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# Spherical and Symmetric Supervarieties

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Alexander C Sherman

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Committee in charge:

Professor Vera Serganova, Chair
Professor Nicolai Reshetikhin
Professor Ori Ganor

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## Abstract

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Alexander C Sherman

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University of California, Berkeley

Professor Vera Serganova, Chair

We develop and study the notion of a spherical supervariety, which is a generalization of the classical notion of a spherical variety in algebraic geometry. Spherical supervarieties are supervarieties admitting an action of a quasi-reductive group with an open orbit of a hyperborel subgroup. Three characterizations of spherical supervarieties are given: one which generalizes the Vinberg-Kimelfeld characterization of affine spherical varieties, another that extends the ideas of the affine case to the quasi-projective case, and finally one in terms of invariant rational functions which applies to any supervariety. Our characterization of affine spherical supervarieties leads to (non-constructive) existence theorems for finite-dimensional highest weight representations admitting certain coinvariants under spherical quasireductive subgroups.

Several interesting examples of spherical supervarieties are given. We present a classification of indecomposable spherical representations (for certain supergroups) and for each the description of its algebra of functions. Adjoint orbits of odd self-commuting elements are shown to be spherical in many cases, in particular for basic simple Lie superalgebras. We study group-graded supergroups and their spherical homogeneous supervarieties, showing in particular that the algebra of functions on an affine homogeneous supervariety is almost never completely reducible for such supergroups.

Finally we study the case of symmetric supervarieties and show that, despite their not always being spherical (in contrast to the classical case), we may under some circumstances guarantee the existence of an Iwasawa decomposition, which implies sphericity. The fixed points of automorphisms of generalized root systems coming from supersymmetric pairs are studied along the way. We use the Iwasawa decomposition to gain partial understanding of the structure of the space of functions as a representation. Finally, the case of a supergroup as a symmetric supervariety is discussed in detail, culminating in a description of the socle filtration and the Loewy layers of its space of functions.

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# Chapter 1

# Introduction

## 1.1 For a general audience

Symmetry plays a pervasive role in modern mathematics. A *symmetry* of an object is a reversible transformation of the object that preserves its structure. For example, if the object is a circle, then a rotation or reflection of the circle is a symmetry of it. The collection of symmetries of an object form what is known as a *group* in abstract algebra. Early in the development of group theory the mathematicians Evariste Galois, Felix Klein, and Sophus Lie (to name a few) made significant progress in the fields of number theory, geometry, and analysis in realizing that symmetry groups often reflect the full nature of, in each case, solutions to polynomial equations, the structure of a geometric space, and solutions to differential equations.

It has been observed over time that a fruitful approach to studying a particular collection of mathematical objects is to organize them by the type of symmetries they have, i.e. by their symmetry groups. The first famous instance of this occured in the early 1800s when Galois proved the insolvability of the quintic equation (in contrast to the quadratic, cubic, and quartic equations where we have a quadratic, cubic, and quartic formula) by studying the possible group of symmetries of the set of solutions. Another famous example that relates to the topic of this thesis is Klein's Erlangen program where types of geometry (e.g. Euclidean, affine, and projective) were paired with symmetry groups according to which quantities (e.g. length, angles) and properties (e.g. parallelism, orientation) should remain invariant under a transformation.

With this context we introduce spherical varieties. A *variety* is a geometric space that is the solution set of a collection of polynomial equations. A *spherical variety* is a variety with a particularly large amount of symmetry, meaning that its symmetry group is large enough in a precise sense. Many spaces of interest (e.g. spheres!) are spherical, and they form a rich and varied class of varieties which motivates their study. On the other hand the restrictive nature of the definition and the large symmetry group has led to a deep understanding of spherical varieties, and even a complete classification. The classification was the culmination of decades of work by many mathematicians and was only finished in the mid-2010s.

For many, the term *supersymmetry* brings up associations with esoteric theories in particle physics. However from a mathematical standpoint, supersymmetry can be regarded as a generalization of symmetry. The objects (or 'superobjects') considered have both 'even' and 'odd' components, and these components can mix with one another in specified ways. We define *supervarieties* to be geometric spaces described by solution sets of superpolynomials, and then we can consider supersymmetries of these spaces. Collections of supersymmetries now form *supergroups*. (A terminology pattern should be emerging at this point.) Therefore we can study supergeometry using supersymmetry.

In this work we develop an extension of the notion of spherical variety to that of spherical supervariety. We characterize spherical supervarieties and provide a rich collection of examples.

## 1.2 Spherical supervarieties

We begin with a brief discussion of spherical varieties before describing spherical supervarieties, and the results of this thesis.

### 1.2.1 Spherical varieties

We work over an algebraically closed field $k$ of characteristic zero. Let $G$ be a reductive algebraic group over $k$, and $X$ a $G$-variety. Then $X$ is spherical if a Borel subgroup of $G$ has an open orbit on $X$. Spherical varieties are a rich and well-studied class which simultaneously generalizes toric varieties (when $G = T$ is a torus), flag varieties, and symmetric varieties. They connect representation theory, combinatorics, and algebraic geometry.

A quest to classify spherical varieties was completed in the mid-2010s thanks to work by Bravi, Brion, Cupit-Foutou, Knop, Losev, Luna, Pezzini, Vust, and others spanning several decades. In some ways it began with the 1985 Luna-Vust classification of embeddings of spherical homogeneous varieties via the combinatorics of colored fans, generalizing the familiar theory of fans in toric geometry. It ended with the classification of homogeneous spherical varieties via a combinatorial description proposed by Luna in the early 2000s and completed a little more than a decade later.

The characterization of affine spherical varieties immediately demonstrates connections of the subject to representation theory.

**Theorem 1.2.1** (Vinberg-Kimelfeld)**.** *Let $X$ be an affine $G$-variety. Then $X$ is spherical if and only if $k[X]$ is a multiplicity-free representation.*

If $K \subseteq G$ is a closed reductive subgroup then $G/K$ is affine, and thus theorem 1.2.1 and Frobenius reciprocity imply that an irreducible representation can have at most one $K$-invariant if and only if $G/K$ is spherical, i.e. the Lie algebra of $K$ has a complementary Borel subalgebra. For instance when $G/K$ is spherical one obtains a collection of polynomials on

$G/K$ indexed by the irreducible representations that appear. This is useful for obtaining a Harish-Chandra homomorphism from the algebra of invariant differential operators to polynomials, e.g. in the case when $G/K$ is a symmetric space.

Another illustration of the importance of theorem 1.2.1 is in multiplicity-free representations, i.e. $G$-modules $V$ for which $S^{\bullet}V$ is completely reducible. In [23], Roger Howe emphasizes such representations as being pervasive in the study of invariant theory. However a multiplicity-free representation is nothing but a spherical variety. Irreducible multiplicity-free representations were originally classified by Kac in [27], and the full classification was later completed in [33], as communicated by F. Knop. Multiplicity-free properties of exterior algebras and even of supersymmetric algebras are natural in this regard, and have been studied by T. Pecher in [42] and [41].

### 1.2.2 Spherical supervarieties

Let $G$ be an algebraic supergroup with $G_0$ reductive, where $G_0$ is the even underlying algebraic group of $G$. We call such supergroups quasi-reductive. We would like to consider supervarieties with actions of such supergroups which have an especially large amount of symmetry; namely, we would like a hyperborel subgroup (see definition 4.3.3) to have an open orbit. For those familiar with Lie superalgebras, the notion of hyperborel subalgebra agrees with the usual notion of Borel subalgebra for many heavily studied cases, apart from queer superalgebras (see remark 2.3.13). We call such supervarietes spherical, generalizing the classical notion to the super world.

It is interesting to ask how the properties of spherical varieties generalize. For affine supervarieties, an extension of the Vinberg-Kimelfeld theorem should be modified to take into account the fact that quasi-reductive groups have non-semisimple representation theory. The socle of the algebra of functions is a natural replacement, and we have:

**Theorem 1.2.2** (Corollary 5.2.5). *If $X$ is an affine spherical supervariety, then the socle of $k[X]$ is multiplicity-free.*

One might hope the converse holds, at least in the case when $k[X]$ is completely reducible. However this is not the case; there are situations in which a $G$-supervariety $X$ is affine, $k[X]$ is completely reducible and multiplicity-free, but $X$ is not spherical (see example 5.2.1). A simple example is the standard representation of $GL(0|n)$. Thus this connection does not generalize nicely to the super world.

However, we do find a characterization of sphericity in terms of the subalgebra of $k[X]$ generated by $B$-highest weight functions, where $B$ is a hyperborel subgroup.

**Theorem 1.2.3** (Theorem 5.2.2). *Let $X$ be an affine spherical supervariety, $B$ a hyperborel subgroup of $G$. Write $N$ for the maximal unipotent subgroup of $B$, and $T$ for a maximal torus of $B$. Then the following are equivalent:*

- *$X$ is spherical;*

- *Every nonzero $B$-eigenfunction in $k[X]$ is non-nilpotent, and* $\dim k[X]^N_\lambda \leq 1$ *for all weights $\lambda$ of $T$;*
- *$k[X]^N$ is an even commutative algebra without nilpotents, and the natural $T$-action is multiplicity-free.*

This characterization, which first appeared in the author's work [59], generalizes the classical fact that an affine $G$-variety $X$ is spherical if and only if $X//N$ is a toric variety for a maximal torus $T$ of a Borel subgroup $B$, where $N$ is the unipotent radical of $B$. We note that a generalization of this theorem to the quasi-projective case is possible, and follows a similar line of argument as in the affine case. We provide it in this thesis, with statement and proof in theorem 5.3.1.

We say a supervariety is graded if it is the exterior algebra of a coherent sheaf on a variety. (Note that the term graded is sometimes used instead of super, as in [31]. Also the term split is sometimes used instead of graded, as in [62].) Remarkably, if a spherical supervariety is graded (which holds for all smooth affine supervarieties) we obtain a non-constructive existence theorem on highest weights. To be precise, let $X$ be a graded affine spherical supervariety with an open $B$-orbit and write $\Lambda^+_B(X)$ for the monoid of highest weights of $k[X]$. Then we have $\Lambda^+_B(X) \subseteq \Lambda^+_{B_0}(X_0)$.

**Theorem 1.2.4** (Corollary 5.2.7)**.** *A generic weight in $\Lambda^+_{B_0}(X_0)$ is also contained in $\Lambda^+_B(X)$.*

The term generic is made more precise in the proof. Since $G/K$ is affine if and only if $K$ is also quasi-reductive, we obtain:

**Corollary 1.2.5.** *If $K$ is a quasi-reductive subgroup of $G$ and $G/K$ has an open $B$-orbit, then for a generic weight $\lambda$ in $\Lambda^+_{B_0}(G_0/K_0)$ there exists an indecomposable $G$-module $V$ admitting a $B$-eigenvector $v$ of weight $\lambda$ and a $K$-coinvariant $\varphi$ such that $\varphi(v) \neq 0$.*

The author began studying examples of spherical supervarieties in [58]. In that work, we found indecomposable spherical representations for a large class of quasi-reductive supergroups and computed the structure of the algebra of functions. We present the classification here once again as examples of spherical supervarieties; see section 5.5.

The Duflo-Serganova functor has played an important role in the theory of Lie superalgebras. It is therefore interesting to ask whether there is a relationship to the notion of sphericity. We have been able to obtain the following result. Let $\widetilde{G}$ be a quasi-reductive supergroup such that its Lie superalgebra $\widetilde{\mathfrak{g}} := \operatorname{Lie} G$ is $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(m|2n)$, or exceptional basic simple. Let $x \in \widetilde{\mathfrak{g}}_{\overline{1}}$ be self-commuting, and write $C(x)$ for its centralizer in $\widetilde{G}$. Let $M$ be a closed normal subgroup of $C(x)$ such that the Lie superalgebra of $M$ is $[x, \widetilde{\mathfrak{g}}]$, and define $G$ to be the supergroup $C(x)/M$. Then $G$ will be quasi-reductive with Lie superalgebra isomorphic to $\widetilde{\mathfrak{g}}_x$.

Now let $K$ be a spherical subgroup of $G$, and consider the $\widetilde{G}$-supervariety $\widetilde{G}/\widetilde{K}$. Then we have:

**Theorem 1.2.6** (Theorem 5.8.1)**.** *$\widetilde{G}/\widetilde{K}$ is a spherical supervariety, and there is a precise description of the Borel subgroups that have an open orbit on it in terms of the Borel subgroups that have an open orbit on $G/K$ and the root vector $x$.*

In particular we obtain the following intriguing result when $G/K$ is a point:

**Corollary 1.2.7.** *Let $x \in \widetilde{\mathfrak{g}}_{\overline{1}}$ be self-commuting. Then $G/C(x)$ is spherical.*

For $x \in \mathfrak{g}_{\overline{1}}$, $G/C(x)$ is the orbit of $x \in \Pi\mathfrak{g}$, so it is an 'odd' adjoint orbit. This says in particular that the self-commuting cone, as defined in [15], is the even subvariety of a spherical supervariety. Note that it was known to be spherical already, see [13].

Part of the motivation for the study of spherical supervarieties comes from the interest in supersymmetric spaces, or symmetric supervarieties as we call them in this thesis. Several authors have studied different aspects of these spaces. In physics, supersymmetric spaces are studied as target spaces of non-linear SUSY $\sigma$-models; see for instance [64].

Mathematically they have also generated significant interest. In [44] and [45], the Capelli eigenvalue problem was studied for supersymmetric pairs coming from simple Jordan superalgebras. In [1] a generalization of the Harish-Chandra isomorphism theorem was given, and in [2] certain facts about the socle of the space of functions was proven, among other things. Further in [55] the combinatorics of root systems from supersymmetric pairs is used to construct new integrable systems.

Classically, symmetric varieties are spherical by the Iwasawa decomposition. In the super setting the Iwasawa decomposition need not hold, and in fact a symmetric supervariety need not be spherical. We shed light on the existence of an Iwasawa decomposition in the case of supersymmetric pairs $(\mathfrak{g}, \mathfrak{k})$ where $\mathfrak{g}$ is a basic simple Lie superalgebra and $\mathfrak{k}$ is the fixed points of an involution $\theta$ preserving the invariant form on $\mathfrak{g}$. We study this situation by passing to the root system which has the structure of a generalized root system (GRS) as defined by Serganova in [48]. Using this framework we are able to prove a strong result on the structure of centralizers of certain tori arising from semisimple automorphisms of $\mathfrak{g}$ (theorem 6.2.9). In particular it implies that:

**Theorem 1.2.8** (Theorem 6.2.8)**.** *If $\mathfrak{g}$ is a basic simple Lie superalgebra and $\theta$ is an involution of $\mathfrak{g}$ preserving the invariant form, then either $\theta$ or $\delta \circ \theta$ admits an Iwasawa decomposition, where $\delta(x) = (-1)^{\overline{x}}x$ is the canonical grading automorphism.*

These results were originally shown by the author in [56]. The perspective of studying involutions via root systems motivates studying the restricted root system. We explain its structure in proposition 6.3.3. We further find (by observation) the following remarkable fact:

**Theorem 1.2.9** (Subsection 6.3.2)**.** *If a restricted root system has more than one real component, then it has the natural structure of a deformed generalized root system, as defined by Sergeev and Veselov in [54].*

This seems to have been known by Sergeev and Veselov in their original work, as well as by other researchers in the field. We explicitly give the Sergeev-Veselov parameters for each eligible supersymmetric pair in section 6.3.2.

Thus many supersymmetric pairs admit an Iwasawa decomposition and using it we can gain some understanding of $k[G/K]$ as a $G$-module. Let $\mathfrak{a}$ be a Cartan subspace, i.e. a maximal toral subalgebra of the $(-1)$-eigenspace of $\theta$. We may extend $\mathfrak{a}$ to a Cartan subalgebra of $\mathfrak{g}$. Then it is not difficult to show that $\Lambda^+(G/K) \subseteq \mathfrak{a}^*$ as in the classical setting. Of more interest is the following statement:

**Theorem 1.2.10** (Theorem 6.4.4)**.** *For a generic weight $\lambda \in \Lambda^+(G/K)$, we have $L(\lambda) \subseteq k[G/K]$.*

This theorem says that generically (to be explained more carefully) a highest weight submodule of $k[G/K]$ generates an irreducible representation. This can be viewed as a strengthening of corollary 1.2.5. Note that a more precise result in the analytic case is proven in [2].

Finally, a case of significant interest is the diagonal supersymmetric pair $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$, corresponding to the symmetric supervariety $G \times G/G \cong G$. For a reductive group, the Peter-Weyl theorem expresses the decomposition of the algebra of functions in simple terms, as matrix coefficients over all the irreducible representations. Without semisimplicity we cannot hope for such a full description. However, in theorem 6.5.11 we describe the socle (Loewy) filtration of $k[G]$ as a $G \times G$-module. This follows from a more general result on the coradical filtration of a coalgebra, originally shown by the author in [57].

We believe that the understanding of spherical supervarieties can enrich both our understanding of the representation theory of supergroups and provide an interesting geometric theory that is worthy of study in its own right.

## 1.3 Structure of thesis

The first chapter introduces linear superalgebra, defining the rigid symmetric monodial category of super vector spaces and progressing to a discussion of Lie superalgebras. The structure of quasi-reductive Lie superalgebras and their representation theory is discussed at some length, as well as the notion of Borel and hyperborel subalgebras, which both make frequent appearances. Certain distinguished quasi-reductive Lie superalgebras are introduced and their root systems are described. Finally we describe a relationship between Schur functors, the parity shift functor, and tensor representations of $\mathfrak{gl}(m|n)$.

Chapter 2 discusses the rudimentary algebraic supergeometry that we will need. Properties of supercommutative algebras are introduced to define affine superschemes, followed by superschemes in general. Supervarieties are defined as superschemes satisfying conditions similar to varieties in the classical world; however subtle differences appear. The notion of being graded as a supervariety is explained, followed by a section on quasi-coherent sheaves, line bundles, and vector bundles. Projective supergeometry is introduced, followed by standard definitions of the tangent space, the tangent sheaf, and the sheaf of relative differentials. Finally, a definition and in-depth characterization of smoothness of supervarieties is given.

The third chapter defines algebraic supergroups and how they relate to Lie superalgebras. Super Harish-Chandra pairs and an associated category are defined, which allows one to replace algebraic supergroups with the pair of an algebraic group and a Lie superalgebra. We show that this replacement can be performed for actions of supergroups as well, simplifying some questions such as whether actions of supergroups restrict to open subsets. Quasi-reductive supergroups, the global versions of quasi-reductive Lie superalgebras, are introduced and a list of distinguished supergroups is given to complement our list of distinguished Lie superalgebras. Next, linearizations of sheaves are defined with a special focus on the case of equivariant line bundles. The notion of orbits is discussed, and the connection between open orbits, pullbacks of sections, and rational invariants is given. Finally, a useful theorem on the existence of equivariant gradings is proven.

In the fourth chapter the definition of spherical supervariety is given and characterizations in the affine and general case are explained. When the spherical supervariety is graded an existence theorem on highest weight functions is proven, and the properties (or lack thereof) of the monoid of highest weights are examined. After dealing with the affine case, we explain the generalization to the quasi-projective case. Next, we delve into several important examples of spherical supervarieties, starting with spherical indecomposable representations of the distinguished Lie superalgebras. The symmetric superalgebras of these representations are computed. Following that we explain the situation when the supergroup is group-graded, and finish with a link between the Duflo-Serganova functor and sphericity. In particular we show that adjoint orbits of odd self-commuting elements are spherical for basic distinguished Lie superalgebras, as well as other cases.

Chapter 5 focuses on the case of symmetric supervarieties. They are defined, and the Iwasawa decomposition is discussed at some length. Although the Iwasawa decomposition need not always hold, we show it often does (in precise sense) for basic distinguished Lie superalgebras. We explain restricted root systems arising from supersymmetric pairs and which Borel subalgebras arise from the Iwasawa decomposition. Some general properties of the representation on the space of functions is then proven, giving in particular that a generic highest weight submodule in the space of functions is irreducible. Finally, the case of the regular representation of a Cartan-even, quasi-reductive supergroup is studied in depth, culminating in an explicit description of its socle filtration and the layers thereof.

The three appendices prove, in order, the classification of spherical indecomposable representations for the distinguished Lie superalgebras (section 5.5), the existence of an Iwasawa decomposition for certain supersymmetric pairs of basic distinguished Lie superalgebras (theorem 6.2.8), and the facts concerning the socle filtration of the regular representation of a Cartan-even quasi-reductive supergroup (proposition 6.5.9 and theorem 6.5.11). These proofs can largely be read independently of the rest of the thesis.

## 1.4 List of notation

Here we give a list of commonly used notation for the benefit of the reader.

- $k$ an algebraically closed field of characteristic zero;
- $\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$;
- $V = V_{\overline{0}} \oplus V_{\overline{1}}$ the parity decomposition of a super vector space;
- $\delta$ the grading automorphism of a super vector space;
- $\Pi V$ the parity shift of $V$ (see definition 2.1.4);
- $\underline{\mathrm{Hom}}(V_1, V_2)$ the internal Hom in $\mathsf{SVect}$ (see definition 2.1.3);
- $S^\bullet V$ the symmetric superalgebra of $V$;
- $\mathfrak{g}$ a Lie superalgebra;
- $\mathfrak{b}$ a (hyper)Borel subalgebra;
- $\mathfrak{n} \subseteq \mathfrak{b}$ the maximal nilpotent subalgebra;
- $\mathfrak{h}_{\overline{0}}$ a Cartan subalgebra of $\mathfrak{g}_{\overline{0}}$;
- $\alpha$ a root of $\mathfrak{g}$;
- $\mathfrak{g}_\alpha$ the root space of $\alpha$ in $\mathfrak{g}$;
- $\Delta$ the set of all roots of $\mathfrak{g}$;
- $Q$ the root lattice of $\mathfrak{g}$;
- $W$ the Weyl group of $\Delta$;
- $\Sigma \subseteq \Delta$ a simple root system;
- $\mathfrak{a}$ a Cartan subspace;
- $\overline{\Delta}$ the restricted root system;
- $\overline{\alpha}$ a restricted root;
- $\Delta^0$ the roots fixed by an automorphism of a root system;
- $\mathfrak{h}$ the centralizer of $\mathfrak{h}_{\overline{0}}$ in $\mathfrak{g}$;
- $\lambda$ a weight of $\mathfrak{h}_{\overline{0}}$;

- $L(\lambda) = L_{\mathfrak{b}}(\lambda)$ the irreducible representation of $\mathfrak{g}$ of (even) dominant highest weight $\lambda$ with respect to $\mathfrak{b}$;
- $L_0(\lambda)$ the irreducible representation of $\mathfrak{g}_{\overline{0}}$ with respect to $\mathfrak{b}_{\overline{0}}$ of (even) dominant highest weight $\lambda$ with respect to $\mathfrak{b}_{\overline{0}}$;
- $r_\alpha$ an odd reflection;
- $r_\alpha(\mathfrak{b})$ the Borel subalgebra obtained from $\mathfrak{b}$ via $r_\alpha$;
- $\mathfrak{b}^\sigma$ the Borel subalgebra defined by the $\epsilon\delta$-sequence $\sigma$ (see section 2.3.4);
- $\mathfrak{g} \ltimes V$ see example 2.3.2;
- $\mathfrak{g}_{(\chi)} = \{X \in \mathfrak{g} : \chi(X) = 0\}$ where $\chi : \mathfrak{g} \to k$ is a morphism of Lie superalgebras;
- $A$ a supercommutative algebra;
- $X, Y$ are supervarieties;
- $|X|$ the underlying topological space of $X$;
- $\mathcal{O}_X$ the structure sheaf of $X$;
- $k[X] = \Gamma(X, \mathcal{O}_X)$;
- $\mathcal{O}_{X,x}$ the stalk of $\mathcal{O}_X$ at $x \in |X|$;
- $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$;
- $k(X)$ rational functions on $X$ (see section 3.2.2);
- $X(A)$ the $\operatorname{Spec} A$-points of $X$ (see section 3.2.4);
- $X_0$ the even subvariey of $X$;
- $\mathcal{N}_X$ the conormal sheaf of $X_0$ in $X$;
- $U$ an open subvariety of $X$;
- $\mathcal{F}$ a quasi-coherent sheaf;
- $\mathcal{L}$ a line bundle;
- $\mathcal{F}_x$ the stalk of $\mathcal{F}$ at $x \in |X|$;
- $T_xX$ the tangent space of $X$ at $x \in X(k)$;
- $\mathcal{T}_X$ the tangent sheaf of $X$;

- $\Omega_{X/k}$ the sheaf of relative differentials;
- $G, H, K$ algebraic supergroups;
- $G_0$ the even algebraic subgroup of $G$;
- $B$ a (hyper)Borel subgroup of a quasi-reductive supergroup;
- $T$ a maximal torus of $G_0$;
- $N$ the maximal unipotent subgroup of $B$;
- $\Lambda^+_{\mathfrak{b}}(X)$ the weights of non-zero $\mathfrak{b}$-eigenfunctions on $X$;
- $\Lambda_{\mathfrak{b}}(X)$ the weights of non-zero rational $\mathfrak{b}$-eigenfunctions on $X$.

# Chapter 2

# Linear superalgebra

Here we develop the basic notions of linear superalgebra we will need. Throughout $k$ denotes an algebraically closed field of characteristic zero. Write $\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$. We let $\mathbb{N} = \{0, 1, 2, \cdots\} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

## 2.1 Super vector spaces

**Definition 2.1.1.** Define the category $\mathsf{SVect}$ to have objects consisting of $\mathbb{Z}_2$-graded $k$-vector spaces $V = V_{\overline{0}} \oplus V_{\overline{1}}$, with morphisms $\mathrm{Hom}(V_1, V_2)$ all degree-preserving linear maps $V_1 \to V_2$. For a super vector space $V$ we will always think of $V_{\overline{0}}$ and $V_{\overline{1}}$ individually as even vector spaces. We write $\mathsf{Svect}$ for the full subcategory of $\mathsf{SVect}$ whose objects are finite-dimensional super vector spaces. For a homogeneous element $v \in V$, we write $\overline{v} \in \mathbb{Z}_2$ for its degree.

*Example* 2.1.2. For $m, n \in \mathbb{Z}_{\geq 0}$, define the super vector space $k^{m|n}$ to have grading

$$(k^{m|n})_{\overline{0}} = k^m \quad \text{and} \quad (k^{m|n})_{\overline{1}} = k^n.$$

**Definition 2.1.3.** Given super vector spaces $V_1, V_2$, define the super vector space $\underline{\mathrm{Hom}}(V_1, V_2)$ by $\underline{\mathrm{Hom}}(V_1, V_2)_{\overline{0}} := \mathrm{Hom}(V_1, V_2)$ and

$$\underline{\mathrm{Hom}}(V_1, V_2)_{\overline{1}} := \mathrm{Hom}((V_1)_{\overline{0}}, (V_2)_{\overline{1}}) \oplus \mathrm{Hom}((V_1)_{\overline{1}}, (V_2)_{\overline{0}}).$$

We write $\underline{\mathrm{End}}(V) := \underline{\mathrm{Hom}}(V, V)$ and $\mathrm{End}(V) := \underline{\mathrm{End}}(V)_{\overline{0}} = \mathrm{Hom}(V, V)$.

### 2.1.1 Rigid symmetric monoidal structure

We give $\mathsf{SVect}$ the structure of a symmetric monoidal category as follows. For super vector spaces $V_1, V_2$, define $V_1 \otimes V_2$ to have $\mathbb{Z}_2$-grading

$$(V_1 \otimes V_2)_i = \bigoplus_{j+k=i} (V_1)_j \otimes (V_2)_k.$$

We define the braiding isomorphism $s_{V_1,V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1$ by

$$s_{V_1,V_2}(v_1 \otimes v_2) = (-1)^{\overline{v_1}\overline{v_2}} v_2 \otimes v_1.$$

We see this braiding satisfies $s_{V_2,V_1} \circ s_{V_1,V_2} = \mathrm{id}_{V_1 \otimes V_2}$.

By abuse of notation we set $k := k^{1|0}$. We have natural isomorphisms $k \otimes V \cong V \otimes k \cong V$, so that $k$ is a unit object in $\mathsf{SVect}$. Given a super vector space $V$, it has a dual object given by $V^* = V_{\overline{0}}^* \oplus V_{\overline{1}}^*$. The pairing $\mathrm{ev} : V^* \otimes V \to k$ is the natural one. For finite-dimensional super vector spaces $V_1$ and $V_2$ we have a natural identification $\underline{\mathrm{Hom}}(V_1, V_2) = V_2 \otimes (V_1)^*$ in the usual way, and the copairing $\mathrm{coev} : k \to V \otimes V^* = \underline{\mathrm{End}}(V)$ sends 1 to the identity morphism. With this structure, we define the superdimension $\operatorname{sdim} V$ of a finite-dimensional super vector space $V$, to be the dimension of $V$ in the rigid symmetric monoidal category $\mathsf{Svect}$. That is, we have a composition of natural maps

$$k \xrightarrow{\mathrm{coev}} V \otimes V^* \xrightarrow{s_{V,V^*}} V^* \otimes V \xrightarrow{\mathrm{ev}} k$$

and we let $\operatorname{sdim} V$ be the scalar that this map defines. With this definition, we have the formula

$$\operatorname{sdim} V = \dim V_{\overline{0}} - \dim V_{\overline{1}}.$$

We also write $\dim^s(V) := (\dim V_{\overline{0}} | \dim V_{\overline{1}})$.

**Definition 2.1.4.** For a super vector space $V$ define the parity shift of $V$ to be

$$\Pi V := k^{0|1} \otimes V.$$

For $n \in \mathbb{N}$ set $\Pi^n V := (k^{0|1})^{\otimes n} \otimes V$. This defines a functor on $\mathsf{SVect}$, see section 2.4.2 for more on this

### 2.1.2 Schur functors

Given a super vector space $V$ we obtain, via the braiding, an action of $S_n$ on $V^{\otimes n}$. These actions are natural in $V$, and hence for a partition $\lambda$ we obtain definitions of the Schur functors $S^\lambda$, giving a super vector space $S^\lambda(V)$. By complete reducibility of the $S_n$ action, we obtain a natural isomorphism

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} S^\lambda(V) \otimes S_\lambda,$$

where $S_\lambda$ is the irreducible $S_n$-representation corresponding to the tableau $\lambda$.

*Example* 2.1.5. Given a super vector space $V$ and $n \in \mathbb{N}$, we have the symmetric powers

$$S^n V := S^{(n)}(V) = V^{\otimes n} / (\sigma \cdot x - x).$$

Then, considering $V_{\bar{0}}$ and $V_{\bar{1}}$ as even vector spaces (as always), we have a natural isomorphism of super vector spaces

$$S^n(V) \cong \bigoplus_{i+j=n} \Pi^j S^i(V_{\bar{0}}) \otimes \Lambda^j(V_{\bar{1}}).$$

Similarly we have the exterior powers

$$\Lambda^n V := S^{(1^n)}V = V^{\otimes n}/(\sigma \cdot x + x).$$

Thus

$$\Lambda^n V \cong \bigoplus_{i+j=n} \Pi^j \Lambda^i(V_{\bar{0}}) \otimes S^j(V_{\bar{1}}).$$

Notice that in contrast with vector spaces, there may be infinitely many non-zero exterior powers of a super vector space. In fact, in section 2.4.2 we exhibit a natural isomorphism

$$\Lambda^n \Pi V \cong \Pi^n S^n V.$$

## 2.2 Superalgebras

A superalgebra is an algebra object in SVect, or equivalently it is a $\mathbb{Z}_2$-graded associative $k$-algebra. If $A$ is a superalgebra then $A_{\bar{0}}$ is an algebra in the usual sense, and $A_{\bar{1}}$ is naturally a bimodule over $A_{\bar{0}}$ equipped with a map $A_{\bar{1}} \otimes_{A_{\bar{0}}} A_{\bar{1}} \to A_{\bar{0}}$ of $A_{\bar{0}}$-modules.

Given a superalgebra $A$, a subalgebra is a subspace $B$ of $A$ as a super vector space, which is closed under the product morphism. When we discuss supercommutative algebras we will also assume $B$ contains the unit element. The term subsuperalgebra might be more appropriate than subalgebra, but for ease of writing and speech we use subalgebra.

*Example* 2.2.1. If $V$ is a super vector space, we may consider the superalgebra $\underline{\mathrm{End}}(V) = \underline{\mathrm{Hom}}(V, V)$ consisting of all $k$-linear endomorphisms of $V$. Notice that $\underline{\mathrm{End}}(V)_{\bar{0}} = \mathrm{End}(V_{\bar{0}}) \times \mathrm{End}(V_{\bar{1}})$.

*Example* 2.2.2. If $V$ is a super vector space, the tensor algebra $\mathcal{T}V$ is a $\mathbb{Z}$-graded superalgebra. If we let $I$ be the ideal of $\mathcal{T}V$ generated by $\sigma \cdot x - x$ for $x \in V^{\otimes 2}$ and $\sigma \in S_2$, the quotient $\mathcal{T}V/I$ is the symmetric algebra $S^\bullet V$ which is $\mathbb{Z}$-graded with homogeneous components $S^n V$.

### 2.2.1 Sign convention

The symmetric monoidal structure on SVect implies that whenever one tensor factor is moved past an adjacent one we obtain a sign in formulas. Thus in the theory of superalgebras many signs appear which depend on the parity of elements involved. We adopt the universally used convention that in any formula where a sign appears we assume all elements are homogeneous and that the formula extends multilinearly to nonhomogeneous elements as well.

For example, if a superalgebra $A$ is supercommutative (definition 3.1.1) then $ab = (-1)^{\bar{a}\bar{b}} ba$. Thus for arbitrary elements $a = a_{\bar{0}} + a_{\bar{1}}$, $b = b_{\bar{0}} + b_{\bar{1}}$ we have

$$ab = (a_{\bar{0}} + a_{\bar{1}})(b_{\bar{0}} + b_{\bar{1}}) = b_{\bar{0}} a + b_{\bar{1}}(a_{\bar{0}} - a_{\bar{1}}).$$

### 2.2.2 Lie superalgebras

**Definition 2.2.3.** A Lie superalgebra is a Lie algebra in the category SVect. Explicitly, it is a super vector space $\mathfrak{g}$ with a map $[-,-] : \Lambda^2\mathfrak{g} \to \mathfrak{g}$ (the Lie bracket) such that

$$[x,[y,z]] = [[x,y],z] + (-1)^{\overline{xy}}[y,[x,z]].$$

A morphism of Lie superalgebras is a map of super vector spaces respecting the Lie bracket.

*Remark* 2.2.4. Given a Lie superalgebra $\mathfrak{g}$, the even part $\mathfrak{g}_{\overline{0}}$ is preserved under the Lie bracket and thus is endowed with the structure of a Lie algebra. Further, $\mathfrak{g}_{\overline{1}}$ naturally obtains the structure of a $\mathfrak{g}_{\overline{0}}$-module under the Lie bracket such that $[-,-] : S^2\mathfrak{g}_{\overline{1}} \to \mathfrak{g}_{\overline{0}}$ is a homomorphism of $\mathfrak{g}_{\overline{0}}$-modules. Conversely, the data of a Lie algebra $\mathfrak{k}$ and a $\mathfrak{k}$-module $V$ with a $\mathfrak{k}$-module morphism $[-,-] : S^2V \to \mathfrak{k}$ defines a Lie superalgebra structure on $\mathfrak{k} \oplus V$ if and only if the equation $[[x,x],x] = 0$ holds for all $x \in V$.

*Example* 2.2.5. If $V$ is a super vector space, we may define a Lie bracket on it by $[v,w] = 0$ for all $v, w \in V$. We say a Lie superalgebra is abelian if it is isomorphic to one of this form.

*Example* 2.2.6. Given a superalgebra $A$, we may naturally give it the structure of a Lie superalgebra by setting the Lie bracket to be the supercommutator, i.e.

$$[a,b] = ab - (-1)^{\overline{a}\overline{b}}ba.$$

*Example* 2.2.7. Given a super vector space $V$, we define $\mathfrak{gl}(V)$ to be the Lie superalgebra obtained from the superalgebra $\mathrm{End}(V)$ under supercommutator. It is finite-dimensional whenever $V$ is. Observe that $\mathfrak{gl}(V)_{\overline{0}} = \mathfrak{gl}(V_{\overline{0}}) \times \mathfrak{gl}(V_{\overline{1}})$ and that if $V$ is finite-dimensional then $\mathfrak{gl}(V)_{\overline{1}} = V_{\overline{0}} \otimes V_{\overline{1}}^* \oplus V_{\overline{1}} \otimes V_{\overline{0}}^*$ as a $\mathfrak{gl}(V)_{\overline{0}}$-module. When $V$ is finite-dimensional, for $u \in \mathfrak{gl}(V)$ we define the supertrace of $u$, $\mathrm{str}(u)$, to be the scalar in $k = \mathrm{End}_k(k)$ defining the map

$$k \xrightarrow{coev} V \otimes V^* \xrightarrow{u\otimes 1} V \otimes V^* \xrightarrow{s_{V,V^*}} V^* \otimes V \xrightarrow{ev} k.$$

Explicitly we have that $\mathrm{str}(u) = \mathrm{tr}(u|_{V_{\overline{0}}}) - \mathrm{tr}(u|_{V_{\overline{1}}})$. The supertrace map satisfies

$$\mathrm{str}(uv) = (-1)^{\overline{uv}}\,\mathrm{str}(vu),$$

and thus $\mathrm{str} : \mathfrak{gl}(V) \to k$ defines a surjective Lie superalgebra homomorphism, where $k$ has the trivial Lie superalgebra structure.

If $V$ is finite-dimensional we define $\mathfrak{sl}(V)$ to be the kernel of str, a Lie superalgebra. When $\mathrm{sdim}(V) = 0$ we have $\mathrm{str}(\mathrm{id}_V) = 0$, so $\mathrm{id}_V \in \mathfrak{sl}(V)$, and in this case we set

$$\mathfrak{psl}(V) := \mathfrak{sl}(V)/k\langle \mathrm{id}_V\rangle.$$

*Example* 2.2.8. If $A$ is a superalgebra, we define a derivation of $A$ to be a $k$-linear map of super vector spaces $D : A \to A$ such that $D(ab) = D(a)b + (-1)^{\overline{aD}}aD(b)$ for all $a, b \in A$. Write $\mathrm{Der}(A)$ for the super vector space of derivations of $A$ viewed as a subspace of $\underline{\mathrm{End}}(V)$. Then $\mathrm{Der}(A)$ is closed under supercommutator and thus is a Lie superalgebra.

*Example* 2.2.9. Let $V$ be a finite-dimensional super vector space with an even non-degenerate supersymmetric bilinear form $(-,-) : S^2V \to k$. We define $\mathfrak{osp}(V)$ to be the Lie subalgebra of $\mathfrak{gl}(V)$ given by those endomorphisms $u \in \mathfrak{gl}(V)$ such that $(uv, w) + (-1)^{\overline{uv}}(v, uw) = 0$ for all $v, w \in V$. Then we have $\mathfrak{osp}(V)_{\bar{0}} = \mathfrak{o}(V_{\bar{0}}) \times \mathfrak{sp}(V_{\bar{1}})$, and $\mathfrak{osp}(V)_{\bar{1}} = V_{\bar{0}} \otimes V_{\bar{1}}$ as an $\mathfrak{osp}(V)_{\bar{0}}$-module.

**Definition 2.2.10.** Given a Lie superalgebra $\mathfrak{g}$, its enveloping algebra $\mathcal{U}\mathfrak{g}$ is the superalgebra $\mathcal{T}\mathfrak{g}/I$, where $I$ is the ideal generated by elements of the form $xy - (-1)^{\overline{xy}}yx - [x, y]$, where $x, y \in \mathfrak{g}$. The superalgebra $\mathcal{U}\mathfrak{g}$ has a natural $\mathbb{N}$-filtration $\mathcal{U}^k\mathfrak{g}$ by degree of monomials in $\mathcal{T}\mathfrak{g}$ such that $\mathcal{U}^0\mathfrak{g} = k$ and $\mathcal{U}^1\mathfrak{g} = k + \mathfrak{g}$. By the PBW theorem for Lie superalgebras, the natural map $\mathfrak{g} \to \mathcal{U}\mathfrak{g}$ is injective, and the natural algebra homomorphism $S\mathfrak{g} \to \mathrm{gr}\,\mathcal{U}\mathfrak{g}$ is an isomorphism.

The universal enveloping algebra has the natural structure of a cocommutative Hopf superalgebra with comultiplication given by $x \mapsto x \otimes 1 + 1 \otimes x$, counit $x \mapsto 0$, and antipode $x \mapsto -x$ for $x \in \mathfrak{g}$, each extended multilinearly.

Just as with Lie algebras we have a natural correspondence between Lie superalgebra maps $\mathfrak{g} \to A$ and superalgebra homomorphisms $\mathcal{U}\mathfrak{g} \to A$ where $A$ is an associative superalgebra.

### 2.2.3 Representations

**Definition 2.2.11.** A representation of a Lie superalgebra $\mathfrak{g}$ on a super vector space $V$ is a map of Lie superalgebras $\mathfrak{g} \to \mathfrak{gl}(V)$. Equivalently, it is a map of superalgebras $\mathcal{U}\mathfrak{g} \to \mathfrak{gl}(V)$. Given a representation $\mathfrak{g} \to \mathfrak{gl}(V)$, we say that $\mathfrak{g}$ acts on $V$, and we will refer to $V$ as a representation of $\mathfrak{g}$, or a $\mathfrak{g}$-module.

The representations of $\mathfrak{g}$ form an abelian category. In addition we have constructions of tensor product, dual, and parity shift representations described below.

- Given a super vector space $V$ we define the trivial action of $\mathfrak{g}$ on $V$ by setting $x \cdot v = 0$ for all $x \in \mathfrak{g}$, $v \in V$.
- Given two representations $V_1, V_2$ of $\mathfrak{g}$ we obtain a tensor product representation $V_1 \otimes V_2$ defined by $x \cdot (v_1 \otimes v_2) = x \cdot v_1 \otimes v_2 + (-1)^{\overline{xv_1}} v_1 \otimes x \cdot v_2$ where $x \in \mathfrak{g}$, $v_i \in V_i$.
- Given a representation $V$ of $\mathfrak{g}$, we define the representation $\Pi V$ as the tensor product representation $k^{0|1} \otimes V$, where $k^{0|1}$ is given the trivial action of $\mathfrak{g}$.
- Given a representation $V$ of $\mathfrak{g}$, we define the dual representation $V^*$ by $(x \cdot \varphi)(v) = -(-1)^{\overline{x\varphi}}\varphi(xv)$ for $x \in \mathfrak{g}$, $\varphi \in V^*$, and $v \in V$.

## 2.3 Quasi-reductive Lie superalgebras

It is not clear what types of Lie superalgebras are the 'correct' generalizations of reductive or semisimple Lie algebras. In [26], Kac classified all simple finite-dimensional Lie superalgebras. However for many purposes there is a wider class of Lie superalgebras that one wants to consider. We will focus on Lie superalgebras with even reductive part (quasi-reductive) and also, for the most part, have an even Cartan subalgebra (Cartan-even). We now give all the necessary definitions along with a brief description of their finite-dimensional representation categories. We refer to [51] for more on quasi-reductive Lie superalgebras.

**Definition 2.3.1.** A finite dimensional Lie superalgebra $\mathfrak{g}$ is called quasi-reductive is $\mathfrak{g}_{\overline{0}}$ is reductive and the adjoint action of $\mathfrak{g}_{\overline{0}}$ on $\mathfrak{g}_{\overline{1}}$ is semisimple.

*Example* 2.3.2. Let $\mathfrak{g}$ be a Lie superalgebra and $V$ a representation of $\mathfrak{g}$. Then we may define a new Lie superalgebra $\mathfrak{g} \ltimes V$ whose underlying super vector space is $\mathfrak{g} \oplus V$, such that $\mathfrak{g}$ is a subalgebra, $V$ is an abelian ideal, and $[(u, 0), (0, v)] = (0, u \cdot v)$.

If $\mathfrak{g} = \mathfrak{g}_{\overline{0}}$ and $V$ is a purely odd representation of $\mathfrak{g}$, we say that $\mathfrak{g}$ is graded. If in addition $\mathfrak{g} = \mathfrak{g}_{\overline{0}}$ is reductive then $\mathfrak{g}$ will be quasi-reductive.

For the rest of this section we assume $\mathfrak{g}$ is quasi-reductive.

**Definition 2.3.3.** Let $\mathfrak{h}_{\overline{0}} \subseteq \mathfrak{g}_{\overline{0}}$ be a Cartan subalgebra of $\mathfrak{g}_{\overline{0}}$. Then a Cartan subalgebra of $\mathfrak{g}$ is a subalgebra of the form $\mathfrak{h} = \mathfrak{c}(\mathfrak{h}_{\overline{0}})$, i.e. it is the centralizer of a Cartan subalgebra of $\mathfrak{g}_{\overline{0}}$.

Since $\mathfrak{g}_{\overline{1}}$ is a semisimple $\mathfrak{g}_{\overline{0}}$-module, we can decompose it into $\mathfrak{h}_{\overline{0}}$-eigenspaces. We call the nonzero weights of the adjoint $\mathfrak{h}_{\overline{0}}$-action on $\mathfrak{g}$ with nonzero weight spaces the roots of $\mathfrak{g}$, and let $\Delta \subseteq \mathfrak{h}_{\overline{0}}^*$ be the set of roots. The root lattice is defined to be the free abelian group $Q = \mathbb{Z}\Delta \subseteq \mathfrak{h}_{\overline{0}}^*$.

Write $\Delta_{\overline{0}} = \{\alpha \in \Delta : (\mathfrak{g}_\alpha)_{\overline{0}} \neq 0\}$ and $\Delta_{\overline{1}} = \Delta \setminus \Delta_{\overline{0}}$. Then $\Delta_{\overline{0}}$ is a reduced root system in its span. Its Weyl group, $W$, acts on $\mathfrak{h}_{\overline{0}}^*$ and preserves $\Delta_{\overline{1}}$. We call $W$ the Weyl group of $\mathfrak{g}$. Because all Cartan subalgebras of $\mathfrak{g}_{\overline{0}}$ are conjugate, the combinatorial data of the roots inside $\mathfrak{h}_{\overline{0}}^*$ is well-defined up to isomorphism. Given a root $\alpha \in \Delta$, we write $\mathfrak{g}_\alpha \subseteq \mathfrak{g}$ for the corresponding root space.

By standard facts about the representation theory of reductive Lie algebras, we have:

**Lemma 2.3.4.** *The following are equivalent if* $\mathfrak{g}_{\overline{0}} \neq 0$*:*

- *A Cartan subalgebra* $\mathfrak{h}_{\overline{0}} \subseteq \mathfrak{g}_{\overline{0}}$ *is self-centralizing in* $\mathfrak{g}$*;*
- *the zero weight space* $\mathfrak{g}_0$ *is of pure parity (and is even);*
- *all root spaces* $\mathfrak{g}_\alpha$ *are of pure parity.*

**Definition 2.3.5.** We say that $\mathfrak{g}$ is Cartan-even if $\mathfrak{g}_{\overline{0}} \neq 0$ and the three equivalent conditions in lemma 2.3.4 hold.

Now fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$.

**Definition 2.3.6.** Given a group homomorphism $\gamma : Q \to \mathbb{R}$ such that $\gamma(\alpha) \neq 0$ for all $\alpha \in \Delta$, we define a Borel subalgebra of $\mathfrak{g}$ to be one of the form $\mathfrak{b} = \bigoplus_{\gamma(\alpha)\geq 0} \mathfrak{g}_\alpha$. We then write $\Delta^\pm = \{\alpha \in \Delta : \pm\gamma(\alpha) > 0\}$, and we call elements of $\Delta^+$ positive roots and elements of $\Delta^-$ negative roots. We refer to such a partition of $\Delta$ into positive and negative roots as a positive system. We set $\mathfrak{n}^\pm = \bigoplus_{\alpha\in\Delta^\pm} \mathfrak{g}_\alpha$, write $\mathfrak{n} = \mathfrak{n}^+$, and call $\mathfrak{n}$ the nilpotent radical of $\mathfrak{b}$.

*Remark* 2.3.7.

- If $\mathfrak{b}$ is a Borel of $\mathfrak{g}$ then $\mathfrak{b}_{\overline{0}}$ is a Borel of $\mathfrak{g}_{\overline{0}}$. Further, if $\mathfrak{b}'_{\overline{0}}$ is a Borel of $\mathfrak{g}_{\overline{0}}$, then $\mathfrak{b}$ is conjugate to a Borel $\mathfrak{b}'$ whose even part if $\mathfrak{b}'_{\overline{0}}$.
- Because all Cartan subalgebras of $\mathfrak{g}_{\overline{0}}$ are conjugate, it is easy to see there are only finitely many conjugacy classes of Borel subalgebras. However, unlike for Lie algebras, there is often more than one conjugacy class of Borels, e.g. for $\mathfrak{gl}(V)$ or $\mathfrak{osp}(V)$ (see section 2.3.4).

### 2.3.1 Representations of $\mathfrak{g}$

Fix a choice of Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. Then we obtain a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. We also obtain the monoids $Q^\pm = \mathbb{N}\Delta^\pm$. Define a partial ordering on $\mathfrak{h}_{\overline{0}}^*$ by

$$\mu \leq \lambda \iff \lambda - \mu \in Q^+.$$

Consider the category $\mathcal{F}(\mathfrak{g})$ of finite-dimensional $\mathfrak{g}$-representations which are semisimple over $\mathfrak{g}_{\overline{0}}$. For the remainder of this section all $\mathfrak{g}$-modules will be in $\mathcal{F}(\mathfrak{g})$. Then each $V \in \mathcal{F}(\mathfrak{g})$ is $\mathfrak{h}_{\overline{0}}$-semisimple and thus admits a weight space decomposition

$$V = \bigoplus_{\lambda\in\mathfrak{h}_{\overline{0}}^*} V_\lambda.$$

We set $\mathcal{P}(V) = \{\lambda \in \mathfrak{h}_{\overline{0}}^* : V_\lambda \neq 0\}$. Using the triangular decomposition we have the following standard result:

**Theorem 2.3.8.** *Let $L$ be a simple $\mathfrak{g}$-module. Then there exists a unique weight $\lambda \in P(L)$, called the highest weight of $L$ (with respect to $\mathfrak{b}$), such that $\mu \leq \lambda$ for all $\mu \in P(L)$. Further, $L_\lambda$ is a simple $\mathfrak{h}$-module. If two simple $\mathfrak{g}$-modules have the same highest weight, then they are isomorphic up to parity shift.*

**Corollary 2.3.9.** *If $L, L'$ are irreducible $\mathfrak{g}$-modules, then either $L \cong L'$ or $L \cong \Pi L'$ if and only if $\mathcal{P}(L) = \mathcal{P}(L')$.*

**Corollary 2.3.10.** *If $\mathfrak{h} = \mathfrak{h}_{\overline{0}}$ and $L$ is a simple $\mathfrak{g}$-module of highest weight $\lambda$, there exists a unique (up to nonzero scalar) nonzero vector $v_\lambda \in L_\lambda$, which we call a highest weight vector of $L$.*

**Definition 2.3.11.** We say a weight $\lambda \in \mathfrak{h}_{\bar{0}}^*$ is dominant with respect to $\mathfrak{b}$ if there is a finite-dimensional irreducible representation $L$ of $\mathfrak{g}$ of highest weight $\lambda$ with respect to $\mathfrak{b}$. We write $\Lambda_{\mathfrak{b}}^+(\mathfrak{g}) \subseteq \mathfrak{h}_{\bar{0}}^*$ (or simply $\Lambda^+(\mathfrak{g})$) for the set of $\mathfrak{b}$-dominant weights. Observe that $\Lambda_{\mathfrak{b}}^+(\mathfrak{g}) \subseteq \Lambda_{\mathfrak{b}_{\bar{0}}}^+(\mathfrak{g}_{\bar{0}})$.

**Notation:**

- If $\mathfrak{h} = \mathfrak{h}_{\bar{0}}$ and $\lambda$ is dominant we define $L_{\mathfrak{b}}(\lambda)$ (or simply $L(\lambda)$ when no confusion arises) to be a simple module of highest weight $\lambda$ with respect to $\mathfrak{b}$ such that the highest weight vector is even.
- Because of subtleties surrounding parity, if $\mathfrak{h} \neq \mathfrak{h}_{\bar{0}}$ we write $L_{\mathfrak{b}}(\lambda)$ (or $L(\lambda)$) for a fixed choice of irreducible representation of highest weight $\lambda$. This case will only arise for us when $\mathfrak{g} = \mathfrak{q}(n)$, and this subtlety will be largely unimportant.
- In any case, we write $L_0(\lambda)$ for the even irreducible representation of $\mathfrak{g}_{\bar{0}}$ of highest weight $\lambda$.

### 2.3.2 Distinguished quasi-reductive Lie superalgebras

We now discuss the most important families of Lie superalgebras we consider in this thesis. We will refer to them as the distinguished Lie superalgebras for the remainder of this thesis.

Over time representation theorists have found these superalgebras have representation theory that exhibits especially interesting behavior and provides for a rich theory, while also avoiding some of the technical deficiencies of certain simple Lie superalgebras that appear in Kac's list in [26]. Further all the simple Lie superalgebras appearing in Kac's list which are quasi-reductive either appear in the following list (i.e. $\mathfrak{osp}(m|2n)$, $\mathfrak{d}(2|1;t)$, $\mathfrak{g}(1|2)$, or $\mathfrak{ab}(1|3)$) or can be gotten from one in the following list by taking a codimension one derived subalgebra and/or taking the quotient by a one-dimensional center.

Each distinguished Lie superalgebra is quasi-reductive. For each we give an explicit description of its root system. Some are also basic Lie superalgebras, meaning that they admit a non-degenerate invariant even supersymmetric form (these are the superalgebras $\mathfrak{gl}(m|n), \mathfrak{osp}(m|2n), \mathfrak{d}(2|1;t), \mathfrak{g}(1|2)$, and $\mathfrak{ab}(1|3)$). For these cases we also describe the induced form on the root system.

- $\mathfrak{gl}(m|n) = \mathrm{End}(k^{m|n})$: we have $\mathfrak{h} = \mathfrak{h}_{\bar{0}}$ and a natural basis $\epsilon_1, \dots, \epsilon_m, \delta_1, \dots, \delta_n$ of $\mathfrak{h}^*$ such that $\Delta_{\bar{0}} = \{\epsilon_i - \epsilon_j, \delta_i - \delta_j : i \neq j\}$ and $\Delta_{\bar{1}} = \{\pm(\epsilon_i - \delta_j)\}$. The inner product is given by $(\epsilon_i, \epsilon_j) = -(\delta_i, \delta_j) = \delta_{ij}$ and $(\epsilon_i, \delta_j) = 0$.
- $\mathfrak{osp}(2m+1|2n)$: we have $\mathfrak{h} = \mathfrak{h}_{\bar{0}}$ and a natural basis $\epsilon_1, \dots, \epsilon_m, \delta_1, \dots, \delta_n$ of $\mathfrak{h}_{\bar{0}}^*$ such that $\Delta_{\bar{0}} = \{\pm\epsilon_i \pm \epsilon_j, \pm\delta_i \pm \delta_j : i \neq j\} \sqcup \{\pm\epsilon_i, \pm 2\delta_i\}$ and $\Delta_{\bar{1}} = \{\pm\epsilon_i \pm \delta_j\} \sqcup \{\pm\delta_i\}$. The inner product is given by $(\epsilon_i, \epsilon_j) = -(\delta_i, \delta_j) = \delta_{ij}$ and $(\epsilon_i, \delta_j) = 0$.

- $\mathfrak{osp}(2m|2n)$: we have $\mathfrak{h} = \mathfrak{h}_{\overline{0}}$ and a natural basis $\epsilon_1, \ldots, \epsilon_m, \delta_1, \ldots, \delta_n$ of $\mathfrak{h}^*$ such that $\Delta_{\overline{0}} = \{\pm\epsilon_i \pm \epsilon_j, \pm\delta_i \pm \delta_j : i \neq j\} \sqcup \{\pm 2\delta_i\}$ and $\Delta_{\overline{1}} = \{\pm\epsilon_i \pm \delta_j\}$. The inner product is given by $(\epsilon_i, \epsilon_j) = -(\delta_i, \delta_j) = \delta_{ij}$ and $(\epsilon_i, \delta_j) = 0$.

- $\mathfrak{d}(2|1;t)$: we have $\mathfrak{h} = \mathfrak{h}_{\overline{0}}$ and a natural basis $\epsilon_1, \epsilon_2, \epsilon_3$ of $\mathfrak{h}^*$ such that $\Delta_{\overline{0}} = \{\pm 2\epsilon_i\}$ and $\Delta_{\overline{1}} = \{\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3\}$. The inner product is given by $(\epsilon_1, \epsilon_1) = \frac{1}{t}(\epsilon_2, \epsilon_2) = -\frac{1}{1+t}(\epsilon_3, \epsilon_3) = 1$.

- $\mathfrak{g}(1|2)$: we have $\mathfrak{h} = \mathfrak{h}_{\overline{0}}$ and a spanning set $\delta, \epsilon_1, \epsilon_2, \epsilon_3$ of $\mathfrak{h}_{\overline{0}}^*$ with the relation $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ such that $\Delta_{\overline{0}} = \{\pm\epsilon_i, \epsilon_i - \epsilon_j : i \neq j\} \sqcup \{\pm 2\delta\}$ and $\Delta_{\overline{1}} = \{\pm\delta\} \sqcup \{\pm\delta \pm \epsilon_j\}$. The inner product is given by $(\epsilon_i, \epsilon_i) = -2(\epsilon_i, \epsilon_j) = -(\delta, \delta) = 2$, where $i \neq j$.

- $\mathfrak{ab}(1|3)$: we have $\mathfrak{h} = \mathfrak{h}_{\overline{0}}$ and a natural basis $\epsilon_1, \epsilon_2, \epsilon_3, \delta$ of $\mathfrak{h}^*$ such that $\Delta_{\overline{0}} = \{\pm\epsilon_i \pm \epsilon_j : i \neq j\} \sqcup \{\pm\epsilon_i\} \sqcup \{\pm\delta\}$ and $\Delta_{\overline{1}} = \{\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \delta)\}$. The inner product is given by $(\epsilon_i, \epsilon_j) = \delta_{ij}$ and $(\delta, \delta) = -3$.

- $\mathfrak{p}(n)$: we have $\mathfrak{h} = \mathfrak{h}_{\overline{0}}$ and a natural basis $\epsilon_1, \ldots, \epsilon_n$ of $\mathfrak{h}^*$ such that $\Delta_{\overline{0}} = \{\epsilon_i - \epsilon_j : i \neq j\}$ and $\Delta_{\overline{1}} = \{\epsilon_i + \epsilon_j\} \sqcup \{-\epsilon_i - \epsilon_j : i \neq j\}$.

- $\mathfrak{q}(n)$: we have $\mathfrak{h} = \mathfrak{h}_{\overline{0}} \oplus k^{0|n}$ such that $\mathfrak{h}_{\overline{0}}$ has a basis $e_1, \ldots, e_n$ and $k^{0|n}$ has a basis $f_1, \ldots, f_n$ with $[f_i, f_j] = \delta_{ij} e_i$. Write $\epsilon_1, \ldots, \epsilon_n \in \mathfrak{h}_{\overline{0}}^*$ for the dual basis to $e_1, \ldots, e_n$. Then we have $\Delta = \Delta_{\overline{0}} = \{\epsilon_i - \epsilon_j : i \neq j\}$, and $\dim \mathfrak{g}_\alpha = (1|1)$ for all $\alpha \in \Delta_{\overline{0}}$.

Note that all distinguished Lie superalgebras are Cartan-even except for $\mathfrak{q}(n)$. We will refer to the Lie superalgebras $\mathfrak{d}(2|1;t)$, $\mathfrak{g}(1|2)$, and $\mathfrak{ab}(1|3)$ as the exceptional basic simple Lie superalgebras.

### 2.3.3 Odd reflections in basic Lie superalgebras

Let $\mathfrak{g}$ be one of the basic distinguished Lie superalgebras in our list above, that is one of $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(m|2n)$, $\mathfrak{g}(1|2)$, $\mathfrak{ab}(1|3)$, or $\mathfrak{d}(2|1;\alpha)$. As previously stated, basic superalgebras admit an even invariant non-degenerate supersymmetric form which induces a symmetric non-degenerate form on $\mathfrak{h}^*$ for a Cartan subalgebra $\mathfrak{h}$ and is described in section 2.3.2. Basic superalgebras have extra structure on their representation categories, and conjugacy classes of Borel subalgebras are well-understood. We state what we will need for this thesis.

Let $\mathfrak{g}$ be a basic distinguished Lie superalgebra and choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. The choices of positive systems, and thus of Borel subalgebras containing $\mathfrak{h}$, are in natural bijection with choices of simple roots in $\Delta$ just like in the case of semisimple Lie algebras. Recall that a subset $\Sigma = \{\alpha_1, \ldots, \alpha_n\} \subseteq \Delta$ is called a base for $\Delta$ if every root $\alpha \in \Delta$ can be uniquely represented as a linear combination

$$\alpha = \sum_i k_i \alpha_i$$

where either all $k_i \in \mathbb{Z}_{\geq 0}$ or all $k_i \in \mathbb{Z}_{\leq 0}$. In this case the elements of $\Sigma$ are called simple roots. Note that for all distinguished basic Lie superalgebras a base $\Sigma$ will be linearly independent.

If $\Sigma \subseteq \Delta$ is a set of simple roots, and $\alpha \in \Sigma$ is isotropic (i.e. $(\alpha, \alpha) = 0$), then we denote by $r_\alpha$ the odd reflection with respect to $\alpha$, which takes $\Sigma$ to a new simple root system $r_\alpha(\Sigma)$. See section 1.4 of [10] or the axiomatic approach in section 1 of [48] for more on how odd reflections change the simple root system.

The main result we will be using regarding simple reflections is the following: let $\mathfrak{b}$ be the Borel corresponding to $\Sigma$, and $r_\alpha(\mathfrak{b})$ the Borel corresponding to $r_\alpha(\Sigma)$. Further let $\lambda \in \mathfrak{h}^*$ be dominant with respect to $\mathfrak{b}$. Then we have:

$$L_{\mathfrak{b}}(\lambda) \cong L_{r_\alpha(\mathfrak{b})}(\lambda) \quad \text{if } (\lambda, \alpha) = 0, \quad \text{and} \quad L_{\mathfrak{b}}(\lambda) \cong \Pi L_{r_\alpha(\mathfrak{b})}(\lambda - \alpha) \quad \text{if } (\lambda, \alpha) \neq 0.$$

See [10], lemma 1.40 for a proof and sections 1.4 and 1.5 for more on odd reflections in highest weight theory.

### 2.3.4 Borel subalgebras for $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$

Here we describe the conjugacy classes of Borel subalgebras for $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$ and make a particular choice of Borel subalgebra for each conjugacy class. The notation developed will be heavily used in appendix A.

Let $\mathfrak{g}$ be either $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(2m|2n)$, or $\mathfrak{osp}(2m+1|2n)$ and let $\mathfrak{h}_{\bar{0}} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Write $(-,-)$ for the restriction of a fixed invariant form to $\mathfrak{h}_{\bar{0}}$. Then as described in section 2.3.2 there is basis of $\mathfrak{h}_{\bar{0}}^*$ given by $\epsilon_1, \dots, \epsilon_m, \delta_1, \cdots, \delta_n$, where

$$(\epsilon_i, \epsilon_j) = -(\delta_i, \delta_j) = \delta_{ij}, \quad (\epsilon_i, \delta_j) = 0.$$

Let $W$ denote the Weyl group of the even part of each root system above. Then the conjugacy classes of Borel subalgebras of $\mathfrak{g}$ are in bijection with a choice of simple roots up to the $W$-action. For each superalgebra, there is a well-known classification of such conjugacy classes in terms of $\epsilon\delta$-sequences (see for instance [10], section 1.3).

For example, for $\mathfrak{gl}(1|2)$ there are three conjugacy classes of Borels, corresponding to the sets of simple roots $\{\epsilon_1 - \delta_1, \delta_1 - \delta_2\}$, $\{\delta_1 - \epsilon_1, \epsilon_1 - \delta_2\}$, and $\{\delta_1 - \delta_2, \delta_2 - \epsilon_1\}$. To each of these we associate, in order, the $\epsilon\delta$-sequence $\epsilon\delta\delta$, $\delta\epsilon\delta$, and $\delta\delta\epsilon$. We occasionally write, for example, $\delta^2\epsilon$ for the sequence $\delta\delta\epsilon$.

Now for each Lie superalgebra $\mathfrak{g}$ that we consider, we would like to define a map:

$$\{\epsilon\delta\text{-sequences}\} \to \{\text{Borels } \mathfrak{b} \subseteq \mathfrak{g}\}, \quad \sigma \mapsto \mathfrak{b}^\sigma$$

such that the conjugacy class of $\mathfrak{b}^\sigma$ corresponds to the positive system defined by $\sigma$.

Since we have already chosen a Cartan subalgebra, and an $\epsilon\delta$-sequence specifies a conjugacy class of Borel, to define $\mathfrak{b}^\sigma$ it suffices to rigidify the $W$-symmetry by making a choice of even Borel subalgebra, or equivalently a choice of simple roots for the even root system. We now do this for each Lie superalgebra $\mathfrak{g}$.

For $\mathfrak{gl}(m|n)$, we choose the even simple roots

$$\{\epsilon_i - \epsilon_{i+1}\}_{1\le i\le m-1} \cup \{\delta_j - \delta_{j+1}\}_{1\le j\le n-1}.$$

For $\mathfrak{osp}(2m|2n)$, we choose the even simple roots

$$\{\epsilon_i - \epsilon_{i+1}, \epsilon_{m-1} + \epsilon_m\}_{1\le i\le m-1} \cup \{\delta_j - \delta_{j+1}, 2\delta_n\}_{1\le j\le n-1}.$$

Finally for $\mathfrak{osp}(2m+1|2n)$, we choose the even simple roots

$$\{\epsilon_i - \epsilon_{i+1}, \epsilon_m\}_{1\le i\le m-1} \cup \{\delta_j - \delta_{j+1}, 2\delta_n\}_{1\le j\le n-1}.$$

For example, if $\mathfrak{g} = \mathfrak{gl}(m|n)$, then $\mathfrak{b}^{\epsilon\cdots\epsilon\delta\cdots\delta} = \mathfrak{b}^{\epsilon^m\delta^n}$ has simple root system

$$\{\epsilon_1 - \epsilon_2, \dots, \epsilon_m - \delta_1, \delta_1 - \delta_2, \cdots \delta_{n-1} - \delta_n\}.$$

Then if $\mathfrak{h}$ is the Cartan subalgebra of diagonal matrices, $\mathfrak{b}^{\epsilon^m\delta^n}$ will be the subalgebra of upper-triangular matrices in $\mathfrak{gl}(m|n)$.

Or, if $\mathfrak{g} = \mathfrak{osp}(4|4)$, then $\mathfrak{b}^{\epsilon(-\epsilon)\delta\delta} = \mathfrak{b}^{\epsilon(-\epsilon)\delta^2}$ has simple root system

$$\{\epsilon_1 + \epsilon_2, -\epsilon_2 - \delta_1, \delta_1 - \delta_2, 2\delta_2\}.$$

### 2.3.5 Hyperborel subalgebras

We introduce another natural generalization of Borel subalgebras for quasi-reductive Lie superalgebras, which we call hyperborel subalgebras. We then give a brief discussion of the definition.

**Definition 2.3.12.** A hyperborel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is a subalgebra that is maximal amongst subalgebras with the following two properties:

- $\mathfrak{b}_{\overline{0}}$ is a Borel of $\mathfrak{g}_{\overline{0}}$ in the usual sense; and
- $[\mathfrak{b}_{\overline{1}}, \mathfrak{b}_{\overline{1}}] \subseteq [\mathfrak{b}_{\overline{0}}, \mathfrak{b}_{\overline{0}}]$.

*Remark* 2.3.13.

- Given a hyperborel subalgebra $\mathfrak{b}$ and a choice of Cartan subalgebra $\mathfrak{h}_{\overline{0}} \subseteq \mathfrak{b}_{\overline{0}}$, we have $\mathfrak{b} = \mathfrak{h}_{\overline{0}} \ltimes \mathfrak{n}$ where $\mathfrak{n}$ is a nilpotent ideal. We call $\mathfrak{n}$ the unipotent radical of $\mathfrak{b}$.
- We may always conjugate a hyperborel $\mathfrak{b}$ by an inner automorphism of $\mathfrak{g}$ so that $\mathfrak{b}_{\overline{0}}$ is a chosen Borel subalgebra of $\mathfrak{g}_{\overline{0}}$.
- If $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{g}$ is Cartan-even, then $[\mathfrak{b}_{\overline{1}}, \mathfrak{b}_{\overline{1}}] \subseteq [\mathfrak{b}_{\overline{0}}, \mathfrak{b}_{\overline{0}}]$, and so $\mathfrak{b}$ is contained in a hyperborel subalgebra of $\mathfrak{g}$.

- If $\mathfrak{l}\subseteq\mathfrak{g}_{\overline{1}}$ is an odd abelian ideal of $\mathfrak{g}$, then $\mathfrak{l}$ is contained in every hyperborel subalgebra of $\mathfrak{g}$.

*Example* 2.3.14. Unlike for Borel subalgebras, in general there may be infinitely many non-conjugate hyperborel subalgebras. For example, one can take a central extension of an odd abelian Lie superalgebra to get an odd Heisenberg superalgebra $\mathfrak{g}$ where $\mathfrak{g}_{\overline{0}}=k\langle c\rangle$ is central, $\mathfrak{g}_{\overline{1}}=V$, and $[-,-]:S^2V\to k\langle c\rangle$ is a non-degenerate symmetric bilinear form. Then hyperborel subalgebras for $\mathfrak{g}$ are in bijection with maximal isotropic subspaces of $V$, and none are conjugate to another.

To obtain an example which is Cartan-even, if we assume $V=V_+\oplus V_-$ where $\dim V_{\pm}=n$ then one may take our Heisenberg superalgebra above and add a grading element $d$ which has that $[d,v]=\pm v$ if $v\in V_{\pm}$. Here the inner automorphism group is a one-dimensional torus, so if $n$ is large enough there will be infinitely many hyperborel subalgebras.

However for both of the examples above, there are still only finitely many hyperborel subalgebras up to automorphism. The author does not know if this is always the case.

*Remark* 2.3.15. By definition, hyperborel subalgebras are solvable and their irreducible representations are all one-dimensional (see lemma 1.37 of [10]). This property is the primary way in which hyperborel subalgebras generalize Borel subalgebras for reductive Lie algebras.

With this in mind, another natural candidate definition for hyperborel subalgebra is a subalgebra $\mathfrak{b}$ which is solvable, has $[\mathfrak{b}_{\overline{1}},\mathfrak{b}_{\overline{1}}]\subseteq[\mathfrak{b}_{\overline{0}},\mathfrak{b}_{\overline{0}}]$, and is maximal with these properties. A hyperborel subalgebra as we have defined it satisfies this definition. However the notions are not equivalent.

A counterexample is given by the Lie superalgebra $\mathfrak{g}$ with $\mathfrak{g}_{\overline{0}}=k\langle s,t_1,t_2\rangle$ abelian and $\mathfrak{g}_{\overline{1}}=k\langle v_1,v_2,w_1,w_2\rangle$. Here $s$ is central and

$$[t_i,v_j]=\delta_{ij}v_j,\quad [t_i,w_j]=-\delta_{ij}w_j,\quad [v_i,w_j]=\delta_{ij}s,\quad [v_i,v_j]=[w_i,w_j]=0.$$

Then $\mathfrak{g}$ is quasi-reductive (and Cartan-even), and if we let $\mathfrak{b}=\langle s,t_1+t_2,v_1+v_2,w_1-w_2\rangle$ then $\mathfrak{b}$ is solvable, and since

$$[v_1+v_2,w_1-w_2]=0,$$

we have $[\mathfrak{b}_{\overline{1}},\mathfrak{b}_{\overline{1}}]\subseteq[\mathfrak{b}_{\overline{0}},\mathfrak{b}_{\overline{0}}]$. However if $t_1$ is added to form a larger subalgebra $\mathfrak{b}'$ then $v_1$ and $w_1$ would also need to be added to $\mathfrak{b}'$ and thus $s\in[\mathfrak{b}'_{\overline{1}},\mathfrak{b}'_{\overline{1}}]$. So $\mathfrak{b}$ is maximal with the properties that it is solvable and $[\mathfrak{b}_{\overline{1}},\mathfrak{b}_{\overline{1}}]\subseteq[\mathfrak{b}_{\overline{0}},\mathfrak{b}_{\overline{0}}]$ even though $\mathfrak{b}_{\overline{0}}$ is not a Borel subalgebra of $\mathfrak{g}_{\overline{0}}$.

We prefer our definition of hyperborel subalgebra due to the importance of having its even part be a Borel subalgebra.

