac{d}{dz} \begin{pmatrix} P-z \\ \gamma \end{pmatrix} = (R_P(z), K - R_P(z)(P-z)K) \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -R_P(z).$$

Now (6.1.7) and the proposition follows directly from (6.1.5). □

6.2 Model Grushin problem

In this section, we shall study the model problem for ordinary differential operators by setting up a suitable Grushin problem. Recall that we have the combined operator (6.1.3)

$$\begin{pmatrix} P-z \\ \gamma \end{pmatrix} : H^2(\mathbb{R}^n \setminus \mathcal{O}) \to L^2(\mathbb{R}^n \setminus \mathcal{O}) \times H^{1/2}(\partial\mathcal{O}),$$

where $P - z$ is given by

$$P - z = h^{-2/3}(-h^2\Delta_\Gamma - w) - z : H^2(\mathbb{R}^n \setminus \mathcal{O}) \to L^2(\mathbb{R}^n \setminus \mathcal{O})$$

and γ is given by

$$\gamma = h^{2/3}(\gamma_1 + k\gamma_0) : H^2(\mathbb{R}^n \setminus \mathcal{O}) \to L^2(\partial\mathcal{O}), \quad u \mapsto h^{2/3}(\partial_\nu u + ku)|_{\partial\mathcal{O}}.$$

In local coordinates $(t = h^{-2/3}x_n, x')$ near the boundary, we have

$$\begin{aligned} P - z = e^{-2\pi i/3}(D_t^2 + 2tQ(h^{2/3}t, x', hD_{x'}; h)) \\ + h^{-2/3}(R(x', hD_{x'}; h) - w) + F(h^{2/3}t, x')h^{2/3}D_t - z, \end{aligned}$$

and

$$\gamma(u) = \gamma_1(u) + h^{2/3}k\gamma_0(u) = (\partial_t u + h^{2/3}ku)(0, \cdot).$$

Therefore we start by ignoring the lower order terms and considering a model operator

$$P_\lambda - z = e^{-2\pi i/3}(D_t^2 + \mu t) + \lambda - z \tag{6.2.1}$$

with $\gamma_1 : u \mapsto u'(0)$, where $\lambda \in \mathbb{R}$, $C^{-1} \leqslant \mu \leqslant C$ and $|\operatorname{Im} z| < C_1$ with C_1 large but fixed. Here we regard λ as $h^{-2/3}(R(x', hD_{x'}) - w)$, and μ as $Q(0, x', hD_{x'})$. The other terms will be small perturbation.

The model above is only necessary for handling the region near the glancing hypersurface $\Sigma_w = \{R(x', \xi') = w\}$. In the situation that $|\lambda| \gg 1 + |\operatorname{Re} z|$, i.e. away from the glancing region, since Q is bounded by R, we can also treat the term $e^{-2\pi i/3}\mu t$ as a perturbation and instead consider the model operator

$$P_\lambda^\# - z = e^{-2\pi i/3}D_t^2 + \lambda - z \tag{6.2.2}$$

with the same γ_1 and $\lambda \in \mathbb{R}, |\operatorname{Im} z| < C_1$. Here we note that (6.2.2) is elliptic as $|\lambda - \operatorname{Re} z| \gg 1$ and thus this model is easier to work with.

In this section, we shall first review some properties of the Airy function and estimates of Airy operators and boundary operators. Next we solve the Grushin problem for the model Airy operators in the case $\mu = 1$. Then we treat the easier model operator (6.2.2) in the same way. Finally we shall show how the additional parameter μ affects our construction and that all the estimates are uniform for μ in a compact subset of $(0, \infty)$.

6.2.1 Some basic estimates

In this part, we give more estimates on the Airy differential operators and the boundary operators.

Consider the Airy operator $D_t^2 + t : B \subset L^2 \to L^2$ and the boundary operators

$$\gamma_0 : B \to \mathbb{C}, \;\; u \mapsto u(0), \quad \gamma_1 : B \to \mathbb{C}, \;\; u \mapsto u'(0).$$

Here $L^2 = L^2(0,\infty)$ and $B = \{u \in L^2 : D_t^2 u, tu \in L^2\}$ is a Banach space equipped with the norm

$$\|u\|_B = \|D_t^2 u\| + \|tu\| + \|u\|, \tag{6.2.3}$$

where we use $\|\cdot\|$ to represent the standard L^2-norm on $(0,\infty)$.

It is clear that $\|(D_t^2+t)u\| \leqslant C\|u\|_B$. More precisely, we have the following identity,

$$\|(D_t^2+t)u\|^2 = \|D_t^2 u\|^2 + \|tu\|^2 + 2\|\sqrt{t}D_t u\|^2 - |\gamma_0 u|^2, \tag{6.2.4}$$

for any $u \in C_0^\infty([0,\infty))$. The proof is based on a simple integration by parts. To see this, let $\langle , \rangle$ be the standard L^2 inner product on $(0,\infty)$. Then

$$\begin{aligned}\|(D_t^2+t)u\|^2 &= \|D_t^2 u\|^2 + \|tu\|^2 + 2\operatorname{Re}\langle D_t^2 u, tu\rangle \\ &= \|D_t^2 u\|^2 + \|tu\|^2 + 2\operatorname{Re}\langle D_t u, D_t(tu)\rangle \\ &= \|D_t^2 u\|^2 + \|tu\|^2 + 2\operatorname{Re}\langle D_t u, tD_t u\rangle + 2\operatorname{Re}\frac{1}{i}\langle D_t u, u\rangle \\ &= \|D_t^2 u\|^2 + \|tu\|^2 + 2\|\sqrt{t}D_t u\|^2 - |\gamma_0 u|^2.\end{aligned}$$

Here in the last step, we use again the integration by parts

$$\langle D_t u, u\rangle = \langle u, D_t u\rangle - i|u(0)|^2 \tag{6.2.5}$$

to get

$$\operatorname{Re}\frac{1}{i}\langle D_t u, u\rangle = \operatorname{Im}\langle D_t u, u\rangle = -\frac{i}{2}|u(0)|^2.$$

Next we give some estimates of γ_0 and γ_1. For any $u \in C_0^\infty([0,\infty))$, by the Cauchy-Schwartz inequality and (6.2.5), we get

$$|\gamma_0 u|^2 \leqslant 2\|D_t u\|\|u\|,$$

and similarly

$$|\gamma_1 u|^2 \leqslant 2\|D_t^2 u\|\|D_t u\|.$$

Another application of integration by parts and the Cauchy-Schwartz inequality also gives

$$\begin{aligned}\|D_t u\|^2 &= \langle D_t^2 u, u\rangle - u(0)u'(0) \\ &\leqslant |\gamma_1 u||\gamma_0 u| + \|D_t^2 u\|\|u\| \\ &\leqslant 2\|D_t^2 u\|^{1/2}\|u\|^{1/2}\|D_t u\| + \|D_t^2 u\|\|u\|\end{aligned}$$

which leads to the standard interpolation estimates

$$\|D_t u\| \leqslant (\sqrt{2}+1)\|D_t^2 u\|^{1/2}\|u\|^{1/2}. \tag{6.2.6}$$

As a consequence, for any $\epsilon > 0$,

$$\begin{aligned}|\gamma_0 u| &\leqslant C\|D_t^2 u\|^{1/4}\|u\|^{3/4} \leqslant \epsilon\|D_t^2 u\| + C_\epsilon\|u\| \\ |\gamma_1 u| &\leqslant C\|D_t^2 u\|^{3/4}\|u\|^{1/4} \leqslant \epsilon\|D_t^2 u\| + C_\epsilon\|u\|.\end{aligned} \tag{6.2.7}$$

Now from (6.2.3) and (6.2.4) we get

$$\|u\|_B \leqslant C(\|u\|_{L^2} + \|(D_t^2 + t)u\|_{L^2}) \tag{6.2.8}$$

and

$$|\gamma_0 u| \leqslant C\|u\|_B, \quad |\gamma_1 u| \leqslant C\|u\|_B. \tag{6.2.9}$$

We finish this part by using these two estimates to show that elements in B can be written in a unique way as a linear combination of the Neumann Airy eigenfunctions $(e'_j)_{j=1}^\infty$ introduced in the previous section and one other element $f \in B$ with $\gamma_1 f \neq 0$. We remark that (e'_j) is not an orthonormal basis in B, so this expression might be different from the orthogonal expansion in L^2.

On one hand, if the sum $\sum_j u_j e'_j$ converges in B to some u, then by (6.2.9) we have $\gamma_1 u = \sum_j u_j \gamma_1 e'_j = 0$. On the other hand, if $u \in B$ satisfies $\gamma_1 u = u'(0) = 0$, then we can consider the L^2-orthogonal expansion

$$u = \sum_j \langle u, e'_j\rangle e'_j. \tag{6.2.10}$$

By (6.2.8), we have for any finite subset J of $\mathbb{Z}_+$,

$$\begin{aligned}
\|\sum_{j\in J}\langle u, e'_j\rangle e'_j\|_B &\leqslant C(\|\sum_{j\in J}\langle u, e'_j\rangle e'_j\| + \|(D_t^2+t)\sum_{j\in J}\langle u, e'_j\rangle e'_j\|) \\
&\leqslant C(\|\sum_{j\in J}\langle u, e'_j\rangle e'_j\| + \|\sum_{j\in J}\zeta'_j\langle u, e'_j\rangle e'_j\|) \\
&\leqslant C(\|\sum_{j\in J}\langle u, e'_j\rangle e'_j\| + \|\sum_{j\in J}\langle u, (D_t^2+t)e'_j\rangle e'_j\|) \\
&\leqslant C(\|\sum_{j\in J}\langle u, e'_j\rangle e'_j\| + \|\sum_{j\in J}\langle (D_t^2+t)u, e'_j\rangle e'_j\|).
\end{aligned}$$

which shows that the sum (6.2.10) converges to u in B since $(D_t^2 + t)u \in L^2$.

Therefore if we fix some $f \in B$ such that $\gamma_1 f = f'(0) \neq 0$, then every $u \in B$ can be uniquely expressed in the form

$$u = u_0 f + \sum_{j=1}^\infty u_j e'_j \tag{6.2.11}$$

where the sum converges in B. We simply choose u_0 first such that $\gamma_1(u - u_0 f) = 0$, then write the orthogonal expansion of $u - u_0 f$ by (e'_j) in L^2, i.e. $u_j = \langle u - u_0 f, e'_j\rangle$.

6.2.2 Model Airy problem

The operator in (6.2.1) (taking $\mu = 1$) combined with the Neumann boundary operator

$$\begin{pmatrix} P_\lambda - z \\ \gamma_1 \end{pmatrix} : B \to L^2 \times \mathbb{C} \tag{6.2.12}$$

may not be invertible for all z with $|\operatorname{Im} z| < C_1$. In fact, let us take $N = N(C_1)$ as the largest number such that

$$|\operatorname{Im} e^{-2\pi i/3}\zeta'_N| \leqslant C_1,$$

so that $e^{-2\pi i/3}\zeta'_j + \lambda - z \neq 0$ for $j \geqslant N+1$. Then (6.2.12) is not invertible precisely when $e^{-2\pi i/3}\zeta'_j + \lambda - z = 0$ for some $j = 1, \ldots, N$ since e'_j is in its kernel. Therefore we need to correct this operator in a suitable way to make it invertible. We shall also modify our spaces by putting an exponential weight. Moreover, we also need to add correct powers of $\langle \lambda - \operatorname{Re} z\rangle$ in the norm.

More precisely, let us consider the following Grushin problem for (6.2.12):

$$\mathcal{P}_\lambda(z) = \begin{pmatrix} P_\lambda - z & R^0_- \\ \gamma_1 & r_- \\ R^0_+ & 0 \end{pmatrix} : \mathcal{B}_{z,\lambda,r} \to \mathcal{H}_{z,\lambda,r} \tag{6.2.13}$$

(Later on we shall always choose $r_- = 0$.) Here the spaces and the norms on the spaces are given by

$$\begin{aligned} \mathcal{B}_{z,\lambda,r} &= B_{z,\lambda,r} \times \mathbb{C}^N, \\ \left\| \begin{pmatrix} u \\ u_- \end{pmatrix} \right\|_{\mathcal{B}_{z,\lambda,r}} &= \|u\|_{B_{z,\lambda,r}} + |u_-|, \\ \mathcal{H}_{z,\lambda,r} &= L^2_r \times \mathbb{C}_{\langle \lambda - \operatorname{Re} z\rangle^{1/4}} \times \mathbb{C}^N_{\langle \lambda - \operatorname{Re} z\rangle}, \\ \left\| \begin{pmatrix} v \\ v_0 \\ v_+ \end{pmatrix} \right\|_{\mathcal{H}_{z,\lambda,r}} &= \|v\|_{L^2_r} + \langle \lambda - \operatorname{Re} z\rangle^{1/4}|v_0| + \langle \lambda - \operatorname{Re} z\rangle |v_+|. \end{aligned} \tag{6.2.14}$$

with $|\cdot|$ fixed norms on $\mathbb{C}$ or $\mathbb{C}^N$ and

$$L^2_r = L^2([0,\infty), e^{rt}dt), B_{z,\lambda,r} = \{u \in L^2_r; D_t^2 u, tu \in L^2_r\}.$$

The norms are given by the standard weighted L^2-norm $\|\cdot\|_{L^2_r}$ and

$$\|u\|_{B_{z,\lambda,r}} = \langle \lambda - \operatorname{Re} z\rangle \|u\|_{L^2_r} + \|D_t^2 u\|_{L^2_r} + \|tu\|_{L^2_r}, \tag{6.2.15}$$

respectively. Moreover, the operators are given by

$$\begin{aligned} P_\lambda - z &: B_r \to L^2_r, \quad u \mapsto (e^{-2\pi i/3}(D_t^2 + t) + \lambda - z)u; \\ \gamma_1 &: B_r \to \mathbb{C}, \quad u \mapsto u'(0); \\ R^0_+ &: B_r \to \mathbb{C}^N, \quad u \mapsto (\langle u, e'_j\rangle)_{1\leqslant j\leqslant N}; \\ R^0_- &: \mathbb{C}^N \to L^2_r, \quad u_- \mapsto \sum_{j=1}^N u_-(j)e'_j; \\ r_- &: \mathbb{C}^N \to \mathbb{C}, \quad u_- \mapsto \sum_{j=1}^N r_j u_-(j). \end{aligned}$$

We remark that the heuristic reason for the weight $\langle\lambda - \operatorname{Re} z\rangle^{1/4}$ in the second component $\mathbb{C}$ on $\mathcal{H}_{z,\lambda,r}$ is that $\langle\lambda - \operatorname{Re} z\rangle$ roughly represents the Laplacian on the boundary $\langle\Delta_{\partial\mathcal{O}}\rangle$ (up to some parameters). Therefore if $u \in H^2(\mathbb{R}^n \setminus \mathcal{O})$, then by the well-known property of boundary operators $\partial_\nu u|_{\partial\mathcal{O}} \in H^{1/2}(\partial\mathcal{O})$ the norm of which corresponds to $\langle\lambda - \operatorname{Re} z\rangle^{1/4}$. We can also see that this is the correct weight by rescaling the estimate (6.2.7). For the same reason, if we wish to work with Dirichlet boundary operator, then we need to replace this weight $\langle\lambda - \operatorname{Re} z\rangle^{1/4}$ by $\langle\lambda - \operatorname{Re} z\rangle^{3/4}$.

Moreover, to handle powers of t which will appear in lower order terms, it is necessary to introduce the exponential weight $e^{rt}, r > 0$ in the definition of spaces $\mathcal{B}_{z,\lambda,r}$ and $\mathcal{H}_{z,\lambda,r}$. This will be explained in full details in the next section.

For $r = 0$, it is clear that the space $B_{z,\lambda,0}$ is just B in the previous section with an equivalent norm (of course not uniformly in z, λ) and $\mathcal{P}_\lambda(z) : \mathcal{B}_{z,\lambda,0} \to \mathcal{H}_{z,\lambda,0}$ is a uniformly bounded operator. Now we look for the inverse of $\mathcal{P}_\lambda(z)$. Let

$$\mathcal{P}_\lambda(z) \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_0 \\ v_+ \end{pmatrix}. \tag{6.2.16}$$

Then explicitly we have

$$\begin{aligned} (P_\lambda - z)u + R_-^0 u_- &= v \\ u'(0) + r_- u_- &= v_0 \\ R_+^0 u &= v_+. \end{aligned}$$

We express v in terms of the orthonormal basis $(e'_j)_{j=1}^\infty$ in L^2:

$$v = \sum_{j=1}^\infty v_j e'_j,$$

and we write $v_+ = (v_+(j))_{1\leqslant j\leqslant N}$. Then we look for solutions with $u \in B$ as in (6.2.11)

$$u = u_0 f + \sum_{j=1}^\infty u_j e'_j$$

and

$$u_- = (u_-(j))_{1\leqslant j\leqslant N}.$$

Let us write

$$f_0 := f'(0), \quad f_j := \langle f, e'_j\rangle, \quad \eta_j := e^{-2\pi i/3}\zeta'_j + \lambda - z.$$

then we have

$$(P_\lambda - z)e'_j = \eta_j e'_j, \quad (P_\lambda - z)^* e'_j = \bar{\eta}_j e'_j.$$

where $(P_\lambda - z)^* = e^{2\pi i/3}(D_t^2 + t) + \lambda - \bar{z}$ is the formal adjoint of $P_\lambda - z$. Moreover,

$$\langle (P_\lambda - z)f, e_j' \rangle = e^{-2\pi i/3} e_j'(0) f_0 + \langle f, (P_\lambda - z)^* e_j' \rangle = e^{-2\pi i/3} e_j'(0) f_0 + \eta_j f_j.$$

Then we can rewrite the system (6.2.16) as an infinite system of linear equations:

$$\begin{aligned} [e^{-2\pi i/3} e_j'(0) f_0 + \eta_j f_j] u_0 + \eta_j u_j + u_-(j) &= v_j, \quad (1 \leqslant j \leqslant N) \\ [e^{-2\pi i/3} e_j'(0) f_0 + \eta_j f_j] u_0 + \eta_j u_j &= v_j, \quad (j \geqslant N+1) \\ f_0 u_0 + \sum_{j=1}^{N} r_j u_-(j) &= v_0 \\ f_j u_0 + u_j &= v_+(j), \quad (1 \leqslant j \leqslant N). \end{aligned} \tag{6.2.17}$$

It is not difficult to see that as long as

$$1 - e^{-2\pi i/3} \sum_{j=1}^{N} r_j e_j'(0) \neq 0,$$

we have a unique solution for (6.2.17),

$$\begin{aligned} u_0 &= \left[1 - e^{-2\pi i/3} \sum_{j=1}^{N} r_j e_j'(0) \right]^{-1} f_0^{-1} \left[v_0 + \sum_{j=1}^{N} r_j (\eta_j v_+(j) - v_j) \right] \\ u_j &= v_+(j) - f_j u_0, \quad (1 \leqslant j \leqslant N) \\ u_j &= \eta_j^{-1} (v_j - (e^{-2\pi i/3} e_j'(0) f_0 + \eta_j f_j) u_0), \quad (j \geqslant N+1) \\ u_-(j) &= v_j - \eta_j v_+(j) - e^{-2\pi i/3} e_j'(0) f_0 u_0, \quad (1 \leqslant j \leqslant N). \end{aligned}$$

For simplicity, henceforth we shall choose $f_0 = 1, r_- = 0$ (though other choices are also possible). Then the solution becomes

$$\begin{aligned} u_0 &= v_0 \\ u_j &= v_+(j) - f_j v_0, \quad (1 \leqslant j \leqslant N) \\ u_j &= \eta_j^{-1} (v_j - e^{-2\pi i/3} e_j'(0) v_0) - f_j v_0, \quad (j \geqslant N+1) \\ u_-(j) &= v_j - e^{-2\pi i/3} e_j'(0) v_0 - \eta_j v_+(j), \quad (1 \leqslant j \leqslant N). \end{aligned} \tag{6.2.18}$$

Now we need to estimate the norm.

Lemma 6.2.1. *The Grushin problem* (6.2.13) *is well-posed for* $r = 0$. *In other words, suppose* (6.2.16), *then we have*

$$\|u\|_{B_{z,\lambda,0}} + |u_-| \leqslant C(\|v\|_{L^2} + \langle \lambda - \operatorname{Re} z \rangle^{1/4} |v_0| + \langle \lambda - \operatorname{Re} z \rangle |v_+|). \tag{6.2.19}$$

where C is independent of λ, z.

Proof. We first observe that for $1 \leqslant j \leqslant N$,

$$|\eta_j| \leqslant C\langle \lambda - \operatorname{Re} z\rangle$$

while for $j \geqslant N+1$

$$|\eta_j| \geqslant C^{-1}(\langle \lambda - \operatorname{Re} z\rangle + \zeta_j').$$

The first inequality just follows the definition $\eta_j = e^{-2\pi i/3}\zeta_j' + \lambda - z$ and the assumption $|\operatorname{Im} z| < C_1$. When $\langle \lambda - \operatorname{Re} z\rangle \geqslant C\zeta_j'$, we can get the second inequality simply by estimating the real part using $|\operatorname{Re}\eta_j| \geqslant |\lambda - z| - C\zeta_j'$. Otherwise we use the imaginary part $\operatorname{Im}\eta_j = -(\sin 2\pi/3)\zeta_j' - \operatorname{Im} z$ which does not vanish from the assumption on N. Therefore $|\operatorname{Im}\eta_j| \geqslant C^{-1}\zeta_j'$ and we also get the second inequality.

