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Authors

Parrish, B
McCarthy, JM

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Identification of a Usable Six-Bar Linkage for Dimensional Synthesis

B. Parrish and J.M. McCarthy

Abstract In this paper we present an algorithm to determine if a six-bar linkage that has been designed as a constrained 3R chain using Burmester Theory is usable. A usable six-bar linkage is one that moves a workpiece smoothly through a given set of task positions with actuation by one joint parameter.

Our algorithm is a two-step process. First the linkage is assembled in each of the task positions and is verified to have the same assembly configuration. Next, a numerical solution of the linkage is tracked between each task position and the assembly is verified to lie on the same branch of one coupler curve. A six-bar linkage that passes this two-part test is usable.

An example using Mathematica demonstrates that this computation can be used to automatically evaluate a large number of design candidates.

Keywords Linkage synthesis • Dixon determinant • Branching defect • Six-bar linkage

1 Introduction

In this paper we present a method to determine if a six-bar linkage moves smoothly through five task positions. We assume the six-bar linkage has been designed as a constrained 3R chain using Burmester theory as described by Soh and McCarthy [1]. A linkage designed in this way may not reach the five task positions in the same linkage assembly configuration.

If the linkage assembly configuration changes at a task position or between task positions, the solution to the design equations has changed branches. Balli and

B. Parrish (✉) • J.M. McCarthy
University of California, Irvine, CA, USA
e-mail: bparrish@uci.edu; jmmccart@uci.edu

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Chand [2] provide an overview of linkage defects that include circuit and branching defects. A usable linkage moves a workpiece smoothly through a given set of task positions with actuation by one joint parameter, therefore a usable linkage must remain on a single branch of a single circuit of the coupler curve. We present a procedure for analyzing a six-bar linkage to ensure that it is usable.

Wang et al. [3] determines the branches of six-bar linkages using algebraic techniques involving the quadratic discriminant of the loop equations. Similarly several papers describe methods for determining all of the branches of a linkage using the four-bar coupler curve and the five-bar joint rotation space. See Ting et al. [4–7] and Dou and Ting [8]. Primrose et al. [9] studied the motion of the joints of the Watt I six-bar linkage and described branches mathematically by considering the range of motion limits. These results provide a more detailed study of the trajectories than are used for our analysis.

Our approach to evaluate the branching condition focuses on identifying the branch within the range of motion that reaches a set of desired task positions and all positions in between. Our approach utilizes the Dixon determinant and the Jacobian to characterize the linkage loops. The selection of the methodology is for the purpose of extending the procedure to Watt and Stephenson six-bar linkage types.

2 Dimensional Synthesis

The dimensional parameters of a planar six-bar linkage can be determined by a synthesis routine based on a set of five task positions. See McCarthy and Soh [10] and Perez and McCarthy [11]. Figure 1 shows an example of a five position task with the synthesized linkage in the first task position. Dimensional synthesis provides a linkage design that reaches the specified five task positions. However, the linkage design may not be useful because it does not move smoothly through the task positions.

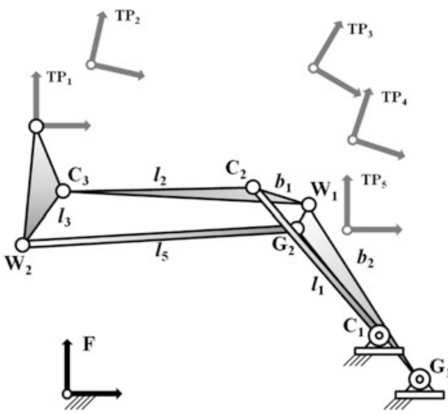

Fig. 1 Five position synthesis

Our goal is to determine if a linkage design is useful, we use a two-step process to determine if all five task positions lie on a single continuous branch. First, the linkage configuration at each of the five task positions is evaluated. If these configurations are the same, then we analyze the linkage to ensure that it does not change configuration between the task positions. This evaluation process can be embedded into a dimensional synthesis procedure similar to that proposed by Plecnik and McCarthy [12] in order to design useful six-bar linkages by iteratively selecting and evaluating task positions within given tolerance zones.

