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Abstract

It is well known that exact averaging of the equations of flow and transport in random porous media presently realized only for a small number of special, occasionally exotic, fields. On the other hand, the properties of approximate averaging methods are not yet fully understood. For example, the convergence behavior and the accuracy of truncated perturbation series are not well known. Furthermore, the calculation of the high-order perturbations is very complicated. These problems for a long time have stimulated attempts to find the answer for the question: Are there in existence some exact general and sufficiently universal forms of averaged equations? If the answer is positive, there arises the problem of the construction of these equations and analyzing them. There exist many publications related to these problems and oriented on different applications: hydrodynamics, flow and transport in porous media, theory of elasticity, acoustic and electromagnetic waves in random fields, etc. We present a method of finding some **general form of exactly averaged** equations for flow and transport in random fields by using (1) an assumption of the existence of Green's functions for appropriate stochastic problems, (2) some general properties of the Green's functions, and (3) the some basic information about the random fields of the conductivity, porosity and flow velocity. We present some general forms of the exactly averaged non-local equations for the following cases. 1. Steady-state flow with sources in porous media with random conductivity. 2. Transient flow with sources in compressible media with random conductivity and porosity. 3. Non-reactive solute transport in random porous media. We discuss the problem of uniqueness and the properties of the non-local averaged equations, for the cases with some types of symmetry (isotropic, transversal isotropic, orthotropic) and we analyze the structure of non-local equations in a general case of stochastically homogeneous fields.

1. INTRODUCTION

Recently the methods of analysis for the flow and transport in random media are finding ever-widening applications in science and technology involving various physical processes. It is possible to select an investigation strategy from the following three approaches.

Exact analytical approach: It is well known that the exact analytical averaging of the equations of flow and transport in random porous media turn out well be realized only for a small number of special, occasionally exotic, fields

Numerical approach: Numerically solve appropriate equations for representative sets of realizations of random fields. This approach is one sometimes called the Monte Carlo technique. The information obtained in this way makes it possible to find the highest moments together with computing the local and mean fields of pressure, velocity, etc. However, the exceptionally large volume of calculation and the difficulty for generalizing the results and finding the relations between the known and the unknown functionals restrict the significance of this approach for our goal.

Use of a perturbation technique: Every so often one can use the series expansion of small parameters, which specifies the deviation of some fields from their mean values. This approach usually utilizes analytical techniques. Although it is possible to find many results, it should be pointed out that it involves significant difficulties. Even in a problem of a comparatively simple structure one can usually find only the first few terms of expansions, because the analytical difficulties grow very quickly with the number of terms. Moreover, the convergence of the expansion is not studied. One approach utilizes the distinctive space scale for fast oscillating fields as a small parameter. This approach, so-called "homogenization", was largely developed for investigating processes with periodical structures. Many rigorous results were obtained that justify the method, although the computation of the results is highly laborious. For random structures, which is the focus of the present paper, some results have been obtained but the constructive theory is still absent.

Generally speaking, direct averaging and defining the functionals and the relations between them are exceptionally complicated. However, the fundamental information contained in the local equations and their structure has not been sufficiently utilized. Later we will show that investigation in this direction leads to, for example, finding the forms for the relations between average fields. We show that this is possible in some cases without actually solving appropriate equations but by presupposing only the existence of the solutions and using their general properties.

The following question has for a long time stimulated attempts to find the answer: Are there in existence some exact general and sufficiently universal forms of averaged equations for transport of mass, moment, energy, etc? If the answer is positive, then there arises a quest to construct the equations and to analyze them.

Many publications can be found related to this subject that discuss various applications. They include hydrodynamics, flow and transport in porous media, theory of elasticity, acoustic and electromagnetic waves in random fields (ex.: Batchelor,1953; Monin and Yaglom, 1965, 1967; Tatarsky, 1967; Saffman, 1971; Klyatskin, 1975 ,1980; Shermergor , 1979; Shvidler, 1985;Dagan, 1989; Bakhvalov and Panasenko, 1989; Zhikov et all, 1993; Neuman and Orr , 1993; Indelman and Abramovich, 1994; Indelman, 1996; Teodorovich, 1997; Shvidler and Karasaki, 1999).

We present a method of finding the general form of exactly averaged equations by using (1) an assumption of the existence of random Green's functions for appropriate stochastic problems, (2) some general properties of the Green`s functions, and (3) the information about the random fields of the conductivity, porosity and flow velocity. We present a general form of the exactly averaged non-local equations for the following cases: 1. Steady-state flow with sources in porous media with random conductivity, 2.

