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ATTEMPT AT A STRONGEST VECTOR TOPOLOGY

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A set A in E is circled if $aA \subset A$ for all scalars a such that $|a| \leq 1$. A set is radial at 0 if it contains a line segment through 0 in each direction. A vector topology for E is a topology such that addition and scalar multiplication are each jointly continuous. A local base is a fundamental system of neighborhoods of 0.

If E is an infinite dimensional real or complex linear space, the family of all circled sets radial at 0 is not a local base for a vector topology for E . This is observed in [1, I, Section 1, no. 9, ex. 6] and the example is repeated in [2, 5D], where the infinite dimensionality of E is essential to the argument.

We offer here the sharper observation that if E is a real or complex linear space, the family of all circled sets radial at 0 is not a local base for a vector topology for E iff $\dim E \geq 2$.

First consider the case $E = E^2$, where E^2 is either real or complex two dimensional Euclidean space and $\{u_1, u_2\}$ is the usual orthonormal basis. For each x in E such that $|x| = 1$, let $f(x)$ be the magnitude $|(x, u_1)|$ of the inner product (x, u_1) . Let A consist of a union of ray segments, one for each x such that $|x| = 1$. If $f(x)$ is 0 or irrational, let $[0, x] \subset A$. If $f(x) = p/q$ is a nonzero rational in lowest terms, let $[0, x/q] \subset A$. The fact that A is circled follows from $|(x, u_1)| = |(e^{i\theta}x, u_1)|$. Clearly A is radial at 0.

There is no set C radial at 0 with $C + C \subset A$. Suppose that there is such a C . Then there is an $\epsilon > 0$ such that $[0, \epsilon u_1]$ and $[0, \epsilon u_2]$ are in C . Therefore $C + C$ contains all ray segments $[0, x]$ where

$$x = t_1 u_1 + t_2 u_2, \quad 0 \leq t_1, t_2 \leq \epsilon.$$

Letting

$$y = \frac{t_1 u_1 + t_2 u_2}{\sqrt{t_1^2 + t_2^2}}, \quad \text{we have} \quad f(y) = \frac{t_1}{\sqrt{t_1^2 + t_2^2}}.$$

Let q be a prime such that $1/q < \epsilon$. If $t_1 = \epsilon/q$ and $t_2 = \epsilon\sqrt{1 - 1/q^2}$, then $f(y) = 1/q$. Hence the ray in A in the y direction is $[0, y/q]$. Now $|y| = \epsilon$ so y is not in A . But y is in $C + C$. Thus $C + C \not\subset A$.

In the general case $\dim E \geq 2$, let e_1 and e_2 be linearly independent vectors in E . Then the subspace $F = \text{sp}(e_1, e_2)$ spanned by e_1 and e_2 is isomorphic to E^2 under the map I such that $Iu_1 = e_1$, $Iu_2 = e_2$. If $B = I(A)$, B as a subspace of F has all the properties proved above for A . Let G be a subspace complementary to F . Then $G + A$ is radial at 0 in E and is circled.

There is no set C in E which is radial at 0 and such that $C + C \subset A + G$ (and thus the circled sets radial at 0 do not yield a vector topology). Suppose that there is such a C . Then $D = C \cap F$, considered as a subset of F , is radial at 0 and $D + D \subset A$. But this contradicts our result for E^2 .

If $\dim E \leq 1$, the family of circled sets radial at 0 is a base for the Euclidean topology.

REMARK. In the real case for E^2 the proof is simpler and more intuitive if we let $f(x)$ be the angle between x and e_1 .

References

1. N. Bourbaki, *Espaces vectoriels topologiques*, Actualités Sci. Ind., 1189 and 1229, Hermann, Paris, 1953 and 1955.
2. J. L. Kelley, *Linear topological spaces*, Van Nostrand, Princeton, N. J., 1963