From the last equation in (6.2.18), we easily get

$$|u_-| \leqslant C(\|v\|_{L^2} + |v_0| + \langle \lambda - \operatorname{Re} z\rangle |v_+|). \tag{6.2.20}$$

To estimate u, we first write its orthogonal expansion in L^2 following the first three equations in (6.2.18)

$$\begin{aligned} u &= u_0 f + \sum_{j=1}^{\infty} u_j e_j' \\ &= v_0\left(f - \sum_{j=1}^{\infty} f_j e_j'\right) + \sum_{j=1}^{N} v_+(j) e_j' + \sum_{j=N+1}^{\infty} \eta_j^{-1}(v_j - e^{-2\pi i/3}e_j'(0)v_0)e_j' \\ &= \sum_{j=1}^{N} v_+(j) e_j' + \sum_{j=N+1}^{\infty} \eta_j^{-1}(v_j - e^{-2\pi i/3}e_j'(0)v_0)e_j' \end{aligned}$$

which shows that

$$\begin{aligned} \|u\|_{L^2}^2 &= \sum_{j=1}^{N} |v_+(j)|^2 + \sum_{j=N+1}^{\infty} |\eta_j|^{-2}|v_j - e^{-2\pi i/3}e_j'(0)v_0|^2 \\ &\leqslant C|v_+|^2 + C\langle \lambda - \operatorname{Re} z\rangle^{-2}\|v\|_{L^2}^2 + C|v_0|^2 \sum_{j=N+1}^{\infty} |\eta_j|^{-2}|e_j'(0)|^2. \end{aligned}$$

To treat the last term, we need a careful study of Airy functions. Recall from (2.2.9)

$$e_j'(0) = \operatorname{Ai}(-\zeta_j')/\|\operatorname{Ai}\|_{L^2(-\zeta_j',\infty)} = (-1)^{j-1}\zeta_j'^{-1/2},$$

and the asymptotic formula (2.2.6)

$$\zeta_j' = (\frac{3}{2}j\pi)^{2/3}(1 + o(1)), \quad j \to \infty$$

As a consequence, we have

$$|e_j'(0)|^2 = (\frac{3}{2}j)^{-2/3}(1+o(1)), \quad j \to \infty$$

Now we can compute

$$\begin{aligned}\sum_{j=N+1}^{\infty} |\eta_j|^{-2}|e_j'(0)|^2 &\leqslant C \sum_{j=N+1}^{\infty} j^{-2/3}(\langle \lambda - \operatorname{Re} z\rangle + \zeta_j')^{-2} \\ &\leqslant C \sum_{j=N+1}^{\infty} j^{-2/3}(\langle \lambda - \operatorname{Re} z\rangle + j^{2/3})^{-2} \\ &\leqslant C \int_1^{\infty} s^{-2/3}(\langle \lambda - \operatorname{Re} z\rangle + s^{2/3})^{-2} ds \\ &\leqslant C\langle \lambda - \operatorname{Re} z\rangle^{-3/2} \int_0^{\infty} t^{-2/3}(1+t^{2/3})^{-2} dt \leqslant C\langle \lambda - \operatorname{Re} z\rangle^{-3/2},\end{aligned}$$

where in the last step we use the change of variable $s = \langle \lambda - \operatorname{Re} z\rangle^{3/2} t$. This gives the following estimate on the L^2-norm of u:

$$\langle \lambda - \operatorname{Re} z\rangle \|u\|_{L^2} \leqslant C(\|v\|_{L^2} + \langle \lambda - \operatorname{Re} z\rangle^{1/4}|v_0| + \langle \lambda - \operatorname{Re} z\rangle |v_+|). \tag{6.2.21}$$

Now since

$$(D_t^2 + t)u = e^{2\pi i/3}(v - R_-^0 u_- - (\lambda - z)u),$$

we have

$$\|(D_t^2 + t)u\|_{L^2} \leqslant C(\|v\|_{L^2} + |u_-| + \langle \lambda - \operatorname{Re} z\rangle \|u\|_{L^2})$$

Now we can use a variation of (6.2.8)

$$\|u\|_{B_{z,\lambda,0}} \leqslant C(\|(D_t^2 + t)u\|_{L^2} + \langle \lambda - \operatorname{Re} z\rangle \|u\|_{L^2})$$

and (6.2.21) to get (6.2.19). □

The next step is to consider adding a small exponential weight, i.e. $r \in (0, r_0)$ for r_0 small.

Lemma 6.2.2. *There exists $r_0 > 0$ such that the Grushin problem* (6.2.13) *is uniformly well-posed for $r \in (0, r_0)$. More precisely, suppose* (6.2.16)*, then we have*

$$\|u\|_{B_{z,\lambda,r}} + |u_-| \leqslant C(\|v\|_{L_r^2} + \langle \lambda - \operatorname{Re} z\rangle^{1/4}|v_0| + \langle \lambda - \operatorname{Re} z\rangle |v_+|). \tag{6.2.22}$$

where C is independent of λ, z and r.

Proof. We introduce

$$\begin{aligned}\mathcal{P}_\lambda^r(z) &= \begin{pmatrix} e^{rt/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathcal{P}_\lambda \begin{pmatrix} e^{-rt/2} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \mathcal{P}_\lambda(z) + \begin{pmatrix} e^{rt/2} P_\lambda e^{-rt/2} - P_\lambda & (e^{rt/2}-1)R_-^0 \\ \gamma_1(e^{-rt/2}-1) & 0 \\ R_+^0(e^{-rt/2}-1) & 0 \end{pmatrix}\end{aligned}$$

By the interpolation estimate (6.2.6), we have

$$D_t = O(\langle \lambda - \operatorname{Re} z\rangle^{-1/2}) : B_{z,\lambda,0} \to L^2,$$

thus

$$e^{rt/2} P_\lambda e^{-rt/2} - P_\lambda = e^{-2\pi i/3}(irD_t - \frac{1}{4}r^2) = O(r\langle \lambda - \operatorname{Re} z\rangle^{-1/2}) : B_{z,\lambda,0} \to L^2.$$

Next, by (6.2.7),

$$\gamma_0 = O(\langle \lambda - \operatorname{Re} z\rangle^{-3/4}) : B_{z,\lambda,0} \to \mathbb{C},$$

so

$$\gamma_1(e^{-rt/2}-1) = -\frac{r}{2}\gamma_0 = O(r\langle \lambda - \operatorname{Re} z\rangle^{-1/2}) : B_{z,\lambda,0} \to \mathbb{C}_{\langle \lambda - \operatorname{Re} z\rangle^{1/4}}.$$

Also by the super exponential decay of e_j', $j = 1, \dots, N$: $\|(e^{-rt/2}-1)e_j'(t)\|_{L^2} = o(1)$ as $r \to 0+$, so

$$R_+^0(e^{-rt/2}-1) = o(1) : B_{z,\lambda,0} \to \mathbb{C}^N_{\langle \lambda - \operatorname{Re} z\rangle}.$$

Similarly, we have $\|(e^{rt/2}-1)e_j'(t)\|_{L^2} = o(1)$, and

$$(e^{rt/2}-1)R_-^0 = o(1) : \mathbb{C}^N \to L^2.$$

We see that $\mathcal{P}_\lambda^r(z)$ is a small perturbation of $\mathcal{P}_\lambda(z)$ in the sense that

$$\mathcal{P}_\lambda^r(z) - \mathcal{P}_\lambda(z) = o(1) : \mathcal{B}_{z,\lambda,0} \to \mathcal{H}_{z,\lambda,0}$$

uniformly in z, λ as $r \to 0+$. Therefore

$$\mathcal{P}_\lambda^r(z) : \mathcal{B}_{z,\lambda,0} \to \mathcal{H}_{z,\lambda,0}$$

is uniformly invertible when $r \in [0, r_0]$ for some small $r_0 > 0$. Now we note that

$$\|u\|_{B_{z,\lambda,r}} \sim \|e^{rt/2}u\|_{B_{z,\lambda,0}}$$

uniformly in z, λ and $r \in [0, r_0]$ which again follows from the interpolation estimate (6.2.6) for D_t. This finishes the proof of the lemma. □

In particular, from (6.2.18), we see that the inverse of $\mathcal{P}_\lambda(z)$ is given by

$$\mathcal{E}_\lambda(z) = \begin{pmatrix} E & K & E_+ \\ E_- & K_- & E_{-+} \end{pmatrix} : \mathcal{H}_{z,\lambda,r} \to \mathcal{B}_{z,\lambda,r},$$

where

$$E_{-+} \in \hom(\mathbb{C}^N, \mathbb{C}^N), \quad (E_{-+})_{1\leqslant j,k\leqslant n} = -\eta_j \delta_{ij}. \tag{6.2.23}$$

6.2.3 Dependence on parameters

Now we shall modify our Grushin problem so that we get nice global symbolic properties. For $0 < \delta \ll 1$, we put

$$e_j'^{\lambda,\delta}(t) = \Lambda^{1/2} e_j'(\Lambda t), \Lambda = \langle \delta\lambda \rangle^{1/2}$$

which also forms an orthonormal basis for $L^2([0,\infty))$. We notice that

$$\partial_\lambda^k \Lambda = O_k(1)\delta^k \Lambda^{1-2k}, \quad \|\partial_\lambda^k e_j'^{\lambda,\delta}\|_{L^2} = O_k(1)\delta^k \Lambda^{-2k}.$$

In particular,

$$\|e_j'^{\lambda,\delta} - e_j'\|_{L^2} \leqslant C\delta|\lambda|.$$

We define $R_+^{\lambda,\delta}$ and $R_-^{\lambda,\delta}$ by replacing e_j' with $e_j'^{\lambda,\delta}$ in the definition of R_+^0 and R_-^0 and then construct

$$\mathcal{P}_\lambda^\delta(z) = \begin{pmatrix} P_\lambda - z & R_-^{\lambda,\delta} \\ \gamma_1 & 0 \\ R_+^{\lambda,\delta} & 0 \end{pmatrix} : \mathcal{B}_{z,\lambda,r} \to \mathcal{H}_{z,\lambda,r}. \tag{6.2.24}$$

Then we have

$$\mathcal{P}_\lambda^\delta(z) - \mathcal{P}_\lambda(z) = \begin{pmatrix} 0 & O(|\lambda|\delta) \\ 0 & 0 \\ O(|\lambda|\delta) & 0 \end{pmatrix} : \mathcal{B}_{z,\lambda,r} \to \mathcal{H}_{z,\lambda,r}.$$

(Here we no longer use the notation $\mathcal{P}_\lambda^r$ and instead abuse the notation to put parameter δ on top.) Thus for $|\lambda|\delta \ll 1$ we get the uniform invertibility of $\mathcal{P}_\lambda^\delta(z)$. To get the same estimate for all λ, we need to assume

$$|\operatorname{Re} z| \ll \frac{1}{\delta}, \tag{6.2.25}$$

so that $|\lambda| \gg 1 + |\operatorname{Re} z|$ and we have the invertibility of $\begin{pmatrix} P_\lambda - z \\ \gamma_1 \end{pmatrix}$ without the correcting terms $R_\pm^{\lambda,\delta}$ given by the following lemma.

Lemma 6.2.3. *For $|\lambda| \gg 1 + |\operatorname{Re} z|$ and $|\operatorname{Im} z| < C_1$, there exists a constant $C > 0$ independent of z and λ such that for any $u \in B_{z,\lambda,0}$,*

$$|\langle (P_\lambda - z)u, u \rangle| + \langle \lambda - \operatorname{Re} z \rangle^{-1/2} |\gamma_1 u|^2 \geqslant C^{-1} \langle \lambda - \operatorname{Re} z \rangle \|u\|_{L^2}^2. \tag{6.2.26}$$

Furthermore, for small r,

$$\begin{pmatrix} P_\lambda - z \\ \gamma_1 \end{pmatrix} u = \begin{pmatrix} v \\ v_0 \end{pmatrix} \quad \Rightarrow \quad \|u\|_{B_{z,\lambda,r}} \leqslant C(\|v\|_{L_r^2} + \langle \lambda - \operatorname{Re} z \rangle^{1/4} |v_0|). \tag{6.2.27}$$

Proof. It is possible to repeat the argument as in Lemma 6.2.1 using orthogonal expansion with respect to (e'_j). We present here another proof by using the Poisson operator $K_\lambda : \mathbb{C} \to B_{z,\lambda,0}$, satisfying

$$P_\lambda K_\lambda = 0, \quad \gamma_1 K_\lambda = \mathrm{Id}\,.$$

This Poisson operator is given by multiplying $f = f_\lambda$ which is the solution to the equation

$$e^{-2\pi i/3}(D_t^2 + t)f + \lambda f = 0, \quad f'(0) = 1.$$

We can give an explicit expression of f in terms of the Airy function:

$$f_\lambda(t) = \mathrm{Ai}'(e^{2\pi i/3}\lambda)^{-1}\,\mathrm{Ai}(t + e^{2\pi i/3}\lambda).$$

Notice that all the zeroes of Ai and Ai$'$ lie on the negative real axis, this expression is well-defined as λ is real.

We shall apply the asymptotic formulas for the Airy function and its derivatives (2.2.5) to study the L^2-norm of f_λ. First we consider the case $\lambda > 0$. Then

$$\mathrm{Ai}'(e^{2\pi i/3}\lambda) = -(2\sqrt{\pi})^{-1} e^{\pi i/6} e^{\lambda^{3/2}} \lambda^{1/4}(1 + O(\lambda^{-3/2})).$$

and

$$\mathrm{Ai}(t + e^{2\pi i/3}\lambda) = (2\sqrt{\pi})^{-1} e^{-\zeta} z^{-1/4}(1 + O(|\zeta|^{-1}))$$

where

$$z = t + e^{2\pi i/3}\lambda, \quad |z| = (t^2 - t\lambda + \lambda^2)^{1/2}, \quad \zeta = \frac{2}{3} z^{3/2}.$$

We change variables by letting $\arg z = \frac{\pi}{2} - \theta$. Then $\theta \in [-\frac{\pi}{6}, \frac{\pi}{2})$ and

$$t = \frac{\lambda}{2} + \frac{\sqrt{3}}{2}\lambda \tan\theta, \quad |z| = \frac{\sqrt{3}}{2}\lambda \sec\theta, \quad \zeta = \frac{\sqrt{3}}{4}\lambda^{3/2} e^{i(3\pi/4 - 3\theta/2)} \sec^{3/2}\theta.$$

We have the following uniform asymptotic formula in λ and θ for $f_\lambda(t)$:

$$f_\lambda(t) = g(\lambda) e^{\lambda^{3/2}\psi(\theta)} e^{-i(7\pi/24 - \theta/4)}(\sec^{-1/4}\theta)(1 + O(\lambda^{-3/2}\sec^{-3/2}\theta)),$$

where

$$g(\lambda) = (\sqrt{3}/2)^{-1/4}\lambda^{-1/2}(1 + O(\lambda^{-3/2})), \quad \psi(\theta) = -\frac{2}{3} - \frac{\sqrt{3}}{4} e^{i(3\pi/4 - 3\theta/2)} \sec^{3/2}\theta.$$

Therefore

$$\|f_\lambda\|^2_{L^2(0,\infty)} = \frac{\sqrt{3}}{2}\lambda |g(\lambda)|^2 \int_{-\pi/6}^{\pi/2} e^{\lambda^{3/2}\varphi(\theta)}(\sec^{3/2}\theta)(1 + O(\lambda^{-3/2}\sec^{-3/2}\theta))d\theta,$$

where

$$\varphi(\theta) = 2\,\mathrm{Re}\,\psi(\theta) = 2\left[-\frac{2}{3} - \frac{\sqrt{3}}{4}\sec^{3/2}\theta \cos(\frac{3\pi}{4} - \frac{3\theta}{2})\right]$$

satisfies

$$\varphi(-\pi/6) = 0, \quad \lim_{\theta \to \pi/2 - 0} \varphi(\theta) = -\infty,$$

and

$$\varphi'(\theta) = -\frac{3\sqrt{3}}{4} \sec^{5/2}\theta \sin\left(\frac{3\pi}{4} - \frac{\theta}{2}\right) < -\frac{3\sqrt{3}}{8} < 0, \quad \theta \in \left[-\frac{\pi}{6}, \frac{\pi}{2}\right).$$

Therefore integration by parts gives us

$$\|f_\lambda\| = O(\lambda^{-3/4}). \tag{6.2.28}$$

Now for every $u \in B_{z,\lambda,0}$, let $v = u - K_\lambda(\gamma_1 u) = u - u'(0) f_\lambda$. We have $v'(0) = 0$. Now we can write

$$\begin{aligned} \langle (P_\lambda - z)u, u\rangle = \langle (P_\lambda - z)v, v\rangle + \overline{\gamma_1 u}\langle (P_\lambda - z)v, f_\lambda\rangle \\ - z(\gamma_1 u)\langle f_\lambda, v\rangle - z|u'(0)|^2 \|f_\lambda\|_{L^2}^2. \end{aligned}$$

For the second term on the right-hand side, we integrate by parts:

$$\begin{aligned} \langle (P_\lambda - z)v, f_\lambda\rangle &= -e^{-2\pi i/3} v(0) + \langle v, (P_\lambda - z)^* f_\lambda\rangle \\ &= -e^{-2\pi i/3} v(0) + (\lambda(1 - e^{2\pi i/3}) - \bar{z})\langle v, f_\lambda\rangle. \end{aligned}$$

Therefore

$$\begin{aligned} \langle (P_\lambda - z)u, u\rangle = e^{-2\pi i/3}\langle (D_t^2 + t)v, v\rangle + (\lambda - z)\|v\|^2 - e^{-2\pi i/3}(\overline{\gamma_1 u})v(0) \\ + \overline{\gamma_1 u}(\lambda(1 - e^{2\pi i/3}) - \bar{z})\langle v, f_\lambda\rangle - z(\gamma_1 u)\langle f_\lambda, v\rangle - z|\gamma_1 u|^2 \|f_\lambda\|_{L^2}^2, \end{aligned}$$

where we notice that $\langle (D_t^2 + t)v, v\rangle$ is always nonnegative. This gives

$$\begin{aligned} |\langle (P_\lambda - z)u, u\rangle| \geqslant\ & \mathrm{Re}(e^{\pi i/3}\langle (P_\lambda - z)u, u\rangle \\ \geqslant\ & \frac{1}{2}\langle (D_t^2 + t)v, v\rangle + C^{-1}\langle \lambda - \mathrm{Re}\, z\rangle \|v\|^2 - \epsilon\langle \lambda - z\rangle^{1/2}|v(0)|^2 \\ & - \epsilon\langle \lambda - \mathrm{Re}\, z\rangle\|v\|^2 - O_\epsilon(\langle \lambda - z\rangle^{-1/2})|\gamma_1 u|^2 \end{aligned}$$

Now by choosing ϵ small enough but fixed and using

$$\langle \lambda - z\rangle^{1/2}|v(0)|^2 \leqslant 2\langle \lambda - z\rangle^{1/2}\|D_t v\|\|v\| \leqslant \|D_t v\|^2 + \langle \lambda - \mathrm{Re}\, z\rangle\|v\|^2$$

and $\langle (D_t^2 + t)v, v\rangle \geqslant \|D_t v\|^2$ to deduce that

$$|\langle (P_\lambda - z)u, u\rangle| \geqslant C^{-1}\langle \lambda - \mathrm{Re}\, z\rangle\|v\|^2 - C\langle \lambda - \mathrm{Re}\, z\rangle^{-1/2}|\gamma_1 u|^2$$

by $\|u\|^2 \leqslant C(\|v\|^2 + \langle \lambda - \mathrm{Re}\, z\rangle^{-3/2}|\gamma_1 u|^2)$, we can conclude the proof of (6.2.26) for $\lambda > 0$. For $\lambda < 0$, we can get similarly $\|f_\lambda\| = O(|\lambda|^{-3/4})$ and then use

$$|\langle (P_\lambda - z)u, u\rangle| \geqslant \mathrm{Re}(-\langle (P_\lambda - z)u, u\rangle)$$

to reproduce the argument above and prove (6.2.26).

Now we prove (6.2.27). For $r=0$, we can see from (6.2.26),

$$\|u\|_{L^2}^2 \leqslant C\langle\lambda-\operatorname{Re}z\rangle^{-1}\|(P_\lambda-z)u\|_{L^2}\|u\|_{L^2}+C\langle\lambda-\operatorname{Re}z\rangle^{-3/2}|\gamma_1 u|^2.$$

Therefore

$$\|u\|_{L^2} \leqslant C\langle\lambda-\operatorname{Re}z\rangle^{-1}\|(P_\lambda-z)u\|+C\langle\lambda-\operatorname{Re}z\rangle^{-3/4}|\gamma_1 u|^2$$

which proves (6.2.27) for $r=0$. For small r, we can simply repeat the conjugation and perturbation argument as in Lemma 6.2.2 to conclude the uniform invertibility. $\square$

Now we give the desired invertibility for the full operator in the Grushin problem.