3 Analysis of a Six-Bar Linkage

The first step of our evaluation procedure determines the joint angles defining the six-bar linkage configuration at each task position. In the second step, the joint angles defining each possible linkage configuration are needed for arbitrary input angles.

A six-bar linkage has two loops that yield two vector loop equations [1]. This yields four component equations, which for the Watt I linkage shown in Fig. 2 are given by

$$\begin{aligned}
Loop_1 x: &\quad l_1\cos\theta_1 + b_1\cos(\theta_2-\gamma) - b_2\cos(\theta_4+\eta) - l_0 = 0,\\
Loop_1 y: &\quad l_1\sin\theta_1 + b_1\sin(\theta_2-\gamma) - b_2\sin(\theta_4+\eta) = 0,\\
Loop_2 x: &\quad l_1\cos\theta_1 + l_2\cos\theta_2 + l_3\cos\theta_3 - l_4\cos\theta_4 - l_5\cos\theta_5 - l_0 = 0,\\
Loop_2 y: &\quad l_1\sin\theta_1 + l_2\sin\theta_2 + l_3\sin\theta_3 - l_4\sin\theta_4 - l_5\sin\theta_5 = 0. \qquad (1)
\end{aligned}$$

We choose θ_1 as our input angle and solve for the remaining angles using the Dixon determinant method described by Wampler [13] and shown in detail in Soh et al. [14] and Soh and McCarthy [1]. We combine the sine and cosine terms into a

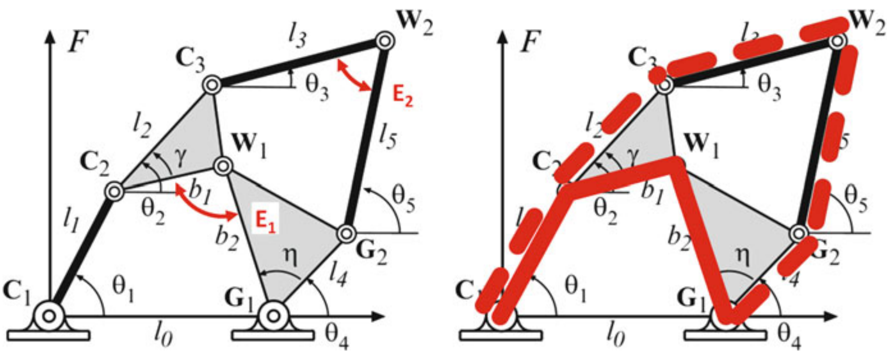

Fig. 2 Watt I Linkage and the two Dixon determinant loops

single variable by converting them to complex vectors. The y direction is taken to be along the imaginary axis so we multiply both y direction equations by i which is defined as $i^2 = -1$. Applying exponential identities and defining the complex variable $\Theta_j = e^{i\theta_j} = \cos\theta_j + i\sin\theta_j$ reduces the system to two equations. Taking the complex conjugate of each equation and defining the conjugate variable $\bar{\Theta}_j = e^{-i\theta_j} = \cos\theta_j - i\sin\theta_j$ produces four complex equations.

$$\begin{aligned}
F_1 &: \quad l_1\Theta_1 + b_1\Theta_2 e^{-i\gamma} - b_2\Theta_4 e^{i\eta} - l_0 = 0,\\
F_1^* &: \quad l_1\bar{\Theta}_1 + b_1\bar{\Theta}_2 e^{i\gamma} - b_2\bar{\Theta}_4 e^{-i\eta} - l_0 = 0,\\
F_2 &: \quad l_1\Theta_1 + l_2\Theta_2 + l_3\Theta_3 - l_4\Theta_4 - l_5\Theta_5 - l_0 = 0,\\
F_2^* &: \quad l_1\bar{\Theta}_1 + l_2\bar{\Theta}_2 + l_3\bar{\Theta}_3 - l_4\bar{\Theta}_4 - l_5\bar{\Theta}_5 - l_0 = 0.
\end{aligned} \tag{2}$$