The notion of hyperborel is most natural for quasi-reductive superalgebras which are Cartan-even. If $\mathfrak{g}$ is Cartan-even then the notion of hyperborel agrees with the definition of Borel subalgebra given in chapter 3 of [39]. Further, the notion of hyperborel subalgebra and Borel subalgebra coincide if $\mathfrak{g}$ is $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$ for $m\neq n$ and $(m,n)\neq(1,1)$, $\mathfrak{psl}(n|n)$ or $\mathfrak{sl}(n|n)$ for $n\geq 3$, $\mathfrak{p}(n)$, $\mathfrak{osp}(m|2n)$, or is one of the exceptional basic simple Lie superalgebras. This is proven in Proposition 4.6.1 of [39]. The case of $\mathfrak{p}(n)$ is not considered there, but one can show the notions agree for this superalgebra as well (although they do not agree for the derived subalgebra of $\mathfrak{p}(n)$).

However if $\mathfrak{g}$ is not Cartan-even, for instance $\mathfrak{g}$ is the queer Lie superalgebra $\mathfrak{q}(n)$, then hyperborels greatly differ from Borels as they may not contain a Cartan subalgebra.

*Remark* 2.3.16. If $\mathfrak{g}$ is quasi-reductive and $\mathfrak{b}$ is a hyperborel subalgebra of $\mathfrak{g}$, then for a finite dimensional irreducible representation $V$ of $\mathfrak{g}$, $\dim V^{(\mathfrak{b})} \geq 1$ by remark 2.3.15. However, it is possible that $\dim V^{(\mathfrak{b})} > 1$, and thus we no longer have a bijective correspondence between certain characters of the Borel and finite dimensional irreducible representations of $\mathfrak{g}$.

Indeed even when $\mathfrak{g}$ is Cartan-even this phenomenon can occur; in [52], a nontrivial central extension of the derived subalgebra of $\mathfrak{p}(4)$ is considered, along with an irreducible representation $V_t$ deforming the standard representation of $\mathfrak{p}(4)$. If $t \neq 0$ it is shown that $\Lambda^2 V_t$ is irreducible. However there is a hyperborel subalgebra given by (in the notation of the paper) $\mathfrak{b} = \mathfrak{g}_{-2} \oplus \mathfrak{b}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{b}_0$ is a Borel subalgebra of $\mathfrak{g}_0$. One can check that $\Lambda^2 V_t^{(\mathfrak{b})}$ is two-dimensional for any $t$.

Nevertheless, if a hyperborel subalgebra $\mathfrak{b}$ contains a Borel subalgebra then $\dim V^{(\mathfrak{b})} = 1$ for an irreducible representation $V$ of $\mathfrak{g}$, by highest weight theory.

## 2.4 Schur-Weyl duality and the parity shift functor

### 2.4.1 Schur-Weyl duality

Let $V$ be a finite-dimension super vector space. Then it has a natural action of the Lie superalgebra $\mathfrak{gl}(V)$, inducing a natural action of $\mathfrak{gl}(V)$ on $V^{\otimes \ell}$. This action commutes with the action of $S_\ell$ on $V^{\otimes \ell}$. Just as in the classical case, these actions are double-centralizing; see theorem 3.10 in [10] for a proof of the following result.

**Proposition 2.4.1.** *Let $\underline{End}_{S_\ell}(V^{\otimes \ell})$ denote the super vector space endomorphisms of $V^{\otimes \ell}$ which commute with $S_\ell$, and $\underline{End}_{\mathcal{U}\mathfrak{gl}(V)}(V^{\otimes \ell})$ the super vector space endomorphisms which commute with $\mathfrak{gl}(V)$. Then we have the double centralizer property for these two actions: the natural maps*

$$k[S_\ell] \to \underline{End}_{\mathcal{U}\mathfrak{gl}(V)}(V^{\otimes \ell}), \qquad \mathcal{U}\mathfrak{gl}(V) \to \underline{End}_{S_\ell}(V^{\otimes \ell})$$

*are surjective.*

Recall from section 2.1.2 that for a partition $\lambda$ of $\ell$ we write $S_\lambda$ for the corresponding irreducible representation of $S_\ell$ and $S^\lambda(V)$ for the Schur functor, which now gives a $\mathfrak{gl}(V)$-representation. Let $\dim^s(V) = (m|n)$, and write $\mathcal{H}(m,n)$ for the set of $(m,n)$ hook partitions $\lambda$, i.e. partitions that satisfy $\lambda_{m+1} \leq n$. Theorem 3.11 of [10] proves the following:

**Corollary 2.4.2.** *For $\lambda \in \mathcal{H}(m,n)$, $S_\lambda(V)$ is an irreducible $\mathfrak{gl}(V)$ representation. Further we have a natural isomorphism of $\mathfrak{gl}(V) \times S_\ell$ modules*

$$V^{\otimes \ell} \cong \bigoplus_{\lambda \vdash \ell} S^\lambda(V) \otimes S_\lambda$$

### 2.4.2 Parity shift functor

We now give a brief discussion of the parity shift functor on Svect and its relationship to the $\mathfrak{gl}(V)$-representations $S^\lambda(V)$.

**Definition 2.4.3.** For a super vector space $V$, recall that we define its parity shift $\Pi V$ to be the super vector space $k^{0|1} \otimes V$. We also define $\Pi^{-1}V := (k^{0|1})^* \otimes V$. For $n \in \mathbb{Z}$, write $\Pi^n V := (k^{0|1})^{\otimes n} \otimes V$, where for $n < 0$ we set $(k^{0|1})^{\otimes n} := ((k^{0|1})^*)^{\otimes -n}$.

*Remark* 2.4.4. We have that $\Pi^n$ is an endofunctor of SVect and $\Pi \circ \Pi^{-1} \cong \Pi^{-1} \circ \Pi \cong \mathrm{id}_{\mathsf{SVect}}$ via the coevaluation morphism. From this we obtain a natural isomorphism of functors $\Pi^n \circ \Pi^m \cong \Pi^{m+n}$ for all $m, n \in \mathbb{Z}$.

We will, throughout, write $\varepsilon$ for a chosen basis element of $k^{0|1}$, and $\varepsilon^{-1}$ for a dual basis element of $(k^{0|1})^*$. In this way we write elements of $\Pi^n V$ as $\varepsilon^n V$, where $v \in V$.

*Remark* 2.4.5. The functor $\Pi^n$ admits, via the braiding, an $S_{|n|}$-action by natural automorphisms. For a super vector space $V$, $\sigma \in S_{|n|}$, $v \in \Pi^n V$, this natural automorphism is given by

$$\sigma \cdot v = \mathrm{sgn}(\sigma) v.$$

This natural action prevents the existence of a natural isomorphism $\mathrm{id}_{\mathsf{SVect}} \to \Pi^2$ which respects the symmetric structure on the category.

For $n \in \mathbb{Z}$ and super vector spaces $V_1, V_2$, we have natural isomorphisms

$$(\Pi^n V_1) \otimes V_2 \cong \Pi^n(V_1 \otimes V_2) \cong V_1 \otimes (\Pi^n V_2)$$

via the braiding isomorphism. Explicitly,

$$(\varepsilon^n v_1) \otimes v_2 \mapsto \varepsilon^n(v_1 \otimes v_2) \mapsto (-1)^{\overline{v}n} v_1 \otimes (\varepsilon^n v_2)$$

We have a natural action of $\mathfrak{gl}(V)$ on $\Pi V$ given explicitly by

$$u(\varepsilon v) = (-1)^{\overline{u}} \varepsilon(uv)$$

Consider $(\Pi V)^{\otimes n}$. The braiding gives a natural isomorphism

$$\Phi : (\Pi V)^{\otimes n} \to \Pi^n V^{\otimes n}$$

which explicitly sends

$$(\varepsilon v_1) \otimes (\varepsilon v_2) \otimes \cdots \otimes (\varepsilon v_n) \mapsto (-1)^{\overline{v_{n-1}} + \overline{v_{n-3}} + \cdots} \varepsilon^n v_1 \otimes v_2 \otimes \cdots \otimes v_n$$

Let $u \in \mathfrak{gl}(V)$. Then,

$$\begin{aligned} u((\varepsilon v_1) \otimes \cdots \otimes (\varepsilon v_n)) &= \sum_{i=1}^n (-1)^{\overline{u}(i + \overline{v_1} + \cdots + \overline{v_{i-1}})} (\varepsilon v_1) \otimes \cdots \otimes \varepsilon u v_i \otimes \cdots \otimes (\varepsilon v_n) \\ &\mapsto (-1)^{\overline{v_{n-1}} + \overline{v_{n-3}} + \cdots} \sum_{i=1}^n (-1)^{\overline{u}(n + \overline{v_1} + \cdots + \overline{v_{i-1}})} \varepsilon^n v_1 \otimes \cdots \otimes u v_i \otimes \cdots \otimes v_n \\ &= u((-1)^{\overline{v_{n-1}} + \overline{v_{n-3}} + \cdots} \varepsilon^n v_1 \otimes \cdots \otimes v_n) \end{aligned}$$

so $\Phi$ is $\mathfrak{gl}(V)$-equivariant. We have a natural $S_n$ action on $(\Pi V)^{\otimes n}$; the transposition $\sigma_i = (i, i+1)$ acts as

$$\sigma_i \cdots \otimes (\varepsilon v_i) \otimes (\varepsilon v_{i+1}) \otimes \cdots = (-1)^{(\overline{v_i}+1)(\overline{v_{i+1}}+1)} \cdots \otimes (\varepsilon v_{i+1}) \otimes (\varepsilon v_i) \otimes \cdots$$

However on $\Pi^n V^{\otimes n}$, we have a natural $S_n \times S_n$ action; the map $\Phi$ is $S_n$-equivariant with respect to the diagonal $S_n$ action on $\Pi^n V^{\otimes n}$. The diagonal action is isomorphic to the action of $S_n$ on $V^{\otimes n}$ tensor the sign representation. Therefore, if we write $p_\lambda$ for the projector onto the isotypic component of $S_\lambda$, we find that

$$\Phi(p_\lambda (\Pi V)^{\otimes n}) = p_\lambda(\Pi^n V^{\otimes n}) = \Pi^n p_{\lambda'} V^{\otimes n}$$

where $\lambda'$ denotes the transposed partition. Since this is an isomorphism of $\mathfrak{gl}(V)$-modules, we obtain the following:

**Proposition 2.4.6.** *We have a natural isomorphism of $\mathfrak{gl}(V)$-modules*

$$S_\lambda(\Pi V) \cong \Pi^{|\lambda|} S_{\lambda'}(V).$$

# Chapter 3

# Supergeometry

Here we develop the algebraic supergeometry that we will need. The treatment almost entirely parallels the development of algebraic geometry via schemes as in [21], where we replace commutative rings with supercommutative rings. However the necessary introduction of nilpotents, even for smooth spaces, leads to differences.

## 3.1 Supercommutative algebra

**Definition 3.1.1.** A superalgebra $A$ over $k$ is supercommutative if $A$ is unital and $ab = (-1)^{\bar{a}\bar{b}}ba$ for all $a, b \in A$. In this case we call $A$ a supercommutative algebra.

Unless otherwise stated, all superalgebras in the rest of this section are assumed to be supercommutative.

*Example* 3.1.2. Given a super vector space $V$, we have defined $S^\bullet V$ to be the polynomial superalgebra on $V$. It is a supercommutative algebra explicitly given by

$$S^\bullet V = S^\bullet V_{\bar{0}} \otimes_k \Lambda^\bullet V_{\bar{1}}.$$

Given a homogeneous basis $x_1, \ldots, x_m, \xi_1, \ldots, \xi_n$ of $V$, where the $x_i$ are even, $\xi_i$ odd, we have $S^\bullet V = k[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n]$, the superalgebra of polynomials in these elements.

### 3.1.1 Ideals

Given a supercommutative algebra $A$ the notions of left, right, and two-sided ideals all coincide. We always assume the ideals are $\mathbb{Z}_2$-graded. A prime ideal $\mathfrak{p} \subseteq A$ is one such that if $ab \in \mathfrak{p}$ then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$; equivalently $A/\mathfrak{p}$ is an integral domain. (In particular it is a commutative ring.) A maximal ideal is an ideal that is maximal amongst all proper ideals, or equivalently has the property that $A/\mathfrak{m}$ is a field.

Given a homogeneous subset $I \subseteq A$, we define $(I) \subseteq A$ to be the ideal generated by $I$, that is the smallest ideal of $A$ containing $I$. The ideal $(A_{\bar{1}})$ is particularly distinguished in $A$.

We observe that every element of $A_{\overline{1}}$ squares to 0, so that every element of $(A_{\overline{1}})$ is nilpotent, and thus every prime ideal contains $(A_{\overline{1}})$.

Define $\overline{A} := A/(A_{\overline{1}})$, a commutative algebra. Then by the lattice isomorphism theorem there is an inclusion-preserving bijection between the prime ideals of $A$ and the prime ideals of $\overline{A}$. On the other hand, if $\mathfrak{p}$ is a prime ideal of $A$, then $\mathfrak{p}_{\overline{0}}$ is a prime ideal of $A_{\overline{0}}$, and conversely if $\mathfrak{q}$ is a prime ideal of $A_{\overline{0}}$ then $\mathfrak{q} + A_{\overline{1}}$ is a prime ideal of $A$. Thus we obtain:

**Lemma 3.1.3.** *There are natural inclusion-preserving bijections between the prime ideals of $A, A_{\overline{0}}$, and $\overline{A}$.*

### 3.1.2 Modules

A left (resp. right) $A$-module is a super vector space $M$ with a morphism of super vector spaces $A \otimes_k M \to M$ (resp. $M \otimes_k A \to M$) satisfying the usual axioms. Note that a left $A$-module $M$ has a natural right $A$-module structure given by

$$m \cdot a = (-1)^{\overline{a}\overline{m}} am.$$

In this way all $A$-modules are naturally $A$-bimodules. It follows in particular that the tensor product of $A$-modules over $A$ is once again an $A$-module.

Given an $A$-module $M$, we may define an $A$-module structure on its parity shift $\Pi M$ by

$$a(\varepsilon m) = (-1)^{\overline{a}} \varepsilon am.$$

In this way, the map $M \to \Pi^2 M$, $m \mapsto \varepsilon^2 m$ becomes an isomorphism of $A$-modules.

Given $A$-modules $M, N$, a morphism of $A$-modules $f : M \to N$ is a map of super vector spaces such that $f(am) = af(m)$ for $a \in A$, $m \in M$. We denote by $\operatorname{Hom}_A(M, N)$ the vector space of $A$-module morphisms. We have a natural map $A_{\overline{0}} \to \operatorname{Hom}_A(M, M)$ given by left multiplication. On the other hand we have a natural map $A_{\overline{1}} \to \operatorname{Hom}_A(M, \Pi M)$ given by

$$a \mapsto (m \mapsto a\varepsilon m).$$

(See section 2.4.2 for an explanation of the notation.) Therefore, we define the $A$-module $\underline{\operatorname{Hom}}_A(M, N)$ by $\underline{\operatorname{Hom}}_A(M, N)_{\overline{0}} = \operatorname{Hom}_A(M, N)$ and $\underline{\operatorname{Hom}}_A(M, N)_{\overline{1}} = \operatorname{Hom}_A(M, \Pi N)$.

Given an $A$-module homomorphism $f : M \to \Pi N$ we may define a map $f' : M \to N$ given by $f'(m) = \varepsilon^{-1} f(m)$. Then $f'(M_i) \subseteq N_{i+\overline{1}}$ for $i \in \mathbb{Z}_2$, and $f'(am) = (-1)^{\overline{a}} af'(m)$. This correspondence defines a natural isomorphism of vector spaces

$$\operatorname{Hom}_A(M, \Pi N) \cong \{f : M \to N : f(M_i) \subseteq N_{i+\overline{1}},\ f(am) = (-1)^{\overline{a}} f(m)\}.$$

Thus we may also define the homogeneous components of $\underline{\operatorname{Hom}}_A(M, N)$ by

$$\underline{\operatorname{Hom}}_A(M, N)_j = \{f : M \to N : f(M_i) \subseteq N_{i+j},\ f(am) = (-1)^{\overline{a}j} f(m)\}.$$

**Definition 3.1.4.** We say that an $A$-module is free if it is isomorphic to $A \otimes_k V$ for a super vector space $V$. We write $A^{m|n}$ for $A \otimes_k k^{m|n}$.

**Definition 3.1.5.** Given an $A$-module $M$, we say it is finitely generated if there exists a surjective map of $A$-modules $A^{m|n} \to M$ for some $m, n \in \mathbb{N}$.

### 3.1.3 Localization and local superalgebras

Given a multiplicatively closed subset $S \subseteq A_{\bar{0}}$ containing 1, we define $S^{-1}A$ as the usual localization of $A$ at the set $S$ with the natural $\mathbb{Z}_2$-grading. If $M$ is an $A$-module then we may similarly localize $M$ to obtain $S^{-1}M$, an $S^{-1}A$-module. We will employ the following standard shorthands: given $f \in A_{\bar{0}}$, we define $A_f = S^{-1}A$, where $S = \{1, f, f^2, \dots\}$. Given a prime ideal $\mathfrak{p}$ of $A$, we write $A_{\mathfrak{p}} = S^{-1}A$, where $S = A_{\bar{0}} \setminus \mathfrak{p}_{\bar{0}}$.

Just as with commutative rings there is a natural inclusion-preserving bijection between the prime ideals of $S^{-1}A$ and the ideals of $A$ not intersecting $S$. Further, if $A$ is Noetherian then any localization of it is Noetherian as well.

If $\mathfrak{p}$ is a prime ideal then $A_{\mathfrak{p}}$ will have a unique maximal ideal given by the localization of $\mathfrak{p}$, and thus every element of $A_{\mathfrak{p}}$ not in the unique maximal ideal will be a unit. We call a supercommutative algebra with a unique maximal ideal a local superalgebra.

*Remark* 3.1.6. Nakayama's lemma and the Krull intersection theorem both hold for Noetherian local superalgebras using the same proofs as in ordinary commutative algebra. We will use both of these facts throughout the thesis.

### 3.1.4 Integral superdomains

We make a definition of the notion of an integral domain in the super setting. Clearly it is not reasonable to demand there be no zero divisors as then the superalgebra would be purely even.

**Definition 3.1.7.** We say a superalgebra $A$ is an integral superdomain if $(A_{\bar{1}})$ is a prime ideal and the localization map $A \to A_{(A_{\bar{1}})}$ is injective. Equivalently $A \setminus (A_{\bar{1}})$ contains no zero divisors.

Observe that if $A$ is an intergral superdomain then $\overline{A}$ is an integral domain.

### 3.1.5 Noetherian and finitely generated superalgebras

A Noetherian superalgebra is one satisfying the usual ascending chain condition on ideals. It is a standard exercise to show that the Noetherian property is preserved under arbitrary localization and quotients.

**Definition 3.1.8.** We say a superalgebra $A$ is finitely-generated if there exists a finite dimensional super vector subspace $V \subseteq A$ such that the induced map $S^{\bullet}V \to A$ is surjective. In this case $A$ is a Noetherian algebra, i.e. all of its ideals are finitely-generated.

### 3.1.6 Graded superalgebras

Let $R$ be a commutative $k$-algebra and $M$ an $R$-module. Then the $R$-linear exterior algebra $A = \Lambda^{\bullet}M$ has the natural structure of a supercommutative algebra such that $\overline{A} = R$.

**Definition 3.1.9.** A graded supercommutative algebra is one of the form $\Lambda^\bullet M$ for some commutative ring $R$ and $R$-module $M$.

If $A$ is a graded supercommutative algebra, we observe that $A \cong \Lambda^\bullet(A_{\bar{1}})/(A_{\bar{1}})^2$, where the exterior algebra is $\overline{A}$-linear. We say a superalgebra $A$ is locally graded if there exist $f_1, \ldots, f_n \in A_{\bar{0}}$ with $(f_1, \ldots, f_n) = A$ such that $A_{f_i}$ is graded for all $i$. Using the ideas in [63] we have the following:

**Proposition 3.1.10.** *A superalgebra is graded if and only if it is locally graded.*

*Remark* 3.1.11 (Caution)*.* The term graded is sometimes used instead of super, meaning that the terms graded manifold or graded group might be used instead of supermanifold and supergorup (see for instance [31]). Another term that has been used to mean graded is split, as in [62]. However others (e.g. [63]) have used split to mean there is an algebra splitting of the surjective morphism $A \to \overline{A}$. We will use the term graded in this thesis and hope no confusion will arise.

For finitely-generated integral superdomains the property of being graded is strong, as we now see.

**Proposition 3.1.12.** *If $A$ is a finitely-generated graded integral superdomain, then $A \cong \Lambda^\bullet M$ where $M$ is a projective $\overline{A}$-module.*

First we prove a lemma.

**Lemma 3.1.13.** *Let $R$ be a finitely generated integral domain and $M$ a finitely generated $R$-module such that $\Lambda^i M$ is a torsion-free $R$-module for all $i$. Then $M$ is projective.*

*Proof.* If $\mathfrak{p}$ is a prime ideal of $R$ it suffices to show that $M_\mathfrak{p}$ is a projective, equivalently free, $R_\mathfrak{p}$-module. Let $m_1, \ldots, m_n$ be a minimal generating set for $M_\mathfrak{p}$ so that in the quotient $M_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} M_\mathfrak{p}$ these elements map to a basis. Then since base change to the residue field commutes with taking exterior powers, we must have that $m_1 \wedge \cdots \wedge m_n$ maps to a nonzero element of $\Lambda^n M_\mathfrak{p}$. Suppose we have a relation $a_1 m_1 + \cdots + a_n m_n = 0$, where necessarily $a_1, \ldots, a_n \in \mathfrak{m}_\mathfrak{p}$, and $\mathfrak{m}_\mathfrak{p}$ is the maximal ideal of $R_\mathfrak{p}$. Then in $\Lambda^n M$ we have the relation $a_1(m_1 \wedge \cdots \wedge m_n) = 0$, so by our torsion-free assumption we must have $m_1 \wedge \cdots \wedge m_n = 0$, a contradiction. It follows that $M_\mathfrak{p}$ must be free on $m_1, \ldots, m_n$ and we are done. □

*Proof of proposition 3.1.12.* Since $A$ is finitely generated and graded, there exists an $\overline{A}$-module $M$ such that $A \cong \Lambda^\bullet M$. Since $A$ is an integral superdomain, for $f \in A_{\bar{0}} \setminus (A_{\bar{1}})_{\bar{0}}$ the localization map $A \to A_f$ is injective and thus the localization maps of $\overline{A}$-modules $\Lambda^i M \to \Lambda^i M_f$ for nonzero $f \in \overline{A}$ are injective for all $i$. This is equivalent to $\Lambda^i M$ being a torsion-free $\overline{A}$-module for all $i$. Now we apply lemma 3.1.13. □

This proposition says that the spectrum of a graded integral superdomain is smooth in the odd directions. In particular, its spectrum will be smooth if and only if the spectrum of $\overline{A}$ is smooth. See section 3.7 for more on smoothness.

### 3.1.7 Module of relative differentials

**Definition 3.1.14.** Given a morphism of superalgebras $B \to A$, the module of relative differentials is the $A$-module $\Omega^1_{A/B}$ defined as follows. It has generators $da$ of parity $\overline{a}$ for all homogeneous $a \in A$, with relations $d(ba) = bda$ for $b \in B$, $d(a_1 + a_2) = da_1 + da_2$, and

$$d(a_1a_2) = a_1da_2 + (-1)^{\overline{a_1a_2}}a_2da_1$$

for $a_1, a_2 \in A$. In particular, $db = 0$ for all $b \in B$.

With the above definition, we obtain that for any $A$-module $M$, $\underline{\mathrm{Hom}}_A(\Omega^1_{A/B}, M) = \mathrm{Der}_B(A, M)$, where

$$\begin{aligned} \mathrm{Der}_B(A, M) \;=\; & \{D : A \to M | D \text{ is } B\text{-linear and} \\ & D(a_1a_2) = (Da_1)a_2 + (-1)^{\overline{a_1}\overline{D}}a_1(Da_2)\} \end{aligned}$$

We call the elements of $\mathrm{Der}_B(A, M)$ the $B$-derivations from $A$ to $M$. Note that they need not preserve parity of elements.

## 3.2 Superschemes

For a locally ringed space $X$, we write $|X|$ for its underlying topological space and $\mathcal{O}_X$ for its structure sheaf. For an open subset $|U| \subseteq |X|$ we write $U \subseteq X$ for the locally ringed space gotten by restricting $\mathcal{O}_X$ to $|U|$. For a sheaf $\mathcal{F}$ on $|X|$ we write either $\Gamma(U, \mathcal{F})$ or $\Gamma(|U|, \mathcal{F})$ for its sections over an open subset $|U| \subseteq |X|$. For a point $x \in |X|$ we write $\mathcal{F}_x$ for the stalk of the sheaf at $x$.

### 3.2.1 Affine superschemes

Let $A$ be a supercommutative algebra. Let $\mathrm{Spec}\, A$ denote the topological space of all prime ideals of $A$, equipped with the usual Zariski topology. By lemma 3.1.3 we may also naturally identify this topological space with $\mathrm{Spec}\, \overline{A}$ and $\mathrm{Spec}\, A_{\overline{0}}$. Further $\mathrm{Spec}\, A$ has a basis of open sets given by $D(f) = \{\mathfrak{p} \in \mathrm{Spec}\, A : f \notin \mathfrak{p}\}$, where $f \in A_{\overline{0}}$.

We define a locally ringed space $(\mathrm{Spec}\, A, \mathcal{O}_A)$ as having underlying topological space $\mathrm{Spec}\, A$, and structure sheaf $\mathcal{O}_A$ defined by $\Gamma(D(f), \mathcal{O}_A) = A_f$. The restriction map

$$\Gamma((D(f), \mathcal{O}_A) \to \Gamma(D(g), \mathcal{O}_X)$$

for $D(g) \subseteq D(f)$ is the natural one coming from localization. Since the open sets $D(f)$ form a base of the topology, this defines a sheaf on $\mathrm{Spec}\, A$. Note that given $\mathfrak{p} \in \mathrm{Spec}\, A$ we have $\mathcal{O}_{A,\mathfrak{p}} = A_{\mathfrak{p}}$.

By abuse of language we will refer to this locally ringed space as $\mathrm{Spec}\, A$, and call it the spectrum of $A$. We say a locally ringed space is an affine superscheme if it is isomorphic to $\mathrm{Spec}\, A$ for some supercommutative algebra $A$.

### 3.2.2 Superschemes

We define a superscheme to be a locally ringed space $X = (|X|, \mathcal{O}_X)$ which is locally isomorphic to an affine superscheme. Write $\mathcal{O}_X = (\mathcal{O}_X)_{\overline{0}} \oplus (\mathcal{O}_X)_{\overline{1}}$ for the parity decomposition of this sheaf. For a point $x \in |X|$, we write $\mathcal{O}_{X,x}$ for the stalk of the sheaf $\mathcal{O}_X$ at $x$, which will be a local superalgebra, and $\mathfrak{m}_x$ for its unique maximal ideal. A morphism of superschemes is a map of locally ringed spaces such that the pullback morphism defines maps of superalgebras on each open set. For an open subset $|U| \subseteq |X|$, we write $U$ for the superscheme obtained by restriction of $X = (|X|, \mathcal{O}_X)$ to $|U|$, and we call $U$ an open subscheme of $X$ (instead of 'subsuperscheme').

We define closed embeddings (or closed immersions) and closed subschemes with the same definition as in section 2.3 of [21]. Note that just as with schemes, there is a one-to-one correspondence between the closed subschemes of a superscheme $X$ and ideal sheaves of $\mathcal{O}_X$.

We write $X_0$ for the even subscheme of a superscheme $X$, that is the closed subscheme determined by the ideal sheaf $\mathcal{J}_X$ generated by $(\mathcal{O}_X)_{\overline{1}}$. Observe that for affine superschemes, $(\operatorname{Spec} A)_0 = \operatorname{Spec} \overline{A}$. The correspondence $X \mapsto X_0$ is functorial. Write $i_X : X_0 \hookrightarrow X$ for the corresponding closed embedding, or simply $i$ if the space is clear from context. Let $\mathcal{N}_X := \mathcal{J}_X/\mathcal{J}_X^2$ be the conormal sheaf, which is a quasi-coherent sheaf on $X_0$.

For a superscheme $X$ such that $|X|$ is Noetherian and irreducible as a topological space, write $k(X)$ for the stalk of $\mathcal{O}_X$ at the generic point of $|X|$. Then for any open subscheme $U$ of $X$ we have a natural map $\Gamma(U, \mathcal{O}_X) \to k(X)$. This map may not be injective, but if $f$ is a section over $|U|$ we will sometimes speak of it as an element of $k(X)$ with the understanding that we are talking about its image under this restriction map.

### 3.2.3 Maps to affine superschemes

Following the same ideas as in [21] we have:

**Proposition 3.2.1.** *For a supercommutative algebra $A$ and a superscheme $X$ we have a canonical isomorphism* $\operatorname{Hom}(X, \operatorname{Spec} A) \cong \operatorname{Hom}(A, \Gamma(X, \mathcal{O}_X))$*, where the first* Hom *is in the category of locally ringed spaces and the second is in the category of supercommutative algebras. In particular the category of affine superschemes is anti-equivalent to the category of supercommutative algebras.*

We obtain the following corollary which highlights a difference with the classical setting.

**Corollary 3.2.2.** *For a superscheme $X$ we have a canonical identification:*

$$\Gamma(X, \mathcal{O}_X) \cong \operatorname{Hom}(X, \mathbb{A}^{1|1}).$$

*Proof.* We write $x$ and $\xi$ for an even and odd coordinate on $\mathbb{A}^{1|1}$ so that $k[\mathbb{A}^{1|1}] = k[x, \xi]$. Then by proposition 3.2.1,

$$\operatorname{Hom}(X, \mathbb{A}^{1|1}) \cong \operatorname{Hom}(k[x, \xi], \Gamma(X, \mathcal{O}_X)) \cong \Gamma(X, \mathcal{O}_X)_{\overline{0}} \oplus \Gamma(X, \mathcal{O}_X)_{\overline{1}} = \Gamma(X, \mathcal{O}_X).$$

□

### 3.2.4 Functor of points

Given two superschemes $X, Y$ we write $X(Y)$ for the set of morphisms of superschemes from $Y$ to $X$. For a superalgebra $A$, we write $X(A)$ for $X(\operatorname{Spec} A)$. In particular, $X(k)$ is the set of $k$-points of $X$, which for most of the spaces we consider will be exactly the closed points of the underlying topological space.

If we fix the superscheme $X$ and let $Y$ vary, we obtain the fully faithful contravariant Yoneda embedding from the category of superschemes to sets.

*Remark* 3.2.3. If we restrict the functor $X(-) : Y \mapsto X(Y)$ to the category of schemes, then the functor we obtain represents the scheme $X_0$.

## 3.3 Supervarieties

**Definition 3.3.1.** We define a supervariety to be an irreducible superscheme $X$ over $k$ such that the following conditions are satisfied:

1. $X$ admits a finite cover by affine open subschemes of the form $\operatorname{Spec} A$, where $A$ is a finitely-generated superalgebra over $k$;
2. For any open subscheme $U \subseteq X$, the map $\Gamma(U, \mathcal{O}_X) \to k(X)$ is injective;
3. The superalgebra $k(X)$ is an integral superdomain;
4. $X$ is separated over $k$, that is the the diagonal morphism $X \to X \times X$ is a closed embedding.

**Definition 3.3.2.** An affine supervariety is an affine superscheme satisfying the conditions of a supervariety. Note that if $X$ is a supervariety then $X$ is affine if and only if $X_0$ is affine (see [65]).

*Example* 3.3.3 (Affine superspace). Let $V$ be a super vector space. Then we will think of $V$ as the affine superscheme $\operatorname{Spec} S^\bullet V^*$, and we call an affine superscheme of this form affine superspace. If $V$ is finite-dimensional then it will be an affine supervariety. In particular we define $\mathbb{A}^{m|n}$ to be the affine superscheme $k^{m|n}$, whose coordinate ring is $k[x_1, \dots, x_m, \xi_1, \dots, \xi_n]$, where $x_1, \dots, x_m, \xi_1, \dots, \xi_n$ is a homogeneous basis of $(k^{m|n})^*$.

A morphism of affine superspaces $V \to W$ is said to be linear if it is induced by a linear map $V \to W$ (see section 3.2.3). We will use later that there is a linear isomorphism of affine superspaces $V \to \Pi^2 V$ given by $v \mapsto \epsilon^2 v$.

For an affine superspace $V$ we have natural maps of affine superschemes

$$a_{ev} : \mathbb{A}^{1|0} \times V \to V \quad \text{and} \quad a_{odd} : \mathbb{A}^{0|1} \times V \to \Pi V,$$

defined as follows. Writing $x$ for the coordinate of $\mathbb{A}^{1|0}$ and $\xi$ for the coordinate of $\mathbb{A}^{0|1}$, we set $a_{ev}^*(\varphi) = x \otimes \varphi$ and $a_{odd}^*(\varepsilon^{-1}\varphi) = \xi \otimes \varphi$, where $\varphi \in V^*$. (See section 2.4.2 for an explanation of the notation.)

*Remark* 3.3.4.
- If $X$ is a supervariety, then for all open subschemes $U, U' \subseteq X$ with $U' \subseteq U$, the restriction map $\Gamma(U, \mathcal{O}_X) \to \Gamma(U', \mathcal{O}_X)$ is injective. This follows from functoriality of restriction.
- If $A$ is a finitely generated $k$-algebra such that $\operatorname{Spec} A$ is a supervariety, condition (2) implies that the zero divisors of $A_{\overline{0}}$ are all nilpotent.

### 3.3.1 Integrality of supervarieties

We now discuss condition (3) of being a supervariety. First we give an equivalent formulation which is straightforward to prove.

**Lemma 3.3.5.** *Let $X$ be an irreducible superscheme over $k$ satisfying conditions (1),(2), and (4). Then condition (3) is equivalent to:*

*(3') For each open subscheme $U \subseteq X$, if $f \in \Gamma(U, \mathcal{O}_X)$ is a zero divisor then $f \in (k(X)_{\overline{1}})$.*

If $X$ is just a scheme, then conditions (2) and (3) imply it is an integral scheme. However it is not true that $\Gamma(X, \mathcal{O}_X)$ need be an integral superdomain if $X$ is a supervariety:

*Example* 3.3.6. Consider the affine superscheme $X$ given by the spectrum of

$$k[x, y, \xi, \eta]/(xy - \xi\eta, y^2, y\xi, y\eta).$$

Here $y$ defines a global section of $\mathcal{O}_X$ which is nilpotent but is not an element of $(\Gamma(X, \mathcal{O}_X)_{\overline{1}})$. However on the open subscheme where $x \neq 0$ we have $y = x^{-1}\xi\eta$, so $y \in (k(X)_{\overline{1}})$.

We can still define integral supervarieties in the following natural way.

**Definition 3.3.7.** We say a supervariety $X$ is integral if condition (3) is replaced by

(3+) For an open subscheme $U \subseteq X$, $\Gamma(U, \mathcal{O}_X)$ is an integral superdomain.

If $X$ is an integral supervariety, then $X_0$ is a variety in the usual sense. But for supervarieties this need not be true, as again example 3.3.6 demonstrates.

**Lemma 3.3.8.** *If $X$ is a supervariety, then there exists a dense open subscheme $U \subseteq X$ such that $U$ is an integral supervariety.*

*Proof.* It suffices to prove this in the case that $X = \operatorname{Spec} A$ is affine, where $A$ is a finitely generated $k$-superalgebra. Write $\mathfrak{n}$ for its nilradical, which contains $(A_{\overline{1}})$, and consider $\mathfrak{n}/(A_{\overline{1}})$. This must be a finitely generated module, and is purely even, so choose $f_1, \dots, f_n \in \mathfrak{n}_{\overline{0}}$ which generate it. Then since $f_1, \dots, f_n$ are zero divisors, condition (3) implies that there exists $g \in A_{\overline{0}}$, not a zero divisor, such that $f_1, \dots, f_n \in ((A_g)_{\overline{1}})$, i.e. such that for some large enough $m$ we have $g^m f_i \in (A_{\overline{1}})$ for all $i$. Thus the open subscheme $D(g)$ is an integral supervariety. ∎

The main reason we consider superschemes that are not integral is that orbit closures of supergroup actions are often not integral (see example 4.5.2). However orbit closures will be supervarieties by the following lemma:

**Lemma 3.3.9.** *Suppose that $X$ is a superscheme which satisfies conditions (1), (2), and (4) of definition 3.3.1. Then $X$ satisfies condition (3) if and only if it has a dense open subscheme $U \subseteq X$ such that $U$ is integral.*

*Proof.* Since $k(X) = k(U)$ this is immediate. □

Now let $X$ be a supervariety, $U$ an open subscheme of $X$ with inclusion morphism $j : U \to X$, and $Y \subseteq U$ a closed subvariety. Then we define the closure of $Y$ in $X$, written $\overline{Y}$, to be the closed subscheme of $X$ determined by the ideal sheaf $\ker(\mathcal{O}_X \to j_*\mathcal{O}_Y)$.

**Proposition 3.3.10.** *The subscheme $\overline{Y}$ is a supervariety.*

*Proof.* The properties (1) and (4) follow from being a closed subscheme of a supervariety. We now check property (2), from which property (3) will follow by lemma 3.3.9 since $Y \subseteq \overline{Y}$ will be an open subscheme.

Choose an affine open covering $U_i = \operatorname{Spec} A_i$ of $X$, and let $I_i \subseteq A_i$ be the ideal of functions vanishing on $Y \cap U_i \subseteq U_i$. Then by definition $I_i$ is exactly the kernel of the morphism $A_i \to \Gamma(U_i, j_*\mathcal{O}_Y)$, or equivalently since $Y$ is a supervariety it is the kernel of $A_i \to k(Y)$. Thus $A_i/I_i \to k(Y)$ is injective by construction. However $A_i/I_i = \Gamma(U_i \cap \overline{Y}, \mathcal{O}_{\overline{Y}})$ and this morphism coincides with restriction to the generic point. Since the open subschemes $\operatorname{Spec} A_i/I_i$ of $\overline{Y}$ form an affine open covering, and property (2) is affine local, we are done. □

### 3.3.2 Dimension

Let $X$ be a supervariety. Then because $k$ is characteristic zero, there is a dense open subscheme $U \subseteq X$ such that $U_0$ is a smooth variety. The sheaf $\mathcal{N}_U$ is coherent and thus a vector bundle over a dense open subscheme of $U_0' \subseteq U_0$, of rank $r$ say. We define the dimension of $X$ to be $\dim^s \mathfrak{m}_x/\mathfrak{m}_x^2$ for any $x \in U_0'$. Then $\dim^s X = (n|r)$ where $n = \dim U_0$. We call $n$ the even dimension and $r$ the odd dimension of $X$.

### 3.3.3 Gradings of supervarieties

We will often be interested in supervarieties that are (locally) graded. If $X_0$ is a scheme and $\mathcal{N}$ is a coherent $\mathcal{O}_X$-module, then we may construct the sheaf of supercommutative algebras on $X_0$ given by $\Lambda^\bullet\mathcal{N}$. Then $(|X_0|, \Lambda^\bullet\mathcal{N})$ is naturally a superscheme.

**Definition 3.3.11.** We say that a supervariety $X$ is graded if there exists a coherent sheaf $\mathcal{N}$ on $X_0$ and an isomorphism $X \cong (|X|, \Lambda^\bullet\mathcal{N})$. We call an isomorphism of $X$ with $(|X|, \Lambda^\bullet\mathcal{N})$ a grading of $X$. We say $X$ is locally graded if $X$ has a covering by open subschemes that are themselves graded.

*Remark* 3.3.12. Observe that if $X$ is graded then $X \cong (|X|, \Lambda^\bullet \mathcal{N}_X)$.

When $X$ is graded, a choice of grading endows its structure sheaf with a $\mathbb{Z}$-grading according to the exterior powers of the conormal sheaf, namely $(\Lambda^\bullet \mathcal{N}_X)_i = \Lambda^i \mathcal{N}_X$. However a graded supervariety $X$ has, in general, many isomorphisms with $(|X|, \Lambda^\bullet \mathcal{N})$ (see for instance [30]).

*Remark* 3.3.13.

- If $X$ is a supervariety, then by remark 3.3.4 there is a dense open subscheme $U \subseteq X$ where $U$ is an integral supervariety. If $X$ is locally graded, then by proposition 3.1.12 we have that $\mathcal{N}_U$ is locally free. Since $U_0$ is a variety and our field is of characteristic zero, there exists a dense open subscheme $U' \subseteq U$ such that $U'_0$ is a smooth variety, and $\mathcal{N}_{U'}$ remains locally free. It follows that $U'$ is a smooth supervariety (see definition 3.7.1), and so a locally graded supervariety is smooth on a dense open subset.

- The property of being locally graded is affine local; that is, if $X$ is an affine superscheme, $X$ is locally graded if and only if it is graded. This is proposition 3.1.10, and follows from the same cohomology argument given in [63] that Stein supermanifolds are graded.

## 3.4 Quasi-coherent sheaves

Let $X = \operatorname{Spec} A$ be an affine superscheme, and let $M$ be an $A$-module. We may construct a sheaf $\widetilde{M}$ on $X$ defined by $\Gamma(D(f), \widetilde{M}) = M_f$ for each $f \in A_{\overline{0}}$. Then $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-modules; we call a sheaf of $\mathcal{O}_X$-modules constructed in this way a quasi-coherent sheaf. If $A$ is finitely generated, then we call $\widetilde{M}$ a coherent sheaf if the corresponding module $M$ is finitely generated over $A$.

**Definition 3.4.1.** Let $X$ be a supervariety. A quasi-coherent sheaf $\mathcal{F}$ on $X$ is a sheaf of $\mathcal{O}_X$-modules which is quasi-coherent on each affine open subscheme of $X$. We say $\mathcal{F}$ is coherent if in addition it is coherent on each affine open subscheme.

*Remark* 3.4.2. We observe that the category of (quasi-)coherent sheaves on a supervariety is abelian and admits the usual bifunctors $(-)\otimes_{\mathcal{O}_X}(-)$ and $\underline{\mathscr{H}om}_{\mathcal{O}_X}(-,-)$, along with pullback and pushforward functors along morphisms of supervarieties. If $V$ is a super vector space and $\mathcal{F}$ is a quasi-coherent sheaf, we sometimes consider $\mathcal{F} \otimes V$, which is the tensor product of $\mathcal{F}$ with the constant sheaf associated to $V$ over the constant sheaf of rings associated to $k$, and is again a quasi-coherent sheaf.

Let $\mathcal{F}$ be a coherent sheaf on $X$. Then for $x \in |X|$ we define the fiber of $\mathcal{F}$ at $x$ to be the super vector space $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$. For an open subscheme $U$ of $X$ containing $x$, we have a restriction morphism $\Gamma(U, \mathcal{F}) \to \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$. Let $s$ be a section of $\mathcal{F}$ over $U$. Then we say $s$ is non-vanishing at $x$ if $s$ restricts to a nonzero element of the fiber. In this case, $s$ is non-vanishing in an open subset containing $x$.

**Definition 3.4.3.** We say a coherent sheaf $\mathcal{F}$ on $X$ is globally generated (or generated by global sections) if the the natural morphism

$$\mathcal{O}_X \otimes \Gamma(X, \mathcal{F}) \to \mathcal{F}$$

is surjective. If $V \subseteq \Gamma(X, \mathcal{F})$ is a subspace such that

$$\mathcal{O}_X \otimes V \to \mathcal{F}$$

is surjective then we say that $V$ generates $\mathcal{F}$.

**Definition 3.4.4.** Let $X$ be a supervariety. A vector bundle on $X$ is a coherent sheaf $\mathcal{E}$ which is locally isomorphic to $\mathcal{O}_X^{m|n} := \mathcal{O}_X \otimes k^{m|n}$ as an $\mathcal{O}_X$-module. A line bundle $\mathcal{L}$ on $X$ is a coherent sheaf which is locally isomorphic to $\mathcal{O}_X = \mathcal{O}_X^{1|0}$ as an $\mathcal{O}_X$-module.

*Remark* 3.4.5. As usual, we may form the abelian group $\mathrm{Pic}(X)$ which is the isomorphism classes of line bundles on $X$ under tensor product. Just as in the classical setting there is an identification $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$.

**Lemma 3.4.6.** *If $\mathcal{E}$ is a vector bundle on a supervariety $X$, and $U, U'$ are open subschemes of $X$ such that $U' \subseteq U$, then $\Gamma(U, \mathcal{E}) \to \Gamma(U', \mathcal{E})$ is injective.*

*Proof.* By assumption, the statement is true when $\mathcal{E} = \mathcal{O}_X^{m|n}$. Covering $X$ by open subschemes on which $\mathcal{E}$ is trivial, the result follows. □

*Remark* 3.4.7. There is an equivalent geometric definition of vector bundles which is as follows. Let $E$ and $X$ be supervarieties with a map $\pi : E \to X$, along with the data of a covering $\{U_i\}$ of $X$ by open subschemes and isomorphisms

$$\psi_i : \pi^{-1}(U_i) \to \mathbb{A}^{m|n} \times U_i.$$

These morphisms $\psi_i$ are required to respect the projections to $U_i$, and on overlaps $U_i \cap U_j$ the morphisms $\psi_j \circ \psi_i^{-1}$ must be linear on the fibers $\mathbb{A}^{m|n}$.

To obtain a sheaf $\mathcal{E}$ from this geometric definition, for an open subscheme $U$ of $X$ we set $\Gamma(U, \mathcal{E})_{\overline{0}}$ to be sections of the morphism $\pi$ over $U$. To obtain the odd sections of $\mathcal{E}$, let $\Pi E$ denote the supervariety obtained by gluing together the supervarieties $\Pi\mathbb{A}^{m|n} \times U_i$ along the isomorphisms

$$\Pi(\psi_j \circ \psi_i^{-1}) : \Pi\mathbb{A}^{m|n} \times U_i \cap U_j \to \Pi\mathbb{A}^{m|n} \times U_i \cap U_j.$$

Then we have a natural projection $\pi' : \Pi E \to X$ making $\Pi E$ into a geometric vector bundle over $X$ such that the fibers of $\Pi E$ are obtain by parity shift from the fibers of $E$. One may call $\Pi E$ the parity shift of $E$. We now set $\Gamma(U, \mathcal{E})_{\overline{1}}$ to be the sections of $\pi'$ over $U$.

To show that $\mathcal{E}$ is an $\mathcal{O}_X$-module, we use the description of sections of $\mathcal{O}_X$ given in corollary 3.2.2, the maps $a_{ev}$ and $a_{odd}$ in example 3.3.3, and with the linear isomorphism $\mathbb{A}^{m|n} \to \Pi^2\mathbb{A}^{m|n}$ described in example 3.3.3.

On the other hand, given a vector bundle $\mathcal{E}$ as a sheaf, we may construct a geometric vector bundle as the relative spectrum of the symmetric algebra of $\mathcal{E}$ as an $\mathcal{O}_X$-module.

### 3.4.1 Rational functions from line bundles

Let $\mathcal{L}$ be a line bundle on $X$, and let $s_1, s_2$ be homogeneous sections of $\mathcal{L}$ over an open subscheme $U$. Assume that $s_2$ is even and non-vanishing at some point of $|U|$. Then we may choose an open subscheme $U'$ of $U$ on which $s_2$ is non-vanishing and with the property that $\mathcal{L}|_{U'} \cong \mathcal{O}_{U'}$. Let $s_1/s_2$ be the section of $\mathcal{O}_{U'}$ determined by any isomorphism $\mathcal{L}|_{U'} \cong \mathcal{O}_{U'}$, and define $s_1/s_2 \in k(X)$ to be the rational function determined in this way. As in the classical case this is well-defined.

## 3.5 Projective supervarieties

Let $S = \bigoplus_{n\geq 0} S_n$ be an $\mathbb{N}$-graded supercommuative algebra, where $S_n$ is a super vector space for all $n$, and write $S_+ = \bigoplus_{n\geq 1} S_n$. We always assume that $S$ is generated by $S_0$ and $S_1$. Then we may define the superscheme $\operatorname{Proj} S$ in the usual way, as follows.

The underlying topological space of $\operatorname{Proj} S$ consists of all homogeneous prime ideals $\mathfrak{p}$ of $S$ which do not contain $S_+$. This space has a basis of open sets given by $D_+(f)$, where $f$ is an even homogeneous element of $S_+$, and $D_+(f)$ is the set of prime homogeneous ideals not containing either $S_+$ or $f$. We define the structure sheaf by declaring that $\Gamma(D_+(f), \mathcal{O}_{\operatorname{Proj} S}) = S_{(f)}$, where $S_{(f)}$ is the collection of degree zero elements of $S_f$. In particular we have $D_+(f) \cong \operatorname{Spec} S_{(f)}$ as an open subscheme of $\operatorname{Proj} S$.

**Definition 3.5.1.** We say a supervariety is projective if it is isomorphic to $\operatorname{Proj} S$ for a $\mathbb{N}$-graded superalgebra $S$ such that $S_0 = k$ and $S_1$ is a finite-dimensional super vector space. We call a supervariety quasi-projective if it is isomorphic to an open subscheme of a projective supervariety.

Note that for a given projective supervariety $X$ there may be different $\mathbb{N}$-graded superalgebras $S$ such that $\operatorname{Proj} S \cong X$.

*Example* 3.5.2. Let $S = k[x_0, \dots, x_m, \xi_1, \dots, \xi_n]$ be the free polynomial superalgebra, and make $S$ into an $\mathbb{N}$-graded superalgebra by declaring that $\deg x_i = \deg \xi_j = 1$ for all $i, j$. Then projective superspace $\mathbb{P}^{m|n}$ is defined be $\operatorname{Proj} S$. We observe that $D_+(x_i) \cong \mathbb{A}^{m|n}$ in a natural way, and thus $\mathbb{A}^{m|n}$ is quasi-projective.

As in the classical setting, we have the following (functorial) description of $\mathbb{P}^{m|n}$: morphisms $X \to \mathbb{P}^{m|n}$ from a supervariety $X$ are in bijective correspondence with (isomorphism classes of) a choice of line bundle $\mathcal{L}$ on $X$ and homogeneous sections $s_0, \dots, s_m, \sigma_1, \dots, \sigma_n$ which globally generate $\mathcal{L}$ (see [8]).

*Remark* 3.5.3. Following the same ideas as in Corollary 2.5.16 of [21], we have that a supervariety $X$ is projective if and only if there exists a $m, n \geq 0$ and a homogeneous ideal $I \subseteq k[x_0, \dots, x_m, \xi_1, \dots, \xi_n]$ such that $X \cong \operatorname{Proj} k[x_0, \dots, x_m, \xi_1, \dots, \xi_n]/I$. Further this realizes $X$ as a closed subvariety of $\mathbb{P}^{m|n}$, and every closed subvariety of $\mathbb{P}^{m|n}$ is of this form.

Given a projective supervariety $X$ presented as $\operatorname{Proj} S$ where $S_0 = k$ and $S_1$ is finite-dimensional, we obtain a correspondence between $\mathbb{Z}$-graded $S$-modules and quasi-coherent sheaves on $X$. Namely, given a $\mathbb{Z}$-graded $S$-module we may define the sheaf $\widetilde{M}$ which has that $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$ for all $f \in (S_+)_{\overline{0}}$. Further, $\widetilde{M}$ is coherent whenever $M$ is finitely generated over $S$.

Write $S(n)$ for the $\mathbb{Z}$-graded $S$-module given by $S(n)_m = S_{m+n}$. Then we write $\mathcal{O}_X(1) := \widetilde{S(1)}$ for the Serre twisting sheaf on $X$, a line bundle. If $X$ is quasi-projective and given as an open subscheme of $\operatorname{Proj} S$, we write $\mathcal{O}_X(1)$ for the restriction of $\widetilde{S(1)}$ to $X$.

**Definition 3.5.4.** If $X$ is quasi-projective we say a line bundle $\mathcal{L}$ is very ample if it is isomorphic to $\mathcal{O}_X(1)$ for some embedding of $X$ as an open subscheme of a projective supervariety.

*Example* 3.5.5. By definition, the restriction of a very ample line bundle to an open subscheme is very ample. The same is true for a closed subscheme using proposition 3.3.10. Since $\mathcal{O}_{\mathbb{P}^{m|n}}(1)|_{\mathbb{A}^{m|n}} \cong \mathcal{O}_{\mathbb{A}^{m|n}}$, it follows that $\mathcal{O}_X$ is very ample whenever a supervariety $X$ is (an open subscheme of) an affine supervariety. In fact, any line bundle on (an open subscheme of) an affine supervariety is very ample.

As in the classical case, we have the following results.

**Lemma 3.5.6.** *Let $X$ be a quasi-projective supervariety and $\mathcal{L}$ a very ample line bundle on $X$. Then for every homogeneous $f \in k(X)$, there exists $n \geq 0$ and homogeneous sections $s_1, s_2 \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $f = s_1/s_2$.*

**Proposition 3.5.7.** *Let $X$ be a quasi-projective supervariety, $\mathcal{L}$ a very ample line bundle on $X$, and $\mathcal{F}$ a coherent sheaf on $X$. Then for some $n \geq 0$, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is globally generated.*

*Remark* 3.5.8. Quasi-projective supervarieties are not as pervasive as quasi-projective varieties in the category of varieties. Indeed, there are many linear algebraic supervarieties of importance which are not projective, including 'most' super Grassmanians. (See [35] for a discussion in the analytic setting, or [8] for the algebraic setting.) This is a nontrivial hindrance in understanding such spaces and how supergroups act on them.

## 3.6 Relative differentials and the tangent sheaf

**Definition 3.6.1.** Given a morphism of supervarieties $X \to Y$, define the sheaf of relative differentials $\Omega_{X/Y}$ to be the sheaf on $X$ given by $\Delta^*(\mathcal{I}/\mathcal{I}^2)$ where $\Delta : X \to X \times_Y X$ is the diagonal morphism and $\mathcal{I}$ the sheaf of ideals given by the kernel of the pullback morphism $\mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X$. If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ then the global sections f orm the module of relative differentials $\Omega_{A/B}$ (see section 3.1.7).

If $X$ is a supervariety over $k$, then we write $\Omega_{X/k}$ for the sheaf of relative differentials of the canonical map $X \to \operatorname{Spec} k$. Then if $x \in X(k)$ we have, by the universal property of the module of relative differentials, $(\Omega_{X/k})_x/\mathfrak{m}_x(\Omega_{X/k})_x \cong \mathfrak{m}_x/\mathfrak{m}_x^2$.

**Definition 3.6.2.** For a supervariety $X$, we define the tangent sheaf $\mathcal{T}_X$ as the unique sheaf defined on an affine open subscheme $U = \operatorname{Spec} A$ of $X$ by $\Gamma(U, \mathcal{T}_X) = \operatorname{Der}(A)$, that is all (not necessarily even) $k$-linear superalgebra derivations of $A$ (see example 2.2.8). Restriction of sections is given by the extension of the corresponding derivation to the localization. In this way $\mathcal{T}_X$ is a coherent sheaf of Lie superalgebras on $X$, and $\Gamma(U, \mathcal{T}_X)$ acts by super derivations on $\Gamma(U, \mathcal{O}_X)$.

**Definition 3.6.3.** For a supervariety $X$, we define the sheaf $\mathcal{D}_X$ of differential operators on $X$ to be the subsheaf of $\underline{\mathscr{E}nd}(\mathcal{O}_X)$ constructed inductively as follows. Let $\mathcal{D}_X^0 = \mathcal{O}_X$ (acting on $\mathcal{O}_X$ by multiplication), $\mathcal{D}_X^1 = \mathcal{O}_X + \mathcal{T}_X$, and for an open subscheme $U$ of $X$ we inductively set

$$\Gamma(U, \mathcal{D}_X^n) = \{D \in \Gamma(U, \underline{\mathscr{E}nd}(\mathcal{O}_X)) : [D, f] \in \Gamma(U, \mathcal{D}_X^{n-1}) \text{ for all } f \in \Gamma(U, \mathcal{O}_X)\}.$$

Then $\mathcal{D}_X = \bigcup_{n \geq 0} \mathcal{D}_X^n$ defines a filtered sheaf of algebras.

### 3.6.1 Tangent spaces

**Definition 3.6.4.** Given $x \in X(k)$, we define the tangent space at $x$ to be the super vector space $T_x X$ given by point derivations $\delta : \mathcal{O}_{X,x} \to k$, i.e. maps of vector spaces such that $\delta(fg) = \delta(f)g(x) + (-1)^{\overline{\delta f}} f(x)\delta(g)$. Note that the minus sign is not strictly necessary since if $\overline{f} = \overline{1}$ then $f(x) = 0$.

*Remark* 3.6.5. We have a natural identification $T_x X \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$.

## 3.7 Smoothness of supervarieties

**Definition 3.7.1.** Let $X$ be a supervariety and $x \in X(k)$. Then there is a natural map of super vector spaces

$$\mathcal{T}_{X,x} \to T_x X$$

given by $D \mapsto (f \mapsto D(f)(x))$. We say that $X$ is smooth at $x$ if this map is surjective.

We seek to give a list of conditions that are equivalent to smoothness at a point. To state our characterization of smoothness, we need to introduce a few notions, most of which should be familiar.

- For $x \in X(k)$ we may view $T_xX$ as the affine superspace Spec $S^\bullet(\mathfrak{m}_x/\mathfrak{m}_x^2)$. Define the tangent cone at $x$, $TC_xX$, to be the closed conical subvariety of $T_xX$ given by

$$TC_xX = \operatorname{Spec}\left(\bigoplus_{n\geq 0}\mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}\right)$$

  The derivations in $\mathcal{T}_{X,x}$ act on both $k[T_xX]$ and $k[TC_x]$ by derivations of degree -1, and the action is equivariant with respect to the above closed embedding.

- For a local supercommutative algebra $A$ with unique maximal ideal $\mathfrak{m}$, we write $\widehat{A}$ for the completion of $A$ with respect to the $\mathfrak{m}$-adic topology.

- Following [46], given a superalgebra $A$ we say that an even element $t \in A_{\overline{0}}$ is $A$-regular if the multiplication map by $t$ is injective. We say an odd element $\xi \in A_{\overline{1}}$ is $A$-regular if the cohomology of the multiplication map by $\xi$ is trivial. Finally, if $(r_1, \dots, r_k)$ is a sequence of homogeneous elements of $A$, we say the sequence is $A$-regular if $r_i$ is regular in $A/(r_1, \dots, r_{i-1})$.

**Definition 3.7.2.** A local supercommutative algebra $A$ is regular if the unique maximal ideal $\mathfrak{m}$ is generated by an $A$-regular sequence.

**Lemma 3.7.3.** *Let $\overline{F}$ be a finitely generated field over $k$ of transcendence degree $m$, and let $F = \overline{F}[\xi_1, \dots, \xi_n]$ for odd variables $\xi_1, \dots, \xi_n$. Then $\Omega_{F/k}$ is a free $F$-module of rank $(m|n)$.*

*Proof.* We have the short exact sequence

$$F \otimes_{\overline{F}} \Omega_{\overline{F}/k} \to \Omega_{F/k} \to \Omega_{F/\overline{F}} \to 0.$$

Since $\Omega_{F/\overline{F}}$ is a free $F$-module of rank $(0|n)$ with generators $d\xi_1, \dots, d\xi_n$, the last map splits which implies that $d\xi_1, \dots, d\xi_n$ generate a free summand of $\Omega_{F/k}$ of rank $(0|n)$. We know that $\Omega_{\overline{F}/k}$ is a free $\overline{F}$-module of rank $(m|0)$ with generators $dt_1, \dots, dt_m$, where $t_1, \dots, t_m$ form a transcendence basis of $\overline{F}$ over $k$. Hence $\Omega_{F/k}$ is generated by $dt_1, \dots, dt_m, d\xi_1, \dots, d\xi_n$, and it suffices to show that $dt_1, \dots, dt_m$ are $F$-linearly independent.

However if we compute $\underline{Hom}_F(\Omega_{F/k}, F)$ we get the module of $k$-linear derivations of $F$, which contains a free submodule of rank $(m|0)$ generated by $\partial_{t_1}, \dots, \partial_{t_m}$. These may be used to show that $dt_1, \dots, dt_m$ are $F$-linearly independent, and we are done.

□

**Proposition 3.7.4.** *For a supervariety $X$ and closed point $x \in X(k)$, let $A := \mathcal{O}_{X,x}$ with maximal ideal $\mathfrak{m} = \mathfrak{m}_x$. Let $t_1, \dots, t_m, \xi_1, \dots, \xi_n \in \mathfrak{m}$ project to a homogeneous basis of $\mathfrak{m}/\mathfrak{m}^2$, where $\overline{t_i} = \overline{0}$ and $\overline{\xi_i} = \overline{1}$. Then the following are equivalent.*

1. $\widehat{A} \cong k[\![t_1, \dots, t_m, \xi_1, \dots, \xi_n]\!]$;

2. *$Gr_{\mathfrak{m}} A := \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} \cong k[\underline{t_1}, \ldots, \underline{t_m}, \underline{\xi_1}, \ldots, \underline{\xi_n}]$, where $\underline{(\cdot)} : \mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2$ is the natural projection;*

3. *$\Omega_{X,x} = \Omega_{A/k}$ is free over $A$;*

4. *$\operatorname{Spec} A \to k$ is a formally smooth morphism;*

5. *$\overline{A} = A/(A_{\bar{1}})$ is a regular local ring, and $A \cong \overline{A}[\xi_1, \ldots, \xi_n]$;*

6. *there exists an affine neighborhood $U = \operatorname{Spec} B$ of $x$ such that $\overline{B} = B/(B_{\bar{1}})$ is regular and $B \cong \Lambda^\bullet \overline{B}^{\oplus n}$;*

7. *$T_x X = TC_x X$;*

8. *the natural map $\mathcal{T}_{X,x} \to T_x X$ is surjective;*

9. *$A$ is a regular local superalgebra;*

10. *$A$ is a graded integral superdomain such that $\overline{A}$ is a regular local ring;*

*Proof.* The equivalence (1) $\iff$ (2) is proven in [18], (2) $\iff$ (7) is clear, (3) $\iff$ (4) is proven in [28], and (5) $\iff$ (9) is proven in [46]. The equivalence (5) $\iff$ (10) follows from proposition 3.1.12.

For (1) $\implies$ (3), we have that $\mathfrak{m}/\mathfrak{m}^2 \cong \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2$ is $(m|n)$-dimensional, so by Nakayama's lemma $\Omega_{A/k}$ is generated by $(m|n)$ elements. Localizing $A$ at the generic point, we obtain a superalgebra $F$ which by our assumptions and the Cohen structure theorem is isomorphic to $\overline{F}[\xi_1, \ldots, \xi_n]$, where $\overline{F}$ is the fraction field of $\overline{A}$. Hence by lemma 3.7.3 $\Omega_{F/k}$, which is the localization of $\Omega_{A/k}$ at the generic point, is free of rank $(m|n)$. It follows that $\Omega_{A/k}$ must itself be free of rank $(m|n)$.

For (3) $\implies$ (8), $dt_1, \ldots, dt_m, d\xi_1, \ldots, d\xi_n$ form a basis of $\Omega_{A/k}$. Then

$$\mathcal{T}_{X,x} = \underline{\operatorname{Hom}}_A(\Omega_{A/k}, A)$$

will be free with basis $\partial_{t_1}, \ldots, \partial_{t_m}, \partial_{\xi_1}, \ldots, \partial_{\xi_n}$ and these derivations map to a basis of $T_x X$, namely the dual basis of $\underline{t_1}, \ldots, \underline{t_m}, \underline{\xi_1}, \ldots, \underline{\xi_n} \in \mathfrak{m}/\mathfrak{m}^2$.

(8) $\implies$ (7): If $TC_x X \neq T_x X$, then the vanishing ideal of $TC_x X$ must be preserved by all derivations from $\mathcal{T}_{X,x}$. By our assumption, we get all coordinate derivations from the derivations of $\mathcal{T}_{X,x}$, so no such non-trivial ideals exist.

For (5) $\iff$ (6), the backward direction follows from localizing. For the forward direction, the isomorphism $\mathcal{O}_{X,x} \to \overline{A}[\xi_1, \ldots, \xi_n]$ may be extended to a morphism of sheaves $\mathcal{O}_X \to \operatorname{gr} X$ on a small enough affine open of $x$. This morphism will be an isomorphism of stalks at $x$, and so using Noetherian and coherent properties, we get an isomorphism in an open neighborhood of $x$.

The implication (5) $\implies$ (1) is clear.

Now we assume (1), and use (3) (which we have so far shown is equivalent to (1)) to prove (5). First, (1) implies that $\overline{A}$ is regular. As noted previously, by (3) we know that $A$ has derivations $\partial_{t_1}, \ldots, \partial_{t_m}, \partial_{\xi_1}, \ldots, \partial_{\xi_n}$. These derivations extend canonically to $\widehat{A}$ as the usual coordinate derivations, and these derivations preserve $A$ as a subalgebra. We have the following diagram:

$$\begin{array}{ccc} A & \longrightarrow & k[\![t_1, \ldots, t_m, \xi_1, \ldots, \xi_n]\!] \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \widehat{\pi}} \\ \overline{A} & \longrightarrow & k[\![t_1, \ldots, t_m]\!] \end{array}$$

where $\pi$ is the natural quotient map. To construct a splitting $\overline{A} \to A$, we observe that $\widehat{\pi}$ has a natural splitting $\widehat{s}$ sending $t_i$ to $t_i$. We would like to show that $\widehat{s}(\overline{A})$ lies in the image of $A$ in the completion.

Let $f \in \overline{A}$, thought of as a power series. Then we may lift $f$ to $\widetilde{f} \in A_{\overline{0}}$. The power series expansion of $\widetilde{f}$ will then be

$$\widetilde{f} = f + \sum_{I \neq \emptyset} f_I \xi_I \in A$$

where $\xi_I = \xi_{i_1} \cdots \xi_{i_k}$ if $I = \{i_1, \ldots, i_k\}$, and $f_I \in k[\![t_1, \ldots, t_m]\!]$. Using the derivations $\partial_{\xi_i}$ for varying $I$, we may show that each function $f_I$ lies in $A$, and so $f$ itself lies in $A$. Therefore we have our splitting, and now it follows that $A \cong \overline{A}[\xi_1, \ldots, \xi_n]$. □

## 3.8 Derivative map; immersions and submersions

Let $X, Y$ be supervarieties and $\phi : X \to Y$ a morphism. Then for $x \in X(k)$, $y \in Y(k)$ such that $\phi(x) = y$, the morphism of local rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ coming from pullback of functions induces a map of super vector spaces $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$. Dualizing this we obtain a map of tangent spaces $d\phi_x : T_xX \to T_yY$, the differential of $\phi$ at $x$.

**Definition 3.8.1.** We say that $\phi$ is an immersion at $x$ if $d\phi_x$ is injective, and a submersion at $x$ if $d\phi_x$ is surjective.

An alternative perspective is to consider the natural morphism $\phi^*\Omega_{Y,y} \to \Omega_{X,x}$. The fiber of this morphism is exactly the map on cotangent spaces $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$. Using this perspective it is easy to show that:

**Lemma 3.8.2.** *The property of being an immersion or submersion at a point is open on the source.*

### 3.8.1 Immersions

**Proposition 3.8.3.** *If $\phi$ is an immersion at $x$, then $\mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ is surjective. In particular, if $\phi$ is an immersion at every point and $|\phi|$ is a homeomorphism of $|X|$ onto a closed subspace of $|Y|$, then $\phi$ is a closed embedding.*

*Proof.* Since $\phi$ is an immersion at $x$, the hypotheses of lemma 7.4 in [21] are satisfied, and thus the desired morphism is surjective. □

**Definition 3.8.4.** We say that $\phi : X \to Y$ is an immersion if $\phi$ factors as $X \xrightarrow{\psi} U \xrightarrow{j} Y$ where $\psi$ is a closed immersion and $j$ the inclusion of an open subscheme.

**Corollary 3.8.5.** *A morphism $\phi$ is an immersion if and only if $\phi$ is an immersion at every point and $|\phi|$ is homeomorphism of $|X|$ onto the closed subspace of an open subspace of $|Y|$.*

### 3.8.2 Submersions

**Lemma 3.8.6.** *Let $\phi : X \to Y$ be a submersion at $x \in X(k)$, and suppose that $X$ is smooth at $x$. Then $\phi_x^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is an injective morphism.*

*Proof.* Let $M \subseteq \mathcal{O}_{Y,y}$ be the kernel of this morphism of local rings. Then choose $n > 0$ such that $M \subseteq \mathfrak{m}_x^n$ but $M \nsubseteq \mathfrak{m}_x^{n+1}$, and choose $f \in M$ such that $f \in \mathfrak{m}_x^n \setminus \mathfrak{m}_x^{n+1}$. Let $x_1, \ldots, x_m, \xi_1, \ldots, \xi_n \in$
$m_y$ be homogeneous such that they project to a basis of $\mathfrak{m}_y/\mathfrak{m}_y^2$. Then by assumption, $\phi_x(x_1), \ldots, \phi_x(x_m), \phi_x(\xi_1), \ldots, \phi_x(\xi_n)$ project to a linearly independent set in $\mathfrak{m}_x/\mathfrak{m}_x^2$. Thus there exists a nonzero homogeneous polynomial $p$ of degree $n$ such that

$$f - p(x_1, \ldots, x_m, \xi_1, \ldots, \xi_n) \in \mathfrak{m}_y^{n+1}.$$

Therefore

$$\phi_x(p(x_1, \ldots, x_m, \xi_1, \ldots, \xi_n) - f) = p(\phi_x(x_1), \ldots, \phi_x(x_m), \phi_x(\xi_1), \ldots, \phi_x(\xi_n)) \in \mathfrak{m}_x^{n+1}.$$

But since $X$ is smooth at $x$ and $d\phi_x$ is injective, we must have

$$p(\phi_x(x_1), \ldots, \phi_x(x_m), \phi_x(\xi_1), \ldots, \phi_x(\xi_n)) \in \mathfrak{m}_x^n \setminus \mathfrak{m}_x^{n+1}.$$

This is a contradiction, and therefore $M = 0$. □

# Chapter 4

# Supergroups and their actions

In this chapter we introduce algebraic supergroups, which are the global versions of Lie superalgebras. Quasi-reductive algebraic supergroups will be of particular importance to us. Actions of algebraic supergroups on supervarieties will be discussed, detailing some important properties.

## 4.1 Supergroups

See sections 8, 9, and 11 of [8] for more on the foundations of (algebraic) supergroups and their actions.

**Definition 4.1.1.** An algebraic supergroup over $k$ is a supervariety $G$ over $k$ equipped with morphisms $m = m_G : G \times G \to G$, $s = s_G : G \to G$, and $e = e_G : \operatorname{Spec} k \to G$ satisfying the usual constraints:

$$m \circ (m \times \mathrm{id}) = m \circ (\mathrm{id} \times m),$$

$$m \circ (e \times \mathrm{id}) = m \circ (\mathrm{id} \times e) = \mathrm{id}_G,$$

and

$$m \circ (\mathrm{id}_G \times s) \circ \Delta_G = m \circ (s \times \mathrm{id}) \circ \Delta_G = e.$$

where $\Delta_G : G \to G \times G$ is the diagonal embedding. In addition, we assume throughout that $G$ is affine. We will use the term Lie supergroup synonymously with algebraic supergroup.

*Remark* 4.1.2. Because we assume our supergroups are affine, we may equivalently define an algebraic supergroup as the spectrum of a finitely generated supercommutative Hopf algebra $A$ over $k$.

By the usual functor of points perspective, we may equivalently think of a supergroup as an affine supervariety whose functor of points admits a factorization through the category of groups.

**Definition 4.1.3.** For $u_e \in T_eG$, construct a right-invariant vector field $u_L$ on $G$ via left infinitesimal translation by the equation

$$u_L(f) = -(u_e \otimes 1)(m^*(f)).$$

Then the value of $u_L$ at $e$ as a tangent vector is $-u_e$. Write $\mathfrak{g} = \operatorname{Lie} G$ for the Lie superalgebra of right-invariant vector fields on $G$. The restriction map $\mathfrak{g} \to T_eG$ is an isomorphism of super vector spaces, so we will freely identify $\mathfrak{g}$ with $T_eG$ when convenient. Given $u_e \in T_eG$, we may also construct a left-invariant vector field on $G$ via right infinitesimal translation given by

$$u_R(f) = (1 \otimes u_e)(m^*(f)).$$

The value of $u_R$ at $e$ is $u_e$. The Lie superalgebra of left-invariant vector fields is canonically isomorphic to the Lie superalgebra of right-invariant vector fields via $u_L \leftrightarrow u_R$.

*Remark* 4.1.4. If $G$ is an algebraic supergroup, then $G_0$ is an algebraic group in the usual sense, and we have a canonical isomorphism $\mathfrak{g}_{\overline{0}} \cong \operatorname{Lie}(G_0)$.

If $H \subseteq G$ is a closed subvariety of $G$ which contains the unit element and is preserved under the multiplication and inversion morphisms, we say that $H$ is a subgroup of $G$. The term subsupergroup might be more appropriate, but like with subsuperalgebra we find it cumbersome to use.

### 4.1.1 Morphisms of supergroups and subgroups

Given two supergroups $G, H$, a morphism of supergroups $\phi : G \to H$ is a morphism of supervarieties which respects the supergroup structure morphisms on $G$ and $H$. This induces a morphism of algebraic groups $\phi_0 : G_0 \to H_0$. In particular $\phi(e_G) = e_H$, and thus we obtain a morphism $d\phi_e : T_eG \to T_eH$ inducing a morphism of Lie superalgebras $\operatorname{Lie}(G) \to \operatorname{Lie}(H)$.