Proposition 6.2.4. *For $|\lambda|\geqslant 1/(C\delta)$ and $|\operatorname{Re}z|\ll 1/\delta$, $r\in[0,r_0]$ with $r_0>0$ small enough,*

$$\mathcal{P}_\lambda^\delta\begin{pmatrix} u\\ u_-\end{pmatrix}=\begin{pmatrix} v\\ v_0\\ v_+\end{pmatrix} \quad\Rightarrow\quad \left\|\begin{pmatrix} u\\ u_-\end{pmatrix}\right\|_{\mathcal{B}_{z,\lambda,r}}\leqslant C\left\|\begin{pmatrix} v\\ v_0\\ v_+\end{pmatrix}\right\|_{\mathcal{H}_{z,\lambda,r}}. \tag{6.2.29}$$

Moreover, we have the following mapping properties of $\mathcal{P}_\lambda^\delta(z)$ and its inverse $\mathcal{E}_\lambda^\delta(z)$: for every k,

$$\begin{aligned}\|\partial_\lambda^k\mathcal{P}_\lambda^\delta(z)\|_{\mathcal{L}(\mathcal{B}_{z,\lambda,r},\mathcal{H}_{z,\lambda,r})}&\leqslant C_k\langle\lambda-\operatorname{Re}z\rangle^{-k},\\ \|\partial_\lambda^k\mathcal{E}_\lambda^\delta(z)\|_{\mathcal{L}(\mathcal{H}_{z,\lambda,r},\mathcal{B}_{z,\lambda,r})}&\leqslant C_k\langle\lambda-\operatorname{Re}z\rangle^{-k}.\end{aligned} \tag{6.2.30}$$

Proof. Again, we start with $r=0$. Let

$$\Pi=R_-R_+:L^2\to(\ker R_+)^\perp=\operatorname{Image}R_-=\bigoplus_{j=1}^N\mathbb{C}e_j'^{\lambda,\delta}$$

be the orthogonal projection. Then since

$$\|D_t^2 e_j'^{\lambda,\delta}\|_{L^2}=O(\langle\delta\lambda\rangle),\quad \|te_j'^{\lambda,\delta}\|_{L^2}=O(\langle\delta\lambda\rangle^{-1/2}),$$

we have $\|(P_\lambda-z)|_{\operatorname{Image}R_-}\|=O(\langle\lambda-\operatorname{Re}z\rangle)$. Also it is easy to see $\|R_+\|=\|R_-\|=1$. Since $\Pi u=R_-R_+u=R_-v_+$, we have

$$\|\Pi u\|_{L^2}\leqslant|v_+|$$

and

$$\|(P_\lambda-z)\Pi u\|_{L^2}\leqslant O(\langle\lambda-\operatorname{Re}z\rangle)|v_+|. \tag{6.2.31}$$

On the other hand, by the previous lemma,

$$\begin{aligned}\|(I-\Pi)u\|_{L^2}^2 \leqslant{}& C\langle\lambda-\operatorname{Re}z\rangle^{-1}|\langle(P_\lambda-z)(I-\Pi)u,(I-\Pi)u\rangle|\\ &+C\langle\lambda-\operatorname{Re}z\rangle^{-3/2}|\gamma_1(I-\Pi)u|^2\end{aligned}$$

For the first term, we have

$$\begin{aligned}\langle (P_\lambda - z)(I-\Pi)u, (I-\Pi)u\rangle &= \langle (I-\Pi)(P_\lambda - z)(I-\Pi)u, u\rangle \\ &= \langle (I-\Pi)(P_\lambda - z)u, u\rangle - \langle (I-\Pi)(P_\lambda - z)\Pi u, u\rangle \\ &= \langle (I-\Pi)(v - R_-u_-), u\rangle - \langle (P_\lambda - z)\Pi u, (I-\Pi)u\rangle \\ &= \langle (I-\Pi)v, u\rangle - \langle (P_\lambda - z)\Pi u, (I-\Pi)u\rangle \\ &= \langle v, (I-\Pi)u\rangle - \langle (P_\lambda - z)\Pi u, (I-\Pi)u\rangle.\end{aligned}$$

For the second term, we use $\gamma_1 \Pi = 0$ to get

$$\gamma_1(I-\Pi)u = \gamma_1 u = v_0.$$

Therefore

$$\begin{aligned}\|(I-\Pi)u\|_{L^2}^2 \leqslant C\langle \lambda - \operatorname{Re} z\rangle^{-1}(\|v\|_{L^2} + \|(P_\lambda - z)\Pi u\|_{L^2})\|(I-\Pi)u\| \\ + C\langle \lambda - \operatorname{Re} z\rangle^{-3/2}|v_0|\end{aligned}$$

and thus

$$\begin{aligned}\|(I-\Pi)u\|_{L^2} &\leqslant C\langle \lambda - \operatorname{Re} z\rangle^{-1}(\|v\|_{L^2} + \|(P_\lambda - z)\Pi u\|_{L^2}) + C\langle \lambda - \operatorname{Re} z\rangle^{-3/4}|v_0| \\ &\leqslant C\langle \lambda - \operatorname{Re} z\rangle^{-1}\|v\|_{L^2} + |v_+| + C\langle \lambda - \operatorname{Re} z\rangle^{-3/4}|v_0|.\end{aligned} \tag{6.2.32}$$

Combining (6.2.31) and (6.2.32), we have

$$\langle \lambda - \operatorname{Re} z\rangle \|u\|_{L^2} \leqslant C(\|v\|_{L^2_r} + \langle \lambda - \operatorname{Re} z\rangle^{1/4}|v_0| + \langle \lambda - \operatorname{Re} z\rangle |v_+|.).$$

Since

$$u_- = R_+R_-u_- = R_+(v - (P_\lambda - z)u) = R_+v - R_+(P_\lambda - z)u,$$

we have

$$|u_-| \leqslant \|v\|_{L^2} + \|R_+(P_\lambda - z)u\|_{L^2} \leqslant \|v\|_{L^2} + C\sum_{j=1}^{N} |\langle (P_\lambda - z)u, e_j'^{\lambda,\delta}\rangle|.$$

To estimate the sum, we integrate by parts and get

$$\langle (P_\lambda - z)u, e_j'^{\lambda,\delta}\rangle = \langle u, (P_\lambda - z)^* e_j'^{\lambda,\delta}\rangle + e^{-2\pi i/3}u'(0)e_j'^{\lambda,\delta}(0).$$

where $(P_\lambda - z)^* = e^{2\pi i/3}(D_t^2 + t) + \lambda - \bar{z}$ is the formal adjoint of $P_\lambda - z$ so

$$\|(P_\lambda - z)^* e_j'^{\lambda,\delta}\|_{L^2} = O(\langle \lambda - \operatorname{Re} z\rangle).$$

In addition, we have $u'(0) = v_0$ and by definition of $e_j'^{\lambda,\delta}$,

$$e_j'^{\lambda,\delta}(0) = O(\langle \delta\lambda\rangle^{1/4}),$$

which shows that

$$|\langle (P_\lambda - z)u, e_j'^{\lambda,\delta}\rangle| \leqslant C\langle \lambda - \operatorname{Re} z\rangle \|u\| + C\langle \lambda - \operatorname{Re} z\rangle^{1/4}|v_0|.$$

As a consequence,

$$|u_-| \leqslant C(\|v\|_{L_r^2} + \langle \lambda - \operatorname{Re} z\rangle^{1/4}|v_0| + \langle \lambda - \operatorname{Re} z\rangle |v_+|).$$

Now as in Lemma 6.2.1, we can use the equation $(P_\lambda - z)u = v - R_- u_-$ to give the estimates on the L^2 norm of $D_t^2 u$ and tu. This finishes the proof of (6.2.29) for $r = 0$.

To extend this to $r \in [0, r_0]$ for some small $r_0 > 0$, we notice that

$$\|(e^{\pm rt/2} - 1)e_j'^{\lambda,\delta}\| = \|(e^{\pm r\langle \delta\lambda\rangle^{-1/2}t/2} - 1)e_j'\| = o(1)$$

uniformly as $r \to 0$ which allow us to repeat the argument in Lemma 6.2.2.

Finally, since for $k > 1$,

$$\partial_\lambda^k \mathcal{P}_\lambda^\delta(z) = \begin{pmatrix} \delta_{1k} & \partial_\lambda^k R_+^{\lambda,\delta} \\ 0 & 0 \\ \partial_\lambda^k R_-^{\lambda,\delta} & 0 \end{pmatrix}$$

and

$$\|\partial_\lambda^k e_j'^{\lambda,\delta}\|_{L_r^2} = O_k(1)\delta^k \langle \delta\lambda\rangle^{-k} = O_k(1)\langle \lambda - \operatorname{Re} z\rangle^{-k},$$

we get the mapping properties of $\mathcal{P}_\lambda^\delta(z)$ in (6.2.30). For its inverse $\mathcal{E}_\lambda^\delta(z)$, (6.2.29) gives the mapping property when $k = 0$. The case $k > 0$ follows directly from the case $k = 0$ and the Leibnitz rule. □

To end this part, we study the $(-+)$-component of $\mathcal{E}_\lambda^\delta$:

Proposition 6.2.5. *For any $\epsilon > 0$, $|\lambda| \leqslant 1/(C\sqrt{\delta}), |\operatorname{Re} z| \ll 1/\sqrt{\delta}$ sufficiently small depending on ϵ,*

$$\|E_{-+}^\delta(z, \lambda) - \operatorname{diag}(z - \lambda - e^{-2\pi i/3}\zeta_j')\| \leqslant \epsilon. \tag{6.2.33}$$

Also, $\det E_{-+}^\delta(z, \lambda) = 0$ *if and only if*

$$z = \lambda + e^{-2\pi i/3}\zeta_j' \tag{6.2.34}$$

for some $j = 1, \ldots, N$. Each zero is simple. Moreover, for $|\lambda| \gg 1 + |\operatorname{Re} z|$,

$$\|E_{-+}^\delta(z, \lambda)^{-1}\|_{\mathcal{L}(\mathbb{C}^N, \mathbb{C}^N)} = O(\langle \lambda - \operatorname{Re} z\rangle^{-1}) \tag{6.2.35}$$

Proof. The (6.2.33) follows from the perturbation

$$\|E_{-+}^\delta(z, \lambda) - \operatorname{diag}(z - \lambda - e^{-2\pi i/3}\zeta_j')\| \leqslant O(\lambda|\delta|)\langle \lambda - \operatorname{Re} z\rangle.$$

Let us recall the general fact, (which is essentially the Schur complement formula, see e.g. [10] or [32] in the setting of Grushin problems),

$$(E_{-+}^{\delta})^{-1} = -R_{+}^{\lambda,\delta}\begin{pmatrix} P_\lambda - z \\ \gamma_1 \end{pmatrix}^{-1}\begin{pmatrix} R_{-}^{\lambda,\delta} \\ 0 \end{pmatrix}.$$

Since $\begin{pmatrix} P_\lambda - z \\ \gamma_1 \end{pmatrix}$ is not invertible precisely when $\eta_j = e^{-2\pi i/3}\zeta_j' + \lambda - z = 0$, (in which case e_j' is in the kernel), the same is true for E_{-+}^{δ}. This gives the zeros of $\det E_{-+}^{\delta}$, (6.2.34). The simplicity of the zeros is a consequence of the fact that each kernel is of one-dimensional. Finally, in the case $|\lambda| \gg 1 + |\operatorname{Re} z|$, by 6.2.3, $\begin{pmatrix} P_\lambda - z \\ \gamma_1 \end{pmatrix}$ is invertible. Therefore $E_{-+}^{\delta} : \mathbb{C}^N_{\langle \lambda - \operatorname{Re} z\rangle} \to \mathbb{C}^N$ is also invertible, which gives (6.2.35). $\square$

6.2.4 The "easy" model

When $|\lambda| \gg 1 + |\operatorname{Re} z|$ and $|\operatorname{Im} z| < C_1$, we can consider an even simpler model problem with the operator (6.2.2) which is already invertible. To obtain the uniform symbolic properties, we shall construct the Grushin problem using the same correction terms $R_{\pm}^{\lambda,\delta}$ as in (6.2.24). We define

$$\mathcal{P}_\lambda^{\#}(z) = \begin{pmatrix} P_\lambda^{\#} - z & R_{-}^{\lambda,\delta} \\ \gamma_1 & 0 \\ R_{+}^{\lambda,\delta} & 0 \end{pmatrix} : \mathcal{B}_{\lambda,r}^{\#} \to \mathcal{H}_{\lambda,r}^{\#}, \tag{6.2.36}$$

where the spaces $\mathcal{B}_{\lambda,r}^{\#}$ and $\mathcal{H}_{\lambda,r}^{\#}$ are defined by

$$\begin{aligned} \mathcal{B}_{\lambda,r}^{\#} &= B_{\lambda,r}^{\#} \times \mathbb{C}^N, B_{\lambda,r}^{\#} = \{u \in L_r^2 : D_t^2 u \in L_r^2\}, \\ \left\| \begin{pmatrix} u \\ u_- \end{pmatrix} \right\|_{\mathcal{B}_{\lambda,r}^{\#}} &= \langle \lambda \rangle \|u\|_{L_r^2} + \|D_t^2 u\|_{L_r^2} + |u_-|, \\ \mathcal{H}_{\lambda,r}^{\#} &= L_r^2 \times \mathbb{C}_{\langle \lambda \rangle^{1/4}} \times \mathbb{C}^N_{\langle \lambda \rangle}, \\ \left\| \begin{pmatrix} v \\ v_0 \\ v_+ \end{pmatrix} \right\|_{\mathcal{H}_{\lambda,r}^{\#}} &= \|v\|_{L_r^2} + \langle \lambda \rangle^{1/4}|v_0| + \langle \lambda \rangle |v_+|. \end{aligned} \tag{6.2.37}$$

Proposition 6.2.6. *For $|\lambda| \gg 1 + |\operatorname{Re} z|$, and $r \in [0, r_0]$ with $r_0 > 0$ small enough, $\mathcal{P}_\lambda^{\#}(z) : \mathcal{B}_{\lambda,r}^{\#} \to \mathcal{H}_{\lambda,r}^{\#}$ is uniformly invertible. We have the mapping properties for $\mathcal{P}_\lambda^{\#}(z)$ and its inverse $\mathcal{E}_\lambda^{\#}(z)$:*

$$\begin{aligned} \|\partial_\lambda^k \mathcal{P}_\lambda^{\#}(z)\|_{\mathcal{L}(\mathcal{B}_{\lambda,r}^{\#}, \mathcal{H}_{\lambda,r}^{\#})} &\leqslant C_k \langle \lambda \rangle^{-k} \\ \|\partial_\lambda^k \mathcal{E}_\lambda^{\#}(z)\|_{\mathcal{L}(\mathcal{H}_{\lambda,r}^{\#}, \mathcal{B}_{\lambda,r}^{\#})} &\leqslant C_k \langle \lambda \rangle^{-k}. \end{aligned} \tag{6.2.38}$$

Moreover, the $(-+)$*-component of* $\mathcal{E}_\lambda^\#$ *satisfies:*

$$E_{-+}^\#(z,\lambda)^{-1} = O(\langle\lambda\rangle^{-1}). \tag{6.2.39}$$

Proof. The proof is almost identical to the Airy model problem we discussed above. To make the argument work, we only need to replace the Poisson operator K_λ by $K_\lambda^\#$ satisfying

$$P_\lambda^\# K_\lambda^\# = 0, \gamma_1 K_\lambda^\# = 0,$$

which is given by multiplying the function

$$f_\lambda^\# = -e^{\pi i/3}\lambda^{-1/2}\exp(-e^{-\pi i/3}\lambda^{1/2}t).$$

When λ is negative, we choose the branch $\lambda^{1/2} = i(-\lambda)^{1/2}$ so $f_\lambda^\#$ has exponential decay. An easy calculation shows that

$$\|f_\lambda\|_{L^2} = O(|\lambda|^{-3/4}),$$

and therefore all our arguments in Lemma 6.2.3, thus in Proposition 6.2.4 and 6.2.5 can be carried out in the same way. We shall omit the details here. $\square$

6.2.5 The μ-dependent construction.

Now we shall put the parameter μ back into the operator and describe the necessary modifications we need to make in the model problem. The idea is to change coordinates $t = \mu^{-1/3}\tilde{t}$ in (6.2.1) which will reduce to the case $\mu = 1$. From our discussion, it will be clear that when μ varies in a compact subset of $(0,\infty)$ all the estimates will be uniform in μ provided that we construct all the operators accordingly and replace the the eigenvalues ζ_j' of Neumann Airy operator $D_t^2 + t$ by $\mu^{2/3}\zeta_j'$. More precisely, we have the following Grushin problem

$$\mathcal{P}_\lambda^\delta(z) = \begin{pmatrix} P_\lambda - z & R_-^{\lambda,\delta,\mu} \\ \gamma_1 & 0 \\ R_+^{\lambda,\delta,\mu} & 0 \end{pmatrix} : \mathcal{B}_{z,\lambda,r} \to \mathcal{H}_{z,\lambda,r} \tag{6.2.40}$$

where the spaces $\mathcal{B}_{z,\lambda,r}, \mathcal{H}_{z,\lambda,r}$ are as before and we reintroduce the additional parameter μ in the operators

$$\begin{aligned} P_\lambda - z &= e^{-2\pi i/3}(D_t^2 + \mu t) + \lambda - z \\ R_+^{\lambda,\delta,\mu} u &= (\langle u, e_{j,\mu}^{\lambda,\delta}\rangle)_{1\leqslant j\leqslant N} \\ R_-^{\lambda,\delta,\mu} u_- &= \sum_{j=1}^N u_-(j) e_{j,\mu}^{\lambda,\delta} \end{aligned}$$

with

$$e_{j,\mu}^{\lambda,\delta}(t) = \mu^{1/6} e_j'^{\lambda,\delta}(\mu^{1/3}t) = \mu^{1/6}\langle\delta\lambda\rangle^{1/4} e_j'(\mu^{1/3}\langle\delta\lambda\rangle^{1/2}t). \tag{6.2.41}$$

In the mean time, we also replace the $R_\pm^{\lambda,\delta}$ in the easy model by $R_\pm^{\lambda,\delta,\mu}$. Then all the previous results hold uniformly in $\mu \in [C^{-1}, C] \subset (0,\infty)$ with possibly a smaller $r_0 > 0$ due to the change of variable $t = \mu^{-1/3}\tilde{t}$.

6.3 Microlocal Grushin problem

6.3.1 Analysis near the glancing hypersurface

We can use $|R(x',\xi')-w|$ as our distance function to the glancing hypersurface Σ_w for which we shall perform the second microlocalization. First, we work near the glancing hypersurface, i.e. $|R(x',\xi')-w| \leqslant 2C^{-1}$. Then

$$\lambda = h^{-2/3}(R(x',\xi')-w) = O(h^{-2/3}).$$

We shall think of this as a perturbation of the principal symbol

$$\begin{pmatrix} P_0 - z \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} e^{-2\pi i/3}(D_t^2+\mu t)+\lambda - z \\ \gamma_1 \end{pmatrix}, \tag{6.3.1}$$

where

$$\mu = 2Q(x',\xi') \in [C^{-1},C].$$

As in the previous section, we set up the Grushin problem by letting $R_\pm = R_\pm^{\lambda,\delta}$ there. Then we have the operator-valued symbol

$$\mathcal{P}_0(z) = \begin{pmatrix} P_0 - z & R_- \\ \gamma_1 & 0 \\ R_+ & 0 \end{pmatrix} \tag{6.3.2}$$

which is uniformly invertible in $\mathcal{L}(\mathcal{B}_{z,\lambda,r},\mathcal{H}_{z,\lambda,r})$ with inverse $\mathcal{E}_0(z)$.

For simplicity, let us pretend for now that Q does not depend additionally in h, then by Taylor expansion with respect to $x_n = h^{2/3}t$, we have

$$\mathcal{P}(z) \equiv \mathcal{P}_0(z) + h^{2/3}\mathcal{K}_0 + \sum_{j=1}^{\infty} h^{2j/3}T^j\mathcal{P}_j + \sum_{j=1}^{\infty} h^{2j/3}T^{j-1}\mathcal{D}_j.$$

Here

$$\mathcal{K}_0 = \begin{pmatrix} 0 & 0 \\ k(x')\gamma_0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathcal{P}_j = \begin{pmatrix} \frac{1}{j!}2e^{-2\pi i/3}t\partial_t^j Q(0,x',\xi') & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathcal{D}_j = \begin{pmatrix} \frac{1}{(j-1)!}\partial_t^{j-1}F(0,x')D_t & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$T = \begin{pmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To find the inverse of such symbols, we shall take an approach similar to Sjöstrand [29, section 1] which is motivated by the work of Boutet de Monvel-Kree [22] on formal analytic symbols. Instead of considering a symbol $q = q(x, \xi; h)$, we deal with the formal operator

$$Q = q(x, \xi + hD_x; h) \equiv \sum_{\alpha \in \mathbb{N}^{n-1}} \frac{1}{\alpha!} \partial_\xi^\alpha q(x, \xi; h)(hD_x)^\alpha.$$

The symbol q itself can be recovered by the formula

$$q = Q(1).$$

The advantage of working with this setting is that the composition formula

$$a \#_h b = \sum_{\alpha \in \mathbb{N}^{n-1}} \frac{1}{\alpha!} (h\partial_\xi)^\alpha a D_x^\alpha b$$

becomes the formal composition of the corresponding formal operators A and B:

$$a \#_h b = A \circ B(1).$$

Therefore finding the inverse of such a symbol is equivalent to finding the inverse of the corresponding formal operator.

For this purpose, we shall consider operators of the form

$$\mathfrak{A} = \sum_{k,\alpha} (h^{2/3} T)^k A_{k,\alpha}(x', \xi, \lambda; h) D_{x'}^\alpha,$$

where

$$A_{k,\alpha} : \mathcal{B}_{z,\lambda,r} \to \mathcal{H}_{z,\lambda,r}.$$

The inverses of such operators should be of the form

$$\mathfrak{B} = \sum_{k,\alpha} (h^{2/3} T)^k B_{k,\alpha}(x', \xi, \lambda; h) D_{x'}^\alpha,$$

where

$$B_{k,\alpha} : \mathcal{H}_{z,\lambda,r} \to \mathcal{B}_{z,\lambda,r}.$$

However, we should notice that the T in the second class of operators should be interpreted as

$$T = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$$

acting on $\mathcal{B}_{z,\lambda,r}$ instead of on $\mathcal{H}_{z,\lambda,r}$. When needed, we shall write this one as $T_{\mathcal{B}}$ and the previous one as $T_{\mathcal{H}}$.

There are several technical issues about these two different operators T that we have to deal with. First, T is not a bounded operator on $\mathcal{B}_{z,\lambda,r}$ or $\mathcal{H}_{z,\lambda,r}$. We can deal with this issue by relaxing the exponentially weighted space.

$$T^k = O(1)C^k k^k (r-r')^{-k} : \mathcal{B}_{z,\lambda,r} \to \mathcal{B}_{z,\lambda,r'}$$

if $r > r'$ and similarly for $\mathcal{H}_{z,\lambda,r} \to \mathcal{H}_{z,\lambda,r'}$. Therefore we can work on the formal level and interpret the formal operators in the end as operators from $\mathcal{B}_{z,\lambda,r}$ to $\mathcal{H}_{z,\lambda,r'}$ (or similar operators with the weight function in the codomain relaxed to r'.)

The second issue comes from the non-commutativity of the operators T with A_k or B_k. When composing two such operators $\mathfrak{A}$ and $\mathfrak{B}$, we are hoping to get operators of the form

$$\mathfrak{C} = \sum_{k,\alpha} (h^{2/3}T)^k C_{k,\alpha}(x', \xi', \lambda) D^{\alpha}_{x'},$$

where

$$C_{k,\alpha} : \mathcal{H}_{z,\lambda,r} \to \mathcal{H}_{z,\lambda,r} \text{ or } \mathcal{B}_{z,\lambda,r} \to \mathcal{B}_{z,\lambda,r},$$

depending on the order of composition. This composition will involve the "commutators" $\operatorname{ad}_T = [T, \cdot]$ which we interpret as

$$\operatorname{ad}_T(A) = T_{\mathcal{H}}A - AT_{\mathcal{B}},$$

$$\operatorname{ad}_T(B) = T_{\mathcal{B}}B - BT_{\mathcal{H}},$$

when it acts on different classes. We shall also need ad_T to act on the two different classes of $\mathfrak{C}$ and we shall interpret it accordingly.

This involves the study of stability of mapping properties of A_k and B_k under the "commutator operation" ad_T. We first consider $\mathcal{P}_0$ to see its mapping properties and then adjust our definition of formal operators in a suitable way.