We now construct the Dixon determinant such that the four constraint equations form the first row of the Dixon determinant. The remaining rows are the same equation except we sequentially replace $\Theta_2, \Theta_4, \Theta_5$ with $\alpha_2, \alpha_4, \alpha_5$.

$$\Delta = \begin{vmatrix}
F_1(\Theta_2,\Theta_4,\Theta_5) & F_1^*(\bar{\Theta}_2,\bar{\Theta}_4,\bar{\Theta}_5) & F_2(\Theta_2,\Theta_4,\Theta_5) & F_2^*(\bar{\Theta}_2,\bar{\Theta}_4,\bar{\Theta}_5)\\
F_1(\alpha_2,\Theta_4,\Theta_5) & F_1^*(\bar{\alpha}_2,\bar{\Theta}_4,\bar{\Theta}_5) & F_2(\alpha_2,\Theta_4,\Theta_5) & F_2^*(\bar{\alpha}_2,\bar{\Theta}_4,\bar{\Theta}_5)\\
F_1(\alpha_2,\alpha_4,\Theta_5) & F_1^*(\bar{\alpha}_2,\bar{\alpha}_4,\bar{\Theta}_5) & F_2(\alpha_2,\alpha_4,\Theta_5) & F_2^*(\bar{\alpha}_2,\bar{\alpha}_4,\bar{\Theta}_5)\\
F_1(\alpha_2,\alpha_4,\alpha_5) & F_1^*(\bar{\alpha}_2,\bar{\alpha}_4,\bar{\alpha}_5) & F_2(\alpha_2,\alpha_4,\alpha_5) & F_2^*(\bar{\alpha}_2,\bar{\alpha}_4,\bar{\alpha}_5)
\end{vmatrix} \tag{3}$$

Performing row operations subtracting Row 2 from Row 1, Row 3 from Row 2, and Row 4 from Row 3 cancels terms in Row 1 through Row 3 that do not contain Θ_2, Θ_4, Θ_5 and the associated α_2, α_4, α_5. Last, factoring out the extraneous roots where $\Theta_j = \alpha_j$ by applying the relationship $\Theta_j - \alpha_j = -\Theta_j\alpha_j(\bar{\Theta}_j - \bar{\alpha}_j)$ yields

$$\Delta = \begin{vmatrix}
-b_1e^{-i\gamma}\Theta_2\alpha_2 & b_1e^{i\gamma} & -l_2\Theta_2\alpha_2 & l_2\\
b_2e^{i\eta}\Theta_4\alpha_4 & -b_2e^{-i\eta} & l_4\Theta_4\alpha_4 & -l_4\\
0 & 0 & l_5\Theta_5\alpha_5 & -l_5\\
r_{41} & r_{42} & r_{43} & r_{44}
\end{vmatrix} = 0 \tag{4}$$

where

$$\begin{aligned}
r_{41} &= l_1\Theta_1 + b_1\alpha_2 e^{-i\gamma} - b_2\alpha_4 e^{i\eta} - l_0,\\
r_{42} &= l_1\bar{\Theta}_1 + b_1\bar{\alpha}_2 e^{i\gamma} - b_2\bar{\alpha}_4 e^{-i\eta} - l_0,\\
r_{43} &= l_1\Theta_1 + l_2\alpha_2 + l_3\Theta_3 - l_4\alpha_4 - l_5\alpha_5 - l_0,\\
r_{44} &= l_1\bar{\Theta}_1 + l_2\bar{\alpha}_2 + l_3\bar{\Theta}_3 - l_4\bar{\alpha}_4 - l_5\bar{\alpha}_5 - l_0.
\end{aligned} \tag{5}$$

Expand this determinant to obtain the polynomial

$$\delta = \mathbf{a}^T\,[W]\,\mathbf{t} \tag{6}$$

where

$$\mathbf{a} = (\alpha_2, \alpha_4, \alpha_5, \alpha_2\alpha_4, \alpha_4\alpha_5, \alpha_2\alpha_5)^T,$$
$$\mathbf{t} = (\Theta_2, \Theta_4, \Theta_5, \Theta_2\Theta_4, \Theta_4\Theta_5, \Theta_2\Theta_5)^T. \quad (7)$$

The values $\mathbf{t}$ that satisfy the loop equations (1) cause the Dixon determinant to be zero independent of the values of $\mathbf{a}$.