Let $K$ denote the affine scheme given by the fiber of $\phi$ over the identity element of $H$. Then $K$ has the structure of an algebraic supergroup and we call it the kernel of $\phi$. Further we have that $\operatorname{Lie}(K) = \ker(d\phi_e)$ and $K_0 = \ker(\phi_0)$. If $K$ is the trivial group then $\phi$ is an immersion. If $\phi$ is a closed embedding, we say that $G$ is a closed subgroup of $H$, and $G_0$ will be a closed subgroup of $H_0$ in the usual sense. On the level of Lie superalgebras, $\operatorname{Lie}(G)$ will be a Lie subalgebra of $\operatorname{Lie}(H)$.

### 4.1.2 Examples of supergroups

Let $V$ be a finite-dimensional super vector space.

*Example* 4.1.5. (General linear supergroup) Let $GL(V)$ be the supergroup with coordinate superalgebra $k[\underline{\operatorname{End}}(V), \det_{\overline{0}}^{\pm 1}, \det_{\overline{1}}^{\pm 1}]$ where $\det_i$ is the determinant polynomial on $\operatorname{End}(V_i) \subseteq \underline{\operatorname{End}}(V)_{\overline{0}}$. In particular we write $GL(m|n) := GL(k^{m|n})$. Then we have $\operatorname{Lie}(GL(V)) = \mathfrak{gl}(V)$ and $GL(V)_{\overline{0}} = GL(V_{\overline{0}}) \times GL(V_{\overline{1}})$. For a superalgebra $A$ the functor of points gives

$$GL(V)(A) = \operatorname{Aut}_A(A \otimes_k V),$$

where $\mathrm{Aut}_A(A\otimes_k V)$ denotes the set of $A$-linear automorphisms of the free $A$-module $A\otimes_k V$.

There is a surjective morphism of algebraic groups $\mathrm{Ber}: GL(V)\to\mathbb{G}_m$, the Berezinian (see [8]), and we write $SL(V)$ for the kernel of this morphism, the special linear supergroup. Then $d\,\mathrm{Ber}_e:\mathfrak{gl}(V)\to k^{1|0}$ is exactly the supertrace morphism on matrices, and so $\mathrm{Lie}(SL(V))=\mathfrak{sl}(V)$.

*Example* 4.1.6. Suppose that $(-,-)$ is a non-degenerate supersymmetric even bilinear form on $V$. We define $OSP(V)$ to be the closed subgroup of $GL(V)$ with functor of points given by, for a superalgebra $A$,

$$OSP(V)(A)=\{\phi\in\mathrm{Aut}_A(A\otimes_k V):(\phi(v),\phi(w))=(v,w)\text{ for all }v,w\in V\}.$$

Then $\mathrm{Lie}(OSP(V))=\mathfrak{osp}(V)$ and $OSP(V)_0=O(V_{\overline{0}})\times SP(V_{\overline{1}})$. If we restrict the Berezinian to $OSP(V)$ we obtain a morphism $OSP(V)\to\mathbb{G}_m$, and we let $SOSP(V)$ denote the kernel, which will be the connected component of the identity of $OSP(V)$.

### 4.1.3 Super Harish-Chandra pairs

**Definition 4.1.7.** A super Harish-Chandra pair (SHCP) $(\overline{G},\mathfrak{g})$ is the following data:

- an even algebraic group $\overline{G}$;
- a finite-dimensional Lie superalgebra $\mathfrak{g}$;
- an action of $\overline{G}$ on $\mathfrak{g}$; and
- an isomorphism $\mathrm{Lie}\,\overline{G}\cong\mathfrak{g}_{\overline{0}}$ such that the adjoint action of $\overline{G}$ on $\mathrm{Lie}\,\overline{G}$ is the restriction of the action of $\overline{G}$ on $\mathfrak{g}$ to $\mathfrak{g}_{\overline{0}}$.

A homomorphism of SHCPs $(\overline{G},\mathfrak{g})\to(\overline{H},\mathfrak{h})$ is the data of homomorphisms $\Phi:\overline{G}\to\overline{H}$ of algebraic groups and $\phi:\mathfrak{g}\to\mathfrak{h}$ of Lie superalgebras, such that $d\Phi=\phi|_{\mathfrak{g}_{\overline{0}}}$ under the identifications $\mathrm{Lie}(\overline{G})=\mathfrak{g}_{\overline{0}}$ and $\mathrm{Lie}(\overline{H})=\mathfrak{h}_{\overline{0}}$. In this way we obtain the category of SHCPs.

There is a natural functor from the category of algebraic supergroups to the category of SHCPs by taking $G$ to $(G_0,\mathrm{Lie}(G))$. The following is well-known and originally proven in [31].

**Theorem 4.1.8.** *The functor $G\mapsto(G_0,Lie(G))$ is an equivalence of categories. An inverse functor is given by constructing from the SHCP $(\overline{G},\mathfrak{g})$ the Hopf superalgebra $\underline{\mathrm{Hom}}_{\mathfrak{g}_{\overline{0}}}(\mathcal{U}\mathfrak{g},k[\overline{G}])$ and taking its spectrum to obtain an algebraic supergroup. In particular, for an algebraic supergroup $G$ we have a natural identification $k[G]\cong\underline{\mathrm{Hom}}_{\mathfrak{g}_{\overline{0}}}(\mathcal{U}\mathfrak{g},k[G_0])$.*

Let us describe more explicitly the Hopf algebra structure on $\underline{\mathrm{Hom}}_{\mathfrak{g}_{\overline{0}}}(\mathcal{U}\mathfrak{g},k[G_0])$ for future purposes. Multiplication is given by, for $\varphi,\psi\in k[G]$, $u\in\mathcal{U}\mathfrak{g}$,

$$(\varphi\psi)(u)=\Delta_{G_0}^*(\varphi\otimes\psi)\Delta(u),$$

where $\Delta_{G_0}$ is the diagonal morphism $G_0 \to G_0 \times_k G_0$, so $\Delta_{G_0}^*$ is multiplication on $k[G_0]$. The counit morphism is $\epsilon : \varphi \mapsto \epsilon_0(\varphi(1))$, where $\epsilon_0$ is the counit for $k[G_0]$. Comultiplication is given by

$$m^*(\varphi)(u \otimes v)(g,h) = \varphi((h^{-1}.u)v)(gh);$$

and the antipode map is

$$(s^*(\varphi)(u))(g) = \varphi(g^{-1}.\widetilde{u})(g),$$

where $\widetilde{u}$ is the antipode of $u$ in $\mathcal{U}\mathfrak{g}$ (see section 2.2.2).

For an open subscheme $U \subseteq G$, we have

$$\Gamma(U, \mathcal{O}_G) = \underline{\operatorname{Hom}}_{\mathfrak{g}_0}(\mathcal{U}\mathfrak{g}, k[U_0]).$$

Under this correspondence, we find that left-invariant and right-invariant vector fields act as follows: for $u \in \mathfrak{g}$,

$$u_R(f)(v) = (-1)^{\overline{u}(\overline{f}+\overline{v})} f(vu)$$

and

$$u_L(f)(v)(g) = (-1)^{\overline{u}\overline{f}} f((g^{-1}.u)v)(g).$$

(See lemma 7.4.11 of [8]. Note that we use the subscripts $L$ and $R$ to refer to left or right infinitesimal translation, while there the authors used it to refer to left or right invariance.)

## 4.2 Actions of supergroups

**Definition 4.2.1.** Let $X$ be a supervariety and $G$ an algebraic supergroup. An action of $G$ on $X$ is a morphism $a : G \times X \to X$ such that

$$a \circ (m_G \times \mathrm{id}_X) = a \circ (\mathrm{id}_G \times a)$$

and

$$a \circ (e \times \mathrm{id}_X) = \mathrm{id}_X .$$

Given an action of $G$ on $X$ we obtain a natural morphism of functors $G \to \operatorname{Aut}(X)$.

**Definition 4.2.2.** Let $X$ be a supervariety and $\mathfrak{g}$ a Lie superalgebra. An action of $\mathfrak{g}$ on $X$ is a Lie superalgebra homomorphism $\mathfrak{g} \to \Gamma(X, \mathcal{T}_X)$. This homomorphism induces a superalgebra homomorphism $\mathcal{U}\mathfrak{g} \to \Gamma(X, \mathcal{D}_X)$.

If $G$ (resp. $\mathfrak{g}$) acts on a supervariety $X$, we will say that $X$ is a $G$-supervariety (resp. $\mathfrak{g}$-supervariety).

*Remark* 4.2.3. An action of a Lie superalgebra $\mathfrak{g}$ on a supervariety $X$ induces an action of $\mathfrak{g}$ on every open subvariety of $X$ by restriction.

Given an action of $G$ on $X$, we obtain a homomorphism $\rho_a : \mathfrak{g} \to \Gamma(X, \mathcal{T}_X)$ inducing an action of $\mathfrak{g}$ on $X$, as follows. For an open set $U \subseteq X$, choose an open subset $U' \subseteq G$ containing the identity such that $a$ sends $U' \times U$ into $U$. Let $f \in \Gamma(U, \mathcal{O}_X)$ and $u \in \mathfrak{g}$. Then define the action of $u$ on $f$ by

$$u(f) = -(u_e \otimes 1)(a^*(f)).$$

We claim this defines a vector field on $X$, and we set $\rho_a(u)$ to be the corresponding element of $\Gamma(X, \mathcal{T}_X)$.

*Remark* 4.2.4. If a Lie supergroup $G$ acts on a supervariety $X$, then $G_0$ acts on $X$ and $X_0$, such that the following diagram commutes:

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ \uparrow & & \uparrow \\ G_0 \times X & \longrightarrow & X \\ \uparrow & & \uparrow \\ G_0 \times X_0 & \longrightarrow & X_0 \end{array}$$

### 4.2.1 Representations of $G$ and $k[G]$-comodules

**Definition 4.2.5.** A representation of a supergroup $G$ on a finite-dimensional super vector space $V$ is a morphism of supergroups $G \to GL(V)$. In general if $V$ is infinite-dimensional, a representation of $G$ on $V$ is an action $a : G \times V \to V$ such that for every superalgebra $A$ the induced morphism $G(A) \to \mathrm{Aut}(V)(A) = \mathrm{Aut}(A \otimes_k V)$ maps elements of $G(A)$ into $A$-linear automorphisms of $A \otimes_k V$.

We will sometimes call $V$ a $G$-representation or $G$-module. We may also say $V$ is a rational $G$-module or a rational $G$-representation if we want to clarify that it is not simply a representation of the group of closed points $G(k)$.

**Definition 4.2.6.** A (left) $k[G]$-comodule $V$, where $V$ is a super vector space, is the data of a morphism $a_V : V \to k[G] \otimes V$ such that $(1 \otimes a_V) \circ a_V = (m_G^* \otimes 1) \circ a_V$.

We may form separately the categories of $G$-modules and $k[G]$-comodules in the natural way from the above definitions. Each category is abelian and admits natural tensor product and dual constructions. As usual there is an equivalence of categories between theses two categories (see [24]). Thus we will often think of $G$-modules as $k[G]$-comodules.

*Remark* 4.2.7. By a standard argument, a $G$-module $V$ is a union of finite-dimensional submodules.

*Remark* 4.2.8. Let $V$ be a $G$-module. Then $V$ is naturally a $\mathfrak{g} = \mathrm{Lie}(G)$-module via $u \cdot v = -(u_e \otimes 1) \circ a_V$.

### 4.2.2 Actions of SHCPs

Just as supergroups act on supervarieties, we can define the action of a SHCP on a supervariety as follows.

**Definition 4.2.9.** An action of a SHCP $(\overline{G}, \mathfrak{g})$ on a supervariety $X$ is a pair of actions $\overline{a} : \overline{G} \times X \to X$ and $\rho : \mathfrak{g} \to \Gamma(X, \mathcal{T}_X)$ such that

- the map $\mathrm{Lie}(\overline{G}) \to \Gamma(X, \mathcal{T}_X)$ determined by $\overline{a}$ agrees with $\rho|_{\mathfrak{g}_{\overline{0}}}$ in the natural sense;
- we have $\rho(\mathrm{Ad}(g)u) = (\overline{a}^{g^{-1}})^* \circ \rho(u) \circ (\overline{a}^g)^*$ for all $g \in \overline{G}(k)$ and $u \in \mathfrak{g}$, where $\overline{a}^g = \overline{a} \circ (i_g \times \mathrm{id}_X)$.

There is a natural definition of a morphism of supervarieties with an action of $(\overline{G}, \mathfrak{g})$ which we leave to the reader.

We observe that the action of an algebraic supergroup $G$ on a supervariety $X$ determines an action of the corresponding SHCP $(G_0, \mathfrak{g})$ on $X$. This is gives a functor from the category of $G$-supervarieties to the category of varieties with an action of $(G_0, \mathfrak{g})$. We prove this functor is an equivalence. The fact is stated for supermanifolds without proof in [14] and a full proof for supermanifolds is given in section 4.5 of [6].

**Theorem 4.2.10.** *Let $G$ be a Lie supergroup with $\mathfrak{g} = \mathrm{Lie}(G)$, and suppose that $X$ is a supervariety. Suppose that $G_0$ acts on $X$ via $a_0 : G_0 \times X \to X$, and that we have a homomorphism of Lie superalgebras $\rho : \mathfrak{g} \to \Gamma(X, \mathcal{T}_X)$ such that*

1. *$\rho|_{\mathfrak{g}_{\overline{0}}}(u) = -(u \otimes 1) \circ a_0^*$ for all $u \in \mathfrak{g}_{\overline{0}}$;*
2. *$\rho(\mathrm{Ad}(g)(u)) = (a_0^{g^{-1}})^* \circ \rho(u) \circ (a_0^g)^*$ for all $g \in G_0(k)$ and $u \in \mathfrak{g}$, where $a_0^g = a_0 \circ (i_g \times \mathrm{id}_X)$, where $i_g : \{g\} \to G_0$ is the natural inclusion.*

*Then there exists a unique action $a : G \times X \to X$ of $G$ on $X$ such that $a|_{G_0} = a_0$ and $\rho_a = \rho$.*

Before we explain how to define an action of $G$ on $X$ given an action of its SHCP on $X$, we observe the following.

**Lemma 4.2.11.** *Suppose that a supergroup $G$ acts on a supervariety $X$ via $a : G \times X \to X$. Then on open subschemes $U_1, U_2 \subseteq X$ and $U' \subseteq G$ such that $a$ takes $U' \times U_2$ into $U_1$, the pullback morphism*

$$a^* : k[U_1] \to \underline{\mathrm{Hom}}_{\mathfrak{g}_{\overline{0}}}(\mathcal{U}\mathfrak{g}, k[U_0']) \otimes k[U_2] \cong \underline{\mathrm{Hom}}_{\mathfrak{g}_{\overline{0}}}(\mathcal{U}\mathfrak{g}, k[U_0'] \otimes k[U_2])$$

*(where $k[U_2]$ is given the trivial $\mathfrak{g}_{\overline{0}}$-action) is given by, for $u \in \mathcal{U}\mathfrak{g}$,*

$$a^*(f)(u) = (-1)^{\overline{u}\overline{f}}(1 \otimes \rho_a(u))a_0^*(f)$$

*Proof.* The statement is clear for $u = 1$. Now suppose it is true for $v$, and we would like to show it holds for $u = vw$, where $w \in \mathfrak{g}$. We have

$$\begin{aligned}
a^*(f)(vw) &= (-1)^{\overline{w}(\overline{v}+\overline{f})}(w_R \otimes 1 \circ a^*(f))(v) \\
&= (-1)^{\overline{w}(\overline{v}+\overline{f})}(1 \otimes w \otimes 1 \circ m^* \otimes 1 \circ a^*(f))(v) \\
&= (-1)^{\overline{w}(\overline{v}+\overline{f})}(1 \otimes w \otimes 1 \circ 1 \otimes a^* \circ a^*(f))(v) \\
&= (-1)^{\overline{w}(\overline{v}+\overline{f})}(1 \otimes \rho_a(w) \circ a^*(f))(v) \\
&= (-1)^{\overline{w}(\overline{v}+\overline{f})}(1 \otimes \rho_a(w))(a^*(f)(v)) \\
&= (-1)^{\overline{w}(\overline{v}+\overline{f})+\overline{v}\overline{f}}(1 \otimes \rho_a(w)\rho_a(v))(a_0^*(f)) \\
&= (-1)^{(\overline{w}+\overline{v})\overline{f}}(1 \otimes \rho_a(vw))(a_0^*(f)).
\end{aligned}$$

□

Now we see how to take an action $(a_0, \rho)$ of the SHCP $(G_0, \mathfrak{g})$ on a supervariety $X$ and use it to define an action of $G$ on $X$. For open subsets $U_1, U_2, U'$ described above, we define

$$a^*(f)(u) := (-1)^{\overline{u}\overline{f}}(1 \otimes \rho(u))a_0^*(f).$$

This map is natural, hence gives a global map on sheaves. Let us check it is $\mathfrak{g}_{\overline{0}}$-linear: for $v \in \mathfrak{g}_{\overline{0}}$, we have

$$\begin{aligned}
a^*(f)(vu) &= (-1)^{\overline{u}\overline{f}}(1 \otimes \rho(vu))a_0^*(f) \\
&= (-1)^{\overline{u}\overline{f}}(1 \otimes \rho(u))(1 \otimes \rho(v))a_0^*(f) \\
&= (-1)^{\overline{u}\overline{f}}(1 \otimes \rho(u))(1 \otimes v \otimes 1) \circ (1 \otimes a_0^*) \circ a_0^*(f) \\
&= (-1)^{\overline{u}\overline{f}}(1 \otimes \rho(u))(1 \otimes v \otimes 1) \circ (\overline{m}^* \otimes 1) \circ a_0^*(f) \\
&= (-1)^{\overline{u}\overline{f}}(1 \otimes v \otimes 1) \circ (\overline{m}^* \otimes 1) \circ (1 \otimes \rho_a(u)) \circ a_0^*(f) \\
&= (-1)^{\overline{u}\overline{f}}(v_R \otimes 1)((1 \otimes \rho(u)) \circ a_0^*(f)) \\
&= (v^L \otimes 1)(a_0^*(f)(u)).
\end{aligned}$$

Next we check $a^*$ respects multiplication. For $f_1, f_2 \in k[U_1]$, we have

$$\begin{aligned}
a^*(f_1 f_2)(u) &= (-1)^{\overline{u}(\overline{f_1}+\overline{f_2})}(1 \otimes \rho(u))a_0^*(f_1 f_2) \\
&= (-1)^{\overline{u}(\overline{f_1}+\overline{f_2})}(1 \otimes \rho(u))a_0^*(f_1)a_0^*(f_2) \\
&= (-1)^{\overline{u}(\overline{f_1}+\overline{f_2})+\overline{u^{(2)}}\overline{f_1}}(1 \otimes \rho(u^{(1)})a_0^*(f_1))(1 \otimes \rho(u^{(2)})a_0^*(f_2)) \\
&= (-1)^{\overline{u^{(1)}}\overline{f_2}\overline{f_1}}a^*(f_1)(u^{(1)})a^*(f_2)(u^{(2)}) \\
&= (a^*(f_1)a^*(f_2))(u).
\end{aligned}$$

It now remains to check that the associative property holds—assume we restrict to appropriate open sets, and let $f$ be a section. Then we have

$$\begin{aligned}
((m^*\otimes 1)\circ a^*(f))(u,v)(g,h) &= a^*(f)(h^{-1}.uv)(gh)\\
&= (-1)^{\overline{uv}+\overline{f}(\overline{u}+\overline{v})}\rho(v)(a^h)^*\rho(u)(a^{h^{-1}})^*a_0^*(f)(gh)\\
&= (-1)^{\overline{uv}+\overline{f}(\overline{u}+\overline{v})}\rho(v)(a^h)^*\rho(u)(a^{h^{-1}})^*(a^{gh})^*(f)\\
&= (-1)^{\overline{uv}+\overline{f}(\overline{u}+\overline{v})}\rho(v)(a^h)^*\rho(u)(a^{g})^*(f)\\
&= (-1)^{\overline{uv}+\overline{f}(\overline{u}+\overline{v})}\rho(v)(a^h)^*\rho(u)a_0^*(f)(g)\\
&= (-1)^{\overline{uv}+\overline{f}\overline{v}}\rho(v)(a^h)^*(a^*(f)(u))(g)\\
&= (-1)^{\overline{uv}+\overline{f}\overline{v}}\rho(v)(1\otimes a_0^*)a^*(f)(u)(g,h)\\
&= ((1\otimes a^*)(a^*(f))(u,v))(g,h).
\end{aligned}$$

It follows we indeed get an action. Finally, observe that this action is the unique one that agrees with the action of the corresponding SHCP—indeed this follows from lemma 4.2.11.

In particular, we get the following useful corollary:

**Corollary 4.2.12.** *Suppose that a Lie supergroup $G$ acts on a supervariety $X$, and that for an open subscheme $U\subseteq X$, $U_0$ is stable under the action of $G_0$. Then the open subvariety $U$ is stable under the action of $G$, i.e. the action of $G$ on $X$ restricts to an action of $G$ on $U$.*

Further, we have the following description of the representations of $G$.

**Corollary 4.2.13.** *The data of a representation $\rho: G\to GL(V)$ is equivalent to the data of compatible representations $G_0\to GL(V_{\overline{0}})\times GL(V_{\overline{1}})$ and $\mathfrak{g}=\mathrm{Lie}(G)\to\mathfrak{gl}(V)$. Thus a representation $V$ of $\mathfrak{g}$ comes from a representation of $G$ if and only if it integrates to $G_0$.*

## 4.3 Quasi-reductive supergroups and hyperborels

**Definition 4.3.1.** A supergroup $G$ is quasi-reductive if $G_0$ is reductive.

By definition, if $G$ is quasi-reductive then $\mathrm{Lie}(G)$ is quasi-reductive. Of course the converse may not hold.

**Definition 4.3.2.** If $G$ is quasi-reductive, we say $G$ is Cartan-even if $\mathrm{Lie}\,G$ is.

**Definition 4.3.3.** If $G$ is quasi-reductive, we call a subgroup $B$ a Borel (resp. hyperborel) subgroup if $B$ is connected and $\mathrm{Lie}(B)$ is a Borel (resp. hyperborel) subalgebra of $\mathrm{Lie}(G)$.

If $\mathfrak{n}$ is the unipotent radical of $\mathfrak{b}$, we write $N$ for the connected subgroup of $B$ it integrates to in $G$ and call it the maximal unipotent subgroup of $B$. Finally, we write $T$ for the connected subgroup of $G$ that a chosen Cartan subalgebra $\mathfrak{h}_{\overline{0}}\subseteq\mathfrak{b}$ integrates to, which will be a maximal torus of $G_0$; we call $T$ a maximal torus of $B$, and of $G$ as well.

| Supergroup | SHCP |
|---|---|
| $GL(m\|n)$ | $(GL(m) \times GL(n), \mathfrak{gl}(m\|n))$ |
| $(S)OSP(m\|2n)$ | $((S)O(m) \times Sp(2n), \mathfrak{osp}(m\|2n))$ |
| $D(2\|1;t)$ | $(SL(2) \times SL(2) \times SL(2), \mathfrak{d}(2\|1;t))$ |
| $G(1\|2)$ | $(SL(2) \times G_2, \mathfrak{sl}(2) \times \mathfrak{g}_2)$ |
| $AB(1\|3)$ | $(SL(2) \times Spin(7), \mathfrak{ab}(1\|3))$ |
| $P(n)$ | $(GL(n), \mathfrak{p}(n))$ |
| $Q(n)$ | $(GL(n), \mathfrak{q}(n))$ |

Table 4.1: Distinguished supergroups

If $B$ is a hyperborel subgroup, or more generally if $G$ is Cartan-even, there is a canonical identification of weights of $T$ with characters of $B$ via the composition of maps $T \to B \to B/N$. If $X$ is a $G$-supervariety, the algebra $k[X]^N$ has a natural $T$-action, and $\Lambda_B^+(X)$ is the collection of weights of this action under this identification. Also observe that neither $\Lambda_B^+(X)$ nor $\Lambda_B(X)$ is a monoid or group in general, due to the presence of nilpotent functions. For example, consider the action of an even torus on a purely odd super vector space.

### 4.3.1 Representation theory

Consider the category of finite-dimensional representations of a supergroup $G$. By the formalism of SHCPs already explained, there is a natural equivalence between this category and the category of finite-dimensional representations of $\mathfrak{g} = \mathrm{Lie}(G)$ which admit a consistent action of the algebraic group $G_0$.

Thus the category of representations of a quasi-reductive supergroup $G$ is equivalent to the category of representations of $\mathfrak{g}$ which are semisimple over $\mathfrak{g}_{\overline{0}}$ and for which the irreducible $\mathfrak{g}_{\overline{0}}$-representations appearing are all integrable to representations of $G_0$. Therefore the statements in section 2.3.1 carry over almost entirely to the category of $G$-representations.

### 4.3.2 Distinguished quasi-reductive supergroups

Just as we had a list of distinguished Lie superalgebras which were of special importance, there are certain supergroups that we will call distinguished. We list them in the table, giving for each the SHCP associated to them.

As we did with their Lie superalgebras, we will call $GL(m|n)$, $(S)OSP(m|2n)$, $D(2|1;t)$, $G(1|2)$, and $AB(1|3)$ basic.

## 4.4 Homogeneous supervarieties and Frobenius reciprocity

### 4.4.1 Homogeneous supervarieties

Suppose that $G$ is a supergroup and $K$ is a closed subgroup of $G$. Then we may construct a supervariety $G/K$ which represents the quotient of $G$ by right translation by $K$. For the technical aspects and properties of $G/K$ in arbitrary characteristic (not equal to 2), see [38] and more recently [37]. We give a description of what we need about $G/K$.

The underlying topological space of the quotient is $|G/K| = |G_0/K_0|$. We have a natural projection map $\pi_0 : G_0 \to G_0/K_0$. For an open set $|U| \subseteq |G_0/K_0|$ the open set $\pi_0^{-1}(|U|)$ is $K_0$-stable under right translation, and thus the corresponding open subscheme of $G$ is $K$-stable by corollary 4.2.12. Therefore we set the structure sheaf on $G/K$ to be $\mathcal{O}_{G/K} = ((\pi_0)*\mathcal{O}_G)^K$. Explicitly,

$$\Gamma(|U|, (\pi_*\mathcal{O}_G)^K) = \Gamma(\pi^{-1}(|U|), \mathcal{O}_G)^K,$$

where $(-)^K$ takes the $K$-invariants of a module, and the action is by right translation. Thus in particular $(G/K)_0 = G_0/K_0$.

There is a natural left translation action of $G$ on $G/K$, and a natural projection morphism $\pi : G \to G/K$ which is equivariant with respect to the left translation actions. We write $eK \in (G/K)(k)$ for the image of the identity element in $G$ under this projection. Then the action of $\mathfrak{g}$ on $G/K$ by infinitesimal translation induces a surjective map $\mathfrak{g} \to T_e(G/K)$ with kernel $\mathfrak{k}$, and we obtain an identification of $K$-modules $T_{eK}(G/K) \cong \mathfrak{g}/\mathfrak{k}$. In particular $G/K$ is a smooth supervariety by proposition 3.7.4.

Here $G/K$ is affine if and only if $G_0/K_0$ is, and in this case we have $k[G/K] = \operatorname{Spec} k[G]^K$, where $K$ acts on $k[G]$ by pullback under right translation on $G$. In general we always have $k[G/K] = k[G]^K$.

### 4.4.2 Frobenius reciprocity

Suppose that $V$ is a $G$-module, and $K$ is a closed subgroup of $G$. The following isomorphism of vector spaces is referred to as Frobenius reciprocity.

**Proposition 4.4.1.** *We have a canonical isomorphism of vector spaces*

$$\operatorname{Hom}_G(V, k[G/K]) \cong \operatorname{Hom}_K(V, k)$$

*where on the RHS $V$ is considered as an $K$-module.*

*Proof.* Given a $G$-module morphism $V \to k[G/K]$ we with the evaluation map at the identity coset $eK$ to obtain an $K$-coinvariant $V \to k$.

Conversely, let $a_V : V \to k[G] \otimes V$ denote the structure morphism as a comodule. Given a $K$-coinvariant $\varphi : V \to k$ we obtain $\phi := (1 \otimes \varphi) \circ a_V : V \to k[G]$. One may check that the image of $\phi$ lands in $k[G]^K$, and further that it is a morphism of $G$-modules.

The proof that the correspondences above are inverse to one another is standard. □

## 4.5 Orbits and stabilizers

Let $G$ be a supergroup and $X$ a $G$-supervariety. For $x \in X(k)$, we have the orbit map at $x$, $a_x : G \to X$, given by $a \circ (\mathrm{id}_G \times i_x)$, where $i_x : \{x\} \to X$ is the natural inclusion. We refer to $a_x^{-1}(x)$, the fiber of this morphism over $x$, as the stabilizer $\mathrm{Stab}_G(x)$ of $x$, a closed subgroup of $G$ (see section 11.8 of [8]). The following lemma is well-known (see for instance Lemma 4 of [62]).

**Lemma 4.5.1.** *For $x \in X(k)$, the differential of the orbit map $a_x$ at the identity of $G$, $(da_x)_e : T_eG \to T_xX$, coincides with the natural evaluation map $\rho_a(\mathfrak{g}) \to T_xX$.*

The Lie superalgebra of $\mathrm{Stab}_G(x)$, which we write as $\mathfrak{stab}_{\mathfrak{g}}(x)$, is then exactly the kernel of the restriction morphism $\rho_a(\mathfrak{g}) \to T_xX$. In this way $T_xX$ is naturally a representation of the supergroup $\mathrm{Stab}_G(x)$, and thus also of $\mathfrak{stab}_{\mathfrak{g}}(x)$.

### 4.5.1 Orbits as homogeneous supervarieties

Suppose that $G$ acts on a supervariety $X$ and $x \in X(k)$. Then there is a factorization of the orbit map at $x$ given by

$$\begin{array}{ccc} G & \xrightarrow{\quad a_x \quad} & X \\ & {\scriptstyle \pi} \searrow \quad \nearrow {\scriptstyle b_x} & \\ & G/\mathrm{Stab}_G(x) & \end{array}$$

Further $b_x : G/\mathrm{Stab}_G(x) \to X$ is an immersion of supervarieties, and the image of its underlying topological space is locally closed in $X$. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be the sheaf of ideals given by the kernel of the pullback map $\mathcal{O}_X \to (a_x)_*\mathcal{O}_G$. We define the orbit closure of $x$ to be the closed subscheme determined by $\mathcal{I}$, and we write this as $\overline{G \cdot x}$. A dense open subscheme of $\overline{G \cdot x}$ is isomorphic to $G/\mathrm{Stab}_G(x)$ and thus is smooth. Therefore $\overline{G \cdot x}$ is a supervariety by proposition 3.3.10. However it need not be an integral supervariety, as the following example shows.

*Example* 4.5.2. The supergroup $GL(1|2)$ naturally acts on $k^{1|2}$ with basis $e, f_1, f_2$ by its tautological representation, and thus we obtain a representation of $GL(1|2)$ on $S^2(k^{1|2})^*$. Write $k[S^2(k^{1|2})^*] = S^\bullet(S^2k^{1|2}) = k[x, y, \xi, \eta]$, where $x$ is $e^2$, $y$ is $f_1f_2$, $\xi$ is $ef_1$, and $\eta$ is $ef_2$. For a superalgebra $A$ the map $GL(1|2)(A) \to GL(S^2(k^{1|2}))(A)$ is given by:

$$\begin{bmatrix} h & \alpha & \beta \\ \gamma & a & b \\ \delta & c & d \end{bmatrix} \mapsto \begin{bmatrix} h^2 & \alpha\beta & h\alpha & h\beta \\ 2\delta\gamma & ad - bc & c\gamma + a\delta & d\gamma + b\delta \\ 2h\gamma & b\alpha - a\beta & ha + \alpha\gamma & hb + \beta\gamma \\ 2h\delta & d\alpha - c\beta & hc + \alpha\delta & hd + \beta\delta \end{bmatrix}$$

where $h, a, b, c, d \in A_{\overline{0}}$ and $\alpha, \beta, \gamma, \delta \in A_{\overline{1}}$, and we also use these letters for the corresponding coordinate functions on $GL(1|2)$. Let us consider the orbit of the point $x = 1, y = 0$. The orbit morphism $a^*_{(1,0)} : k[X] \to k[G]$ sends

$$x \mapsto h^2, \quad y \mapsto \alpha\beta, \quad \xi \mapsto h\alpha, \quad \eta \mapsto h\beta$$

The orbit closure is the vanishing set of the ideal generated by this morphism. The generators of this ideal are $y^2, xy - \xi\eta, y\xi, y\eta$, so we see that the supervariety is not integral by example 3.3.6.

## 4.6 Linearization of quasi-coherent sheaves

**Definition 4.6.1.** Let $G$ be a supergroup, $X$ a $G$-supervariety, and $\mathcal{F}$ a quasi-coherent sheaf on $X$. Write $a : G \times X \to X$ for the action morphism and $p : G \times X \to X$ for the natural projection. A $G$-linearization of $\mathcal{F}$ is a choice of isomorphism of $\mathcal{O}_{G\times X}$-modules $\varphi_{\mathcal{F}} : a^*\mathcal{F} \cong p^*\mathcal{F}$ such that the following cocycle condition is satisfied:

$$(m_G \times \mathrm{id}_X)^*\varphi_{\mathcal{F}} = p_{23}^*\varphi_{\mathcal{F}} \circ (\mathrm{id}_G \times a)^*\varphi_{\mathcal{F}},$$

where $p_{23} : G \times G \times X \to G \times X$ is the projection onto the second and third factor. This equality is a slight abuse of notation, as it requires using canonical identifications of the functors $(\phi_1 \circ \phi_2)^*$ and $\phi_2^* \circ \phi_1^*$ for composable morphisms of supervarieties $\phi_1$ and $\phi_2$.

We will also refer to a linearized quasi-coherent sheaf as an equivariant quasi-coherent sheaf. A morphism of $G$-equivariant quasi-coherent sheaves is a morphism of quasi-coherent sheaves that respects their equivariant structure.

*Remark* 4.6.2. If we think in terms of geometric vector bundles, a linearization of a vector bundle $\pi : E \to X$ is an action of $G$ on $E$ as a supervariety, such that $\pi$ is a $G$-equivariant morphism and the induced morphism on fibers is linear.

We briefly recall the representation-theoretic constructions we obtain from a linearization, which work in the same way as they do classically. Given a $G$-equivariant quasi-coherent sheaf $\mathcal{F}$ on $X$, we obtain the structure of a $G$-representation on $\Gamma(X, \mathcal{F})$ with comodule morphism

$$\Gamma(X, \mathcal{F}) \xrightarrow{a^*} \Gamma(G \times X, a^*\mathcal{F}) \xrightarrow{\varphi_{\mathcal{F}}} \Gamma(G \times X, p^*\mathcal{F}) = k[G] \otimes \Gamma(X, \mathcal{F}).$$

The axioms of a comodule are satisfied exactly due to the cocycle conditions on $\varphi$.

We also obtain an action of $\mathfrak{g} = \operatorname{Lie} G$ on sections of $\mathcal{F}$ as follows. Let $U$ be an open subscheme of $X$. Then there exists an open subscheme $U'$ of $G$ containing $e_G$ such that $a$ maps $U' \times U$ into $U$, so that in particular $U' \times U \subseteq a^{-1}(U)$. For $u_e \in T_eG$ define the action by $u$ on $\Gamma(U, \mathcal{F})$ by

$$\Gamma(U, \mathcal{F}) \xrightarrow{a^*} \Gamma(a^{-1}(U), a^*\mathcal{F}) \xrightarrow{\varphi_{\mathcal{F}}} \Gamma(a^{-1}(U), p^*\mathcal{F}) \to \Gamma(U' \times U, p^*\mathcal{F}) \xrightarrow{-u_e \otimes 1} \Gamma(U, \mathcal{F}).$$

This action satisfies the following Liebniz property: for $u \in \mathfrak{g}$, $s$ a section of $\mathcal{F}$, and $f$ a section of $\mathcal{O}_X$, we have

$$u(fs) = u(f)s + (-1)^{\overline{f}\overline{u}} f u(s).$$

**Definition 4.6.3.** Let $\mathfrak{g}$ be a Lie superalgebra and $X$ a $\mathfrak{g}$-supervariety. Then a $\mathfrak{g}$-equivariant structure on a quasi-coherent sheaf $\mathcal{F}$ on $X$ is a morphism $\mathfrak{g} \to \underline{\mathrm{End}}(\mathcal{F})$ such that for $u \in \mathfrak{g}$, $s$ a section of $\mathcal{F}$, and $f$ a section of $\mathcal{O}_X$, we have

$$u(fs) = u(f)s + (-1)^{\overline{f}\overline{u}} f u(s).$$

Clearly a $G$-equivariant structure induces a $\operatorname{Lie} G$-equivariant structure.

*Example* 4.6.4. The structure sheaf $\mathcal{O}_X$ always has a natural equivariant structure such that the actions of $G$ and $\mathfrak{g}$ on sections agrees with the natural action coming from pullback of functions. Given a section $f$ of $\mathcal{O}_X$ and $u \in \mathfrak{g}$, we will write $u(f)$ for the corresponding action of $u$ on $f$ if no other $G$-linearization of $\mathcal{O}_X$ is present.

### 4.6.1 Action on line bundles

Let $\mathcal{L}$ be a $G$-equivariant line bundle on $X$. Everything we state will also work for $\mathfrak{g}$-equivariant line bundles. Let $U$ be an open subscheme on which $\mathcal{L}|_U$ is trivial under an isomorphism $\psi : \mathcal{L}|_U \to \mathcal{O}_U$. Then let $s = \psi^{-1}(1)$, so that every other sections is an $\mathcal{O}_U$-multiple of $s$. For a section $f$ of $\mathcal{O}_U$ and $u \in \mathfrak{g}$ we have

$$u(fs) = u(f)s + (-1)^{\overline{u}\overline{f}} f u(s).$$

Thus the action restricted to $U$ is determined by the sections $u(s)$ for $u \in \mathfrak{g}$. Using the isomorphism $\psi$, we may transport the action of $\mathfrak{g}$ on $\mathcal{L}|_U$ to an action of $\mathfrak{g}$ on $\mathcal{O}_U$, and for a section $f$ of $\mathcal{O}_U$ and $u \in \mathfrak{g}$ write $u_\psi(f)$ for this transported action. Then we obtain the formula

$$u_\psi(f) = u(f) + (-1)^{\overline{u}\overline{f}} f u_\psi(1). \tag{4.6.1}$$

Suppose that $s_1, s_2$ are homogeneous sections of $\mathcal{L}$ over $U$ such that $s_2$ is even and non-vanishing. Write $f = s_1/s_2 \in k(X)$ for the corresponding rational function on $X$. Then for $u \in \mathfrak{g}$ we have

$$u \cdot f = \frac{(u \cdot s_1)s_2 - (-1)^{\overline{u s_1}} s_1(u \cdot s_2)}{s_2^2},$$

where the action on the LHS comes from the natural action of $\mathfrak{g}$ on rational functions, and the action on the RHS comes from the described action of $\mathfrak{g}$ on rational section of $\mathcal{L}$. Further, on the RHS we take the quotient of the sections

$$(u \cdot s_1)s_2 - (-1)^{\overline{u s_1}} s_1(u \cdot s_2), \qquad s_2^2$$

of $\mathcal{L}^{\otimes 2}$. This formula is most easily shown by passing to a trivialization and using eq. (4.6.1).

### 4.6.2 Functorial constructions

Note that the following constructions may also be performed for $\mathfrak{g}$-equivariant sheaves.

Let $X$ be a $G$-supervariety, and suppose that $\mathcal{F}, \mathcal{G}$ are $G$-equivariant quasi-coherent sheaves on $X$. Then their tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ admits a natural equivariant structure as the tensor product of the isomorphisms $\varphi_{\mathcal{F}}$ and $\varphi_{\mathcal{G}}$.

Now suppose $Y$ is another $G$-supervariety and that $\phi : X \to Y$ is a $G$-equivariant morphism. Let $\mathcal{F}$ be a $G$-equivariant quasi-coherent sheaf on $Y$. Then the quasi-coherent sheaf $\phi^*\mathcal{F}$ has a natural $G$-equivariant structure given by

$$a_Y^*\phi^*\mathcal{F} \cong (1 \times \phi)^* a_X^* \mathcal{F} \xrightarrow{(1\times\phi)^*\varphi_{\mathcal{F}}} (1 \times \phi)^* p_X^* \mathcal{F} \cong p_Y^* \phi^* \mathcal{F}.$$

Assume that $\mathcal{G}$ is a $G$-equivariant quasi-coherent sheaf on $X$. By flat base change, we have natural isomorphisms $a_Y^*\phi_*\mathcal{G} \cong (1 \times \phi)^* a_X^* \mathcal{G}$, and $p_Y^*\phi_*\mathcal{G} \cong (1 \times \phi)^* p_X^* \mathcal{G}$. Thus $\phi_*\mathcal{G}$ has a natural $G$-equivariant structure given by

$$a_Y^*\phi_*\mathcal{G} \cong (1 \times \phi)_* a_X^* \mathcal{G} \xrightarrow{(1\times\phi)_*\varphi_{\mathcal{G}}} (1 \times \phi)_* p_X^* \mathcal{G} \cong p_Y^* \phi_* \mathcal{G}.$$

Further, these equivariant structures are such that for a $G$-equivariant quasi-coherent sheaf $\mathcal{F}$ on $Y$, the natural morphism

$$\mathcal{F} \to \phi_*\phi^*\mathcal{F}$$

is a morphism of $G$-equivariant sheaves.

### 4.6.3 Equivariant vector bundles on homogeneous supervarieties

Let $G$ be an algebraic supergroup, $K$ a closed subgroup, and consider the homogeneous space $G/K$. Then we have:

**Lemma 4.6.5.** *The category of $G$-equivariant vector bundles on $G/K$ is equivalent to the category of finite-dimensional $K$-modules.*

*Proof.* We think in terms of geometric vector bundles. Given a $G$-equivariant vector bundle $\pi : E \to G/K$, we obtain an action of $K$ on $\pi^{-1}(eK)$, defining a $K$-module.

Conversely, given $K$-module $V$ we may construct a vector bundle $\pi : G \times_K V := (G \times V)/K \to G/K$, where the action of $K$ on $G \times V$ is diagonal, and the morphism is the natural one coming from the universal property of the quotient. This construction is formally carried out in a book in progress on supergeometry by Musson and Serganova.

The fact that this correspondence respects morphisms is straightforward from the setup. □

*Remark* 4.6.6. Let $X_0$ be a normal variety with an action of an algebraic group $G_0$. Then there is a well-known theorem (see for instance [29]) which states that for any line bundle $\mathcal{L}$ on $X_0$ there exists $n > 0$ such that $\mathcal{L}^{\otimes n}$ admits a $G_0$-linearization. This implies, in particular, that if $X_0$ is quasi-projective it admits a very ample $G_0$-linearized line bundle, and thus $X_0$ has a $G_0$-equivariant embedding in $\mathbb{P}V$ for some $G_0$-module $V$.

Unfortunately this need not be true even for smooth supervarieties. Consider the projective superspace $\mathbb{P}^{m|n}$. Then there is a natural linear action of $G = GL(m+1|n)$ on it which realizes $\mathbb{P}^{m|n}$ as the homogeneous supervariety $G/P$ for a parabolic subgroup $P$. This parabolic subgroup has character group $\mathbb{Z}$, and thus by lemma 4.6.5 these characters index the $G$-equivariant line bundles on $\mathbb{P}^{m|n}$. On the other hand, in [7] it was shown that for $n \geq 2$ we have

$$\mathrm{Pic}(\mathbb{P}^{1|n}) = \mathbb{Z} \oplus \mathbb{C}^{2^{n-2}(n-2)+1}.$$

Thus for $\mathbb{P}^{1|n}$ there are many line bundles for which no tensor power is $G$-linearizable. In particular, given a quasi-projective $G$-supervariety $X$, a priori one may not be able to find a very ample linearizable line bundle on $X$; however the author has not yet found an example where this occurs.

## 4.7 Open orbits and invariants

### 4.7.1 Open orbits

**Definition 4.7.1.** Suppose that $G$ acts on $X$. We say that the action is a submersion at a point $x \in X(k)$ if the map $a_x : G \to X$ is a submersion at $e_G \in G(k)$ (or equivalently at every $k$-point of $G$). In this case, the locus of points where the action is a submersion will be an open subset of $|X|$, and we refer to the open subvariety defined by this locus as an open orbit of $G$. If all of $X$ is an open orbit of $G$, we say that $X$ is a homogeneous $G$-supervariety, and in this case we indeed have a $G$-equivariant isomorphism $X \cong G/K$ for some closed subgroup $K$ of $G$.

*Remark* 4.7.2. Note that an action is a submersion at $x$ if and only if the evaluation map $\mathfrak{g} \to \Gamma(|X|, \mathcal{T}_X) \to T_xX$ is surjective, by lemma 4.5.1. Further, by proposition 3.7.4, an open orbit of $G$ must be smooth.

**Proposition 4.7.3.** *Let $X$ be a supervariety, and let $a : G \times X \to X$ be an action of an algebraic supergroup $G$ on $X$. Then for $x \in X(k)$ the following are equivalent:*

1. *$a_x$ is a submersion;*
2. *the pullback morphism of sheaves $a_x^* : \mathcal{O}_X \to (a_x)_*\mathcal{O}_G$ is injective;*
3. *there exists a line bundle $\mathcal{L}$ such that the pullback morphism $a_x^* : \mathcal{L} \to (a_x)_*a_x^*\mathcal{L}$ is injective.*
4. *for all line bundles $\mathcal{L}$ on $X$, the pullback morphism $a_x^* : \mathcal{L} \to (a_x)_*a_x^*\mathcal{L}$ is injective.*

*Proof.* We first prove the equivalence (1) $\iff$ (2). Let $K$ be the stabilizer of $x$, and write $\pi : G \to G/K$ for the natural projection. Then the natural map of sheaves $\mathcal{O}_{G/K} \to \pi_*\mathcal{O}_G$ is injective. There is an induced $G$-equivariant immersion $b_x : G/K \to X$ and this map factors the orbit map $a_x$. Therefore if $a_x$ is a submersion, $G/K \to X$ is too, and hence it induces an isomorphism of $G/K$ onto an open subset of $X$. By our assumption that restriction of functions is injective on supervarieties, the map

$$\mathcal{O}_X \to b_*\mathcal{O}_{G/K} \to b_*\pi_*\mathcal{O}_G = (a_x)_*\mathcal{O}_G$$

is injective.

If $a_x$ is not a submersion, first suppose that the underlying image of $G/K$ in $X$ is not open. Then we may choose a non-nilpotent function on $X$ which vanishes on the underlying closed subscheme defined by its image, so that some power of this function will vanish under pullback, and thus $a_x^*$ is not injective. Therefore assume $G/K$ has an underlying open image, say $|U| \subseteq |X|$. Then we may restrict to the open subscheme $U$ of $X$, and there the morphism $G/K \to U$ will be an isomorphism on closed points and an immersion, but not a submersion. One may then show that this map is a closed embedding, by considering the map on local rings and using Nakayama's lemma. Hence the map on stalks is surjective, and so if it were also injective the map would be an isomorphism on this open set $U$, contradicting the fact that $a_x$ is not submersive.

Now we show that (2) $\iff$ (3) $\iff$ (4). For any line bundle $\mathcal{L}$ on $X$, we may cover $X$ with open sets $U_i$ for which $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$. Then by functoriality, over $U_i$ the pullback morphism $\mathcal{L} \to (a_x)_*(a_x)^*\mathcal{L}$ is identified with $\mathcal{O}_{U_i} \to (a_x)_*(a_x)^*\mathcal{O}_{U_i} \cong (a_x)_*\mathcal{O}_{U_i}$, and one is injective if and only if the other is. Since injectivity of a morphism of sheaves is a local property, we are done.

□

**Proposition 4.7.4.** *Let $X$ be a $G$-supervariety. If $X$ is homogeneous then for any $G$-equivariant line bundle $\mathcal{L}$, a non-zero $G$-submodule $V$ of $\Gamma(X, \mathcal{L})$ generates $\mathcal{L}$ in the sense of definition 3.4.3. Conversely, if $X$ is quasi-projective and admits a very ample $G$-equivariant line bundle $\mathcal{L}$, then the converse also holds.*

*Proof.* Let $\mathcal{L}$ be a $G$-equivariant line bundle on $X$, and let $V \subseteq \Gamma(X, \mathcal{L})$ be a nonzero $G$-submodule. Then if $V$ does not generate $\mathcal{L}$, there must exist a point $x \in X(k)$ such that when we pass to the stalk of $\mathcal{L}$ at $x$ we find that $V \subseteq \mathfrak{m}_x\mathcal{L}_x$. Therefore there exists a maximal positive integer $n$ such that $V \subseteq \mathfrak{m}_x^n\mathcal{L}_x$.

Let $s \in V$ be in $\mathfrak{m}_x^n\mathcal{L}_x \setminus \mathfrak{m}_x^{n+1}\mathcal{L}_x$. Choose a trivialization $\psi : \mathcal{L}_x \cong \mathcal{O}_x$ and write $f = \psi(s)$ so that $f \in \mathfrak{m}_x^n \setminus \mathfrak{m}_x^{n+1}$. Then because $X$ is homogeneous there exists $u \in \mathfrak{g}$ such that $u(f) \in \mathfrak{m}_x^{n-1} \setminus \mathfrak{m}_x^n$. Therefore

$$u_\psi(f) = u(f) + (-1)^{\overline{f}\overline{u}} f u_\psi(1) \in \mathfrak{m}_x^{n-1} \setminus \mathfrak{m}_x^n,$$

so in particular $u(s) \in \mathfrak{m}_x^{n-1}\mathcal{L}_x \setminus \mathfrak{m}_x^n\mathcal{L}_x$ and $u(s) \in V$, a contradiction. Therefore instead $V$ must generate $\mathcal{L}$.

Now let $X$ be quasi-projective admitting a very ample $G$-equivariant line bundle $\mathcal{L}$, and suppose that $X$ is not homogeneous. Then there exists $x \in X(k)$ such that $a_x$ is not a submersion. By proposition 4.7.3, the pullback morphism $\mathcal{O}_X \to (a_x)_*\mathcal{O}_X$ is not injective. Write $\mathcal{K}$ for its kernel, so that we obtain an exact sequence of $G$-equivariant sheaves

$$0 \to \mathcal{K} \to \mathcal{O}_X \to (a_x)_*\mathcal{O}_X.$$

Since $\mathcal{L}$ is $G$-equivariant and flat as an $\mathcal{O}_X$-module, we may twist by it and obtain for each $n \in \mathbb{N}$ an exact sequence of $G$-equivariant sheaves

$$0 \to \mathcal{K}(n) \to \mathcal{L}^{\otimes n} \to (a_x)_*\mathcal{O}_X \otimes \mathcal{L}^{\otimes n},$$

where we write $\mathcal{K}(n) := \mathcal{K} \otimes \mathcal{L}^{\otimes n}$. Since $\mathcal{L}$ is very ample, by proposition 3.5.7 there exists $n > 0$ such that $\mathcal{K}(n)$ is globally generated, and so in particular $\Gamma(X, \mathcal{K}(n)) \neq 0$. However by left exactness of global sections, $\Gamma(X, \mathcal{K}(n)) \subseteq \Gamma(X, \mathcal{L}^{\otimes n})$ is a non-zero $G$-submodule. Since the morphism $\mathcal{O}_X \to (a_x)_*\mathcal{O}_x$ is not zero, necessarily $\Gamma(X, \mathcal{K}(n))$ cannot generate $\mathcal{L}^{\otimes n}$, a contradiction, and we are done. □

*Remark* 4.7.5. Let $\mathfrak{g}$ be a Lie superalgebra and suppose it acts homogeneously on a supervariety $X$ (see definition 5.4.1 for the meaning of this). Then the proof of proposition 4.7.4 shows that if $\mathcal{L}$ is a $\mathfrak{g}$-equivariant line bundle and $V$ is a non-zero $\mathfrak{g}$-stable submodule of $\Gamma(X, \mathcal{L})$ then $V$ generates $\mathcal{L}$.

**Corollary 4.7.6.** *If $X$ is an affine $G$-supervariety and $\mathcal{L}$ is a $G$-equivariant line bundle on $X$, then $\Gamma(X, \mathcal{L})$ admits a non-zero $G$-stable $\Gamma(X, \mathcal{O}_X)$-submodule if and only if $X$ is not homogeneous. In particular, $k[X]$ has no nontrivial $G$-stable ideals if and only if $X$ is homogeneous.*

*Proof.* We apply proposition 4.7.4, using that if $X$ is affine the global sections functor is exact. □

## 4.7.2 Rational invariants

In the classical world, if an algebraic group $G_0$ acts on a normal variety $X_0$, then it admits an open orbit if and only if $k(X_0)^{\mathrm{Lie}\, G_0} = k$. In the super world, this general principle no longer holds.

*Example* 4.7.7. Consider the action of $G = GL(0|n)$ on $X = k^{0|n}$ by the standard representation of $GL(0|n)$. This supervariety has one point, and the orbit of that point is not open, so $G$ does not have an open orbit on $X$. However we have $k(X) = \Lambda^\bullet(k^n)^*$, and this is a multiplicity-free representation of $\mathfrak{g} = \mathfrak{gl}(n)$, so in particular $k(X)^{\mathfrak{g}} = k$.

We do have the forward direction:

**Proposition 4.7.8.** *If a Lie supergroup $G$ acts on a supervariety $X$ with an open orbit, we have $k(X)^{\mathfrak{g}} = k$.*

*Proof.* Let $f \in k(X)^{\mathfrak{g}}$ be non-zero, and choose an affine open subvariety $\operatorname{Spec} A$ of $X$ contained in the open orbit of $G$ on which $f$ is regular. Then $A$ has no non-trivial $\mathfrak{g}$-stable ideals by corollary 4.7.6 and the remark following it. Therefore $(f) = A$, so $f$ is non-vanishing on $A$. However, if $x \in \operatorname{Spec} A(k)$, then $f - f(x)$ is $\mathfrak{g}$-fixed and vanishes at $x$, i.e. is not invertible, so $(f - f(x))$ is a $\mathfrak{g}$-stable ideal not equal to $A$. Thus it must be trivial, i.e. $f = f(x)$, so $f$ is a constant function. □

We may state a converse for certain algebraic subgroups. Suppose that $\mathfrak{b}$ is a solvable Lie superalgebra such that $[\mathfrak{b}_{\overline{1}}, \mathfrak{b}_{\overline{1}}] \subseteq [\mathfrak{b}_{\overline{0}}, \mathfrak{b}_{\overline{0}}]$. Then by lemma 1.37 of [10], every finite-dimensional irreducible representation of $\mathfrak{b}$ is one-dimensional.

If $V$ is a representation of $\mathfrak{b}$, we write $V^{(\mathfrak{b})}$ for the span of the $\mathfrak{b}$-eigenvectors of $V$, which will be a semisimple representation of $\mathfrak{b}$. Write $\Lambda_{\mathfrak{b}}(V)$ for the collection of characters $\lambda$ of $\mathfrak{b}$ such that there is a $\mathfrak{b}$-eigenvector of weight $\lambda$ in $V$. Finally, if $\mathfrak{b}$ acts on a supervariety $X$, set

$$\Lambda_{\mathfrak{b}}^{+}(X) := \Lambda_{\mathfrak{b}}(k[X]), \qquad \Lambda_{\mathfrak{b}}(X) := \Lambda_{\mathfrak{b}}(k(X)).$$

Observe that if $A$ is a superalgebra on which $\mathfrak{b}$ acts by derivations, then $A^{(\mathfrak{b})}$ is a subalgebra of $A$.

**Definition 4.7.9.** If $G$ is quasi-reductive, $X$ a $G$-supervariety, and $B$ a hyperborel of $G$, we set $\Lambda_B^{+}(X) := \Lambda_{\mathfrak{b}}^{+}(X)$ and $\Lambda_B(X) := \Lambda_{\mathfrak{b}}(X)$ (or simply $\Lambda^{+}(X)$, resp. $\Lambda(X)$ when there is no confusion).

**Proposition 4.7.10.** *Let $B$ be a solvable connected algebraic supergroup such that $[\mathfrak{b}_{\overline{1}}, \mathfrak{b}_{\overline{1}}] \subseteq [\mathfrak{b}_{\overline{0}}, \mathfrak{b}_{\overline{0}}]$ where $\mathfrak{b} = \operatorname{Lie}(B)$. Let $X$ be a $B$-supervariety. If $k(X)^{(\mathfrak{b})}$ is a multiplicity-free $\mathfrak{b}$-representation such that every non-zero $f \in k(X)^{(\mathfrak{b})}$ is non-nilpotent, then $X$ has an open $B$-orbit. Equivalently, $k(X)^{\mathfrak{b}} = k$ and $k(X)^{(\mathfrak{b})}$ is an integral domain.*

*Proof.* Write $\Lambda$ for the character lattice of $B$, a finitely generated free abelian group. By our assumptions, $k(X)^{(\mathfrak{b})}$ is isomorphic to the group algebra of a subgroup $\Lambda(X)$ of $\Lambda$, and hence $\Lambda(X)$ is free of some rank, say $n \in \mathbb{N}$. Choose rational $B$-eigenfunctions $f_1, \dots, f_n \in k(X)^{(\mathfrak{b})}$ that such that their weights form a $\mathbb{Z}$-basis of $\Lambda(X)$. Then by removing the divisors of zeroes and poles of $f_1, \dots, f_n$, there exists a $B$-stable open subvariety $U$ of $X$ where $f_1, \dots, f_n$ are regular and non-vanishing, and hence $k(X)^{(\mathfrak{b})} \subseteq k[U]$. We may shrink $U$ further to assume that $U_0$ is normal, and we still may assume $U$ is $B$-stable. Now apply Sumihiro's theorem (see for instance [29]) using normality of $U_0$ and corollary 4.2.12 to find a $B$-stable affine open subvariety $U'$ of $U$.

Now we claim that $U'$ is a homogeneous $B$-supervariety. Indeed, if $I \subseteq k[U']$ is a nontrivial $B$-stable ideal, then it admits a $B$-eigenfunction $f \in I$. Then $f \in k(X)^{(\mathfrak{b})}$, so by assumption $f$ is invertible on $U$, and $k[U'] = (f) = I$. We conclude by corollary 4.7.6. □

## 4.8 $G_0$-equivariant gradings of supervarieties

**Definition 4.8.1.** Let $G_0$ be an algebraic group. If $X$ is a $G_0$-supervariety, then we say it has a $G_0$-equivariant grading if there exists a $G_0$-equivariant sheaf $\mathcal{M}$ on $X_0$ and a $G_0$-equivariant isomorphism $X \cong (X_0, \Lambda^\bullet\mathcal{M})$.

We seek to prove:

**Theorem 4.8.2.** *Let $G_0$ be a reductive group and $X$ a $G_0$-supervariety. If $X$ is graded, then $X$ admits a $G_0$-equivariant grading. In particular, if $G$ is quasi-reductive and $X$ is a $G$-supervariety which is graded, then $X$ admits a $G_0$-equivariant grading.*

This question was considered by Rothstein in [43], in the analytic setting. We adapt the ideas of his proof to the algebraic setting.

*Proof.* Since $X$ is graded, we assume that $X = (X_0, \Lambda^\bullet\mathcal{M})$ where $\mathcal{M}$ is a rank $n$ vector bundle on $X_0$. Then the structure sheaf $\mathcal{O}_X$ admits a natural $\mathbb{Z}$-grading by degree of exterior power.

First we study the group $\mathrm{Aut}(X)$. We write $\mathrm{Aut}(\mathcal{M})$ for the vector bundle automorphisms of $\mathcal{M}$, which for us will mean the data of a pair $(\phi, \psi)$ where $\phi$ is an automorphism the scheme $X_0$ and $\psi : \phi^*\mathcal{M} \to \mathcal{M}$ is an isomorphism of coherent sheaves. Then given $\Phi = (\phi, \psi) \in \mathrm{Aut}(\mathcal{M})$ we obtain an automorphism of $X$ by $\Lambda^\bullet\Phi$, which acts by $\phi$ on the underlying scheme $X_0$ and by $\Lambda^\bullet\psi$ on the sheaf of algebras, where $\Lambda^0\psi = \phi^*$. This isomorphism preserves the $\mathbb{Z}$-grading on the sheaf of algebras $\mathcal{O}_X$. Hence we have defined an inclusion $\mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(X)$.

On the other hand, given an automorphism $\phi$ of $X$, $\phi$ preserves $(\mathcal{O}_{X,\bar{1}})$, and thus every power of it as well, so it induces an automorphism $\widetilde{\phi}$ of the vector bundle $\mathcal{N}_X = \mathcal{M}$. The map $\phi \mapsto \widetilde{\phi}$ splits the inclusion $\mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(X)$. Given $\phi \in \mathrm{Aut}(X)$, we denote by abuse of notation $\widetilde{\phi} \in \mathrm{Aut}(X)$ to be the map gotten by applying these two morphisms.

Now let $\mathcal{T}_X^+ \subseteq \mathcal{T}_X$ be the subsheaf of $\mathcal{T}_X$ consisting of those derivations which increase the $\mathbb{Z}$-grading of a section of $\mathcal{O}_X$. This is a subsheaf of Lie superalgebras. Note we have a decomposition $\mathcal{T}_X^+ = \bigoplus \mathcal{T}_X^i$, where a section $u$ of $\mathcal{T}_X^i$ sends degree one elements into degree $i+1$. In particular only even degrees can show up. Then given $u \in \Gamma(X, \mathcal{T}_X^+)$ we obtain an automorphism $e^u := \exp(u) \in \mathrm{Aut}(X)$ defined in the natural way. Let $N$ be the subgroup of $\mathrm{Aut}(X)$ generated by these exponentials, which is precisely the exponentials themselves since the group is unipotent.

**Lemma 4.8.3.** *We have a short exact sequence of groups*

$$1 \to N \to \mathrm{Aut}(X) \to \mathrm{Aut}(\mathcal{M}) \to 1.$$

*Proof.* Clearly the first map is injective and the last map is surjective, and their composition is trivial. So it suffices to show that if $\phi \in \mathrm{Aut}(X)$ such that $\widetilde{\phi}$ is the identity automorphism of $\mathcal{M}$, then $\phi \in N$. But in this case, $\phi^* - \mathrm{id} : \phi^*\mathcal{O}_X \to \mathcal{O}_X$ is nilpotent, hence $\log(\phi) = \log(1 - (1 - \phi)) \in \Gamma(X, \mathcal{T}_X^+)$. Taking the exponential gives $\phi$, and we are done. □

Using the splitting $\mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(X)$ we obtain:

**Corollary 4.8.4.** *We have a decomposition* $\mathrm{Aut}(X) = N \ltimes \mathrm{Aut}(\mathcal{M})$.

Now let $G_0$ be an algebraic group, and suppose it acts on a supervariety $X$, inducing a map of groups $T : G_0(k) \to \mathrm{Aut}(X)$. Then composing with $\mathrm{Aut}(X) \to \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(X)$, we get a new action, written $\widetilde{T}$, of $G_0$ on $X$, which is still algebraic. Further, the action of elements under $\widetilde{T}$ preserves the $\mathbb{Z}$-grading on $X$. Hence it suffices to show that $T$ is conjugate to $\widetilde{T}$ by an element of $\mathrm{Aut}(X)$. We will in fact show that under our assumption that $G_0$ is reductive, $T$ is conjugate to $\widetilde{T}$ by an element of $N$ due to the following proposition.

**Proposition 4.8.5.** *If* $\mathrm{Ext}^1_{G_0}(k, \Gamma(X, \mathcal{T}_X^+)) = 0$*, i.e. if there are no nontrivial extensions of* $\Gamma(X, \mathcal{T}_X^+)$ *by the trivial module in the category of rational representations of* $G_0$*, then* $T$ *is conjugate to* $\widetilde{T}$ *by an element of* $N$*.*

Note that $\Gamma(X, \mathcal{T}_X^+)$ is a rational $G_0$-module, being a $G_0$-submodule of the global sections of the tangent sheaf, which is a $G_0$-equivariant vector bundle.

*Proof.* Recall that for a rational $G_0$-module $V$ there is an identification between $\mathrm{Ext}^1_{G_0}(k, V)$ and algebraic cocycles modulo coboundaries. An algebraic cocycle for us means a morphism of varieties $\psi : G_0 \to V$ that satisfies $\psi(gh) = \psi(g) + g \cdot \psi(h)$ for $g, h \in G_0(k)$, and a coboundary is such a morphism given by $\psi(g) = g \cdot v - v$ for $g \in G_0(k)$ and for some $v \in V(k)$.

Before constructing a cocyle, we make a definition. Given $g \in G_0(k)$ and an $N$-conjugate $T'$ of $T$, write $T'(g) = e^{u(g)}\widetilde{T}(g)$, using our splitting, for a unique element $u(g) \in \Gamma(X, \mathcal{T}_X^+)$. Define the order of $T'(g)$ to be the largest $\ell$ such that $u_j(g) = 0$ for $j \leq \ell$. Then define the order of $T'$ the minimal order of $T'(g)$ for $g \in G_0(k)$. Define the order of the $N$-conjugacy class of $T$ to be the supremum of the orders over all $N$-conjugates of $T$. Then $T$ is conjugate to $\widetilde{T}$ by an element of $N$ if and only if the order of $T$ is infinite.

Therefore suppose the order of $T$ is finite, and let $T'$ be a chosen $N$-conjugate of maximal order $\ell$. Then for this conjugate we obtain a well-defined function $u_{\ell+1} : g \mapsto u_{\ell+1}(g) \in \Gamma(X, \mathcal{T}_X^{\ell+1})$, which does not vanish by definition. This defines a morphism of varieties $G_0 \to \Gamma(X, \mathcal{T}_X^{\ell+1})$, as $u_{\ell+1}(g)$ is obtained as a projection of $T'(g)\widetilde{T}'(g)^{-1}$ onto a subspace of $\mathrm{End}(V)$.

We notice that

$$T'(gh) = e^{u(g)}\widetilde{T}(g)e^{u(h)}\widetilde{T}(h) = e^{u(g)}\widetilde{T}(g)e^{u(h)}\widetilde{T}(g)^{-1}\widetilde{T}(g)\widetilde{T}(h),$$

so

$$u_{\ell+1}(gh) = u_{\ell+1}(g) + \widetilde{T}(g)u_{\ell+1}(h)\widetilde{T}(g)^{-1},$$

hence $u_{\ell+1}$ is a an algebraic cocycle of $G_0$ with values in $\Gamma(X, \mathcal{T}_X^{\ell+1})$.

Now suppose that $v_{\ell+1} \in \Gamma(X, \mathcal{T}_X^{\ell+1})$. Then conjugating $T'$ by $e^{v_{\ell+1}}$, we get

$$e^{v_{\ell+1}}e^{u(g)}\widetilde{T}(g)e^{-v_{\ell+1}} = (e^{v_{\ell+1}}e^{u(g)}\widetilde{T}(g)e^{-v_{\ell+1}}\widetilde{T}(g)^{-1})\widetilde{T}(g).$$

It is not hard to check that the order of $e^{v_{\ell+1}}T'e^{-v_{\ell+1}}$ cannot be less than $\ell$, so by definition of $\ell$ the order must remain $\ell$. Hence we find that the cocycle defined by this conjugate of $T$ is given by $g \mapsto u_{\ell+1}(g) + v_{\ell+1} - \widetilde{T}(g)v_{\ell+1}\widetilde{T}(g)^{-1}$, i.e. it differs from the cocycle determined by $T'$ by the coboundary determined by $v_{\ell+1}$.

Now we may finish the proof of the proposition. We have seen that $u_{\ell+1}$ is an algebraic one-cocyle of $G_0$ with values in $\Gamma(X, \mathcal{T}_X^{\ell+1})$. By our assumption that $\mathrm{Ext}^1_{G_0}(k, \Gamma(X, \mathcal{T}_X^+)) = 0$, we also have $\mathrm{Ext}^1_{G_0}(k, \Gamma(X, \mathcal{T}_X^{\ell+1})) = 0$ since it is a direct summand. Therefore there exists $v_{\ell+1} \in \Gamma(X, \mathcal{T}_X^{\ell+1})$ such that $e^{v_{\ell+1}}T'e^{-v_{\ell+1}}$ has order larger than $\ell$. This contradicts the definition of $\ell$, so we must have instead that $\ell = n$, i.e. $T' = \widetilde{T}$, and we are done. □

Now we finish the proof of the theorem. If $G_0$ is reductive, then its category of rational representations is semisimple, so $\mathrm{Ext}^1_{G_0}(k, \Gamma(X, \mathcal{T}_X^+)) = 0$. We may now apply proposition 4.8.5, and we are done. □

# Chapter 5

# Spherical supervarieties

In this chapter we define the notion of a spherical supervariety. Three characterizations are given, one in the general case, one in the affine case, and one in the quasi-projective case. Several important examples are then discussed.

## 5.1 Spherical supervarieties

Let $G$ be a quasi-reductive supergroup.

**Definition 5.1.1.** We say a $G$-supervariety $X$ is spherical if there exists a hyperborel subgroup $B$ of $G$ with an open orbit on $X$. If a hyperborel subgroup $B$ has an open orbit on $X$, we say that $X$ is $B$-spherical.

*Remark* 5.1.2.

- If a $G$-supervariety $X$ is spherical, then the $G_0$-variety $X_0$ is also spherical.
- Note that a spherical supervariety need not be spherical with respect to every hyperborel; in fact if $G$ is basic and distinguished this occurrence would be a degeneracy.

**Definition 5.1.3.** If $K$ is a closed subgroup of $G$ such that $G/K$ is spherical, then we say that $K$ is a spherical subgroup of $G$. Similarly, if $\mathfrak{k}$ is a Lie subalgebra of $\operatorname{Lie} G$ such that there exists a hyperborel subalgebra $\mathfrak{b}$ with $\mathfrak{b}+\mathfrak{k}=\mathfrak{g}$, then we say $\mathfrak{k}$ is a spherical subalgebra of $\mathfrak{g}$.

**Theorem 5.1.4.** *Let $G$ be quasi-reductive, $B$ a hyperborel of $G$, and $X$ a $G$-supervariety. Then $X$ is $B$-spherical if and only if $k(X)^{(\mathfrak{b})}$ is a multiplicity-free $\mathfrak{b}$-module whose nonzero elements are non-nilpotent, where $\mathfrak{b}=\operatorname{Lie} B$. Equivalently, $k(X)^{\mathfrak{b}}=k$ and $k(X)^{(\mathfrak{b})}$ is an integral domain.*

*Proof.* This follows immediately from proposition 4.7.10. □

## 5.2 Affine spherical supervarieties

In the classical case we have a characterization of affine spherical varieties which states that $X$ is spherical if and only if $k[X]$ is a multiplicity-free representation (theorem 1.2.1). One might hope that this generalizes to the super case. Of course there is a first issue that for supergroups complete reducibility is a rare phenomenon to begin with. But one might hope that perhaps $k[X]^N$ being multiplicity-free as a $T$-module is sufficient, where $N$ is the unipotent radical of a hyperborel subgroup $B$ and $T$ is a maximal torus of $B$. This turns out to not be the case as the next examples demonstrate.

*Example* 5.2.1.

- Consider the action of $GL(0|n)$ on $k^{0|n}$ by the standard representation. The algebra of functions is $\Lambda^\bullet(k^n)^*$, which is completely reducible and multiplicity-free. However, there is only one closed point and the orbit of it under the whole group is itself, so this space is not spherical.
- An example which has a nontrivial even part is given by considering $G = OSP(1|2)$ and letting $X = OSP(1|2)/T$, where $T$ is a maximal torus of $G_0$. By the representation theory of $OSP(1|2)$ and Frobenius reciprocity, $k[X] \cong \bigoplus_{n\geq 0} \Pi^n L(n)$, where $L(n)$ is the irreducible representation of highest weight $n$ with even highest weight vector. Hence $k[X]$ is completely reducible and multiplicity-free. However, no hyperborel admits an open orbit since the odd dimension of $X$ is 2 while the odd dimension of any hyperborel subgroup is 1.

The next theorem demonstrates that the issue with the above two spaces is that some of the highest weight functions are nilpotent.

**Theorem 5.2.2.** *Let $X$ be an affine $G$-supervariety, $B$ a hyperborel of $G$ with maximal unipotent subgroup $N$ and maximal torus $T$. Then the following are equivalent:*

1. *$X$ is spherical for $B$.*
2. *$X_0$ is spherical for $B_0$, and every nonzero $B$-highest weight function in $k[X]$ is non-nilpotent.*
3. *Every nonzero $B$-highest weight function in $k[X]$ is non-nilpotent, and $\dim k[X]^N_\lambda \leq 1$ for all weights $\lambda$ of $T$.*
4. *$k[X]^N$ is an even commutative algebra without nilpotents, and the natural $T$-action is multiplicity-free.*

*Proof.* (1) $\Longrightarrow$ (2): Let $x \in X(k)$ be such that $a_x : B \to X$ is a submersion, so that $a_x^*$ is injective. In $k[B]$, all $B$-highest weight functions are non-nilpotent, and therefore the same must be true of the functions on $X$.