Lemma 6.3.1. *For $|\operatorname{Re} z| \ll 1/\delta$, we have*

$$\operatorname{ad}_T^k \mathcal{P}_0 = O_k(\delta^{-k/2}\langle \lambda - \operatorname{Re} z\rangle^{-k/2}) : \mathcal{B}_{z,\lambda,r} \to \mathcal{H}_{z,\lambda,r}. \tag{6.3.3}$$

Proof. We have seen in the last section that this is true for $k = 0$. A simple calculation gives

$$\operatorname{ad}_T^k \mathcal{P}_0 = \begin{pmatrix} \operatorname{ad}_t^k(P_0 - z) & t^k R_- \\ (-1)^k \gamma_1 t^k & 0 \\ (-1)^k R_+ t^k & 0 \end{pmatrix},$$

where $\operatorname{ad}_t = [t, \cdot]$ is the commutator with multiplying t. For $k = 1$,

$$\operatorname{ad}_t(P_0 - z) = 2ie^{-2\pi i/3} D_t = O(\langle \lambda - \operatorname{Re} z\rangle^{-1/2}) : \mathcal{B}_{z,\lambda,r} \to L^2_r.$$

For $k=2$,
$$\mathrm{ad}_t^2(P_0-z) = -2e^{-2\pi i/3} = O(\langle\lambda-\operatorname{Re} z\rangle^{-1}) : \mathcal{B}_{z,\lambda,r} \to L_r^2.$$
For $k>2$,
$$\mathrm{ad}_t^k(P_0-z) = 0.$$
For $k=1$,
$$(-1)^k\gamma_1 t^k = \gamma_0 = O(\langle\lambda-\operatorname{Re} z\rangle^{-1/2}) : \mathcal{B}_{z,\lambda,r} \to \mathbb{C}_{\langle\lambda-\operatorname{Re} z\rangle^{1/4}}.$$
For $k>1$,
$$(-1)^k\gamma_1 t^k = 0.$$
Also for $k\geqslant 1$, we have
$$R^+t^k = O_k(\delta^{-k/2}\langle\lambda-\operatorname{Re} z\rangle^{-k/2}) : \mathcal{B}_{z,\lambda,r} \to \mathbb{C}^N;$$
$$(-1)^k t^k R^- = O_k(\delta^{-k/2}\langle\lambda-\operatorname{Re} z\rangle^{-k/2}) : \mathbb{C}^N \to L_r^2.$$
Combining all these estimates together, we get the desired mapping properties for $\mathrm{ad}_T^k \mathcal{P}_0$. $\square$

On the other hand, we also need the stability for $\mathcal{P}_0(z)$ under differentiation in x', ξ', λ which will give the second microlocal symbol class which is simply
$$\mathcal{P}_0(z) \in S_{\Sigma_w,2/3}(\partial\mathcal{O}; 1, \mathcal{L}(\mathcal{B}_{z,\lambda,r}, \mathcal{H}_{z,\lambda,r})).$$
We shall combine the two types of mapping properties together to get
$$\partial_{x'}^\alpha \partial_{\xi'}^\beta \partial_\lambda^l \,\mathrm{ad}_T^k \mathcal{P}_0(z) = O(\delta^{-k/2}\langle\lambda-\operatorname{Re} z\rangle^{-l-k/2}) : \mathcal{B}_{z,\lambda,r} \to \mathcal{H}_{z,\lambda,r},$$
with the constants depending on k, l, α, β. Now each of $\partial_{x'}, \partial_{\xi'}, \partial_\lambda$ and ad_T is a derivation provided we interpret ad_T suitably. We get similar estimates for the inverse:
$$\partial_{x'}^\alpha \partial_{\xi'}^\beta \partial_\lambda^l \,\mathrm{ad}_T^k \mathcal{E}_0(z) = O(\delta^{-k/2}\langle\lambda-\operatorname{Re} z\rangle^{-l-k/2}) : \mathcal{H}_{z,\lambda,r} \to \mathcal{B}_{z,\lambda,r},$$
since we have seen the estimates for $k=l=0, \alpha=\beta=0$ in last section. We can replace $\langle\lambda-\operatorname{Re} z\rangle$ by $\langle\lambda\rangle$ at the expense of δ-dependent constants.

Also we have the symbol properties for $\mathcal{P}_j$, $\mathcal{D}_j$ and $\mathcal{K}_0$:
$$\partial_{x'}^\alpha \partial_{\xi'}^\beta \partial_\lambda^l \,\mathrm{ad}_T^k \mathcal{P}_j(z) = O(\langle\lambda\rangle^{-l-k/2}) : \mathcal{B}_{z,\lambda,r} \to \mathcal{H}_{z,\lambda,r}, \tag{6.3.4}$$
$$\partial_{x'}^\alpha \partial_{\xi'}^\beta \partial_\lambda^l \,\mathrm{ad}_T^k \mathcal{D}_j(z) = O(\langle\lambda\rangle^{-1/2-l-k/2}) : \mathcal{B}_{z,\lambda,r} \to \mathcal{H}_{z,\lambda,r}, \tag{6.3.5}$$
and
$$\partial_{x'}^\alpha \partial_{\xi'}^\beta \partial_\lambda^l \,\mathrm{ad}_T^k \mathcal{K}_0(z) = O(\langle\lambda\rangle^{-1/2-l-k/2}) : \mathcal{B}_{z,\lambda,r} \to \mathcal{H}_{z,\lambda,r}. \tag{6.3.6}$$
We remark that we neglect a number of simplifying features here, for example, for $\partial_{x'}^\alpha \partial_{\xi'}^\beta \partial_\lambda^l \,\mathrm{ad}_T^k \mathcal{K}_0 = 0$, unless β, k and l are all zero.

Now we can introduce the suitable class of formal operators: these take the form

$$\mathfrak{A} = \sum_{\alpha\in\mathbb{N}^{n-1},j,k,l,m\in\mathbb{N}} (h^{2/3}T)^j (h^{2/3}\langle\lambda\rangle^{-1/2})^k (h^{1/3}\langle\lambda\rangle^{-1})^l h^m \mathcal{A}_{\alpha,j,k,l,m}(x',\xi',\lambda,z) D_{x'}^{\alpha}, \tag{6.3.7}$$

with the mapping properties for $\mathcal{A}_{\alpha,j,k,l,m}$

$$\partial_{x'}^{\tilde\alpha}\partial_{\xi'}^{\tilde\beta}\partial_{\lambda}^{\tilde l}\operatorname{ad}_T^{\tilde k}\mathcal{A}_{\alpha,j,k,l,m} = O(\langle\lambda\rangle^{-\tilde l-\tilde k/2}) : \mathcal{B}_{z,\lambda,r}\to\mathcal{H}_{z,\lambda,r}. \tag{6.3.8}$$

Now we restore the dependence on h of Q and rewrite the operator $\mathcal{P}$ as

$$\mathcal{P}(z) = h^{2/3}\mathcal{K}_0(x') + \sum_{j=0}^{\infty} h^{2j/3}T^j(\mathcal{P}_j(x',\xi',\lambda,z;h) + h^{2/3}\mathcal{D}_{j+1}(x';h)),$$

where $\mathcal{K}_0$ is the same as above, while $\mathcal{P}_j$ and $\mathcal{D}_j$ satisfy the same symbol properties (6.3.4) and (6.3.5).

Then the associated formal operator $\mathfrak{P}$ is given by

$$\begin{aligned}
\mathfrak{P} &= \sum_{\alpha\in\mathbb{N}^{n-1}} \frac{1}{\alpha!}\partial_{\xi'}^{\alpha}(\mathcal{P}(x',\xi',\lambda,z;h))(hD_{x'})^{\alpha}\\
&= \sum_{\alpha\in\mathbb{N}^{n-1}} \frac{1}{\alpha!}[\partial_{\xi''}^{\alpha''}(\partial_{\xi_1}+h^{-2/3}\partial_\lambda)^{\alpha_1}\mathcal{P}](x',\xi',\lambda,z;h)(hD_{x'})^{\alpha}\\
&= \sum_{\alpha\in\mathbb{N}^{n-1}} \frac{1}{\alpha!}[(h\partial_{\xi''})^{\alpha''}(h\partial_{\xi_1}+h^{1/3}\partial_\lambda)^{\alpha_1}\mathcal{P}](x',\xi',\lambda,z;h)D_{x'}^{\alpha}\\
&= h^{2/3}\mathcal{K}_0 + \sum_{j=1}^{\infty} h^{2j/3}T^{j-1}\mathcal{D}_j\\
&\quad + \sum_{\alpha\in\mathbb{N}^{n-1}} \frac{1}{\alpha!}\sum_{j\in\mathbb{N}} h^{2j/3}T^j[(h\partial_{\xi''})^{\alpha''}(h\partial_{\xi_1}+h^{1/3}\partial_\lambda)^{\alpha_1}\mathcal{P}_j](x',\xi',\lambda,z;h)D_{x'}^{\alpha'}
\end{aligned}$$

It is of the form (6.3.7) with terms satisfying (6.3.8), and has principal term $\mathcal{P}_0(x',\xi,\lambda,z) = \mathcal{P}_0(z)$. Here we write $\alpha' = (\alpha_1,\alpha'')$.

For the inverse, we consider operators $\mathfrak{B}$ of the same form as $\mathfrak{A}$ with $\mathcal{A}_{\alpha,j,k,l,m}$ replaced by $\mathcal{B}_{\alpha,j,k,l,m}$ satisfying

$$\partial_{x'}^{\tilde\alpha}\partial_{\xi'}^{\tilde\beta}\partial_{\lambda}^{\tilde l}\operatorname{ad}_T^{\tilde k}\mathcal{B}_{\alpha,j,k,l,m} = O(\langle\lambda\rangle^{-\tilde l-\tilde k/2}) : \mathcal{H}_{z,\lambda,r}\to\mathcal{B}_{z,\lambda,r}. \tag{6.3.9}$$

Then the composition of $\mathfrak{A}$ and $\mathfrak{B}$,

$$\mathfrak{C} = \mathfrak{A}\circ\mathfrak{B}, \quad (\text{or } \mathfrak{B}\circ\mathfrak{A}),$$

is of the same form as $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathcal{A}_{\alpha,j,k,l,m}$ or $\mathcal{B}_{\alpha,j,k,l,m}$ replaced by $\mathcal{C}_{\alpha,j,k,l,m}$ satisfying

$$\partial_{x'}^{\tilde\alpha}\partial_{\xi'}^{\tilde\beta}\partial_{\lambda}^{\tilde l}\operatorname{ad}_T^{\tilde k}\mathcal{B}_{\alpha,j,k,l,m} = O(\langle\lambda\rangle^{-\tilde l-\tilde k/2}) : \mathcal{H}_{z,\lambda,r}\to\mathcal{H}_{z,\lambda,r}. \tag{6.3.10}$$

(or $\mathcal{B}_{z,\lambda,r} \to \mathcal{B}_{z,\lambda,r}$.)

Now the construction of the formal inverses is through the standard techniques of Neumann series.

Lemma 6.3.2. *Let $\mathfrak{A}$ be of the form* (6.3.7) *with terms satisfying* (6.3.8)*, and with $\mathcal{A}_0$ invertible. Suppose that $\mathcal{B}_0 = \mathcal{A}_0^{-1}$ satisfies*

$$\mathcal{B}_0 = O(1) : \mathcal{H}_{z,\lambda,r} \to \mathcal{B}_{z,\lambda,r}.$$

Then there exists $\mathfrak{B}$ as above with the principal term $\mathcal{B}_0$ such that

$$\mathfrak{A} \circ \mathfrak{B} = \mathrm{Id}, \quad \mathfrak{B} \circ \mathfrak{A} = \mathrm{Id}\,.$$

Proof. Let $\mathfrak{C} = \mathfrak{A} \circ \mathfrak{B}_0$ where $\mathfrak{B}_0 = \mathcal{B}_0$, then $\mathfrak{C}$ is as above with $\mathcal{C}_0 = \mathcal{A}_0 \circ \mathcal{B}_0 = \mathrm{Id}$. Therefore we can form the formal Neumann series

$$\mathfrak{D} = \mathrm{Id} + (\mathrm{Id} - \mathfrak{C}) + (\mathrm{Id} - \mathfrak{C}) \circ (\mathrm{Id} - \mathfrak{C}) + \cdots$$

which again gives a formal operator as above. Then we can simply take $\mathfrak{B} = \mathcal{B}_0 \circ \mathfrak{D}$ to get the right inverse. The left inverse can be constructed in the same way and the standard argument shows that the two must have the same formal expansions. And it is clear from the construction that the principal term of $\mathfrak{B}$ is $\mathcal{B}_0$. □

Now applying this lemma to $\mathfrak{P}$, we get an inverse $\mathfrak{E}$. Letting $\mathcal{E} = \mathfrak{E}(1)$, we get a parametrix for $\mathcal{P}(z)$ in the region $|R(x', \xi') - w| \leqslant 2C^{-1}$:

$$\mathcal{E}(x', \xi', \lambda, z; h) = \sum_{j,k,l,m \in \mathbb{N}} (h^{2/3} T)^j (h^{2/3} \langle \lambda \rangle^{-1/2})^k (h^{1/3} \langle \lambda \rangle^{-1})^l h^m \mathcal{E}_{0,j,k,l,m}(x', \xi', \lambda, z) \quad (6.3.11)$$

with

$$\partial_{x'}^{\tilde{\alpha}} \partial_{\xi'}^{\tilde{\beta}} \partial_{\lambda}^{\tilde{l}} \operatorname{ad}_T^{\tilde{k}} \mathcal{E}_{0,j,k,l,m} = O(\langle \lambda \rangle^{-\tilde{l} - \tilde{k}/2}) : \mathcal{H}_{z,\lambda,r} \to \mathcal{B}_{z,\lambda,r}. \quad (6.3.12)$$

In particular, the principal term is exactly $\mathcal{E}_0$ as we constructed in the previous section.

6.3.2 Analysis away from the glancing hypersurface

Now we deal with the region $|R(x', \xi') - w| > C^{-1}$. In this case, $Q \ll |\lambda| = h^{-2/3} |R - w|$ so that we are working with the second model operator in last section where we regard $t Q(h^{2/3} t, x', \xi')$ also as a perturbation. Let

$$P_0^{\#} = e^{-2\pi i/3} D_t^2 + \lambda, \quad \lambda = h^{-2/3}(R(x', \xi') - w),$$

and $R_\pm$ as before. The operator-valued symbol

$$\mathcal{P}_0^{\#}(z) = \begin{pmatrix} P_0^{\#} - z & R_- \\ \gamma_1 & 0 \\ R_+ & 0 \end{pmatrix} : \mathcal{B}_{\lambda,r}^{\#} \to \mathcal{H}_{\lambda,r}^{\#} \quad (6.3.13)$$

is uniformly invertible with inverse $\mathcal{E}_0^{\#}(z)$ since $|\lambda| \geqslant h^{-2/3}/C \gg |\operatorname{Re} z|$. Moreover,

$$\mathcal{P}_0^{\#}(z) \in S_{\Sigma_w,2/3}(\partial\mathcal{O}; 1, \mathcal{L}(\mathcal{B}^{\#}_{\lambda,r}, \mathcal{H}^{\#}_{\lambda,r})).$$

Recall the definition for the symbol class that away from the glancing hypersurface, the symbol behaves classically and we do not need to specify the derivative in λ. However, we need to consider the possibility that ξ' may get large. More precisely, the symbol properties for $\mathcal{P}_0^{\#}$ and $\mathcal{E}_0^{\#}$ are given by

$$\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta} \operatorname{ad}_T^k \mathcal{P}_0^{\#}(z) = O(\langle\xi'\rangle^{-|\beta|}\langle\lambda\rangle^{k/2}) : \mathcal{B}^{\#}_{\lambda,r} \to \mathcal{H}^{\#}_{\lambda,r};$$

$$\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta} \operatorname{ad}_T^k \mathcal{P}_0^{\#}(z) = O(\langle\xi'\rangle^{-|\beta|}\langle\lambda\rangle^{-k/2}) : \mathcal{H}^{\#}_{\lambda,r} \to \mathcal{B}^{\#}_{\lambda,r};$$

where we notice that $|\lambda|^{-k/2} \sim (h^{-1/3}\langle\xi'\rangle)^{-k}$ and $Q(0,x',\xi') = O(h^{2/3})|\lambda|$. For the lower order term in the expansion

$$\mathcal{P}(z) \equiv h^{2/3}\mathcal{K}_0 + \sum_{j=0}^{\infty}(h^{2/3}T)^j\mathcal{P}_j^{\#}(x,\xi,z;h)$$

with T, $\mathcal{K}_0$ as before and

$$\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta} \operatorname{ad}_T^k \mathcal{P}_j^{\#} = O(1)\langle\xi'\rangle^{-|\beta|}(h^{1/3}\langle\xi'\rangle^{-1})^k : \mathcal{B}^{\#}_{\lambda,r} \to \mathcal{H}^{\#}_{\lambda,r}.$$

We proceed exactly as before to define the associated formal operator

$$\mathfrak{P}^{\#} = \sum_{\alpha\in\mathbb{N}^{n-1}} \frac{1}{\alpha!}((h\partial_{\xi'})^{\alpha}\mathcal{P})D_{x'}^{\alpha}.$$

This motivates us to consider the general class of formal operators of the form

$$\mathfrak{A}^{\#} = \sum_{\alpha\in\mathbb{N}^{n-1},j,k\in\mathbb{N}} (h^{2/3}T)^j(h\langle\xi'\rangle)^k\mathcal{A}^{\#}_{\alpha,j,k}(x',\xi',z;h)D_{x'}^{\alpha} \tag{6.3.14}$$

with

$$\partial_{x'}^{\tilde\alpha}\partial_{\xi'}^{\tilde\beta} \operatorname{ad}_T^{\tilde k} \mathcal{A}_{\alpha,j,k} = O(1)\langle\xi'\rangle^{-|\tilde\beta|}(h^{1/3}\langle\xi'\rangle^{-1})^{\tilde k} : \mathcal{B}^{\#}_{\lambda,r} \to \mathcal{H}^{\#}_{\lambda,r}. \tag{6.3.15}$$

So we see that $\mathfrak{P}^{\#}$ is in this class. The same argument as in the case near the glancing hypersurface shows that $\mathfrak{P}^{\#}$ has a formal inverse $\mathfrak{E}^{\#}$ of the same form satisfying the estimates with $\mathcal{H}^{\#}$ and $\mathcal{B}^{\#}$ exchanged. Therefore we have an inverse of $\mathcal{P}(z)$ in the region $|R(x',\xi') - w| \geqslant C^{-1}$,

$$\mathcal{E}^{\#}(x',\xi',z;h) = \mathfrak{E}^{\#}(1) = \sum_{j,k\in\mathbb{N}}(h^{2/3}T)^j(h\langle\xi'\rangle)^k\mathcal{E}^{\#}_{j,k}(x',\xi',z;h) \tag{6.3.16}$$

with the following mapping properties

$$\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta} \operatorname{ad}_T^{\tilde k} \mathcal{E}^{\#}_{j,k} = O(1)\langle\xi'\rangle^{-|\beta|}(h^{1/3}\langle\xi'\rangle)^k : \mathcal{H}^{\#}_{\lambda,r} \to \mathcal{B}^{\#}_{\lambda,r}. \tag{6.3.17}$$

6.3.3 Analysis in the intermediate region

In the intermediate region $C^{-1} \leqslant |R(x',\xi') - w| \leqslant 2C^{-1}$, we observe that both cases reduce to simpler expansions that coincide with each other. The key point is that in this region both λ and ξ' will be irrelevant. In fact, $|\xi'|$ is bounded and $\lambda \sim h^{-2/3}$. Therefore we have the expansions

$$\mathcal{E}(x',\xi',z;h) = \sum_{j,k\in\mathbb{N}} (h^{2/3}T)^j h^k \mathcal{E}_{j,k}(x',\xi',z;h)$$

where

$$\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta} \operatorname{ad}_T^{\tilde{k}} \mathcal{E}_{j,k} = O(h^{k/3}) : \mathcal{H}_{z,\lambda,r} \to \mathcal{B}_{z,\lambda,r};$$

and

$$\mathcal{E}^{\#}(x',\xi',z;h) = \sum_{j,k\in\mathbb{N}} (h^{2/3}T)^j h^k \mathcal{E}^{\#}_{j,k}(x',\xi',z;h)$$

where

$$\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta} \operatorname{ad}_T^{\tilde{k}} \mathcal{E}^{\#}_{j,k} = O(h^{k/3}) : \mathcal{H}^{\#}_{\lambda,r} \to \mathcal{B}^{\#}_{\lambda,r}.$$

Of course the same is true for $\mathcal{P}$ with $\mathcal{B}$ and $\mathcal{H}$ exchanged. Therefore we introduce spaces $\mathcal{B}$ and $\mathcal{H}$ which agree with $\mathcal{B}_{z,\lambda,r}$ and $\mathcal{H}_{z,\lambda,r}$ microlocally in $|R(x',\xi') - w| < 2C^{-1}$, and also agree with $\mathcal{B}^{\#}_{\lambda,r}$ and $\mathcal{H}^{\#}_{\lambda,r}$ microlocally in $|R(x',\xi') - w| > C^{-1}$. Then this coincidence on the intermediate region shows that the symbol $\mathcal{P}$ and $\mathcal{E}$ satisfy the global construction at least near the boundary.

6.4 Estimates away from the boundary

In this section, we review the proof of the following proposition which provides a parametrix away from the boundary in [30]. We write

$$D(\alpha) = \{x \in \mathbb{R}^n \setminus \mathcal{O} : d(x,\partial\mathcal{O}) > \alpha\}.$$

Proposition 6.4.1. *Let $0 < \epsilon < \frac{2}{3}$, $|\operatorname{Re} z| \leqslant L$, $|\operatorname{Im} z| \leqslant C$, then there exists $h_0 = h_0(L)$ such that for $0 < h < h_0(L)$, there exists maps E_ϵ, K_ϵ defined on $C_c^\infty(D(h^\epsilon))$, with the properties $(P-z)E_\epsilon = I + K_\epsilon$ and*

$$\begin{aligned} E_\epsilon &= O(h^{2/3-\epsilon}) : L^2(D(h^\epsilon)) \to H_h^2(\mathbb{R}^n \setminus \mathcal{O}), \\ K_\epsilon &= O(e^{-C^{-1}h^{-1+\frac{3\epsilon}{2}}}) : L^2(D(h^\epsilon)) \to H_h^k(\mathbb{R}^n \setminus \mathcal{O}), \quad \forall k \in \mathbb{R}. \end{aligned} \tag{6.4.1}$$

Moreover, for any fixed $\gamma \in (0,1)$, we can construct E_ϵ and K_ϵ such that for $u \in C_c^\infty(D(h^\epsilon))$, $E_\epsilon u$ and $K_\epsilon u$ are supported in $D((1-\gamma)h^\epsilon)$.