Thus, the configurations $\mathbf{t}$ of the six-bar satisfy

$$[W]\,\mathbf{t} = 0. \quad (8)$$

This expands to yield the generalized eigenvalue problem

$$[M\Theta_3 - N]\,\mathbf{t} = 0, \quad (9)$$

where

$$M = \begin{bmatrix} D_1 & 0 \\ A & -D_2^* \end{bmatrix}, \quad N = \begin{bmatrix} -D_2 & -A^T \\ 0 & D_1^* \end{bmatrix}. \quad (10)$$

The 3×3 matrices D_1, D_2, D_1^*, D_2^*, A, and A^T are constants defined by the linkage dimensions and the input angle Θ_1 and its conjugate $\bar{\Theta}_1$.

Solutions for $\mathbf{t}$ of this generalized eigenvalue problem yield the angles Θ_j, $j = 2, 3, 4, 5$ that define the configuration of the six-bar linkage. Equation (9) can have as many as six roots $\mathbf{t}_i$, $i = 1, \ldots, 6$, which means that for a given input value Θ_1, there can be as many as six assembly configurations for the six-bar linkage.

4 Step I: Evaluation of Branching at the Task Positions

We now use the loop equations to distinguish between the linkage assembly configurations. A usable linkage must retain a consistent assembly configuration.

Watt I Linkage: For a known input angle the Watt I has four assembly configurations which can be distinguished by the set of elbow angles, E_1 and E_2, as shown in Fig. 2. For convenience, an angle between 0 and π is called $+$ and an angle between π and 2π is called $-$. The available combinations are shown in Table 1. The Watt I shown in Fig. 2 is $++$. For our first evaluation criterion we calculate the set of elbow angles directly from the synthesis results for each task position and compare them.

Table 1 Elbow angle sets for the four Watt I linkage assembly configurations

Elbow angle set	$++$	$+-$	$--$	$-+$
E_1	$+$	$+$	$-$	$-$
E_2	$+$	$-$	$-$	$+$

Watt and Stephenson Six-bar Linkages: The linkage assembly configuration can be determined from the signs of the two Jacobian determinants constructed from the loop equations (1). First, note that Chase and Mirth [15] show the sign of the determinant of the Jacobian of the loop equations distinguishes between two assembly configurations of a four-bar linkage when that linkage is on one circuit. $Loop_1$ describes a four-bar with a Jacobian, J_1.

$$J_1 = \begin{bmatrix} -b_1 \sin(\theta_2 - \gamma) & b_2 \sin(\theta_4 + \eta) \\ b_1 \cos(\theta_2 - \gamma) & -b_2 \cos(\theta_4 + \eta) \end{bmatrix} \tag{11}$$

Similarly, the Jacobian of $Loop_2$, J_2, is

$$J_2 = \begin{bmatrix} 0 & -b_1 \sin(\theta_2 - \gamma) & b_2 \sin(\theta_4 + \eta) & 0 \\ 0 & b_1 \cos(\theta_2 - \gamma) & -b_2 \cos(\theta_4 + \eta) & 0 \\ -l_3 \sin\theta_3 & -l_2 \sin\theta_2 & l_4 \sin\theta_4 & l_5 \sin\theta_5 \\ l_3 \cos\theta_3 & l_2 \cos\theta_2 & -l_4 \cos\theta_4 & -l_5 \cos\theta_5 \end{bmatrix} \tag{12}$$

Notice that when the links located by $(\theta_2 - \gamma)$ and $(\theta_4 + \eta)$ are collinear, J_1 and J_2 are both singular. The columns of J_2 involving θ_3 and θ_5 are independent of the columns involving θ_2 and θ_4 for all angles. Therefore J_2 is also singular when the links located by θ_3 and θ_5 are collinear.