(2) $\Longrightarrow$ (3): Since $X_0$ is spherical for $B_0$, we have $\dim k[X_0]^{(\mathfrak{b}_{\overline{0}})}_\lambda \leq 1$ for all $\lambda$. Since the $B$-highest weight functions are non-nilpotent, the restriction map $k[X]^{(\mathfrak{b})} \to k[X_0]^{(\mathfrak{b}_{\overline{0}})}$ is injective, and we are done.

(3) $\Longrightarrow$ (4): We see that $k[X]^N$ is the subalgebra generated by the $B$-highest weight functions, so this is clear.

(4) $\Longrightarrow$ (1): Let $S$ be the submonoid of the character lattice of $T$ determined by the weights of $k[X]^N$. Then because the group generated by $S$ is finitely generated, of rank say $m$, there exists weights $\lambda_1, \ldots, \lambda_m \in S$ such that the monoid generated by $S$ and $-\lambda_1, \ldots, -\lambda_m$ is a group. Then if we let $U = D(f_{\lambda_1} \cdots f_{\lambda_m})$, all $B$-eigenfunctions in $k[U]$ will be invertible. Further, this open subscheme $U$ will be $B$-stable. Choose a point $x \in U(k)$, and consider the orbit map $a_x : B \to X$. Since all $f_\lambda$ become units on $U$, they must not be in the kernel of $a_x^*$. But if $a_x^*$ is not injective, the kernel will contain a $B$-highest weight function, a contradiction. Therefore $a_x$ must be a submersion by proposition 4.7.3, and so $X$ is spherical. □

**Definition 5.2.3.** If a $G$-supervariety $X$ is $B$-spherical, define the rank of $X$ to be the rank of the lattice $\Lambda_B(X)$.

A corollary of the proof of the above proposition is the following.

**Corollary 5.2.4.** *If $X$ is $B$-spherical of rank $m$, there exists $m$ $B$-highest weight functions $f_{\lambda_1}, \ldots, f_{\lambda_m} \in k[X]$ such that their common non-vanishing set is the open $B$-orbit.*

**Corollary 5.2.5.** *If $X$ is spherical, the socle of $k[X]$ is multiplicity-free.*

*Proof.* Suppose that an irreducible representation $L$ shows up with multiplicity greater than 1. If $B$ is a hyperborel for which $X$ is $B$-spherical, there will be two $B$-eigenfunctions of the same weight in $k[X]$. This contradicts (3) of theorem 5.2.2. □

Now suppose that $X$ is an affine $B$-spherical supervariety and $U$ is the open $B$-orbit. By the reasoning given in the proof of proposition 4.7.10, we know that all rational $\mathfrak{b}$-eigenfunctions will be regular (and in fact non-vanishing) on $U$. Hence $k[U]^N = k(X)^{(\mathfrak{b})}$, and because these functions are all non-nilpotent we have

$$k(X)^{(\mathfrak{b})} = k[U]^N \cong k[U_0]^{N_0} = k(X_0)^{\mathfrak{b}_{\overline{0}}}$$

by restriction of functions. Further, these algebras are all isomorphic to group algebras on $\Lambda_B(X)$, a finitely generated free abelian subgroup of the character lattice of $T$.

Now on all of $X$, restriction induces an injective map $k[X]^N \to k[X_0]^{N_0}$, and hence an inclusion $\Lambda_B^+(X) \subseteq \Lambda_{B_0}^+(X_0)$, and thus $\Lambda_B^+(X)$ will be a submonoid of $\Lambda_{B_0}^+(X_0)$. Note that $k[X]^N$ is the monoid algebra on $\Lambda_B^+(X)$ and $k[X_0]^{N_0}$ is the monoid algebra on $\Lambda_{B_0}^+(X_0)$.

It is a classical fact about spherical varieties that $k[X_0]^{N_0}$ is finitely generated, so choose generators $g_1, \ldots, g_n$ which are $\mathfrak{b}_{\overline{0}}$-eigenfunctions. Note that $U_0$ is precisely the non-vanishing locus of these functions. We may uniquely lift these to $\mathfrak{b}$-eigenfunctions $f_1, \ldots, f_n$ on $U$.

Let us now assume in addition that $X$ is graded (equivalently locally graded since $X$ is affine). By theorem 4.8.2 we may choose a $G_0$-equivariant grading of $X$. Thus we may write $k[X] = \Lambda^\bullet M$, where $M$ is a finitely generated $G_0$-equivariant $k[X_0]$-module. Let us assume the largest non-zero exterior power of $M$ is $q$. Then we may write

$$f_i = g_i + m_{i1} + \cdots + m_{iq} \text{ where } m_{ij} \in \Lambda^j M_{g_1 \cdots g_n}.$$

Here $M_{g_1\cdots g_n}$ is the localization of $M$ to the non-vanishing locus of $g_1,\dots,g_n$. We may do this because since $f_i$ is a $\mathfrak{b}$-eigenvector, each $m_{ij}$ must be a $\mathfrak{b}_{\overline{0}}$-eigenvector and it must be regular on the open $B_0$-orbit. Now the obstruction to regularity of $f_i$ is the poles of $m_{ij}$ along $g_1\cdots g_n=0$. For each $m_{ij}$, there exists $q_{ij}\in\mathbb{N}$ such that $(g_1\cdots g_k)^{q_{ij}}m_{ij}\in\Lambda^j M$. By choosing an integer $p$ larger than $q+\max\limits_{i,j}q_{ij}$, we now have:

**Proposition 5.2.6.** *If $X$ is graded then there exists an integer $p>0$ such that $f_1^{r_1}\cdots f_n^{r_n}$ is regular whenever $r_1,\dots,r_n\geq p$.*

*Proof.* Expanding out the product, one sees that for any integer $p$ chosen as described in the paragraph before the proposition, the poles will be resolved. □

**Corollary 5.2.7.** *If $X$ is graded then the set $\Lambda_B^+(X)$, which is a submonoid of $\Lambda_{B_0}^+(X_0)$, generates $\Lambda_B(X)=\Lambda_{B_0}(X_0)$ as a group. Further it is Zariski dense in the vector space spanned by its weights.*

*Proof.* By proposition 5.2.6, $\Lambda_B^+(X)$ contains the lattice points of a translated orthant of $\mathbb{R}\otimes_{\mathbb{Z}}\Lambda_B(X)$, and so the results follow. □

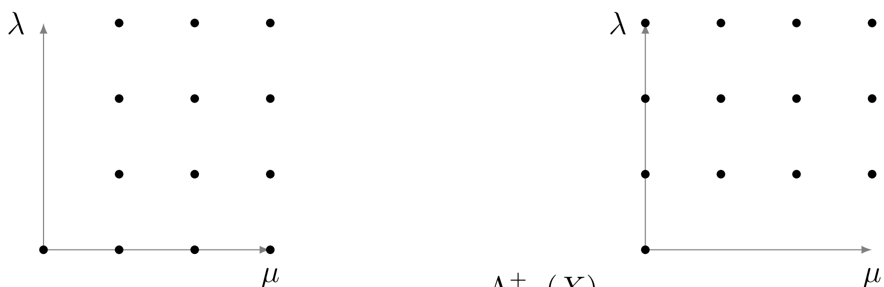
Write $X//N:=\operatorname{Spec}k[X]^N$. Then by (4) of theorem 5.2.2, $X//N$ is an even variety and admits a natural $T$-action such that $k[X//N]$ is a multiplicity-free $T$-module. In particular, $X//N$ has an open $T$-orbit, hence is essentially a toric variety but that it need not be normal or Noetherian. Indeed, we observe it is isomorphic to the group algebra of $\Lambda_B^+(X)$, so being normal is equivalent to this monoid being saturated, and being Noetherian is equivalent to the monoid being finitely generated. We now present examples showing how these properties can fail.

*Example* 5.2.8. Consider the action of $G=GL(1|2)$ on $X=S^2k^{1|2}$ as the second symmetric power of the standard representation. This is a spherical supervariety as one can check (see appendix A), and is spherical exactly with respect to the hyperborels $B^+$ and $B^-$ of upper and lower triangular matrices, respectively. The coordinate ring $k[X]$ is a supersymmetric polynomial algebra given by $S^\bullet(S^2(k^{1|2})^*)$ as both an algebra and a $G$-module.

As a $G_0=GL(1)\times GL(2)$-representation $X_0$ is a sum of two one-dimensional representations of distinct weights. Therefore the $B_0$-highest weight functions of $X_0$ are the monomials in two $G_0$-eigenfunctions $x,y$, where we let $x$ have weight $\lambda$ and $y$ have weight $\mu$. Let $\xi,\eta\in(S^2k^{1|2})^*_{\overline{1}}$ be odd weight vectors of weights $\alpha,\beta$. Then $k[X]=k[x,y,\xi,\eta]$. We have that $\xi\eta$ is a $G_0$-eigenvector of weight $\lambda+\mu$, and so one can show that for any hyperborel $B$ the rational $B$-eigenfunctions on $X$ are, up to scalar, all of the form:

$$f_{ij}=x^iy^j+c_{ij}x^iy^j\frac{\xi\eta}{xy}$$

where $i,j\in\mathbb{Z}$ and $c_{ij}\in k$ is a coefficient in $k$ to be determined depending on the choice of hyperborel. For the hyperborel $B^+$, we find that $c_{ij}=i$ and for $B^-$ we find that $c_{ij}=-j$. These values for $c_{ij}$ tell us which rational $B$-eigenfunctions are regular on all of $X$, or equivalently tell us what $\Lambda^+_{B^\pm}(X)$ are. We draw the two monoids below to visualize the result:




$\Lambda^+_{B^+}(X)$ $\Lambda^+_{B^-}(X)$

Figure 5.1: Weight monoids for two different hyperborels

For comparison, the monoid $\Lambda^+_{B_0}(X)$ for any Borel subgroup $B_0$ of $G$ consists of all the lattice points that are a nonnegative linear combination of $\lambda$ and $\mu$. This example demonstrates that $\Lambda^+_B(X)$ need not be finitely generated as neither of the above monoids are finitely generated.

*Example* 5.2.9. Consider the action of $G = OSP(3|4) \times OSP(3|4)$ on $X = OSP(3|4)$ by left and right multiplication. The notion of Borel and hyperborel coincide for both $OSP(3|4)$ and $OSP(3|4) \times OSP(3|4)$. If $B$ is a Borel of $OSP(3|4)$, then $X$ is $B \times B^-$-spherical where $B^-$ is the opposite Borel of $B$. By lemma 6.5.7, $\Lambda^+_{B\times B^-}(X)$ will be naturally isomorphic to the monoid of $B$-dominant weights of $OSP(3|4)$. Now if we choose the Borel determined by the simple roots $\delta_1 - \delta_2, \delta_2 - \epsilon_1, \epsilon_1$ as described in section 1.3.3 of [10], then by theorem 2.11 of [10] the weight $\lambda = \epsilon_1 + \epsilon_2 + \delta_1 + \delta_2$ is not dominant while $k\lambda$ is dominant for $k \geq 2$. Thus $\Lambda^+(X)$ is will not be saturated in this case.

## 5.3 Spherical quasi-projective supervarieties

We now state a more general characterization of spherical supervarieties in the quasi-projective setting. However in the rest of this thesis we will mostly use the affine characterization.

Let $G$ be quasi-reductive, $X$ a quasi-projective $G$-supervariety, and $B$ a hyperborel subgroup. As observed in remark 4.6.6, there may exist line bundles on $X$ for which no power is $B$-linearizable. Thus in order to obtain a representation-theoretic characterization for such spaces, it is necessary to assume the existence of a very ample $B$-linearizable line bundle on $X$.

**Theorem 5.3.1.** *Let $X$ be a quasi-projective $G$-supervariety, $B$ a hyperborel subgroup of $G$ with unipotent radical $N$ and maximal torus $T$. Assume that there exists a very ample $B$-equivariant line bundle on $X$. Then the following are equivalent:*

1. *$X$ is spherical for $B$.*

2. *$X_0$ is spherical for $B_0$, and for every $B$-equivariant line bundle $\mathcal{L}$ the non-zero elements of $\Gamma(X,\mathcal{L})^N$ are non-vanishing at some point.*

3. *For every $B$-equivariant line bundle $\mathcal{L}$, the non-zero elements of $\Gamma(X,\mathcal{L})^N$ are non-vanishing at some point, and $\dim\Gamma(X,\mathcal{L})^N_\lambda\leq 1$ for all weights $\lambda$ of $T$.*

*Proof.* (1) $\Longrightarrow$ (2) Let $\mathcal{L}$ be a $B$-equivariant line bundle on $X$. Write $U$ for the open $B$-orbit on $X$. Then if $\sigma\in\Gamma(X,\mathcal{L})$ is a $B$-eigenvector, restricting to $U$ it spans a nonzero $B$-submodule of $\Gamma(U,\mathcal{L})$. Since $U$ is a homogeneous $B$-supervariety, by proposition 4.7.4 $\sigma$ must generate $\mathcal{L}|_U$, and thus must be non-vanishing on $U$.

(2) $\Longrightarrow$ (3) It suffices to check that for a $B$-equivariant line bundle $\mathcal{L}$, $\dim\Gamma(X,\mathcal{L})^N_\lambda\leq 1$ for all weights $\lambda$ of $T$. But since $X_0$ is spherical for $B_0$ and $i:X_0\to X$ is $G_0$-equivariant, $i^*\mathcal{L}$ is a $B_0$-equivariant line bundle and the pullback morphism $\Gamma(X,\mathcal{L})\to\Gamma(X_0,i^*\mathcal{L})$ is $B_0$-equivariant. By assumption, $\Gamma(X,\mathcal{L})^N\to\Gamma(X_0,i^*\mathcal{L})^{N_0}$ must be injective, and since $X_0$ is spherical we have $\dim\Gamma(X_0,i^*\mathcal{L})^{N_0}_\lambda\leq 1$ for all weights $\lambda$.

(3) $\Longrightarrow$ (1) Let $\mathcal{L}$ be a $B$-equivariant very ample line bundle on $X$. Then if $X$ does not have an open $B$-orbit, it must follow that $k(X)^{(\mathfrak{b})}$ either has a nilpotent function or is not multiplicity-free. If $f\in k(X)^{(\mathfrak{b})}$ is homogeneous, there exists $n>0$ and a homogeneous global section $s\in\Gamma(X,\mathcal{L}^{\otimes n})$ such that $fs\in\Gamma(X,\mathcal{L}^{\otimes n})$ is also a global section. Let $V\subseteq\Gamma(X,\mathcal{L}^{\otimes n})$ be the subspace of sections $s$ such that $fs\in\Gamma(X,\mathcal{L}^{\otimes n})$. Then $V$ is a $B$-submodule of $\Gamma(X,\mathcal{L}^{\otimes n})$ and thus admits a non-zero $B$-eigenvector $s_2$. Let $s_1:=fs_2$. Then by construction $s_1$ is also a $B$-eigenvector. In particular, $s_1$ and $s_2$ both are non-vanishing at some point by assumption, and thus $f=s_1/s_2$ is also non-vanishing at some point and therefore cannot be nilpotent.

If $f,g\in\mathfrak{k}(X)^{(\mathfrak{b})}$ are $\mathfrak{b}$-eigenvectors with the same weight for the action of $\mathfrak{b}$, then $f/g$ will be $\mathfrak{b}$-invariant. Thus by our construction there exists an $n>0$ and $s_1,s_2\in\Gamma(X,\mathcal{L}^{\otimes n})^{(\mathfrak{b})}$ such that $f=s_1/s_2$, and thus $s_1$ and $s_2$ have the same weight for $\mathfrak{b}$. But by assumption $\Gamma(X,\mathcal{L}^{\otimes n})^N$ is a multiplicity-free $T$-module, so we obtain a contradiction. This completes the proof. □

## 5.4 Spherical supervarieties for Lie superalgebra actions

Let $\mathfrak{g}$ be an arbitrary Lie superalgebra and $X$ a supervariety.

**Definition 5.4.1.** If $\mathfrak{g}$ acts on $X$, then we say $\mathfrak{g}$ has an open orbit on $X$ if there exists a point $x\in X(k)$ such that the natural restriction map $\mathfrak{g}\to T_xX$ is a surjection. In this case, the locus of points where $\mathfrak{g}\to T_xX$ is surjective is open, and we call this open set an open orbit of $\mathfrak{g}$. We say $X$ is a homogeneous $\mathfrak{g}$-supervariety if all of $X$ is an open orbit.

An open orbit of $\mathfrak{g}$ will be smooth by proposition 3.7.4. Also observe that an open subvariety of a homogeneous supervariety is still homogeneous for the natural restricted action.

**Proposition 5.4.2.** *Suppose that $X$ is a homogeneous $\mathfrak{g}$-supervariety. If $\mathcal{L}$ is a $\mathfrak{g}$-equivariant line bundle on $X$, then a $\mathfrak{g}$-submodule of $\Gamma(X, \mathcal{L})$ generates $\mathcal{L}$. In particular, if $X$ is affine, $k[X]$ has no non-trivial $\mathfrak{g}$-invariant ideals.*

*Proof.* See remark 4.7.5. □

Now assume that $\mathfrak{g}$ is quasi-reductive.

**Definition 5.4.3.** A $\mathfrak{g}$-supervariety $X$ is said to be spherical if there exists a hyperborel subalgebra $\mathfrak{b}$ in $\mathfrak{g}$ such that $\mathfrak{b}$ has an open orbit on $X$. In this case we say that $X$ is $\mathfrak{b}$-spherical.

*Remark* 5.4.4. If $G$ is quasi-reductive and acts on a supervariety $X$, and $B$ is a hyperborel subgroup of $G$, then $X$ is $B$-spherical if and only if $X$ is $\mathfrak{b}$-spherical for the induced action of $\mathfrak{g}$ on $X$.

**Theorem 5.4.5.** *Let $X$ be a $\mathfrak{g}$-supervariety, $\mathfrak{b}$ a hyperborel subalgebra of $\mathfrak{g}$ and $\mathfrak{h}_{\bar{0}} \subseteq \mathfrak{b}$ a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$. If $X$ is $\mathfrak{b}$-spherical then for a $\mathfrak{b}$-equivariant line bundle $\mathcal{L}$ on $X$, $\Gamma(X, \mathcal{L})^{(\mathfrak{b})}$ is a multiplicity-free $\mathfrak{h}_{\bar{0}}$-module and if $s \in \Gamma(X, \mathcal{L})^{(\mathfrak{b})}$ is non-zero then it is non-vanishing.*

*Proof.* Suppose that $s \in \Gamma(X, \mathcal{L})^{(\mathfrak{b})}$ is a non-zero weight vector of $\mathfrak{b}$. If we restrict $s$ to the open orbit $U$ of $\mathfrak{b}$, by proposition 5.4.2 it must generate $\mathcal{L}|_U$ since it cannot restrict to zero (since restriction of sections is injective by lemma 3.4.6). This implies the restriction of $s$ to $U$ must be non-vanishing.

Now if $s_1, s_2 \in \Gamma(X, \mathcal{L})^{(\mathfrak{b})}$ are non-zero weight vectors for $\mathfrak{b}$ of the same weight, then $f = s_1/s_2$ is a rational $\mathfrak{b}$-invariant function. Since $s_2$ is non-vanishing on $U$, $f$ is regular on $U$. We may assume by further restriction that $U$ affine. Then since it is $\mathfrak{b}$-homogeneous, $k[U]$ has no nontrivial $\mathfrak{b}$-invariant ideals by proposition 5.4.2. However for $x \in U(k)$, $(f - f(x))$ will be an invariant ideal which is not equal to $k[U]$ since it is contained in $\mathfrak{m}_x$. Therefore $f - f(x) = 0$, so $f$ is constant, and thus $s_1$ and $s_2$ are proportional. This completes the proof. □

### 5.4.1 Contragredient case

Let $\mathfrak{g}$ be a contragredient Lie superalgebra (as in 2.5 of [26]– in particular this includes the basic distinguished Lie superalgebras) and $V$ a $\mathfrak{b}$-highest weight module for a Borel $\mathfrak{b}$ of highest weight $\lambda$. Let $\alpha$ be a simple isotropic root with $e_{-\alpha}$ a root vector of weight $-\alpha$. Then if $(\lambda, \alpha) = 0$ and $v \in V$ is the highest weight vector then either $e_{-\alpha}v = 0$ or $e_{-\alpha}v$ is a $\mathfrak{b}$-highest weight vector of opposite parity of $v$.

**Proposition 5.4.6.** *Let $X$ be a supervariety and $\mathfrak{g}$ a contragredient Lie superalgebra acting on $X$. Let $X$ be $\mathfrak{b}$-spherical for a Borel $\mathfrak{b}$ of $\mathfrak{g}$, and $f_\lambda \in k[X]$ a $\mathfrak{b}$-highest weight vector of weight $\lambda$. Then if $\alpha$ is a simple isotropic root vector of $\mathfrak{b}$ with $(\lambda, \alpha) = 0$ then $e_{-\alpha} f_\lambda = 0$.*

*Proof.* This follows from the fact that $e_{-\alpha}f_\lambda$ is an odd $\mathfrak{b}$-highest weight function and theorem 5.4.5. □

Of course one may generalize proposition 5.4.6 to equivariant line bundles as well, which we do not state here.

## 5.5 Spherical representations

Irreducible spherical representations of reductive algebraic groups were originally classified by Kac in [27]. We seek to classify all indecomposable spherical representations of the Cartan-even distinguished quasi-reductive groups. The case of $Q(n)$ is also looked at, however the more interesting question in that case is to find representations with open orbits under a Borel subgroup.

We state the problem studied more precisely. For greater generality, representations of the underlying Lie superalgebra were looked at instead. Therefore let $\mathfrak{g}$ be one of the distinguished Lie superalgebras. Recall that for all cases except $\mathfrak{q}(n)$ the notions of Borel and hyperborel coincide. We would like to find all indecomposable $\mathfrak{g}$-representations $V$, up to equivalence, such that the action of $\mathfrak{g}\times k\langle\mathcal{E}\rangle$ on $V$ as a supervariety is spherical, where $\mathcal{E}$ is the Euler vector field on $V$, i.e. the infinitesimal generator of scaling on $V$. This extra scaling action is added for greater generality. This is equivalent to asking if there exists a vector $v\in V_{\overline{0}}$ and a Borel subalgebra $\mathfrak{b}\subseteq\mathfrak{g}$ such that $\mathfrak{b}\cdot v+k\langle v\rangle=V$. Thus we obtain a triple $(V,\mathfrak{g},\rho)$, where $\rho:\mathfrak{g}\to\mathfrak{gl}(V)$ defines the representation.

We now state precisely the notion of equivalence we work with. We say that two spherical representations $(V,\mathfrak{g},\rho)$ and $(V',\mathfrak{g}',\rho')$ are equivalent if there exists an isomorphism of super vector spaces $\psi:V\to V'$ such that if $\Psi:\mathfrak{gl}(V)\to\mathfrak{gl}(V')$ is the induced map, then $\rho'(\mathfrak{g}')+k\,\mathrm{id}_{V'}=\Psi(\rho(\mathfrak{g}))+k\,\mathrm{id}_{V'}$.

### 5.5.1 Explanation of classification proof

The classification of spherical indecomposables for the Lie superalgebras we considered is written out fully in appendix A. The proof goes case by case, and we briefly describe how the proof works. For each superalgebra $\mathfrak{g}$ we fix a Cartan subalgebra and a Borel subalgebra $\mathfrak{b}$ containing it, and thus a correspondence between simple $\mathfrak{g}$-modules up to parity and $\mathfrak{b}$-dominant weights.

If $V$ is a spherical $\mathfrak{g}$-module then $V_{\overline{0}}$ is spherical for $\mathfrak{g}_{\overline{0}}$. Further, if $V$ is spherical we clearly must have $\dim V_{\overline{1}}\leq\dim\mathfrak{b}_{\overline{1}}$ (for $\mathfrak{p}(n)$ we should take the maximum over all Borel subalgebras on the RHS). If $V'$ is a subquotient of a spherical representation $V$, then we must have $V'_{\overline{0}}$ is spherical and again we have $\dim V'_{\overline{1}}\leq\dim\mathfrak{b}_{\overline{1}}$. Using this as motivation, we define a representation $V$ of $\mathfrak{g}$ to be numerically spherical if $V_{\overline{0}}$ is $\mathfrak{g}_{\overline{0}}$-spherical and $\dim V_{\overline{1}}\leq\mathfrak{b}_{\overline{1}}$.

The proof then goes by determining, for each Lie superalgebra $\mathfrak{g}$, the simple numerically spherical modules. Then any indecomposable spherical representation must have all its

| $V$ | $\dim^s V$ | $S^\bullet V^*$ Completely Reducible? |
|---|---|---|
| $GL_{m\|n}$ | $(m\|n)$ | Yes |
| $S^2 GL_{m\|n}$ | $(\frac{n(n-1)}{2}+\frac{m(m+1)}{2}\|mn)$ | Yes |
| $\Pi S^2 GL_{n\|n}$ | $(n^2\|n^2)$ | Yes |
| $\Pi S^2 GL_{n\|n+1}$ | $(n(n+1)\|n(n+1))$ | Yes |
| $OSP_{m\|2n}$, $m \geq 2$ | $(m\|2n)$ | Iff $m$ is odd or $m > 2n$ |
| $\Pi OSP_{m\|2n}$ | $(2n\|m)$ | Yes |
| $\Pi P_{n\|n}$ | $(n\|n)$ | No |
| $Q_{n\|n}$ | $(n\|n)$ | Yes |

Table 5.1: Infinite families of spherical representations

composition factors numerically spherical, and thus all possible candidates are produced and studied individually.

### 5.5.2 Classification

We found there are a few infinite families of irreducible spherical representations, along with certain small exceptional cases, some of which are not irreducible. First we give a table of the infinite families. We write $GL_{m|n}$, $OSP_{m|2n}$, $P_{n|n}$, and $Q_{n|n}$ respectively for the standard representations of $GL(m|n)$, $OSP(m|2n)$, $P(n)$, and $Q(n)$ respectively. We also state the dimension of the representation and whether the algebra of functions on it is completely reducible (see section 5.6 for the full description of these symmetric algebras as $\mathfrak{g}$-modules). Note also that in appendix A we describe exactly which Borel subalgebras each representation is spherical with respect to.

In addition to the above infinite families, we have some small cases of spherical indecomposable representations. They are given in the table shown. We give an explicit description of these modules in the following list.

- If we consider the nontrivial $(1|1)$-dimensional representation of $\mathfrak{q}(1)$ which is annihilated by the center and takes the odd operator to $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ we obtain a spherical representation which we call $Q^{1|1}$. It may also be described as the restriction of the standard representation of $P(1|1)$ to its derived subalgebra.

- For the Lie superalgebra $\mathfrak{gl}(1|2)$, the Kac modules $K_{1|2}(t\epsilon_1)$ are spherical for all $t$, and their parity shifts $\Pi K_{1|2}(t\epsilon_1)$ are spherical for $t \neq 0$.

| $V$ | $\dim^s V$ | $S^\bullet V^*$ Completely Reducible? |
|---|---|---|
| $Q^{1\|1} = \mathrm{Res}_{[\mathfrak{p}(1),\mathfrak{p}(1)]} P_{1\|1}$ | $(1\|1)$ | No |
| $K_{1\|2}(t\epsilon_1)$ | $(2\|2)$ | Iff $t \notin \mathbb{Q} \cap [0,1]$ |
| $\Pi K_{1\|2}(t\epsilon_1)$, $t \neq 0$ | $(2\|2)$ | Iff $t \neq 1$ |
| $L_{\mathfrak{b}^{\epsilon\delta\delta}}(\delta_1+\delta_2-\epsilon_1)$ | $(6\|4)$ | Yes |
| $\nabla(\omega)$ | $(4\|4)$ | No |
| $\mathrm{rad}\,\nabla(\omega)$ | $(3\|4)$ | No |
| $\nabla(\omega)/\,\mathrm{soc}\,\nabla(\omega)$ | $(4\|3)$ | No |
| $(Q_{2\|2})_t, t \neq -1/2$ | $(2\|2)$ | Always |
| $\Pi(Q_{2\|2})_t, t \neq -1/2$ | $(2\|2)$ | Always |
| $\mathrm{Res}_{[\mathfrak{p}(2),\mathfrak{p}(2)]} P_{2\|2}$ | $(2\|2)$ | No |

Table 5.2: Exceptional indecomposable spherical representations

- For $\mathfrak{osp}(2|4)$ consider the Borel $\mathfrak{b}^{\epsilon\delta\delta}$ with simple roots $\epsilon_1-\delta_1, \delta_1-\delta_2, 2\delta_2$. Then $L_{\mathfrak{b}^{\epsilon\delta\delta}}(\delta_1+\delta_2-\epsilon_1)$, the irreducible module of highest weight $\delta_1+\delta_2-\epsilon_1$ with respect to this Borel, is spherical.

- For $\mathfrak{p}(3)$, let $\omega = \epsilon_1+\epsilon_2+\epsilon_3$. Let $\nabla(\omega)$ denote the thin Kac module of highest weight $\omega$ (see [5] for a definition). Then this module, along with its radical ($\mathrm{rad}\,\nabla(\omega)$) and quotient by its socle ($\nabla(\omega)/\,\mathrm{soc}\,\nabla(\omega)$) are all spherical.

- Finally, for $\mathfrak{q}(2)$, we may take $Q_{2|2} = L(\epsilon_1)$ with respect to the standard Borel, and twist the highest weight by multiples of $\epsilon_1+\epsilon_2$, and we will still have a dominant weight. So we consider $(Q_{2|2})_t = L(\epsilon_1 + t(\epsilon_1+\epsilon_2))$ for $t \in k$. If $t \neq -1/2$, then $(Q_{2|2})_t$ and $\Pi(Q_{2|2})_t$ are both spherical.

  When $t = -1/2$, $(Q_{2|2})_{-1/2}$ is the representation obtained via the isomorphism

  $$\mathfrak{q}(2)/kI_{2|2} \to [\mathfrak{p}(2), \mathfrak{p}(2)]$$

  and is the restriction of the standard representation of $\mathfrak{p}(2)$. We will therefore write this representation as $\mathrm{Res}_{[\mathfrak{p}(2),\mathfrak{p}(2)]} P_{2|2}$.

Note that there is some redundancy in the above list, in that some of the Kac modules for $\mathfrak{gl}(1|2)$ are equivalent and also some of the modules for $\mathfrak{q}(2)$ showing up are equivalent.

*Remark* 5.5.1. Some observations about the above classification:

- All cases with quasi-reductive stabilizer appear in [45]. These are exactly the cases in which the open $G$-orbit is affine.
- The number of irreducible $\mathfrak{g}_{\overline{0}}$-components of any spherical irreducible representation is always less than or equal to 3; in an indecomposable spherical representation, there may be up to 4 $\mathfrak{g}_{\overline{0}}$-components.
- No exceptional basic lie superalgebras admit any non-trivial spherical representations.

Recall from section 4.7.2, for an supervariety $X$, the definition of $\Lambda_{\mathfrak{b}}^{+}(X)$,. It is the monoid of $\mathfrak{b}$-highest weights in $k[X]$, where $X$ is spherical with respect to $\mathfrak{b}$. In the case when $X = V$ is representation, we have $k[X] = S^{\bullet}V^*$. Below we present a table with these monoids for almost all of the spherical modules found, with the choice of Borel specified.

We introduce some notation. We give the name $\zeta$ for the character of the action of $\mathcal{E}$. Then $\mathcal{E}$ acts on $S^dV^*$ by the weight $d\zeta$ so that multiple of $\zeta$ in a weight records which symmetric power it appears in.

We also define certain 'fundamental' weights:

$$\omega_i = \epsilon_1 + \cdots + \epsilon_i \text{ for } i \leq m, \quad \eta_i = \delta_1 + \cdots + \delta_i \text{ for } 1 \leq i \leq n$$

$$\gamma_i = \omega_i + \eta_i$$

We now present the table of monoids below in terms of a generating set. For a subset $S \subseteq \Lambda_{\mathfrak{b}}^{+}$, we write $\langle S \rangle$ to mean the submonoid generated by $S$ (here $\Lambda_{\mathfrak{b}}^{+}$ is the set of all dominant weights of $\mathfrak{g}$ with respect to $\mathfrak{b}$).

## 5.6 Computations of symmetric powers

It is interesting to study the structure of $k[X]$ as a $G$-module when $X$ is spherical. Thus we compute the structure of $k[V] = S^{\bullet}V^*$ for each spherical representation in our list above, proving complete reducibility in the cases we have claimed it and giving a more thorough description of the structure in the other cases. We will go through the different symmetric algebras one by one.

We begin by stating:

**Proposition 5.6.1.** *Let $Q^{1|1}$ be the representation of the one-dimensional odd abelian algebra as described in the introduction. Then $S^d(Q^{1|1})^* \cong \Pi Q^{1|1}$ for all $d$.*

*Proof.* Omitted. □

### 5.6.1 $\mathfrak{gl}(m|n)$:

The computations of $S^d(V^*)$ for a spherical irreducible of $\mathfrak{gl}(m|n)$ from proposition A.3.2 use Schur-Weyl duality as in [23]. Most computations were also discussed in [45]. We do not rewrite them here.

| Rep | Borel | $\Lambda^+$(Rep) |
|---|---|---|
| $GL_{m\|n}$ | $\mathfrak{b}^{\delta^n\epsilon^m}$ | $\langle -\delta_n + \zeta \rangle$ |
| $S^2GL_{m\|n}$ | $(\mathfrak{b}^{\delta^n\epsilon^m})^{op}$ | $\langle -2\omega_i - i\zeta, -2j\omega_m - \eta_j + m(j+1)\zeta \rangle_{1\leq i\leq m, 1\leq j\leq \lfloor \frac{n}{2} \rfloor}$ |
| $\Pi S^2GL_{n\|n}$ | $(\mathfrak{b}^{\delta\epsilon\delta\epsilon\cdots\delta\epsilon})^{op}$ | $\langle -(\gamma_1 + \cdots + \gamma_i) + i\zeta \rangle_{1\leq i\leq n}$ |
| $\Pi S^2GL_{n\|n+1}$ | $(\mathfrak{b}^{\delta\epsilon\delta\epsilon\cdots\epsilon\delta})^{op}$ | $\langle -(\gamma_1 + \cdots + \gamma_i) + i\zeta \rangle_{1\leq i\leq n}$ |
| $OSP_{m\|2n}$, $m \geq 2$ | $\mathfrak{b}^{\epsilon\cdots\epsilon\delta\cdots\delta}$ | $\langle \epsilon_1 + \zeta, 2\zeta \rangle$ |
| $\Pi OSP_{m\|2n}$ | $\mathfrak{b}^{\delta\cdots\delta\epsilon\cdots\epsilon}$ | $\langle \delta_1 + \zeta \rangle$ |
| $K(\lambda)$ | $\mathfrak{b}^{\epsilon\delta\delta}$ | $\langle (2-(j+1)t)\epsilon_1 - (\delta_1+\delta_2) + (j+1)\zeta \rangle_{j\in\mathbb{Z}_{\geq 0}}$ |
| $\Pi K(\lambda)$ | $\mathfrak{b}^{\delta\epsilon\delta}$ | $\langle (1-t)\epsilon_1 - \delta_2 + \zeta \rangle$ |
| $L(\delta_1 + \delta_2 - \epsilon_1)$ | $\mathfrak{b}^{(-\epsilon)\delta\delta}$ | $\langle \epsilon_1 + \delta_1 + \delta_2 + \zeta, 2\epsilon_1 + 2\zeta, (3s-4)\epsilon_1 + s\zeta \rangle_{s\in\mathbb{Z}_{\geq 2}}$ |
| $Q_{n\|n}$ | $\mathfrak{b}^{st}$ | $\langle -\epsilon_m + \zeta \rangle$ |
| $\Pi P_{n\|n}$ | $\epsilon_i + \epsilon_j > 0$ for all $i, j$ | $\langle \epsilon_1 + \zeta \rangle$ |
| $\nabla(\omega)$ | $2\epsilon_2, \epsilon_1 + \epsilon_3,$ $-\epsilon_2 - \epsilon_3 > 0$ | $\langle \epsilon_1 + \zeta, -\omega + \zeta \rangle$ |
| $(Q_{2\|2})_t, t \neq -1/2$ | $(\mathfrak{b}^{st})^{op}$ | $\langle -(t+1)\epsilon_1 - t\epsilon_2 + \zeta \rangle$ |
| $\Pi(Q_{2\|2})_t, t \neq -1/2$ | $(\mathfrak{b}^{st})^{op}$ | $\langle -(t+1)\epsilon_1 - t\epsilon_2 + \zeta \rangle$ |

Table 5.3: Weight monoid of spherical representations

### 5.6.2 $\mathfrak{gl}(1|2)$:

First we compute $S^d(K(t\epsilon_1)^*)$ for $d \geq 1$ as a $\mathfrak{gl}(1|2)$-module. We have that $K(t\epsilon_1)^* \cong K((2-t)\epsilon_1 - \delta_1 - \delta_2)$. Then with respect to $\mathfrak{b}^{\delta\delta\epsilon}$, the highest weights functions in $S^d(K(t\epsilon_1)^*)$ have weight

$$\lambda_{i,j} = (2i - (i+j)t)\epsilon_1 - i(\delta_1 + \delta_2)$$

where $i + j = d$, $i \geq 0$, and $j > 0$. The weight $\lambda_{i,j}$ is then atypical if and only if

$$t = \frac{i}{i+j} = \frac{i}{d} \quad \text{or} \quad t = \frac{i+1}{i+j} = \frac{i+1}{d}$$

This can only happen if $t \in \mathbb{Q} \cap [0,1]$. Therefore, if $t \notin \mathbb{Q} \cap [0,1]$, we have

$$S^d(K(t\epsilon_1)^*) = \bigoplus_{\substack{i+j=d \\ i\geq 0, j>0}} L_{\mathfrak{b}^{\delta\delta\epsilon}}(\lambda_{i,j}) \tag{5.6.1}$$

Now suppose that $t = \frac{m}{n}$ with $m, n \in \mathbb{Z}$, $0 \leq m \leq n$, and $(m, n) = 1$. Then if $n$ does not divide $d$, we get the decomposition as in 5.6.1.

If $n$ divides $d$, we may write $d = nk$, where $k > 0$. If $i + j = d$, then $\lambda_{i,j}$ is atypical if and only if $i = mk$ or $i = mk - 1$. If $t \neq 0, 1$ then $S^d((K(\frac{m}{n}\epsilon_1)^*)$ is projective, since it is a summand of the projective module $(K(\frac{m}{n}\epsilon_1)^*)^{\otimes d}$. Therefore in this case we find that

$$S^{nk}(K(\frac{m}{n}\epsilon_1)^*) = P(mk\,\mathrm{Ber}) \oplus \bigoplus_{\substack{i+j=nk \\ 0 \leq i < nk, i \neq mk, mk+1}} K(\lambda_{i,j})$$

where $P(mk\,\mathrm{Ber})$ is the projective cover of the one-dimensional even module of weight $mk\,\mathrm{Ber}$.

If $t = 0$ then $m = 0$ and $n = 1$. The weight $\lambda_{i,j} = 2i\epsilon_1 - i(\delta_1 + \delta_2)$ is atypical only when $i = 0$, $j = k$. Therefore we get

$$S^k(K(0)^*) = K_{1|2}(0)^* \oplus \bigoplus_{\substack{i+j=k \\ 0 < i < k}} L_{\mathfrak{b}^{\delta\delta\epsilon}}(\lambda_{i,j})$$

The first summand is obtained by using that that $K(0)^*$ has an even $\mathfrak{g}$-invariant vector, so multiplication by powers of it define injective homomorphisms $K(0)^* \to S^k(K(0)^*)$ for all $k$.

Finally, if $t = 1$ then $m = n = 1$. The weight $\lambda_{i,j} = (i - j)\epsilon_1 - i(\delta_1 + \delta_2)$ is atypical if and only if $j = 1, i = k - 1$. Therefore we get

$$S^k(K(\epsilon_1)^*) = (K_{1|2}(\epsilon_1)^*)_{(k-1)Ber} \oplus \bigoplus_{\substack{i+j=k \\ 0 \leq i < k-1}} L_{\mathfrak{b}^{\delta\delta\epsilon}}(\lambda_{i,j})$$

The first summand can be obtained by observing we have an even $\mathfrak{g}$ semi-invariant derivation on functions coming from the even semi-invariant element of $K(\epsilon_1)$. Powers of it define surjective $\mathfrak{g}$ semi-equivariant homomorphisms $S^k(K(\epsilon_1)^*) \to K(\epsilon_1)^*$ for all $k$.

### 5.6.3 $S^d(\Pi K(t\epsilon_1)^*)$, $t \neq 0$ and $d > 0$

Here, with respect to the $\delta\epsilon\delta$-Borel, the highest weight is $(1 - t)\epsilon_1 - \delta_2$, and the $d$th power of the highest weight vector will be a highest weight vector of weight

$$\mu_d = d(1 - t)\epsilon_1 - d\delta_2$$

When $t \neq 1$, these weight are typical. It follows that

$$S^d(\Pi K(t\epsilon_1)^*) \cong L_{\mathfrak{b}^{\delta\epsilon\delta}}(\mu_d)$$

When $t = 1$, $\Pi K(t\epsilon_1)$ has socle $\Pi k_{-\mathrm{Ber}}$. Therefore, there is an odd $\mathfrak{g}$ semi-invariant derivation on functions, which defines non-zero $\mathfrak{g}$ semi-equivariant homomorphisms $S^d(\Pi K(t\epsilon_1)^*) \to S^{d-1}(\Pi K(t\epsilon_1)^*)$ such that the composition of two is zero. Since $S^d(\Pi K(t\epsilon_1)^*)$ also contains a unique highest weight vector, of weight $\mu_d = -d\delta_2$, the socle must be $L_{\mathfrak{b}^{\delta\epsilon\delta}}(-d\delta_2)$, and the derivation must vanish on the socle. By the structure of projectives for an atypical block of $\mathfrak{gl}(1|2)$-modules, it follows we must have

$$S^d(\Pi K(t\epsilon_1)^*) \cong K_{1|2}(\epsilon_1 - \delta_1 - d\delta_2)$$

### 5.6.4 $\mathfrak{osp}(m|2n)$

- The representation $\Lambda^d(OSP_{m|2n})$ is irreducible for all $d$ as a $\mathfrak{osp}(m|2n)$-module (see the remarks at the end of chapter 2 in [10]). Since

  $$S^d(\Pi OSP_{m|2n}) \cong \Pi^d\Lambda^d(OSP_{m|2n}),$$

  all symmetric powers of $\Pi OSP_{m|2n}$ are irreducible.

- The computation of $S^dL(\delta_1 + \delta_2 - \epsilon_1)^*$ is given in [45].

### 5.6.5 Computation of $S^\bullet(OSP_{m|2n})$

Write $V = OSP_{m|2n}$ for the standard representation of $\mathfrak{g} = \mathfrak{osp}(m|2n)$. For each $d \geq 0$ we find the structure of $S^dV$ as a $\mathfrak{g}$-module. This is related to the notion of skew-symmetric harmonic polynomials, and the proof uses ideas from the classical theory of harmonic polynomials as functions on the sphere. For more on the classical story of harmonic polynomials, see Chapter III of [22]. See also [12] for more on supersymmetric harmonic polynomials and their representation theory.

### 5.6.6 Setup

Let $(-,-) \in S^2V^*$ be a non-degenerate $\mathfrak{g}$-invariant supersymmetric form on $V$. Then we have an induced isomorphism of $\mathfrak{g}$-modules $V \cong V^*$ and a corresponding dual element to the form, $\omega \in S^2V$. The form $(-,-)$ gives rise to a non-degenerate, supersymmetric $\mathfrak{g}$-invariant form on each symmetric power $S^dV$, which we also denote by $(-,-)$. Let $\Omega \in \mathrm{End}(S^\bullet V)$ be the adjoint to left multiplication by $\omega$, i.e.

$$(\Omega x, y) = (x, \omega y)$$

for all $x, y \in S^\bullet V$. This is the Laplacian operator on functions. Let $H = [\Omega, -\omega]$. Then for each $d$ we have $\mathfrak{g}$-module endomorphisms

$$H : S^dV \to S^dV, \quad \Omega : S^dV \to S^{d-2}V, \quad L_\omega : S^dV \to S^{d+2}V$$

where we denote left multiplication by $\omega$ as $L_\omega$. Further, these three endomorphisms form an $\mathfrak{sl}_2$-triple:

$$[H, \Omega] = 2\Omega, \quad [H, -L_\omega] = 2L_\omega.$$

Therefore we have an action of $\mathfrak{sl}_2 \times \mathfrak{g}$ on $S^\bullet V$. The operator $H$ takes the specific form

$$H = (n - r) - \mathcal{E}$$

where $r := m/2$ and $\mathcal{E}$ is the Euler vector field, i.e. the operator which acts as scalar multiplication by $d$ on $S^d V$.

### 5.6.7 $\mathfrak{sl}_2 \times \mathfrak{g}_{\overline{0}}$ structure:

By the theory of harmonic polynomials on $V_{\overline{0}} = k^m$, we have as an $\mathfrak{sl}_2 \times \mathfrak{o}(m)$-module

$$S^\bullet V_{\overline{0}} = \bigoplus_{\ell \geq 0} M_0(-r - \ell) \boxtimes H_\ell^{ev},$$

where $H_\ell^{ev}$ is the irreducible $\mathfrak{so}(m)$-module of harmonic polynomials of degree $\ell$, and we write $M_0(s)$ for the $\mathfrak{sl}_2$ Verma module of highest weight $s$. Here, we have

$$M_0(-r - \ell)_{-r-\ell} \otimes H_\ell^{ev} \subseteq S^\ell V_{\overline{0}}.$$

By the theory of skew-symmetric harmonic polynomials on $k^{2n}$, we have as an $\mathfrak{sl}_2 \times \mathfrak{sp}(2n)$-module

$$\Lambda^\bullet V_{\overline{1}} = \bigoplus_{0 \leq j \leq n} L_0(n - j) \boxtimes E_j$$

where $E_j$ is the $j$th fundamental representation of $\mathfrak{sp}(2n)$ for $j \geq 1$ and $E_0$ is the trivial representation, and we write $L_0(s)$ for the irreducible $\mathfrak{sl}_2$-module of highest weight $s$. Here, we have

$$L_0(n - j)_{n-j} \otimes E_j \subseteq \Lambda^j V_{\overline{1}}$$

Hence we have, as an $\mathfrak{sl}_2 \times \mathfrak{so}(m) \times \mathfrak{sp}(2n)$-module,

$$S^\bullet V = S^\bullet V_{\overline{0}} \otimes \Lambda^\bullet V_{\overline{1}} = \bigoplus_{\substack{\ell \geq 0 \\ 0 \leq j \leq n}} (M_0(-r - \ell) \otimes L_0(n - j)) \boxtimes H_\ell^{ev} \boxtimes E_j.$$

**Lemma 5.6.2.** *As a $\mathfrak{g}_{\overline{0}}$-module, $S^d V / L_\omega S^{d-2} V$ is multiplicity-free, self-dual, and every irreducible summand is isomorphic to a module of the form $H_i^{ev} \boxtimes E_j$, where $0 \leq j \leq n$.*

*Proof.* The irreducible factors of $S^d V$ are all isomorphic to a module of the form $H_i^{ev} \boxtimes E_j$ for some $0 \leq j \leq n$, and these are all self-dual $\mathfrak{g}_{\overline{0}}$-modules. So it remains to prove $S^d V / L_\omega S^{d-2} V$ is multiplicity-free.

By our decomposition as an $\mathfrak{sl}_2 \times \mathfrak{g}_{\overline{0}}$-module, we can write

$$S^dV = \bigoplus_{i+j=d} \left( \bigoplus_{0 \le \ell \le \lfloor \frac{i}{2} \rfloor} r_0^\ell H^{ev}_{i-2\ell} \right) \otimes \left( \bigoplus_{\max(0,j-n) \le k \le \lfloor \frac{j}{2} \rfloor} \omega_0^k E_{j-2k} \right)$$

where $\omega = r_0 + \omega_0$, $r_0 \in S^2V_{\overline{0}}$, $\omega_0 \in \Lambda^2 V_{\overline{1}}$. The extra lower bound condition on $k$ comes from the finite-dimensional structure of the corresponding $\mathfrak{sl}_2$-module. It follows that each summand can be written uniquely as $r_0^\ell H^{ev}_s \otimes \omega_0^k E_t$, with $2\ell + s + 2k + t = d$.

Suppose another summand isomorphic to this one shows up, e.g. $r_0^p H^{ev}_s \otimes \omega_0^q E_t$, with $2p + s + 2q + t = d$. Then we must have $q \neq k$. Without loss of generality suppose $q < k$. Then $r_0^\ell H^{ev}_s \otimes \omega_0^{k-1} E_t$ will be a $\mathfrak{g}_{\overline{0}}$ summand of $S^{d-2}V$. Multiplying it by $\omega = r_0 + \omega_0$, we learn that modulo $L_\omega S^{d-2}V$, $r_0^\ell H^{ev}_s \otimes \omega_0^k E_t$ is identified with $r_0^{\ell+1} H^{ev}_s \otimes \omega_0^{k-1} E_t$.

By induction on $|k-q|$, we can identify $r_0^\ell H^{ev}_s \otimes \omega_0^k E_t$ with $r_0^p H^{ev}_s \otimes \omega_0^q E_t$ modulo $L_\omega S^{d-2}V$. This proves the quotient is multiplicity-free. □

**Corollary 5.6.3.** *As a $\mathfrak{g}$-module, $S^dV/L_\omega S^{d-2}V$ is multiplicity-free and each composition factor is self-dual.*

### 5.6.8 Frobenius reciprocity

If we consider the action of the supergroup $G = OSP(m|2n)$ on $V$ as a supervariety, the stabilizer of an even vector of length 1 will be $K = OSP(m-1|2n)$, which gives rise to a closed embedding of $G/K$ into $V$. In fact we obtain an identification

$$G/K \cong \operatorname{Spec}\ S^\bullet V/(1-\omega)$$

Frobenius reciprocity tells us, in this case, that for an integrable $\mathfrak{g}$-module $W$ we have

$$\operatorname{Hom}_G(W, k[G/K]) \cong \operatorname{Hom}_K(W, k) = (W^*)^K \tag{5.6.2}$$

This will be heavily used in what follows. In particular, we observe that for any $d \in \mathbb{Z}_{\ge 0}$, we have a natural injective map:

$$S^dV \hookrightarrow S^\bullet V/(1-\omega) = k[G/K],$$

and because $L_\omega$ is injective, we have an isomorphism of $\mathfrak{g}$-modules

$$k[G/K] \cong \varinjlim S^{2d}V \oplus \varinjlim S^{2d+1}V \tag{5.6.3}$$

### 5.6.9 $\mathfrak{sl}_2$-module structure:

Let $I = \{n-j, n-j-2, \ldots, j-n\}$. By Prop. 3.12 of [16], we have:

$$M_0(-r-\ell) \otimes L_0(n-j) = \quad M_0 \oplus \bigoplus_{\substack{t\in I \\ -(r+\ell)+t\in\mathbb{Z}_{\geq 0} \\ (r+\ell-t)-2+r+\ell\in I}} P_0(-(r+\ell)+t) \tag{5.6.4}$$

where for $k \geq 0$ we denote by $P_0(k)$ the big projective in the block of category $\mathcal{O}$ for $\mathfrak{sl}_2$ containing $L_0(k)$, and $M_0$ is a direct sum of Verma modules. The structure of $P_0(k)$ is such that the highest weight is $k$, and the endomorphism

$$\Omega L_\omega : P_0(k)_j \to P_0(k)_j$$

is an isomorphism if $j \neq -k$, and is the zero map when $j = -k$.

**Corollary 5.6.4.** *The map $\Omega : S^d V \to S^{d-2} V$ is surjective for all $d$.*

*Proof.* Follows from the surjectivity of the weight raising operator on $P_0(k)$ and all Verma modules. Or more simply because it is adjoint to an injective linear map. □

**Corollary 5.6.5.** *If $r$ is a half integer or $r > n$, then as an $\mathfrak{sl}_2$-module $S^\bullet V$ is a direct sum of irreducible Verma modules. In particular, it is semisimple.*

*Proof.* Our conditions imply that $-(r+\ell)+t \notin \mathbb{Z}_{\geq 0}$ for any integer $t \leq n$. By 5.6.4, this implies $M_0(-r-\ell) \otimes L_0(n-j)$ is a direct sum of Verma modules of either negative or half-integer highest weight. □

**Notation:** Write $H_d := \ker(\Omega : S^d V \to S^{d-2} V)$ for the space of 'harmonic superpolynomials'. Note that since $\Omega$ is never injective, $H_d \neq 0$ for all $d \geq 0$.

**Corollary 5.6.6.** *If $n - r \in \mathbb{Z}_{\geq 0}$, we have*

$$S^\bullet V = M_0 \oplus \bigoplus_{d \leq n-r} P_0(n-r-d) \otimes H_d$$

*where $M_0$ is a direct sum of Verma modules of negative highest weight.*

*Proof.* By 5.6.4, it suffices to prove that if $-(r+\ell)+t \geq 0$ for $t \in I = \{n-j, n-j-2, \ldots, j-n\}$, then

$$(r+\ell-t)-2+r+\ell \in I$$

or, equivalently

$$-(n-j) \leq 2(r+\ell)-2-t \leq n-j.$$

These inequalities follow from the following two inequalities:

$$-(n-j) \leq (r+\ell)-t-1 < 0, \qquad 0 \leq r+\ell-1 \leq n-j$$

where we are using that $r \geq 1$. □

We now break down our analysis of $S^\bullet V$ into two cases: that when it is a semisimple $\mathfrak{sl}_2$-module, i.e. $n-r \notin \mathbb{Z}_{\geq 0}$, and that when it is not, i.e. $n-r \in \mathbb{Z}_{\geq 0}$.

### 5.6.10 Semisimple case

We now suppose that either $r$ is a half integer or $r > n$. Then by corollary 5.6.5, we get that $\Omega L_\omega : S^dV \to S^dV$ is an isomorphism for all $d$, and therefore we have

$$S^dV = H_d \oplus L_\omega H_{d-2} \oplus L_\omega^2 H_{d-4} \oplus \cdots .$$

**Claim**: $H_d$ is irreducible for all $d \geq 0$.

To see this, first observe that

$$S^dV = H_d \oplus L_\omega S^{d-2}V$$

and therefore by lemma 5.6.2, all composition factors of $H_d$ have multiplicity one and are self-dual. But we also observe by $\mathfrak{sl}_2$-semisimplicity that the form on $S^dV$ is non-degenerate when restricted to $L_\omega S^{d-2}V$, and therefore the form must also be non-degenerate when restricted to the complement $H_d$. Therefore $H_d$ itself is self-dual as a $\mathfrak{g}$-module. Because it is also multiplicity-free, by a standard argument this implies $H_d$ is completely reducible.

To show that $H_d$ is actually irreducible, observe that we have shown

$$k[G/K] = S^\bullet V/(1-\omega) \cong \bigoplus_{d\geq 0} H_d,$$

and that $H_d$ is completely reducible. By (5.6.2) $H_d$ is irreducible if and only if

$$(H_d^*)^K \cong H_d^K = 1$$

As a $K$-module, we have $V = V' \oplus k$, where $V'$ is the standard $K$-module, and $k$ is the one-dimensional even trivial module. Therefore, we get the $K$-module decomposition

$$S^d(V) = S^dV' \oplus S^{d-1}V' \oplus \cdots$$

By Cor 5.3 of [34], the dimension of the space of $K$-invariants in $S^aV'$ is 1 if $a$ is even, 0 if $a$ is odd (where we use here that necessarily $m > 2$), and hence $\dim S^d(V)^K = \lfloor \frac{d}{2} \rfloor + 1$. On the other hand,

$$S^d(V) \cong H_d \oplus H_{d-2} \oplus \cdots$$

Since we must have $\dim H_j^K \geq 1$ for each $j \geq 0$, we obtain that $\dim H_d = 1$. Hence $H_d$ is irreducible.

### 5.6.11 Non-semisimple case: $n - r \in \mathbb{Z}_{\geq 0}$

**Lemma 5.6.7.** *The map*

$$\Omega L_\omega : S^d V \to S^d V$$

*is an isomorphism if and only if $d < n - r$ or $d > 2(n - r)$.*

*If $n - r + 2 \leq d \leq 2(n - r) + 2$, write $s = d - (n - r) - 1$. Then*

$$\ker(\Omega L_\omega : S^{d-2} V \to S^{d-2} V) = L_\omega^{s-1} H_{d-2s}$$

*In particular $L_\omega^s H_{d-2s} \subseteq H_d$.*

*Proof.* This follows from corollary 5.6.6 and the structure of the $\mathfrak{sl}_2$-modules $P(k)$ for $k \geq 0$. □

It follows that we have, for $0 \leq d \leq n - r + 1$,

$$S^d V = H_d \oplus L_\omega H_{d-2} \oplus \cdots .$$

Further, by following the same proof as in the semisimple case we can again show that each such $H_d$ for $0 \leq d \leq n - r + 1$ is a simple $\mathfrak{g}$-module.

Now suppose $n - r + 2 \leq d \leq 2(n - r) + 2$. Using the $\mathfrak{sl}_2$ structure, we may write

$$S^{d-2} V = L_\omega^{s-1} H_{d-2s} \oplus W_{d-2}$$

for some complementary $\mathfrak{g}$-submodule $W_{d-2}$ with $\Omega L_\omega : W_{d-2} \to W_{d-2}$ an isomorphism. Hence we may write

$$S^d V = A_d \oplus L_\omega W_{d-2}$$

where $A_d$ is a $\mathfrak{g}$-module complement to $L_\omega W_{d-2}$. In particular, $H_d \subseteq A_d$.

For $d > 2(n - r) + 2$, $\Omega L_\omega : S^{d-2} V \to S^{d-2} V$ is an isomorphism, so $H_d$ splits off from $S^d V$, and we get a decomposition

$$S^d V = \bigoplus_{i=0}^{t} L_\omega^i H_{d-2i} \oplus \bigoplus_{j=t+1}^{\lfloor \frac{d-(n-r)-1}{2} \rfloor} L_\omega^j A_{d-2j}$$

where

$$t = \left\lfloor \frac{d - 2(n - r) - 3}{2} \right\rfloor, \qquad A_{n-r+1} := H_{n-r+1}$$

Again using Cor. 5.3 of [34] and arguments as before, one can show that $H_d$ is irreducible for $d > 2(n - r) + 2$ when $r \geq 2$, while $H_d$ is the sum of two irreducibles with highest weights $d\epsilon_1$ and $(2n - d)\epsilon_1$ with respect to $\mathfrak{b}^{st}$ for $r = 1$. It therefore remains to understand the structure of $A_d$.

### 5.6.12 Structure of $A_d$

We now assume $(n-r)+2 \leq d \leq 2(n-r)+2$. Recall our decomposition

$$S^dV = A_d \oplus L_\omega W_{d-2}$$

Since the form will be non-degenerate when restricted to $L_\omega W_{d-2}$, it will also be non-degenerate on $A_d$, and therefore $A_d$ is self-dual. Further, we observe that $\operatorname{Im} L_\omega \cap A_d = L_\omega^s H_{d-2s}$, and in fact we have

$$S^dV/L_\omega S^{d-2}V \cong A_d/L_\omega^s H_{d-2s},$$

so $A_d/L_\omega^s H_{d-2s}$ is multiplicity-free.

By construction, $L_\omega^s H_{d-2s} \subseteq H_d \subseteq A_d$. We get short exact sequences

$$0 \to H_d \to A_d \xrightarrow{\Omega} L_\omega^{s-1} H_{d-2s} \to 0$$

$$0 \to L_\omega^s H_{d-2s} \to H_d \to Q_d \to 0$$

where we have defined $Q_d$ as the quotient $H_d/L_\omega^s H_{d-2s}$.

Using self-duality of $A_d$ we find that $Q_d$ is self-dual, and since $Q_d$ is a submodule of $A_d/L_\omega^s H_{d-2s}$ we get that it is multiplicity-free and each composition factor is self-dual. Therefore it must be completely reducible.

Again, by Cor 5.3 of [34], we learn that the $(A_d)^K$ is two-dimensional (even for the case $r=1$), and since $A_d$ is self-dual we have by 5.6.2

$$\dim \operatorname{Hom}(A_d, k[G/K]) = 2.$$

Two such linearly independent maps are

$$\phi : A_d \subseteq S^dV \hookrightarrow S^\bullet V/(1-\omega) \quad \text{and} \quad \psi : A_d \xrightarrow{\Omega} L_\omega^{s-1} H_{d-2s} \subseteq S^{d-2}V \hookrightarrow S^\bullet V/(1-\omega)$$

**Claim:** $A_d$ is indecomposable, with irreducible head and socle isomorphic to $H_{d-2s}$.

To prove the claim, first notice that the map $A_d \xrightarrow{\Omega} L_\omega^{s-1} H_{d-2s}$ cannot split, for otherwise $H_{d-2s}^{\oplus 2}$ would be a submodule of $S^dV$, which would contradict corollary 5.2.5.

Now suppose that $A_d$ split, i.e. we have $A_d = M_1 \oplus M_2$ for two non-trivial submodules $M_1$ and $M_2$, and write $p_{M_1}, p_{M_2}$ for the projections onto $M_1$ and $M_2$ respectively. Then $\phi \circ p_{M_1}$, $\phi \circ p_{M_2}$ and $\psi$ would be three linearly independent maps $A_d \to k[G/K]$, a contradiction. Therefore $A_d$ is indecomposable.

The fact that $\operatorname{soc}(A_d) \cong H_{d-2s}$ follows from the fact that $Q_d$ is multiplicity-free and each summand is self-dual. Since $A_d$ is self-dual, the head must also be isomorphic to $H_{d-2s}$. We now have the following picture of $A_d$, with its socle filtration illustrated:

$$A_d = \begin{array}{|c|}\hline H_{d-2s} \\ \hline Q_d \\ \hline H_{d-2s} \\ \hline \end{array}$$

It remains to understand $Q_d$. We split our analysis into the cases of $r \geq 2$ and $r = 1$.

**Case when $r \geq 2$.** By our description of

$$S^dV/L_\omega S^{d-2}V \cong A_d/L_\omega^s H_{d-2s}$$

as a $\mathfrak{g}_{\overline{0}}$-module in lemma 5.6.2, we see that the $\mathfrak{g}_{\overline{0}}$-dominant weights with respect to the standard even Borel of $\mathfrak{g}_{\overline{0}}$ are all of the form $t\epsilon_1 + \delta_1 + \cdots + \delta_i$ for some $t \geq 0$ and some $i$.

If we choose the Borel corresponding to the $\epsilon\delta$-sequence $\delta \cdots \delta\epsilon \cdots \epsilon$, we see that the only such weights which are $\mathfrak{g}$-dominant with respect to this Borel are ones of the form $\delta_1 + \cdots + \delta_n + t\epsilon_1$ for some $t \geq 0$. When we change via odd reflections to the Borel with $\epsilon\delta$ sequence $\epsilon \cdots \epsilon\delta \cdots \delta$, this highest weight becomes $(t+n)\epsilon_1$. Call this latter Borel $\mathfrak{b}$.

We learn therefore, that with respect to the Borel $\mathfrak{b}$, the irreducibles which can appear in $Q_d$ must all have highest weight $t\epsilon_1$ for some $t \geq 0$.

Since $A_d$ is indecomposable, any irreducible factors which show up in it must have the same central character. The module $H_{d-2s}$ has highest weight $(d-2s)\epsilon_1$ with respect to $\mathfrak{b}$, and the only other highest weight of the form $t\epsilon_1$ with the same central character is $d\epsilon_1$. It follows that either $Q_d = L_{\mathfrak{b}}(d\epsilon_1)$ or $Q_d = 0$. But again, if $Q_d = 0$ then $A_d$ would give a non-trivial extension of $H_{d-2s}$ by itself which does not exist. This gives the structure of $A_d$ when $r \geq 2$:

$$A_d = \begin{array}{|c|}\hline L_{\mathfrak{b}}((d-2s)\epsilon_1) \\ \hline L_{\mathfrak{b}}(d\epsilon_1) \\ \hline L_{\mathfrak{b}}((d-2s)\epsilon_1) \\ \hline \end{array}$$

**Case when $r = 1$.** Notice that our decomposition of $S^dV$ for $d$ large implies, by 5.6.3,

$$k[G/K] = \bigoplus_{i=n-r+1}^{2(n-r)+2} A_d \oplus \bigoplus_{d>2(n-r)+2} H_d$$

so $A_d$ is a direct summand of $k[G/K]$. On the other hand, $k[G/K] = \operatorname{Ind}_K^G k$, where here $K = OSP_{1|2n}$. Since the category of finite-dimensional representations of $K$ is semi-simple, by Frobenius reciprocity we obtain that $k[G/K]$ must be a direct sum of injective $\mathfrak{g}$-modules.

In particular, $A_d$ must be itself be a sum of injective modules. Because we have shown it is indecomposable with socle $L_{\mathfrak{b}^{st}}((d-2s)\epsilon_1)$ with respect to the standard Borel $\mathfrak{b}^{st}$ of $\mathfrak{osp}(2|2n)$, it must be the injective hull of this irreducible module. The socle filtration of this module is:

$$A_d = \begin{array}{|c|} \hline L_{\mathfrak{b}^{st}}((d-2s)\epsilon_1) \\ \hline L_{\mathfrak{b}^{st}}(d\epsilon_1) \oplus L_{\mathfrak{b}^{st}}(-\epsilon_1 + \delta_1 + \cdots + \delta_{d-2s+1}) \\ \hline L_{\mathfrak{b}^{st}}((d-2s)\epsilon_1) \\ \hline \end{array}$$

### 5.6.13 Computation of $S^\bullet P_{n|n}$

In this section we compute the algebra of functions on $\Pi P_{n|n}$ as a $\mathfrak{p}(n)$-module.

Write $V = P_{n|n}$. We choose for Borel $\mathfrak{b} = \begin{bmatrix} A & B \\ 0 & -A^t \end{bmatrix}$, where $A$ is upper triangular and $B$ is symmetric. Recall that $\Pi V$ is spherical with respect to $\mathfrak{b}$. Therefore, we will compute highest weights of $S^\bullet(\Pi V^*)$ with respect to $\mathfrak{b}$.

We have $\Pi V^* \cong V \cong L_{\mathfrak{b}}(\epsilon_1)$, so $S^d(\Pi V^*) \cong S^d(V)$. Our non-degenerate odd $\mathfrak{p}(n)$-invariant form, which we view as an odd $\mathfrak{p}(n)$-module homomorphism

$$q : S^2V \to k,$$

gives odd $\mathfrak{p}(n)$-module homomorphisms

$$q : S^dV \to S^{d-2}V$$

for all $d \geq 2$. We also have

$$(S^dV)^* \cong S^dV^* \cong S^d\Pi V \cong \Pi^d\Lambda^dV$$

and hence we have the element $c \in \Pi^2\Lambda^2V^* = (S^2V)^*$ which is adjoint to $q$, i.e. for $x \in S^dV$, $y \in \Pi^{d-2}\Lambda^{d-2}V$, we have

$$(q(x), y) = (-1)^{|x|}(x, cy)$$

where $(-,-)$ denotes the pairing of dual vector spaces. Since $c$ is odd, $c^2 = 0$, so we get that $q^2 = 0$.

As a $\mathfrak{g}_{\bar{0}}$-module we have the decomposition

$$S^dV = \bigoplus_{\substack{i+j=d \\ j \leq n}} S^iL_0(\epsilon_1) \otimes \Lambda^jL_0(-\epsilon_n)$$

By Pieri's rule, we get

$$S^iL_0(\epsilon_1) \otimes \Lambda^jL_0(-\epsilon_n) = L_0(i\epsilon_1 - \epsilon_{n-j+1} - \cdots - \epsilon_n) \oplus L_0((i-1)\epsilon_1 - \epsilon_{n-j+2} - \cdots - \epsilon_n)$$

when $1 \leq j < n$, and if $j = n$ we only get the first factor.

For $j \neq n$, as a $\mathfrak{g}_{\overline{0}}$-module the kernel of $q$ on $S^iV_{\overline{0}} \otimes \Lambda^j V_{\overline{1}}^*$ will be

$$L_0(i\epsilon_1 - \epsilon_{n-j+1} - \cdots - \epsilon_n)$$

For $j = n$, the kernel of $q$ on $S^iL_0(\epsilon_1) \otimes \Lambda^j L_0(-\epsilon_n)$ will be everything if $d = n$ (equivalently $i = 0$), or the kernel will be trivial if $d > n$ (equivalently $i > 0$). Hence, as a $\mathfrak{g}_{\overline{0}}$-module, the kernel of $q$ on $S^dV$ will be

$$\bigoplus_{0 \leq i < \min(d,n)} L_0((d-i)\epsilon_1 - \epsilon_{n-i+1} - \cdots - \epsilon_n).$$

for $d \neq n$, and for $d = n$ we get

$$\bigoplus_{0 \leq i \leq n} L_0((n-i)\epsilon_1 - \epsilon_{n-i+1} - \cdots - \epsilon_n).$$

We can write $S^\bullet V = k[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$ where $x_i$ has weight $\epsilon_i$, $\xi_i$ weight $-\epsilon_i$. Then $x_1$ is our highest weight vector, and so $x_1^d$ will be a highest weight vector for all $d \geq 0$. As a $\mathfrak{g}_{\overline{0}}$-module, $x_1^d$ generates $S^dV_{\overline{0}}$. In $\mathfrak{g}_{\overline{1}}$ we have all differential operators $\xi_i\partial_{x_j} - \xi_j\partial_{x_i}$, for $i < j$. Applying to $x_1^d$, sequentially,

$$\xi_n\partial_{x_1} - \xi_1\partial_{x_n}, \quad \xi_{n-1}\partial_{x_1} - \xi_1\partial_{x_{n-1}}, \quad \ldots, \quad \xi_2\partial_{x_1} - \xi_1\partial_{x_2}$$

we get, up to scalar,

$$\xi_n x_1^{d-1}, \quad \xi_{n-1}\xi_n x_1^{d-2}, \quad \ldots, \quad \xi_2 \cdots \xi_n x_1^{d-n+1}$$

these are exactly the generators of the irreducible $\mathfrak{g}_{\overline{0}}$-summands of the kernel of $q$ for $d \neq n$. For $d = n$, we also get the one-dimensional $\mathfrak{g}_{\overline{0}}$-module generated by

$$\xi_1 \cdots \xi_n.$$

However, this element is not in $\mathcal{U}\mathfrak{g} \cdot x_1^n$; indeed, $x_1^n$ is in the $\mathfrak{g}$-submodule given by the image of $q : S^{n+2}V \to S^nV$, but $\xi_1 \cdots \xi_n$ is not in the image of $q$. Further, by applying operators $x_i\partial_{\xi_i}$ from the $\mathfrak{p}(n)$-action we can get back to $S^dV_{\overline{0}}$ from any of the above elements $\xi_i \cdots \xi_n x_1^{d-n+i}$. It follows from the above analysis that $\ker q \cong L_{\mathfrak{b}}(d\epsilon_1)$ for $d \neq n$. If $d = n$, $\ker q$ is a nontrivial extension of $L_{\mathfrak{b}}(-\omega)$ by $L_{\mathfrak{b}}(n\epsilon_1)$.

Now we claim that $S^0(V) = k$, $S^1V = V$, and for $d \geq 2$, $d \neq n$ we have the following socle filtrations:

$$S^dV = \boxed{\begin{array}{c} \Pi L((d-2)\epsilon_1) \\ \hline L(d\epsilon_1) \end{array}},$$

unless $d = n$, in which case we get

$$S^nV = \begin{array}{|c|} \hline \Pi L((n-2)\epsilon_1) \\ \hline \Pi^n L(-\omega) \\ \hline L(n\epsilon_1) \\ \hline \end{array}.$$

Further, in each of the above cases, the radical of the module is equal to $\ker q$.

Proof of claim: we have already computed the kernel of $q$ in each case and have shown it is indeed the radical of each module shown above. Further, our analysis shows that as a $\mathfrak{g}_{\overline{0}}$-module, we get an isomorphism (for $d \geq 2$)

$$q : S^dV/\ker q \to L_{\mathfrak{b}}((d-2)\epsilon_1)$$

It follows that this must be an isomorphism of $\mathfrak{g}$-modules. To finish the proof, we need to show the socle filtration is as advertised. We see that $q(x_1\xi_1 \cdots \xi_{n-1}) \neq$, and further

$$(\xi_n\partial_{x_1} - \xi_1\partial_{x_n})(x_1\xi_1\cdots\xi_{n-1}) = (-1)^{n-1}\xi_1\cdots\xi_n.$$

This completes the proof. We obtain the following corollary on the cohomology of $q$.

**Corollary 5.6.8.** *The cohomology of the operator $q$ on $S^\bullet V$ is one-dimensional and spanned by $\xi_1 \cdots \xi_n$. As a $\mathfrak{p}(n)$-module it is isomorphic to $\Pi^n L(-\omega)$.*

### 5.6.14 Symmetric algebras for $\mathfrak{q}(n)$-modules

By Schur-Sergeev duality (see [10] chapter 3), we have that $S^m(Q_{n|n}^*)$ is irreducible of highest weight $-m\epsilon_n$ with respect to the standard Borel for all $m, n$.