We remark that we can not use Neumann series and this proposition to give an inverse of $P - z$ since the support of $K_\epsilon u$ is larger than that of u in general.

In a fixed distance away from the obstacle, we get the following better results which gives a parametrix of $P - z$.

Lemma 6.4.2. *If $v_1 \in C_c^\infty(\mathbb{R}^n \setminus \mathcal{O})$ satisfies $\operatorname{supp}(v_1) \subset D(\delta)$ where $\delta > 0$, then there exists $u_1 \in C^\infty(\mathbb{R}^n \setminus \mathcal{O})$ and $v_0^1 \in C_c^\infty(\mathbb{R}^n \setminus \mathcal{O})$ such that $u_1 = 0$ near $\partial\mathcal{O}$ and*

$$(P - z)u_1 = v_1 + v_0^1, \tag{6.4.2}$$

$$\begin{aligned} &\|u_1\|_{H^2(\mathbb{R}^n\setminus\mathcal{O})} + e^{C/h}\|u_1\|_{H^k(\mathbb{R}^n\setminus D((1-\gamma)\delta)} \\ &+ e^{C/h}\|v_0^1\|_{H^k(\mathbb{R}^n\setminus\mathcal{O})} \leqslant C_\gamma h^{2/3}\|v_1\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})}. \end{aligned} \tag{6.4.3}$$

for any $k, \gamma > 0$.

For the region $D(h^\epsilon) \setminus D(\delta) = \{h^\epsilon < d(x, \partial\mathcal{O}) < \delta\}$ which is h^ϵ-near to the obstacle, we have the following lemma.

Lemma 6.4.3. *If $v_2 \in C_c^\infty(\mathbb{R}^n \setminus \mathcal{O})$, $\operatorname{supp}(v_2) \subset \{x : h^\epsilon < d(x, \partial\mathcal{O}) < \delta\}$, then there exists $u_2 \in C_c^\infty(\mathbb{R}^n \setminus \mathcal{O})$ such that $u_2 = 0$ near $\partial\mathcal{O}$ and*

$$(P - z)u_2 = v_2 + v_0^2, \tag{6.4.4}$$

$$\begin{aligned} &\|u_2\|_{H^2(\mathbb{R}^n\setminus\mathcal{O})} + e^{Ch^{-1+3\epsilon/2}}\|u_2\|_{H^k(\mathbb{R}^n\setminus D((1-\gamma)h^\epsilon)} \\ &+ e^{C/h}\|u_2\|_{H^k(D(1+\gamma)\delta)} + e^{Ch^{-1+3\epsilon/2}}\|v_0^2\|_{H^k(\mathbb{R}^n\setminus\mathcal{O})} \\ \leqslant\ & h^{2/3-\epsilon}\|v_2\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})} \end{aligned} \tag{6.4.5}$$

The proof of the proposition then follows from the two lemmas by decomposing any $v \in C_c^\infty(\mathbb{R}^n \setminus \mathcal{O})$ with $\operatorname{supp} v \subset D(h^\epsilon)$ into

$$\begin{aligned} v = v_1 + v_2,\ \ v_1, v_2 \in C_c^\infty(\mathbb{R}^n \setminus \mathcal{O}),\ \ \|v_1\|_{L^2} + \|v_2\|_{L^2} \leqslant 2\|v\|_{L^2} \\ \text{and } \operatorname{supp} v_1 \subset D(\delta/2), \quad \operatorname{supp} v_2 \subset D(h^\epsilon) \setminus D(\delta). \end{aligned} \tag{6.4.6}$$

6.5 Global Grushin problem

6.5.1 Setting up for global Grushin problems

To study the global Grushin problem, we introduce the spaces for $w \in W \Subset (0, \infty)$, $0 < \delta \ll 1$, $0 \leqslant r \leqslant r_0$:

$$\begin{aligned} \mathcal{B}_{w,r,\delta} &= H^2(\mathbb{R}^n \setminus \mathcal{O}) \times L^2(\partial\mathcal{O}; \mathbb{C}^N), \\ \mathcal{H}_{w,r} &= L^2(\mathbb{R}^n \setminus \mathcal{O}) \times H^{1/2}(\partial\mathcal{O}) \times H^2(\partial\mathcal{O}; \mathbb{C}^N). \end{aligned}$$

with the norms microlocally compatible with the ones introduced in section 6.2, (6.2.14), (6.2.37) in each of the regions we considered. We need to translate the norms to x_n-coordinates by the relation $x_n = h^{2/3}t$.

Let

$$\begin{aligned}
\left\| \begin{pmatrix} u \\ u_- \end{pmatrix} \right\|_{\mathcal{B}_{w,r,\delta}} &= h^{-2/3} \| e^{r\psi(x_n)/2h^{2/3}} (hD_{x_n})^2 u \|_{L^2(\mathbb{R}^n \setminus \mathcal{O})} \\
&\quad + h^{-2/3} \| e^{r\psi(x_n)/2h^{2/3}} \chi(x_n/\delta) x_n u \|_{L^2(\mathbb{R}^n \setminus \mathcal{O})} \\
&\quad + \| e^{r\psi(x_n)/2h^{2/3}} \langle x_n \rangle^{-2} \langle h^{-2/3}(-h^2 \Delta_{\partial \mathcal{O}} - w) \rangle u \|_{L^2(\mathbb{R}^n \setminus \mathcal{O})} \\
&\quad + h^{-2/3} \| e^{r\psi(x_n)/2h^{2/3}} (1 - \chi(x_n/\delta)) u \|_{H_h^2(\mathbb{R}^n \setminus \mathcal{O})} \\
&\quad + h^{1/3} \| u_- \|_{L^2(\partial \mathcal{O}; \mathbb{C}^N)}, \\
\left\| \begin{pmatrix} v \\ v_0 \\ v_+ \end{pmatrix} \right\|_{\mathcal{H}_{w,r}} &= \| e^{r\psi(x_n)/2h^{2/3}} v \|_{L^2(\mathbb{R}^n \setminus \mathcal{O})} \\
&\quad + h^{1/3} \| \langle h^{-2/3}(-h^2 \Delta_{\partial \mathcal{O}} - w) \rangle^{1/4} v_0 \|_{L^2(\partial \mathcal{O})} \\
&\quad + h^{1/3} \| \langle h^{-2/3}(-h^2 \Delta_{\partial \mathcal{O}} - w) \rangle v_+ \|_{L^2(\partial \mathcal{O}; \mathbb{C}^N)},
\end{aligned} \tag{6.5.1}$$

where the weight function $\psi \in C^\infty([0,\infty);[0,1])$ satisfying $\psi(t) = t$ for $t < \frac{1}{2}$ and $\psi(t) = 1$ for $t \geqslant 1$; and the cut-off function $\chi \in C^\infty([0,\infty);[0,1])$ satisfying $\chi(t) = 1$ for $t < 1$ and $\chi(t) = 0$ for $t > 2$. Here we still use the geodesic normal coordinates $(x', x_n) \in \partial\mathcal{O} \times (0, \infty)$ for $\mathbb{R}^n \setminus \mathcal{O}$ as introduced before.

First we claim that

$$\begin{pmatrix} P - z & 0 \\ \gamma_1 & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{B}_{w,r} \to \mathcal{H}_{w,r}.$$

In fact, we can decompose $u \in H^2(\mathbb{R}^n \setminus \mathcal{O})$ as $u = u_1 + u_2$ where $\operatorname{supp} u_1 \subset \{x_n \leqslant 3\delta\}$ and $\operatorname{supp} u_2 \subset \{x_n \geqslant 2\delta\}$. Then we see that

$$\left\| \begin{pmatrix} u \\ 0 \end{pmatrix} \right\|_{\mathcal{B}_{w,r,\delta}} \sim \left\| \begin{pmatrix} u_1 \\ 0 \end{pmatrix} \right\|_{\mathcal{B}_{w,r,\delta}} + \left\| \begin{pmatrix} u_2 \\ 0 \end{pmatrix} \right\|_{\mathcal{B}_{w,r,\delta}}.$$

We notice that

$$\begin{aligned}
\left\| \begin{pmatrix} u_1 \\ 0 \end{pmatrix} \right\|_{\mathcal{B}_{w,r,\delta}} &\sim h^{-2/3} \| e^{rx_n/2h^{2/3}} (hD_{x_n})^2 u_1 \|_{L^2(\mathbb{R}^n \setminus \mathcal{O})} \\
&\quad + h^{-2/3} \| e^{rx_n/2h^{2/3}} \chi(x_n/\delta) x_n u \|_{L^2(\mathbb{R}^n \setminus \mathcal{O})} \\
&\quad + \| e^{rx_n/2h^{2/3}} \langle h^{-2/3}(-h^2 \Delta_{\partial \mathcal{O}} - w) \rangle u \|_{L^2(\mathbb{R}^n \setminus \mathcal{O})},
\end{aligned}$$

so the estimate

$$\| e^{r\psi(x_n)/2h^{2/3}} (P - z) u_1 \|_{L^2} \leqslant \left\| \begin{pmatrix} u_1 \\ 0 \end{pmatrix} \right\|_{\mathcal{B}_{w,r,\delta}}$$

follows from the change of variable $x_n = h^{2/3}t$ and Lemma 6.2.2. Also notice that

$$\left\| \begin{pmatrix} u_2 \\ 0 \end{pmatrix} \right\|_{\mathcal{B}_{w,r,\delta}} \sim h^{-2/3} \| e^{r\psi(x_n)/2h^{2/3}} u_2 \|_{H_h^2(\mathbb{R}^n \setminus \mathcal{O})},$$

so we can easily deduce that

$$\| e^{r\psi(x_n)/2h^{2/3}} (P - z) u_2 \|_{L^2} \leqslant \left\| \begin{pmatrix} u_2 \\ 0 \end{pmatrix} \right\|_{\mathcal{B}_{w,r,\delta}}.$$

Finally we need to estimate γu. We shall use the fact that

$$\gamma_0 = O(h^{-1/2}) : H_h^2(\mathbb{R}^n \setminus \mathcal{O}) \to H_h^{3/2}(\partial\mathcal{O})$$

and

$$h\gamma_1 = O(h^{-1/2}) : H_h^2(\mathbb{R}^n \setminus \mathcal{O}) \to H_h^{1/2}(\partial\mathcal{O})$$

which follows from the estimates of non-semiclassical restriction operators. Therefore we have

$$\begin{aligned} & h^{1/3} \| \langle h^{-2/3}(-h^2\Delta_{\partial\mathcal{O}} - w) \rangle^{1/4} (\gamma u) \|_{L^2(\partial\mathcal{O})} \\ \leqslant\ & h^{1/6} \| \gamma u \|_{H_h^{1/2}(\partial\mathcal{O})} \leqslant h^{1/6} \| h^{2/3} \gamma_1 u \|_{H_h^{1/2}(\partial\mathcal{O})} + h^{1/6} \| h^{2/3} k \gamma_0 u \|_{H_h^{1/2}(\partial\mathcal{O})} \\ \leqslant\ & h^{5/6} \| \gamma_1 u \|_{H_h^{1/2}(\partial\mathcal{O})} + C h^{5/6} \| \gamma_0 u \|_{H_h^{3/2}(\partial\mathcal{O})} \\ \leqslant\ & C h^{-2/3} \| u \|_{H_h^2(\mathbb{R}^n \setminus \mathcal{O})}. \end{aligned}$$

Now we need to correct this operator with

$$R_{+,w} : H^2(\mathbb{R}^n \setminus \mathcal{O}) \to L^2(\partial\mathcal{O}; \mathbb{C}^N),$$

and

$$R_{-,w} : L^2(\partial\mathcal{O}; \mathbb{C}^N) \to L^2(\mathbb{R}^n \setminus \mathcal{O}).$$

They are obtained by quantizing the symbols that appeared in section 6.2. Let $e_{j,\mu}^{\lambda,\delta}$ be as in (6.2.41), then we shall define

$$R_{+,w} = \mathrm{Op}_{\Sigma_w,h}(\tilde{e}_w^\delta) : L^2(\mathbb{R}^n \setminus \partial\mathcal{O}) \to L^2(\partial\mathcal{O}; \mathbb{C}^N), \tag{6.5.2}$$

where

$$\tilde{e}_w^\delta \in S_{\Sigma_w,2/3}(\partial\mathcal{O}; 1, \mathcal{L}(L^2[0,\infty); \mathbb{C}^N))$$

is given by

$$\tilde{e}_w^\delta(j) u(p) = \int_0^\infty h^{-1/3} \chi(x_n) e_{j,\mu}^{\lambda,\delta}(h^{-2/3} x_n) u(x_n) dx_n, p \in T^*\partial\mathcal{O}$$

with $\lambda = h^{-2/3}(R(p) - w), \mu = Q(0, p)$. Similarly, the operator $R_{-,w}$ can be defined as the formal adjoint of $R_{+,w}$ or more precisely,

$$R_{-,w} = \mathrm{Op}_{\Sigma_w,h}((\tilde{e}_w^\delta)^*) : L^2(\mathbb{R}^n \setminus \partial\mathcal{O}) \to L^2(\partial\mathcal{O}; \mathbb{C}^N),$$

where

$$(\tilde{e}_w^\delta)^* \in S_{\Sigma_w,2/3}(\partial\mathcal{O}; 1, \mathcal{L}(\mathbb{C}^N; L^2([0,\infty)))$$

is given by

$$\tilde{e}_w^\delta u_-(p) = \sum_{j=1}^N h^{-1/3}\chi(x_n)e_{j,\mu}^{\lambda,\delta}(h^{-2/3}x_n)u_-(j), p \in T^*\partial\mathcal{O}.$$

Then we have the Grushin problem for

$$\mathcal{P}_w(z) = \begin{pmatrix} P_w - z & R_{-,w} \\ \gamma_1 & 0 \\ R_{+,w} & 0 \end{pmatrix} : \mathcal{B}_{w,r} \to \mathcal{H}_{w,r}. \tag{6.5.3}$$

Our goal is to construct an inverse of $\mathcal{P}_w(z)$ for all h small depending on δ,

$$\mathcal{E}_w(z) = \begin{pmatrix} E_w(z) & K_w(z) & E_{w,+}(z) \\ E_{w,-}(z) & K_{w,-}(z) & E_{w,-+}(z) \end{pmatrix} : \mathcal{H}_{w,r} \to \mathcal{B}_{w,r} \tag{6.5.4}$$

where $E_{w,-+}(z)$ has nice properties that will be specified later.

6.5.2 Construction of the inverse operator

To construct the inverse operator, we first divide phase space into three different parts: near the boundary and glancing hypersurface, near the boundary away from the glancing hypersurface and away from the boundary. In this section, we again work with $w = 1$ for simplicity and it will be clear that the analysis is uniform for w in a fixed compact subset of $(0, \infty)$.

We consider the case near the boundary and glancing hypersurface first. Let us translate the space $\mathcal{B}_{z,\lambda,r}$ and $\mathcal{H}_{z,\lambda,r}$ in section 6.2 into the x_n-coordinates and scale it by $h^{1/3}$ due to the change of coordinates. In this stage, we drop the dependence on z and introduce the same weight function ψ as previously.

$$\begin{aligned}\left\| \begin{pmatrix} u \\ u_- \end{pmatrix} \right\|_{\mathcal{B}_{\lambda,r}} &= h^{-2/3}\|e^{r\psi(x_n)/2h^{2/3}}(hD_{x_n})^2u\|_{L^2([0,\infty))} + h^{-2/3}\|e^{r\psi(x_n)/2h^{2/3}}x_nu\|_{L^2([0,\infty))} \\ &\quad + \langle\lambda\rangle\|e^{r\psi(x_n)/2h^{2/3}}u\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})} + h^{1/3}|u_-|_{\mathbb{C}^N}, \\ \left\| \begin{pmatrix} v \\ v_0 \\ v_+ \end{pmatrix} \right\|_{\mathcal{H}_{\lambda,r}} &= \|e^{r\psi(x_n)/2h^{2/3}}v\|_{L^2([0,\infty))} + h^{1/3}\langle\lambda\rangle^{1/4}|v_0|_{\mathbb{C}} + h^{1/3}\langle\lambda\rangle|v_+|_{\mathbb{C}^N}.\end{aligned}$$

Lemma 6.5.1. *Let $0 < \epsilon < 2/3$, $\chi_1 \in \Psi^{0,0}(\partial\mathcal{O})$ be such that $\mathrm{WF}_h(\chi_1 - \mathrm{Id}) \subset \{m : d(m,\Sigma) \geqslant C\}$ and $\mathrm{WF}_h(\chi_1) \subset \{m : d(m,\Sigma) \leqslant 2C\}$. Then there exists $\mathcal{E}_1^L(z), \mathcal{E}_1^R(z) \in \Psi_{\Sigma,2/3}(\partial\mathcal{O}; 1, \mathcal{L}(\mathcal{H}_{\lambda,r}, \mathcal{B}_{\lambda,r}))$ such that*

$$\mathcal{E}_1^L(z)\mathcal{P}(z) = \chi_1 \begin{pmatrix} \chi(x_n/h^\epsilon) & 0 \\ 0 & \mathrm{Id} \end{pmatrix} + \mathcal{R}_1^L(z),$$

$$\mathcal{P}(z)\mathcal{E}_1^R(z) = \chi_1 \begin{pmatrix} \chi(x_n/h^\epsilon) & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix} + \mathcal{R}_1^R(z),$$

where the remainder terms satisfy

$$\mathcal{R}_1^L(z) \in \Psi_{\Sigma,2/3}(\partial\mathcal{O}; h^N\langle\lambda\rangle^{-N}, \mathcal{L}(\mathcal{B}_{\lambda,r}, \mathcal{B}_{\lambda,r}))$$
$$\mathcal{R}_1^R(z) \in \Psi_{\Sigma,2/3}(\partial\mathcal{O}; h^N\langle\lambda\rangle^{-N}, \mathcal{L}(\mathcal{H}_{\lambda,r}, \mathcal{H}_{\lambda,r}))$$

for any N.

Proof. From section 6.3.1, we can construct an operator $\tilde{\mathcal{E}}_1 \in \Psi_{\Sigma,2/3}(\partial\mathcal{O}; 1, \mathcal{L}(\mathcal{H}_{\lambda,r}, \mathcal{B}_{\lambda,r}))$ with $\mathrm{WF}_h(\tilde{\mathcal{E}}_1) \subset \{m : d(m,\Sigma) \leqslant 2C\}$ such that

$$\tilde{\mathcal{E}}_1(z)\mathcal{P}(z) = \mathrm{Id} + \tilde{R}_1^L(z), \quad \mathcal{P}(z)\tilde{\mathcal{E}}_1(z) = \mathrm{Id} + \tilde{R}_1^R(z).$$

Here the remainder term $\tilde{R}_1^L$ satisfies that for any $A \in \Psi^{0,0}(\partial\mathcal{O})$ with $\mathrm{WF}_h(A) \subset \{m : d(m,\Sigma) \leqslant C\}$ and any k,

$$A\tilde{R}_1^L = \begin{pmatrix} x_n^k & 0 \\ 0 & 0 \end{pmatrix} B_k^L + h^k A_k^L,$$

with

$$A_k^L, B_k^L \in \Psi_{\Sigma,2/3}(\partial\mathcal{O}; 1, \mathcal{L}(\mathcal{B}_{\lambda,r}, \mathcal{B}_{\lambda,r})).$$

We notice that for $0 < \epsilon < 2/3$, the operator

$$\begin{pmatrix} \chi(x_n/h^\epsilon) & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

is bounded on $\mathcal{B}_{\lambda,r}$. In fact, in t coordinates, this becomes $\chi(h^{2/3-\epsilon}t)$ whose derivatives are all bounded. Therefore we can set

$$\mathcal{E}_l^L(z) = \chi_1 \begin{pmatrix} \chi(x_n/h^\epsilon) & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \tilde{\mathcal{E}}_1(z).$$

Since $\langle\lambda\rangle = O(h^{-2/3})$, it is clear that this operator satisfies the condition. Similarly, we can construct

$$\mathcal{E}_l^R(z) = \tilde{\mathcal{E}}_1(z)\chi_1 \begin{pmatrix} \chi(x_n/h^\epsilon) & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}.$$

□

Now for the case near the boundary but away from the glancing hypersurface, the spaces $\mathcal{H}^{\#}_{\lambda,r}$ and $\mathcal{B}^{\#}_{\lambda,r}$ become

$$\left\| \begin{pmatrix} u \\ u_- \end{pmatrix} \right\|_{\mathcal{B}^{\#}_{\lambda,r}} = h^{-2/3} \| e^{r\psi(x_n)/2h^{2/3}} (hD_{x_n})^2 u \|_{L^2([0,\infty))} + \langle \lambda \rangle \| e^{r\psi(x_n)/2h^{2/3}} u \|_{L^2(\mathbb{R}^n \setminus \mathcal{O})} + h^{1/3} |u_-|_{\mathbb{C}^N},$$

$$\left\| \begin{pmatrix} v \\ v_0 \\ v_+ \end{pmatrix} \right\|_{\mathcal{H}_{\lambda,r}} = \| e^{r\psi(x_n)/2h^{2/3}} v \|_{L^2([0,\infty))} + h^{1/3} \langle \lambda \rangle^{1/4} |v_0|_{\mathbb{C}} + h^{1/3} \langle \lambda \rangle |v_+|_{\mathbb{C}^N},$$

in the x_n-coordinates. In this situation, we have

Lemma 6.5.2. *Let $0 < \epsilon < 2/3$, $\chi_2 \in \Psi^{0,0}(\partial\mathcal{O})$ be such that $\mathrm{WF}_h(\chi_2 - \mathrm{Id}) \subset \{m : d(m, \Sigma) \leqslant C\}$ and $\mathrm{WF}_h(\chi_2) \subset \{m : d(m,\Sigma) \geqslant \frac{1}{2}C\}$. Then there exists $\mathcal{E}_2^L(z), \mathcal{E}_2^R(z) \in \Psi_{\Sigma,2/3}(\partial\mathcal{O}; 1, \mathcal{L}(\mathcal{H}_{\lambda,r}, \mathcal{B}_{\lambda,r}))$ such that*

$$\mathcal{E}_2^L(z)\mathcal{P}(z) = \chi_2 \begin{pmatrix} \chi(x_n/h^\epsilon) & 0 \\ 0 & \mathrm{Id} \end{pmatrix} + \mathcal{R}_2^L(z),$$

$$\mathcal{P}(z)\mathcal{E}_2^R(z) = \chi_2 \begin{pmatrix} \chi(x_n/h^\epsilon) & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix} + \mathcal{R}_2^R(z),$$

where the remainder terms satisfy

$$\mathcal{R}_2^L(z) \in \Psi_{\Sigma,2/3}(\partial\mathcal{O}; h^N \langle\lambda\rangle^{-N}, \mathcal{L}(\mathcal{B}^{\#}_{\lambda,r}, \mathcal{B}^{\#}_{\lambda,r}))$$
$$\mathcal{R}_2^R(z) \in \Psi_{\Sigma,2/3}(\partial\mathcal{O}; h^N \langle\lambda\rangle^{-N}, \mathcal{L}(\mathcal{H}^{\#}_{\lambda,r}, \mathcal{H}^{\#}_{\lambda,r}))$$

for any N.