For these determinants to have different signs at the five task positions the Jacobian must be singular at a point between the task positions. For our first evaluation criterion we compare the sign of the determinant of the Jacobian for each of the loop equations at each task position. A consistent linkage configuration will have a consistent sign for each of these Jacobian determinants.

Both the Watt and the Stephenson six-bar linkages contain a four-bar sub-linkage which can be defined as $Loop_1$. Doing so results in two Jacobian matrices with similar singularity conditions, therefore this dual Jacobian approach is applicable to both the Watt and the Stephenson family of linkages.

5 Step II: Evaluation of Branching Between the Task Positions

A usable linkage moves smoothly through the five task positions. In order to determine the linkage trajectory between the task positions, we solve the loop equations using the Dixon determinant. We parameterize the input angle Θ_1, incrementally advance Θ_1 such that the linkage moves through the five task positions, and analyze the solution for branching defects at each increment.

For the Watt I linkage we evaluate the Dixon determinant solutions for the appropriate set of elbow angles E_1 and E_2 at each incrementally advanced input Θ_1. The elbow angle set determined in our first evaluation step must exist

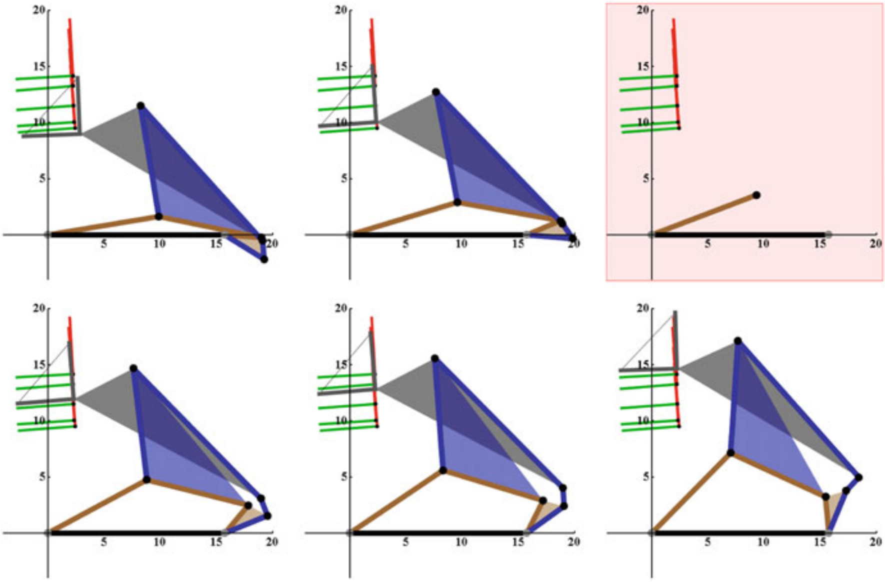

Fig. 3 Visualization of a Watt I moving through the task positions

simultaneously in one of the solutions of the Dixon determinant, otherwise, the linkage has branched at this input angle.

For the Watt and the Stephenson six-bar linkages we evaluate the Dixon determinant solutions for the appropriate sign of both Jacobian determinants. If the required Jacobian sign combination does not exist simultaneously in one of the solutions of the Dixon determinant, then the linkage has branched at this input angle.

A usable linkage has been found when branching does not exist for all of the parameterized input angles which achieve the five task positions and the positions in between. If however there exists a branch at any of these parameterized input angles, then the linkage is not usable.

The resolution of this evaluation approach is as fine as the step size we choose for the input angle. Modeled in Mathematica, Fig. 3 shows an example linkage moving through five task positions where the first image shows the linkage just before reaching the first task position. At the input angle of the third image, the linkage has branched.

6 Conclusions

In this paper we present a procedure to evaluate if a six-bar linkage that has been designed using Burmester-style dimensional synthesis has a branch defect. Our approach uses a two step procedure to determine if the linkage moves through the task positions in the same assembly configuration.

This procedure is vulnerable to some of special cases of the six-bar coupler curve described by Chase and Mirth [15], which means a final visual examination of the linkage is necessary. Numerical experiments for the Watt I linkage yielded no false positives for non-branching linkages. However, this is an area of research that needs further work.

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