For the family of typical spherical modules for $\mathfrak{q}(2)$, we compute the symmetric algebras of the dual with respect to the Borel $(\mathfrak{b}^{st})^{op}$. This leads to a canonical identification $L_{\mathfrak{b}^{st}}(\lambda)^* \cong L_{(\mathfrak{b}^{st})^{op}}(-\lambda)$ when the length of $\lambda$ is 2 (the length is the number of non-zero entries of $\lambda$ with respect to the basis $\epsilon_1, \dots, \epsilon_n$ of $\mathfrak{h}_{\overline{0}}^*$ described in section 2.3.2). We then have for $t \neq -1/2$, by the character formula for typical $\mathfrak{q}(2)$-modules,

$$S^d(L((t+1)\epsilon_1 + t\epsilon_2)^*) \cong \Pi^d L_{(\mathfrak{b}^{st})^{op}}(-d(t+1)\epsilon_1 - dt\epsilon_2)$$

and

$$S^d(\Pi L((t+1)\epsilon_1 + t\epsilon_2)^*) \cong L_{(\mathfrak{b}^{st})^{op}}(-d(t+1)\epsilon_1 - dt\epsilon_2)$$

## 5.7 Group-graded supergroups and actions

Next we study spherical varieties for an especially well-understood class of quasi-reductive supergroups, those which are group-graded. We first give a definition along with a brief discussion of group-graded supergroups and their actions. This will closely follow the definitions and theorems of section 4 of [62], except that we are working in the algebraic category and not the complex analytic category.

Introduce the category GSV whose objects are supervarieties of the form $X = (|X_0|, \Lambda^\bullet \mathcal{N})$ where $X_0$ is a variety and $\mathcal{N}$ is a coherent sheaf on $X_0$. In other words the objects are graded supervarieties with a given choice of grading. This endows all objects of GSV with a canonical $\mathbb{Z}$-grading on their structure sheaf. We then define morphisms in this category to be those morphisms of supervarieties that preserve the given $\mathbb{Z}$-gradings.

There is a natural functor gr from the category of locally graded supervarieties to GSV. On objects it is given by

$$\operatorname{gr} X = (|X|, \bigoplus_{i\geq 0} \mathcal{J}_X^i/\mathcal{J}_X^{i+1}),$$

so that in particular $|X| = |\operatorname{gr} X|$ and $X(k) = |\operatorname{gr} X(k)|$. Note that the natural map $\Lambda^i \mathcal{J}_X/\mathcal{J}_X^2 \to \mathcal{J}_X^i/\mathcal{J}_X^{i+1}$ is an isomorphism because of our assumption that supervarieties are locally graded. For a morphism $\psi : X \to Y$ we let $\operatorname{gr}\psi : \operatorname{gr} X \to \operatorname{gr} Y$ be the same map of underlying topological spaces and set

$$(\operatorname{gr}\psi)^* : \bigoplus_{i\geq 0} \mathcal{J}_Y^i/\mathcal{J}_Y^{i+1} \to (\operatorname{gr}\psi)_* \bigoplus_{i\geq 0} \mathcal{J}_X^i/\mathcal{J}_X^{i+1}$$

to be

$$(\operatorname{gr}\psi)^*(f + \mathcal{J}_Y^i) = \psi^*(f) + \mathcal{J}_X^i$$

where $f$ is a section of $\mathcal{J}_Y^{i-1}$.

If $X$ and $Y$ are locally graded supervarieties, then $X \times Y$ is a locally graded supervariety in a natural way, and $\mathcal{J}_{X\times Y} = p_X^*\mathcal{J}_X + p_Y^*\mathcal{J}_Y$, where $p_X, p_Y$ are the natural projection maps. On the other hand, given two graded supervarieties $X' = (|X'|, \Lambda^\bullet \mathcal{N}_{X'})$, $Y' = (|Y'|, \Lambda^\bullet \mathcal{N}_{Y'})$, we define their direct product in GSV to be the direct product of supervarieties $X' \times Y'$ with the natural splitting $\mathcal{O}_{X'\times Y'} = \Lambda^\bullet(p_{X_0'}^*\mathcal{N}_{X'} \oplus p_{Y_0'}^*\mathcal{N}_{Y'})$. Then there is a canonical isomorphism in GSV $\operatorname{gr}(X \times Y) \cong \operatorname{gr} X \times \operatorname{gr} Y$ coming from the fact that taking tensor product commutes with taking associated graded for filtered vector spaces with finite filtrations.

If $G$ is an algebraic supergroup, then using the canonical isomorphism $\operatorname{gr}(G\times G) \cong \operatorname{gr} G \times \operatorname{gr} G$ we have that $\operatorname{gr} G$ with the maps $\operatorname{gr} m_G, \operatorname{gr} e_g$ and $\operatorname{gr} s_G$ forms a algebraic supergroup. If $\mathfrak{g} = \operatorname{Lie} G$ we write $\mathfrak{g}^{\mathrm{gr}} := \operatorname{Lie}\operatorname{gr} G$. Further, if $a : G \times X \to X$ is an action of a Lie supergroup on a locally supervariety $X$, then $\operatorname{gr} a : \operatorname{gr}(G \times X) \cong \operatorname{gr} G \times \operatorname{gr} X \to \operatorname{gr} X$ defines an action of $\operatorname{gr} G$ on $\operatorname{gr} X$.

**Definition 5.7.1.** If $G$ is a supergroup, we call $\operatorname{gr} G$ the group-graded supergroup gotten from $G$, and we say $G$ is a group-graded supergroup if $G \cong \operatorname{gr} G$ as supergroups. If $a : G \times X \to X$ is an action of $G$ on a locally graded supervariety $X$, we call $\operatorname{gr} a$ the graded action of $\operatorname{gr} G$ on $\operatorname{gr} X$, and we say that $a$ is a graded action if it is isomorphic to $\operatorname{gr} a$ in the natural sense.

*Remark* 5.7.2. As algebraic supergroups are smooth affine supervarieties, they are always graded. The property of being group-graded is stronger in that it requires the multiplication and inversion morphisms to respect some grading.

We give an explicit construction of $\mathrm{gr}\, G$. Being affine the supergroup $G$ is graded, so fix a grading of $G$ so that its structure sheaf is equipped with a $\mathbb{Z}$-grading. We call $G$ with this chosen grading $\mathrm{gr}\, G$, and we think of it as an object of $\mathsf{GSV}$. This choice of grading determines a grading of $G \times G$, and thus we may write

$$m_G^* = \bigoplus_{i \geq 0} (m_G^*)_i, \qquad s_G^* = \bigoplus_{i \geq 0} (s_G^*)_i$$

where $(m_G^*)_i$, respectively $(s_G^*)_i$ increase the $\mathbb{Z}$-grading of an element by exactly $i$. We set $m_{\mathrm{gr}\, G}^* = (m_G^*)_0$, $s_{\mathrm{gr}\, G}^* = (s_G^*)_0$, and $e_{\mathrm{gr}\, G}^* = e_G^*$, and these are all algebra homomorphisms. In this way, the induced maps on the supervariety $G$ given by $m_{\mathrm{gr}\, G}$, $s_{\mathrm{gr}\, G}$, and $e_{\mathrm{gr}\, G}$ become morphisms in $\mathsf{GSV}$ and define the structure of a supergroup on $\mathrm{gr}\, G$, and thus this supergroup is group-graded. It follows in particular that we may identify $(\mathrm{gr}\, G)_0$ and $G_0$ as algebraic groups.

Now since we have constructed $\mathrm{gr}\, G$ so that it is the same supervariety as $G$ (the only difference being that it has a chosen $\mathbb{Z}$-grading on its structure sheaf), we have an identification $T_eG = T_e\, \mathrm{gr}\, G$. Thus we may canonically identify $\mathfrak{g} \cong \mathfrak{g}^{\mathrm{gr}}$ as super vector spaces. Given $u_e \in T_eG$, we write $u_L$ (resp. $u_R$) for the corresponding $G$ right-invariant (resp. $G$ left-invariant) vector field on $G$, and $\mathrm{gr}\, u_L$ (resp. $\mathrm{gr}\, u_R$) for the corresponding $\mathrm{gr}\, G$ right-invariant (resp. $\mathrm{gr}\, G$ left-invariant) vector field on $G$. Using the $\mathbb{Z}$-grading on $k[G]$ we may write $u_L = \sum\limits_{i \in \mathbb{Z}} (u_L)_i$ (resp. $u_R = \sum\limits_{i \in \mathbb{Z}} (u_R)_i$) as endomorphisms of $k[G]$, where $(u_L)_i$ (resp. $(u_R)_i$) changes the $\mathbb{Z}$-grading by $i$.

**Lemma 5.7.3.** *Let $u_e \in T_eG$. If $u_e$ is even then $\mathrm{gr}\, u_L = (u_L)_0$ and $\mathrm{gr}\, u_R = (u_R)_0$, and if $u_e$ is odd then $\mathrm{gr}\, u_L = (u_L)_{-1}$ and $\mathrm{gr}\, u_R = (u_R)_{-1}$.*

*Proof.* We prove this for right-invariant vectors, with the case of left-invariant vector fields being similar. have

$$u_L = -(u_e \otimes 1) \circ (m_G^*) = \bigoplus_{i \geq 0} -(u_e \otimes 1) \circ (m_G^*)_i.$$

For $f \in k[G]_k$, $(m_G^*)_i(f) \in \bigoplus\limits_j k[G]_j \otimes k[G]_{k+i-j}$. If $u_e$ is even, then $u_e$ vanishes on $k[G]_i$ for $i > 0$, so

$$-(u_e \otimes 1) \circ (m_G^*)_i = (u_L)_i,$$

so $\mathrm{gr}\, u_L = (u_L)_0$. If $u_e$ is odd, then $u_e$ vanishes on $k[G]_i$ for $i \neq 1$, so

$$-(u_e \otimes 1) \circ (m_G^*)_i = (u_L)_{i-1},$$

so $\mathrm{gr}\, u_L = (u_L)_{-1}$. □

**Corollary 5.7.4.** *We have* $[\mathfrak{g}_{\bar{1}}^{\mathrm{gr}}, \mathfrak{g}_{\bar{1}}^{\mathrm{gr}}] = 0$*, i.e.* $\mathfrak{g}^{\mathrm{gr}}$ *is graded (see example 2.3.2). In fact a supergroup* $G$ *is group-graded if and only if* $\mathfrak{g}$ *is graded, where* $\mathfrak{g} = \operatorname{Lie} G$*.*

*Proof.* For the first statement, the supercommutator of two degree (-1)-maps is of degree (-2) with respect to the $\mathbb{Z}$-grading. However there are no vector fields of degree (-2) on a graded supervariety, thus the supercommutator must be zero. A proof of the second statement is given in proposition 4.4 of [62]. □

Now $G_0 \times G_0$ acts on $G$ by left and right translation. Using Koszul's realization of $k[G]$ as a coinduced algebra on $k[G_0]$ (see [31]), which gives a natural grading of $G$, we obtain a natural $G_0 \times G_0$-equivariant grading (this does not require that $G_0$ is reductive; if $G_0$ is reductive we could also use theorem 4.8.2 to find a $G_0 \times G_0$-equivariant grading). Thus if we constructed $\operatorname{gr} G$ as above, then using the $G_0 \times G_0$-equivariant grading we would have that if $u_e$ is even, $u_L = (u_L)_0$ and $u_R = (u_R)_0$ since they will preserve the $\mathbb{Z}$-grading. Thus we have shown:

**Lemma 5.7.5.** *If we construct* $\operatorname{gr} G$ *by using a* $G_0 \times G_0$*-equivariant grading of* $G$*, then for an even tangent vector* $u_e \in T_eG$*,* $u_L = \operatorname{gr} u_L$ *and* $u_R = \operatorname{gr} u_R$*. In particular* $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_{\bar{0}}^{\mathrm{gr}}$ *as Lie algebras of vector fields on* $G$*. Further, the natural isomorphism of super vector spaces* $\mathfrak{g}_{\bar{1}} \cong \mathfrak{g}_{\bar{1}}^{\mathrm{gr}}$ *induced from this grading is an isomorphism of* $\mathfrak{g}_{\bar{0}}$*-modules.*

*Proof.* It remains to show the second statement. For this, we observe that for $u \in \mathfrak{g}_{\bar{0}}$, $v \in \mathfrak{g}_{\bar{1}}$, $[u, v]_i = [u, v_i]$. Since $\operatorname{gr} v = v_{-1}$, the statement follows. □

We now move on to the study of graded actions. For the rest of the section we assume that $X$ is locally graded.

**Lemma 5.7.6.** *Suppose* $G$ *is a supergroup which acts on a supervariety* $X$*, and consider the action of* $\operatorname{gr} G$ *on* $\operatorname{gr} X$*. Then for* $u \in \mathfrak{g}_{\bar{0}}^{\mathrm{gr}}$*,* $u$ *preserves the* $\mathbb{Z}$*-grading on* $\mathcal{O}_{\operatorname{gr} X}$*, and for* $u \in \mathfrak{g}_{\bar{1}}^{\mathrm{gr}}$*,* $u$ *acts by degree* $-1$ *on* $\mathcal{O}_{\operatorname{gr} X}$*.*

*Proof.* For $f \in (\mathcal{O}_{\operatorname{gr} X})_i$, we have

$$u(f) = -(u_e \otimes 1) \circ (\operatorname{gr} a)^*(f).$$

Now since $\operatorname{gr} a$ preserves the $\mathbb{Z}$-grading, we have $(\operatorname{gr} a)^*(f) \in \bigoplus_{0 \le j \le i} (\mathcal{O}_{\operatorname{gr} G})_j \otimes (\mathcal{O}_{\operatorname{gr} X})_{i-j}$. If $u \in \mathfrak{g}_{\bar{0}}^{\mathrm{gr}}$, then $u_e$ vanishes on $(\mathcal{O}_G)_i$ for $i > 0$, and if $u \in \mathfrak{g}_{\bar{1}}^{\mathrm{gr}}$ then $u_e$ vanishes on $(\mathcal{O}_G)_i$ for $i \neq 1$. The result follows. □

Now if $K$ is a closed subgroup of $G$ via the inclusion $\phi : K \to G$, then the $\mathbb{Z}$-gradings induced on $k[G]$ and $k[K]$ from Koszul's realization make the natural pullback surjection $\phi^* : k[G] \to k[K]$ into a graded map. Thus the kernel of this map, $I_K \subseteq k[G]$, becomes a graded ideal. Further, if we consider the group-graded supergroup structure on $K$ and $G$ from these gradings, $\phi$ will be a homomorphism of supergroups $\operatorname{gr} K \to \operatorname{gr} G$. Thus $\operatorname{gr} \phi = \phi$, and so $I_K = I_{\operatorname{gr} K}$.

**Lemma 5.7.7.** *If $X$ is a supervariety and $x \in X(k)$, $\mathrm{Stab}_{\mathrm{gr}\,G}(x) = \mathrm{Stab}_G(x)$ as closed subvarieties of $G$.*

*Proof.* Write $K = \mathrm{Stab}_G(x)$, $\mathcal{I}_x$ for the maximal ideal sheaf of $x \in X(k)$ and $\mathcal{I}_x^{\mathrm{gr}}$ for the maximal ideal sheaf of $x \in \mathrm{gr}\,X(k)$. Then by assumption we have $(a_x)^*(\mathcal{I}_x) = I_K$. But with respect to the $\mathbb{Z}$-grading from Koszul's realization, $I_K$ is a graded ideal and thus $(\mathrm{gr}\,a_x)^*(\mathcal{I}_x^{\mathrm{gr}}) = I_K = I_{\mathrm{gr}\,K}$, and we are done. □

**Corollary 5.7.8.** *If $X$ is a homogeneous $G$-supervariety isomorphic to $G/K$, then $\mathrm{gr}\,X$ is a homogeneous $\mathrm{gr}\,G$-supervariety isomorphic to $\mathrm{gr}\,G/\,\mathrm{gr}\,K$.*

### 5.7.1 $G$ a quasi-reductive group-graded supergroup

Let $G$ be a quasi-reductive supergroup, and write $\mathfrak{g} = \mathrm{Lie}\,G$ as always.

**Lemma 5.7.9.** *If $\mathfrak{l} \subseteq \mathfrak{g}_{\overline{1}}$ is an abelian ideal of $\mathfrak{g}$, then $\mathfrak{l}$ is contained in every hyperborel subalgebra of $\mathfrak{g}$.*

*Proof.* If $\mathfrak{b}$ is a hyperborel subalgebra, then $\mathfrak{b} + \mathfrak{l}$ is a subalgebra that still satisfies the first two properties of being a hyperborel, and thus by maximality $\mathfrak{b} = \mathfrak{b} + \mathfrak{l}$. □

**Corollary 5.7.10.** *Let $G$ be a group-graded quasi-reductive supergroup. Then every hyperborel of $\mathfrak{g}$ is of the form $\mathfrak{b}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$, where $\mathfrak{b}_{\overline{0}}$ is a Borel subalgebra of $\mathfrak{g}_{\overline{0}}$. In particular $G$ has only one hyperborel subalgebra up to conjugacy.*

*Proof.* In this case $\mathfrak{g}_{\overline{1}}$ is an abelian ideal of $\mathfrak{g}$, so we use lemma 5.7.9 to get that every hyperborel must contain $\mathfrak{g}_{\overline{1}}$, and thus they are all of this form. If $\mathfrak{b}, \mathfrak{b}'$ are two hyperborels, then conjugating $\mathfrak{b}_{\overline{0}}$ to $\mathfrak{b}'_{\overline{0}}$ will conjugate $\mathfrak{b}$ to $\mathfrak{b}'$. □

In fact, we have

**Proposition 5.7.11.** *If $\mathfrak{g}$ is quasi-reductive and $\mathfrak{g}_{\overline{1}}$ is contained in a hyperborel subalgebra, then $\mathfrak{g}$ is graded.*

*Proof.* Since $[\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}] \subseteq \mathfrak{g}_{\overline{0}}$ is a submodule of the adjoint representation, if it is nonzero it must intersect any Cartan subalgebra nontrivially. Thus if $\mathfrak{g}_{\overline{1}}$ is contained in a hyperborel subalgebra we must have $[\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}] = 0$, i.e. $\mathfrak{g}$ is graded. □

The following lemma now follows easily from what we have shown so far.

**Lemma 5.7.12.** *If $G$ is a quasi-reductive supergroup, and $B$ is a hyperborel subgroup of $G$, then $\mathrm{gr}\,G$ is quasi-reductive and $\mathrm{gr}\,B$ is a subgroup of a hyperborel of $\mathrm{gr}\,G$.*

We can now prove that the functor gr preserves sphericity.

**Corollary 5.7.13.** *Suppose that $G$ is quasi-reductive and $X$ is a spherical $G$-supervariety. Then $\mathrm{gr}\,X$ is a locally graded spherical $\mathrm{gr}\,G$-supervariety under the graded action.*

*Proof.* Let $B$ be a hyperborel of $G$ with an open orbit on $X$. Then by corollary 5.7.8, $\operatorname{gr} B$ has an open orbit on the same underlying open subset of $|X|$. By lemma 5.7.12, $\operatorname{gr} B$ is contained in a hyperborel of $\operatorname{gr} G$, and the hyperborel of $\operatorname{gr} G$ containing $\operatorname{gr} B$ has an open orbit at $x$. Thus $\operatorname{gr} X$ is spherical. $\square$

For the rest of this section we assume that $G$ is a group-graded quasi-reductive supergroup.

**Proposition 5.7.14.** *Suppose that $X$ is a locally graded spherical $G$-supervariety. Then $\operatorname{soc} k[X]$ is a subalgebra of $k[X]$ and the restriction of $i_X^*$ to $\operatorname{soc} k[X]$ is injective. In particular, $\operatorname{soc} k[X]$ is an even subalgera of $k[X]$ without nilpotents.*

*Proof.* A semisimple representation of $G$ is exactly the pullback of a semisimple representation of $G_0$ under the natural surjection $G \to G_0$. Therefore $\operatorname{soc} k[X]$ can be thought of as a sum of simple $G_0$-representations, and thus the tensor product of two subrepresentations of $\operatorname{soc} k[X]$ is again a semisimple $G_0$-representation. Since multiplication is $G$-equivariant, it follows that $\operatorname{soc} k[X]$ is a subalgebra of $k[X]$.

Recall that $i_X$ is a $G_0$-equivariant map of algebras. If $\operatorname{soc} k[X] \cap \ker i_X^* \neq 0$, then it must contain a simple subrepresentation $L$. Let $f \in L$ be the $B$-highest weight vector for some hyperborel $B$ of $G$. Then by theorem 5.1.4, $f$ is non-nilpotent and thus $i_X^*(f) \neq 0$, a contradiction. This completes the proof. $\square$

**Corollary 5.7.15.** *If $X$ is a locally graded affine spherical $G$-supervariety, then $k[X]$ is completely reducible if and only if $X = X_0$.*

*Proof.* If $X = X_0$ then $G$ acts via the quotient to $G_0$ so $k[X]$ is completely reducible.

On the other hand, the condition that $k[X]$ is completely reducible is equivalent to $k[X] = \operatorname{soc} k[X]$. By proposition 5.7.14, this condition implies that $i_X^*$ is an isomorphism, so $X = X_0$. $\square$

We now focus on the case of homogeneous spherical supervarieties for $G$.

**Lemma 5.7.16.** *If $X$ is a homogeneous $G$-supervariety, then $X$ is graded, and the action $a : G \times X \to X$ is isomorphic to the graded action $\operatorname{gr} a$.*

*Proof.* This follows directly from corollary 5.7.8. $\square$

**Proposition 5.7.17.** *If $X$ is a homogeneous $G$-supervariety, then $X$ is spherical if and only if $X_0$ is a spherical $G_0$-variety.*

*Proof.* If $X = G/K$, then we want to determine when $\mathfrak{k} = \operatorname{Lie} K$ has a complementary hyperborel subalgebra in $\mathfrak{g} = \operatorname{Lie} G$. By corollary 5.7.10, the hyperborels of $\mathfrak{g} = \operatorname{Lie} G$ are all of the form $\mathfrak{b}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ for a Borel subalgebra $\mathfrak{b}_{\overline{0}}$ of $\mathfrak{g}_{\overline{0}}$. Thus it is equivalent to find a Borel subalgebra $\mathfrak{b}_{\overline{0}}$ complementary to $\mathfrak{k}_{\overline{0}}$ in $\mathfrak{g}_{\overline{0}}$. Since $X_0 = G_0/K_0$, this completes the proof. $\square$

**Proposition 5.7.18.** *If $X$ is a homogeneous spherical $G$-supervariety, then there exists a grading of $X$ for which $k[X]_0 = \operatorname{soc} k[X]$. In particular, if $B$ is a hyperborel of $G$, then $\Lambda_B^+(X) = \Lambda_{B_0}^+(X_0)$.*

*Proof.* By lemma 5.7.16, there exists a grading of $X$ for which the action of $G$ is graded. With respect to this action, $\mathfrak{g}_{\bar{1}}$ acts by degree $-1$ derivations on $\mathcal{O}_X$. Thus $k[X]_0 \subseteq k[X]^{\mathfrak{g}_{\bar{1}}} = \operatorname{soc} k[X]$. On the other hand, by proposition 5.7.14, $i_X : \operatorname{soc} k[X] \to k[X_0]$ is injective. Since $i_X : k[X]_0 \to k[X_0]$ is an isomorphism we must have $k[X]_0 = \operatorname{soc} k[X]$. □

In the case of homogeneous affine spaces, we have the following strengthening of corollary 5.7.15. Note that a homogeneous space $G/K$ is affine if and only if $K_0$ is reductive, i.e. $K$ is quasi-reductive.

**Proposition 5.7.19.** *If $X = G/K$ is a homogeneous affine $G$-space, then the following are equivalent.*

1. *$X = X_0$.*
2. *$k[X]$ is completely reducible.*
3. *$k$ splits of from $k[X]$ as a $G$-module.*

Before proving this, we first state a lemma.

**Lemma 5.7.20.** *Suppose that $G$ is quasi-reductive and that $\mathfrak{g} = \operatorname{Lie}(G)$ has an odd abelian ideal $\mathfrak{l} \subseteq \mathfrak{g}_{\bar{1}}$. Then if $K \subseteq G$ is a quasi-reductive subgroup, $k$ splits off from $k[G/K]$ only if $\mathfrak{l} \subseteq \mathfrak{k} = \operatorname{Lie}(K)$.*

*Proof.* Suppose that $\mathfrak{l}$ is not contained in $\mathfrak{k}$. Let $\mathfrak{m} = \mathfrak{k} \cap \mathfrak{l}$, and let $\mathfrak{r}$ be a $\mathfrak{k}_{\bar{0}}$-invariant complement to $\mathfrak{m}$ in $\mathfrak{l}$, where we are using that $K_0$ is reductive. Write $L$,$M$, and $R$ for the purely even vector spaces with $L = \mathfrak{l}_{\bar{1}}, M = \mathfrak{m}_{\bar{1}}$, and $R = \mathfrak{r}_{\bar{1}}$. We may naturally view $L$ as a $\mathfrak{g}_{\bar{0}}$-module according to the restriction of the adjoint action of $\mathfrak{g}_{\bar{0}}$ to $\mathfrak{l}$, using that $\mathfrak{l}$ is an ideal of $\mathfrak{g}$.

Now consider the following $\mathfrak{g}$-module $V$. As a $\mathfrak{g}_{\bar{0}}$-module, $V = L \otimes L^* \oplus \Pi L^*$. Choose a $\mathfrak{g}_{\bar{0}}$-invariant complement $\mathfrak{l}'$ to $\mathfrak{l}$ in $\mathfrak{g}_{\bar{1}}$. Then we say that for $u \in \mathfrak{l}'$, $u$ acts by 0 on $V$, and for $u \in \mathfrak{l}$, $u$ acts by 0 on $V_0 = L \otimes L^*$, while for $\varphi \in V_{\bar{1}} = \Pi L^*$, we set $u \cdot \varphi := u \otimes \varphi \in V_{\bar{0}}$. Then this defines a representation of $\mathfrak{g}$ on $V$. Further, the span of the element $v_L \in V_{\bar{0}} = L \otimes L^*$ which correspond to the identity map on $L$ defines an even trivial subrepresentation $k\langle v_L \rangle$ of $V$. This subrepresentation does not split off of $V$, as we see that if $u_1, \dots, u_n$ is a basis of $L$ and $\varphi_1, \dots, \varphi_n$ is a the parity shift of a dual basis in $\Pi L^*$, then we have the following equation in $V$:

$$\sum_{i=1}^{n} u_i \cdot \varphi_i = \sum_{i=1}^{n} u_i \otimes \varphi_i = v_L$$

Consider the element $\psi \in V^*$ corresponding to the trace form on $R \otimes R^* \subseteq L \otimes L^*$. Then as an element of $V^*$, $\psi$ is $\mathfrak{k}_{\bar{0}}$-invariant since $R$ is a $\mathfrak{k}_{\bar{0}}$-submodule. If $u \in \mathfrak{k}_{\bar{1}}$ and $\varphi \in V_{\bar{1}}$, then $u \cdot \varphi = u \otimes \varphi \in M \otimes L^*$, and thus $\psi(u \otimes \varphi) = 0$. It follows that $\psi \in (V^*)^{\mathfrak{k}}$, i.e. it defines an even coinvariant of $V$, so by Frobenius reciprocity it defines a $G$-module morphism $\Psi : V \to k[G/K]$. Further, since $\psi(v_L) \neq 0$ and $v_L$ is $G$-fixed, $\Psi(v_L)$ is a non-zero constant function on $G/K$. We see that

$$\sum_{i=1}^{n} u_i \cdot \Psi(\varphi_i) = \Psi\left(\sum_{i=1}^{n} u_i \cdot \varphi_i\right) = \Psi(v_L).$$

It follows that $k$ does not split off from $k[G/K]$, and we are done. □

Now we prove proposition 5.7.19.

*Proof.* Since $\mathfrak{g}_{\overline{1}}$ if an odd abelian ideal of $\mathfrak{g}$, if $K \subseteq G$ is a quasi-reductive subgroup, $k$ splits off from $k[G/K]$ only if $\mathfrak{g}_{\overline{1}} \subseteq \mathfrak{k}$ by lemma 5.7.20, and in this case $G/K$ is a purely even variety. This shows (3) $\implies$ (1). Both (1) $\implies$ (2) and (2) $\implies$ (3) are obvious. □

## 5.8 Sphericity and the Duflo-Serganova functor

Let $\mathfrak{g}$ be a Lie superalgebra and $x \in \mathfrak{g}_{\overline{1}}$ a self-commuting element (i.e. $[x, x] = 0$). Then $\operatorname{Im} \operatorname{ad}(x) = [x, \mathfrak{g}]$ is an ideal of $\ker \operatorname{ad}(x)$, so we may define a new Lie superalgebra $\mathfrak{g}_x := \ker \operatorname{ad}(x)/\operatorname{Im} \operatorname{ad}(x)$. Further, if $V$ is a representation of $\mathfrak{g}$, then $V_x := \ker(x)/\operatorname{Im}(x)$ will be a representation of $\mathfrak{g}_x$. This defines a functor $DS_x$ from the category of $\mathfrak{g}$-modules to the category of $\mathfrak{g}_x$-modules, called the Duflo-Serganova functor, which was originally studied in [15]. It is a tensor functor but it is not exact.

### 5.8.1 Formulation of question

Let $\widetilde{G}$ be a quasi-reductive supergroup with Lie superalgebra $\widetilde{\mathfrak{g}}$, and let $x \in \widetilde{\mathfrak{g}}_{\overline{1}}$ be a self-commuting element. Let $C(x)$ be the stabilizer of $x$ under the parity shift of the adjoint representation, i.e. the action of $\widetilde{G}$ on $\Pi\widetilde{\mathfrak{g}}$. In particular

$$\operatorname{Lie} C(x) = \mathfrak{c}(x) = \ker \operatorname{ad}(x),$$

the centralizer of $x$ in $\widetilde{\mathfrak{g}}$. Let $M \subseteq C(x)$ be a normal subgroup with $\operatorname{Lie} M = [x, \widetilde{\mathfrak{g}}] = \operatorname{Im} \operatorname{ad}(x)$. Now let $G := C(x)/M$ so that $\mathfrak{g} := \operatorname{Lie} G \cong \widetilde{\mathfrak{g}}_x$. Finally, let $K$ be a closed subgroup of $G$ with $\mathfrak{k}$ its Lie superalgebra.

Consider the homogeneous supervariety $\widetilde{G}/\widetilde{K}$ where $\widetilde{K}$ is a closed subgroup of $C(x)$ with $\widetilde{\mathfrak{k}} := \operatorname{Lie} \widetilde{K}$ the preimage of $\mathfrak{k}$ under the projection $\mathfrak{c}(x) \to \mathfrak{g}$. Notice that a special case of this is when $K = G$ and $\widetilde{K} = C(x)$ so that we get $\widetilde{G}/C(x)$. This is exactly the odd adjoint orbit of $x$ in $\Pi\widetilde{\mathfrak{g}}$.

We will give a answer to the following question for certain supergroups $G$, certain subgroups $K$, and certain self-commuting elements $x$:

(Q1) If $G/K$ is spherical, is $\widetilde{G}/\widetilde{K}$ also spherical? Equivalently is $\widetilde{\mathfrak{k}}$ is a spherical subalgebra of $\widetilde{\mathfrak{g}}$?

As already indicated, (Q1) includes in particular the question:

(Q2) Is $\widetilde{G}/C(x)$ spherical? In other words, are odd adjoint orbits of self-commuting elements of $\widetilde{\mathfrak{g}}_{\overline{1}}$ spherical?

Given the importance of adjoint orbits in representation theory, the second question is of clear interest.

Both (Q1) and (Q2) are false in general: if $\mathfrak{g}_{\overline{0}}$ is reductive and $V$ is a $\mathfrak{g}_{\overline{0}}$-module that is not spherical then the centralizer of a general odd element in $\mathfrak{g}_{\overline{0}} \ltimes V$ will not be a spherical subalgebra. However in many situations, in particular for distinguished basic Lie superalgebras, the answer is yes:

**Theorem 5.8.1.** *Suppose that $\widetilde{G}$ is a quasi-reductive supergroup such that* Lie $\widetilde{G}$ *is distinguished basic. Then the answer to (Q1), and thus also (Q2), is always yes. Further we have a characterization of which Borels (up to conjugacy) of $\widetilde{\mathfrak{g}}$ are complementary to $\widetilde{\mathfrak{k}}$ in $\widetilde{\mathfrak{g}}$.*

To clarify, a subspace $V \subseteq \widetilde{\mathfrak{g}}$ is complementary to $\widetilde{\mathfrak{k}}$ if $V + \widetilde{\mathfrak{k}} = \widetilde{\mathfrak{g}}$. If $\widetilde{\mathfrak{g}}$ is not basic distinguished then our results are not as strong, but are still of interest.

Our approach to understanding sphericity in this problem will be via Borel subalgebras, which are always contained in hyperborel subalgebras if $\mathfrak{g}$ is Cartan even, which we will assume. Thus if a subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ has a complementary Borel subalgebra, and $\mathfrak{g}$ is Cartan even, it will also have a complementary hyperborel subalgebra.

### 5.8.2 General case

Suppose that $\widetilde{\mathfrak{g}}$ is a Cartan-even quasi-reductive Lie superalgebra, $\mathfrak{h} \subseteq \widetilde{\mathfrak{g}}$ a Cartan subalgebra, and $x \in \widetilde{\mathfrak{g}}_{\overline{1}}$ is an odd self-commuting root vector of weight $\alpha$. Finally, choose a Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ which is complementary to $\mathfrak{k}$. Then we suppose that there exists a Borel subalgebra $\widetilde{\mathfrak{b}}$ of $\widetilde{\mathfrak{g}}$ such that

(1) $\alpha$ is the smallest root, i.e. if $\widetilde{\mathfrak{b}}$ is determined by the homomorphism $\gamma : Q \to \mathbb{R}$, then $\gamma(\alpha) \leq \gamma(\beta)$ for all $\beta \in \Delta$; and

(2) $\mathfrak{b} \subseteq (\mathfrak{c}(x) \cap \widetilde{\mathfrak{b}})/([x, \mathfrak{g}] \cap \widetilde{\mathfrak{b}})$.

Writing $p_x : \mathfrak{c}(x) \to \mathfrak{g}$ for the natural projection, we claim that $\widetilde{\mathfrak{k}} = p_x^{-1}(\mathfrak{k})$ has $\widetilde{\mathfrak{k}} + \widetilde{\mathfrak{b}} = \widetilde{\mathfrak{g}}$, so in particular $\widetilde{\mathfrak{k}}$ is a spherical subalgebra. Indeed, if $y$ is a root vector of weight $\beta$ and $y \notin \mathfrak{c}(x)$, then $[x, y] \neq 0$ so $\alpha + \beta$ is a root. Since $\alpha$ is the smallest root, we must have $\beta \in \Delta^+$, i.e. $\beta$ must be positive. Therefore $y \in \widetilde{\mathfrak{b}}$. On the other hand if $y \in \mathfrak{c}(x)$ then by our second assumption on $\widetilde{\mathfrak{b}}$ we have that $\widetilde{\mathfrak{b}} \cap \mathfrak{c}(x) + \widetilde{\mathfrak{k}} = \mathfrak{c}(x)$ so $y \in \widetilde{\mathfrak{b}} + \widetilde{\mathfrak{k}}$. Thus we have shown that $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{b}} + \widetilde{\mathfrak{k}}$. We state this as a theorem:

**Theorem 5.8.2.** *In the context of section 5.8.1, suppose that $\widetilde{G}$ is Cartan-even, $x$ is a root vector of weight $\alpha$, and $\widetilde{\mathfrak{k}}$ is spherical with respect to some Borel subalgebra $\mathfrak{b}$. Then if there exists a Borel subalgebra $\widetilde{\mathfrak{b}}$ of $\widetilde{\mathfrak{g}}$ such that $\alpha$ is the smallest root and $\mathfrak{b} \subseteq (\mathfrak{c}(x) \cap \widetilde{\mathfrak{b}})/([x, \mathfrak{g}] \cap \widetilde{\mathfrak{b}})$, then $\widetilde{G}/\widetilde{K}$ is spherical with respect to $\widetilde{\mathfrak{b}}$.*

This result has the following nice corollary.

**Corollary 5.8.3.** *If $\widetilde{G}$ is a Cartan-even quasi-reductive Lie supergroup and $x \in \widetilde{\mathfrak{g}}_{\bar{1}}$ is root vector with weight $\alpha$ such that there exists a Borel subalgebra $\widetilde{\mathfrak{b}}$ of* Lie $\widetilde{G}$ *where $\alpha$ is the smallest root, then $\widetilde{G}/C(x)$ is spherical with respect to this Borel.*

### 5.8.3 Basic distinguished case

The Duflo-Serganova functor has been heavily studied for basic distinguished Lie superalgebras. For the remainder of this section, let $\mathfrak{g}$ be one of basic distinguished Lie superalgebras and choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. We will use many of the results proven in [15], including the next lemma.

**Lemma 5.8.4.** *Let $x \in \mathfrak{g}_{\bar{1}}$ be self-commuting. Then there exists a Cartan subalgebra $\mathfrak{h}$ and mutually orthogonal isotropic roots $\alpha_1, \ldots, \alpha_k$ such that $x = x_1 + \cdots + x_k$ where $x_i \in \mathfrak{g}_{\alpha_i}$ is a nonzero root vector.*

**Definition 5.8.5.** Given a self-commuting element $x \in \mathfrak{g}_{\bar{1}}$, the integer $k$ in the above lemma is an invariant of $x$ and we call it the rank of $x$. We define the defect of $\mathfrak{g}$ to be the maximal rank of all self-commuting elements of $\mathfrak{g}_{\bar{1}}$.

Define the self-commuting cone $X_0$ of $\mathfrak{g}$ to be the variety $X_0 = \{x \in \mathfrak{g}_{\bar{1}} : [x, x] = 0\}$. Then $X_0$ is stratified by rank. Further, $X_0$ is $G_0$-stable under the adjoint action of $G_0$ on $\mathfrak{g}_{\bar{1}}$, and this action preserves the rank of an element. In [15] it shown that there are only finitely many $G_0$-orbits on $X_0$, and the orbits are in natural bijection with the orbits of the Weyl group acting on the set of all subsets of $\Delta_{\bar{1}}$ consisting of mutually orthogonal isotropic roots.

The exceptional Lie superalgebras are all of defect one, so for our purposes they will mostly fall under theorem 5.8.2. On the other hand, $\mathfrak{gl}(m|n)$ is of defect $\min(m, n)$ and $\mathfrak{osp}(m|2n)$ is of defect $\min(\lfloor \frac{m}{2} \rfloor, n)$, so these cases will be more complicated. For them we will use the following lemma which is easy to prove.

**Lemma 5.8.6.** *For $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$, all odd self-commuting elements are $G_0$-conjugate to one of the form $x = x_1 + \cdots + x_k$, where $x_i$ is a root vector of weight $c_i(\epsilon_i - \delta_i)$, where $c_i = \pm 1$.*

We now give a table which explains the application of the Duflo-Serganova functor on $\mathfrak{g}$ for each case we consider. Note that if $x$ is rank 0 we have $x = 0$ and $\mathfrak{g}_x = \mathfrak{g}$, so we don't consider this case.

| $\mathfrak{g}$ | Defect | Rank of $x$ | $\mathfrak{g}_x$ |
|---|---|---|---|
| $\mathfrak{gl}(m\|n)$ | $\min(m,n)$ | $k$ | $\mathfrak{gl}(m-k\|n-k)$ |
| $\mathfrak{osp}(2m\|2n)$ | $\min(m,n)$ | $k$ | $\mathfrak{osp}(2(m-k)\|2(n-k))$ |
| $\mathfrak{osp}(2m+1\|2n)$ | $\min(m,n)$ | $k$ | $\mathfrak{osp}(2(m-k)+1\|2(n-k))$ |
| $\mathfrak{d}(2\|1;t)$ | 1 | 1 | $\mathfrak{sl}_2$ |
| $\mathfrak{ab}(1\|3)$ | 1 | 1 | $\mathfrak{sl}_3$ |
| $\mathfrak{g}(1\|2)$ | 1 | 1 | $\mathfrak{sl}_2$ |

Table 5.4: Duflo-Serganova functor for basic distinguished Lie superalgebras

Now we briefly explain how to split the surjection $\mathfrak{c}(x) \to \mathfrak{g}_x$ in these cases, as is described in [15]. Let $x \in \mathfrak{g}_{\overline{1}}$ be self-commuting and write $x = x_1 + \cdots + x_k$ as in lemma 5.8.4 for odd mutually orthogonal isotropic roots $A = \{\alpha_1, \dots, \alpha_k\}$. Let

$$\Delta_x = \{\beta \in \Delta : (\beta, \alpha_i) = 0 \text{ for } i = 1, \dots k\} \setminus \{\pm\alpha_1, \dots, \pm\alpha_k\}.$$

Let $\mathfrak{h}_A = \operatorname{span}(h_{\alpha_1}, \dots, h_{\alpha_k}) \subseteq \mathfrak{h}$, where $h_{\alpha_i}$ is the coroot of $\alpha_i$. Then $\mathfrak{h}_A \subseteq \mathfrak{h}_A^{\perp}$. If we let $\mathfrak{h}_{\Delta_x} = \operatorname{span}(h_\beta : \beta \in \Delta_x)$ then $\mathfrak{h}_{\Delta_x} \subseteq \mathfrak{h}_A^{\perp}$, and $\mathfrak{h}_{\Delta_x} \cap \mathfrak{h}_A = 0$. Now choose a splitting $\mathfrak{h}_A^{\perp} = \mathfrak{h}(x) \oplus \mathfrak{h}_A$ such that $\mathfrak{h}_{\Delta_x} \subseteq \mathfrak{h}(x)$. Then $\mathfrak{g}_x$ may be described as the following subalgebra of $\mathfrak{g}$:

$$\mathfrak{g}_x = \mathfrak{h}(x) \oplus \bigoplus_{\beta \in \Delta_x} \mathfrak{g}_\beta.$$

With a little more care in how one chooses $\mathfrak{h}(x)$, one may perform the above process of producing a splitting for $\mathfrak{g}_{x_1+\cdots+x_i}$ for each $1 \leq i \leq k$ so that we have $\mathfrak{g}_{x_1+\cdots+x_{i-1}} \subseteq \mathfrak{g}_{x_1+\cdots+x_i}$ for all $i$. Let us suppose we have done this, so that we have a chain of subalgebras

$$\mathfrak{g}_x = \mathfrak{g}_{x_1+\cdots+x_k} \subseteq \mathfrak{g}_{x_1+\cdots+x_{k-1}} \subseteq \cdots \subseteq \mathfrak{g}_{x_1} \subseteq \mathfrak{g}.$$

### 5.8.4 $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$

We now deal with the cases of $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$. Let us state a few lemmas. We remain in the setup of the above construction for $\mathfrak{g}_x$ as a subalgebra of $\mathfrak{g}$.

**Lemma 5.8.7.** *If $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$, then $\mathfrak{b} \cap \mathfrak{g}_x$ is a Borel subalgebra of $\mathfrak{g}_x$ containing $\mathfrak{h}(x)$.*

*Proof.* Using the description of $\mathfrak{g}_x$ as a subalgebra of $\mathfrak{g}$ spanned by certain root spaces and a torus, and that the roots of $\mathfrak{b}$ are those which are positive with respect to some homomorphism $Q \to \mathbb{R}$, this is immediate via restriction of said homomorphism □

**Lemma 5.8.8.** *Let $\mathfrak{g}$ be the Lie superalgebra $\mathfrak{gl}(m|n)$ or $\mathfrak{osp}(m|2n)$. Let $\mathfrak{b}(x)$ be a Borel subalgebra of $\mathfrak{g}_x$ with $\mathfrak{h}(x) \subseteq \mathfrak{b}(x)$. Then there exists a Borel $\mathfrak{b}$ of $\mathfrak{g}$ such that*

*(1)* $\alpha_1$ *is the lowest root*

*(2)* $\mathfrak{b} \cap \mathfrak{g}_{x_1+\cdots+x_{i-1}}$ *is a Borel subalgebra of* $\mathfrak{g}_{x_1+\cdots+x_{i-1}}$, *and* $\alpha_i$ *is the lowest root of of this Borel.*

*(3)* $\mathfrak{b} \cap \mathfrak{g}_x = \mathfrak{b}(x)$.

*Proof.* Using lemma 5.8.6, let us assume that $\alpha_i = c_i(\epsilon_i - \delta_i)$ where $c_i = \pm 1$.

Note that for $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$ Borel subalgebra $\mathfrak{b}$ may be constructed via a homomorphism $Q \to \mathbb{R}$ given by pairing with a particular integral coweight $h \in \mathfrak{h}$, which we now construct. If $\mathfrak{h}(x)$ is the Cartan of $\mathfrak{b}(x)$, a subalgebra of $\mathfrak{h}$ by our setup, let $h_x \in \mathfrak{h}(x)$ be an integral coweight which defines $\mathfrak{b}(x)$. Using the explicit description of the root system given in chapter 1, define $h_i$ to be dual to $\epsilon_i$, $h_{\overline{i}}$ dual to $\delta_i$. Then let

$$h' = c_1(a_1h_1 - b_1h_{\overline{1}}) + \cdots + c_k(a_kh_k - b_kh_{\overline{k}}),$$

where

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \ll 0.$$

Here we we want $b_k$ to have the property that $b_k < \min\limits_{\alpha}(h_x, \alpha)$. Then let $h = h' + h_x$. The element $h'$ will give conditions (1)-(2), and $h_x$ condition (3), using that $h'$ is orthogonal to all the roots in $\Delta_x$. ☐

We continue with our setup in the next proposition.

**Proposition 5.8.9.** *Let* $\mathfrak{b}$ *be a Borel subalgebra containing* $\mathfrak{h}$. *Then* $\mathfrak{b} + \mathfrak{c}(x) = \mathfrak{g}$ *if and only if* $\mathfrak{b}$ *satisfies the conditions (1) and (2) in lemma 5.8.8 for some reordering of* $\alpha_1, \ldots, \alpha_k$.

First we have a lemma:

**Lemma 5.8.10.** *Let* $\mathfrak{g}$ *be either* $\mathfrak{gl}(m|n)$ *or* $\mathfrak{osp}(m|2n)$. *Then if* $\beta$ *is any root, and* $\alpha_1, \ldots, \alpha_k$ *is a set of mutually orthogonal isotropic roots, then* $\beta$ *is not orthogonal to at most two of the* $\alpha_i$.

*Proof.* This is easily seen from the root systems of each algebra. ☐

*Proof of proposition 5.8.9.* Suppose $\mathfrak{b}$ satisfies the conditions. Let $y \in \mathfrak{g}_\beta$ be a root vector of $\mathfrak{g}$. If $\beta = \pm\alpha_i$ for some $i$, then clearly $y \in \mathfrak{b} + \mathfrak{c}(x)$. Hence we may assume this is not the case. If $y$ is orthogonal to all $\alpha_i$, then $y \in \mathfrak{c}(x)$. Therefore let $\alpha_i, \alpha_j$ with $i \leq j$ be such that $(\beta, \alpha_i), (\beta, \alpha_j) \neq 0$, and $(\beta, \alpha_l) = 0$ for $l \neq i, j$ (note that we may have $i = j$). By our assumptions on $y$, we know it must commute with $x_l$ for $l \neq i, j$.

If $y$ commutes with $x_i$ and $x_j$, then $y \in \mathfrak{c}(x)$. By assumption we have $y \in \mathfrak{g}_{x_1+\cdots+x_{i-1}}$, so if $[y, x_i] \neq 0$ we use that $\alpha_i$ is the smallest root of $\mathfrak{b} \cap \mathfrak{g}_{x_1+\cdots+x_{i-1}}$ to obtain that $\beta$ must be positive in $\mathfrak{g}_{x_1+\cdots+x_{i-1}}$, and thus $y \in \mathfrak{b}$.

Finally, we suppose $i < j$, $[y, x_i] = 0$ and $[y, x_j] \neq 0$. Write $x_{-i} \in \mathfrak{g}_{-\alpha_i}$ for a nonzero root vector of weight $-\alpha_i$. Then we have $[x_{-i}, y] = z \neq 0$, and $z$ is a root vector of weight

$\beta-\alpha_i$ possessing the same properties as $\beta$ with respect to the roots $\alpha_1,\dots,\alpha_k$. In particular, $[x_l,z]=0$ for $l\neq i,j$. Up to a nonzero scalar we can write $y=[x_i,z]$, and therefore

$$y=[x_i,z]=[x_i+x_j,z]-[x_j,z]=[x,z]-[x_j,z]$$

Since $[x,z]\in\mathfrak{c}(x)$, it suffices to show that $[x_j,z]\in\mathfrak{b}+\mathfrak{c}(x)$, which is a root vector of weight $\beta-\alpha_i+\alpha_j$, and thus again has the same orthogonality properties as $\beta$ with respect to the roots $\alpha_1,\dots,\alpha_k$. Further we see that it commutes with $x_j$, and so by repeating our above argument with $y=[x_j,z]$ we will find that $y\in\mathfrak{c}(x)+\mathfrak{b}$. Since $\mathfrak{h}\subseteq\mathfrak{b}$, we have shown that $\mathfrak{g}=\mathfrak{c}(x)+\mathfrak{b}$.

Now suppose that $\mathfrak{b}$ is a root Borel subalgebra such that $\mathfrak{b}+\mathfrak{c}(x)=\mathfrak{g}$. Suppose none of the roots $\alpha_1,\dots,\alpha_k$ are lowest. Then choose one which is minimal, call it $\alpha_1$ up to reordering. Then there exists a negative root $\beta$ such that $[e_\beta,x_1]\neq0$, where $e_\beta\in\mathfrak{g}_\beta$ is a nonzero root vector. However by assumption we may write $e_\beta=b+c$, for $b\in\mathfrak{b}$, $c\in\mathfrak{c}(x)$. Then we have $[e_\beta,x]=[b,x]$. But the lowest non-zero weight vector in $[b,x]$ will be larger than that in $[e_\beta,x]$ in the ordering that defines $\mathfrak{b}$, a contradiction. Therefore we have that $\alpha_1$ is the smallest root.

Now suppose, up to reordering of our indices, we have shown that $\alpha_1,\dots,\alpha_i$ satisfy the correct minimality properties. Suppose up to reordering that $\alpha_{i+1}$ is minimal amongst $\alpha_{i+1},\dots,\alpha_k$ in $\mathfrak{g}_{x_1+\dots+x_{i-1}}$, but $\alpha_i$ is not minimal in $\mathfrak{g}_{x_1+\dots+x_i}$. Then by the same argument just described, there exists a nonzero root vector $e_\beta\in\mathfrak{g}_{x_1+\dots+x_i}$ such that $[e_\beta,x_{i+1}]$ is a lower root vector in $\mathfrak{g}_{x_1+\dots+x_i}$. Suppose $e_\beta=b+c$, with $b\in\mathfrak{b}$ and $c\in\mathfrak{c}(x)$. Then we have

$$[e_\beta,x_{i+1}+\dots+x_k]=[b,x_{i+1}+\dots+x_k]-[c,x_1+\dots+x_i]$$

Now since $[e_\beta,x_{i+1}+\dots+x_k]\in\mathfrak{g}_{x_1+\dots+x_i}$, and $[c,x_1+\dots+x_i]\in[x_1+\dots+x_i,\mathfrak{g}]$, we must have that the all root vectors with non-zero coefficient in $[e_\beta,x_{i+1}+\dots+x_k]$ appear in $[b,x_{i+1}+\dots+x_k]$. But this is impossible, since $b$ will only increase weights in our ordering. Hence $\alpha_{i+1}$ must be minimal. Induction proves our claim. □

**Proposition 5.8.11.** *Let $\mathfrak{g}$ and $x$ be as in the above proposition, and let $\mathfrak{k}(x)\subseteq\mathfrak{g}_x$ be a subalgebra such that $\mathfrak{b}(x)+\mathfrak{k}(x)=\mathfrak{g}_x$, for a Borel $\mathfrak{b}(x)$ with Cartan subalgebra $\mathfrak{h}(x)$ (we may always assume this up to conjugacy). Let $\mathfrak{k}=[x,\mathfrak{g}]+\mathfrak{k}(x)$. Then $\mathfrak{k}$ is spherical, and for a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$ we have $\mathfrak{b}+\mathfrak{k}=\mathfrak{g}$ if and only if $\mathfrak{b}$ satisfies (1)-(3) of lemma 5.8.8 for some Borel subalgebra $\mathfrak{b}(x)$ complementary to $\mathfrak{k}(x)$.*

*Proof.* Let $\mathfrak{b}$ be a Borel of $\mathfrak{b}$ satisfying conditions (1)-(3) with respect to $\alpha_1,\dots,\alpha_k$ and the Borel $\mathfrak{b}(x)$. Then by proposition 5.8.9 we have $\mathfrak{b}+\mathfrak{c}(x)=\mathfrak{g}$. On the other hand, $\mathfrak{c}(x)=\mathfrak{b}(x)+\mathfrak{k}$. Therefore $\mathfrak{g}=\mathfrak{b}+\mathfrak{k}$.

Conversely, if $\mathfrak{b}+\mathfrak{c}(x)=\mathfrak{g}$, by proposition 5.8.9 we have $\mathfrak{b}$ that satisfies conditions (1)-(2). The third condition is clear. □

### 5.8.5 The case of exceptional algebras

Now let $\mathfrak{g}$ be an exceptional basic distinguished Lie superalgebra, and choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$.

**Lemma 5.8.12.** *The self-commuting cone of $\mathfrak{g}$ has two orbits, $\{0\}$ and $G_0 \cdot e_\alpha$ where $e_\alpha$ is a non-zero root vector of weight $\alpha$ where $(\alpha, \alpha) = 0$.*

*Proof.* We know that $\mathfrak{g}$ has defect one, and further all isotropic roots are in the same Weyl group orbit, so the statement follows. □

Using lemma 5.8.12, we now assume that $x = e_\alpha$, where $e_\alpha$ is an odd isotropic root $\alpha$.

**Proposition 5.8.13.** *A Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ which contains $\mathfrak{h}$ is complementary to $\mathfrak{c}(x)$ if and only if $\alpha$ is the smallest root. Further, such Borel subalgebras always exist.*

*Proof.* For the first statement we can use an argument identical to proposition 5.8.9. The second statement can be checked case by case. □

**Proposition 5.8.14.** *Let $\mathfrak{k}(x) \subseteq \mathfrak{g}_x$ be a spherical subalgebra of $\mathfrak{g}_x$. Then $\mathfrak{k} := \mathfrak{k}(x) + [x, \mathfrak{g}]$ is a spherical subalgebra of $\mathfrak{g}$. A Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$ is complementary to $\mathfrak{k}$ if and only if $\alpha$ is the lowest root of $\mathfrak{b}$ and $\mathfrak{b} \cap \mathfrak{g}_x$ is complementary to $\mathfrak{k}(x)$.*

*Proof.* The same proof as in proposition 5.8.11 applies. □

From our work we have now finished proving theorem 5.8.1.

# Chapter 6

# Symmetric supervarieties

## 6.1 Symmetric supervarieties

### 6.1.1 Symmetric supervarieties

Let $G$ be quasi-reductive. By an involution $\theta$ of $G$ we mean a (nontrivial) homomorphism of $\theta : G \to G$ of supergroups such that $\theta^2 = \mathrm{id}_G$. Then $\theta$ induces an involution on $G_0$ and $\mathfrak{g}$, which by abuse of notation we also write as $\theta$. Write $G_0^\theta$ for the fixed points of $\theta$ on $G_0$ and $\mathfrak{k}$ the fixed points of $\theta$ on $\mathfrak{g}$, and define $G^\theta$ to be the closed subgroup of $G$ corresponding to the SHCP $(G_0^\theta, \mathfrak{k})$.

Now let $K \subseteq G$ be a closed subgroup such that $(G^\theta)^0 \subseteq K \subseteq G^\theta$, where $(G^\theta)^0$ is the connected component of the identity of $G^\theta$. In particular we have $\mathrm{Lie}(K) = \mathfrak{k}$. Note that since $G_0$ is reductive $G_0^\theta$ is also, so $K$ and thus also $\mathfrak{k}$ are quasi-reductive.

**Definition 6.1.1.** The homogeneous space $G/K$ is called a symmetric supervariety. We call the pair $(\mathfrak{g}, \mathfrak{k})$ a supersymmetric pair.

We will often study supersymmetric pairs coming from involutions on Lie superalgebras where no associated Lie supergroup has been chosen.

*Example* 6.1.2. Every Lie supergroup admits a canonical central involution $\delta$ which acts by the identity on $G_0$ and by the grading operator $\delta(x) = (-1)^{\overline{x}}x$ on $\mathfrak{g}$. In particular since it is central it induces an order two bijection on the set of involutions of $G$ with itself, up to any notion of conjugacy, given by $\theta \mapsto \delta \circ \theta$.

The associated symmetric supervariety $G/G_0$ has underlying space consisting of just a point with coordinate superalgebra is $k[G/G_0] = \mathrm{Hom}_{\mathcal{U}\mathfrak{g}_{\overline{0}}}(\mathcal{U}\mathfrak{g}, k)$. Thus it is is isomorphic to $\Pi\mathfrak{g}_{\overline{1}}$ as a $G_0$-supervariety. Note that it is spherical if and only if there exists a hyperborel containing $\mathfrak{g}_{\overline{1}}$, equivalently if and only if $\mathfrak{g}$ is graded by corollary 5.7.4.

**Definition 6.1.3.** Given a supersymmetric pair $(\mathfrak{g}, \mathfrak{k})$ coming from an involution $\theta$, we write $(\mathfrak{g}, \mathfrak{k}')$ for the supersymmetric pair coming from the involution $\delta \circ \theta$. Writing $\mathfrak{p}$ for the $(-1)$-eigenspace of $\theta$, we write $\mathfrak{p}'$ for the $(-1)$-eigenspace for $\delta \circ \theta$.

Notice that $\mathfrak{k}' = \mathfrak{k}_{\overline{0}} \oplus \mathfrak{p}_{\overline{1}}$ and $\mathfrak{p}' = \mathfrak{p}_{\overline{0}} \oplus \mathfrak{k}_{\overline{1}}$.

## 6.2 Iwasawa decomposition

The example 6.1.2 demonstrates that symmetric supervarieties need not be spherical. In the classical world, symmetric varieties for reductive groups are always spherical, as follows from the Iwasawa decomposition of the supersymmetric pair $(\mathfrak{g}, \mathfrak{k})$. Thus it must be the case that the Iwasawa decomposition fails for symmetric supervarieties in general. We recall how this decomposition works now, generalizing it to the super case.

Let $\mathfrak{g}$ be a quasi-reductive Lie superalgebra, $\theta$ an involution of $\mathfrak{g}$, and $\mathfrak{k}$ the fixed points of $\theta$. Write $\mathfrak{p}$ for the $(-1)$-eigenspace of $\theta$ so that $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. Now let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal toral subalgebra of $\mathfrak{p}$, i.e. a maximal abelian subspace of $\mathfrak{p}_{\overline{0}}$ with the property that the elements of $\mathfrak{a}$ are semisimple in $\mathfrak{g}_{\overline{0}}$. We refer to $\mathfrak{a}$ as a Cartan subspace of the supersymmetric pair $(\mathfrak{g}, \mathfrak{k})$. Then $\mathfrak{a}$ is also a Cartan subspace of the supersymmetric pair $(\mathfrak{g}, \mathfrak{k}')$. Note that it is a classical fact that $\mathfrak{a} \neq 0$ if and only if $\mathfrak{g}_{\overline{0}} \not\subseteq \mathfrak{k}$, and any two Cartan subspaces are conjugate. Then we may decompose $\mathfrak{g}$ into weight spaces under the adjoint action of $\mathfrak{a}$.

**Definition 6.2.1.** Write $\overline{\Delta} \subseteq \mathfrak{a}^*$ for the set of non-zero weights under the action of $\mathfrak{a}$ on $\mathfrak{g}$.

If we extend $\mathfrak{a}$ to a $\theta$-stable Cartan subalgebra $\mathfrak{h}_{\overline{0}} \subseteq \mathfrak{g}_{\overline{0}}$, then $\overline{\Delta}$ will exactly consist of the set of non-zero restrictions of roots to $\mathfrak{a}$. Explicitly, we have a natural projection $\mathfrak{h}^* \to \mathfrak{a}^*$ coming from restriction, sending $\lambda \mapsto \overline{\lambda} = (\lambda - \theta\lambda)/2$, and $\overline{\Delta} = \{\overline{\alpha} : \alpha \in \Delta\} \setminus \{0\}$.

**Definition 6.2.2.** Define $\Delta^0 \subseteq \Delta$ to be the roots $\alpha \in \Delta$ such that $\overline{\alpha} = 0$, i.e. such that $\alpha = \theta\alpha$.

We call $\overline{\Delta} \subseteq \mathfrak{a}^*$ the restricted root system of $\theta$ and we call the elements of $\overline{\Delta}$ restricted roots. Write $\overline{Q} := \mathbb{Z}\overline{\Delta} \subseteq \mathfrak{a}^*$ for the restricted root lattice. Then for a choice of homomorphism $\overline{\phi} : \overline{Q} \to \mathbb{R}$ such that $\overline{\phi}(\overline{\alpha}) \neq 0$ for all $\overline{\alpha} \in \overline{\Delta}$ we obtain subsets $\overline{\Delta}^{\pm} \subseteq \overline{\Delta}$ of positive and negative restricted roots. We call such a partition of $\Delta$ into positive and negative roots that arises in this way a positive system for $\overline{\Delta}$. Define

$$\mathfrak{n} = \bigoplus_{\overline{\alpha} \in \overline{\Delta}^+} \mathfrak{g}_{\overline{\alpha}}.$$

Write $\mathfrak{c}(\mathfrak{a})$ for the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Observe that

$$\mathfrak{c}(\mathfrak{a}) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^0} \mathfrak{g}_{\alpha}$$

Then $\mathfrak{c}(\mathfrak{a})$ is $\theta$-stable, and so $\mathfrak{c}(\mathfrak{a}) = \mathfrak{c}(\mathfrak{a}) \cap \mathfrak{k} \oplus \mathfrak{c}(\mathfrak{a}) \cap \mathfrak{p}$.

**Proposition 6.2.3.** *The condition* $\mathfrak{c}(\mathfrak{a})\cap\mathfrak{p}=\mathfrak{a}$ *is equivalent to the following decomposition of* $\mathfrak{g}$*:*

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}.$$

*We call such a decomposition an Iwasawa decomposition of the supersymmetric pair* $(\mathfrak{g},\mathfrak{k})$ *(or of the involution* $\theta$*).*

*Proof.* Suppose the Iwasawa decomposition holds, and let $x\in\mathfrak{c}(\mathfrak{a})\cap\mathfrak{p}$. Write $x=k+a+n$ for a unique $k\in\mathfrak{k}$, $a\in\mathfrak{a}$, and $\mathfrak{n}\in\mathfrak{n}$. Then

$$-k-a-n=-x=\theta x=k-a+\theta n.$$

Thus we find that $k=-k$ and $n=-\theta n$. This forces $k=0$, and since $n$ and $\theta n$ have different weights for the action of $\mathfrak{a}$, we must have $n=0$. Thus $x\in\mathfrak{a}$.

Conversely, suppose $\mathfrak{c}(\mathfrak{a})\cap\mathfrak{p}=\mathfrak{a}$. Then $\mathfrak{c}(\mathfrak{a})\subseteq\mathfrak{a}+\mathfrak{k}$ so we check that $\mathfrak{g}_{\overline{\alpha}}\subseteq\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ for all $\overline{\alpha}\in\overline{\Delta}$. Let $x\in\mathfrak{g}_{\overline{\alpha}}$. If $\overline{\alpha}\in\overline{\Delta}^+$ we have $x\in\mathfrak{n}$, so assume instead $\overline{\alpha}$ is negative. Then $x=(x+\theta x)-\theta x$, and $x+\theta x\in\mathfrak{k}$ while $\theta x\in\mathfrak{n}$, so we are done. □

One reason for the importance of the Iwasawa decomposition, as we already stated, is the following.

**Proposition 6.2.4.** *If* $(\mathfrak{g},\mathfrak{k})$ *admits an Iwasawa decomposition, then there exists a Borel subalgebra* $\mathfrak{b}$ *such that* $\mathfrak{b}+\mathfrak{k}=\mathfrak{g}$*. In particular an associated symmetric supervariety* $G/K$ *is spherical.*

*Proof.* Let $\overline{\phi}:\overline{Q}\to\mathbb{R}$ determine a positive system for $\overline{\Delta}$. We have a natural surjection $Q\to\overline{Q}$ induced by restriction of roots to $Q$. Since $\overline{Q}$ is a free abelian group, we may write $Q=\overline{Q}\oplus Q'$ for a free-abelian group $Q'$. Now extend $\overline{\phi}$ to a homomorphism $\phi:Q\to\mathbb{R}$ such that $\phi(\alpha)\neq 0$ for all $\alpha\in\Delta$ and $\phi(\alpha)>0$ whenever $\overline{\phi}(\overline{\alpha})>0$. Then the Borel subalgebra determined by $\phi$ contains $\mathfrak{a}\oplus\mathfrak{n}$ and thus is complementary to $\mathfrak{k}$ in $\mathfrak{g}$. □

**Definition 6.2.5.** Given an involution $\theta$ which admits an Iwasawa decomposition, we obtain a group homomorphism $\phi:\mathbb{Z}\Delta\to\mathbb{R}$ as constructed in the proof of proposition 6.2.4, giving rise to a positive system of $\Delta$. We call a positive system of $\Delta$ constructed in this way an Iwasawa positive system and a Borel subalgebra arising from an Iwasawa positive system will be called an Iwasawa Borel subalgebra of $\mathfrak{g}$.

It is a well-known theorem that if $\mathfrak{g}=\mathfrak{g}_{\overline{0}}$ is reductive then every symmetric pair has an Iwasawa decomposition (see for instance section 26.4 of [61]). It follows that we always have $\mathfrak{c}(\mathfrak{a})\cap\mathfrak{p}_{\overline{0}}=\mathfrak{a}$. Therefore we obtain:

**Corollary 6.2.6.** *If* $(\mathfrak{g},\mathfrak{k})$ *is a supersymmetric pair with Cartan subspace* $\mathfrak{a}$ *then* $\mathfrak{g}$ *admits an Iwasawa decomposition if and only if* $\mathfrak{c}(\mathfrak{a})\cap\mathfrak{p}_{\overline{1}}=0$*, i.e.* $\mathfrak{c}(\mathfrak{a})_{\overline{1}}\subseteq\mathfrak{k}_{\overline{1}}$*.*

In the super world example 6.1.2 shows this need not be true, in particular it is possible for $\mathfrak{c}(\mathfrak{a})\cap\mathfrak{p}_{\overline{1}}\neq 0$. In general, if $\mathfrak{c}(\mathfrak{a})_{\overline{1}}\neq 0$ then necessarily at least one of $(\mathfrak{g},\mathfrak{k})$ or $(\mathfrak{g},\mathfrak{k}')$ must not admit an Iwasawa decomposition. Further in this case one of these two supersymmetric pairs admits an Iwasawa decomposition if and only if $\mathfrak{c}(\mathfrak{a})_{\overline{1}}\subseteq\mathfrak{k}_{\overline{1}}$ or $\mathfrak{c}(\mathfrak{a})_{\overline{1}}\subseteq\mathfrak{p}_{\overline{1}}$. This need not happen, as the following example illustrates.

*Example* 6.2.7. Consider the involution $\theta$ on $\mathfrak{gl}(n|n)$ given explicitly by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} -d^t & b^t \\ -c^t & -a^t \end{bmatrix}$$

This gives rise to the supersymmetric pair $(\mathfrak{gl}(n|n),\mathfrak{p}(n))$. A Cartan subspace is given by

$$\mathfrak{a}=\left\{\begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} : d \text{ is diagonal}\right\}.$$

Hence

$$\mathfrak{c}(\mathfrak{a})=\left\{\begin{bmatrix} d & d' \\ d' & d \end{bmatrix} : d, d' \text{ are diagonal}\right\}\cong\mathfrak{sl}(1|1)\times\cdots\times\mathfrak{sl}(1|1).$$

We see that $\theta|_{\mathfrak{c}(\mathfrak{a})_{\overline{1}}}\neq\pm\mathrm{id}_{\mathfrak{c}(\mathfrak{a})_{\overline{1}}}$, so neither $(\mathfrak{g},\mathfrak{k})$ nor $(\mathfrak{g},\mathfrak{k}')$ admit an Iwasawa decomposition.

However despite the failure of having an Iwasawa decomposition, $\mathfrak{p}(n)$ is still a spherical subalgebra of $\mathfrak{gl}(n|n)$, i.e. there is a complementary Borel subalgebra of $\mathfrak{p}(n)$ in $\mathfrak{gl}(n|n)$. In particular the Borel subalgebra $\mathfrak{b}^{\delta\epsilon\delta\cdots\delta\epsilon}$ with simple roots

$$\delta_1-\epsilon_1,\epsilon_1-\delta_2,\delta_2-\epsilon_2,\dots,\epsilon_{n-1}-\delta_n,\delta_n-\epsilon_n$$

is complementary to $\mathfrak{p}(n)$ (in fact this is the only Borel subalgebra with this property up to conjugacy).

An issue, as we will see, with the supersymmetric pair in example 6.2.7 is that the involution does not preserve the non-degenerate invariant form on $\mathfrak{gl}(n|n)$. Indeed, for basic distinguished Lie superalgebras we have a nice situation if we assume that the involution preserves the invariant form.

**Theorem 6.2.8.** *If $\mathfrak{g}$ is a basic distinguished Lie superalgebra and $\theta$ is an involution that preserves the invariant bilinear form on $\mathfrak{g}$, then either $\theta$ or $\delta\circ\theta$ has an Iwasawa decomposition.*

The proof of this theorem is given in appendix B and goes by way of studying (mild generalizations of) generalized root systems (GRSs) as developed by Serganova in [48]. Indeed, the root system $\Delta\subseteq\mathfrak{h}^*$ satisfies the axioms of a GRS (see appendix B for definitions) and the involution $\theta$ induces an automorphism of this GRS that preserves the bilinear form on it when $\theta$ preserves the form on $\mathfrak{g}$. In appendix B we study the possibilities for $\Delta^0$, the set of roots fixed by $\theta$, in order to prove theorem 6.2.8.

In fact theorem 6.2.8 follows from the following more general theorem: let $\mathfrak{g}$ be a distinguished basic Lie superalgebra and let $\theta$ be a semisimple automorphism of $\mathfrak{g}$ which preserves the non-degenerate invariant form. Let $\mathfrak{h}$ be a $\theta$-stable Cartan subalgebra and write $\mathfrak{a} \subseteq \mathfrak{h}$ for the sum of eigenspaces of $\theta$ on $\mathfrak{h}$ with eigenvalue not 1. Then we have

**Theorem 6.2.9.** *The Lie superalgebra $\mathfrak{c}(\mathfrak{a})$ is an extension of an abelian Lie superalgebra by the product of ideals $\mathfrak{a} \times \widetilde{\mathfrak{l}} \times \mathfrak{l}$, where $\mathfrak{l}$ is an even semisimple Lie algebra and $\widetilde{\mathfrak{l}}$ is isomorphic to either a simple basic Lie superalgebra, $\mathfrak{sl}(n|n)$ for some $n \geq 1$, or is trivial.*

Applying theorem 6.2.9 to the case where $\mathfrak{h}$ is a Cartan subalgebra containing a Cartan subspace $\mathfrak{a}$, we learn that $\mathfrak{c}(\mathfrak{a})_{\overline{1}}$ is contained in a distinguished Lie superalgebra and thus it is not hard to show from this that $\theta|_{\mathfrak{c}(\mathfrak{a})_{\overline{1}}} = \pm \mathrm{id}_{\mathfrak{c}(\mathfrak{a})_{\overline{1}}}$, which is what we need.

We now list all supersymmetric pairs $(\mathfrak{g}, \mathfrak{k})$ where $\mathfrak{g}$ is basic distinguished and the associated involution preserves the form on $\mathfrak{g}$. When $m \neq n$ we assume the involution fixes the center of $\mathfrak{gl}(m|n)$. The induced automorphism of the GRS via the action on a $\theta$-stable Cartan subalgebra containing a Cartan subspace is also described. In each case we are describing the action of the involution on basis elements of $\mathfrak{h}^*$ where we omit any basis elements that are fixed by the involution. See section 2.3.2 for the descriptions of root systems for each. For cases (1) and (3) we are giving the GRS automorphism when $r \leq m/2$ and $s \leq n/2$.

For all cases not of type $\mathfrak{g}(1|2)$ or $\mathfrak{ab}(1|3)$ we refer to Serganova's classification in [53]. The cases for $\mathfrak{g}(1|2)$ and $\mathfrak{ab}(1|3)$ were communicated to the author by Serganova.

Note that by lemma 6.3.1, $\mathfrak{osp}(1|2)$ does not admit a nontrivial involution preserving an invariant form, and therefore by remark B.3.3 has no nontrivial involutions. Further, lemma 6.3.1 also implies there is never an involution that acts by (-1) on a Cartan subalgebra and preserves the form. This may seem surprising given the existence of the Chevalley involution for reductive Lie algebras. The following remark seeks to contextualize this.

*Remark* 6.2.10. A complex Kac-Moody Lie algebra $\mathfrak{g}$ always admits a nontrivial involution $\omega$, the Chevalley involution, that acts by $(-1)$ on a Cartan subalgebra (see [25] chapter 1). If one modifies this involution to make it complex antilinear as in chapter 2 of [25], one can construct a Cartan involution of $\mathfrak{g}$, i.e. an involution whose fixed points are a compact real form of $\mathfrak{g}$. For finite type complex Kac-Moody algebras one can use Cartan involutions to set up a bijection between real forms of $\mathfrak{g}$ and complex linear involutions of $\mathfrak{g}$, as originally shown by Cartan.