Proof. We can repeat the same argument with the standard semiclassical calculus and notice that $\langle\lambda\rangle = O(h^{-2/3}\langle\xi'\rangle^2)$ to get the properties of the remainder. $\square$

Now combining the two lemmas above, we get the approximated inverse near the boundary. More precisely,

Proposition 6.5.3. *There exists $\mathcal{E}^L(z), \mathcal{E}^R(z) : \mathcal{H}_r \to \mathcal{B}_{r,\epsilon}$ such that*

$$\mathcal{E}^L(z)\mathcal{P}(z) = \begin{pmatrix} \chi(x_n/h^\epsilon) & 0 \\ 0 & \mathrm{Id} \end{pmatrix} + \mathcal{R}^L(z),$$

$$\mathcal{P}(z)\mathcal{E}^R(z) = \begin{pmatrix} \chi(x_n/h^\epsilon) & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix} + \mathcal{R}^R(z),$$

where the remainder terms satisfy

$$\langle h^2\Delta_{\partial\mathcal{O}}\rangle^N\mathcal{R}_3^L(z)\langle h^2\Delta_{\partial\mathcal{O}}\rangle^N = O(h^N) : \mathcal{B}_{r,\epsilon} \to \mathcal{B}_{r,\epsilon}$$
$$\langle h^2\Delta_{\partial\mathcal{O}}\rangle^N\mathcal{R}_3^R(z)\langle h^2\Delta_{\partial\mathcal{O}}\rangle^N = O(h^N) : \mathcal{H}_r \to \mathcal{H}_r,$$

for any N. Here $\langle h^2\Delta_{\partial\mathcal{O}}\rangle^N$ applies to all the components and the spaces $\mathcal{B}_{r,\epsilon}$ are defined as $\mathcal{B}_{r,\delta}$ further truncated to the h^ϵ-neighborhood of the boundary by $\chi(x_n/h^\epsilon)$.. Moreover, the $-+$-components for the approximate inverses satisfy

$$E_{-+}^L(z) \equiv E_{-+}^R(z) \in \Psi_{\Sigma,2/3}^{0,1,2}(\partial\mathcal{O};\mathcal{L}(\mathbb{C}^N,\mathbb{C}^N)).$$

Proof. We can simply choose χ_1 and χ_2 such that $\chi_1+\chi_2 = 1$ and set $\mathcal{E}^\cdot(z) = \mathcal{E}_1^\cdot(z)+\mathcal{E}_2^\cdot(z)$, $\cdot = L, R$. To prove the last statement, we notice that from the construction,

$$E_{-+}^L = \chi_1\tilde{E}_{-+1} + \chi_2\tilde{E}_{-+2}, \quad E_{-+}^R = \tilde{E}_{-+1}\chi_1 + \tilde{E}_{-+2}\chi_2.$$

Near the glancing hypersurface, $\{m : d(m,\Sigma) \leqslant \frac{1}{2}C\}$, $\chi_1 \equiv \mathrm{Id}$ while $\chi_2 \equiv 0$. Away from the glancing hypersurface $\{m : d(m,\Sigma) \geqslant 2C\}$, $\chi_1 \equiv 0$ while $\chi_2 \equiv \mathrm{Id}$. In the intermediate region, $E_{-+1} \equiv E_{-+2}$ from our discussion in section 6.3.3. Therefore E_{-+}^L and E_{-+}^R are essentially the same in the $\Psi_{\Sigma,2/3}^{0,1,2}(\partial\mathcal{O};\mathcal{L}(\mathbb{C}^N,\mathbb{C}^N))$. □

Finally, we can combine this with the estimate away from the boundary to get the inverse.

Proposition 6.5.4. *Let $0 < \epsilon < 2/3$, $0 < h < h_0(\delta)$, there exists $\mathcal{E}_w(z) : \mathcal{H}_{w,0} \to \mathcal{B}_{w,0,\epsilon}$ such that*

$$\mathcal{P}_w(z)\mathcal{E}_w(z) = \mathrm{Id}, \quad \mathcal{E}_w(z)\mathcal{P}_w(z) = \mathrm{Id}$$

and $E_{w,-+} \in \Psi_{\Sigma_w,2/3}^{0,1,2}(\partial\mathcal{O};\mathcal{L}(\mathbb{C}^N,\mathbb{C}^N))$.

Proof. Let us begin with an approximate right inverse

$$\mathcal{E}^R(z) = \mathcal{E}^R(z)\begin{pmatrix} \tilde{\chi}(x_n/h^\epsilon) & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix} + \begin{pmatrix} E_\epsilon(1-\tilde{\chi}(x_n/h^\epsilon)) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here $\tilde{\chi} \in C^\infty([0,\infty))$ supported in $\{\chi = 1\}$. Then we can compute

$$\mathcal{P}(z)\tilde{\mathcal{E}}^R(z) = \mathrm{Id} + \mathcal{K}^R(z)$$

where the remainder is given by

$$\mathcal{K}^R(z) = \mathcal{R}^R(z)\begin{pmatrix} \tilde{\chi}(x_n/h^\epsilon) & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix} + \begin{pmatrix} K_\epsilon(1-\tilde{\chi}(x_n/h^\epsilon)) & 0 & 0 \\ \gamma E_\epsilon(1-\tilde{\chi}(x_n/h^\epsilon)) & 0 & 0 \\ R_+E_\epsilon(1-\tilde{\chi}(x_n/h^\epsilon)) & 0 & 0 \end{pmatrix}.$$

Since $E_\epsilon(1-\tilde{\chi})u$ is supported away from the boundary, we have $\gamma E_\epsilon(1-\tilde{\chi}(x_n/h^\epsilon)) = 0$. Moreover, for any smooth u, since $(1-\tilde{\chi}(x_n/h^\epsilon))u$ is supported in $D(h^\epsilon)$, $E_\epsilon(1-\tilde{\chi}(x_n/h^\epsilon))u$ is supported in $(D(1-\gamma)h^\epsilon)$, so by the super-exponential decay of $e^{\lambda,\delta}_{j,\mu}$, we have

$$\tilde{e}^\delta_w(j)u(p,x_n) = \int_0^\infty h^{-1/3}\chi(x_n)e^{\lambda,\delta}_{j,\mu}(h^{-2/3}x_n)u(p,x_n)dx_n = O(h^\infty) \tag{6.5.5}$$

which gives $R_+E_\epsilon(1-\tilde{\chi}(x_n/h^\epsilon)) = O(h^\infty)$. Therefore we get $\mathcal{K}^R = O(h^\infty) : \mathcal{H}_0 \to \mathcal{H}_0$ and hence for h small enough, $(\mathrm{Id}+\mathcal{K}^R)^{-1} = \mathrm{Id}+\mathcal{A}$ where $\mathcal{A} = O(h^\infty) : \mathcal{H}_0 \to \mathcal{H}_0$. We can now put

$$\mathcal{E}(z) = \mathcal{E}^R(z)(\mathrm{Id}+\mathcal{A}(z))$$

Suppose

$$\mathcal{A}(z) = \begin{pmatrix} A_{11}(z) & A_{12}(z) & A_{13}(z) \\ A_{21}(z) & A_{22}(z) & A_{23}(z) \\ A_{31}(z) & A_{32}(z) & A_{33}(z) \end{pmatrix}$$

then from the formula of $\mathcal{K}^R$, we see it is lower triangular and thus the same is true for $\mathcal{A}$. Therefore

$$E_{-+}(z) = E^R_{-+}(z) + E^R_{-+}(z)A_{33}(z)$$

Here $A_{33}(z) \in \Psi^{-\infty,-\infty}(\partial\mathcal{O};\mathcal{L}(\mathbb{C}^N,\mathbb{C}^N))$ since it comes entirely from $\mathcal{R}^R_3$. Therefore $E_{-+}(z) \in \Psi^{0,1,2}_{\Sigma_w,2/3}(\partial\mathcal{O};\mathcal{L}(\mathbb{C}^N,\mathbb{C}^N))$ is essentially the same as E^R_{-+} (and also as E^L_{-+}). $\square$

6.5.3 Reduction to E_{-+}

Now we state the main result of this section.

Theorem 5. *Assume that W is a fixed compact subset of $(0,\infty)$ and $\epsilon \ll 1$. For every $w \in W$ and $z \in \mathbb{C}$ such that $|\operatorname{Re} z| \ll 1/\delta$, $|\operatorname{Im} z| \leqslant C_1$, there exists*

$$E_{w,-+}(z) \in \Psi^{0,1,2}_{\Sigma_w,2/3} \tag{6.5.6}$$

where $\Sigma_w = \{p \in T^*\partial\mathcal{O} : R(p) = w\}$, $N = N(C_1)$ such that for $0 < h < h_0$ and some large $C > 0$:

(i) The multiplicity of resonances are given by

$$m_{\mathcal{O}}(h^{-2}(w+h^{2/3}z)) = \frac{1}{2\pi i}\operatorname{tr}\oint_{|\tilde{z}-z|=\epsilon} E_{w,-+}(\tilde{z})^{-1}\frac{d}{d\tilde{z}}E_{w,-+}(\tilde{z})d\tilde{z} \tag{6.5.7}$$

(ii) If $E^0_{w,-+}(z;p,h) = \sigma_{\Sigma,h}(E_{w,-+}(z))(p;h)$, $p \in T^*\partial\mathcal{O}$, then

$$E^0_{w,-+}(z,p,h) = O(\langle\lambda - \operatorname{Re} z\rangle) : \mathbb{C}^N \to \mathbb{C}^N, \tag{6.5.8}$$

where $\lambda = h^{-2/3}(R(p)-w)$.

(iii) For $|\lambda| \leqslant 1/C\sqrt{\delta}$,

$$\|E^0_{w,-+}(z;p,h) - \mathrm{diag}(z - \lambda - e^{-2\pi i/3}\zeta'_j(p))\|_{\mathcal{L}(\mathbb{C}^N,\mathbb{C}^N)} \leqslant \epsilon. \tag{6.5.9}$$

Moreover, $\det E^0_{w,-+}(z;p,h) = 0$ *if and only if*

$$z = \lambda + e^{-2\pi i/3}\zeta'_j(p) \tag{6.5.10}$$

for some $1 \leqslant j \leqslant N$ *and all zeroes are simple. Here* $\zeta'_j(p) = \zeta'_j(2Q(p))^{2/3}$.

(iv) For $|\lambda| \geqslant 1/C\sqrt{\delta}$, $E^0_{w,-+}$ *is invertible and*

$$E^0_{w,-+}(z,p,h)^{-1} = O(\langle \lambda - \mathrm{Re}\, z\rangle^{-1}) : \mathbb{C}^N \to \mathbb{C}^N, \tag{6.5.11}$$

Proof. The statement (i) follows from the formula

$$\begin{pmatrix} h^{-2/3}(P(h) - w) - z \\ \gamma \end{pmatrix}^{-1} = (E_w(z), K_w(z)) - E_{w,+}(z)E_{w,-+}(z)^{-1}(E_{w,-}(z), K_{w,-}(z)).$$

The other statements follow directly from our construction of $\mathcal{E}_w$: (ii) follows from proposition 6.2.4 and the symbolic construction of $\mathcal{E}$ and $\mathcal{E}^\#$ in section 6.3; (iii) follows from proposition 6.2.5 and the fact that when $|\lambda| \leqslant 1/C\sqrt{\delta}$, the only contribution in the symbol comes from $\mathcal{E}$; (iv) follows from proposition 6.2.5 and 6.2.6. □

6.6 Proof of the theorem

6.6.1 Resonance Bands

We first prove Theorem 3. Under the pinched curvature condition, we have

$$K\zeta'_j < \kappa\zeta'_{j+1}, \quad 1 \leqslant j \leqslant j_0$$

which can be translated to

$$\max_{p\in\Sigma} \zeta'_j(p) < \min_{p\in\Sigma} \zeta'_{j+1}(p), \quad 1 \leqslant j \leqslant j_0.$$

Suppose λ is a resonance which satisfies that for some $1 \leqslant j \leqslant j_0$,

$$K\zeta'_j(\mathrm{Re}\,\lambda)^{1/3} + C \leqslant -\mathrm{Im}\,\lambda \leqslant \kappa\zeta'_{j+1}(\mathrm{Re}\,\lambda)^{1/3} - C.$$

Let $\zeta = \lambda^2 = h^{-2}(1 + h^{2/3}z)$ and $h = (\mathrm{Re}\,\lambda)^{-1}$, then we have

$$K\zeta'_j h^{1/3} + C \leqslant -\mathrm{Im}\,\lambda \leqslant \kappa\zeta'_{j+1}h^{1/3} - C$$

and

$$\mathrm{Re}\, z = h^{-2/3}(h^2\,\mathrm{Re}\,\zeta - 1) = O(h^{2/3}).$$

$$-\operatorname{Im} z = h^{-2/3}(-h^2 \operatorname{Im} \zeta) = -2h^{1/3} \operatorname{Im} \lambda \in [2K\zeta'_j + Ch^{1/3}, 2\kappa\zeta'_{j+1} - Ch^{1/3}].$$

Therefore for $p \in \Sigma_1$, i.e. $R(p) = 1$,

$$\operatorname{Im}[z - \lambda - e^{-2\pi i/3}\zeta'_k(p)] = \operatorname{Im} z + \zeta'_k(2Q(p))^{2/3} \cos(\pi/6) \in [\operatorname{Im} z + 2\kappa\zeta'_k, \operatorname{Im} z + 2K\zeta'_k]$$

thus for at most one of $k \in \{j, j+1\}$,

$$|\operatorname{Im}[z - \lambda - e^{-2\pi i/3}\zeta'_k(p)]| \geqslant Ch^{1/3}$$

while for all other $k \in \{1, \dots, j_0\}$,

$$|\operatorname{Im}[z - \lambda - e^{-2\pi i/3}\zeta'_k(p)]| \geqslant \frac{1}{O(1)}.$$

Therefore we can decompose

$$E_{-+}(z) := E_{1,-+}(z) = A(z)G_{-+}(z)B(z)$$

where

$$A(z), B(z) \in \Psi^{0,0,0}_{\Sigma_1,2/3}(\partial\mathcal{O}; \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N))$$

are invertible and

$$G_{-+}(z) \in \Psi^{0,1,2}_{\Sigma_1,2/3}(\partial\mathcal{O}; \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N))$$

has principal symbol $G^0_{-+}(z)$, such that, near Σ_1,

$$\operatorname{Im} G^0_{-+}(z) \geqslant C_0 h^{1/3} \operatorname{Id}_{\mathbb{C}^N}$$

while away from Σ_1,

$$\operatorname{Im} G^0_{-+}(z) \geqslant \frac{1}{O(1)} h^{-2/3}\langle\xi\rangle^2.$$

Now we choose C_0 large enough, then we see that the imaginary part of the total symbol of $G_{-+}(z)$ is bounded below by a positive symbol in $S^{-1/3,0,2}_{\Sigma_1,2/3}$. The sharp Gårding's inequality gives

$$\|E_{-+}(z)u\|_{L^2} \geqslant C\|G_{-+}(z)u\|_{L^2} \geqslant Ch^{1/3}\|u\|_{L^2}, \quad \forall u \in C^\infty(\partial\mathcal{O}; \mathbb{C}^N).$$

Therefore $E_{-+}(z)$ is invertible for $0 < h \leqslant h_0$. Therefore when $\operatorname{Re} \lambda \geqslant C = h_0^{-1}$, it cannot be a resonance.

6.6.2 Weyl's Law

In this part, we sketch the proof of Theorem 4. See [30, Section 9-10] for details of the proof.

Heuristically, we want to use the symbol of $E_{w,-+}(z)$ to compute its trace, then use (6.5.7) to count the number of resonances. However, this operator is not in the trace class. The first step is to construct a finite-rank approximation $\tilde{E}_{w,-+}(z) \in \Psi^{0,1,2}_{\Sigma_w,2/3}(\partial\mathcal{O};\mathcal{L}(\mathbb{C}^N,\mathbb{C}^N))$ which is invertible and such that

$$\tilde{E}_{w,-+}(z)^{-1},\ \ (\Lambda_w^{-1}\tilde{E}_{w,-+}(z))^{-1},\ \ \tilde{E}_{w,-+}(z)^{-1}E_{w,-+}(z) = O(1) : L^2(\partial\mathcal{O};\mathbb{C}^N) \to L^2(\partial\mathcal{O};\mathbb{C}^N)$$

where $\Lambda_w = \langle h^{-2/3}(-h^2\Delta_{\partial\mathcal{O}} - w)\rangle \in \Psi^{0,1,2}_{\Sigma_w,2/3}$ is elliptic. Moreover, we have $E_{w,-+}(z) - \tilde{E}_{w,-+}(z)$ is independent of z and of rank $M = O(Lh^{1-n+2/3})$. Microlocally $\tilde{E}$ is only different from E on the the glancing region where E is not invertible.

From this finite-rank approximation, we can solve another Grushin problem to reduce $E_{w,-+}$ to a finite matrix. More precisely, we consider

$$\mathcal{Q}_w(z) = \begin{pmatrix} \Lambda^{-1}E_{w,-+}(z) & R_{w,-}(z) \\ R_{w,+}(z) & 0 \end{pmatrix} : L^2(\partial\mathcal{O};\mathbb{C}^N)\times\mathbb{C}^M \to L^2(\partial\mathcal{O};\mathbb{C}^N)\times\mathbb{C}^M, \quad (6.6.1)$$

with bounded inverse

$$\mathcal{F}_w(z) = \begin{pmatrix} F_w(z)\Lambda & F_{w,+}(z) \\ F_{w,-}(z) & F_{w,-+}(z) \end{pmatrix} : L^2(\partial\mathcal{O};\mathbb{C}^N)\times\mathbb{C}^M \to L^2(\partial\mathcal{O};\mathbb{C}^N)\times\mathbb{C}^M.$$

The construction of the Grushin problem is as follows: Let $e_1,\dots,e_M$ be an orthonormal basis of the image of $\Lambda_w^{-1}(E_{w,-+}(z) - \tilde{E}_{w,-+}(z))^*$, then we set

$$R_{w,+}u(j) = \langle u, e'_j\rangle, \quad 1\leqslant j\leqslant M; \quad R_{w,-}(z)u_- = \Lambda^{-1}\tilde{E}_{w,-+}(z)R^*_{w,+}u_-.$$

The inverse is given by

$$\begin{aligned} F_w(z) &= (I - R^*_{w,+}R_{w,+})\tilde{E}_{w,-+}(z)^{-1}, \\ F_{w,+}(z) &= R^*_{w,+} - (I - R^*_{w,+}R_{w,+})\tilde{E}_{w,-+}(z)^{-1}E_{w,-+}(z)R^*_{w,+}, \\ F_{w,-}(z) &= R_{w,+}\tilde{E}_{w,-+}(z)^{-1}, \\ F_{w,-+}(z) &= -R_{w,+}\tilde{E}_{w,-+}(z)^{-1}E_{w,-+}(z)R^*_{w,+}. \end{aligned}$$

With these preparations, we can prove a local trace formula on the scale 1 in the z variable for every w. This is on the scale $h^{2/3}$ for the semiclassical variable $w + h^{2/3}z$ which is the square of the resonances $h^2\lambda^2$. We remark that this is the largest scale that we can work with for each fixed w since the whole microlocal framework is built exactly on such a scale.

For the j_0-th band of the resonances, we consider a domain

$$W = \left\{-\frac{1}{2}L < \operatorname{Re} z < \frac{1}{2}L, A_- < -\operatorname{Im} z < A_+\right\}$$

where

$$2K\zeta'_{j_0-1} < A_- < 2\kappa\zeta'_{j_0} \leqslant 2K\zeta'_{j_0} < A_+ < 2\kappa\zeta'_{j_0+1}.$$

Let $\partial W = \gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ be the boundary of W, where γ_1 and γ_3 are the horizontal segments while γ_2 and γ_4 are the vertical segments. If we write $\mathrm{Res}_w(h) = \{z : m_{\mathcal{O}}(h^{-2}(w + h^{2/3}z)) > 0\}$, then we have the local trace formula

$$\begin{aligned}\sum_{z\in \mathrm{Res}_w(h)\cap W} f(z) = \sum_{j=1,3} \operatorname{tr} \frac{1}{2\pi i}\int_{\gamma_j} f(z)\Bigg[&E_{w,-+}(z)^{-1}\frac{d}{dz}E_{w,-+}(z) \\ &-\tilde{E}_{w,-+}(z)^{-1}\frac{d}{dz}\tilde{E}_{w,-+}(z)\Bigg]dz + O(Lh^{1-n+2/3})\end{aligned} \tag{6.6.2}$$

for any holomorphic function f defined near W such that $|f(z)| \leqslant 1$ near $\gamma_2 \cup \gamma_4$. (In fact, to make this argument work, we need to choose a slightly larger rectangular contour around W and f holomorphic in an even larger domain. Also we need to the contour does not pass through any of the poles of $E_{w,-+}^{-1}$. These technical issues are handled in [30].)