Transient flow with sources in compressible media with random conductivity and porosity, and 3. Non-reactive solute transport in random porous media. In this paper we discuss the properties of the non-local averaged equations. The case 1 is presented in detail and for other cases we present only the basic results.

We discuss the problem of uniqueness and the properties of the non-local averaged equations for the cases with some type of symmetry (isotropic, transversally isotropic and orthotropic). We also present and analyze the non-local equations in a general case of stochastically homogeneous fields.

2. STEADY-STATE FLOW WITH SOURCES

We consider the steady flow with sources in a heterogeneous porous unbounded domain. The condition of continuity is given by the equation:

$$\frac{\partial \; v_l(\mathbf{x})}{\partial \; x_l} = f(\mathbf{x}) \tag{1}$$

Here $\mathbf{x} = (x_1, x_2, x_3)$ is a 3-dimensional vector with components $x_l \; (l = 1, 2, 3)$, the function $f(\mathbf{x})$ is the density of sources and we assume that it is a locally integrable function, and $\mathbf{v}(\mathbf{x})$ is the Darcy‘s velocity vector.

The velocity and pressure (or hydraulic head) $u(\mathbf{x})$ obey the Darcy‘s Law

$$\mathbf{v}(\mathbf{x}) = -\boldsymbol{\sigma}(\mathbf{x})\nabla u(\mathbf{x}) \tag{2}$$

We assume that $\boldsymbol{\sigma}(\mathbf{x}) = \{\sigma_{lm}(\mathbf{x})\}$ is the second rank conductivity tensor symmetric by subscripts and is a positive definite and local tensor, i.e., for any $\boldsymbol{x}$ and vector $\boldsymbol{\xi}$, the elliptic condition $\xi_m \sigma_{lm}(\mathbf{x})\xi_l \geq \theta|\boldsymbol{\xi}|^2, \quad (\theta > 0)$ is satisfied.

In this case the unique and positive defined tensor $\mathbf{r}(\mathbf{x}) = \boldsymbol{\sigma}^{-1}(\mathbf{x})$ exist and we can write the conservative form of Darcy‘s Law as condition for momentum balance

$$\mathbf{r}(\mathbf{x})\mathbf{v}(\mathbf{x}) = -\nabla u(\mathbf{x}) \tag{3}$$

It is evidently from equations (2) and (3) that the fields $\mathbf{v}(\mathbf{x})$ and $\nabla u(\mathbf{x})$ are one-to-one.

For the pressure or head $u(\mathbf{x})$ we assume the condition:

$$u(\mathbf{x}) = 0 \;\text{ for } |\mathbf{x}| \to \infty \tag{4}$$

3. STOCHASTIC FORMULATION

We assume that the tensor $\boldsymbol{\sigma}(\mathbf{x})$ is a stochastically homogeneous random field. That is, for any vector $\mathbf{x}$ and for an arbitrary vector $\mathbf{h}$, all the finite-dimensional probability distributions for the random field $\boldsymbol{\sigma}(\mathbf{x}+\mathbf{h})$ doesn't depend on the arbitrary vector **h**. Let $f(\mathbf{x})$ be a non-random source density function. We introduce the random Green's function $g(\mathbf{x},\mathbf{y})$ for the problem described (1), (2) and (4), so that for almost all realizations of field $\boldsymbol{\sigma}(\mathbf{x})$ the function $g(\mathbf{x},\mathbf{y})$ satisfies the following equations:

$$\frac{\partial}{\partial x_l}\left[\sigma_{lm}(\mathbf{x})\frac{\partial g(\mathbf{x},\mathbf{y})}{\partial x_m}\right]=-\delta(\mathbf{x}-\mathbf{y}) \tag{5}$$

$$g(\mathbf{x},\mathbf{y})=0 \text{ for } |\mathbf{x}|\to\infty \tag{6}$$

In the general case we can now write the solution for the problem (1), (2) and (4):

$$u(\mathbf{x})=\int g(\mathbf{x},\mathbf{y})f(\mathbf{y})dy^3 \tag{7}$$

where $d\,y^3=d\,y_1\,d\,y_2\,d\,y_3$ and the integration is over the entire unbounded 3-D space.