For complex Kac-Moody Lie superalgebras the natural generalization of the Chevalley involution, which we write as $\widetilde{\omega}$, is of order 4. In fact $\widetilde{\omega}^2 = \delta$, so it is of order 2 on $\mathfrak{g}_{\overline{0}}$ and order 4 on $\mathfrak{g}_{\overline{1}}$. Write $\mathrm{Aut}_{2,4}(\mathfrak{g})$ for the complex linear automorphisms $\theta$ of $\mathfrak{g}$ which are order 2 on $\mathfrak{g}_{\overline{0}}$ and order 4 on $\mathfrak{g}_{\overline{1}}$. If $\mathfrak{g}$ a finite-dimensional contragredient Lie superalgebra then there is a bijection between the real forms of $\mathfrak{g}$ and conjugacy classes in $\mathrm{Aut}_{2,4}(\mathfrak{g})$ as shown in [11].

**Proposition 6.2.11.** *Let $\theta$ be an involution as in theorem 6.2.8 which admits an Iwasawa decomposition and suppose that $\mathfrak{b}$ is an Iwasawa Borel subalgebra of $\mathfrak{g}$. Then the simple roots of $\mathfrak{b}$ that are fixed by $\theta$ generate all fixed roots of $\theta$. In particular, $\mathfrak{c}(\mathfrak{a})$ is generated by $\mathfrak{h} \sqcup \{e_\gamma, e_{-\gamma}\}_{\gamma \in I}$, where $I$ is the set of positive simple roots fixed by $\theta$.*

| Supersymmetric Pair | Iwasawa Decomposition? | GRS Automorphism |
|---|---|---|
| $(\mathfrak{gl}(m\|n),$ $\mathfrak{gl}(r\|s)\times\mathfrak{gl}(m-r\|n-s))$ | Iff $(m-2r)(n-2s)\geq 0$ | $\epsilon_i\leftrightarrow\epsilon_{m-i+1}, 1\leq i\leq r,$ $\delta_j\leftrightarrow\delta_{n-j+1}, 1\leq j\leq s$ |
| $(\mathfrak{gl}(m\|2n),\mathfrak{osp}(m\|2n))$ | Yes | $\epsilon_i\leftrightarrow-\epsilon_i,\quad\delta_i\leftrightarrow-\delta_{2n-i+1}$ |
| $(\mathfrak{osp}(m\|2n),$ $\mathfrak{osp}(r\|2s)\times$ $\mathfrak{osp}(m-r,2n-2s))$ | Iff $(m-2r)(n-2s)\geq 0$ | $\epsilon_i\leftrightarrow-\epsilon_i, 1\leq i\leq r$ $\delta_i\leftrightarrow\delta_{n-i+1}, 1\leq i\leq s$ |
| $(\mathfrak{osp}(2m\|2n),\mathfrak{gl}(m\|n))$ | Yes | $\delta_i\leftrightarrow-\delta_i,\ \epsilon_i\leftrightarrow\epsilon_{m-i+1}$ |
| $(\mathfrak{d}(2\|1;\alpha),\mathfrak{osp}(2\|2)\times\mathfrak{so}(2))$ | Yes | $\epsilon\leftrightarrow-\epsilon,\delta\leftrightarrow-\delta$ |
| $(\mathfrak{ab}(1\|3),\mathfrak{gosp}(2\|4))$ | Yes | $\epsilon_1\leftrightarrow-\epsilon_1,\delta\leftrightarrow-\delta$ |
| $(\mathfrak{ab}(1\|3),\mathfrak{sl}(1\|4))$ | Yes | $\epsilon_1\leftrightarrow-\epsilon_1,\epsilon_2\leftrightarrow-\epsilon_2,\delta\leftrightarrow-\delta$ |
| $(\mathfrak{ab}(1\|3),\mathfrak{d}(2\|1;2))$ | Yes | $\epsilon_i\leftrightarrow-\epsilon_i$ for all $i$ |
| $(\mathfrak{g}(1\|2),\mathfrak{d}(2\|1;3))$ | Yes | $\epsilon_i\leftrightarrow-\epsilon_i$ for all $i$ |
| $(\mathfrak{g}(1\|2),\mathfrak{osp}(3\|2)\times\mathfrak{sl}_2)$ | No | $\epsilon_i\leftrightarrow-\epsilon_i$ for all $i$ |

Table 6.1: Supersymmetric pairs for basic distinguished Lie superalgebras

*Proof.* If $\beta$ is a positive root then we may write

$$\beta=\sum_{\alpha\notin I}c_\alpha\alpha+\sum_{\gamma\in I}d_\gamma\gamma$$

where the first sum is over simple roots $\alpha$ not fixed by $\theta$, and $c_\alpha, d_\gamma\in\mathbb{Z}_{\geq 0}$. If $\theta\beta=\beta$ then we obtain that

$$\beta=\sum_{\alpha\notin I}c_\alpha\theta\alpha+\sum_{\gamma\in I}d_\gamma\gamma.$$

But $\theta\alpha$ is a negative root for $\alpha\notin I$, and thus $c_\alpha=0$. □

In terms of roots we obtain the following.

**Corollary 6.2.12.** *In the context of proposition 6.2.11, a base for $\Delta^0$ is given by the set of simple roots of $\Delta$ (for an Iwasawa positive system) which are fixed by $\theta$.*

### 6.2.1 Satake diagrams

Let $(\mathfrak{g}, \mathfrak{k})$ be a supersymmetric pair where $\mathfrak{g}$ is a distinguished basic Lie superalgebra and which comes from an involution preserving the form. Then a choice of simple roots of its root system can be encoded in a Dynkin-Kac diagram, and one obtains a bijection between Dynkin-Kac diagrams and choices of simple roots up to Weyl group symmetries for a given superalgebra (see [26]). Just as in the classical case, if one chooses an Iwasawa positive system one can construct a Satake diagram from it using the results of the following lemma, which are standard. Note we use that the simple roots are linearly independent for the superalgebras we consider.

**Lemma 6.2.13.** *Let $\Sigma$ be the set of simple roots coming from an Iwasawa positive system. Then if $\alpha$ is a simple root such that $\theta\alpha \neq \alpha$, then*

$$-\theta\alpha = \alpha' + \sum_{\gamma \in I} d_\gamma \gamma$$

*where $\alpha'$ is a simple root and $I \subseteq \Sigma$ is the set of simple roots fixed by $\theta$. The correspondence $\alpha \mapsto \alpha'$ defines a bijection of order 1 or 2 of $\Sigma \setminus I$ with itself. In particular, for distinct simple roots $\alpha, \beta$, we have $\overline{\alpha} = \overline{\beta}$ if and only if $\beta = \alpha'$.*

*Proof.* Write $\{\alpha_i\}_i$ for the set of simple roots not fixed by $\theta$. Then $-\theta\alpha_i$ is a positive root for all $i$, and thus we may write

$$-\theta\alpha_i = \sum_j c_{ij}\alpha_j + \sum_{\gamma \in I} d^i_\gamma \gamma$$

for some $d^i_\gamma \in \mathbb{Z}_{\geq 0}$, where $C = (c_{ij})$ is square and has nonnegative integer entries. Applying $(-\theta)$ to this equation once again, we obtain that

$$\alpha_i = \sum_{j,k} c_{ij}c_{jk}\alpha_k + \sum r^i_\gamma \gamma$$

for some $r^i_\gamma \in \mathbb{Z}$. Since $\alpha_i$ is simple, this forces $C^2$ to be the identity matrix, which implies that $C$ is in fact a permutation matrix. This permutation matrix defines our autobijection.

For the last statement, if $\overline{\alpha} = \overline{\beta}$, then $\alpha - \theta\alpha = \beta - \theta\beta$, so there exists $\gamma_\alpha, \gamma_\beta$ in the span of fixed simple roots such that

$$\alpha + \alpha' + \gamma_\alpha = \beta + \beta' + \gamma_\beta.$$

By linear independence of our base, we must have that $\{\alpha, \alpha'\} = \{\beta, \beta'\}$, so we are done. □

Using the above result, we may construct a Satake diagram from $(\mathfrak{g}, \mathfrak{k})$ as follows: choosing an Iwasawa positive system, we get a Dynkin-Kac diagram for $\mathfrak{g}$. Now draw an arrow between two distinct simple roots if they are related by the involution constructed in lemma 6.2.13.

Finally, draw a solid black line over a node if the corresponding simple root $\alpha$ is fixed by $\theta$. Classically one would color the node black, but unfortunately Dynkin-Kac diagrams may already have black nodes as they represent non-isotropic odd simple roots. Neither option feels particularly pleasing to this author, however.

We call the result a Satake diagram for the corresponding supersymmetric pair. Note that it is not unique– proposition 6.2.14 shows that it is determined exactly up to choices of positive systems for $\overline{\Delta}$ and $\Delta^0$ (see definition 6.2.2). Others have given examples of such diagrams, such as in [40]. In that paper nodes are drawn black if the corresponding simple root is fixed by $\theta$. The author has drawn all possibilities elsewhere but did not see the use in listing them here.

Before we state the proposition, we define a positive system of $\Delta^0$ to be a choice of positive and negative roots in $\Delta^0$ arising from a group homomorphism $\psi : \mathbb{Z}\Delta^0 \to \mathbb{R}$ such that $\psi(\gamma) \neq 0$ for all $\gamma \in \Delta^0$.

**Proposition 6.2.14.** *There is a natural bijection between Iwasawa positive systems and choices of positive systems for* $\overline{\Delta}$ *and* $\Delta^0$.

*Proof.* The simple roots of any positive root system form a $\mathbb{Z}$-basis of $Q$. Thus by proposition 6.2.11 we have that $\mathbb{Z}\Delta^0$ splits off from $Q$, so we can write $Q = \mathbb{Z}\Delta^0 \oplus Q'$. Write $\pi : Q \to \mathbb{Z}\overline{\Delta}$ for the canonical projection, and observe that $\mathbb{Z}\Delta^0 \subseteq \ker \pi$. Therefore the restricted map $Q' \to \mathbb{Z}\overline{\Delta}$ is surjective, so we may split it and write $Q' = \mathbb{Z}\overline{\Delta} \oplus Q''$, so that $Q' = \mathbb{Z}\Delta^0 \oplus \mathbb{Z}\overline{\Delta} \oplus Q''$.

Now let $\phi : Q \to \mathbb{R}$ be a group homomorphism determining an Iwasawa positive system coming from $\overline{\phi} : \mathbb{Z}\overline{\Delta} \to \mathbb{R}$ as in proposition 6.2.4. Write $\psi : \mathbb{Z}\Delta^0 \to \mathbb{R}$ for the restriction of $\phi$ to $\mathbb{Z}\Delta^0$. Then since $\psi(\gamma) \neq 0$ for all $\gamma \in \Delta^0$, $\psi$ determines a positive system for $\Delta^0$. Thus the Iwasawa positive system gives rise to positive systems of $\overline{\Delta}$ and $\Delta^0$ respectively from $\overline{\phi}$ and $\psi$.

Conversely, given positive systems of $\overline{\Delta}$ and $\Delta^0$ coming from group homomorphisms $\overline{\phi} : \mathbb{Z}\overline{\Delta} \to \mathbb{R}$ and $\psi : \mathbb{Z}\Delta^0 \to \mathbb{R}$, the map $\phi : \mathbb{Z}\Delta \to \mathbb{R}$ defined by $\phi = \psi \oplus \overline{\phi} \oplus 0 : \mathbb{Z}\Delta^0 \oplus \mathbb{Z}\overline{\Delta} \oplus Q'' \to \mathbb{R}$ determines an Iwasawa positive system. The described correspondences are seen to be bijective and thus we are done. $\square$

## 6.3 Restricted root systems

Consider one of the supersymmetric pairs $(\mathfrak{g}, \mathfrak{k})$ from the table of section 6.2 which admits an Iwasawa decomposition. Write $\theta$ for the involution, and by abuse of notation also write $\theta$ for the induced involution on the root system $\Delta \subseteq \mathfrak{h}^*$ coming from the dual of a Cartan subalgebra $\mathfrak{h}$ containing a Cartan subspace $\mathfrak{a}$. Continue writing $Q = \mathbb{Z}\Delta \subseteq \mathfrak{h}^*$ for the root lattice, $\Delta^0 \subseteq \Delta$ for the roots fixed by $\theta$ and $\overline{\Delta}$ for the restricted roots. We make a few notes about differences between the super case and the purely even case.

For an even symmetric pair there are often roots $\alpha$ for which $\theta(\alpha) = -\alpha$. In the super case this cannot hold for odd roots.

**Lemma 6.3.1.** *If $\alpha$ is an odd root, then $\theta(\alpha) \neq -\alpha$.*

*Proof.* Suppose $\alpha$ is odd and satisfies $\theta(\alpha) = -\alpha$. Write $h_\alpha \in \mathfrak{h}$ for the coroot of $\alpha$, i.e. $h_\alpha$ satisfies $(h_\alpha, -) = \alpha$ as an element of $\mathfrak{h}^*$. Then we may assume $\theta e_\alpha = e_{-\alpha}$ and $\theta e_{-\alpha} = e_\alpha$ where $e_\alpha \in \mathfrak{g}_\alpha$, $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ are nonzero and $[e_\alpha, e_{-\alpha}] = [e_{-\alpha}, e_\alpha] = h_\alpha$. But then

$$\theta h_\alpha = \theta[e_\alpha, e_{-\alpha}] = [\theta e_\alpha, \theta e_{-\alpha}] = [e_{-\alpha}, e_\alpha] = h_\alpha.$$

However the action of $\theta$ on $\mathfrak{h}^*$ is dual to the action of $\theta$ on $\mathfrak{h}$, so since $\alpha$ and $h_\alpha$ are dual to one another we must have $\theta h_\alpha = -h_\alpha$, a contradiction. □

The following lemma is well-known from the even case, and is proven in [3].

**Lemma 6.3.2.** *If $\alpha$ is an even root, then $\theta\alpha + \alpha$ is not a root.*

However that the corresponding statement for odd roots is false in many cases, for instance it's never true for odd roots in the cases of $(\mathfrak{gl}(m|2n), \mathfrak{osp}(m|2n))$, $(\mathfrak{osp}(2m|2n), \mathfrak{gl}(m|n))$, $(D(2,1;\alpha), \mathfrak{osp}(2|2) \times \mathfrak{so}(2))$, and for $(\mathfrak{osp}(m|2n), \mathfrak{osp}(r|2s) \times \mathfrak{osp}(m-r|2(n-s)))$ it is not true for roots of the form $\pm\epsilon_i \pm \delta_j$ where $1 \leq i \leq r$.

### 6.3.1 Structure of $\overline{\Delta}$

Classically, $\overline{\Delta}$ defines a (potentially non-reduced) root system in $\mathfrak{a}^*$, the restricted root system of the symmetric pair. Each restricted root $\overline{\alpha}$ has a positive integer multiplicity attached to it given by $m_{\overline{\alpha}} := \dim \mathfrak{g}_{\overline{\alpha}}$. The data of the restricted root system with multiplicities completely determines the corresponding symmetric pair.

In the super case it is less clear what type of object the restricted root system is. Even and odd roots can restrict to the same element of $\mathfrak{a}^*$, so the natural replacement of the multiplicity of a restricted root is (a multiple of) the superdimension of the corresponding weight space. In many cases the object obtained behaves like the root system of a basic distinguished superalgebra from a combinatorial perspective, however the bilinear form is deformed. We discuss this situation later on, but first state what can be proven in general.

Let $\overline{\Delta}_{re} = \{\overline{\alpha} : (\alpha, \alpha) \neq 0, \overline{\alpha} \neq 0\}$, the real restricted roots, and let $\overline{\Delta}_{im} = \overline{\Delta} \setminus \overline{\Delta}_{re}$, the imaginary restricted roots.

**Proposition 6.3.3.** *The set $\overline{\Delta} \subseteq \mathfrak{a}^*$ with the restricted bilinear form satisfies the following properties:*

1. $\operatorname{span} \overline{\Delta} = \mathfrak{a}^*$;
2. *The form is non-degenerate;*
3. *Given $\overline{\alpha} \in \overline{\Delta}_{re}$, we have $k_{\overline{\alpha},\overline{\beta}} := 2\frac{(\overline{\alpha},\overline{\beta})}{(\overline{\alpha},\overline{\alpha})} \in \mathbb{Z}$ and $r_{\overline{\alpha}}(\overline{\beta}) = \overline{\beta} - k_{\overline{\alpha},\overline{\beta}}\overline{\alpha} \in \overline{\Delta}$.*
4. *Given $\overline{\alpha} \in \overline{\Delta}_{im}$, $\overline{\beta} \in \overline{\Delta}$ with $\overline{\beta} \neq \pm\overline{\alpha}$, if $(\overline{\alpha}, \overline{\beta}) \neq 0$ then at least one of $\overline{\beta} \pm \overline{\alpha} \in \overline{\Delta}$.*

*5.* $\overline{\Delta} = -\overline{\Delta}$;

*Further,* $\overline{\Delta}_{re} \subseteq \mathfrak{a}^*$ *is an even (potentially non-reduced) root system and* $\overline{\Delta}_{im}$ *is invariant under its Weyl group.*

*Proof.* Properties (1) and (5) are obvious, and (2) follows from the fact that we are only considering Lie superalgebras with non-degenerate invariant forms and our involution preserves the form. The statement (3) is proven just as in the classical case. For (4), since $(\overline{\alpha}, \overline{\beta}) \neq 0$, either $(\alpha, \beta) \neq 0$ or $(-\theta\alpha, \beta) \neq 0$ so either $\beta \pm \alpha$ or $\beta \pm (-\theta\alpha)$ is a root, and restricting gives the desired statement.

That $\overline{\Delta}_{re}$ is a root system is classical (see for instance chapter 26 of [61]), and it's easy to see that $\overline{\Delta}_{im}$ is Weyl group invariant. □

*Remark* 6.3.4. Although we use the notation $\overline{\Delta}_{im}$, it is not true in general that $(\overline{\alpha}, \overline{\alpha}) = 0$ for $\overline{\alpha} \in \overline{\Delta}_{im}$, and this is the main way that a restricted root systems differs from a GRS.

Using proposition 6.3.3 we may now decompose $\overline{\Delta}_{re}$ into a union of irreducible real root systems, $\overline{\Delta}_{re} = \overline{\Delta}_{re}^1 \sqcup \cdots \sqcup \overline{\Delta}_{re}^k$. Write $W_i$ for the Weyl group of $\overline{\Delta}_{re}^i$, and let $W = W_1 \times \cdots \times W_k$. Since $\Delta$ was irreducible we know that $k \leq 3$ by proposition B.1.5. We may decompose $\mathfrak{a}^*$ as $\mathfrak{a}^* = V_0 \oplus V_1 \oplus \cdots \oplus V_k$, where $V_i = \operatorname{span}(\overline{\Delta}_{re}^i)$, and we set $V_0 = (\sum_{i \geq 1} V_i)^{\perp}$. Write $q_i : \mathfrak{a}^* \to V_i$ for the projection maps. The following result is obvious.

**Lemma 6.3.5.** *A real component* $\overline{\Delta}_{re}^i$ *of* $\overline{\Delta}$ *is either gotten by*

*(1) the restriction of nonisotropic roots in a real component of* $\Delta_{re}$ *preserved by* $\theta$*, or*

*(2) is obtained as a diagonal subspace of two isomorphic real components of* $\Delta$ *that are identified by* $\theta$*.*

From lemma 6.3.5 we can prove:

**Proposition 6.3.6.** *For each* $i > 0$, $q_i(\overline{\Delta}_{im}) \setminus \{0\}$ *is a union of small* $W_i$*-orbits.*

*Proof.* Let $\overline{\alpha}, \overline{\beta} \in \overline{\Delta}_{im}$ such that $q_i(\overline{\alpha}), q_i(\overline{\beta}) \neq 0$ and they lie in the same $W_i$-orbit. Let $\alpha, \beta \in \Delta$ be lifts of $\overline{\alpha}$ and $\overline{\beta}$.

If $\overline{\Delta}_{re}^i$ falls into the second case of lemma 6.3.5, then if we write $p$ for the projection from $\mathfrak{h}^*$ onto one of the real components being folded into $\overline{\Delta}_{re}^i$ then $p\alpha$ and $p\beta$ must be conjugate under the Weyl group for that real component too, so we can apply proposition B.1.7.

Suppose on the other hand that $\overline{\Delta}_{re}^i$ falls into the first case of lemma 6.3.5. Write $p$ for the projection from $\mathfrak{h}^*$ onto the corresponding real component giving $\overline{\Delta}_{re}^i$. Then if $p\alpha$ and $p\beta$ are conjugate under the Weyl group we can apply proposition B.1.7. If they are not conjugate under the Weyl group, $\Delta$ must have two imaginary components (see appendix B.1). If $\theta$ preserves the imaginary components, then $\alpha$ and $-\theta\beta$ will lie in the same imaginary component and project to $\overline{\alpha}, \overline{\beta}$ still. If $\theta$ permutes the imaginary components, then the supersymmetric pair is either $(\mathfrak{gl}(m|2n), \mathfrak{osp}(m|2n))$ or $(\mathfrak{osp}(2|2n), \mathfrak{osp}(1|2n-2r), \mathfrak{osp}(1|2r))$.

In the first case $\overline{\Delta}_{re} = A_{m-1} \sqcup A_{n-1}$ so that $\alpha$ and $\beta$ cannot be in distinct imaginary components of $\Delta$, and in the second case $p(\Delta_{im})$ is a single small Weyl group orbit anyway. □

### 6.3.2 Deformed restricted root systems

In the case when $\overline{\Delta}_{re}$ has more than one component, it turns out that the restricted root system is a deformed GRS, as introduced in [54]. There, the authors introduce generalized root systems as more a general object than in [48] by relaxing condition (4) in definition B.1.1 to (we change notation here to use $\Delta$ instead of $R$)

(4') If $\alpha, \beta \in \Delta$ and $(\alpha, \alpha) = 0$, then if $(\alpha, \beta) \neq 0$ at least one of $\beta \pm \alpha \in \Delta$.

It is also assumed that the inner product is non-degenerate. It is shown in [48] that in a GRRS only one of $\beta \pm \alpha$ can be in $\Delta$. Following [19], we will call the notion of GRS in the sense of [54] a weak GRS (WGRS). Serganova classified all WGRSs in section 7 of [48]; there are two cases that do not appear in the classification of GRSs (see theorem B.1.9 for the classifications of GRSs):

- $C(m,n)$, $m, n \geq 1$: $\Delta_{re}^1 = C_m$, $\Delta_{re}^2 = C_n$, $\Delta_{im} = W(\omega_1^{(1)} + \omega_1^{(2)})$
- $BC(m,n)$, $m, n \geq 1$: $\Delta_{re}^1 = BC_m$, $\Delta_{re}^2 = C_n$, $\Delta_{im} = W(\omega_1^{(1)} + \omega_1^{(2)})$.

Sergeev and Veselov define a deformed WGRS as the data of a WGRS with a deformed inner product determined by a nonzero parameter $t \in k^\times$, along with Weyl-group invariant multiplicities $m(\alpha) \in k$ for each root $\alpha \in \Delta$. These multiplicities are required to satisfy certain polynomial relations and that $m(\alpha) = 1$ for an isotropic (with respect to the non-deformed bilinear form) root $\alpha$.

We now explain when and how we can realize $\overline{\Delta}$ as a deformed WGRS. For each of the supersymmetric pairs we consider where $\overline{\Delta}_{re}$ has more than one component the deformation parameter $t$ is determined by the restriction of the form. In this case $\overline{\Delta}_{im} \neq \emptyset$, and the multiplicity of every $\overline{\alpha} \in \overline{\Delta}_{im}$ is $-\ell$ for some positive integer $\ell$. We define the multiplicities of a restricted root $\overline{\alpha} \in \overline{\Delta}$ to be $m(\overline{\alpha}) = -\frac{1}{\ell}\operatorname{sdim}\mathfrak{g}_{\overline{\alpha}}$. Then we claim that we obtain a deformed WGRS in this way. This can be checked case by case, and we do this in the table below. Note that this fact has been known to several researchers for some time (most of whom knew before the author). We give this information here for the benefit of the reader.

In the table below we list, for each supersymmetric pair we consider in which $\overline{\Delta}_{re}$ has more than one component, the corresponding Sergeev-Veselov deformation parameters. Note that in [54], the letter $k$ is used instead of $t$; we have changed it to avoid confusion with the name of our base field. No confusion should arise with respect to the $t$ in $\mathfrak{d}(2|1;t)$, as they fortunately agree for its supersymmetric pair.

Note that for the third supersymmetric pair we assume $(m, r, s) \neq (2, 2, 0)$ since this case is special and dealt with later in the table.

| Supersymmetric Pair | $t$ | $p$ | $q$ | $r$ | $s$ |
|---|---|---|---|---|---|
| $(\mathfrak{gl}(m\|n),$ $\mathfrak{gl}(r\|s) \times \mathfrak{gl}(m-r\|n-s))$ | $-1$ | $(n-m)+$ $2(r-s)$ | $-\frac{1}{2}$ | $(m-n)+2(s-r)$ | $-\frac{1}{2}$ |
| $(\mathfrak{gl}(m\|2n), \mathfrak{osp}(m\|2n))$ | $-\frac{1}{2}$ | 0 | 0 | 0 | 0 |
| $(\mathfrak{osp}(2m\|2n), \mathfrak{osp}(r\|2s)\times$ $\mathfrak{osp}(2m-r, 2(n-s)))$ | $-\frac{1}{2}$ | $(r-m)+$ $(n-2s)$ | 0 | $-2(n-2s)+2(m-r)$ | $-\frac{3}{2}$ |
| $(\mathfrak{osp}(2m+1\|2n),$ $\mathfrak{osp}(r\|2s)\times$ $\mathfrak{osp}(2m+1-r, 2(n-s)))$ | $-\frac{1}{2}$ | $(r-m)+$ $(n-2s)-\frac{1}{2}$ | 0 | $1-2(n-2s)+2(m-r)$ | $-\frac{3}{2}$ |
| $(\mathfrak{osp}(2m\|2n), \mathfrak{gl}(m\|n))$ | $-2$ | 0 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ |
| $(\mathfrak{d}(2\|1;t),$ $\mathfrak{osp}(2\|2) \times \mathfrak{so}(2))$ | $t$ | 0 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ |
| $(\mathfrak{osp}(4\|2n),$ $\mathfrak{osp}(2\|2n) \times \mathfrak{so}(2)$ | 1 | 0 | $-\frac{1}{2n}$ | 0 | $-\frac{1}{2n}$ |
| $(\mathfrak{ab}(1\|3), \mathfrak{gosp}(2\|4))$ | $-3$ | 0 | $-\frac{5}{4}$ | 0 | $-\frac{1}{4}$ |
| $(\mathfrak{ab}(1\|3), \mathfrak{sl}(1\|4))$ | $-\frac{3}{2}$ | 0 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ |

Table 6.2: Sergeev-Veselov parameters from supersymmetric pairs

As a matter of explanation, the meaning of the parameters is as follows. In the root system $BC(m,n)$, each real component has three Weyl group orbits determined by the length of the root. In the first component, the multiplicity $m(\alpha)$ of a short root $\alpha$ is $p$, of the next longest root is $t$, and of the longest root is $q$. In the second real component, the multiplicity of the short root is $r$, the next longest root $t^{-1}$ and the longest root $s$. As already stated isotropic roots are required to have multiplicity one.

The deformed bilinear form is given by $B_1+tB_2$, where $B_1, B_2$ are the standard Euclidean inner products on the root system $BC$. Now each of our restricted root systems can be viewed as $BC(m,n)$ with some multiplicities being set to zero.

### 6.3.3 Supersymmetric pairs of distinguished superalgebras

Below we list all supersymmetric pairs (up to conjugacy) for the distinguished superalgebras which do not appear in the table of section 6.2. For each we state whether or not the pair is spherical as well as whether it admits an Iwasawa decomposition.

| Supersymmetric Pair | Spherical? | Iwasawa Decomposition? |
|---|---|---|
| $(\mathfrak{g}, \mathfrak{g}_{\overline{0}})$ | Iff $\mathfrak{g} = \mathfrak{g}_{\overline{0}}$ | Iff $\mathfrak{g} = \mathfrak{g}_{\overline{0}}$ |
| $(\mathfrak{gl}(n\|n), \mathfrak{p}(n))$ | Yes | No |
| $(\mathfrak{gl}(n\|n), \mathfrak{q}(n))$ | Yes | Yes |
| $(\mathfrak{p}(n), \mathfrak{p}(r) \times \mathfrak{p}(n-r))$ | Iff $r = 1$ | No |
| $(\mathfrak{p}(n), \mathfrak{gl}(r\|n-r))$ | Iff $n = 2, 3$ | No |

Table 6.3: Remaining supersymmetric pairs for distinguished Lie superalgebras

## 6.4 $k[G/K]$ in general

Here we discuss some general aspects of what can be proven about the $G$-module structure of $k[G/K]$. In the next section we look more specifically at the structure of $k[G]$ as a $G \times G$-module, corresponding to the symmetric supervariety $G \times G/G$ where $G$ is embedded diagonally.

Suppose that $G/K$ is a symmetric supervariety, where $G$ is connected, coming from an involution $\theta$ for which the supersymmetric pair $(\mathfrak{g}, \mathfrak{k})$ admits an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. In particular $G/K$ is a spherical supervariety. Let $\mathfrak{h}$ be a $\theta$-stable Cartan subalgebra containing $\mathfrak{a}$ and write $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$, where $\mathfrak{t}$ consists of the fixed points of $\theta$ on $\mathfrak{h}$. Write $M = C_K(\mathfrak{a})$ for the centralizer of $\mathfrak{a}$ in $K$ so that $\mathrm{Lie}(M) = \mathfrak{m} = \mathfrak{c}_{\mathfrak{k}}(\mathfrak{a})$.

Recall that by Frobenius reciprocity, an embedding of a $G$-module $V$ in $k[G/K]$ is equivalent to the data of a $K$-module homomorphism $V \to k$, i.e. an element of $(V^*)^K$.

Suppose that $V$ is a highest weight module with respect to the Borel $\mathfrak{b}$, and let $v \in V^{(\mathfrak{b})}$ be a nonzero element of weight $\lambda$. Then since $\mathfrak{k} + \mathfrak{b} = \mathfrak{g}$, we must have that $\mathcal{U}\mathfrak{k} \cdot v = V$. Let $\varphi : V \to k$ be the $K$-coinvariant corresponding to the embedding $V \subseteq k[G/K]$. Then $\varphi(v) \neq 0$ for else $\varphi$ will be zero on all $V$. If $h \in \mathfrak{t} \subseteq \mathfrak{k}$, then $\varphi(hv) = 0$, while on the other hand $h \cdot v = \lambda(h)v$. Thus $\lambda|_{\mathfrak{t}} = 0$, so $\lambda \in \mathfrak{a}^*$.

Recall from proposition 6.2.11 that $\mathfrak{m}$ is generated by $\mathfrak{t}$ and $\{e_\alpha, e_{-\alpha}\}$ where $\alpha$ runs over the simple positive roots fixed by $\theta$. Clearly $e_{\pm\alpha}v = 0$ if $\alpha$ is even since the root subalgebra of $\alpha$ will generate $\mathfrak{sl}_2$. If $\alpha$ is odd but nonisotropic, then its roots subalgebra is $\mathfrak{osp}(1|2)$ so again $e_{\pm\alpha}v = 0$. Finally, if $\alpha$ is isotropic then since $G/K$ is $\mathfrak{b}$-spherical $e_{-\alpha}v = 0$ by proposition 5.4.6. Thus we have shown that

**Lemma 6.4.1.** *If $V$ is a $\mathfrak{b}$-highest weight submodule of $k[G/K]$, then $V^{(\mathfrak{b})}$ is annihilated by $\mathfrak{m}$. In particular, $\Lambda_{\mathfrak{b}}(G/K) \subseteq \mathfrak{a}^*$.*

We would like to determine to some extent both $\Lambda_{\mathfrak{b}}^+(G/K)$ and $\operatorname{soc} k[G/K]$. The monoid $\Lambda_{\mathfrak{b}_{\overline{0}}}^+(G_0/K_0)$ is well-known and described for instance in section 26 of [61]. Of course by proposition 5.2.6 a dense subset of the weights of $\Lambda_{\mathfrak{b}_{\overline{0}}}^+(G_0/K_0)$ lift to $\Lambda_{\mathfrak{b}}^+(G/K)$. Here is a more precise statement giving a sufficient condition for lifting a weight, the proof and statement of which is easily adapted from the classical setting.

**Proposition 6.4.2.** *Let $\lambda \in \mathfrak{a}^*$ be a $\mathfrak{b}$-dominant weight. Then there exists a highest weight module $V$ of highest weight $2\lambda$ admitting a nonzero $\mathfrak{k}$-coinvariant.*

*Proof.* Consider the $\mathfrak{g}$-module $L(\lambda)^\theta$, which as a super vector space is $L(\lambda)$ but with action twisted by $\theta$. Taking the dual, we obtain the irreducible representation $(L(\lambda)^\theta)^*$, whose weights are exactly the set $\{-\theta\mu : \mu \text{ is a weight of } L(\lambda)\}$, and since $-\theta\lambda = \lambda$ we find that $(L(\lambda)^\theta)^* \cong L(\lambda)$.

Now the identity map $L(\lambda) \to L(\lambda)^\theta$ is a $\mathfrak{k}$-equivariant isomorphism, so we obtain a canonical $\mathfrak{k}$-coinvariant in $L(\lambda) \otimes (L(\lambda)^\theta)^* \cong L(\lambda)^{\otimes 2}$. Writing $v_\lambda \in L(\lambda)$ for a highest weight vector, the $\mathfrak{k}$-coinvariant will be nonzero on $v_\lambda \otimes v_\lambda$. Let $V \subseteq L(\lambda)^{\otimes 2}$ be the $\mathfrak{g}$-submodule generated by $v_\lambda \otimes v_\lambda$, a $\mathfrak{b}$-highest weight module. Then if we restrict our $\mathfrak{k}$-coinvariant to $V$ it is nonzero. □

The above proof provides a sufficient condition for a weight $\lambda$ to be in $\Lambda_{\mathfrak{b}}^+(G/K)$ up to the action of the finite group $K_0/(K_0)^0$, where $(K_0)^0$ is the connected component of the identity of $K_0$. In particular, if $G_0$ is simply connected then $G_0^\theta$ will be connected and so proposition 6.4.2 implies that $2\lambda \in \Lambda_{\mathfrak{b}}^+(G/K)$ if $\lambda \in \mathfrak{a}^*$ is $\mathfrak{b}$-dominant.

Now we explain that for generic $\lambda \in \Lambda_{\mathfrak{b}}^+(G/K)$, $L(\lambda)$ is a submodule of the socle of $k[G/K]$. First recall that weight $\lambda \in \mathfrak{h}^*$ is typical for $\mathfrak{b}$ if $(\lambda+\rho, \alpha) \neq 0$ for all isotropic roots $\alpha$, where $\rho$ is the Weyl vector (see chapter 1 of [10]). The significance of typical weights is that a dominant weight $\lambda \in \Lambda_{\mathfrak{b}}^+(\mathfrak{g})$ is typical if and only if $L(\lambda)$ is projective in $\mathcal{F}(\mathfrak{g})$.

**Lemma 6.4.3.** *If $\Delta^0$ contains no isotropic roots, then a dense open subset of $\mathfrak{a}^*$ consists of typical weights. In particular, a generic $\lambda \in \Lambda_{\mathfrak{b}}^+(G/K)$ will be typical and thus $L(\lambda)$ will be in the socle of $k[G/K]$.*

*Proof.* Since $\Delta^0$ has no isotropic roots, $\overline{\alpha} \neq 0$ for all $\alpha$ isotropic. Thus there is clearly a dense open subset of $\mathfrak{a}$ satisfying our conditions. □

Now we consider the case when $\Delta^0$ contains isotropic roots. In this case, let $x \in \mathfrak{m}_{\overline{1}}$ be self-commuting of rank equal to the defect of $\mathfrak{m}$, so that $\mathfrak{m}_x$ contains no self-commuting odd elements, and thus has no isotropic roots in its root system. In particular $\mathfrak{m}_x$ will be a product of a reductive Lie algebra with $\mathfrak{osp}(1|2n)$ for some $n$ (potentially $n = 0$). Let us assume that $(\mathfrak{g}, \mathfrak{k})$ admits an Iwasawa decomposition, i.e. $\mathfrak{m} \subseteq \mathfrak{k}$, so that we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Then observe that since $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$ are all $\mathfrak{m}$-stable under the adjoint action and $\mathfrak{a}$ commutes with $\mathfrak{m}$, we have $\mathfrak{g}_x = \mathfrak{k}_x \oplus \mathfrak{a} \oplus \mathfrak{n}_x$. Thus $(\mathfrak{g}_x, \mathfrak{k}_x)$ is a supersymmetric pair that also admits an Iwasawa decomposition.

Let $\mathfrak{b}$ be an Iwasawa Borel that arises from this decomposition and contains $x$. Then by lemma 6.4.3, a dense open subset of dominant integral weights in $\mathfrak{a}^*$ are typical for $\mathfrak{g}_x$. Let $\lambda \in \mathfrak{a}^*$ be such a weight, and suppose that $\lambda \in \Lambda_{\mathfrak{b}}^+(G/K)$. Write $V$ for the highest weight submodule of $k[G/K]$ that $\lambda$ gives rise to. Then its socle will be irreducible of highest weight $\mu$, say, and by lemma 6.4.1 we have $\mu \in \mathfrak{a}^*$. Thus the central character defined by $\lambda$ and $\mu$ must be the same. However by applying the Duflo-Serganova functor given by $x$, we see that this implies that $\lambda$ and $\mu$ are dominant weights for $\mathfrak{g}_x$ and must define the same central character for $\mathfrak{g}_x$ (see for instance [50]). However since $\lambda$ is typical and $\mu$ is dominant this implies that $\lambda = \mu$, i.e. $V = L(\lambda)$. Thus we have proven:

**Theorem 6.4.4.** *If $\lambda \in \Lambda^+(G/K)$ is typical for the supersymmetric pair $(\mathfrak{g}_x, \mathfrak{k}_x)$ where $x \in \mathfrak{m}$ is self-commuting of maximal rank, then $L(\lambda)$ is isomorphic to a submodule of $k[G/K]$. In particular for a generic dominant weight $\lambda \in \Lambda^+(G_0/K_0)$ the socle of $k[G/K]$ contains an irreducible component isomorphic to $L(\lambda)$.*

## 6.5 $G$ as a spherical supervariety

Let $G$ be a quasi-reductive supergroup. Then $G \times G$ acts homogeneously on $G$ by left and right translation, and this identifies $G$ as a symmetric supervariety with respect to the involution $\theta$ of $G \times G$ which swaps the factors.

Some is already known about the structure of $k[G]$ as a representation. For instance, in [51], the structure as a $G$-module under left translation was computed and was shown to be a sum of injective modules. In [32] a filtration of $k[GL(m|n)]$ as a $G \times G$-module was constructed following the ideas of Donkin and Koppinen in the modular case, using the highest weight category structure of representations of $GL(m|n)$. Serganova's result on the structure of $k[G]$ under left translation also follows from Green's work on coalgebras in [20], generalized to the setting of supercoalgebras. We state some further results on $k[G]$ looking at its structure as a $G \times G$-module that are straightforward extensions of results found in [20], in particular on indecomposable block summands and the socle of $k[G]$. Then we state a result that describes the Loewy layers of the socle filtration of $k[G]$ (theorem 6.5.11) which the author has not found in the literature. This description is proven in greater generality in appendix C.

**Theorem 6.5.1.** *Let $\mathfrak{g}$ be a quasi-reductive Lie superalgebra and consider the supersymmetric pair $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$ defined by the involution $\theta$ of $\mathfrak{g} \times \mathfrak{g}$ which swaps the factors. Then this supersymmetric pair admits an Iwasawa decomposition if and only if $\mathfrak{g}$ is Cartan-even.*

*Proof.* In this case a maximal toral subalgebra of the (-1)-eigenspace is given by $\mathfrak{a} = \{(h, -h) : h \in \mathfrak{h}_{\overline{0}}\}$ where $\mathfrak{h}_{\overline{0}} \subseteq \mathfrak{g}_{\overline{0}}$ is a Cartan subalgebra of $\mathfrak{g}_{\overline{0}}$. Therefore the centralizer of $\mathfrak{a}$ is just the centralizer of $\mathfrak{h}_{\overline{0}} \times \mathfrak{h}_{\overline{0}}$ in $\mathfrak{g} \times \mathfrak{g}$. This is equal to $\mathfrak{h}_{\overline{0}} \times \mathfrak{h}_{\overline{0}}$ if and only if $\mathfrak{h}_{\overline{0}}$ is a Cartan subalgebra of $\mathfrak{g}$, i.e. $\mathfrak{g}$ is Cartan-even. □

**Proposition 6.5.2.** *If $G$ is Cartan-even, then the finite-dimensional irreducible representations of $G \times G$ are exactly those of the form $L_1 \boxtimes L_2$ for finite-dimensional irreducible representations $L_1, L_2$ of $G$.*

*Proof.* A representation of this form is irreducible because $\mathrm{End}_G(L_i) \cong k$ for each $i$ and the Jacobson density theorem. Conversely, if $L$ is an irreducible representation of $G \times G$ then after choosing a Borel subgroup, it has a highest weight $\lambda_1 + \lambda_2$, where $\lambda_i$ is a weight of $i$th copy of $G$ in the direct product. Thus $L = L_B(\lambda_1) \boxtimes L_B(\lambda_2)$. □

**Definition 6.5.3.** Let $V$ be a finite-dimensional $G$-module corresponding to the coaction $V \to k[G] \otimes V$. Define $\epsilon_V : V \boxtimes V^* \to k[G]$ to be the canonical $G \times G$-equivariant map corresponding to the coaction. Notice that it is always nonzero if $V$ is nonzero. Equivalently, $\epsilon_V$ may be defined by Frobenius reciprocity; it is the unique element of $\mathrm{Hom}_{G\times G}(V \boxtimes V^*, k[G])$ that corresponds to the natural pairing $V \otimes V^* \to k$ under the isomorphism

$$\mathrm{Hom}_{G\times G}(V \boxtimes V^*, k[G]) \cong \mathrm{Hom}_G(V \otimes V^*, k)$$

*Remark* 6.5.4. If $V$ is a finite-dimensional $G$-representation then there is a canonical isomorphism of $G \times G$-modules $V \boxtimes V^* \cong (\Pi V) \boxtimes (\Pi V)^*$, and this map factors $\epsilon_V$ through $\epsilon_{\Pi V}$. In particular, $\mathrm{Im}\,\epsilon_V = \mathrm{Im}\,\epsilon_{\Pi V}$.

For the rest of this section we will assume that $G$ is Cartan-even. Given an irreducible representation $L$ of $G$, the map $\epsilon_L : L \boxtimes L^* \to k[G]$ is injective by irreducibility and the fact that $\epsilon_L$ is not the zero map. In this way we obtain a natural inclusion

$$\bigoplus_L L \boxtimes L^* \subseteq \mathrm{soc}(k[G]),$$

where the sum runs over all irreducible representations of $G$ up to parity. We now go about showing this is the entire socle.

Let $B'$ be a Borel subgroup of $G$ (as defined in [51]) and $(B')^-$ its opposite Borel. Let $B$ be a hyperborel subgroup containing $B'$ and $B^-$ a hyperborel subgroup containing $(B')^-$. Then $B \times B^-$ is a hyperborel of $G \times G$, and $G$ is is $B \times B^-$-spherical. Further, $(B^-)_0$ is the opposite Borel subgroup of $B_0$ in $G_0$.

**Lemma 6.5.5.** *If $L$ is an irreducible representation of $G$, then $L^{(B)} = L^{(B')}$.*

*Proof.* Indeed $L^{(B)} \subseteq L^{(B')}$ but by remark 2.3.15 $1 \leq \dim L^{(B)} \leq \dim L^{(B')} = 1$. □

**Definition 6.5.6.** For a hyperborel subgroup $B$ of $G$, we say an integral weight $\lambda$ is $B$-dominant if there exists an irreducible representation $L$ of $G$ such that $\Lambda_B(L) = \{\lambda\}$.

Recall that (for instance by the Peter-Weyl theorem),

$$\Lambda^+_{B_0\times(B^-)_0}(G_0) = \{(\lambda, -\lambda) : \lambda \text{ is a } B_0\text{-dominant weight}\}.$$

**Lemma 6.5.7.** *We have*

$$\Lambda^+_{B\times B^-}(G) = \{(\lambda, -\lambda) : \lambda \text{ is a } B\text{-dominant weight}\}.$$

*Proof.* By the inclusion $\Lambda^+_{B\times B^-}(G) \subseteq \Lambda^+_{B_0\times(B^-)_0}(G_0)$ we know that $\Lambda^+_{B\times B^-}(G)$ must be contained in the RHS. However our socle computation above shows that $L(\lambda)\boxtimes L(\lambda)^* \subseteq k[G]$ for all $B$-dominant weights $\lambda$, and this is exactly the $G\times G$ irreducible representation of highest weight $(\lambda, -\lambda)$. □

**Corollary 6.5.8.** $\operatorname{soc}(k[G]) \cong \bigoplus\limits_{L} L\boxtimes L^*$, *where the sum runs over all irreducible representations of $G$ up to parity.*

In corollary 6.5.8, up to parity means that we construct a partition on the set of isomorphism classes of irreducible representations of $G$ by identifying $L$ with $\Pi V$ for each irreducible representation $L$ of $G$. Then the direct sum runs over the elements of this partition and we choose a representative of each equivalence class.

We explain further the structure of $k[G]$. Let $\mathsf{Rep}(G)$ denote the category of finite-dimensional representations of $G$. Then we may decompose $\mathsf{Rep}(G)$ into a sum of simple blocks, where a block $\mathcal{B}$ is an abelian subcategory of $\mathsf{Rep}(G)$ such that if $\mathcal{B}'$ is another block distinct from $\mathcal{B}$, then $\operatorname{Ext}^i(V, V') = \operatorname{Ext}^i(V', V) = 0$ for all $i$ and all objects $V$ of $\mathcal{B}$ and $V'$ of $\mathcal{B}'$. A block $\mathcal{B}$ is simple if it cannot be decomposed into a sum of smaller, nontrivial blocks. Notice that every block must contain an irreducible representation.

Given a block $\mathcal{B}$ of $G$, we denote by $\Pi\mathcal{B}$ the block consisting of all $G$-modules $\Pi V$ where $V$ is in $\mathcal{B}$. If we write $\mathrm{Bl}_G$ for the set of blocks of $G$, we want to consider the set $\mathrm{Bl}_G/\sim$ where $\sim$ is the equivalence relation on blocks generated by $\mathcal{B}\sim\Pi\mathcal{B}$ for all blocks $\mathcal{B}$. For $\mathcal{B}\in\mathrm{Bl}_G/\sim$, we write $\operatorname{Irr}\mathcal{B}$ for the set of irreducible representations that appear in $\mathcal{B}$ up to parity. The following is an analogue of theorem (1.5g) part (ii) and theorem (1.6a) in [20].

**Proposition 6.5.9.** *We have as a $G\times G$-module*

$$k[G] = \bigoplus_{\mathcal{B}\in Bl_G/\sim} M_{\mathcal{B}}$$

*where $M_{\mathcal{B}}$ is an indecomposable $G\times G$-module given by*

$$M_{\mathcal{B}} = \sum_{V\in\mathcal{B}} \operatorname{Im}\epsilon_V.$$

*Further,*

$$\operatorname{soc}(M_{\mathcal{B}}) = \bigoplus_{L\in\operatorname{Irr}\mathcal{B}} L\boxtimes L^*.$$

*Remark* 6.5.10. It follows that the module $M_{\mathcal{B}}$ is finite-dimensional if and only if $\operatorname{Irr}\mathcal{B}$ is finite. This example shows another phenomenon that may occur in the super case: given a spherical $G$-supervariety $X$, $k[X]$ need not be a direct sum of finite-dimensional $G$-modules.

We can say more about the socle filtration of $M_{\mathcal{B}}$, and thus of $k[G]$. Recall that for a finite-dimensional $G$-module $V$, the Loewy length of $V$, which we write as $\ell\ell(V)$, is defined to be the length of a minimal semisimple filtration of $V$ (or equivalently the length of the socle or radical filtration of $V$). The first of the following results is an analogue of what was essentially known in [20] for coalgebras. We provide a full proof in appendix C, generalizing the result to coalgebras in certain tensor categories.

**Theorem 6.5.11.** *For each block $\mathcal{B} \in \mathrm{Bl}_G / \sim$ we have:*

- $$\operatorname{soc}^k M_{\mathcal{B}} = \sum_{V \in \mathcal{B},\ \ell\ell(V) \leq k} \operatorname{Im} \epsilon_V$$
- *For simple $G$-modules $L, L'$ which lie in a block of the equivalence class $\mathcal{B}$, we have*

$$\begin{aligned} [\operatorname{soc}^k M_{\mathcal{B}} / \operatorname{soc}^{k-1} M_{\mathcal{B}} : L' \boxtimes L^*] &= [L' : \operatorname{soc}^k I(L) / \operatorname{soc}^{k-1} I(L)] \\ &= \dim \underline{Hom}_G(\mathcal{P}(L'), \operatorname{soc}^k I(L) / \operatorname{soc}^{k-1} I(L)) \end{aligned}$$

### 6.5.1 The case $G = GL(1|1)$

Let $G = GL(1|1)$, and $\mathfrak{g} = \operatorname{Lie} G$. We give a very explicit description of the $\mathfrak{g} \times \mathfrak{g}$ action on $k[G]$. In this case, there is only one block of $\mathsf{Rep}(G)$ which is not semisimple, the principal block $\mathcal{B}_0$, and it contains the irreducible representations where the center of $\mathfrak{gl}(1|1)$ acts trivially. We draw a picture depicting the local structure of $M_{\mathcal{B}_0}$ below. Note that $M_{\mathcal{B}_0}$ is

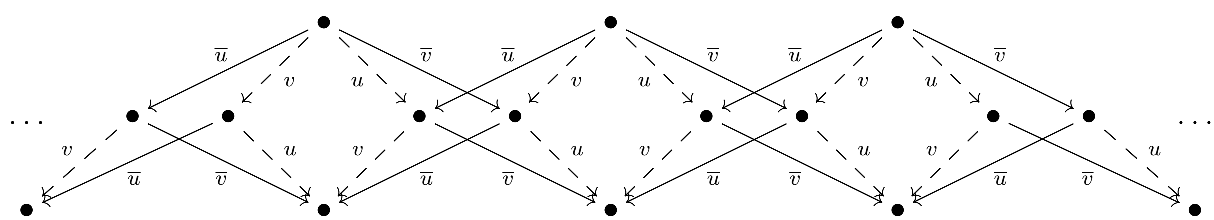



Figure 6.1: Regular representation of $GL(1|1)$

infinite-dimensional since there are infinitely many simple modules in $\mathcal{B}_0$. Each dot in the picture represents a weight vector, with the bottom and top rows having even parity and the middle row having odd parity. We write $u, v$ for the action of the odd weight vectors of $\mathfrak{gl}(1|1)$ by left translation, and $\overline{u}, \overline{v}$ for the action of the odd weight vectors by right translation. One can see rather explicitly here that if we restrict the action to only left or only right translation, then this is just a sum of injective modules.

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# Appendix A

# Indecomposable spherical representations

In this appendix we present the full classification of spherical indecomposable representation of the distinguished Lie superalgebras.

## A.1 Spherical representations and their properties

### A.1.1 Spherical representations

**Definition A.1.1.** Let $V$ be a super vector space, $\mathfrak{g}$ a distinguished Lie superalgebra, $\mathfrak{b} \subseteq \mathfrak{g}$ a Borel subalgebra of $\mathfrak{g}$, and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$. Then we say the triple $(V, \mathfrak{g}, \rho)$ is spherical with respect to $\mathfrak{b}$ if $V$ is a non-zero vector space, and it is a spherical variety for the Lie superalgebra $\rho(\mathfrak{g}) + k\,\mathrm{id}_V$ with respect to $\rho(\mathfrak{b}) + k\,\mathrm{id}_V$.

Equivalently, there exists a vector $v \in V_{\bar{0}}$ such that $(\rho(\mathfrak{b}) + k\,\mathrm{id}_V) \cdot v = V$. A vector $v$ satisfying this condition will be called a spherical vector.

In general we say a $\mathfrak{g}$-module $V$ is spherical if there exists a Borel $\mathfrak{b} \subseteq \mathfrak{g}$ such that $(V, \mathfrak{g}, \rho)$ is spherical with respect to $\mathfrak{b}$, where $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ defines the $\mathfrak{g}$-action.

*Remark* A.1.2. By abuse of language, we will often refer to a super vector space $V$ as being spherical when the algebra which acts on it is clear from context. We may also omit the representation $\rho$ and just say that $(V, \mathfrak{g})$ is spherical, where $V$ is a $\mathfrak{g}$-module and the action is clear from context.

If $V$ is spherical of even (resp. odd) highest weight $\lambda$ with respect to $\mathfrak{b}$, then we will say that $\lambda$ is an even (resp. an odd) spherical weight for $\mathfrak{g}$ with respect to $\mathfrak{b}$, or simply that $\lambda$ is spherical when the choice of Borel and parity of the highest weight is clear from context.

*Remark* A.1.3. As with supervarieties a spherical $\mathfrak{g}$-module is, in general, not spherical for all Borels. However, if $V$ is spherical for $\mathfrak{g}$ with respect to $\mathfrak{b}$, then it is also spherical with respect to any conjugate of $\mathfrak{b}$.

**Lemma A.1.4.** *Let* $(V, \mathfrak{g}, \rho)$ *be a spherical representation with respect to* $\mathfrak{b}$*.*

1. *If* $V'$ *is a super vector space and* $\psi : V \to V'$ *is an isomorphism, let* $\Psi : \mathfrak{gl}(V) \to \mathfrak{gl}(V')$ *denote the induced isomorphism of algebras. Then* $(V', \mathfrak{g}, \Psi \circ \rho)$ *is spherical with respect to* $\mathfrak{b}$*.*

2. *Let* $\tau$ *be an automorphism of* $\mathfrak{g}$*. Then* $(V, \mathfrak{g}, \rho \circ \tau)$ *is spherical with respect to* $\tau^{-1}(\mathfrak{b})$*.*

*Proof.* The proof is straightforward. □

By lemma A.1.4, sphericity is determined by the image of a Lie superalgebra under the representation. Therefore we make the following definition.

**Definition A.1.5.** We say that two spherical representations $(V, \mathfrak{g}, \rho)$ and $(V', \mathfrak{g}', \rho')$ are equivalent if there exists an isomorphism of super vector spaces $\psi : V \to V'$ such that if $\Psi : \mathfrak{gl}(V) \to \mathfrak{gl}(V')$ is the induced map, then $\rho'(\mathfrak{g}') + k\,\mathrm{id}_{V'} = \Psi(\rho(\mathfrak{g})) + k\,\mathrm{id}_{V'}$.

### A.1.2 Properties of spherical representations

Here we collect some properties of spherical modules. First a definition.

**Definition A.1.6.** We say $(V, \mathfrak{g}, \rho)$ is numerically spherical if $V_{\overline{0}}$ is a spherical $\mathfrak{g}_{\overline{0}}$-module and $\dim V_{\overline{1}} \leq \max_{\mathfrak{b}} \dim \mathfrak{b}_{\overline{1}}$.

**Lemma A.1.7.** *Spherical modules are numerically spherical. Any subquotient of a spherical module is numerically spherical.*

*Proof.* For the first statement, if $(V, \mathfrak{g}, \rho)$ is spherical with respect to $\mathfrak{b}$, then if $v \in V_{\overline{0}}$ is a spherical vector we have

$$(\mathfrak{b} + k\,\mathrm{id}_V) \cdot v = (\mathfrak{b}_{\overline{0}} + k\,\mathrm{id}_V + \mathfrak{b}_{\overline{1}}) \cdot v = (\mathfrak{b}_{\overline{0}} + k\,\mathrm{id}_V) \cdot v \oplus \mathfrak{b}_{\overline{1}} \cdot v = V_{\overline{0}} \oplus V_{\overline{1}}.$$

The second statement is straightforward. □

*Remark* A.1.8. Note that if $\mathfrak{g}$ is basic, then all Borels of $\mathfrak{g}$ have the same odd dimension, while in general this is no longer true. In particular, for $\mathfrak{p}(n)$ Borel subalgebras can have odd dimension between $\frac{n(n-1)}{2}$ and $\frac{n(n+1)}{2}$.

**Lemma A.1.9.** *The quotient of a spherical representation remains spherical.*

*Proof.* The image of a spherical vector under the quotient map provides a spherical vector in the quotient. □

**Lemma A.1.10.** *If* $\mathfrak{g}$ *is basic, then an irreducible representation* $V$ *is spherical if and only if* $V^*$ *is. If* $(V, \mathfrak{g})$ *is spherical, then* $(V^*, \mathfrak{g})$ *is equivalent to it.*

*Proof.* In this case, there exists an automorphism $\tau$ of $\mathfrak{g}$ which acts by multiplication by $(-1)$ on a Cartan subalgebra. Since highest weights spaces of irreducible representations are one-dimensional for a basic algebra, if $V$ is irreducible we have, $V^\tau \cong V^*$. The result now follows from lemma A.1.4. □

*Remark* A.1.11. If we drop the condition of irreducibility in lemma A.1.10, the argument breaks down, since we no longer have $V^\tau = V^*$.

For example, let $\mathfrak{g} = \mathfrak{gl}(1|1)$ and consider the representation of $\mathfrak{g}$ on $k^{1|1} = kv \oplus kw$, where $v$ is even and $w$ odd, as follows. Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ act by 0, $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ act by 1 on $w$ and $-1$ on $v$, and $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ send $v$ to $w$ and $y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ act by 0.

Then this is representation is spherical (with respect to the Borel $\mathfrak{b} = k\langle h, I, x\rangle$), but the dual is not spherical with respect to any Borel subalgebra.

*Remark* A.1.12. By lemma A.1.10, when $\mathfrak{g}$ is basic it suffices to consider irreducible representations up to their dual. Note that lemma A.1.10 does not apply to the non-basic algebras $\mathfrak{p}(n)$ or $\mathfrak{q}(n)$. Although $\mathfrak{q}(n)$ does admit an automorphism $\tau$ which acts by $(-1)$ on $\mathfrak{h}_0$, because highest weight spaces of irreducible representations need not be one-dimensional, there are situations when $V$ is irreducible but $V^\tau \ncong V^*$.

**Lemma A.1.13.** *Let $\mathfrak{r}$ be a linear Lie superalgebra, i.e. $\mathfrak{r} \subseteq \mathfrak{gl}(V)$ for a super vector space $V$. Suppose we have $\mathfrak{r}_{\bar{0}} v = V_{\bar{0}}$. If there exists $w \in V_{\bar{0}}$ such that $\mathfrak{r} \cdot w = V$, then $\mathfrak{r} \cdot v = V$.*

*Proof.* Exponentiate $\mathfrak{r}_{\bar{0}}$ to $R_{\bar{0}} = \exp(\mathfrak{r}_{\bar{0}}) \subseteq GL(V)_{\bar{0}}$. Then by assumption, the orbits of both $v$ and $w$ under $R_{\bar{0}}$ are open in $V_{\bar{0}}$ and since we work in the Zariski topology they must lie in the same orbit. Therefore there exists $x \in R_{\bar{0}}$ such that $v = x \cdot w$. Hence

$$\begin{aligned} \mathfrak{r} \cdot v &= \{Xv : X \in \mathfrak{r}\} \\ &= \{X(xw) : X \in \mathfrak{r}\} \\ &= \{x \operatorname{Ad}(x^{-1})(X)w) : X \in \mathfrak{r}\} \\ &= \{x(Xw) : X \in \mathfrak{r}\} \\ &= x(\mathfrak{r} \cdot w) = V \end{aligned}$$

□

**Corollary A.1.14.** *Let $V$ be a $\mathfrak{g}$-module, $\mathfrak{b}$ a Borel of $\mathfrak{g}$ such that $V_{\bar{0}}$ is spherical for $\mathfrak{g}_{\bar{0}}$. Then if $v \in V_{\bar{0}}$ is any spherical vector for $\mathfrak{b}_{\bar{0}}$ and $\mathfrak{b} \cdot v \neq V$, then $V$ is not spherical with respect to $\mathfrak{b}$.*

*Proof.* This follows from lemma A.1.13 by letting $\mathfrak{r} = (\rho(\mathfrak{b}) + k \operatorname{id}_V)$, where $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is the representation giving the $\mathfrak{g}$-action. □

**Lemma A.1.15.** *Let $V$ be an irreducible spherical $\mathfrak{g}$-module of $\mathfrak{b}$-highest weight $\lambda$, such that $\dim V_\lambda = (0|1)$. We do not assume $V$ is spherical with respect to $\mathfrak{b}$. Then there exists an odd negative root $\alpha$ such that $\lambda+\alpha$ is a $\mathfrak{b}_{\overline{0}}$-highest weight of $V_{\overline{0}}$. In particular, $\lambda+\alpha$ is $\mathfrak{g}_{\overline{0}}$-spherical.*

*Proof.* Let $v \in V_\lambda$ be a non-zero highest weight vector. Consider the set

$$S = \{\alpha \in \Delta^- : x_\alpha \cdot v \neq 0 \text{ for some } x_\alpha \in \mathfrak{g}_\alpha\}$$

Then we claim $S \neq \emptyset$. If not, $v$ will be annihilated by $\mathfrak{n}_{\overline{1}}^+$ and $\mathfrak{n}_{\overline{1}}^-$. The condition that $\dim V = (0|1)$ implies further that $v$ is annihilated by $\mathfrak{h}_{\overline{1}}$, so we find then that $v$ is annihilated by $\mathfrak{g}_{\overline{1}}$. Since $V = \mathcal{U}\mathfrak{g}\cdot v$, this in turn implies that $V_{\overline{0}} = 0$, a contradiction.

Since $S$ is not empty, we may choose $\alpha \in S$ which is maximal with respect to the Bruhat order. Then for $z \in \mathfrak{n}_{\overline{0}}^+$ we have

$$zx_\alpha v = [z, x_\alpha]v + x_\alpha z v = [z, x_\alpha]v$$

However, $[z, x_\alpha]$ is of weight strictly larger than $\alpha$ in the Bruhat order. If $[z, x_\alpha] \in \mathfrak{n}^-$, then by maximality of $\alpha$ we have $[z, x_\alpha]v = 0$. If $[z, x_\alpha] \in \mathfrak{h}_{\overline{1}} \oplus \mathfrak{n}_{\overline{1}}^+$, then we also have $[z, x_\alpha]v = 0$. So $x_\alpha v$ is an even $\mathfrak{b}_{\overline{0}}$-highest weight vector. □

**Lemma A.1.16.** *Suppose $\mathfrak{g}$ is distinguished and Cartan-even. Let $\lambda$ be a dominant highest weight. If $L(\lambda)$ is numerically spherical, then the following hold:*

1. *$L_0(\lambda)$ is a spherical $\mathfrak{g}_{\overline{0}}$-module;*

2. *If $\mathfrak{g}$ is basic, and $\alpha$ is a simple isotropic root such that $(\lambda, \alpha) \neq 0$, then* $\dim L_0(\lambda-\alpha) \leq \dim \mathfrak{b}_{\overline{1}}$.

*Proof.* (1) follows from the observation that $L(\lambda)_{\overline{0}}$ must be a spherical $\mathfrak{g}_{\overline{0}}$-module, and that $L_0(\lambda)$ is a $\mathfrak{g}_{\overline{0}}$-submodule of $L(\lambda)_{\overline{0}}$. For (2), by the theory of odd reflections we will have that $L_0(\lambda-\alpha)$ is a $\mathfrak{g}_{\overline{0}}$-submodule of $L(\lambda)_{\overline{1}}$, and the statement follows. □

**Lemma A.1.17.** *In the context of lemma A.1.16, if $\Pi L(\lambda)$ is numerically spherical, then the following hold:*

1. $\dim L_0(\lambda) \leq \max_{\mathfrak{b}} \dim \mathfrak{b}_{\overline{1}}$*;*

2. *If $\mathfrak{g}$ is basic, and $\alpha$ is a positive simple isotropic root such that $(\lambda, \alpha) \neq 0$, then $L_0(\lambda-\alpha)$ is a spherical $\mathfrak{g}_{\overline{0}}$-module.*

3. *If $\mathfrak{g}$ is basic and $\lambda$ is not a character, there exists a positive odd isotropic root $\alpha$ such that $(\lambda, \alpha) \neq 0$ and $L_0(\lambda-\alpha)$ is a spherical $\mathfrak{g}_{\overline{0}}$-module.*

*Proof.* (1) follows from by definition of numerically spherical modules.

For (2), by the theory of odd reflections $L_0(\lambda-\alpha)$ will be a $\mathfrak{g}_{\overline{0}}$-submodule of $L(\lambda)_{\overline{0}}$, and the statement follows.

For (3), since $\lambda$ is not character, if we consider all possible sequences of odd reflections we can apply to our Borel, there will be some sequence of odd reflections $r_{\alpha_s},\cdots,r_{\alpha_1}$ giving rise to a new Borel such that $(\lambda,\alpha_1)=\cdots=(\lambda,\alpha_{s-1})=0$ and $(\lambda,\alpha_s)\neq 0$.

Let $\alpha=\alpha_s$. By the theory of odd reflections, $\lambda-\alpha$ will be an even highest weight vector with respect this new Borel of $\mathfrak{g}$. Therefore $L_0(\lambda-\alpha)$ will be a spherical $\mathfrak{g}_{\overline{0}}$-module. □

*Remark* A.1.18. Characters of a Lie superalgebra, including the trivial one, are all spherical and equivalent to $(k^{1|0},\mathbf{0})$ where $\mathbf{0}$ denotes the trivial Lie algebra. Further, if $\chi$ is a character of $\mathfrak{g}$ and $(V,\mathfrak{g},\rho)$ is a spherical representation, then $(V,\mathfrak{g},\rho\otimes\chi)$ is also spherical and is equivalent to $(V,\mathfrak{g},\rho)$. It follows we may work with equivalence classes of spherical representations up to twists by characters.

We also observe that all one-dimensional odd modules are numerically spherical.

## A.2 Explanation of procedure for proof

To classify spherical indecomposable representations, we will work case by case with various Lie superalgebras.

For a chosen algebra $\mathfrak{g}$, we will first list any needed notation and setup. Then we will determine all (numerically) spherical irreducible representations. If $\mathfrak{g}$ has that $\mathfrak{h}=\mathfrak{h}_{\overline{0}}$ (i.e. for all cases but $\mathfrak{g}=\mathfrak{q}(n)$), we proceed according to the following steps:

1. We choose a fixed, 'standard' Borel for $\mathfrak{g}$, which we write as $\mathfrak{b}^{st}$.

2. We state all $\mathfrak{g}_{\overline{0}}$-dominant weights which are spherical with respect to $\mathfrak{b}^{st}_{\overline{0}}$. We will simply quote the results found by Kac in [27].

3. From (1) and (2), we write a list of candidate $\mathfrak{b}^{st}$-dominant weights $\lambda$ for which $L(\lambda)$ could be numerically spherical. By remark A.1.18, we may find candidate weights up to twists by characters of $\mathfrak{g}$.

   Such $\lambda$ must have the properties that they are dominant with respect to $\mathfrak{b}^{st}$ and spherical with respect to $\mathfrak{g}_{\overline{0}}$. Further, by lemma A.1.16, if $\mathfrak{g}$ is basic and $\alpha$ is a simple positive isotropic root such that $(\lambda,\alpha)\neq 0$, then we must have $\dim L_0(\lambda-\alpha)\leq\dim\mathfrak{b}_{\overline{1}}$. This will be used heavily.

4. Determine whether $L(\lambda)=L_{\mathfrak{b}^{st}}(\lambda)$ is (numerically) spherical for each $\lambda$ from (3).

5. Create a candidate list of $\mathfrak{b}^{st}$-dominant weights $\lambda$ for which $\Pi L(\lambda)$ could be numerically spherical. Again, we can work up to twists by characters. By lemma A.1.17, such a $\lambda$ must have that

$$\dim L_0(\lambda)\leq\max_{\mathfrak{b}}\dim\mathfrak{b}_{\overline{1}}.$$

If $\mathfrak{g}$ is basic, then lemma A.1.17 says that $\lambda-\alpha$ falls into the list in (2) for some positive (with respect to $\mathfrak{b}^{st}$) isotropic root $\alpha$ with $(\lambda, \alpha) \neq 0$. Further, if $\alpha$ is a simple positive isotropic root such that $(\lambda, \alpha) \neq 0$, then $\lambda - \alpha$ falls into the list in (2). This will be used heavily.

If $\mathfrak{g} = \mathfrak{p}(n)$, then by lemma A.1.15 there must exist an odd negative root $\alpha$ for which $\lambda + \alpha$ is spherical for $\mathfrak{g}_{\overline{0}}$.

6. Determine whether $\Pi L(\lambda)$ is (numerically) spherical for each $\lambda$ from (5).

If $\mathfrak{g} = \mathfrak{q}(n)$, then we proceed as above, except we make a single list of candidate weights for which $L(\lambda)$ or $\Pi L(\lambda)$ could be (numerically) spherical, and then make a check. This is because unless $\lambda = 0$, we will have $L(\lambda)_\lambda = (k|k)$ where $k > 0$, so some conditions in both (3) and (5) will apply to $\lambda$.

The above steps will give all (numerically) spherical irreducible representations. By lemma A.1.7 and lemma A.1.9, an indecomposable spherical representation must have numerically spherical composition factors and a spherical head. To determine all spherical indecomposables, we will take the modules from our above list and compute extensions between them, and check if any extensions are spherical.

## A.3 Spherical $\mathfrak{gl}(m|n)$-modules

Set $\mathfrak{g} = \mathfrak{gl}(m|n)$. Using $\mathfrak{gl}(m|n) \cong \mathfrak{gl}(n|m)$, in finding all spherical representations we may assume without loss of generality that $m \leq n$. For more on this algebra, see [39] or [10]. We refer to section 2.3.2 and section 2.3.4 for an explanation of our notation for the root system and Borels of $\mathfrak{g}$.

**Notation:** Write

$$\det{}_\epsilon := \epsilon_1 + \cdots + \epsilon_m, \quad \det{}_\delta := \delta_1 + \cdots + \delta_n$$

For a $\mathfrak{g}_{\overline{0}}$-dominant weight $\lambda$, we will write $K_{m|n}(\lambda)$ for the Kac module

$$K_{m|n}(\lambda) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}(\mathfrak{g}_0 \oplus \mathfrak{g}_1)} L_0(\lambda).$$

Here we use the usual $\mathbb{Z}$-grading on $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

We write $GL_{m|n}$ for the standard $\mathfrak{gl}(m|n)$-module structure on $k^{m|n}$.

**Choice of standard Borel:** $\mathfrak{b}^{st} = \mathfrak{b}^{\epsilon^m\delta^n}$, i.e. the Borel of upper triangular matrices of $\mathfrak{gl}(m|n)$. Observe that for every Borel $\mathfrak{b}$ of $\mathfrak{g}$, we have $\dim \mathfrak{b}_{\overline{1}} = mn$.

**Characters of $\mathfrak{g}$:** The characters of $\mathfrak{g}$ are exactly the multiples of the Berezinian weight Ber, defined by

$$\mathrm{Ber} = \epsilon_1 + \cdots + \epsilon_m - \delta_1 - \cdots - \delta_n = \det{}_\epsilon - \det{}_\delta$$

**Spherical weights for $\mathfrak{g}_{\overline{0}} = \mathfrak{gl}(m) \times \mathfrak{gl}(n)$:**

$$0, \quad \epsilon_1, \quad -\epsilon_m, \quad \delta_1, \quad -\delta_n$$

$$2\epsilon_1, \quad -2\epsilon_m, \quad 2\delta_1, \quad -2\delta_n \tag{A.3.1}$$

$$\epsilon_1+\epsilon_2, \quad -\epsilon_{m-1}-\epsilon_m, \quad \delta_1+\delta_2, \quad -\delta_{n-1}-\delta_n$$

$$\epsilon_1+\delta_1, \quad \epsilon_1-\delta_n, \quad -\epsilon_m+\delta_1, \quad -\epsilon_m-\delta_n$$

We also may add to any of the above weights an arbitrary linear combination of $\det_\epsilon$ and $\det_\delta$, and get another spherical weight.

### A.3.1 The case of $\mathfrak{gl}(1|1)$

We deal with $\mathfrak{g}=\mathfrak{gl}(1|1)$ separately. We have

$$\mathfrak{g}_{\bar{1}} = k\langle u_+, u_-\rangle, \text{ where } u_\pm \in \mathfrak{g}_{\pm(\epsilon_1-\delta_1)}$$

There are exactly two Borel subalgebras, and they are non-conjugate:

$$\mathfrak{b}^{st} := \mathfrak{b}^{\epsilon\delta} = \mathfrak{g}_{\bar{0}} \oplus k\langle u_+\rangle, \quad \mathfrak{b}^{\delta\epsilon} = \mathfrak{g}_{\bar{0}} \oplus k\langle u_-\rangle$$

Every weight $s\epsilon_1 + t\delta_1$ is dominant. These are all $\mathfrak{g}_{\bar{0}}$-spherical, and all differ with $t\epsilon_1$ by some multiple of the Berezinian, so without loss of generality we can restrict our attention to weights of the form $t\epsilon_1$.