The main idea to prove this local trace formula is to change the trace of $E_{-+}^{-1}E_{-+}' - \tilde{E}_{-+}^{-1}\tilde{E}_{-+}'$ to the trace of $F_{-+}^{-1}F_{-+}' = \log\det F_{-+}$ by using the Grushin problem (6.6.1) constructed above. We observe that F_{-+} is an $M\times M$ matrix which is $O(1) : \mathbb{C}^M \to \mathbb{C}^M$ under the standard norm. This shows that $\log\det F_{-+} = O(M) = O(Lh^{1-n+2/3})$ and thus all the contributions from the two vertical segment can be controlled by $O(Lh^{1-n+2/3})$ using lower modulus theorem. Notice that this characterization of resonances by the poles of F_{-+}^{-1} also gives a local upper bound on the number of the resonances

$$\sum_{|\operatorname{Re}\zeta-1|\leqslant Ch^{2/3},0<-\operatorname{Im}\zeta<Ch^{2/3}} m_{\mathcal{O}}(\zeta) = O(h^{1-n+2/3}). \tag{6.6.3}$$

In the local trace formula (6.6.2), we use the (second microlocalization) symbol to compute the trace on the right-hand side and get

$$\begin{aligned}\sum_{z\in \mathrm{Res}_w(h)\cap W} f(z) =&\frac{h^{1-n+2/3}}{(2\pi)^{n-1}}\int_{\Sigma_w\times\mathbb{R}} f(\lambda + e^{-2\pi i/3}\zeta_{j_0}'(q))1_{I(q)}(s)L_{\Sigma_w}(dq)ds \\ &+ O(Lh^{1-n+2/3}) + O_{f,L}(h^{2-n})\end{aligned} \tag{6.6.4}$$

where $(q,s)\in\Sigma_w\times\mathbb{R}$ is a local coordinates for a neighborhood of $\Sigma_w \in T^*\partial\mathcal{O}$ such that $s|_{\Sigma_w} = 0$, $L_{\Sigma_w}(dq)ds$ is the Liouville measure on T^*X, and

$$I(q) = \{s\in\mathbb{R} : s + e^{-2\pi i/3}\zeta_{j_0}'(q) \in W\}.$$

For fixed L (and say $f = 1$), this does not give a better description of resonances than the upper bound (6.6.3). However, if we make L large (which does not change the principal symbol in our construction, but may potentially affect the lower order terms), and choose f suitably, we can get a better estimate than (6.6.3). The idea is to let f to be very large in W away from the $\gamma_2\cup\gamma_4$ but remain bounded ($|f|\leqslant 1$ as required from the assumption in (6.6.2)) near $\gamma_2\cup\gamma_4$. A standard choice is the Gaussian functions

$$f_\epsilon(z) = ((1+O(\epsilon L))e^{-\epsilon L^2/2})^{-1}e^{-\epsilon(z-z_0)^2},\ \ z_0 = -\frac{1}{2}i(A_- + A_+),\ \ \epsilon L \ll 1,\ \epsilon L^2 \gg \log\frac{1}{\epsilon}.$$

Then from (6.6.4) we obtain

$$\sum_{z\in \mathrm{Res}_w(h)\cap W}\sqrt{\frac{\epsilon}{2\pi}}e^{-\epsilon(\mathrm{Re}(z-z_0))^2/2}=(1+O(\epsilon L))\frac{h^{1-n+2/3}}{(2\pi)^{n-1}}\int_{\Sigma_w}L_{\Sigma_w}(dq)+O_{\epsilon,L}(h^{2-n}).$$

Finally, we let $L=\epsilon^{-2/3}$ and integrate in w to get the Weyl's law in the semiclassical setting

Proposition 6.6.1. *(see [30, Proposition 10.1]) For* $0<a<b$, *let*

$$N_h([a,b];j)=\sum_{a<\mathrm{Re}\, z<b, 2\kappa\zeta_j' h^{2/3}<-\mathrm{Im}\, z<2K\zeta_j' h^{2/3}} m_{\mathcal{O}}(h^{-2}z).$$

Then under the assumption of 3, we have

$$N_h([a,b];j)=(1+O(\epsilon))\frac{h^{1-n}}{(2\pi)^{n-1}}\int_{a\leqslant|\xi'|^2_{x'}\leqslant b}dx'd\xi'+O_\epsilon(h^{1-n+1/3}) \tag{6.6.5}$$

for any $1\leqslant j\leqslant j_0$ *and* $\epsilon>0$.

Now the Weyl law (6.0.3) follows from a dyadic decomposition of the interval $|\lambda|\leqslant r$ and applying (6.6.5) for each dyadic piece of the interval.

Chapter 7

Asymptotically Euclidean case

7.1 General set up

7.1.1 Dynamical assumptions

Let M be a real analytic manifold which is diffeomorphic to $\mathbb{R}^n$ and equipped with a real analytic metric g that is asymptotic Euclidean. More precisely, let

$$Q = -h^2\Delta_g = \sum_{|\alpha|\leqslant 2} a_\alpha(x;h)(hD_x)^\alpha, \tag{7.1.1}$$

then we assume that the coefficients $a_\alpha(x;h)$ satisfies the following properties:

(1) for $|\alpha| = 2$, $a_\alpha = a_\alpha(x)$ is independent of h;

(2) $a_\alpha(x;h) = a_{\alpha,0}(x) + O(h)$ has an holomorphic extension to the sector $S = \{x \in \mathbb{C}^n : |\operatorname{Im} x| < C^{-1}\langle \operatorname{Re} x\rangle\}$, bounded in C^∞ with respect to h in S;

(3) a_α satisfies the following conditions:

$$\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geqslant \frac{1}{C}|\xi|^2$$

and

$$\sum_{|\alpha|\leqslant 2} a_\alpha(x;h)\xi^\alpha \to \xi^2$$

as $|x| \to \infty$ uniformly in h.

Moreover, we assume that the geodesic flow on the manifold M is non-trapping. So if we let

$$q(x,\xi) = \sum_{|\alpha|\leqslant 2} a_{\alpha,0}(x)\xi^\alpha, \tag{7.1.2}$$

then there are no trapped H_q-trajectories in $q^{-1}(1)$.

Now let $\mathcal{O}$ be a convex obstacle in M which we assume to be also totally convex, in the sense that

(1) for any $x, y \in \partial\mathcal{O}$, the unique geodesic connecting x and y lies in $\mathcal{O}$;

(2) $\partial\mathcal{O}$ is a C^∞ hypersurface in M with strictly positive principal curvatures.

In particular, the assumptions imply that the dynamics of the geodesic flow on $q^{-1}(1)$ is the same as the one in the Euclidean case: all the geodesics escape to infinity in both directions. Moreover, all the geodesics that leave the obstacle escape to infinity and never come back to the obstacle.

Without loss of generality, we assume that $0 \in \mathcal{O} \subset \{|x| < R\}$. Then $P_0 = -\Delta|_{(M\setminus\mathcal{O},g)}$ with the Dirichlet/Neumann/Robin boundary conditions on $\partial\mathcal{O}$ satisfies the assumption in section 2.1.2 for the blackbox perturbation of Q. Therefore we can define the resonances of $-\Delta|_{(M\setminus\mathcal{O},g)}$ in a sector

$$\mathrm{Res}(P_0) \subset \{z : 0 < -\arg z < \theta_0\}.$$

As in the case of the Euclidean metric, we can find the normal geodesic coordinates near $\partial\mathcal{O}$ on the exterior domain $M \setminus \mathcal{O}$:

$$x = (x', x_n) \mapsto \exp_{x'}(x_n \nu(x')), x' \in \partial\mathcal{O}, x_n = d(x, \partial\mathcal{O}),$$

where $\nu(x')$ is the exterior unit normal vector to $\mathcal{O}$ at x':

$$\nu(x') \in N_{x'}\partial\mathcal{O}, \ \ \|\nu(x')\| = 1.$$

Then near $\partial\mathcal{O}$, say in the region $d(x, \partial\mathcal{O}) < L^{-1}$,

$$P_0 = D_{x_n}^2 + R(x', D_{x'}) - 2x_n Q(x_n, x', D_{x'}) + G(x_n, x')D_{x_n},$$

where $R(x', D_{x'})$, $Q(x_n, x', D_{x'})$ are second order operators on $\partial\mathcal{O}$:

$$R(x', D_{x'}) = -\Delta_{\partial\mathcal{O}} = (\det(g^{ij}))^{1/2} \sum_{i,j=1}^{n-1} D_{y_i} (\det(g_{ij}))^{1/2} g^{ij} D_{y_j}$$

is the Laplacian with respect to the induced metric $g = (g_{ij})$ on $\partial\mathcal{O}$ and $Q(x', D_{x'}) = Q(0, x', D_{x'})$ is of the form

$$\det(g^{ij})^{1/2} \sum_{i,j=1}^{n-1} D_{y'_j} (\det(g_{ij}))^{1/2} a_{ij} D_{y'_i}$$

in any local coordinates such that the principal symbol of Q is the second fundamental form of $\partial\mathcal{O}$ lifted by the duality to $T^*\partial\mathcal{O}$:

$$Q(x', \xi') = \sum_{i,j=1}^{n-1} a_{ij}(x') \xi_i \xi_j.$$

Thus the principal curvatures of $\partial\mathcal{O}$ are the eigenvalues of the quadratic form $Q(x', \xi')$ with respect to the quadratic form $R(x', \xi')$.

7.1.2 Complex scaling contours

In this section, we construct a complex contour based on a slight modification of the one in the Euclidean case. Consider the complex contour given by

$$\mathbb{R}^n \setminus \mathcal{O} \ni x \mapsto z = x + i\theta(x) f'(x) \in \Gamma \subset \mathbb{R}^n \setminus \mathcal{O} + i\mathbb{R}^n,$$

where the function f is a smooth function given by

$$f(x) = \frac{1}{2} d(x, \partial\mathcal{O})^2 \chi(x) + \frac{1}{2}|x|^2(1 - \chi(x))$$

and $\chi \in C^\infty(\mathbb{R}^n; [0,1])$ satisfies $\chi = 1$ when $d(x, \partial\mathcal{O}) < L^{-1}$ and $\chi = 0$ when $|x| > R_1$, where $L, R_1 > 0$ are chosen later to be large but fixed.

Near the boundary, we scale by the angle $\pi/3$ as before

$$\frac{1 + i\theta(x)}{|1 + i\theta(x)|} = e^{i\pi/3}, \quad d(x, \partial\mathcal{O}) < (L_1)^{-1}, \quad L_1 \gg 2L$$

and then we connect to the scaling with a smaller angle $\theta(x) = \theta_0 = \arctan \epsilon_0$ when $d(x, \partial\mathcal{O}) > (2L)^{-1}$.

Then near infinity, say when $|x| > R$, Γ coincides with $\Gamma_0 \subset \mathbb{C}^n$ given by

$$\operatorname{Im} x = \epsilon_0 \operatorname{Re} x. \tag{7.1.3}$$

On the other hand, when $d(x, \partial\mathcal{O}) < L^{-1}$, we use normal geodesic coordinates $(x', x_n) \in \partial\mathcal{O} \times (0, L^{-1})$, then Γ is given by

$$(x', x_n) \mapsto \exp_{x'}(g(x_n)\nu) \tag{7.1.4}$$

where $g : [0, L^{-1}) \to \mathbb{C}$ is smooth and injective such that $g(t) = e^{i\pi/3}$ when $x_n < L_1^{-1}$; $g(t) = e^{i\theta_0}$ when $(2L)^{-1} < x_n < L^{-1}$ and

$$\theta_0 \leqslant \arg g(t) \leqslant \pi/3, \quad \theta_0/2 \leqslant \arg g'(t) \leqslant \pi/3. \tag{7.1.5}$$

We define scaled operator $P = -h^2\Delta_\Gamma$ as the restriction of the holomorphic Laplacian on $\mathbb{C}^n$

$$Q = -h^2\Delta_{g,z} = \sum_{|\alpha| \leqslant 2} a_\alpha(x; h)(hD_z)^\alpha$$

to Γ. Whenever there is no confusion, we shall identify Γ with $\mathbb{R}^n \setminus \mathcal{O}$. Then when $d(x, \partial\mathcal{O}) < L^{-1}$, we have the following formula for the semiclassical complex scaled operator

$$P = \frac{1}{(g'(x_n))^2}(hD_{x_n})^2 + R(x', hD_{x'}) - 2g(x_n)Q(x_n, x', hD_{x'}) + hF(x_n, x')hD_{x_n}. \tag{7.1.6}$$

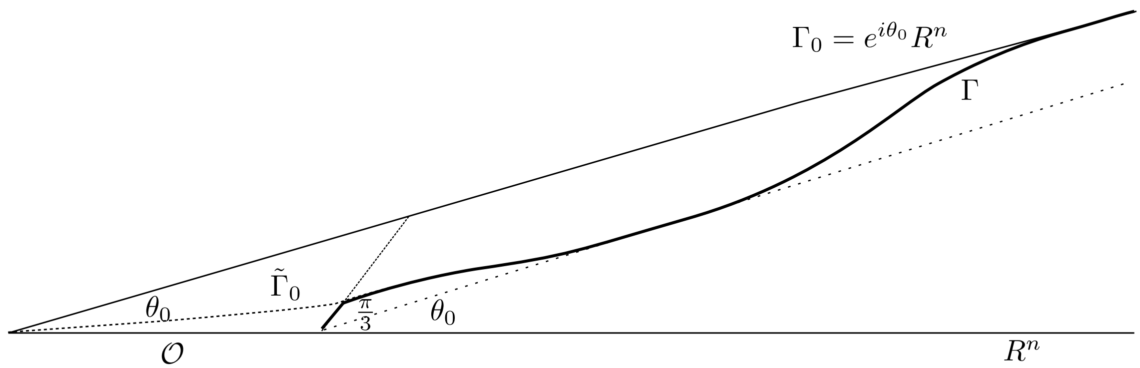

Figure 7.1: The contours for complex scaling on asymptotically Euclidean manifolds.

In particular, when $d(x, \partial\mathcal{O}) < L_1^{-1}$,

$$P = e^{-2\pi i/3}((hD_{x_n})^2 + 2x_n Q(x_n, x', hD_{x'})) + R(x', hD_{x'}) + hF(x_n, x')hD_{x_n}.$$

We also need to consider complex scaling contours on the whole space. We first extend the function f restricted to $d(x, \partial\mathcal{O}) > (2L)^{-1}$ to a function $\tilde{f}$ on $\mathbb{R}^n$ by

$$\tilde{f}(x) = \frac{1}{2} d(x, \partial\mathcal{O})^2 \tilde{\chi}(x) + \frac{1}{2}|x|^2(1 - \tilde{\chi}(x)),$$

where $\tilde{\chi} \in C^\infty(\mathbb{R}^n; [0,1])$ equals to 1 when $d(x, \partial\mathcal{O}) \in ((2L)^{-1}, L^{-1})$ and $\chi = 0$ when $|x| > R_1$ or $d(x, \partial\mathcal{O}) < L_1^{-1}$. Then we have the contour

$$\mathbb{R}^n \ni x \mapsto z = x + i\theta_0 f'(x) \in \tilde{\Gamma}_0 \subset \mathbb{C}^n$$

which agrees with Γ_0 away from a compact set and agrees with Γ when $d(x, \partial\mathcal{O}) > (2L)^{-1}$. Moreover $\tilde{\Gamma}_0$ is ϵ_0-close to Γ_0.

7.1.3 Escape functions

Now we construct an escape function $G(x, \xi)$ associated to the geodesic flow of (M, g). From the dynamical assumption, we see that on $\Sigma_q^\epsilon = q^{-1}([1-\epsilon, 1+\epsilon])$, the function $G_0(x, \xi) = f'(x) \cdot \xi$ satisfies

$$H_q G_0 > C_0^{-1} \tag{7.1.7}$$

when $|x| > R$ as well as when $L_1^{-1} < d(x, \partial\mathcal{O}) < L^{-1}$. In fact, when $|x| > R$, we have $q(x, \xi) = q_0(x, \xi) + o(1)$ where $q_0(x, \xi) = \xi^2$ and

$$|(\partial_\xi q - \partial_\xi q_0)(x, \xi)| = o(1), \quad |(\partial_x q - \partial_x q_0)(x, \xi)| = o(1)\langle x\rangle^{-1},$$

as $|x| \to \infty$ while $\partial_x G_0$ and $\langle x\rangle^{-1}\partial_\xi G_0$ remains bounded, so

$$|H_q G_0 - H_{q_0} G_0(x,\xi)| \to 0,$$

when $\Sigma_q^\epsilon \ni (x,\xi) \to \infty$. Now when $R > R_0$,

$$f(x) = \frac{1}{2}|x|^2, \quad G_0 = x\cdot\xi$$

so $H_{q_0}G_0 = 2|\xi|^2$ and thus we have (7.1.7).

When $L_1^{-1} < d(x,\partial\mathcal{O}) < L^{-1}$, we use the normal geodesic coordinates (x', x_n). Then

$$q(x,\xi) = \xi_n^2 - 2x_n Q(x_n, x', \xi') + R(x', \xi'),$$

and $f(x) = \frac{1}{2}x_n^2$, $G_0 = x_n \cdot \xi_n$. Therefore

$$H_q(G_0) = 2\xi_n^2 + 2x_n Q(x_n, x', \xi') + 2x_n^2 \partial_{x_n} Q(x_n, x', \xi').$$

Since $Q(0, x', \xi')$ is positive definite, we get (7.1.7) provided L is large enough.

Therefore by the non-trapping assumption, we can modify G_0 by a compact supported function $G_1 \in C_c^\infty$ to get an escape function $G = G_0 + G_1$ on the whole space T^*M:

$$H_q G > 1/C_0 \text{ on } q^{-1}(1). \tag{7.1.8}$$

Moreover, $G_1 = 0$ when $d(x,\partial\mathcal{O}) \in [L_1^{-1}, L^{-1}]$.

7.1.4 FBI transform and microlocally weighted space

Now we consider the FBI transform

$$T : L^2(\tilde{\Gamma}_0) \to H_{\tilde{\Phi}_0}(T^*\tilde{\Gamma}_0) \tag{7.1.9}$$

where we replace the contour Γ_0 by $\tilde{\Gamma}_0$. Since $\tilde{\Gamma}_0$ is ϵ_0-close to Γ_0 and agrees with Γ_0 away from the compact set, $T^*\tilde{\Gamma}_0$ is an IR-manifold in an ϵ_0-conic neighborhood of $T^*\Gamma_0$ and thus the associated canonical transformation $\kappa_T : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ maps $T^*\tilde{\Gamma}_0$ to an IR-manifold in an ϵ_0-conic neighborhood of $\kappa_T(T^*\Gamma_0) = \Lambda_{\Phi_0}$, thus of the form $\Lambda_{\tilde{\Phi}_0}$. As before, we identify $\tilde{\Gamma}_0$ with $\mathbb{R}^n$ and $T^*\tilde{\Gamma}_0$ with $\mathbb{C}^n$ naturally.

We consider the IR-manifold

$$\Lambda_{\epsilon_0 G} : \mathrm{Im}(x,\xi) = \epsilon_0 H_G(\mathrm{Re}(x,\xi)).$$

We see that $\Lambda_{\epsilon_0 G}$ coincides with $T^*\tilde{\Gamma}_0$ except on a compact set and is ϵ_0-close to $T^*\tilde{\Gamma}_0$. Therefore the IR-manifold $\kappa_T(\Lambda_{\epsilon_0 G})$ is a small perturbation of $\kappa_T(T^*\tilde{\Gamma}_0)$ and can be written as $\Lambda_{\Phi_{\epsilon_0 G}}$ where $\Phi_{\epsilon_0 G} = \tilde{\Phi}_0$ outside a bounded set. In particular,

$$\Phi_{\epsilon_0 G}(x) = \tilde{\Phi}_0(x) + \epsilon_0 g(x, \epsilon_0)$$

where g is smooth and has compact x-support.

Then as in section 3.3.2, for any order function m, we can define the microlocally weighted Hilbert space $H_{\Phi_{\epsilon_0 G},m}$ which coincides with $H_{\tilde{\Phi}_0}$ and has equivalent norm but not uniformly in h. The theory of pseudodifferential operators with holomorphic symbols in section 3.3.2 also holds. In particular, we remark that for $u \in L^2(\tilde{\Gamma}_0)$ such that $\operatorname{supp} u \subset \{(2L)^{-1} < d(x, \partial\mathcal{O}) < L^{-1}\}$,

$$\|Tu\|_{H_{\Phi_{\epsilon_0 G},1}} \sim \|u\|_{L^2(\tilde{\Gamma}_0)}$$

since $\Lambda_{\epsilon_0 G}$ coincides with $T^*\tilde{\Gamma}_0$ when $(2L)^{-1} < d(x, \partial\mathcal{O}) < L^{-1}$. We shall write

$$\|u\|_{H(\tilde{\Gamma}_0, \Phi_{\epsilon_0 G}, m)} = \|Tu\|_{H_{\Phi_{\epsilon_0 G},m}}. \tag{7.1.10}$$

7.2 Lower bounds on scaled operators and resonance free region

7.2.1 Lower bounds on the scaled operator on M

We first study the scaled operator $Q = -h^2\Delta_g|_{\tilde{\Gamma}_0}$ on the space $H(\tilde{\Gamma}_0, \Phi_{\epsilon_0 G})$. We consider the symbol $q_{\epsilon_0} = q|_{\Lambda_{\epsilon_0 G}}$. Using Taylor expansion, we have

$$q_{\epsilon_0}(x,\xi) = q((x,\xi) + i\epsilon_0 H_G(x,\xi)) = q(x,\xi) - i\epsilon_0 H_q G(x,\xi) + O(\epsilon_0^2\langle\xi\rangle^2).$$

Therefore

$$\begin{aligned} \operatorname{Re} q_{\epsilon_0}(x,\xi) &= q(x,\xi) + O(\epsilon_0\langle\xi\rangle^2), \\ \operatorname{Im} q_{\epsilon_0}(x,\xi) &\leqslant -\epsilon_0/C_1 + C\epsilon_0|q(x,\xi) - 1|. \end{aligned}$$

In particular, let $\omega_0 = 1 + ir_0$, then if ϵ_0 is chosen small, we have

$$\begin{aligned} |q_{\epsilon_0}(x,\xi) - \omega_0|^2 - r_0^2 &= |q_{\epsilon_0} - 1|^2 - 2r_0 \operatorname{Im} q_{\epsilon_0} \\ &\geqslant |q_{\epsilon_0} - 1|^2 + 2\epsilon_0/C_1 - 2C\epsilon_0|q_{\epsilon_0} - 1| \geqslant \epsilon_0/C_1. \end{aligned}$$

Therefore on $q^{-1}[1-\epsilon, 1+\epsilon]$,

$$|q_{\epsilon_0}(x,\xi) - \omega_0|^2 \geqslant r_0^2 + \epsilon_0/C_1. \tag{7.2.1}$$

Away from $q^{-1}[1-\epsilon, 1+\epsilon]$, we even have

$$|q_{\epsilon_0}(x,\xi) - \omega_0|^2 \geqslant r_0^2 + \epsilon_0/C_1\langle\xi\rangle^2. \tag{7.2.2}$$

We write $Q = \operatorname{Op}_h(q)(x, hD; h)$ where

$$q(x,\xi;h) = q|_{T^*\tilde{\Gamma}_0}(x,\xi;h) = q_0(x,\xi) + O(h\langle\xi\rangle^2),$$

and $\tilde{Q} = T \circ Q \circ T^{-1} = \mathrm{Op}_h(\tilde{q})(x, hD_x; h)$, where

$$(\tilde{q} \circ \kappa_T(x, \xi; h)) = p(x, \xi; h).$$

Let m be the order function with $m \circ \kappa_T = \langle \xi \rangle^2$ and $\tilde{q}_0$ be given by $\tilde{q}_0 \circ \kappa_T = q_0$. Then the symbol $\tilde{q}$ is holomorphic in a tubular neighborhood of $\Lambda_{\Phi^0_{\epsilon_0}} = \kappa_T(T^*\tilde{\Gamma}_0)$ of the form $\Lambda_{\Phi^0_{\epsilon_0}} + W$, and in this neighborhood we have

$$\tilde{q}(x, \xi; h) = O(m(x, \xi)), \text{ and } \tilde{q}(x, \xi; h) = \tilde{q}_0(x, \xi) + O(hm).$$

Now since

$$\tilde{q}_0|_{\Lambda_{\Phi_{\epsilon_0}}} \circ \kappa_T(x, \xi) = q_{\epsilon_0}(x, \xi),$$

using (7.2.1) and (7.2.2), we have the following lemma.