We introduce the averaged fields over the ensemble of realizations of the random function $\boldsymbol{\sigma}(\mathbf{x})$:

$$U(\mathbf{x})=\langle u(\mathbf{x})\rangle,\ \mathbf{V}(\mathbf{x})=\langle \mathbf{v}(\mathbf{x})\rangle,\ G(\mathbf{x},\mathbf{y})=\langle g(\mathbf{x},\mathbf{y})\rangle \tag{8}$$

As long as $\boldsymbol{\sigma}(\mathbf{x})$ is a stochastically homogeneous field, the mean Green's function $G(\mathbf{x},\mathbf{y})$ is invariant over translation in space, and therefore, depends only on the difference $\mathbf{x}-\mathbf{y}$. Hence, after averaging the equation (7) over the ensemble, we have:

$$U(\mathbf{x})=\int G(\mathbf{x}-\mathbf{y})f(\mathbf{y})dy^3 \tag{9}$$

Then we can write the averaged equation over the ensembles of equation (1):

$$\frac{\partial V_l(\mathbf{x})}{\partial x_l}=f(\mathbf{x}) \tag{10}$$

After averaging the equation (2), we have:

$$V_l(\mathbf{x})=\int \Gamma_l(\mathbf{x}-\mathbf{y})\,f(\mathbf{y})dy^3 \tag{11}$$

where

$$\Gamma_l(\mathbf{x}-\mathbf{y})=-\left\langle \sigma_{lj}(\mathbf{x})\ \frac{\partial g(\mathbf{x},\mathbf{y})}{\partial x_j}\right\rangle \tag{12}$$

We shall call the vector $\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})$ the mean Green's velocity. By substituting (11) into equation (10) we find the relation of compatibility for the components $\Gamma_l(\mathbf{x}-\mathbf{y})$:

$$\frac{\partial \Gamma_l(\mathbf{x}-\mathbf{y})}{\partial x_l}=\delta(\mathbf{x}-\mathbf{y}) \tag{13}$$

Therefore, the mean pressure $U(\mathbf{x})$ and the mean velocity $V_l(\mathbf{x})$ are convolution integrals of the source density $f(\mathbf{x})$ and the mean Green's function $G(\mathbf{x}-\mathbf{y})$ and the vector-function $\mathbf{\Gamma}(\mathbf{x}-\mathbf{y})$, respectively.

The equations (9), (10) and (11) make up the closed system of equations for the averaged fields $U(\mathbf{x})$ and $\mathbf{V}(\mathbf{x})$. This system contains the kernels, $G(\mathbf{x}-\mathbf{y})$ and $\mathbf{\Gamma}(\mathbf{x}-\mathbf{y})$ that are non-random functional from the random conductivity field $\boldsymbol{\sigma}(\mathbf{x})$ and the random Green's function $g(\mathbf{x},\mathbf{y})$. Of course, the explicit definitions of the functionals G and $\mathbf{\Gamma}$ are very difficult to obtain in the general case (for any random field $\boldsymbol{\sigma}(\mathbf{x})$). For now the existence of these functionals in itself is significant. There is a possibility to determine some of their features that help find a general form of the averaged equations, of which the equations (9), (10) and (11) are a part of. Later we will find them in different forms.

4. FOURIER ANALYSIS

We consider the Fourier transform T_F and inverse Fourier transform T_F^{-1} for analyzing the equations with convolutions in all space.

$$T_F\left[\varphi(\mathbf{x})\right]=\bar{\varphi}(\mathbf{k})=\int \exp\left[-2\pi j(\mathbf{x}\cdot\mathbf{k})\right]\varphi(\mathbf{x})dx^3 \tag{14}$$

$$T_F^{-1}\left[\bar{\varphi}(\mathbf{k})\right]=\varphi(\mathbf{x})=\int \exp\left[2\pi j(\mathbf{k}\cdot\mathbf{x})\right]\bar{\varphi}(\mathbf{k})dk^3 \tag{15}$$

where $j=\sqrt{-1}$.

Applying T_F to equations (9), (10), (11) and (13) we have the linear algebraic equations in **k**-space:

$$\bar{U}(\mathbf{k})=\bar{G}(\mathbf{k})\bar{f}(\mathbf{k}) \tag{16}$$

$$2\pi j k_l \bar{V}_l(\mathbf{k})=\bar{f}(\mathbf{k}) \tag{17}$$

$$\bar{V}_l(\mathbf{k})=\bar{\Gamma}_l(\mathbf{k})\bar{f}(\mathbf{k}) \tag{18}$$

$$2\pi j k_l \bar{\Gamma}_l(\mathbf{k})=1 \tag{19}$$

The equations (16)-(18) are a closed system with respect to functions $\bar{U}(\mathbf{k})$ and $\bar{V}_l(\mathbf{k})$.