**Proposition A.3.1.** *For $\mathfrak{gl}(1|1)$, all non-trivial indecomposable spherical modules (up to equivalence) fall into the following list:*

1. *The standard module $GL_{1|1}$, which is spherical with respect to $\mathfrak{b}^{\delta\epsilon}$ and has stabilizer $(\mathfrak{b}^{\epsilon\delta})_{(\epsilon_1)}$*

2. *The $(1|1)$-dimensional module $K_{1|1}(0)$, which is spherical with respect to $\mathfrak{b}^{\delta\epsilon}$, and has stabilizer $\mathfrak{b}^{\epsilon\delta}$. This module is equivalent to the $\mathfrak{p}(1)$-module $P_{1|1}$ (see appendix A.6).*

*Proof.* The proof is straightforward and hence omitted. □

### A.3.2 The case of $\mathfrak{gl}(1|2)$

**Candidate even weights:**

$$t\epsilon_1 \ (t\neq 0), \qquad -\epsilon_1+\delta_1, \qquad -2\epsilon_1+2\delta_1$$

**Check for (numerical) sphericity of $L(\lambda)$:**

- First suppose that $\lambda = t\epsilon_1$, with $t\neq 0$. In this case, $L(\lambda)$ is a quotient of $K_{1|2}(\lambda)$, which is $(2|2)$-dimensional. It's a straightforward check that $K_{1|2}(\lambda)$ is always spherical, and so $L(\lambda)$ is also.

- If $\lambda = -\epsilon_1 + \delta_1$, then $\lambda$ differs with the Berezinian by $-\delta_2$, which gives $\Pi GL_{1|2}^*$ which is spherical (this will be shown later).
- Finally suppose $\lambda = -2\epsilon_1 + 2\delta_1$. Then $\lambda$ differs by a multiple of the Berezinian with $-2\delta_2$, which is the highest weight of $\Lambda^2 GL_{1|2}^*$. We will see below that the second exterior power of the standard module for $\mathfrak{gl}$ is always spherical, so this is too.

**Candidate odd weights:**

$$t\epsilon_1 \ (t \neq 0), \qquad \lambda = t\epsilon_1 + \delta_1$$

**Check for (numerical) sphericity of $\Pi L(\lambda)$:**

- Suppose $\lambda = t\epsilon$ with $t \neq 0$. Then if $t = 1$, we get the parity shift of the standard module, which is spherical (shown below). If $t \neq 1$ then $\lambda$ is a typical weight, so that $\Pi L(\lambda) \cong \Pi K_{1|2}(\lambda)$. One can check that this module is spherical exactly with respect to the Borel $\delta\epsilon\delta$.
- If $\lambda = t\epsilon_1 + \delta_1$, then if $t = -1$, $\Pi L(\lambda)$ is equivalent up to the Berezinian with $GL_{1|2}^*$, which is spherical. If $t = 1$, then $\lambda = \epsilon_1 + \delta_1$ which gives $\Lambda^2 GL_{1|2}$, and as already mentioned this is spherical. If $t \neq \pm 1$, then the sequence of odd reflections $r_{\epsilon_1 - \delta_1}$ followed by $r_{\epsilon_1 - \delta_2}$ both change the weight, which forces the odd dimension of the module to be larger than 2, so it cannot be numerically spherical.

**(Numerically) Spherical irreducibles for $\mathfrak{gl}(1|2)$:** Along with $(\Pi)k$ we have the following, up to equivalence:

$$(\Pi)GL_{1|2}, \quad (\Pi)K_{1|2}(t\epsilon_1) \ (t \neq 0, 1) \quad \Lambda^2 GL_{1|2}$$

**Indecomposables** For dimension reasons, the only possible extensions which are spherical are of an even one-dimensional module with a spherical irreducible or of $\Pi GL_{1|2}$ with a module equivalent to it or an odd one-dimensional module.

The trivial module for $\mathfrak{gl}(1|2)$ admits non-trivial extensions only with $(GL_{1|2})_{\text{Ber}}$ and $(GL_{1|2})^*_{-\text{Ber}}$, and these extensions are exactly $K_{1|2}(0)$, $K_{1|2}(0)^* \cong (K_{1|2}(\epsilon_1))_{\text{Ber}}$ along with two modules that are geometrically equivalent to these. It has already been noted that $K_{1|2}(0)$ and $K_{1|2}(\epsilon_1)$ are spherical, so we get two new indecomposable spherical modules in this way. The module $K_{1|2}(0)$ is spherical exactly with respect to $\delta\delta\epsilon$ and $\epsilon\delta\delta$, while $K_{1|2}(\epsilon_1)$ is spherical exactly with respect to only $\delta\delta\epsilon$. No further extensions can be constructed which remain spherical.

The module $\Pi GL_{1|2}$ has extensions exactly with $\Pi k_{-\text{Ber}}$ and $\Lambda^2 GL_{1|2}$. The extensions by the latter is not numerically spherical, and hence not spherical. But the extension with $\Pi k_{-\text{Ber}}$ with $\Pi GL_{1|2}$ as the quotient is exactly $\Pi K_{1|2}(\epsilon_1)$, which is spherical exactly with respect to the Borel $\delta\epsilon\delta$. Note the opposite extension of the two has an odd one-dimensional quotient, so cannot be spherical. No further extensions can be constructed which remain spherical.

| Rep | $\dim^s$ | Borels | Stabilizer |
|---|---|---|---|
| $K_{1\|2}(t\epsilon_1)$, $t \neq 0, 1$ | $(2\|2)$ | $\epsilon\delta\delta$, $\delta\delta\epsilon$ | $\mathfrak{osp}(1\|2)$ |
| $K_{1\|2}(0)$ | $(2\|2)$ | $\epsilon\delta\delta$, $\delta\delta\epsilon$ | $k \times \mathfrak{osp}(1\|2)$ |
| $K_{1\|2}(\epsilon_1)$ | $(2\|2)$ | $\delta\delta\epsilon$ | $\mathfrak{sp}(2) \ltimes \Pi SP_2$ |
| $\Pi K_{1\|2}(t\epsilon_1)$, $t \neq 0$ | $(2\|2)$ | $\delta\epsilon\delta$ | $(\mathfrak{b}^{-\delta\epsilon\delta})_{((1-t)\epsilon-\delta_2)}$ |

Table A.1: Spherical representations of $\mathfrak{gl}(1|2)$

Our work shows we get the following list of spherical indecomposables for $\mathfrak{gl}(1|2)$ which do not come from the standard module, along with stabilizers of spherical vectors and Borels for which sphericity is achieved. All modules arising from the standard module will be dealt with in the next subsection.

### A.3.3 Some spherical irreducibles for $\mathfrak{gl}(m|n)$

From our above work we may now assume either $m \geq 2$ or $n \geq 3$.

We present the (numerically) spherical modules for $\mathfrak{gl}(m|n)$ that arise naturally from the standard module, and prove they are spherical. We write $v_1, \dots, v_m, w_1, \dots, w_n$ for a homogeneous basis for $GL_{m|n}$. Here $v_i$ is even with weight $\epsilon_i$ and $w_j$ is odd with weight $\delta_j$.

**Proposition A.3.2.** *Suppose $1 \leq m \leq n$ and either $m \geq 2$ or $n \geq 3$. Then of all the modules $(\Pi)S^dGL_{m|n}$ and $(\Pi)\Lambda^dGL_{m|n}$, where $d \geq 1$, the numerically spherical ones are exactly those in the following list, and they are all spherical:*

$$(\Pi)GL_{m|n}, \quad S^2GL_{m|n}, \quad \Lambda^2GL_{m|n}, \quad \Pi S^2GL_{n|n} \cong (\Pi\Lambda^2GL_{n|n})^{\Pi}, \quad \Pi S^2GL_{n|n+1}$$

*Further, these modules are equivalent to one showing up in the following table, where we allow for $m, n$ to be arbitrary. The table also lists the conjugacy classes of Borels each is spherical with respect to along with the stabilizer of a spherical vector $v$.*

By $\mathfrak{osp}(GL^*_{m|n}, v)$ we mean the superalgebra of matrices preserving the form on $GL^*_{m|n}$ induced by the spherical vector $v$. By $\mathfrak{p}(n)^{\Pi}$, we mean the algebra gotten by applying the parity shift automorphism to $\mathfrak{p}(n)$. In the last stabilizer, the extra copy of $k$ is acting by the character $-1$ (this extra copy of $k$ is spanned by the diagonal element dual to the weight $\delta_1$, hence the notation).

*Proof.* Observe that $GL_{m|n} \cong \Pi GL_{n|m}$, and so $\Lambda^dGL_{m|n} \cong \Pi^dS^dGL_{n|m}$ for all $d$. It therefore suffices to study $GL_{m|n}$ and $(\Pi)S^dGL_{m|n}$ for $d \geq 2$, where we now only require that $m, n \geq 1$ and that either $m = n = 2$ or $\max(m, n) \geq 3$. We have the following cases; let $V = GL_{m|n}$.

1. $V$: We observe that $V$ is spherical if we take $v_m$ for our spherical vector, exactly with respect to Borels for which $\delta_i - \epsilon_m$ is positive for all $i$.

| Rep | $\dim^s$ | Borels | Stabilizer |
|---|---|---|---|
| $GL_{m\|n}$ | $(m\|n)$ | $\cdots\epsilon$ | $\mathfrak{gl}(m-1\|n)\ltimes$ $GL^*_{m-1\|n}$ |
| $S^2GL_{m\|n}$ | $\left(\frac{n(n-1)}{2}+\frac{m(m+1)}{2}\middle\|mn\right)$ | $\delta^{i_1}\epsilon^{j_1}\delta^{2i_2}\epsilon^{j_2}\delta^{2i_3}\cdots\epsilon^{j_k}\delta^{2i_l}$ | $\mathfrak{osp}(GL^*_{m\|n},v)$ |
| $(\Pi\Lambda^2GL_{n\|n})^\Pi\cong$ $\Pi S^2GL_{n\|n}$ | $(n^2\|n^2)$ | $\epsilon\delta\epsilon\delta\cdots\epsilon\delta$ | $\mathfrak{p}(n)^\Pi$ |
| $\Pi S^2GL_{n\|n+1}$ | $(n(n+1)\|n(n+1))$ | $\delta\epsilon\delta\epsilon\cdots\epsilon\delta$ | $(\mathfrak{p}(n)^\Pi\times k)\ltimes$ $(\Pi P_{n\|n})_{-\delta_1}$ |

Table A.2: Spherical tensor representations of $\mathfrak{gl}(m|n)$

2. *Excluding $S^kV$ and $\Pi S^kV$ for $k>2$:* If $S^kV$ is spherical then $S^kV_{\bar{0}}$ is $\mathfrak{g}_0$-spherical, which implies $m=1$- but if $m=1$ the odd part of $S^kV$ will be too large. If $\Pi S^kV$ is spherical, then $S^{k-1}V_{\bar{0}}\otimes V_{\bar{1}}$ is $\mathfrak{g}_0$-spherical, which implies either $m=1$ or $n=1$. In both of these cases, however, the odd part of $\Pi S^kV$ will be too large.

   We are left to look at $(\Pi)S^2V$.

3. *$S^2V$ is always spherical:* A homogeneous basis for $S^2V$ is

$$\text{Even}: \ v_iv_j \ \ i\leq j, \ \ w_iw_j \ \ i<j; \ \ \text{Odd}: \ \ v_iw_j$$

   The vector $v=v_1^2+\cdots+v_m^2+w_1w_2+\cdots+w_{n-1}w_n$ is a $\mathfrak{g}_0$-spherical vector, so by corollary A.1.14, $S^2V$ is spherical with respect to a Borel $\mathfrak{b}$ defined by an $\epsilon\delta$ string if and only if $\mathfrak{b}\cdot v=V$.

   Now a Borel $\mathfrak{b}$ defined by an $\epsilon\delta$ string will contain the odd operators $w_i\partial_{v_j}$ exactly when $\delta_i-\epsilon_j$ is positive, and will have odd operators $v_i\partial_{w_j}$ exactly when $\epsilon_i-\delta_j$ is positive. Further, we see that $\frac{1}{2}w_i\partial_{v_j}(v)=w_iv_j$ and $v_i\partial_{w_j}(v)=v_i(w_{j-1}+w_{j+1})$, where we let $w_0=w_{n+1}=0$.

   Now fix an $i$ with $1\leq i\leq m$. Because of our choice of even Borel, we observe that if $\epsilon_i-\delta_j$ is positive, then so is $\epsilon_i-\delta_k$ for $k>j$. Also, if $\delta_j-\epsilon_i$ is positive, so is $\delta_k-\epsilon_i$ for $k<j$. Hence we may choose $j$ so that for a $k\leq j$, $\delta_k-\epsilon_i$ is positive, and for $j<k$, $\epsilon_i-\delta_k$ is positive. Then in $\mathfrak{b}_{\bar{1}}\cdot v$ we will get all monomials $v_iw_k$ for all $k\leq j$ along with all terms $v_i(w_{k-1}+w_{k+1})$ for $j<k\leq n$. It is now a linear algebra exercise to show that these monomials span the subspace spanned by $\{v_iw_k : 1\leq k\leq n\}$ if and only if $n-j$ is even. Therefore, $\mathfrak{b}_{\bar{1}}\cdot v=(S^2V)_{\bar{1}}$ if and only if the number of $\delta$'s appearing after any $\epsilon$ in the $\epsilon\delta$-string is even- i.e. the $\epsilon\delta$ string must be of the form

$$\delta^{i_1}\epsilon^{j_1}\delta^{2i_2}\epsilon^{j_2}\delta^{2i_3}\cdots\epsilon^{j_k}\delta^{2i_l}$$

4. *Examination of $\Pi S^2V$:* The even part is $V_{\bar{0}}\otimes V_{\bar{1}}$, and its odd part is $S^2V_{\bar{0}}\oplus\Lambda^2V_{\bar{1}}$. Therefore the dimension of the odd part is $\frac{n^2+m^2+m-n}{2}$. This is less than or equal to

$nm$ if and only if $0 \leq n - m \leq 1$. This leaves the cases $V = GL_{n|n}$ and $V = GL_{n|n+1}$. We show these modules are indeed spherical.

For the case of $V = GL_{n|n}$, a $\mathfrak{g}_0$-spherical vector is given by $v = v_1 w_1 + \cdots + v_m w_m$. Let $\mathfrak{b}$ be a Borel defined by an $\epsilon\delta$ string. Then $\Pi S^2 V$ is spherical for a Borel $\mathfrak{b}$ if and only if $\mathfrak{b}_{\overline{1}} \cdot v = (\Pi S^2 V)_{\overline{1}}$. The Borel has odd operators exactly as described in the case for $S^2 V$. Here we see that $v_i \partial_{w_j}(v) = v_i v_j$ and $w_i \partial_{v_j}(v) = w_i w_j$. Now in order to get $v_1^2$, we must have $\epsilon_1 - \delta_1$ is positive, so the $\epsilon\delta$ string must start with $\epsilon$. In order to get $w_1 w_2$, we must have $\delta_1 - \epsilon_2$ is positive, so the $\epsilon\delta$ string must start with $\epsilon\delta$. Similarly, to get $v_2^2$ we must have $\epsilon_2 - \delta_2$ positive, and so on, so that continuing this way we find that the Borel must have $\epsilon\delta$ string $\epsilon\delta\epsilon\delta \cdots \epsilon\delta$. Further, the representation is spherical with respect to this Borel.

For $V = GL_{n|n+1}$, a $\mathfrak{g}_0$-spherical vector is given by $v = v_1 w_2 + v_2 w_3 + \cdots + v_m w_{m+1}$. Let $\mathfrak{b}$ be a Borel defined by an $\epsilon\delta$ string. We see that $v_i \partial_{w_j}(v) = v_i v_{j-1}$, and $w_i \partial_{v_j}(v) = w_i w_{j+1}$. Following the same idea as in the previous case, in order to get $w_1 w_2$ we must have $\delta_1 - \epsilon_1$ positive, and to get $v_1^2$ we must have $\epsilon_1 - \delta_2$ positive, and continuing on like this we find that our Borel must have $\epsilon\delta$ string $\delta\epsilon\delta\epsilon \cdots \epsilon\delta$. Further, this representation is spherical with respect to this Borel.

□

*Remark* A.3.3. The representation $S^2 GL_{m|2n}$ and its relation to the Capelli problem for the symmetric pair $(\mathfrak{gl}(m|2n), \mathfrak{osp}(m|2n))$ is studied in [44].

### A.3.4 $\mathfrak{gl}(1|n)$

**Candidate even weights:**

$$t\epsilon_1 \ (t \neq 0), \quad -\epsilon_1 + \delta_1, \quad -\delta_n$$

$$-2\epsilon_1 + 2\delta_1, \quad -2\delta_n, \quad -\epsilon_1 + \delta_1 + \delta_2, \quad -\delta_{n-1} - \delta_n$$

where $t \in k$.

**Check for (numerical) sphericity of $L(\lambda)$:** In the following table we go through each possible case, grouping them appropriately according to how we either prove they do not give a (numerically) spherical representation or explain why they are covered by proposition A.3.2. The technique is the exact same as what was used when studying $\mathfrak{gl}(1|2)$.

| $\lambda$ | Action | Conclusion |
|---|---|---|
| $t\epsilon_1$, $t \notin \mathbb{Z}_{>0}$ | Weight is typical; apply three odd reflections, get two odd highest weights | Odd part too large |
| $t\epsilon_1$, $t \in \mathbb{Z}_{>0}$<br>$-\epsilon_1 + \delta_1$<br>$-\delta_n$<br>$-2\delta_n$<br>$-\epsilon_1 + \delta_1 + \delta_2$<br>$-\delta_{n-1} - \delta_n$ | After adding multiple of Berezinian get something of the form $(\Pi)S^k GL_{1\|n}^{(*)}$ or $(\Pi)\Lambda^k GL_{1\|n}^{(*)}$ | Falls under cases considered by proposition A.3.2 |
| $-2\epsilon_1 + 2\delta_1$ | Apply $r_{\epsilon_1-\delta_2} \circ r_{\epsilon_1-\delta_1}$ get odd highest weight $-3\epsilon_1 + 2\delta_1 + \delta_2$ | Odd part too large |

**Candidate odd weights:**

$$t\epsilon_1 \ (t \neq 0), \quad t\epsilon_1 + \delta_1, \quad -\delta_n$$

**Check for (numerical) sphericity of $\Pi L(\lambda)$:**

| $\lambda$ | Action | Conclusion |
|---|---|---|
| $t\epsilon_1$, $t \in \mathbb{Z}_{>0}$<br>$t\epsilon_1 + \delta_1$, $t = \pm 1$<br>$-\delta_n$ | After adding multiple of Berezinian get something of the form $(\Pi)S^k GL_{1\|n}^{(*)}$ or $(\Pi)\Lambda^k GL_{1\|n}^{(*)}$ | Falls under cases considered by proposition A.3.2 |
| $t\epsilon_1$, $t \notin \mathbb{Z}_{>0}$<br>$t\epsilon_1 + \delta_1$, $t \neq \pm 1$ | Apply $r_{\epsilon_1-\delta_2} \circ r_{\epsilon_1-\delta_1}$; get new odd highest weight | Odd part too large |

**(Numerically) Spherical irreducibles for $\mathfrak{gl}(1|n)$, $n \geq 3$:** Along with $(\Pi)k$, we have the following numerically spherical irreducibles:

$$(\Pi)GL_{1|n}, \quad S^2 GL_{1|n}, \quad \Lambda^2 GL_{1|n}$$

**Indecomposable Spherical Modules:** The only extensions which could be spherical in this case are extensions of the trivial even module by a numerically spherical module, or an extension of $\Pi GL_{1|n}$ by a numerically spherical module which is geometrically equivalent to either $\Pi k$ or $\Pi GL_{1|n}$.

In this case, the trivial even module has non-trivial extensions only with $\Pi^n(S^{n-1}V^*)_{-\,\mathrm{Ber}}$ and $\Pi^n(S^{n-1}V)_{\mathrm{Ber}}$. These modules are never numerically spherical for $n \geq 3$.

The module $\Pi GL_{1|n}$ has non-trivial extensions only with $\Pi^{n-1}(S^{n-2}V^*)_{-\,\mathrm{Ber}}$ and $L(n\epsilon_1 - \delta_2 - \cdots - \delta_n)$ (up to a parity shift). The latter module is not numerically spherical. Although the former is numerically spherical, its extensions with $\Pi GL_{1|n}$ will never be numerically spherical. It follows that we get no new spherical indecomposables.

### A.3.5 The general case $\mathfrak{gl}(m|n)$, $m \geq 2$

**Candidate even weights:**

$$t\mathrm{det}_\epsilon \ (t \neq 0), \quad \epsilon_1, \quad \mathrm{det}_\epsilon - \epsilon_m, \quad 2\epsilon_1, \quad 2\mathrm{det}_\epsilon - 2\epsilon_m \ (m > 2)$$

$$\epsilon_1 + \epsilon_2 \ (m > 2), \quad \mathrm{det}_\epsilon - \epsilon_{m-1} - \epsilon_m \ (m > 2)$$

$$-\mathrm{det}_\delta + \delta_1, \quad -\delta_n, \quad -2\mathrm{det}_\delta + 2\delta_1 \ (n > 2) \quad -2\delta_n$$

$$-\mathrm{det}_\delta + \delta_1 + \delta_2 \ (n > 2), \quad -\delta_{n-1} - \delta_n \ (n > 2)$$

$$-\mathrm{det}_\epsilon + \epsilon_1 + \delta_1, \quad \epsilon_1 - \delta_n, \quad -\epsilon_m + \delta_1, \quad \mathrm{det}_\epsilon - \epsilon_m - \delta_n$$

**Check for (numerical) sphericity of $L(\lambda)$:**

| $\lambda$ | Action | Conclusion |
|---|---|---|
| $t\det_\epsilon,\ t=0,\pm1$<br>$\epsilon_1$<br>$\det_\epsilon -\epsilon_m$<br>$2\epsilon_1$<br>$\epsilon_1+\epsilon_2,\ m>2$<br>$\det_\epsilon -\epsilon_{m-1}-\epsilon_m,\ m>2,$<br>$-\det_\delta +\delta_1$<br>$-\delta_n$<br>$-2\delta_n$<br>$-\det_\delta +\delta_1+\delta_2,\ n>2$<br>$-\delta_{n-1}-\delta_n,\ n>2$ | After adding multiple of Berezinian $L(\lambda)$ becomes something of the form $(\Pi)S^kGL_{m\|n}^{(*)}$ or $(\Pi)\Lambda^kGL_{m\|n}^{(*)}$ | Falls under cases considered by proposition A.3.2 |
| $2\det_\epsilon -2\epsilon_m,\ m>2$ | Apply $r_{\epsilon_{m-1}-\delta_1}\circ r_{\epsilon_m-\delta_1}$ get odd highest weight $2\epsilon_1+\cdots+2\epsilon_{m-2}+$ $\epsilon_{m-1}+\delta_1$ | Odd part too large |
| $-2\det_\delta +2\delta_1,\ n>2$ | Apply $r_{\epsilon_m-\delta_2}\circ r_{\epsilon_m-\delta_1}$ get odd highest weight $-\epsilon_m-\delta_2-$ $2\delta_3-\cdots-2\delta_n$ | Odd part too large |

The final cases to consider for $\lambda$ are:

$$-\det_\epsilon+\epsilon_1+\delta_1,\quad \epsilon_1-\delta_n,\quad -\epsilon_m+\delta_1,\quad \det_\epsilon-\epsilon_m-\delta_n$$

- If $\lambda = -\det_\epsilon +\epsilon_1+\delta_1$, then applying $r_{\epsilon_m-\delta_2}\circ r_{\epsilon_m-\delta_1}$ we get odd highest weight $-\det_\epsilon +\epsilon_1-\epsilon_m+\delta_1+\delta_2$. If $n>2$, then this shows the odd part is too large.

  If $n=m=2$, then $\lambda=-\epsilon_2+\delta_1$. Applying $r_{\epsilon_2-\delta_2}\circ r_{\epsilon_2-\delta_1}$ gives odd highest weight $-2\epsilon_2+\delta_1+\delta_2$, generating a 3-dimensional submodule of the odd part. On the other hand, if we instead applied $r_{\epsilon_1-\delta_1}\circ r_{\epsilon_2-\delta_1}$, we'd get odd highest weight $-\epsilon_1-\epsilon_2+2\delta_1$, giving a distinct 3-dimension submodule of the odd part. Therefore, the odd part has to be at least 6-dimensional, which is too large.

- If $\lambda=\epsilon_1-\delta_n$, this is the parity shift of the adjoint module, which has too large an odd part.

- If $\lambda = -\epsilon_m + \delta_1$, then applying $r_{\epsilon_m - \delta_1}$ followed by either $r_{\epsilon_m - \delta_2}$ or $r_{\epsilon_{m-1} - \delta_1}$ gives odd highest weights $-\epsilon_{m-1} - \epsilon_m + 2\delta_1$ and $-2\epsilon_m + \delta_1 + \delta_2$. This shows the odd part will be too large.

- Finally, if $\lambda = \det_\epsilon - \epsilon_m - \delta_n$, then applying $r_{\epsilon_{m-1}-\delta_1} \circ r_{\epsilon_m - \delta_1}$ gives odd highest weight $\epsilon_1 + \cdots + \epsilon_{m-2} + \delta_1 - \delta_n$, which is of dimension $n^2 - 1 + m(m-1)/2$, which is bigger than $nm$ whenever $n > 2$, $m \geq 2$.

  Therefore we can assume $n = m = 2$, in which case $n^2 - 1 + m(m-1)/2 = 4$. Then we can also apply $r_{\epsilon_2 - \delta_2} \circ r_{\epsilon_2 - \delta_1}$ to get distinct odd highest weight $\epsilon_1 - \epsilon_2$, making the odd part too large.

**Candidate odd weights:**

$$t\det_\epsilon \ (t \neq 0), \quad \epsilon_1, \quad \epsilon_1 + \epsilon_2 \ (m \geq 3), \quad 2\epsilon_1$$

$$-\delta_n, \quad -\delta_{n-1} - \delta_n \ (n \geq 3), \quad -2\delta_n$$

$$\epsilon_1 + \delta_1 - \det_\epsilon, \quad \epsilon_1 - \delta_n, \quad -\epsilon_m + \delta_1, \quad -\epsilon_m - \delta_n + \det_\epsilon$$

**Check for (numerical) sphericity of $\Pi L(\lambda)$:**

| $\lambda$ | Action | Conclusion |
|---|---|---|
| $t\det_\epsilon$, $t = 0, \pm 1$<br>$\epsilon_1 + t\det_\epsilon$, $t = 0$<br>$2\epsilon_1 + t\det_\epsilon$, $t = 0$<br>$\epsilon_1 + \epsilon_2 + t\det_\epsilon$, $m > 2$, $t = 0$<br>$-\delta_n + t\det_\delta$, $t = 0$<br>$-2\delta_n + t\det_\delta$, $t = 0$<br>$-\delta_{n-1} - \delta_n + t\det_\delta$, $n > 2$ $t = 0$ | After adding multiple of Berezinian $\Pi L(\lambda)$ becomes something of the form $(\Pi)S^n GL_{m\|n}^{(*)}$ or $(\Pi)\Lambda^n GL_{m\|n}^{(*)}$ | Falls under cases considered by proposition A.3.2 |

This leaves us to consider the cases for $\lambda$:

$$t\det_\epsilon \ t \neq 0, \pm 1, \quad -\det_\epsilon + \epsilon_1 + \delta_1, \quad \epsilon_1 - \delta_n, \quad -\epsilon_m + \delta_1, \quad -\det_\epsilon - \epsilon_m - \delta_n$$

- If $\lambda = t\det_\epsilon$ $t \neq 0, \pm 1$, then applying $r_{\epsilon_m - \delta_1}$ we get even highest weight $t\det_\epsilon - \epsilon_m + \delta_1$. Next we may apply either $r_{\epsilon_{m-1}-\delta_1}$ or $r_{\epsilon_m - \delta_2}$, giving odd highest weights $t\det_\epsilon - \epsilon_{m-1} - \epsilon_m + 2\delta_1$, or $t\det_\epsilon - 2\epsilon_m + \delta_1 + \delta_2$. It follows the odd part will be too large.

- If $\lambda = -\det_\epsilon + \epsilon_1 + \delta_1$ and $m = n = 2$, then $\lambda$ differs by a multiple of the Berezinian with $\epsilon_1 - \delta_2$, giving the adjoint representation which is not numerically spherical. If $n > 2$, then applying $r_{\epsilon_m - \delta_2} \circ r_{\epsilon_m - \delta_1}$ gives an even weight which is not $\mathfrak{g}_{\overline{0}}$-spherical.

- If $\lambda = \epsilon_1 - \delta_n$ we get the adjoint module, which is not numerically spherical.
- If $\lambda = -\epsilon_m + \delta_1$, then if $m = n = 2$ we again get a twist of the adjoint module, while if $n > 2$, applying $r_{\epsilon_m-\delta_2} \circ r_{\epsilon_m-\delta_1}$ gives an even weight which is not $\mathfrak{g}_{\overline{0}}$-spherical.
- Finally, if $\lambda = \det_\epsilon -\epsilon_m - \delta_n$ then when $n = m = 2$ we get the adjoint module, while if $n > 2$ then applying $r_{\epsilon_{m-1}-\delta_1} \circ r_{\epsilon_m-\delta_1}$ we get an even weight which is not $\mathfrak{g}_{\overline{0}}$-spherical.

**Spherical irreducibles for $\mathfrak{gl}(m|n)$, $2 \leq m \leq n$:** Apart from $(\Pi)k$, numerically spherical irreducibles are all spherical in this case.

$$GL_{m|n}, \quad \Pi GL_{m|n}, \quad S^2GL_{m|n}, \quad \Lambda^2GL_{m|n}, \quad \Pi S^2GL_{n|n}, \quad \Pi S^2GL_{n|n+1}$$

We note however that if we remove the condition that $m \leq n$, then $\Pi GL_{m|n}$ is equivalent to $GL_{n|m}$ and $\Lambda^2GL_{m|n}$ is equivalent to $S^2GL_{n|m}$.

**Spherical indecomposables for $\mathfrak{gl}(m|n)$, $2 \leq m \leq n$:**

In this case, $S^2GL_{m|n}$, $\Lambda^2GL_{m|n}, \Pi S^2GL_{m|m}$, and $\Pi S^2GL_{m|m+1}$ all have odd dimension equal to the odd dimension of a Borel. Hence the only possible spherical extensions these modules could have are by a one-dimensional even module. However there are no such extensions. In fact, in this case the trivial module has no non-trivial extensions by any (numerically) spherical irreducibles, and nor does its parity shift.

This leaves us to look at the standard module. However neither it nor its parity shift admits non-trivial extensions by numerically spherical modules. It follows that all spherical indecomposable $\mathfrak{gl}(m|n)$-modules for $2 \leq m \leq n$ are all irreducible.

## A.4 $\mathfrak{osp}(m|2n)$ case

We now study the case $\mathfrak{g} = \mathfrak{osp}(m|2n)$, with $m, n > 0$, $(m, n) \neq (2, 1)$ (since $\mathfrak{osp}(2|2) \cong \mathfrak{sl}(1|2)$). For more about $\mathfrak{osp}(m|2n)$ and a description of $\mathfrak{b}^{st}$-dominant weights, see [10] and [39]. We refer to section 2.3.2 and section 2.3.4 for a discussion of our notation for Borels and root systems.

### A.4.1 Modules from the standard representation

Write $OSP_{m|2n}$ for the standard representation of $\mathfrak{osp}(m|2n)$ on $k^{m|2n}$.

**Proposition A.4.1.** *1. $OSP_{m|2n}$ is spherical if and only if $m \geq 2$. In this case it is spherical exactly with respect to Borels with $\epsilon\delta$ strings of the form $\epsilon\cdots$, or $(\pm\epsilon)\delta^n$ when $m = 2$.*

*2. $\Pi OSP_{m|2n}$ is always spherical exactly with respect to Borels with $\epsilon\delta$ strings of the form $\delta\cdots$.*

*Proof.* For the first statement, $OSP_{1|2n}$ is not spherical because the odd dimension of any Borel of $\mathfrak{osp}(1|2n)$ is $n$, which is too small. The rest is straightforward. □

### A.4.2 $\mathfrak{osp}(2|2n)$, $n \geq 2$

**Standard Borel:** $\mathfrak{b}^{st} = \mathfrak{b}^{\epsilon\delta^n}$. The odd dimension of every Borel is $2n$.

**Spherical weights for $\mathfrak{g}_{\bar{0}} = \mathfrak{o}(2) \times \mathfrak{sp}(2n)$:**

$$\delta_1 + s\epsilon_1, \quad s\epsilon_1 \ (\nu \neq 0), \quad \delta_1 + \delta_2 + s\epsilon_1 \ (n = 2) \tag{A.4.1}$$

**Candidate even weights**

$$\delta_1 - \epsilon_1, \quad s\epsilon_1 \ (s \neq 0), \quad \delta_1 + \delta_2 - \epsilon_1 (n = 2)$$

**Candidate odd weights**

$$s\epsilon_1 \ (s \neq 0), \quad -\epsilon_1 + \delta_1$$

**Check for (numerical) sphericity of $(\Pi)L(\lambda)$:**

| $\lambda$ | Parity | Action | Conclusion |
|---|---|---|---|
| $\delta_1 - \epsilon_1$<br>$s\epsilon_1$, $n > 2$, $s \neq 0, 1$<br>$s\epsilon_1$, $s \neq 0, 1$ | even<br>even<br>odd | Apply $r_{\epsilon_1-\delta_2} \circ r_{\epsilon_1-\delta_1}$; get new even or odd highest weight | Odd part too large or even part not spherical |
| $s\epsilon_1$, $s = 1$<br>$s\epsilon_1$, $s = 1$ | even<br>odd | Do nothing | Isomorphic to standard module up to parity shift. Covered by proposition A.4.1 |
| $s\epsilon_1$, $n = 2$, $s = 3$<br>$\delta_1 + \delta_2 - \epsilon_1$, $n = 2$ | even<br>even | Do nothing | Isomorphic to $L(\delta_1 + \delta_2 - \epsilon)$ or its dual- a spherical module |
| $s\epsilon_1$, $n = 2$, $s \neq 1, 3$<br>$\delta_1 - \epsilon_1$, $n = 2$ | even<br>odd | Apply the odd reflections $r_{\epsilon_1-\delta_1}$, $r_{\epsilon_1-\delta_2}$, $r_{\epsilon_1+\delta_2}$ Gives two odd highest weights | Odd part too large |
| $\delta_1 - \epsilon_1$, $n > 2$ | odd | Apply $r_{\epsilon_1-\delta_3} \circ r_{\epsilon_1-\delta_2} \circ r_{\epsilon_1-\delta_1}$ get even highest weight vector | Even part not spherical |

**(Numerically) Spherical irreducibles for $\mathfrak{osp}(2|2n)$, $n \geq 2$:** Apart from $\Pi k$, all numerically spherical irreducibles are spherical.

$$OSP_{2|2n}, \quad \Pi OSP_{2|2n}, \quad L_{\mathfrak{b}^{st}}(\delta_1 + \delta_2 - \epsilon_1)$$

### A.4.3 $\mathfrak{osp}(2m|2n)$, $m \geq 2$

**Standard Borel** $\mathfrak{b}^{st} = \mathfrak{b}^{\delta^n \epsilon^m}$.

**Spherical weights for** $\mathfrak{g}_{\overline{0}} = \mathfrak{o}(2m) \times \mathfrak{sp}(2n)$**:**

| $(m, n)$ | Weights |
|---|---|
| Any $m, n$ | $\delta_1, \epsilon_1$ |
| Any $n$; $m = 2, 3, 5$ | $\frac{1}{2}(\epsilon_1 + \cdots \pm \epsilon_m)$ |
| Any $n$; $m = 2, 3$ | $\epsilon_1 + \cdots \pm \epsilon_m$ |
| Any $n$ and $m = 2$, or $n \leq 2$, $m = 3$ | $\frac{1}{2}(\epsilon_1 + \cdots \pm \epsilon_m) + \delta_1$ |
| Any $m$; $n = 2$ | $\delta_1 + \delta_2$ |
| Any $m$; $n = 1$ | $2\delta_1$ |

**Candidate even weights:** Only $\lambda = \delta_1$. Although $\delta_1 + \delta_2$ when $n = 2$ and $2\delta_1$ when $n = 1$ are dominant, applying a simple odd reflection gives an odd highest weight which is too large.

**Candidate odd weights:** Only $\lambda = \delta_1$.

**Check for (numerical) sphericity of** $(\Pi)L(\lambda)$**:**

| Weight | Parity | Action | Conclusion |
|---|---|---|---|
| $\delta_1$ | even | Do nothing | Parity shift of standard module; covered by proposition A.4.1 |
| $\delta_1$ | odd | | |

**(Numerically) Spherical irreducibles for** $\mathfrak{osp}(2m|2n)$, $m \geq 2$**:** Along with $(\Pi)k$, we have two spherical irreducibles:

$$OSP_{2m|2n}, \quad \Pi OSP_{2m|2n}$$

### A.4.4 $\mathfrak{osp}(2m+1|2n)$

**Standard Borel:** $\mathfrak{b}^{st} = \mathfrak{b}^{\delta^n \epsilon^m}$.

**Spherical weights for $\mathfrak{g}_{\bar{0}} = \mathfrak{o}(2m+1) \times \mathfrak{sp}(2n)$:**

| $(m, n)$ | Weights |
|---|---|
| Any $m, n$ | $\delta_1, \epsilon_1$ |
| Any $n$; $m \leq 4$ | $\frac{1}{2}(\epsilon_1 + \cdots + \epsilon_m)$ |
| $(1,1)$, $(1,2)$, or $(2,1)$ | $\frac{1}{2}(\epsilon_1 + \cdots + \epsilon_m) + \delta_1$ |
| Any $m$; $n = 2$ | $\delta_1 + \delta_2$ |
| Any $m$; $n = 1$ | $2\delta_1$ |

**Candidate even weights:** Only $\delta_1$. Again, the other $\mathfrak{b}^{st}$-dominant weights appearing the above table have too large an odd part as seen by applying the odd simple reflection $r_{\delta_n - \epsilon_1}$.

**Candidate odd weights:** Only $\lambda = \delta_1$.

**Check for sphericity of $(\Pi)L(\lambda)$:**

| Weight | Parity | Action | Conclusion |
|---|---|---|---|
| $\delta_1$ | even | Do nothing | Parity shift of standard module; covered by proposition A.4.1 |
| $\delta_1$ | odd | | |

**Spherical indecomposables for $\mathfrak{osp}(m|2n)$, $(m, n) \neq (2, 1)$:** The trivial module has a distinct central character from $OSP_{m|2n}$ for all $(m, n)$, and therefore there are no new spherical indecomposables for $\mathfrak{osp}(m|2n)$ when $(m, n) \neq (2, 2)$.

When $m = n = 2$, the central character of $L_{\mathfrak{b}^{st}}(\delta_1 + \delta_2 - \epsilon_1)$ is the same as that of $OSP_{2|4}$, and therefore it has no extensions with a trivial module. Any extension of $L_{\mathfrak{b}^{st}}(\delta_1 + \delta_2 - \epsilon_1)$ with $OSP_{2|4}$ will have too large an odd part, and therefore cannot be spherical. So $\mathfrak{osp}(2|2n)$ also has no new spherical indecomposables.

Below is a table of all spherical irreducibles for $\mathfrak{osp}(m|2n)$, as well as Borels for which sphericity is achieved, and stabilizers of spherical vectors:

## A.5 Exceptional basic simple algebras

Here we consider when $\mathfrak{g} = \mathfrak{g}(1|2)$, $\mathfrak{ab}(1|3)$, and $\mathfrak{d}(2|1; t)$. We show that none of these algebras have nontrivial spherical modules (unless $\mathfrak{d}(2|1; t) \cong \mathfrak{osp}(2|4)$). We refer to L. Martirosyan's thesis [36] for more on $\mathfrak{g}(1|2)$ and $\mathfrak{ab}(1|3)$. For the description of the root system of $\mathfrak{d}(2|1; t)$ used below, see [48], and for the parametrization of dominant weights we refer to example 10.7 of [47].

| Rep | $\dim^s$ | Borels | Stabilizer |
|---|---|---|---|
| $OSP_{m\|2n}$, $m \neq 1$ | $(m\|2n)$ | $\epsilon \cdots$ | $\mathfrak{osp}(m-1\|2n)$ |
| $\Pi OSP_{m\|2n}$ | $(2n\|m)$ | $\delta \cdots$ | $\mathfrak{osp}(m-1\|2n-2) \ltimes (k \oplus k^{m-1\|2n-2})$ |
| $L(\delta_1+\delta_2-\epsilon_1)$, $m=n=2$ | $(6\|4)$ | $(\pm\epsilon)\delta\delta, \delta\delta\epsilon$ | $\mathfrak{osp}(1\|2) \times \mathfrak{osp}(1\|2)$ |

Table A.3: Spherical representations of $\mathfrak{osp}(m|2n)$

### A.5.1 $\mathfrak{g}(1|2)$ case

See section 2.3.2 for a description of the root system. The following table goes through our usual to-do list, and shows there are no non-trivial numerically spherical irreducibles.

| | |
|---|---|
| $\mathfrak{b}^{st}$ | Borel with simple roots $\delta+\epsilon_3, \epsilon_1, \epsilon_2-\epsilon_1$. |
| $\mathfrak{g}_{\bar{0}}$-spherical weights | $\delta$, $2\delta$, $\epsilon_1+\epsilon_2$ |
| Candidate even weights | None; $2\delta$ is dominant, but applying $r_{\delta+\epsilon_3}$ gives too large an odd part |
| Candidate odd weights | None |

### A.5.2 $\mathfrak{ab}(1|3)$ case

**Root system:** Again see section 2.3.2 for the root system. Once more the following table shows there are no non-trivial numerically spherical irreducibles.

| | |
|---|---|
| $\mathfrak{b}^{st}$ | Borel with simple roots $\frac{1}{2}(-\epsilon_1-\epsilon_2-\epsilon_3+\delta), \epsilon_3, \epsilon_2-\epsilon_3, \epsilon_1-\epsilon_2$ |
| $\mathfrak{g}_{\bar{0}}$-spherical weights | $\frac{1}{2}\delta$, $\delta$, $\epsilon_1$, $\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3)$ |
| Candidate even weights | None; $\delta$ is dominant, but applying $r_{\frac{1}{2}(-\epsilon_1-\epsilon_2-\epsilon_3+\delta)}$ gives too large an odd part |
| Candidate odd weights | None |

### A.5.3 $\mathfrak{d}(2|1;t)$ case

The root system is described in section 2.3.2.

**Standard Borel:** $\mathfrak{b}^{st}$ is a Borel with three simple isotropic roots $\alpha_1, \alpha_2, \alpha_3$ (such a Borel is unique up to conjugacy). Then the principal roots are $\beta_i = \alpha_j + \alpha_k$, where $\{i, j, k\} = \{1, 2, 3\}$, i.e. they are even roots which are simple after application of some number of odd reflections to the simple root system $\mathfrak{b}^{st}$.

**$\mathfrak{b}^{st}$-dominant weights:** Since the principal roots are a basis of the $\mathfrak{h}_{\bar{0}}^*$, we may parametrize a weight $\lambda$ by $(c_1, c_2, c_3)$, where $c_i = \lambda(h_{\beta_i})$. The conditions that $\lambda$ is dominant integral with respect to this Borel is that $c_1, c_2, c_3 \in \mathbb{Z}_{\geq 0}$, and one of the following holds:

1. $c_1, c_2, c_3 \in \mathbb{Z}_{>0}$
2. $c_1 = (t + 1)c_2 + c_3 = 0$
3. $c_2 = -tc_1 + c_3 = 0$
4. $c_3 = -tc_1 + (t + 1)c_2 = 0$

**Spherical weights for $\mathfrak{g}_{\bar{0}}$:**

$$(x, 0, 0), \quad (0, x, 0), \quad (0, 0, x) \quad \text{where } x = 1, 2$$

and

$$(a, b, c) \quad \text{where two out of } a, b, c \text{ are 1, and the other is 0.}$$

**Candidate even weights:**

$$(0, 1, 1) \text{ with } t = -2, \quad (1, 0, 1) \text{ with } t = 1$$

**Check sphericity of $L(\lambda)$:** We have $\mathfrak{d}(2|1; 1) \cong \mathfrak{d}(2|1; -2) \cong \mathfrak{osp}(2|4)$, so these fall under appendix A.4.

**Candidate odd weights:**

$$(0, 1, 1) \text{ with } t = -2, \quad (1, 0, 1) \text{ with } t = 1$$

**Check sphericity of $\Pi L(\lambda)$:** The cases are again covered by appendix A.4.

**(Numerically) Spherical irreducibles for $\mathfrak{d}(2|1; \tilde{)}$:** None, unless $\mathfrak{d}(2|1; t) \cong \mathfrak{osp}(2|4)$. Therefore there are no new spherical indecomposable modules.

## A.6 The case $\mathfrak{p}(n)$

Let $\mathfrak{g} = \mathfrak{p}(n)$, $n \geq 2$. We refer the reader to [5], in particular for computations of dual representations, as well as [9] and [49] for more on the representation theory of this algebra.

**Notation:** A matrix presentation for $\mathfrak{p}(n)$, under the representation of the standard module, is

$$\begin{bmatrix} A & B\, k & -A^t \end{bmatrix} \tag{A.6.1}$$

where $B^t = B$, $C^t = -C$.

We have a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$, $\mathfrak{g}_1$ are matrices with $A = C = 0$ and $\mathfrak{g}_{-1}$ are matrices with $A = B = 0$.

Write $P_{n|n}$ for the standard module of $\mathfrak{p}(n)$, and $q \in (S^2 P_{n|n}^*)_{\bar{1}}$ for a non-degenerate odd form on $P_{n|n}$ preserved by $\mathfrak{p}(n)$. Then $q$ induces an isomorphism $P_{n|n}^* \cong \Pi P_{n|n}$.

**Root system:** Write $\mathfrak{h}$ for the (even) Cartan subalgebra of diagonal matrices. A description of the root system is given in section 2.3.2.

**Standard Borel:** $\mathfrak{b}^{st} = \mathfrak{b}_{\bar{0}}^{st} \oplus \mathfrak{g}_{-1}$, where $\mathfrak{b}_{\bar{0}}^{st}$ is the Borel of $\mathfrak{g}_{\bar{0}}$ with simple roots $\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n$.

**Characters for $\mathfrak{g}$:** Recall $[\mathfrak{p}(n), \mathfrak{p}(n)]$ is a codimension-one ideal of $\mathfrak{p}(n)$ with one-dimensional even quotient. The even irreducible representations of this quotient are indexed by the complex numbers, and pullback to multiples of the representation of highest weight $\omega = \epsilon_1 + \cdots + \epsilon_n$.

**Proposition A.6.1.** *The standard module $P_{n|n}$ is never spherical. The parity shift $\Pi P_{n|n}$ is spherical exactly with respect to (up to conjugacy) Borels $\mathfrak{b}$ with $\mathfrak{b}_{\bar{0}} = \mathfrak{b}_{\bar{0}}^{st}$, and $\epsilon_1 + \epsilon_i$ positive for all $i$.*

*Proof.* This is seen from the matrix presentation in (A.6.1). □

We may now put aside the case when $n = 1$:

**Proposition A.6.2.** *Up to equivalence, the only non-trivial indecomposable spherical module for $\mathfrak{p}(1)$ is $\Pi P_{1|1}$. The stabilizer of a spherical vector is trivial.*

*Proof.* The proof is straightforward and thus omitted. □

We now assume $n \geq 2$.

**Proposition A.6.3.** *The module $S^2 P_{n|n}$ is indecomposable. If $n > 2$, it has simple socle $L_{\mathfrak{b}^{st}}(-\epsilon_{n-1} - \epsilon_n)$ with a one-dimensional odd quotient. The form $q$ induces an isomorphism $(S^2 P_{n|n})^* \cong \Lambda^2 V$, and we have $L(-\epsilon_{n-1} - \epsilon_n)^* \cong L(-2\epsilon_n)$.*

*In particular, if $n > 2$, $\dim^s L(-\epsilon_{n-1} - \epsilon_n) = \dim^s L(-2\epsilon_n) = (n^2 | n^2 - 1)$, and neither module nor its parity shift is spherical.*

*Proof.* For the statement about $S^2 P_{n|n}$, see Lemma 2.1.2 of [5] or the end of section 5.6.

We have $S^2 P_{n|n}^* \cong S^2 \Pi P_{n|n} \cong \Lambda^2 P_{n|n}$. We compute directly that the highest weight of $\Lambda^2 P_{n|n}$ is $-2\epsilon_n$.

Finally, the statement about dimensions is clear. Because the maximum of $\dim \mathfrak{b}_{\bar{1}}$ over all Borel subalgebras is $n(n+1)/2$, and we have $(n(n+1)/2) < n^2 - 1$ for $n > 2$, the odd part of $(\Pi)L(-\epsilon_{n-1} - \epsilon_n)$ or $(\Pi)L(-2\epsilon_n)$ will always be too large to be spherical. □

Now we assume $n \geq 3$.

**Even spherical dominant weights for $\mathfrak{g}_{\bar{0}} \cong \mathfrak{gl}(n)$:**

$$\epsilon_1, \quad 2\epsilon_1, \quad -\epsilon_n, \quad -2\epsilon_n, \quad \epsilon_1+\epsilon_2 \quad -\epsilon_{n-1}-\epsilon_n$$

**Candidate even weights:** Same as above.
**Check for (numerical) sphericity of $L(\lambda)$:**

- The cases of $\lambda = -\epsilon_n, -2\epsilon_n, -\epsilon_{n-1}-\epsilon_n$ were dealt with in proposition A.6.1 and proposition A.6.3.
- If $\lambda = \epsilon_1$, we compute $L(\lambda)^* \cong L(-n\epsilon_n - \omega)$. Hence for $n \geq 4$ the even part cannot be spherical, while when $n = 3$ the dimension is the same as that of $L(-2\epsilon_n)$, and it cannot be spherical.
- If $\lambda = 2\epsilon_1$, then $L(\lambda)^* \cong L(-(n+1)\epsilon_n - \omega)$, so the even part is never spherical for $n \geq 3$.
- If $\lambda = \epsilon_1 + \epsilon_2$, then $L(\lambda)^* = L((1-n)\epsilon_n - \omega)$. Hence for $n \geq 5$ the even part is not spherical, while for $n = 4$ the dimension is the same as that of $L(-2\epsilon_1)$, so it cannot be spherical. If $n = 3$, $\epsilon_1 + \epsilon_2 - \omega = -\epsilon_3$, so this falls under proposition A.6.1.

**Candidate odd highest weights:**

$$-\epsilon_1, \quad -2\epsilon_n, \quad -\epsilon_{n-1}-\epsilon_n, \quad -2\epsilon_{n-1}-2\epsilon_n (n=3)$$

$$-\epsilon_{n-2}-\epsilon_{n-1}-\epsilon_n \ (n \geq 4) \quad -\epsilon_{n-3}-\epsilon_{n-2}-\epsilon_{n-1}-\epsilon_n \ (n \geq 5)$$

**Check for (numerical) sphericity of $\Pi L(\lambda)$:**

- The cases of $\lambda = -\epsilon_n, -2\epsilon, -\epsilon_{n-1}-\epsilon_n$ where dealt with in proposition A.6.1 and proposition A.6.3.
- If $\lambda = -2\epsilon_{n-1} - 2\epsilon_n$ with $n = 3$, this is equivalent to $\Pi L(2\epsilon_1)$, which was already dealt with.
- If $\lambda = -\epsilon_{n-2}-\epsilon_{n-1}-\epsilon_n$ with $n \geq 4$. This is equivalent to the module $\Pi L(\epsilon_1+\cdots+\epsilon_{n-3})$, whose dual is $\Pi L(-4\epsilon_n - \omega)$. The odd part is then too large to be spherical.
- If $\lambda = -\epsilon_{n-3}-\epsilon_{n-2}-\epsilon_{n-1}-\epsilon_n$ with $n \geq 5$, this is equivalent to the module $\Pi L(\epsilon_1 + \cdots + \epsilon_{n-4})$, whose dual is $\Pi L(-5\epsilon_n - \omega)$. Again the odd part is too large.

### A.6.1 $\mathfrak{p}(2)$

**Candidate even weights:**

$$-\epsilon_2, \quad \epsilon_1 - \epsilon_2$$

**Check for (numerical) sphericity of $L(\lambda)$:**

- The cases $L(-\epsilon_2), \Pi L(-\epsilon_2)$ were covered in proposition A.6.1.
- The module $L(\epsilon_1 - \epsilon_2)$ is the socle of the adjoint representation, represented explicitly as the matrices
$$\begin{bmatrix} A & B \\ 0 & -A^t \end{bmatrix}$$
where $\mathrm{tr}(A) = 0$ and $B^t = B$. An explicit computation shows that neither this nor its parity shift is spherical. Hence neither $L(\epsilon_1 - \epsilon_2)$ nor $\Pi L(\epsilon_1 - \epsilon_2)$ is spherical.

**Candidate odd weights:**

$$-\epsilon_2, \quad \epsilon_1 - \epsilon_2$$

**Check for (numerical) sphericity of $\Pi L(\lambda)$:** The case $\lambda = -\epsilon_2$ is covered in proposition A.6.1, and $\lambda = \epsilon_1 - \epsilon_2$ was discussed in the even weight check above.

**(Numerically) spherical irreducibles for $\mathfrak{p}(n)$, $n \geq 2$:** The only non-trivial spherical irreducible is $\Pi P_{n|n}$. The stabilizer is $\mathfrak{k} = \mathfrak{p}(n-1) \ltimes \Pi L(-\epsilon_{n-1})$. The numerically spherical irreducibles also include $\Pi k$ and $P_{n|n}$.

**Spherical Indecomposables:** For $\mathfrak{p}(2)$, up to equivalence there is one non-trivial extension of one-dimensional modules, which is equivalent to the $\mathfrak{p}(1)$-module $P_{1|1}$. There are no extensions of $(\Pi)P_{2|2}$ by a one-dimensional module, and any extensions of $(\Pi)P_{2|2}$ by a twist of $(\Pi)P_{2|2}$ have too large an odd part to be spherical. This deals with the case $n = 2$.

There are no non-trivial extensions of 1-dimensional modules, or of two modules equivalent to $(\Pi)P_{n|n}$ for $\mathfrak{p}(n)$ when $n \geq 3$. Therefore we need to determine when there are extensions between $(\Pi)P_{n|n}$ and a one-dimensional module.

The weights of $(\Pi)P_{n|n}$ are $\pm\epsilon_i$, while the weights of any one-dimensional module are multiples of $\omega$. Because our odd roots are all of the form $\pm(\epsilon_i + \epsilon_j)$, the only time such an extension could exist is when $n = 3$. Further, the extension would need to appear in either the thin Kac module $\nabla(\omega)$ or the thick Kac module $\Delta(-\omega)$ by our weight restrictions (see [5] for more on these modules).

We see that neither $P_{3|3}$ nor its parity shift appear in the thick Kac module

$$\Delta(-\omega) = \mathcal{U}\mathfrak{p}(3) \otimes_{\mathcal{U}(\mathfrak{p}(3)_{-1} \oplus \mathfrak{p}(3)_0)} k_{-\omega}$$

On the other hand, the thin Kac module on $k_\omega$, i.e.

$$\nabla(\omega) = \mathcal{U}\mathfrak{p}(3) \otimes_{\mathcal{U}(\mathfrak{p}(3)_0 \oplus \mathfrak{p}(3)_1)} k_\omega$$

provides us with a module with the following socle filtration:

$$\nabla(\omega) = \begin{array}{|c|}\hline k_\omega \\ \hline \Pi P_{3|3} \\ \hline \Pi k_{-\omega} \\ \hline \end{array}.$$

Since the space of extensions between any two simple modules for $\mathfrak{p}(3)$ is always at most one-dimensional, the only extensions of $(\Pi)P_{3|3}$ by a one-dimensional module will appear in a subquotient of the above module or its parity shift. Of those, the ones which are non-irreducible and spherical are $\nabla(\omega)$, $\operatorname{rad}\nabla(\omega)$, and $\nabla(\omega)/\operatorname{soc}\nabla\omega$. We note that $\nabla(\omega)$ is in fact a restriction of $P_{4|4}$. Information about Borels for which sphericity is achieved and stabilizers of spherical vectors is not very revealing, and thus is omitted.

## A.7 $\mathfrak{q}(n)$ case

Let $\mathfrak{g} = \mathfrak{q}(n)$. For a more in-depth treatment of this algebra we refer the reader to [10].

**Notation:** We present $\mathfrak{g}$ as the subalgebra of $\mathfrak{gl}(n|n)$ consisting of matrices

$$\mathfrak{g} = \left\{ \begin{bmatrix} A & B \\ B & A \end{bmatrix} : A, B \in \mathfrak{gl}(n) \right\}$$

Let

$$\mathfrak{h} = \begin{bmatrix} D & D' \\ D' & D \end{bmatrix}$$

where $D, D'$ are arbitrary diagonal matrices.

We write $Q_{n|n}$ for the standard module of $\mathfrak{q}(n)$.

**Root system:** See section 2.3.2.

**Standard Borel:** Choose for positive system $\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n$. Write $\mathfrak{b}^{st}$ for the corresponding Borel subalgebra. Note that for $\mathfrak{q}(n)$ all Borel subalgebras are conjugate. Notice also that $\dim \mathfrak{b}_{\overline{1}} = n(n+1)/2 < n^2$ for $n \geq 2$.

### A.7.1 Spherical weights for $\mathfrak{g}_{\overline{0}}$:

$$\epsilon_1, \quad 2\epsilon_1, \quad -\epsilon_n, \quad -2\epsilon_n$$

$$\epsilon_1 + \epsilon_2, \quad -\epsilon_{n-1} - \epsilon_n$$

### A.7.2 $\mathfrak{q}(1)$

**Proposition A.7.1.** *For $\mathfrak{q}(1)$, the non-trivial spherical indecomposable modules are*

$$\operatorname{Ind}_{\mathfrak{q}(1)_{\overline{0}}}^{\mathfrak{q}(1)} k_{t\epsilon_1}, \ \textit{where } t \in k \textit{ is arbitrary.}$$

*For $t \neq 0$ these modules are all equivalent to $Q_{1|1}$, and the stabilizer of a spherical vector is trivial. When $t = 0$, $\mathrm{Ind}_{\mathfrak{q}(1)_{\bar{0}}}^{\mathfrak{q}(1)} k_0$ is equivalent to $U^{1|1}$.*

*Proof.* Omitted. □

### A.7.3 $\mathfrak{q}(n)$, $n \geq 3$

**Candidate weights:** (all weights below are length one, hence satisfy $\Pi L(\lambda) \cong L(\lambda)$)

$$\epsilon_1, \quad 2\epsilon_1, \quad -\epsilon_n, \quad -2\epsilon_n$$

**Check for (numerical) sphericity of $L(\lambda)$:** We have $L(\epsilon_1) \cong Q_{n|n}$ and $L(-\epsilon_n) \cong Q_{n|n}^*$, and each are spherical by a straightforward check- in fact, they are equivalent.. The module $L(2\epsilon_1)$ is $S^2(Q_{n|n})$, so has odd dimension $n^2$, which is too large. Similarly $L(-2\epsilon_n)$ is $S^2(Q_{n|n}^*)$, so cannot be spherical.

**Numerical spherical irreducibles for $\mathfrak{q}(n)$, $n \geq 3$:** Up to equivalence, the only non-trivial numerically spherical irreducible is $Q_{n|n}$ and it is spherical. The stabilizer of a spherical vector is $\mathfrak{q}(n-1) \ltimes (Q_{n-1|n-1})^*$.

**Spherical indecomposables for $\mathfrak{q}(n)$, $n \geq 3$:** For $\mathfrak{q}(n)$, there are no extensions of $k$ by $k$, however the surjective algebra homomorphism $\mathfrak{q}(n) \to k^{0|1}$ gives rise to a non-trivial extension of $\Pi k$ by $k$. This module is spherical and equivalent to $U^{1|1}$.

Since $k$, $Q_{n|n}$, and $Q_{n|n}^*$ have distinct central characters, there are no extensions between them. Any extension of $Q_{n|n}$ by itself cannot be spherical because its even part will not be spherical. Hence we have found all spherical indecomposables of $\mathfrak{q}(n)$ for $n \geq 3$.

### A.7.4 $\mathfrak{q}(2)$

**Candidate weights:**

$$(t+1)\epsilon_1 + t\epsilon_2, \quad (t+2)\epsilon_1 + t\epsilon_2, \text{ where } t \in k$$

We need to consider the irreducible representations with these given highest weights as well as their parity shifts when the length of the weight is two. Note that the above weights are always typical except for $\frac{1}{2}\epsilon_1 - \frac{1}{2}\epsilon_2$ and $\epsilon_1 - \epsilon_2$.

**Check for (numerical) sphericity of $L(\lambda)$ and $\Pi L(\lambda)$:**

- If $\lambda = \epsilon_1 - \epsilon_2$ then up to parity shift, $L(\epsilon_1 - \epsilon_2) \cong [\mathfrak{q}(2), \mathfrak{q}(2)]/kI_{2|2}$, which is $(3|3)$ dimensional. By a direct computation neither this nor its parity shift is spherical.
- Suppose $\lambda = \frac{1}{2}\epsilon_1 - \frac{1}{2}\epsilon_2$. Then $L(\lambda)$ is isomorphic up to a parity shift to the representation coming from the map of algebras $\mathfrak{q}(2) \to \mathfrak{p}(2)$ which induces the following exact sequence:

  $$0 \to kI_{2|2} \to \mathfrak{q}(2) \to [\mathfrak{p}(2), \mathfrak{p}(2)] \to 0$$

  Therefore we can understand this representation, up to a parity shift, as the restriction of the action of $\mathfrak{p}(2)$ on $P_{2|2}$ to its derived subalgebra.

| Rep | $\dim^s$ | Stabilizer |
|---|---|---|
| $L((t+1)\epsilon_1 + t\epsilon_2)$, $t \neq -1/2$ | $(2\|2)$ | $\mathfrak{q}(1) \ltimes \mathrm{Ind}_{\mathfrak{q}(1)_{\bar{0}}}^{\mathfrak{q}(1)} k_{2t+1}$ |
| $\Pi L((t+1)\epsilon_1 + t\epsilon_2)$, $t \neq -1/2$ | $(2\|2)$ | $\mathfrak{q}(1) \ltimes \mathrm{Ind}_{\mathfrak{q}(1)_{\bar{0}}}^{\mathfrak{q}(1)} k_{2t+1}$ |
| $\mathrm{Res}_{[\mathfrak{p}(2),\mathfrak{p}(2)]} P_{2\|2}$ | $(2\|2)$ | $\mathfrak{q}(1) \ltimes \Pi(\mathrm{Ind}_{\mathfrak{q}(1)_{\bar{0}}}^{\mathfrak{q}(1)} k_0)$ |

Table A.4: Spherical representations of $\mathfrak{q}(2)$

- If $\lambda = (t+2)\epsilon_1 + t\epsilon_2$ for $t \neq -1$, then the character formula for $L(\lambda)$ tells us that it will be $(4|4)$ dimensional, so neither it nor its parity shift can be spherical, having too large an odd part.

- Finally, suppose $\lambda = (t+1)\epsilon_1 + t\epsilon_2$, with $t \neq -1/2$. Then by direct computation both $L(\lambda)$ and $\Pi L(\lambda)$ are spherical (note that $L(\lambda) \cong \Pi L(\lambda)$ if and only if $t = 0$ or $t = -1$).

**Spherical irreducibles for $\mathfrak{q}(2)$:** We compute $S^d(Rep^*)$ with respect to $(\mathfrak{b}^{st})^{op}$. This leads to a canonical identification $L_{\mathfrak{b}^{st}}(\lambda)^* \cong L_{(\mathfrak{b}^{st})^{op}}(-\lambda)$ when the length of $\lambda$ is 2.

**Spherical indecomposables for $\mathfrak{q}(2)$:** Again, the quotient map $\mathfrak{q}(2) \to k^{0|1}$ gives rise to a module equivalent to $U^{1|1}$. The modules $(\Pi)L((t+1)\epsilon_1 + t\epsilon_2)$ for $\lambda \neq 0, -1/2, , -1$ are projective hence have no extensions with other modules. For $\lambda = -1/2$, it has the same central character as the trivial module but its weights prevent any extensions between them. Finally, $Q_{2|2}$ and $Q_{2|2}^*$ and the trivial module again have distinct central characters, so we get new spherical indecomposable representations.

# Appendix B

# Generalized roots systems and the Iwasawa decomposition

Here we prove theorem 6.2.8, via the more general theorem 6.2.9. Note that we change our standard notations for root systems, replacing $\Delta$ with $R$, in order to be consistent with commonly used notation.

### B.0.1 Structure of proof

In section 1 we recall the definition of finite GRRSs, state the classification of finite irreducible GRRSs, and prove a few facts we will need later on about them. In section 2 we introduce automorphisms of GRRSs and prove theorem 6.2.9. Section 3 interprets the results from section 2 into statements about centralizers of tori proving theorem 6.2.9. Finally, section 4 proves theorem 6.2.8.

## B.1 Generalized reflection root systems

### B.1.1 Definitions and properties

In [48] the notion of a generalized root system (GRS) was introduced, and GRSs were completely classified. In [19], this notion was generalized to that of a generalized reflection root system (GRRS) that was designed to encompass root systems of affine Lie superalgebras. Finite GRRSs come from root systems of certain (almost) simple Lie superalgebras and we have found they are a natural object to look at for the problem we consider.

The proofs of properties of GRSs stated in [48] carry over almost entirely to finite GRRSs. We will restate some of these results without proof with this understanding.

**Definition B.1.1.** Let $V$ be a finite-dimensional $k$-vector space equipped with a symmetric bilinear form $(\cdot, \cdot)$ (not necessarily non-degenerate). A finite generalized reflection root system (GRRS) is a nonempty finite set $R \subseteq V \setminus \{0\}$ satisfying the following axioms:

1. $\operatorname{span}(R) = V$;
2. for $\alpha \in R$, $(\alpha, -) \neq 0$ as an element of $V^*$.
3. for $\alpha, \beta \in R$ with $(\alpha, \alpha) \neq 0$ we have $k_{\alpha,\beta} := \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ and $r_\alpha(\beta) := \beta - k_{\alpha,\beta}\alpha \in R$;
4. for $\alpha \in R$ such that $(\alpha, \alpha) = 0$ there exists a bijection $r_\alpha : R \to R$ such that $r_\alpha(\beta) = \beta$ if $(\alpha, \beta) = 0$, and $r_\alpha(\beta) = \beta \pm \alpha$ if $(\alpha, \beta) \neq 0$;
5. $R = -R$.

We call the elements of $R$ roots.

For the rest of this appendix we will call a finite GRRS $R$ just a GRRS with the understanding that it is finite. We will not consider infinite GRRSs.

*Remark* B.1.2.
- A GRS, as defined in [48], is a GRRS where the form $(,)$ is assumed to be non-degenerate.
- We note that (2) is equivalent to saying that for all $\alpha \in R$ the bijection $r_\alpha : R \to R$ is nontrivial.
- Another notion of a GRS was given in definition 7.1 in [48]. If one defines $\alpha^\vee = \frac{2}{(\alpha,\alpha)}(\alpha, -)$ for a non-isotropic root $\alpha$ and $\alpha^\vee = (\alpha, -)$ for an isotropic root $\alpha$, then a GRRS is a GRS in the sense of definition 7.1 of [48] if and only if $\alpha^\vee \neq \beta^\vee$ for all odd isotropic roots $\alpha, \beta$. We will see this is the case for all irreducible GRRSs except for $\widetilde{A}(1,1)$, which is defined below.

**Lemma B.1.3.** *Let $R \subseteq V$ be a GRRS and suppose $S \subseteq R$ is a subset of $R$ such that*

- *$S = -S$;*
- *for each $\alpha \in S$ there exists $\beta \in S$ such that $(\alpha, \beta) \neq 0$;*
- *for each $\alpha \in S$, $r_\alpha(S) = S$.*

*Then $S \subseteq \operatorname{span}(S)$ is a GRRS.*

*Proof.* This follows from the definition. □

**Definition B.1.4.** If $R$ is a GRRS we define the subset of real (non-isotropic) and imaginary (isotropic) roots as

$$R_{re} = \{\alpha \in R : (\alpha, \alpha) \neq 0\} \qquad R_{im} = \{\alpha \in R : (\alpha, \alpha) = 0\}.$$

Further, we call $\alpha \in R$ odd if $\alpha \in R_{im}$ or $2\alpha \in R_{re}$. Otherwise we say a root is even.

By construction $R_{re} \subseteq \operatorname{span}(R_{re}) = U$ will be a (potentially non-reduced) root system in the usual sense and in particular the form is non-degenerate when restricted to $U$. Thus we can decompose $U$ as $U = V_1 \oplus \cdots \oplus V_k$, where $R^i_{re} := R_{re} \cap V_i \subseteq V_i$ is irreducible and $R_{re} = \coprod_i R^i_{re}$. Let $W_i$ denote the Weyl group of $R^i_{re}$, and let $W = W_1 \times \cdots \times W_k$, the Weyl group of $R_{re} \subseteq U$. Then $W$ acts naturally on $V$ and preserves $R$ and $(-,-)$. Finally let $V_0$ be the orthogonal complement to $U$ in $V$ so that

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_k,$$

where $R_{re} \cap V_0 = \emptyset$. We write $p_i : V \to V_i$ $i = 0, 1, \ldots, k$ for the projection maps. Note that $(-,-)$ may be degenerate when restricted to $V_0$.

A GRRS $R$ is reducible if we can write $R = R' \coprod R''$, where $R'$ and $R''$ are nonempty and orthogonal to one another. In this case each of $R'$ and $R''$ will form GRRSs in the respective subspaces they span. A GRRS $R$ is irreducible if it is not reducible. Every GRRS can be decomposed into a finite direct sum of irreducible GRRSs.

**Proposition B.1.5.** *(Proposition 2.6 of [48]) For an irreducible GRRS R, either* $\dim V_0 = 1$ *and* $k \leq 2$ *or* $\dim V_0 = 0$ *and* $k \leq 3$*. If* $V_0 \neq 0$*, then* $p_0(R_{im}) = \{\pm v\}$ *for some nonzero vector* $v \in V_0$*.*

*Remark* B.1.6. The proposition B.1.5 in particular implies that if $V_0 = 0$ then $(,)$ is non-degenerate. If $V_0 \neq 0$ then $(,)$ is degenerate if and only if it restricts to the zero form on $V_0$.

For the irreducible root system $R^i_{re} \subseteq V_i$, we write $P_i = \{x \in V_i : \frac{2(x,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha \in R^i_{re}\}$ for the weight lattice of $V_i$. A $W_i$-orbit $X \subseteq P_i$ is small if $x - y \in R^i_{re}$ for any $x, y \in X$, where $x \neq \pm y$.

**Proposition B.1.7.** *(Proposition 3.5 of [48]) Let R be a GRRS. Then* $p_i(R_{im})$ *is a subset of* $P_i \setminus \{0\}$*, and is the union of small* $W_i$*-orbits. In particular* $(p_i(\alpha), p_i(\alpha)) \neq 0$ *for all* $\alpha \in R_{im}$ *and* $i > 0$*.*

Let $R$ be a GRRS. Then $R_{im}$ is $W$-invariant, and thus we may break it up into its orbits

$$R_{im} = R^1_{im} \sqcup \cdots \sqcup R^m_{im}.$$

We call the orbits imaginary components of $R$.

**Lemma B.1.8.** *Let R be an irreducible GRRS. If* $\alpha, \beta$ *are isotropic roots that lie in the same imaginary component of R, and* $p_i(\alpha) = \pm p_i(\beta)$ *for all i, then either* $\alpha = \pm\beta$ *or* $\alpha \pm \beta = 2p_\ell(\alpha) \in R^\ell_{re}$ *for some* $\ell \in \{1, \ldots, k\}$*.*

*Proof.* For ease of notation, for a vector $v \in V$ write $v^2 := (v, v)$, and write $p_i(\beta) = \epsilon_i p_i(\alpha)$, where $\epsilon_i = \pm 1$. Then by assumption we have that

$$0 = (\alpha, \alpha) = \sum_i p_i(\alpha)^2.$$

Suppose that $\alpha \neq \pm\beta$. Since there are at most three terms in the above sum, there must be an $\ell$ such that $\epsilon_\ell$ is distinct from $\epsilon_i$ for all $i \neq \ell$. We see that in this notation,

$$(\alpha, \beta) = \sum_i \epsilon_i p_i(\alpha)^2.$$

If this quantity is 0, then we may add it to $\epsilon_\ell(\alpha, \alpha)$ and find that $2\epsilon_\ell p_\ell(\alpha)^2 = 0$, hence $p_\ell(\alpha)^2 = 0$. However this contradicts proposition B.1.7. So we must instead have $(\alpha, \beta) \neq 0$, so that by axiom (2) of a GRS, either $\alpha + \beta$ or $\alpha - \beta$ is a root. It must be real in either case, and therefore cannot have a component in $V_0$ and can only have a nonzero component in one $V_i$ for some $i > 0$. It now follows whichever of $\alpha \pm \beta$ is a root, it will be equal to $2p_i(\alpha)$ for some $i > 0$, and we are done. □

### B.1.2 Classification

Theorem 5.10 of [48] classified irreducible GRSs. However from an analysis of the proof one see that it also classifies GRRSs, and only on extra family of GRRS arises that are not already GRSs and this is the family $\widetilde{A}(n, n)$. This is verified in [19] as well. In terms of Lie superalgebras, $\widetilde{A}(n, n)$ is the root system of $\mathfrak{pgl}(n|n) = \mathfrak{gl}(n|n)/kI_{n|n}$. To be precise, if we write $\mathfrak{h} \subseteq \mathfrak{gl}(n|n)$ for the subalgebra of diagonal matrices, then $\mathfrak{h}^*$ has a non-degenerate inner product from the supertrace form. If we take the subspace of $\mathfrak{h}^*$ spanned by roots of $\mathfrak{gl}(n|n)$ and restrict the form to it, we get the GRRS $\widetilde{A}(n, n)$.