Lemma 7.2.1. *Let $\Phi_{\epsilon_0 G}$ be as above. For any $u \in C_c^\infty(\tilde{\Gamma}_0)$, $\omega_0 = 1 + ir_0$ where $r_0 > 0$, we have*

$$\|(Q - \omega_0)u\|^2_{H(\tilde{\Gamma}_0, \Lambda_{\epsilon_0 G})} \geqslant (r_0 + 1/C_0 - O(h))^2 \|Tu\|^2_{H(\tilde{\Gamma}_0, \Lambda_{\epsilon_0 G})}. \tag{7.2.3}$$

Proof. Since $T(Q - \omega_0)u = (\tilde{Q} - \omega_0)Tu$, we have

$$\begin{aligned}
\|(\tilde{Q} - \omega_0)Tu\|^2_{H_{\Phi_{\epsilon_0 G}}} &= \int_{\Lambda_{\Phi_{\epsilon_0 G}}} |\tilde{q}_0 - 1 - ir|^2 |Tu|^2 e^{-2\Phi_{\epsilon_0 G}/h} dx + O(h)\|Tu\|^2_{H_{\Phi_{\epsilon_0 G}}} \\
&\geqslant r_0^2 \|Tu\|^2_{H_{\Phi_{\epsilon_0 G}}} + (1/C_0 - O(h))\|Tu\|^2_{H_{\Phi_{\epsilon_0 G}, m}} \\
&\geqslant (r_0 + 1/C_0 - O(h))^2 \|Tu\|^2_{H_{\Phi_{\epsilon_0 G}}}.
\end{aligned}$$

This finishes the proof. $\square$

In particular, this lemma implies that there exists a constant $h_0 > 0$ such that for $0 < h < h_0$, there are no resonances for Q in $D(1, 1/C_0)$.

7.2.2 Lower bounds on the scaled operator in the exterior region

Now we define the space X to be the space $L^2(\Gamma)$ equipped with an equivalent norm $\|\cdot\|_X$ but not uniformly in h. Let $\chi_1, \chi_2 \in C^\infty(\mathbb{R}^n)$ be such that $\chi_1^2 + \chi_2^2 = 1$ and $\chi_1 = 1$ when $d(x, \mathcal{O}) < (2L)^{-1}$ and $\chi_1 = 0$ when $d(x, \partial\mathcal{O}) > L^{-1}$. We write

$$\|u\|_X^2 = \|\chi_1 u\|^2_{L^2(\Gamma)} + \|\chi_2 u\|^2_{H(\tilde{\Gamma}_0, \Lambda_{\epsilon_0 G})}. \tag{7.2.4}$$

Here $\chi_2 u$ can be viewed as a function on $\tilde{\Gamma}_0$, so we can define its $H(\tilde{\Gamma}_0, \Lambda_{\epsilon_0 G})$-norm with no ambiguity. We consider $P = -h^2\Delta_\Gamma : D \to X$ to be the operator which has domain D equipped with the norm

$$\|u\|_D^2 = \|\chi_1 u\|^2_{H^2(\Gamma)} + \|T(\chi_2 u)\|^2_{H(\tilde{\Gamma}_0, \Lambda_{\epsilon_0 G}, m)}. \tag{7.2.5}$$

Then we have the following lower bounds on P:

Proposition 7.2.2. *For any $u \in C_c^\infty(\Gamma)$ with Neumann or Robin boundary condition, $\omega_0 = 1 + ir_0$ where $r_0 > 0$, we have*

$$\|(P - \omega_0)u\|_X^2 \geqslant |r_0 + S(\operatorname{Re}\omega_0)^{2/3}h^{2/3} - O(h)|^2\|u\|_X^2. \tag{7.2.6}$$

Here $S = \kappa\zeta_1'$ is given by (5.0.2).

Proof. From theorem 5.2.4, we have the following estimates near the obstacle

$$\|(P - \omega_0)\chi_1 u\|_{L^2}^2 \geqslant |r_0 + S(\operatorname{Re}\omega_0)^{2/3}h^{2/3} - O(h)|^2\|u\|_{L^2}^2.$$

Using lemma 7.2.1 and the fact that $(Q - \omega_0)\chi_2 = (P - \omega_0)\chi_2$, we have

$$\|(P - \omega_0)\chi_2 u\|_{H(\tilde{\Gamma}_0,\Lambda_{\epsilon_0 G})}^2 \geqslant |r_0 + C_0^{-1} - O(h)|^2\|\chi_2 u\|_{H(\tilde{\Gamma}_0,\Lambda_{\epsilon_0 G})}^2.$$

Therefore we can repeat the argument in theorem 5.2.4 to get

$$\begin{aligned}\|(P - \omega_0)u\|_X^2 \geqslant{}& \|(P - \omega_0)\chi_1 u\|_{L^2}^2 + \|(P - \omega_0)\chi_2 u\|_{H(\tilde{\Gamma}_0,\Lambda_{\epsilon_0 G})}^2 \\ &- \sum_{j=1,2}\|[\chi_j, P]u\|_{L^2}^2 - 2\sum_{j=1,2}\|\chi_j(P - \omega_0)u\|_X^2\|[\chi_j, P]u\|_{L^2}.\end{aligned}$$

Here the commutator $[\chi_j, P]u$ can be estimated by

$$\|[\chi_j, P]u\|_{L^2} \leqslant O(h)(\|(P - \omega_0)u\|_{L^2(\Omega_1)} + \|u\|_{L^2(\Omega_1)}) \leqslant O(h)(\|(P - \omega_0)u\|_X + \|u\|_X).$$

We use the fact that $\operatorname{supp}[\chi_j, P]u \subset \Omega_1 = \{x : d(x, \partial\mathcal{O}) \in ((2L)^{-1}, L^{-1})\}$ and thus the X-norm and L^2-norm are equivalent. □

7.2.3 Resonance free region

Now the cubic resonance free region follows from proposition 7.2.2 in the same way as in section 5.3 and we omit the details.

7.3 Global Grushin problem and resonances bands

Now we follow the notation in section 6.5 to write

$$P - z = h^{-2/3}(-h^2\Delta_\Gamma - w) - z,$$

where $w \in W \Subset (0, \infty)$ and $|\operatorname{Re} z| \leqslant L$, $|\operatorname{Im} z| \leqslant C$ where C fixed. Again we also use the notation

$$D(\alpha) = \{x \in M \setminus \mathcal{O} : d(x, \partial\mathcal{O}) > \alpha\}.$$

7.3.1 Further estimates away from the obstacle

In this section, we prove the following lemma which allows us to glue the estimates away from the obstacle with the Grushin problem considered in section 6.3.

Lemma 7.3.1. *Assume that $0 < h < h_0$. If $v_1 \in C_c^\infty(M \setminus \mathcal{O})$ is supported in $D(\delta)$ where $\delta = (2L_1)^{-1}$, then there exists $u_1 \in C^\infty(M \setminus \mathcal{O})$ supported away from the obstacle and $v_0^1 \in C_c^\infty(M \setminus \mathcal{O})$ such that*

$$(P - z)u_1 = v_1 + v_0^1, \tag{7.3.1}$$

and

$$\|u_1\|_D + e^{C/h}\|u_1\|_{H_h^k(D((1-\gamma)\delta))} + e^{C/h}\|v_0^1\|_{H_h^k(M\setminus\mathcal{O})} \leqslant Ch^{2/3}\|v_1\|_X. \tag{7.3.2}$$

for any $k, \gamma > 0$.

Proof. First since $w + h^{2/3}z$ is not a resonance for $Q = -h^2\Delta_{\tilde{\Gamma}_0}$, we can solve

$$(h^{-2/3}(-h^2\Delta|_{\tilde{\Gamma}_0} - w) - z)\tilde{u} = v_1$$

such that

$$\|\tilde{u}\|_{H(\tilde{\Gamma}_0,\Lambda_{\epsilon_0 G},m)} \leqslant h^{2/3}\|v_1\|_{H(\tilde{\Gamma}_0,\Lambda_{\epsilon_0 G})}.$$

By the standard weighted estimates for the resolvent of $-h^2\Delta|_{\tilde{\Gamma}_0}$, we have

$$|\partial_x^\alpha \partial_y^\beta [(-h^2\Delta|_{\tilde{\Gamma}_0} - \zeta)](x,y) \leqslant C_{\alpha,\beta} e^{d|_{\tilde{\Gamma}_0}(x,y)/C},$$

when $d_{\tilde{\Gamma}_0}(x,y) > \epsilon$ and $\zeta = w + h^{2/3}z$ as above. If we truncate $\tilde{u}$ by a cutoff function equal to 1 on $D((1-2\gamma)\delta)$ and 0 near $\mathcal{O}$, we get u_1 satisfying the boundary condition and an error term v_0^1 satisfies the estimates. $\square$

Now we can follow the same argument as in 6.4 to obtain the following proposition.

Proposition 7.3.2. *Let $0 < \epsilon < \frac{2}{3}$, $|\operatorname{Re} z| \leqslant L$, $|\operatorname{Im} z| \leqslant C$, then there exists $h_0 = h_0(L)$ such that for $0 < h < h_0(L)$, there exists maps E_ϵ, K_ϵ defined on $C_c^\infty(D(h^\epsilon))$, with the properties $(P - z)E_\epsilon = I + K_\epsilon$ and*

$$\begin{aligned} &E_\epsilon = O(h^{2/3-\epsilon}) : X \cap L^2(D(h^\epsilon)) \to D, \\ &K_\epsilon = O(e^{-C^{-1}h^{-1+\frac{3\epsilon}{2}}}) : X \cap L^2(D(h^\epsilon)) \to H_h^k(\mathbb{R}^n \setminus \mathcal{O}), \ \ \forall k \in \mathbb{R}. \end{aligned} \tag{7.3.3}$$

Moreover, for any fixed $\gamma \in (0,1)$, we can construct E_ϵ and K_ϵ such that for $u \in C_c^\infty(D(h^\epsilon))$, $E_\epsilon u$ and $K_\epsilon u$ are supported in $D((1-\gamma)h^\epsilon)$.

7.3.2 Construction of the global Grushin problem

Assume that W is a fixed compact subset of $(0,\infty)$ and $\epsilon \ll 1$. For every $w \in W$ and $z \in \mathbb{C}$ such that $|\operatorname{Re} z| \ll 1/\delta$, $|\operatorname{Im} z| \leqslant C_1$, the same argument as in section 6.5 shows that we can construct the global Grushin problem

$$\mathcal{P}_w(z) = \begin{pmatrix} P_w - z & R_{-,w} \\ \gamma_1 & 0 \\ R_{+,w} & 0 \end{pmatrix} : \mathcal{B}_{w,r} \to \mathcal{H}_{w,r}.$$

with inverse

$$\mathcal{E}_w(z) = \begin{pmatrix} E_w(z) & K_w(z) & E_{w,+}(z) \\ E_{w,-}(z) & K_{w,-}(z) & E_{w,-+}(z) \end{pmatrix} : \mathcal{H}_{w,r} \to \mathcal{B}_{w,r}.$$

Here

$$\begin{aligned} \left\| \begin{pmatrix} u \\ u_- \end{pmatrix} \right\|_{\mathcal{B}_{w,r,\delta}} &= h^{-2/3} \| e^{r\psi(x_n)/2h^{2/3}} (hD_{x_n})^2 u \|_X \\ &+ h^{-2/3} \| e^{r\psi(x_n)/2h^{2/3}} \chi(x_n/\delta) x_n u \|_{L^2(\mathbb{R}^n \setminus \mathcal{O})} \\ &+ \| e^{r\psi(x_n)/2h^{2/3}} \langle x_n \rangle^{-2} \langle h^{-2/3}(-h^2\Delta_{\partial\mathcal{O}} - w) \rangle u \|_X \\ &+ h^{-2/3} \| e^{r\psi(x_n)/2h^{2/3}} (1 - \chi(x_n/\delta)) u \|_D \\ &+ h^{1/3} \| u_- \|_{L^2(\partial\mathcal{O};\mathbb{C}^N)}, \\ \left\| \begin{pmatrix} v \\ v_0 \\ v_+ \end{pmatrix} \right\|_{\mathcal{H}_{w,r}} &= \| e^{r\psi(x_n)/2h^{2/3}} v \|_X \\ &+ h^{1/3} \| \langle h^{-2/3}(-h^2\Delta_{\partial\mathcal{O}} - w) \rangle^{1/4} v_0 \|_{L^2(\partial\mathcal{O})} \\ &+ h^{1/3} \| \langle h^{-2/3}(-h^2\Delta_{\partial\mathcal{O}} - w) \rangle v_+ \|_{L^2(\partial\mathcal{O};\mathbb{C}^N)}, \end{aligned}$$

where the weight function $\psi \in C^\infty([0,\infty);[0,1])$ satisfying $\psi(t) = t$ for $t < \frac{1}{2}$ and $\psi(t) = 1$ for $t \geqslant 1$; and the cut-off function $\chi \in C^\infty([0,\infty);[0,1])$ satisfying $\chi(t) = 1$ for $t < 1$ and $\chi(t) = 0$ for $t > 2$. Here we still use the geodesic normal coordinates $(x', x_n) \in \partial\mathcal{O} \times (0,\infty)$ for $\mathbb{R}^n \setminus \mathcal{O}$ as introduced before.

Moreover, the effective Hamiltonian $E_{w,-+}(z) \in \Psi^{0,1,2}_{\Sigma_w,2/3}$ satisfies the same properties as stated in theorem 5.

7.3.3 Resonance band and Weyl's law

The proof of the existence of resonance bands and Weyl's law for counting functions in each band now follows from the same argument as in 6.6 and we shall not repeat it here.

Bibliography

[1] Joseph. Aguilar and Jean-Michel Combes. "A class of analytic perturbations for one-body Schrödinger Hamiltonians". In: *Comm. Math. Phys.* 22 (1971), pp. 269–279.

[2] V. M. Babich and N. S. Grigoreva. "The analytic continuation of the resolvent of the exterior three-dimensional problem for the Laplace operator to the second sheet". In: *Funktsional. Anal. i Prilozhen.* 8 (1974), pp. 71–74.

[3] Erik Balslev and Jean-Michel Combes. "Spectral properties of many-body Schrödinger operators with dilatation-analytic interactions". In: *Comm. Math. Phys.* 22 (1971), pp. 280–294.

[4] Claude Bardos, Gilles Lebeau, and Jeffery Rauch. "Scattering frequencies and Gevrey 3 singularities". In: *Invent. Math.* 90 (1987), pp. 77–114.

[5] Tanya Christiansen. "Lower bounds for resonance counting functions for obstacle scattering". arXiv:1408.7608.

[6] Tanya Christiansen. "Several complex variables and the order of growth of the resonance counting function in Euclidean scattering". In: *Int. Math. Res. Not.* (2006), Art. ID 43160.

[7] Mouez Dimassi and Johannes Sjöstrand. *Spectral asymptotics in the semi-classical limit*. Vol. 268. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1999.

[8] Semyon Dyatlov and Maciej Zworski. "Mathematical theory of scattering resonances". book in preparation.

[9] Thierry Hargé and Gilles Lebeau. "Diffraction par un convexe". In: *Invent. Math.* 118 (1994), pp. 161–196.

[10] Bernard Helffer and Johannes Sjöstrand. "Résonances en limite semi-classique". In: *Mém. Soc. Math. France (N.S.)* 24-25 (1986).

[11] Lars Hörmander. *The analysis of linear partial differential operators. I.* Vol. 256. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Distribution theory and Fourier analysis. Springer-Verlag, Berlin, 1983.

[12] Lars Hörmander. *The analysis of linear partial differential operators. III.* Vol. 274. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Pseudodifferential operators. Springer-Verlag, Berlin, 1985.

[13] Long Jin. “Resonance-free region in scattering by a strictly convex obstacle”. In: *Ark. Mat.* 52 (2014), pp. 257–289.

[14] Long Jin. “Scattering resonances of convex obstacles for general boundary conditions”. In: *Comm. Math. Phys.* 335 (2015), pp. 759–807.

[15] Bernard Lascar and Richard Lascar. “FBI transforms in Gevrey classes”. In: *J. Anal. Math.* 72 (1997), pp. 105–125.

[16] Peter D. Lax and Ralph S. Phillips. “A logarithmic bound on the location of the poles of the scattering matrix”. In: *Arch. Rational Mech. Anal.* 40 (1971), pp. 268–280.

[17] Peter D. Lax and Ralph S. Phillips. “Decaying modes for the wave equation in the exterior of an obstacle.” In: *Comm. Pure Appl. Math.* 22 (1969), pp. 737–787.

[18] Gilles Lebeau. “Régularité Gevrey 3 pour la diffraction”. In: *Comm. Partial Differential Equations* 9.15 (1984), pp. 1437–1494.

[19] André Martinez. *An introduction to semiclassical and microlocal analysis.* Universitext. Springer-Verlag, New York, 2002.

[20] Richard B. Melrose. *Geometric scattering theory.* Stanford Lectures. Cambridge University Press, Cambridge, 1995.

[21] Richard B. Melrose. “Polynomial bound on the distribution of poles in scatterign by an obstacle”. In: *Exp. No. II. Soc. Math. France* (1984). in Journée Équations aux Dérivées Partielles (Saint-Jean-de-Monts, 1984).

[22] Louis Boutet de Monvel and Paul Krée. “Pseudo-differential operators and Gevrey classes”. In: *Ann. Inst. Fourier (Grenoble)* 17 (1967), pp. 295–323.

[23] Herch M. Nussenzveig. “High-frequency scattering by an impenetrable sphere”. In: *Ann. Physics* 34 (1965), pp. 23–95.

[24] Frank W. J. Olver. “The asymptotic expansion of Bessel functions of large order”. In: *Philos. Trans. Roy. Soc. London. Ser. A.* 247 (1954), pp. 328–368.

[25] Georgi Popov. “Some estimates of Green’s functions in the shadow”. In: *Osaka J. Math.* 24 (1987), pp. 1–12.

[26] J. Sjöstrand. “A trace formula and review of some estimates for resonances”. In: *Microlocal analysis and spectral theory (Lucca, 1996).* Vol. 490. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. Kluwer Acad. Publ., Dordrecht, 1997, pp. 377–437.

[27] J. Sjöstrand and M. Zworski. “Estimates on the number of scattering poles near the real axis for strictly convex obstacles”. In: *Ann. Inst. Fourier (Grenoble)* 43 (1993), pp. 769–790.

[28] Johannes Sjöstrand. "Density of resonances for strictly convex analytic obstacles". In: *Canad. J. Math.* 48 (1996). With an appendix by M. Zworski, pp. 397–447.

[29] Johannes Sjöstrand. *Singularités analytiques microlocales.* Vol. 95. Astérisque. Soc. Math. France, Paris, 1982, pp. 1–166.

[30] Johannes Sjöstrand and Maciej Zworski. "Asymptotic distribution of resonances for convex obstacles". In: *Acta Math.* 183 (1999), pp. 191–253.

[31] Johannes Sjöstrand and Maciej Zworski. "Complex scaling and the distribution of scattering poles". In: *J. Amer. Math. Soc.* 4 (1991), pp. 729–769.

[32] Johannes Sjöstrand and Maciej Zworski. "Elementary linear algebra for advanced spectral problems". In: *Ann. Inst. Fourier (Grenoble)* 57 (2007). Festival Yves Colin de Verdière, pp. 2095–2141.

[33] Johannes Sjöstrand and Maciej Zworski. "Lower bounds on the number of scattering poles". In: *Comm. Partial Differential Equations* 18 (1993), pp. 847–857.

[34] Johannes Sjöstrand and Maciej Zworski. "Lower bounds on the number of scattering poles. II". In: *J. Funct. Anal.* 123 (1994), pp. 336–367.

[35] Johannes Sjöstrand and Maciej Zworski. "The complex scaling method for scattering by strictly convex obstacles". In: *Ark. Mat.* 33 (1995), pp. 135–172.

[36] Plamen Stefanov. "Sharp upper bounds on the number of the scattering poles". In: *J. Funct. Anal.* 231 (2006), pp. 111–142.

[37] Boris R. Vainberg. "Exterior elliptic problems that depend polynomially on the spectral parameter, and the asymptotic behavior for large values of the time of the solutions of nonstationary problems". In: *Mat. Sb. (N.S.)* 92(134) (1973), pp. 224–241, 343.

[38] Georgi Vodev. "Sharp bounds on the number of scattering poles in even-dimensional spaces". In: *Duke Math. J.* 74 (1994), pp. 1–17.

[39] George N. Watson. "Diffraction of electric waves by the Earth". In: *Proc. Roy. Soc. (London)* A95 (1918), pp. 83–99.

[40] Jared Wunsch and Maciej Zworski. "The FBI transform on compact C^∞manifolds". In: *Trans. Amer. Math. Soc.* 353 (2001), pp. 1151–1167.

[41] A. B. Zayaev and Vyacheslav B. Filippov. "Rigorous justification of asymptotic solutions of sliding wave type". In: *J. Soviet Math.* 30 (1985), pp. 2395–2406.

[42] Maciej Zworski. "Counting scattering poles". In: *Spectral and scattering theory (Sanda, 1992).* Vol. 161. Lecture Notes in Pure and Appl. Math. Dekker, New York, 1994, pp. 301–331.

[43] Maciej Zworski. *Semiclassical analysis.* Vol. 138. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2012.