After eliminating $\bar{f}(\mathbf{k})$ from the equations (16) and (18) we find the equation that bind the scalar field $\bar{U}(\mathbf{k})$ and the vector field $\mathbf{V}(\mathbf{k})$:

$$\bar{V}_l(\mathbf{k})=\bar{\Pi}_l(\mathbf{k})\;\bar{U}(\mathbf{k}) \tag{20}$$

$$\bar{\Pi}_l(\mathbf{k})=\bar{\Gamma}_l(\mathbf{k})\left[\bar{G}(\mathbf{k})\right]^{-1} \tag{21}$$

From (18) and (21) we obtain the condition of compatibility for the components vector $\bar{\Pi}(\mathbf{k})$:

$$2\pi i k_l \bar{\Pi}_l(\mathbf{k}) = \left[\bar{G}(\mathbf{k})\right]^{-1} \tag{22}$$

The scalar function $G(\mathbf{x}-\mathbf{y}) = G(\mathbf{y}-\mathbf{x})$ and the vector-function $\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y}) = -\boldsymbol{\Gamma}(\mathbf{y}-\mathbf{x})$ and their Fourier transformations $\bar{G}(\mathbf{k})$ and $\bar{\Gamma}(\mathbf{k})$ are real-even and imaginary-odd functions of k, respectively.

The equations (16) and (20) are also a closed system with respect to functions $\bar{U}(\mathbf{k})$ and $\bar{V}_l(\mathbf{k})$. By applying the T_F^{-1} transform to (20), we can write the convolution equation:

$$V_l(\mathbf{x}) = -\int \Pi_l(\mathbf{x}-\mathbf{y}) U(\mathbf{y}) dy^3 \tag{23}$$

Thus the equation (23) connect $\mathbf{V}(\mathbf{x})$, the mean velocity at point *x*, and the distribution of $U(\mathbf{y})$ in all space with the help of the vector-operator $\boldsymbol{\Pi}(\mathbf{x}-\mathbf{y})$:

$$\Pi_l(\mathbf{x}) = \int \Gamma_l(\mathbf{x}-\mathbf{y}) Q(\mathbf{y}) dy^3 \ , \ Q(\mathbf{x}) = \mathrm{T}_F^{-1}\left\{\left[G(\mathbf{k})\right]^{-1}\right\} \tag{24}$$

The function $Q(\mathbf{x})$ is an even and real. It is obvious that scalar-operator $\Pi_l(\mathbf{x})$ is an odd and real. Obviously, the kernel-vector $\boldsymbol{\Pi}(\mathbf{x})$ is a non-random mapping for the random field $\boldsymbol{\sigma}(\mathbf{x})$ and does not depend on the source density, $f(\mathbf{x})$. If the vector field of the mean velocity $\mathbf{V}(\mathbf{x})$ and the scalar field $U(\mathbf{x})$ describe one averaged flow process in one unbounded stochastic homogeneous system, the kernel-vector $\boldsymbol{\Pi}(\mathbf{x}-\mathbf{y})$ is the **unique** operator that realizes (describes) in *x*-space the linear relation between them in the form of (23). In *k*-space it becomes the vector-function $\bar{\boldsymbol{\Pi}}(\mathbf{k})$.

It easy to see that the exact averaged equation (20) is reversible. If we know the scalar field $\bar{U}(\mathbf{k})$, from (20) we can directly define the vector field $\bar{\mathbf{V}}(\mathbf{k})$ and vice versa. If we know the field $\bar{V}_l(\mathbf{k})$, we can write $\bar{U}(\mathbf{k}) = \bar{V}_l(\mathbf{k}) / \bar{\Pi}_l(\mathbf{k})$ for any *l* using (20). Note that in the singular case of non-random conductivity $\sigma_{lm}(\mathbf{x}) = \sigma\delta_{lm}$, where $\sigma = const$, δ_{lm} is the Kronecker symbol, the $G(\mathbf{k})$ and $\bar{\Gamma}_l(\mathbf{k})$, the vector $\bar{\Pi}_l(\mathbf{k})$ and kernel-vector $\Pi_l(\mathbf{x}-\mathbf{y})$ are:

$$\bar{G}(\mathbf{k}) = \left(4\sigma\pi^2 k^2\right)^{-1}, \bar{\Gamma}_l(\mathbf{k}) = -\sigma\, jk_l / 2\pi k^2, \tag{25}$$

$$\bar{\Pi}_l(\mathbf{k}) = -2\pi\sigma\, jk_l \ , \ \Pi_l(\mathbf{x}-\mathbf{y}) = -\sigma \frac{\partial \delta(\mathbf{x}-\mathbf{y})}{\partial x_l} \tag{26}$$

It is interesting to compare the approaches for solving the direct and inverse problems when we use the local description of flow introduced with the system of equations (1)-(4) and the averaged description presented above with the system of equations (16)-(19) or (20)-(22).