**Theorem B.1.9.** *The irreducible GRRSs with $R_{im} \neq 0$ are as follows.*

*(0)* $\widetilde{A}(n, n)$, $n \geq 1$: $R_{re} = A_n \sqcup A_n$, $R_{im} = (W\omega_1 + v) \sqcup (W\omega_n - v)$.

1. $A(0, n)$, $n \geq 1$: $R_{re} = A_n$, $R_{im} = (W\omega_1 + v) \sqcup (W\omega_n - v)$
2. $C(0, n)$, $n \geq 2$: $R_{re} = C_n$, $R_{im} = (W\omega_1 + v) \sqcup (W\omega_1 - v)$
3. $A(m, n)$: $m \neq n, m \geq 1$: $R^1_{re} = A_m$, $R^2_{re} = A_n$, $R_{im} = (W(\omega^{(1)}_1 + \omega^{(2)}_n) + v) \sqcup (W(\omega^{(1)}_m + \omega^{(2)}_1) - v)$
4. $A(n, n)$, $n \geq 2$: $R^1_{re} = A_n$, $R^2_{re} = A_n$, $R_{im} = W(\omega^{(1)}_1 + \omega^{(2)}_n) \sqcup W(\omega^{(1)}_n + \omega^{(2)}_1)$
5. $B(m, n)$, $m, n \geq 1$: $R^1_{re} = B_m$, $R^2_{re} = BC_n$, $R_{im} = W(\omega^{(1)}_1 + \omega^{(2)}_1)$
6. $G(1, 2)$: $R^1_{re} = BC_1$, $R^2_{re} = G_2$, $R_{im} = W(\omega^{(1)}_1 + \omega^{(2)}_1)$
7. $D(m, n)$, $m > 2$, $n \geq 1$: $R^1_{re} = D_m$, $R^2_{re} = C_n$, $R_{im} = W(\omega^{(1)}_1 + \omega^{(2)}_1)$
8. $AB(1, 3)$: $R^1_{re} = A_1$, $R^2_{re} = B_3$, $R_{im} = W(\omega^{(1)}_1 + \omega^{(2)}_3)$
9. $D(2, n)$, $n \geq 1$: $R^1_{re} = A_1$, $R^2_{re} = A_1$, $R^3_{re} = C_n$, $R_{im} = W(\omega^{(1)}_1 + \omega^{(2)}_1 + \omega^{(3)}_1)$

10. $D(2,1;\lambda)$: $R^1_{re} = A_1$, $R^2_{re} = A_1$, $R^3_{re} = A_1$, $R_{im} = W(\omega_1^{(1)} + \omega_1^{(2)} + \omega_1^{(3)})$

*The only GRRS which is not a GRS (i.e. for which the inner product is degenerate) is* $\widetilde{A}(n,n)$.

*In cases (0)-(3),* $v \in V_0$ *is some nonzero vector, and each inner product is determined up to proportionality, except for* $D(2,1;\lambda)$ *where we get a family of distinct inner products parametrized by* $\lambda \neq 0, -1$ *modulo an action of* $S_3$. *Further the inner products on two distinct real components of* $D(2,1;\lambda)$ *agree if and only if* $D(2,1;\lambda) \cong D(2,1)$, *which is when* $\lambda = 1, -2$, *or* $-1/2$.

*Remark* B.1.10. The cases (1)-(10) are each the root system of a unique simple basic Lie superalgebra. The only basic simple Lie superalgebra that is left out in the above classification is $\mathfrak{psl}(2|2)$. This is due to having root spaces of dimension bigger than one. However using GRRSs we do get $\widetilde{A}(1,1)$, which as already stated corresponds to $\mathfrak{pgl}(2|2)$, whose derived subalgebra is $\mathfrak{psl}(2|2)$.

**Corollary B.1.11.** *let* $\alpha, \beta$ *be linearly independent isotropic roots in an irreducible GRRS* $R$. *Then for some* $i > 0$, *one of two things must occur:*

1. $p_i(\alpha)$ *and* $p_i(\beta)$ *are orthogonal and either* $p_i(\alpha) + p_i(\beta) \in R^i_{re}$ *or* $p_i(\alpha) - p_i(\beta) \in R^i_{re}$.

2. $2p_i(\alpha) = \pm 2p_i(\beta) \in R^i_{re}$.

*Proof.* If $\alpha$ and $\beta$ lie in the same imaginary component of $R$, then $p_i(\alpha)$ and $p_i(\beta)$ lie in the same small $W_i$-orbit. If $p_i(\alpha) \neq \pm p_i(\beta)$ for some $i$, then $p_i(\alpha)$ is orthogonal to $p_i(\beta)$ and by proposition B.1.7 $p_i(\alpha) - p_i(\beta) \in R^i_{re}$ so we are done. Otherwise, we are in the situation of lemma B.1.8, giving $2p_i(\alpha) = \pm 2p_i(\beta) \in R^i_{re}$ for some $i$, and we are done.

If $\alpha$ and $\beta$ lie in distinct imaginary components, then we have $R$ is one of the GRRSs listed in (0)-(4) above. But we see that in each case there are two imaginary components and they are swapped under negation. Thus $\alpha$ and $-\beta$ are in the same imaginary component, so we may apply the argument just given to finish the proof. □

## B.2 Automorphisms of weak generalized root systems

Let $R \subseteq V$ be an irreducible GRRS and $\theta$ an automorphism of $R$, meaning that $\theta : V \to V$ is a linear isomorphism preserving the bilinear form, with $\theta(R) = R$. Write $S \subseteq R$ for the roots fixed by $\theta$. By linearity, we have that $S = -S$ and if $\alpha, \beta \in S$ and $\alpha + \beta \in R$, then $\alpha + \beta \in S$. We now prove the main technical result of the appendix.

**Proposition B.2.1.** *Let* $\alpha, \beta$ *be linearly independent odd roots of* $S$. *Then there exists a real root* $\gamma \in R_{re}$ *with* $\theta(\gamma) = \gamma$ *(i.e.* $\gamma \in S$*) such that* $(\gamma, \alpha) \neq 0$ *and* $(\gamma, \beta) \neq 0$.

*Proof.* We break the proof up into two cases.

*Case 1: $\alpha, \beta$ are isotropic*:

In general, $\theta$ will either preserve all $R^i_{re}$ or will permute some of the $R^i_{re}$. We first deal with the latter case. If $\theta$ permutes $R^i_{re}$ and $R^j_{re}$, then in particular these root systems must be isomorphic. Looking at our list, this leaves only (0), (4), (9), and (10) as possibilities. However, in the cases of (0) and (4) the inner product on each factor of $A_n$ is negative the other, so no such $\theta$ can exist that permutes them. Further, in the case of (10) such a permutation could only exist if two of the underlying real root systems are isomorphic, i.e. their inner products agree, which would give $D(2,1)$. So it remains to deal with case (9).

For the case of (9), we may assume that $R^3_{re}$ is preserved by $\theta$. If $p_3\alpha \neq \pm p_3\beta$ then necessarily $p_3\alpha$ and $p_3\beta$ are orthogonal because of what the orbit of $\omega_1^{(3)}$ is. By smallness of the orbit of $\omega_1$ in $C_n$ we will have $\gamma = p_3\alpha - p_3\beta \in R^3_{re}$ is fixed by $\theta$, and this will not be orthogonal to $\alpha$ or $\beta$ so that $(\gamma, \alpha) \neq 0$ and $(\gamma, \beta) \neq 0$. If $p_3\alpha = \pm p_3\beta$ then $\gamma = 2p_3\alpha \in R^3_{re}$ works.

If instead $\theta$ preserves each $R^i_{re}$, then each $p_i\alpha$ is fixed by $\theta$. We then apply corollary B.1.11 to get that there exists an $i$ such that some linear combination of $p_i(\alpha)$ and $p_i(\beta)$ is in $R^i_{re}$ which is not orthogonal to $\alpha$ or $\beta$ and is fixed by $\theta$.

*Case 2: one of $\alpha, \beta$ non-isotropic*

If $\alpha$ is non-isotropic, then one real component of $R$ must be $BC_n$ for some $n$, hence either $R = G(1,2)$ or $R = B(m,n)$. If $R = G(1,2)$, then $\alpha = \pm\omega_1^{(1)}$. Hence if $\beta$ is isotropic then $(p_1(\beta), \alpha) \neq 0$ so we can take $\gamma = \alpha$. If $\beta$ is non-isotropic then $\beta = \pm\omega_1^{(1)}$ as well, so clearly $(\alpha, \beta) \neq 0$ and we can again take $\gamma = \alpha$.

If $R = B(m,n)$ and $\beta$ is isotropic, then $p_2\beta = \sigma\omega_1^{(2)}$ for some $\sigma$ in the Weyl group of $BC_n$. Hence either $p_2\beta = \pm\alpha$, in which case we can take $\gamma = \alpha$, otherwise $\gamma = p_2\beta + \alpha \in BC_n$ works. If $\beta$ is non-isotropic then either $\beta = \pm\alpha$, in which case we take $\gamma = \alpha$, and otherwise $\gamma = \beta + \alpha \in BC_n$ works. □

**Corollary B.2.2.** *If $S$ contains linearly independent odd roots or no odd roots at all, then $S \subseteq \operatorname{span}(S)$ is a GRRS.*

*Proof.* We may apply lemma B.1.3 along with proposition B.2.1 to obtain the result. □

*Remark* B.2.3. Note that we could have $S = \{\pm\alpha\}$ for an isotropic root $\alpha$. For example if we consider $A(0,2)$, the automorphism given by a simple reflection of the Weyl group of $A_2$ will give rise to such a situation.

Now let $T \subseteq S$ be the smallest subset of $S$ containing all odd roots of $S$ and such that if $\alpha \in T$, $\beta \in S$ and $(\alpha, \beta) \neq 0$, then $\beta \in T$. Then $T$ will be orthogonal to $T' := S \setminus T$, and $T'$ will consist of only even roots by proposition B.2.1.

**Proposition B.2.4.** *$T' \subseteq \operatorname{span}(T')$ is a reduced root system. Further, we have the following possibilities for $T$:*

1. $T = \emptyset$.

2. $T = \{\pm\alpha\}$ *for an isotropic root* $\alpha$.

3. $T \subseteq \mathrm{span}(T)$ *is an irreducible GRRS containing at least one odd root.*

*In all cases,* $T$ *is orthogonal to* $T'$ *and we have both* $S \cap \mathrm{span}(T) = T$ *and* $S \cap \mathrm{span}(T') = T'$.

*Proof.* The first statement is clear. For the second statement, if $S \cap R_{im} = \{\pm\alpha\}$ for some $\alpha$, then we claim $T = \{\pm\alpha\}$. This is because if not then there exists $\beta \in T \setminus \{\pm\alpha\}$ such that $\beta$ must is real and $(\alpha, \beta) \neq 0$. Thus $r_\beta\alpha$ would be another isotropic root in $T$.

If $S \cap R_{im} \neq \{\pm\alpha\}$ for some $\alpha$ then either it is empty, or contains two linearly independent isotropic roots. In the former case $T$ will either be empty or a non-reduced root system which is irreducible (by proposition B.2.1) and thus is $BC_n$. In the latter case $T \subseteq \mathrm{span}(T)$ is an irreducible GRRS with $T_{im} \neq \emptyset$ by proposition B.2.1 and lemma B.1.3.

Now in each possibility for $T$ we always have that the span of the odd roots is everything, as this is true for any irreducible GRRS. It follows that $\mathrm{span}(T')$ is orthogonal to $\mathrm{span}(T)$. Since the inner product restricted to $\mathrm{span}(T')$ will be non-degenerate we must have $S \cap \mathrm{span}(T') = T'$. On the other hand if $\alpha \in T' \cap \mathrm{span}(T)$ we would have that $\alpha$ is a null vector, a contradiction. □

**Corollary B.2.5.** *Either* $S \subseteq \mathrm{span}(S)$ *is a GRRS or* $S = T' \sqcup \{\pm\alpha\}$ *where* $T' \subseteq \mathrm{span}(T')$ *is an even reduced root system and* $\alpha$ *is an isotropic root orthogonal to* $T'$.

## B.3 Applications to centralizers of some tori

**Lemma B.3.1.** *Suppose that* $\mathfrak{g}$ *is a Lie superalgebra such that*

1. $\mathfrak{g}_{\overline{0}}$ *is reductive;*

2. *If* $\mathfrak{h} \subseteq \mathfrak{g}_{\overline{0}}$ *is a Cartan subalgebra (CSA) of* $\mathfrak{g}_{\overline{0}}$*, then it is self-centralizing in* $\mathfrak{g}$.

3. *For any nonzero weight* $\alpha$ *of a CSA* $\mathfrak{h}$ *we have* $\dim \mathfrak{g}_\alpha \leq 1$.

*Then* $\theta \in \mathrm{Aut}(\mathfrak{g})$ *is semisimple if and only if* $\theta|_{\mathfrak{g}_{\overline{0}}}$ *is semisimple. In particular,* $\theta$ *is semisimple if and only if it preserves a Cartan subalgebra of* $\mathfrak{g}_{\overline{0}}$.

*Remark* B.3.2. Property (2) is equivalent to asking that for any root decomposition of $\mathfrak{g}$, each weight space (including the trivial weight space) is of pure parity.

*Proof.* It is known that an automorphism of a reductive Lie algebra is semisimple if and only if it preserves a Cartan subalgebra. Therefore if $\theta|_{\mathfrak{g}_{\overline{0}}}$ is semisimple, it preserves a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}_{\overline{0}}$, and thus must act by a permutation on the roots. Since the root spaces are one-dimensional, it follows that some power of $\theta$ must act by a scalar on each weight space, and thus $\theta$ must be semisimple. □

Suppose that $\mathfrak{g}$ is either a simple basic Lie superalgebra not equal to $\mathfrak{psl}(2|2)$ or is $\mathfrak{gl}(m|n)$ for some $m, n$ so that $\mathfrak{g}$ satisfies the hypothesis of lemma B.3.1. Let $\theta \in \mathrm{Aut}(\mathfrak{g})$ be a semisimple automorphism of $\mathfrak{g}$ which preserves a non-degenerate invariant form on $\mathfrak{g}$. We get an orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the fixed subalgebra of $\theta$, and $\mathfrak{p}$ is the sum of the nonzero eigenspaces of $\theta$.

*Remark* B.3.3. The Killing form is non-degenerate for $\mathfrak{sl}(m|n)$ with $m \neq n$, $\mathfrak{osp}(m|2n)$ when $m - 2n \neq 2$ and $m + 2n \geq 2$, and on $G(1,2)$ and $AB(1,3)$. Thus every automorphism of these superalgebras necessarily preserves the form.

Now suppose $\mathfrak{h} \subseteq \mathfrak{g}_{\overline{0}}$ is a Cartan subalgebra which is $\theta$-invariant. Write $\mathfrak{h} = \tilde{\oplus}\mathfrak{a}$, where $\tilde{=}\mathfrak{k} \cap \mathfrak{h}$ and $\mathfrak{a} = \mathfrak{p} \cap \mathfrak{h}$. Then $\theta$ induces an automorphism of $\mathfrak{h}^*$ preserving the set of roots, $R$, and thus induces an automorphism of the GRRS $R \subseteq V = \mathrm{span}(R)$. In the case of $\mathfrak{gl}(m|n)$, $R \subseteq \mathrm{span}(R)$ will either be $A(m-1, n-1)$ if $m \neq n$ or $\widetilde{A}(n-1, n-1)$ if $m = n \neq 1$, and this is the GRRS we consider. If $m = n = 1$, we are not looking at a GRRS but the following result will be obvious anyway.

We keep the notations as above for $S$, $T$, and $T'$. Write $\mathfrak{c}(\mathfrak{a})$ for the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Notice that we have $\mathfrak{c}(\mathfrak{a}) = \mathfrak{h} + \bigoplus_{\alpha \in S} \mathfrak{g}_\alpha$

**Lemma B.3.4.** *Let $\mathfrak{l} \subseteq \mathfrak{g}$ be the subalgebra of $\mathfrak{g}$ generated by the roots $\{e_\alpha : \alpha \in T'\}$, and write $\widetilde{\mathfrak{l}}$ for the subalgebra of $\mathfrak{g}$ generated by $\{e_\alpha : \alpha \in T\}$. Then $\mathfrak{l}$ is a semisimple Lie algebra, and either $\widetilde{\mathfrak{l}}$ is isomorphic to a simple basic Lie superalgebra, isomorphic to $\mathfrak{sl}(n|n)$ for some $n \geq 1$, or is trivial.*

*Further, the natural map $\mathfrak{a} \times \widetilde{\mathfrak{l}} \times \mathfrak{l} \to \mathfrak{c}(\mathfrak{a})$ is an injective Lie algebra homomorphism. This realizes $\mathfrak{a} \times \widetilde{\mathfrak{l}} \times \mathfrak{l}$ as an ideal of $\mathfrak{c}(\mathfrak{a})$.*

*Proof.* Since $T'$ is a reduced even root system, the subalgebra $\mathfrak{l}$ is a Kac-Moody algebra of finite-type and thus is semisimple. If $T \neq \emptyset$ then we apply proposition B.2.4: either $T = \{\pm\alpha\}$ for an odd isotropic root $\alpha$, in which case $\widetilde{\mathfrak{l}} \cong \mathfrak{sl}(1|1)$, or $T$ is an irreducible GRRS. The only possibilities for $\widetilde{\mathfrak{l}}$ in the latter case are then either a simple basic Lie superalgebra or $\mathfrak{sl}(n|n)$ for $n \geq 2$.

Using proposition B.2.4 we see that $[\mathfrak{l}, \widetilde{\mathfrak{l}}] = 0$, and these algebras commute with $\mathfrak{a}$. Hence we obtain a natural map $\mathfrak{a} \times \widetilde{\mathfrak{l}} \times \mathfrak{l} \to \mathfrak{c}(\mathfrak{a})$ and it is injective again by proposition B.2.4. □

**Proposition B.3.5.** *The algebra $\mathfrak{c}(\mathfrak{a})$ is an extension of an abelian algebra by the product of ideals $\mathfrak{a} \times \widetilde{\mathfrak{l}} \times \mathfrak{l}$. In particular $\mathfrak{c}(\mathfrak{a})_{\overline{1}} + [\mathfrak{c}(\mathfrak{a})_{\overline{1}}, \mathfrak{c}(\mathfrak{a})_{\overline{1}}]$ is an ideal of $\mathfrak{c}(\mathfrak{a})$ isomorphic to either a basic simple Lie superalgebra or $\mathfrak{sl}(n|n)$ for some $n$.*

*Proof.* The quotient is surjected onto by $\mathfrak{h}$, hence is abelian. □

*Remark* B.3.6. The proposition B.3.5 implies that the structure of $\mathfrak{c}(\mathfrak{a})$ is determined by abelian algebras of outer derivations of $\widetilde{\mathfrak{l}} \times \mathfrak{l}$ that act semisimply and preserve both $\mathfrak{l}$ and $\widetilde{\mathfrak{l}}$. Since a semisimple Lie algebra has no outer derivations, we only need to consider outer derivations of $\widetilde{\mathfrak{l}}$. For this, the only algebras with outer derivations are $\mathfrak{psl}(n|n)$ and $\mathfrak{sl}(n|n)$.

These algebras all have a one-dimensional algebra of outer derivations except for $\mathfrak{psl}(2|2)$, whose outer derivations are isomorphic to $\mathfrak{sl}(2)$. However since we only consider semisimple outer derivations, up to symmetry there is only one outer derivation up to scalar.

Thus the possibilities for nontrivial extensions for $\mathfrak{c}(\mathfrak{a})$ that could arise from proposition B.3.5 would be of the form $\mathfrak{a}\times\mathfrak{gl}(n|n)\times\mathfrak{l}$ or $\mathfrak{a}\times\mathfrak{pgl}(n|n)\times\mathfrak{l}$.

## B.4 Involutions and the Iwasawa decomposition

Let us now assume that $\mathfrak{g}$ is either simple basic or is $\mathfrak{gl}(m|n)$ for some $m,n\in\mathbb{N}$, and that $\theta$ is an involution preserving the non-degenerate invariant form on $\mathfrak{g}$. Then in our decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ we have that $\mathfrak{p}$ is the $(-1)$-eigenspace of $\theta$. Recall that on a Lie superalgebra $\mathfrak{g}=\mathfrak{g}_{\overline{0}}\oplus\mathfrak{g}_{\overline{1}}$ there is a canonical involution $\delta\in\operatorname{Aut}(\mathfrak{g})$ defined by $\delta=\operatorname{id}_{\mathfrak{g}_{\overline{0}}}\oplus(-\operatorname{id}_{\mathfrak{g}_{\overline{1}}})$. This involution is central $\operatorname{Aut}(\mathfrak{g})$.

**Lemma B.4.1.** *If $\theta\neq\operatorname{id}_{\mathfrak{g}},\delta$, then $\mathfrak{p}_{\overline{0}}\neq 0$.*

*Proof.* If $\mathfrak{p}_{\overline{0}}=0$, then we have $\mathfrak{g}_{\overline{0}}$ is fixed by $\theta$. Then $\theta$ fixes a Cartan subalgebra $\mathfrak{h}\subseteq\mathfrak{g}_{\overline{0}}$, and hence $\theta$ must preserve the root spaces with respect to this Cartan, and so by the order 2 condition it acts by $\pm1$ on each odd root space of $\mathfrak{g}$. Now $\mathfrak{g}_{\overline{1}}$ is a $\mathfrak{g}_{\overline{0}}$-module, and $\theta$ will be an intertwiner for this module structure. By general theory of simple Lie superalgebras (see chapter 1 of [39]), $\mathfrak{g}_{\overline{1}}$ is either irreducible or breaks into a sum of two non-isomorphic irreducible $\mathfrak{g}_{\overline{0}}$-representations $\mathfrak{g}_{\overline{1}}',\mathfrak{g}_{\overline{1}}''$ such that $[\mathfrak{g}_{\overline{1}}',\mathfrak{g}_{\overline{1}}'']=\mathfrak{g}_{\overline{0}}$ (or $[\mathfrak{g}_{\overline{1}}',\mathfrak{g}_{\overline{1}}'']$ is a codimension 1 subalgebra of $\mathfrak{g}_{\overline{0}}$ in the case of $\mathfrak{gl}(m|n)$). In the former case, $\theta$ must act by $\pm1$ on $\mathfrak{g}_{\overline{1}}$.

In the latter case, if $\theta$ does not act by $\pm1$ on all of $\mathfrak{g}_{\overline{1}}$ then WLOG it will act by $(-1)$ on $\mathfrak{g}_{\overline{1}}'$ and by 1 on $\mathfrak{g}_{\overline{1}}''$, and thus $[\mathfrak{g}_{\overline{1}}',\mathfrak{g}_{\overline{1}}'']\subseteq\mathfrak{p}_{\overline{0}}=0$, a contradiction. □

### B.4.1 Iwasawa decomposition

Since we have an involution on $\mathfrak{g}_{\overline{0}}$ preserving the non-degenerate form on it, by classical theory we may choose a maximal toral subalgebra $\mathfrak{a}\subseteq\mathfrak{p}_{\overline{0}}$ that can be extended to a $\theta$-invariant Cartan subalgebra of $\mathfrak{g}$, which we will call $\mathfrak{h}$. We obtain a decomposition $\mathfrak{h}=\mathfrak{t}\oplus\mathfrak{a}$, where $\mathfrak{t}$ is the fixed subspace of $\theta$. We again write $\mathfrak{c}(\mathfrak{a})$ for the centralizer of $\mathfrak{a}$. Notice that $\mathfrak{a}$ is also a maximal toral subalgebra of the $(-1)$-eigenspace of the involution $\delta\circ\theta$.

We already deduced the structure of $\mathfrak{c}(\mathfrak{a})$ as an algebra in proposition B.3.5, and in particular we saw that $\mathfrak{c}(\mathfrak{a})_{\overline{1}}\subseteq\widetilde{\mathfrak{l}}_{\overline{1}}$. Now $\theta$ restricts to an automorphism of $\mathfrak{c}(\mathfrak{a})$ preserving $\widetilde{\mathfrak{l}}$, and by classical theory we have $\mathfrak{c}(\mathfrak{a})_{\overline{0}}\cap\mathfrak{p}=\mathfrak{a}$. Thus by lemma B.4.1 either $\theta|_{\widetilde{\mathfrak{l}}}=\operatorname{id}_{\widetilde{\mathfrak{l}}}$ or $\theta|_{\widetilde{\mathfrak{l}}}=\delta_{\widetilde{\mathfrak{l}}}$.

**Definition B.4.2.** For $\lambda\in\mathfrak{h}^*$ write $\overline{\lambda}:=(\lambda-\theta\lambda)/2\in\mathfrak{a}^*$ for the orthogonal projection of $\lambda$ to $\mathfrak{a}^*$ (equivalently the restriction to $\mathfrak{a}$), and write $\overline{R}$ for the restriction of roots in $R$ to $\mathfrak{a}^*$ which are nonzero. We call $\overline{R}\subseteq\mathfrak{a}^*$ the restricted root system, and elements of $\overline{R}$ we call restricted roots.

Let $\mathbb{Z}\overline{R} \subseteq \mathfrak{a}^*$ be the $\mathbb{Z}$-module generated by $\overline{R}$, and then choose a group homomorphism $\overline{\phi} : \mathbb{Z}\overline{R} \to \mathbb{R}$ such that $\overline{\phi}(\overline{\alpha}) \neq 0$ for all $\overline{\alpha} \in \overline{R}$. Let $\overline{R}^{\pm} = \{\overline{\alpha} \in \overline{R} : \pm\overline{\phi}(\overline{\alpha}) > 0\}$ so that we obtain a partition of the restricted roots $\overline{R} = \overline{R}^+ \sqcup \overline{R}^-$. We call $\overline{R}^+$ the positive restricted roots, and we call a partition of $\overline{R}$ arising in this way a choice of positive system for $\overline{R}$. Write $\mathfrak{n}^{\pm} = \bigoplus\limits_{\overline{\alpha}\in\overline{R}^{\pm}} \mathfrak{g}_{\overline{\alpha}}$ (where $\mathfrak{g}_{\overline{\alpha}}$ is the weight space of $\overline{\alpha} \in \mathfrak{a}^*$ with respect to the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$), and $\mathfrak{n} = \mathfrak{n}^+$ as a shorthand.

**Theorem B.4.3.** *If $\theta|_{\mathfrak{c}(\mathfrak{a})_{\overline{1}}} = \mathrm{id}$, then we get an Iwasawa decomposition of $\mathfrak{g}$:*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

*Proof.* The proof is identical to the classical case. We see that for $\overline{\alpha} \in \overline{R}$, we have linear isomorphisms $\theta : \mathfrak{g}_{\overline{\alpha}} \to \mathfrak{g}_{-\overline{\alpha}}$, so that $\mathfrak{g}_{\overline{\alpha}} \cap \mathfrak{k} = \mathfrak{g}_{\overline{\alpha}} \cap \mathfrak{p} = 0$. Hence if $y \in \mathfrak{g}_{\overline{\alpha}}$ is nonzero and $y = y_0 + y_1$ where $y_0 \in \mathfrak{k}$ and $y_1 \in \mathfrak{p}$, then $y_0 \neq 0$ and $y_1 \neq 0$, and we have $\theta(y) = y_0 - y_1$. From this it is clear that $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ contains $\mathfrak{n}^-$, and it is also clear that it contains $\mathfrak{h}$. We see $\mathfrak{c}(\mathfrak{a})$ is complementary to $\mathfrak{a} + \mathfrak{n} + \mathfrak{n}^-$, and by our assumption on $\theta$ we have $\mathfrak{c}(\mathfrak{a}) \subseteq \mathfrak{k} + \mathfrak{a}$, which shows that $\mathfrak{k} + \mathfrak{a} + \mathfrak{n} = \mathfrak{g}$.

To show the sum is direct, if we have $x + h + y = 0$, where $x \in \mathfrak{k}$, $h \in \mathfrak{a}$, and $y \in \mathfrak{n}$, then applying $[h', \cdot]$ for $h' \in \mathfrak{a}$ we find that $[h', y] = -[h', x] \in \mathfrak{p}$. Hence $\theta([h', y]) = -[h', y] \in \mathfrak{n}$, while $[\theta(h'), \theta(y)] = -[h', \theta(y)] \in \mathfrak{n}^-$. Hence $[h', y] = 0$ for all $h' \in \mathfrak{a}$ implying $y = 0$. It follows that $x + h = 0$, and since $x \in \mathfrak{k}$ and $h \in \mathfrak{p}$ this implies $x = h = 0$, and we are done. □

Before stating the next corollary, we make a definition.

**Definition B.4.4.** Let $R$ be a GRRS and let $Q = \mathbb{Z}R \subseteq \mathfrak{h}^*$ be the root lattice. Given a group homomorphism $\phi : Q \to \mathbb{R}$ such that $\phi(\alpha) \neq 0$ for all $\alpha \in R$, we obtain a partition $R = R^+ \sqcup R^-$ where $R^{\pm} = \{\alpha \in R : \pm\phi(\alpha) > 0\}$. We call $R^+$ the positive roots of $R$, and any partition of $R$ arising in this way is called a positive system.

Positive systems for $R$ are equivalent to choices of Borel subalgebras of a corresponding Lie superalgebra $\mathfrak{g}$ containing $\mathfrak{h}$, where the Borel subalgebra is given by $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus\limits_{\alpha\in R^+} \mathfrak{g}_{\alpha}$ (in fact we define Borel subalgebras to be subalgebras arising in this way).

**Corollary B.4.5.** *If $\theta$ is an involution on a simple basic superalgebra or $\mathfrak{gl}(m|n)$ such that $\theta$ preserves the non-degenerate invariant form, then either $\theta$ or $\delta \circ \theta$ admits an Iwasawa decomposition. In particular, either the fixed points of $\theta$ or the fixed points of $\delta \circ \theta$ have a complementary Borel subalgebra.*

*Proof.* If $\theta = \delta$ or $\theta = \mathrm{id}$, the statement is obvious. Otherwise, we may assume we are in the hypothesis of theorem B.4.3. If $\mathfrak{g} = \mathfrak{psl}(2|2)$ we reference the classification of involutions in [53].

To find a complementary Borel subalgebra, let $\overline{\phi} : \mathbb{Z}\overline{R} \to \mathbb{R}$ be a group homomorphism determining a positive system for $\overline{R}$. Split the natural surjection of free abelian groups $\mathbb{Z}R \to \mathbb{Z}\overline{R}$ so that $\mathbb{Z}R \cong \mathbb{Z}\overline{R} \oplus K$. Then construct $\phi : \mathbb{Z}R \to \mathbb{R}$ which is an extension of $\overline{\phi}$ with respect to the inclusion $\mathbb{Z}\overline{R} \to \mathbb{Z}R$ such that both $\phi(\alpha) \neq 0$ if $\alpha \in R$ and $\phi(\alpha) > 0$ if $\overline{\phi}(\overline{\alpha}) > 0$ for $\alpha \in R$. Then the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\phi(\alpha)>0} \mathfrak{g}_\alpha$ contains $\mathfrak{a} \oplus \mathfrak{n}$ and thus is complementary to $\mathfrak{k}$ by the Iwasawa decomposition. $\square$

# Appendix C

# Layers of the coradical filtration

In the appendix we prove theorem 6.5.11. The result is naturally stated and proven in the more general context of a coalgebra object in a rigid monoidal tensor category satisfying certain restrictions.

## C.1 Statement of main result[1]

Let $k$ be a field and $\mathcal{C}$ a semisimple pointed tensor category over $k$ (precise definitions are given in appendix C.2). Recall that pointed means that every simple object of $\mathcal{C}$ is invertible. For instance $\mathcal{C}$ could be the category of finite-dimensional (super) vector spaces. Let $C$ be a coalgebra object in the cocompletion of $\mathcal{C}$. Then $C$ is a bicomodule over itself via its comultiplication morphism. We prove a result on one aspect of this structure.

As a $C$-bicomodule, $C$ has an ascending Loewy series, i.e. its socle filtration

$$0 = \sigma_0(C) \subseteq \sigma_1(C) \subseteq \sigma_2(C) \subseteq \cdots .$$

This is often called the coradical filtration of $C$, and it is in fact a filtration of $C$ by coalgebras. We seek to describe the layers of this filtration. But we need a few assumptions on $C$.

Before we state the assumptions, we recall a few constructions. First, every simple right comodule $L$ of $C$ has an injective envelope $I(L)$, which is a right comodule that is an object of the cocompletion of $\mathcal{C}$. It too has a socle filtration $\sigma_\bullet(I(L))$ as a right comodule. Next, given a right $C$-comodule $V$ and an object $S$ of $\mathcal{C}$, the tensor product $S \otimes V$ has the natural structure of a right comodule. Finally, if $V$ is a right $C$-comodule, then its right dual $V^*$ is a left $C$-comodule and if $W$ is a right comodule then the tensor product $V^* \otimes W$ has the natural structure of a bicomodule, which we denote by writing $V^* \boxtimes W$.

Here are the assumptions we place on $C$: for the third assumption we need to fix $n \in \mathbb{N}$ with $n \geq 1$.

[1]In a discussion with Pavel Etingof the author has been informed that this result is known for finite-dimensional coalgebras and can be proven for infinite dimensional coalgebras using the equivalence between the category of comodules and the category of rational modules over the dual algebra.

**(C1)** If $L$ is a simple right $C$-comodule and $S$ is a simple object of $\mathcal{C}$, then $L$ and $S \otimes L$ are not isomorphic as comodules unless $S \cong \mathbf{1}$.

**(C2)** If $L, L'$ are right $C$-comodules then $L^* \boxtimes L'$ is a simple bicomodule and further every simple bicomodule is of this form up to isomorphism.

**(C3-$n$)** If $L, L'$ are simple right comodules, then, $[\sigma_n(I(L)) : L'] < \infty$.

The above conditions hold for many examples. Our motivating example is where $\mathcal{C}$ is the category of finite-dimensional super vector spaces over $k$ and $C$ is the coalgebra of polynomial functions on a quasi-reductive supergroup with an even Cartan subgroup. However if $\mathcal{C}$ is the category of finite-dimensional vector spaces, then (C1) automatically holds, and if $C$ is a coalgebra over an algebraically closed field $k$, then (C1)-(C2) hold[2]. More generally, if $G$ is a group, $k$ an algebraically closed field of characteristic zero or characteristic $p$ where $p$ is coprime to the order of each finite subgroup of $G$, and $\mathcal{C}$ is the category of $G$-graded vector spaces over $k$, then (C1) and (C2) become equivalent. This follows as a corollary of the main results of [4], that a finite-dimensional $G$-graded simple algebra $B$ is a matrix algebra over $k$ if and only if the center of $B$ is $k$.

Finally we observe that (C3-$n$) holds for all $n$ if for all simple right comodules $L, L'$, $\mathrm{Ext}^1(L, L')$ is always finite-dimensional and for a fixed $L$ vanishes for all but finitely many $L'$ (up to isomorphism). For in this case the Ext quiver of the category of right comodules will be suitably finite.

The main theorem we prove is:

**Theorem C.1.1.** *Assuming (C1)-(C3-n), if $i \leq n$ then for simple right comodules $L, L'$ we have*
$$[\sigma_i(C)/\sigma_{i-1}(C) : L^* \boxtimes L'] = [\sigma_i(I(L))/\sigma_{i-1}(I(L)) : L'].$$
*In particular if (C3-n) holds for all $n$, then the above equality holds for all $i$.*

The following statement is clearly equivalent.

**Theorem C.1.2.** *Assuming (C1)-(C3-n), if $i \leq n$ then for simple right comodules $L, L'$ we have*
$$[\sigma_i(C) : L^* \boxtimes L'] = [\sigma_i(I(L)) : L'].$$
*In particular if (C3-n) holds for all $n$, then the above equality holds for all $i$.*

In the case of $i = 2$ we obtain a generalization of a corollary of the Taft-Wilson theorem for pointed coalgebras over a field.

**Corollary C.1.3.** *Assuming (C1)-(C2), if $L, L'$ are simple right comodules with* $\mathrm{Ext}^1(L, L')$ *is finite-dimensional then we have*
$$[\sigma_2(C)/\sigma_1(C) : L^* \boxtimes L'] = \dim \mathrm{Ext}^1(L', L).$$

[2]Thank you to Nicolás Andruskiewitsch for explaining why this is true.

The finiteness assumption in (C3-$n$) is clearly necessary in order to state the theorems. The assumptions (C1) and (C2) are necessary for obtaining a clear description of the simple bicomodules of $C$. If $A$ is a simple finite-dimensional $G$-graded algebra over an algebraically closed field of characteristic zero for a group $G$, and the center of $A$ contains a non-scalar element, then the assumptions (C1) and (C2) will fail for $C = A^*$ the dual coalgebra of $A$, as an object of $G$-graded vector spaces.

An outline of the appendix is as follows. In section 2 we state formal constructions related to coalgebras and comodules in tensor categories, with [17] being our main reference. In sections 3 we state basic results about the matrix coefficient morphism. Section 4 goes into the existence and structure of injective comodules, and section 5 explains the structure of the coalgebra as a right comodule. The statements and proofs of these results are known and go back to [20]. Finally section 6 examines the structure of $C$ as a bicomodule, concluding with theorem C.6.6.

## C.2 Setup and preliminaries

### C.2.1

We follow the definitions and terminology from [17]. Let $k$ be a field and $\mathcal{C}$ a semisimple pointed tensor category over $k$. In other words we assume:

1. $\mathcal{C}$ is a locally finite semisimple $k$-linear abelian category;
2. $\mathcal{C}$ is rigid monoidal such that $(-)\otimes(-)$ is a biexact bilinear bifunctor, and $\mathrm{End}(\mathbf{1}) \cong k$;
3. every simple object of $\mathcal{C}$ is invertible.

Such categories are always isomorphic (as monoidal categories) to a category $\mathsf{vec}(G, \omega)$, the category of finite-dimensional $G$-graded vector spaces (where $G$ is a group) with associativity isomorphism determined by the 3-cocycle $\omega \in Z^3(G, k^\times)$. Note that we do not assume $(\mathcal{C}, \otimes)$ is braided.

### C.2.2

For an object $V$ of $\mathcal{C}$, we write $V^*$ for its right dual, $\mathrm{ev}_V : V^* \otimes V \to \mathbf{1}$ for the evaluation morphism and $\mathrm{coev}_V : \mathbf{1} \to V \otimes V^*$ for the coevaluation morphism. If $W$ is a subobject of $V$, we write $W^\perp$ for the subobject of $V^*$ given by the kernel of the epimorphism $V^* \to W^*$. If $f : W \to V$ is an arbitrary morphism then we have a commutative diagram which will be used later on:

$$\begin{array}{ccc} W^* \otimes W & \xrightarrow{\mathrm{ev}_W} & \mathbf{1} \\ \uparrow & & \uparrow{\scriptstyle \mathrm{ev}_V} \\ V^* \otimes W & \longrightarrow & V^* \otimes V \end{array} \tag{C.2.1}$$

### C.2.3

We consider the cocomplete abelian category $\widehat{\mathcal{C}}$ constructed from $\mathcal{C}$, as described in [60]. Note that here if $\mathcal{C} \cong \mathsf{vec}(G, \omega)$, then $\widehat{\mathcal{C}} \cong \mathbf{Vec}(G, \omega)$ which is the category of $G$-graded vector spaces of arbitrary dimension. We have a fully faithful embedding $\mathcal{C} \to \widehat{\mathcal{C}}$ admitting the usual universal property. Further, in this case $\widehat{\mathcal{C}} \times \widehat{\mathcal{C}}$ is a cocomplete abelian category with a natural fully faithful functor $\mathcal{C} \times \mathcal{C} \to \widehat{\mathcal{C}} \times \widehat{\mathcal{C}}$ that satisfies the desired universal property. Thus in particular $\otimes$ extends to a biexact bilinear functor $\widehat{\mathcal{C}} \times \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ which we continue to write as $\otimes$ by abuse of notation.

### C.2.4

Let $C$ be a coalgebra object in $\widehat{\mathcal{C}}$. This means $C$ comes equipped with morphisms

$$\Delta : C \to C \otimes C \quad \text{and} \quad \epsilon : C \to \mathbf{1}$$

such that

$$(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta, \qquad (\epsilon \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \epsilon) \circ \Delta = \mathrm{id}_C .$$

By thinking of $\widehat{\mathcal{C}}$ as $\mathbf{Vec}(G, \omega)$, a standard argument shows that $\widehat{\mathcal{C}}$ is a direct limit of sub-coalgebras objects of $\mathcal{C}$.

### C.2.5

An object $V \in \widehat{\mathcal{C}}$ is said to be a right $C$-comodule (resp. left $C$-comodule) if it is equipped with a morphism $a_V = a : V \to V \otimes C$ (resp. $a_V = a : V \to C \otimes V$) such that

$$(a \otimes \mathrm{id}_C) \circ a = (\mathrm{id}_V \otimes \Delta) \circ a, \qquad (\text{resp. } (\mathrm{id}_C \otimes a) \circ a = (\Delta \otimes \mathrm{id}_V) \circ a)$$

and

$$(\mathrm{id}_V \otimes \epsilon) \circ a = \mathrm{id}_V, \qquad (\text{resp. } (\epsilon \otimes \mathrm{id}_V) \circ a = \mathrm{id}_V).$$

An object $V \in \widehat{\mathcal{C}}$ is a $C$-bicomodule if it is both a left and right comodule with comodule structure morphisms $a_{V,l}$ and $a_{V,r}$ such that $(\mathrm{id}_C \otimes a_{V,r}) \circ a_{V,l} = (a_{V,l} \otimes \mathrm{id}_C) \circ a_{V,r}$. Observe that $C$ is naturally a left and right comodule via $a_{C,r} = a_{C,l} = \Delta$ such that it obtains the structure of a $C$-bicomodule.

Again by a standard argument any $C$-(bi)comodule $V$ will be a sum of sub-(bi)comodule objects in $\mathcal{C}$. In particular, simple (bi)comodules are always objects of $\mathcal{C}$.

### C.2.6

Consider the category $\mathbf{Mod}_C$ (resp. ${}_C\mathbf{Mod}$) of right $C$-comodules (resp. left $C$-comodules) with morphisms between two objects $V, W$ being morphisms in $\widehat{\mathcal{C}}$ respecting comodule structure morphisms. Let ${}_C\mathsf{mod}$ (resp. $\mathsf{mod}_C$) denote the full subcategory of right $C$-comodules (resp. left $C$-comodules) in $\mathcal{C}$. We also have the categories ${}_C\mathbf{Mod}_C$ and ${}_C\mathsf{mod}_C$ of $C$-bicomodules in $\widehat{\mathcal{C}}$ and $\mathcal{C}$ respectively. By our assumption that $\otimes$ is biexact, these categories are all abelian. Further, ${}_C\mathsf{mod}$, $\mathsf{mod}_C$ and ${}_C\mathsf{mod}_C$ are locally finite, and thus the Jordan-Holder and Krull-Schmidt theorems are valid. The categories ${}_C\mathbf{Mod}$, $\mathbf{Mod}_C$ and ${}_C\mathbf{Mod}_C$ are cocomplete, and we have natural inclusion functors $\mathsf{mod}_C \to \mathbf{Mod}_C$, ${}_C\mathsf{mod} \to {}_C\mathbf{Mod}$, and ${}_C\mathsf{mod}_C \to {}_C\mathbf{Mod}_C$ that have the usual universal properties as cocompletions.

## C.2.7

Given a right (resp. left) $C$-comodule $V$ and an object $S \in \widehat{\mathcal{C}}$, we may construct a new right (resp. left) $C$-comodule $S \otimes V$ (resp. $V \otimes S$) with comodule morphism $a_{S\otimes V} = \mathrm{id}_S \otimes a_V$ (resp. $a_{V\otimes S} = a_V \otimes \mathrm{id}_S$). This defines an endofunctor of the categories $\mathbf{Mod}_C$ and ${}_C\mathbf{Mod}$, and it preserves $\mathsf{mod}_C$ and ${}_C\mathsf{mod}$ if $S$ is in $\mathcal{C}$. We observe that if $S$ is simple (and thus invertible) then this functor defines automorphisms of these abelian categories, and thus it takes simple comodules to simple comodules.

## C.2.8

Given a right $C$-comodule $V$ and left $C$-comodule $W$ we may construct a $C$-bicomodule $V \boxtimes W$ which is $V \otimes W$ as an object of $\widehat{\mathcal{C}}$ and has left and right comodule structures as described in C.2.7. This satisfies the necessary commutativity condition to be a bicomodule.

**Lemma C.2.1.** *Suppose that $V$ is a right $C$-comodule, $W$ a left $C$-comodule, and $S$ is an object of $\widehat{\mathcal{C}}$. Then we have a canonical isomorphism of bicomodules*

$$(W \otimes S) \boxtimes V \cong W \boxtimes (S \otimes V).$$

*Proof.* Indeed, the associativity isomorphism coming from the monoidal structure of $\widehat{\mathcal{C}}$ provides us with such an isomorphism. □

**Corollary C.2.2.** *With the same hypotheses as lemma C.2.1 and assuming that $S$ is a simple object of $\mathcal{C}$, we have a canonical isomorphism*

$$(W \otimes S^*) \boxtimes (S \otimes V) \cong W \boxtimes V.$$

*Proof.* Apply lemma C.2.1 and the invertibility isomorphism $S^* \otimes S \cong \mathbf{1}$. □

## C.2.9

Let $V$ be an object in $\mathsf{mod}_C$ and $V^*$ its right dual in $\mathcal{C}$. Then $V^*$ has the natural structure of a left $C$-comodule by

$$a_{V^*} = (\mathrm{ev}_V \otimes \mathrm{id}_C \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes a_V \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes \mathrm{coev}_V).$$

This construction is functorial, so that we have a contravariant functor $(-)^* : \mathsf{mod}_C \to {}_C\mathsf{mod}$. This functor is an antiequivalence with inverse taking the left dual of a comodule, $V \mapsto {}^*V$. We observe that if $W$ is a right subcomodule of $V$ then $W^\perp$ is naturally a left subcomodule of $V^*$.

# C.3 Matrix coefficients

## C.3.1

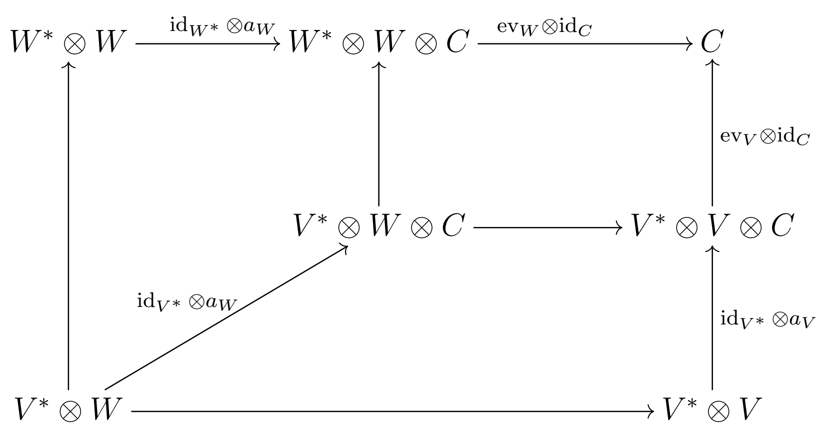
For this section, all objects are assumed to be in $\mathcal{C}$, i.e. they are of finite length. Given an object $V$ of $\mathsf{mod}_C$, by C.2.9 and C.2.8 we obtain a $C$-bicomodule given by $V^* \boxtimes V$. Define the matrix coefficients morphism $c_V : V^* \boxtimes V \to C$ by

$$c_V = (\mathrm{ev}_V \otimes \mathrm{id}_C) \circ (\mathrm{id}_{V^*} \otimes a_V) = (\mathrm{id}_C \otimes \mathrm{ev}_V) \circ (a_{V^*} \otimes \mathrm{id}_V).$$

**Lemma C.3.1.** *Suppose that $f : W \to V$ is a morphism of right $C$-comodules. Then we have the following commutative diagram:*

$$\begin{CD} V^* \boxtimes V @>{c_V}>> C \\ @AAA @AA{c_W}A \\ V^* \boxtimes W @>>> W^* \boxtimes W \end{CD}$$

*Proof.* Indeed, this follows from the commutativity of the following diagram:




The top left square is obviously commutative. The bottom right square is commutative because $f : W \to V$ is a morphism of comodules. The top right square is simply C.2.1 tensored with $C$, and thus is commutative. □

**Corollary C.3.2.** *Suppose that $W$ is a right sub-comodule of $V$. Then $W^\perp \boxtimes W \subseteq \ker c_V$.*

*Proof.* Clear from previous lemma. □

**Lemma C.3.3.** *Suppose that $V$ is a right $C$-comodule, with $W$ a sub-comodule of $V$ and $U$ a quotient of $V$. Then $\operatorname{Im} c_W$ and $\operatorname{Im} c_U$ are sub-bicomodules of $\operatorname{Im} c_V$.*

*Proof.* Apply the commutative squares obtained from C.3.1 for the two morphisms $W \to V$ and $V \to U$. □

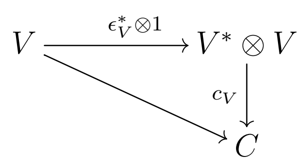
**Corollary C.3.4.** *If $V$ is a right $C$-comodule and $W$ is a subquotient of $V$ as a comodule, then $\operatorname{Im} c_W$ is a sub-bicomodule of $\operatorname{Im} c_V$.*

**Lemma C.3.5.** *Let $V$ be a right $C$-comodule, and suppose that $W_1, W_2$ are subcomodules such that $W_1 + W_2 = V$. Then $\operatorname{Im} c_V = \operatorname{Im} c_{W_1} + \operatorname{Im} c_{W_2}$. Similarly, if $U_1, U_2$ are quotients comodules of $V$ such that the map $V \to U_1 \oplus U_2$ is injective, then $\operatorname{Im} c_V = \operatorname{Im} c_{U_1} + \operatorname{Im} c_{U_2}$.*

*Proof.* We apply lemma C.3.3 to the epimorphism $W_1 \oplus W_2 \to V$ and monomorphism $V \to U_1 \oplus U_2$, and use C.3.2 to find that $c_{V_1 \oplus V_2}$ (resp. $c_{U_1 \oplus U_2}$) factors through $c_{V_1} \oplus c_{V_2}$ (resp. $c_{U_1} \oplus c_{U_2}$). □

### C.3.2

Given a finite-length right $C$-subcomodule $V$ of $C$, let $\epsilon_V : V \to \mathbf{1}$ be the restriction of $\epsilon$ to $V$ and $\epsilon_V^* : \mathbf{1} \to V^*$ its dual. Then the following is a commutative diagram of right $C$-comodules:




Thus $V$ is a right subcomodule of the image of $c_V$ in $C$. Since $C$ is the sum of its finite length right sub-comodules, it follows that $C = \sum \operatorname{Im} c_V$, where the sum runs over all right $C$-comodules in $\mathcal{C}$.

## C.4 Socle filtration and injectives

### C.4.1

The objects of $\mathbf{Mod}_C$, ${}_C\mathbf{Mod}$, and ${}_C\mathbf{Mod}_C$ admit socle filtrations. Using the same notation as Green in [20], we write $\sigma_i(V)$ for the $i$ term in the socle filtration of an object $V$. In this case we have that $V$ is the direct limit of its socle filtration. If the socle filtration of an object $V$ is finite (which happens in particular if $V$ is of finite-length, i.e. is in $\mathcal{C}$), then we write $\ell\ell(V)$ for the length of the socle filtration, the Loewy length of $V$. In this case, $\ell\ell(V)$ is the length of every minimal semisimple filtration of $V$. Further, then $V$ also has a radical filtration which is a descending filtration whose $i$th term we write as $\rho^i(V)$, and whose length is also $\ell\ell(V)$. Recall that $\rho^1(V) := \rho(V)$ is defined to be the minimal subcomodule of $V$ such that $V/\rho(V)$ is semisimple, and we define the filtration inductively by $\rho^i(V) = \rho(\rho^{i-1}(V))$.

**Lemma C.4.1.** *If $V$ is of finite length, then $\sigma_i(V)^\perp = \rho^i(V^*)$ and $\rho^i(V)^\perp = \sigma_i(V^*)$.*

*Proof.* Follows from the fact that dualizing is an antiequivalence of comodule categories. □

## C.4.2

The socle filtration on $C$ as a $C$-bicomodule is often called the coradical filtration of $C$, and is sometimes written $C_i := \sigma_i(C)$. The goal of this appendix is to give a description of the layers of the coradical filtration of $C$.

## C.4.3

Define the functor $F_C : \widehat{\mathcal{C}} \to \mathbf{Mod}_C$ by $F_C(S) = S \otimes C$. This is one of the endofunctors described in C.2.7.

**Lemma C.4.2.** *The functor $F_C$ is right adjoint to the forgetful functor $\mathbf{Mod}_C \to \widehat{\mathcal{C}}$.*

*Proof.* The proof follows the same ideas as in (1.5a) of [20]. □

## C.4.4

**Lemma C.4.3.** *The categories ${}_C\mathbf{Mod}$ and $\mathbf{Mod}_C$ have enough injectives.*

*Proof.* Given a right $C$-comodule $V$, $F_C(V)$ is injective by lemma C.4.2 and the morphism $a_V : V \to F_C(V)$ is a monomorphism of right $C$-comodules. □

**Lemma C.4.4.** *The direct sum of injective comodules is injective.*

*Proof.* The proof in (1.5b) of [20] carries through to our case. □

Given a right $C$-comodule $V$, an injective envelope of $V$ is the data of an injective right comodule $I$ with a monomorphism $V \to I$ which induces an isomorphism $\sigma(V) \xrightarrow{\sim} \sigma(I)$. An injective envelope is unique up to isomorphism if it exists, in the usual way. Using Brauer's idempotent lifting process as described in [20], we can prove that injective envelopes always exists. Choose for each simple right comodule $L$ an injective envelope $I(L)$. We now have:

**Corollary C.4.5.** *The indecomposable injective right comodules are exactly those of the form $I(L)$ for a simple right comodule $L$. Thus the injective right comodules are exactly the direct sums of injective indecomposables $I(L)$.*

## C.5 Structure of $C$ as a right comodule

We now make some assumptions on $C$ and its right comodule category.

**(C1)** We suppose that if $L$ is a simple right $C$-comodule and $S$ is a simple object of $\mathcal{C}$, then $L$ and $S \otimes L$ are not isomorphic as comodules unless $S \cong \mathbf{1}$.

**(C2)** We assume that if $L, L'$ are right $C$-comodules then $L^* \boxtimes L'$ is a simple bicomodule and further every simple bicomodule is of this form.

*Remark* C.5.1. Assumption (C2) implies that every semisimple bicomodule is semisimple as a right comodule. In particular, if $V$ is a bicomodule of finite length then its Loewy length as a right comodule is less than or equal to its Loewy length as a bicomodule.

Assumption (C1) is saying that the action of the Picard group of $\mathcal{C}$ on the set of simple comodules is free. Thus we may, and do, choose representatives of each orbit, $\{L_\alpha\}_\alpha$. In other words the simple right comodules $L_\alpha$ have the property that if $L_\alpha \cong S \otimes L_\beta$ for a simple object $S$ of $\mathcal{C}$ then $\alpha = \beta$ and $S \cong \mathbf{1}$, and further if $L$ is a simple right comodule then there exists an $\alpha$ and a simple object $S$ of $\mathcal{C}$ such that $L \cong S \otimes L_\alpha$.

**Lemma C.5.2.** *Every simple bicomodule may be written as $L_\alpha^* \boxtimes L$ for a unique $\alpha$ and simple right comodule $L$.*

*Proof.* By (C2) the simple bicomodules are all of the form $L^* \boxtimes L'$ for some simple right comodules $L, L'$. Choose $\alpha$ such that $L \cong S \otimes L_\alpha$. Then by lemma C.2.1 $L^* \boxtimes L' \cong (L_\alpha^* \otimes S) \boxtimes L' \cong L_\alpha^* \boxtimes (S \otimes L')$.

The proof of uniqueness of $L$ is a little trickier. We prove the following statement which implies it: if $L$ is a simple right comodule, $L'$ a simple left comodule, and $S$ is a simple object of $\mathcal{C}$, then if $(L' \otimes S) \boxtimes L \cong L' \boxtimes L$ then $S \cong \mathbf{1}$. Write $G = \mathrm{Pic}(\mathcal{C})$ for the Picard group of $\mathcal{C}$, that is the group of simple objects of $\mathcal{C}$ up to isomorphism under tensor product. For each $g \in G$, choose a representative simple object $S_g$, and let $h \in G$ be the class of $S$ so that $S \cong S_h$. Finally, write $\phi : (L' \otimes S) \boxtimes L \to L' \boxtimes L$ for a given isomorphism of bicomodules.

Now write as objects of $\mathcal{C}$ isotypic decompositions $L' = \bigoplus_g T_g$, where $T_g \cong S_g^{\oplus n_g}$ and $L = \bigoplus_g U_g$ where $U_g \cong S_g^{\oplus m_g}$. The our isomorphism $\phi$ of bicomodules gives rise to an isomorphism of right comodules

$$\bigoplus_g (T_g \otimes S) \otimes L \cong \bigoplus_g T_g \otimes L$$

and an isomorphism of left comodules

$$\bigoplus_g L' \otimes (S \otimes U_g) \cong \bigoplus_g L' \otimes U_g.$$

By (C1), this must induce isomorphisms of right comodules

$$(T_g \otimes S) \otimes L \cong T_{gh} \otimes L, \tag{C.5.1}$$

i.e. $\phi$ must take $(T_g \otimes S) \otimes L$ into $T_{gh} \otimes L$ for all $g \in G$, and similarly of left comodules

$$L' \otimes (S \otimes U_g) \cong L' \otimes U_{hg}, \tag{C.5.2}$$

i.e. $\phi$ must take $L' \otimes (S \otimes U_g)$ into $L' \otimes U_{hg}$ for all $g \in G$. However for $g, h, k \in G$, C.5.1 implies that $\phi$ induces an isomorphism

$$T_g \otimes S \otimes U_k \cong T_{gh} \otimes U_k$$

while C.5.2 implies that $\phi$ induces an isomorphism

$$T_{gh} \otimes S \otimes U_{h^{-1}k} \cong T_{gh} \otimes U_k.$$

It follows that we must have $T_g \otimes S \otimes U_k = T_{gh} \otimes S \otimes U_{h^{-1}k}$, i.e. $gh = g$ and $h^{-1}k = k$ i.e. $h$ must be the identity, and so $S \cong \mathbf{1}$ as desired.

□

**Lemma C.5.3.** *If $S$ is a simple object of $\mathcal{C}$ and $V$ is in $\mathsf{mod}_C$, then $\operatorname{Im} c_V = \operatorname{Im} c_{S \otimes V}$. In particular if $L$ is a simple right comodule and $L \cong S \otimes L_\alpha$, then $\operatorname{Im} c_L = \operatorname{Im} c_{L_\alpha}$.*

*Proof.* By corollary C.2.2 we have $V^* \boxtimes V \cong (S \otimes V)^* \boxtimes (S \otimes V)$, and this isomorphism of bicomodules respects the matrix coefficient morphisms. □

**Proposition C.5.4.** *We have $\sigma(C) := \sigma_1(C) = \bigoplus_\alpha L_\alpha^* \boxtimes L_\alpha$ as bicomodules.*

*Proof.* For each $\alpha$ we have a nonzero, and thus injective, morphism $c_{L_\alpha} : L_\alpha^* \boxtimes L_\alpha \to \sigma(C)$. Since the simple bicomodules $L_\alpha^* \boxtimes L_\alpha$, $L_\beta^* \boxtimes L_\beta$ are non-isomorphic for distinct $\alpha, \beta$ by lemma C.5.2, we obtain an inclusion $\bigoplus_\alpha L_\alpha^* \boxtimes L_\alpha \subseteq \sigma(C)$. Conversely, a simple subbicomodule $W$ of $C$ must be semisimple as a right $C$-comodule by remark C.5.1, and thus if $W = \bigoplus_i L_i$ for simple right comodules $L_i$ then $W \subseteq \sum_i \operatorname{Im} c_{L_i}$. Now lemma C.5.3 completes the proof. □

**Corollary C.5.5.** *We have an isomorphism of right comodules:*

$$C \cong \bigoplus_\alpha L_\alpha^* \otimes I(L_\alpha)$$

*Proof.* By proposition C.5.4 these right comodules have isomorphic socles. Since injectives are determined by their socles, we are done. □

## C.6 Layers of the coradical filtration

### C.6.1

We would like to prove that if $V$ is a finite-length right $C$-comodule then $\ell\ell(\operatorname{Im} c_V) = \ell\ell(V)$, i.e. the Loewy length of $\operatorname{Im} c_V$ as bicomodule is equal to the Loewy length of $V$ as a right comodule. First we prove a lemma.

**Lemma C.6.1.** *Suppose that $W$ is a right comodule of finite length with simple socle $L$. Choose a splitting $\widetilde{L} \subseteq W^*$ (in $\mathcal{C}$) of the epimorphism $W^* \to L^*$ so that we obtain a right subcomodule $\widetilde{L} \otimes W$ of $W^* \boxtimes W$. Then the restriction of $c_W$ to $\widetilde{L} \otimes W$ is injective.*

*Proof.* Since this restriction defines a morphism of right comodules $\widetilde{L} \otimes W \to C$, it suffices to show that it is injective on the socle $\sigma(\widetilde{L} \otimes W) = \widetilde{L} \otimes L$. However $\widetilde{L} \otimes L$ is a splitting of the head of the bicomodule $W^* \boxtimes L$, and the restriction of $c_W$ to $W^* \boxtimes L$ has $L^\perp \boxtimes L = \rho(W^* \boxtimes L)$ in its kernel by corollary C.3.2 , and thus factors through $(W^* \boxtimes L)/(L^\perp \boxtimes L) \cong L^* \boxtimes L \xrightarrow{c_L} C$. In summary we have a commutative diagram

$$
\begin{array}{ccccc}
\widetilde{L} \otimes L & \longrightarrow & W^* \otimes L & \longrightarrow & L^* \boxtimes L \\
& {\scriptstyle c_V|_{\widetilde{L} \otimes L}} \searrow & & \swarrow {\scriptstyle c_L} & \\
& & C & &
\end{array}
$$

Since the composition $\widetilde{L} \otimes L \to L^* \boxtimes L$ is an isomorphism and $c_L$ is injective we are done. □

**Lemma C.6.2.** *Let $V$ be a finite-length comodule with $\ell\ell(V) = n$. Then*

$$F_k = \sum_{i+j=k} \rho^{n-i}(V^*) \boxtimes \sigma^j(V) = \sum_{i+j=k} \sigma^{n-i}(V)^\perp \boxtimes \sigma^j(V)$$

*is a semisimple filtration of $V^* \boxtimes V$ such that $F_1 = 0$ and $F_{2n} = V^* \boxtimes V$.*

*Proof.* The tensor product of semisimple filtrations is again a semisimple filtration. □

We now observe that $F_n = \sum_i \sigma_i(V)^\perp \boxtimes \sigma_i(V) \subseteq \ker c_V$ and thus $F_\bullet$ induces a semisimple filtration of $\operatorname{Im} c_V$ of length at most $n = \ell\ell(V)$. It follows that $\ell\ell(\operatorname{Im} c_V) \leq \ell\ell(V)$.

On the other hand $V$ contains a subquotient $W$ with $\ell\ell(W) = \ell\ell(V)$ such that $W$ has a simple socle. Since $\operatorname{Im} c_W \subseteq \operatorname{Im} c_V$, if we can show that $\ell\ell(\operatorname{Im} c_W) \geq \ell\ell(W) = \ell\ell(V)$ then we will have that $\ell\ell(\operatorname{Im} c_V) = \ell\ell(V)$.

By lemma C.6.1 we know that $\operatorname{Im} c_W$ contains a right subcomodule of the form $\widetilde{L} \otimes W$ for an object $\widetilde{L}$ of $\mathcal{C}$, and thus its Loewy length as a right comodule is at least $\ell\ell(W)$, which by remark C.5.1 implies its Loewy length as a bicomodule is at least $\ell\ell(W)$. We have now finished showing:

**Proposition C.6.3.** *For a finite length right comodule $V$, $\ell\ell(V) = \ell\ell(\operatorname{Im} c_V)$.*

**Proposition C.6.4.** *We have*

$$\sigma_i(C) = \sum_{\ell\ell(V)\leq i} \operatorname{Im} c_V.$$

*Proof.* By proposition C.6.3, $\operatorname{Im} c_V$ has Loewy length equal to that of $V$, so if $\ell\ell(V) \leq i$ then $\operatorname{Im} c_V = \sigma_i(\operatorname{Im} c_V) \subseteq \sigma_i(C)$. Conversely if $V \subseteq \sigma_i(C)$ is a right sub-comodule then by remark C.5.1 $\ell\ell(V) \leq i$ and so $V \subseteq \operatorname{Im} c_V \subseteq \sigma_i(C)$. Since $\sigma_i(C)$ is the sum of its right subcomodules, we are done. □

## C.6.2

Fix $n \in \mathbb{N}$ with $n \geq 1$. We make a finiteness assumption on the comodule category $\mathsf{mod}_C$.

**(C3-$n$)** If $L, L'$ are simple right comodules, then $[L' : \sigma_n(I(L))] < \infty$.

Note that (C3-$n$) implies (C3-$m$) whenever $m \leq n$.

## C.6.3

For each pair of simple right comodules $L, L'$ and for each $i \leq n$ we define $H^i_{L',L}$ to be the right subcomodule of $\sigma_i(I(L'))$ that is generated by a splitting of the isotypic component of $L$ in $\sigma_i(I(L'))/\sigma_{i-1}(I(L'))$. In particular it is zero if and only if $[L : \sigma_i(I(L'))/\sigma_{i-1}(I(L'))] = 0$. By (C3-$n$), $H^i_{L',L}$ is a finite length right comodule. Further we have for $i \leq n$

$$\sigma_i(I(L')) = \sum_{L \text{ simple}} \sum_{j\leq i} H^j_{L',L}. \tag{C.6.1}$$

Write $H^i_{\alpha,L} := H^i_{L_\alpha,L}$.

**Lemma C.6.5.** *For $i \leq n$,*

$$\sigma_i(C) = \sum_{\alpha} \sum_{L \,\text{simple}} \sum_{j\leq i} \operatorname{Im} c_{H^j_{\alpha,L}}$$

*Proof.* Since $\ell\ell(H^j_{\alpha,L}) \leq j \leq i$, by proposition C.6.4 it suffices to show that $\operatorname{Im} c_V$ is contained in the RHS whenever $V$ is a right comodule of Loewy length less than or equal to $i$. However in this case $\operatorname{Im} c_V = \sum_W \operatorname{Im} c_W$ where the sum runs over quotients of $V$ with simple socles. Note that $\ell\ell(W) \leq \ell\ell(V) \leq i$ for all such $W$. On the other hand, if $W$ has a simple socle $L'$ then after potentially twisting $W$ by a simple object $S$ (which won't change $\operatorname{Im} c_W$) we may assume $L' \cong L_\alpha$ for some $\alpha$, and then $I(L_\alpha)$ is the injective envelope of $W$. If $\ell\ell(W) \leq i$ then $W \subseteq \sigma_i(I(L_\alpha))$ under an embedding of $W$ in $I(L_\alpha)$. Therefore by C.6.1,

$$W \subseteq \sum_{L \text{ simple}} \sum_{j\leq i} H^j_{\alpha,L}.$$

and so there exists finitely many simple right comodules $L_1, \ldots, L_n$ such that

$$W \subseteq \sum_{k,\ j \leq i} H^j_{\alpha, L_k}$$

and hence

$$\operatorname{Im} c_W \subseteq \sum_{k,\ j \leq i} \operatorname{Im}_{H^j_{\alpha, L_k}}.$$

□

## C.6.4

We may now state the main theorem.

**Theorem C.6.6.** *For $i \leq n$,*

$$[\sigma_i(C)/\sigma_{i-1}(C) : L^* \boxtimes L'] = [\sigma_i(I(L))/\sigma_{i-1}(I(L)) : L'].$$

*Proof.* The case of $i = 1$ is proposition C.5.4. If $n = 1$ then the theorem is proven.

Otherwise if $n > 1$ we consider the case $i > 1$. We use lemma C.6.5 and study the contribution of $\operatorname{Im} c_{H^i_{\alpha,L}}$ for a fixed simple comodule $L$. Write $V_i = H^i_{\alpha,L}$ and $V_{i-1} = \sigma_{i-1}(H^i_{\alpha,L})$ so that $V_i/V_{i-1}$ is a sum of copies of $L$. Consider the sub-bicomodule $W = V_i^* \boxtimes V_{i-1} + (V_i/L_\alpha)^* \boxtimes V_i$ of $V_i^* \boxtimes V_i$. We from the arguments of lemma C.3.1 that

$$c_{V_i}(W) \subseteq \operatorname{Im} c_{V_{i-1}} + \operatorname{Im} c_{V_i/L_\alpha},$$

and since $\ell\ell(V_{i-1}), \ell\ell(V_i/L) \leq i-1$, we find that $c_{V_i}(W) \subseteq \sigma_{i-1}(C)$. We have

$$(V_i^* \boxtimes V_i)/W \cong L_\alpha^* \boxtimes V_i/V_{i-1}$$

and so we have epimorphisms

$$L_\alpha^* \boxtimes V_i/V_{i-1} \cong V_i^* \boxtimes V_i/W \to \operatorname{Im} c_{V_i}/c_{V_i}(W) \to \operatorname{Im} c_{V_i}/(\sigma_{i-1}(C) \cap \operatorname{Im} c_{V_i}). \qquad (*)$$

We aim to show this composition (*) is in fact an isomorphism. To this end, choose a splitting $\widetilde{L_\alpha}$ of $V_i^* \to L_\alpha^*$ so that we get a right subcomodule $\widetilde{L_\alpha} \otimes V_i$ of $V_i^* \otimes V_i$. By lemma C.6.1, the restriction of $c_{V_i}$ to $\widetilde{L_\alpha} \otimes V_i$ will be injective. Further, as a right comodule we have

$$\sigma_{i-1}(\widetilde{L_\alpha} \otimes V_i) = \widetilde{L_\alpha} \otimes V_{i-1}.$$

Thus by remark C.5.1

$$\sigma_{i-1}(C) \cap c_{V_i}(\widetilde{L_\alpha} \otimes V_i) \subseteq c_{V_i}(\widetilde{L_\alpha} \otimes V_{i-1}).$$

Conversely $\widetilde{L_\alpha} \otimes V_{i-1} \subseteq W$ and therefore

$$c_{V_i}(\widetilde{L_\alpha} \otimes V_{i-1}) \subseteq \sigma_{i-1}(C) \cap c_{V_i}(\widetilde{L_\alpha} \otimes V_i)$$

which implies these are equal. It follows that we obtain an injection of right comodules

$$L_\alpha^* \otimes V_i/V_{i-1} \to \operatorname{Im} c_{V_i}/(\sigma_{i-1}(C) \cap \operatorname{Im} c_{V_i})$$

and so (*) is an isomorphism. What this shows is that the contribution of $\operatorname{Im} c_{H^i_{\alpha,L}}$ to $\sigma_i(C)/\sigma_{i-1}(C)$ is exactly $L_\alpha^* \boxtimes V_i/V_{i-1}$. By lemma C.5.2 it follows that

$$\sigma_i(C)/\sigma_{i-1}(C) = \bigoplus_\alpha L_\alpha^* \boxtimes \sigma_i(I(L_\alpha))/\sigma_{i-1}(I(L_\alpha)).$$

Thus we have proven the theorem whenever $L \cong L_\alpha$ for some $\alpha$. For the general case we write $L \cong S \otimes L_\alpha$ for some $\alpha$ and some simple object $S$ of $\mathcal{C}$ and derive the result using lemma C.2.1. □

We now obtain a generalization of the following corollary of the of the Taft-Wilson theorem for pointed coalgebras over a field.

**Corollary C.6.7.** *Assume (C1)-(C2) and that $L, L'$ are simple right comodules such that $\dim \operatorname{Ext}^1(L, L') < \infty$. Then*

$$[\sigma_2(C)/\sigma_1(C) : L^* \boxtimes L'] = \dim \operatorname{Ext}^1(L, L').$$