It is evident that when we know the non-random density-function $f(\mathbf{x})$ and a second rank random tensor-function $\boldsymbol{\sigma}(\mathbf{x})$, we can find a unique scalar field $u(\mathbf{x})$ and vector-field $\mathbf{v}(\mathbf{x})$ for almost all realizations. The inverse problem of finding the density-function $f(\mathbf{x})$ and the tensor $\boldsymbol{\sigma}(\mathbf{x})$ is more complicated and requires a special approach. If we know some velocity-field $\mathbf{v}_1(\mathbf{x})$, we can compute the function $f_1(\mathbf{x})$ from the equation (1). The tensor $\boldsymbol{\sigma}(\mathbf{x})$ is symmetric and we have nine unknown components in three-dimensional space and three conditions of symmetry $\sigma_{lm}(\mathbf{x})=\sigma_{ml}(\mathbf{x})$. Both fields $u_1(\mathbf{x})$ and $\mathbf{v}_1(\mathbf{x})$ depend on the same density- function $f_1(\mathbf{x})$, which makes it possible to use the *Darcy's* Law in (2) to obtain a system of three scalar linear equations. Each of them contains three unknown components. Thus we have an underdetermined system of 6 linrar algebraic equations with nine unknown components. Obviously that if we use two linearly independent pairs, $\{u_1(\mathbf{x}),\mathbf{v}_1(\mathbf{x})\}$ and $\{u_2(\mathbf{x}),\mathbf{v}_2(\mathbf{x})\}$, we can add to the system three independent equations, that are *Darcy's* Law for the second pair of fields. In this case we have a closed system of nine equations for nine components. Since the local fields $u_i(\mathbf{x})$ and $\mathbf{v}_i(\mathbf{x})$ depend on the density-function $f_i(\mathbf{x})$ in all x-space, in order for the pairs $\{u_i(\mathbf{x}),\mathbf{v}_i(\mathbf{x})\}$ to be linearly independent, the functions $f_i(\mathbf{x})$ must be linearly independent.

When analyzing the averaged description, the direct problem is to define the fields $\bar{U}(\mathbf{k})$ and $\bar{\mathbf{V}}(\mathbf{k})$ under a known scalar function $\bar{f}(\mathbf{k})$ and a **vector**-function $\bar{\mathbf{\Pi}}(\mathbf{k})$. It is evident that we can find from equation (22) the function $\bar{G}(\mathbf{k})$ and then find the field $\bar{U}(\mathbf{k})$ from equation (16). At last we find the field $\bar{\mathbf{V}}(\mathbf{k})$ from equation (20). Thus the direct problem is fully defined. Remember that to fully define the direct local problem we need to use the scalar function $f(\mathbf{x})$ and the **tensor**-function $\sigma_{lm}(\mathbf{x})$.

The inverse problem under the averaged descriptions is to define the scalar function $\bar{f}(\mathbf{k})$ and vector-function $\bar{\mathbf{\Pi}}(\mathbf{k})$. It is evident that if we know the scalar-field $\bar{U}(\mathbf{k})$ and the vector-field $\bar{\mathbf{V}}(\mathbf{k})$, the appropriate function $\bar{f}(\mathbf{k})$ can be found from the equation (17) and vector $\bar{\mathbf{\Pi}}(\mathbf{k})$ from equation (20). Note that if we only know the vector-field $\bar{\mathbf{V}}(\mathbf{k})$, we can find the function $\bar{f}(\mathbf{k})$ only. If in addition we know the scalar-function $\bar{G}(\mathbf{k})$, we can find the field $\bar{U}(\mathbf{k})$ from equation (16) after computing $\bar{f}(\mathbf{k})$.

It is easy to see that if we only know the fields $\bar{U}(\mathbf{k})$ and $\bar{G}(\mathbf{k})$, we can find the function $\bar{f}(\mathbf{k})$ only. Thus, in contrary to a local model, the inverse problem is full defined if we know one pair of fields $\{\bar{U}(\mathbf{k}),\bar{\mathbf{V}}(\mathbf{k})\}$.

Notice that the equations for an averaged steady-state flow connect non-random functionals of random fields and thus are not as much detailed as the local models. Similar to any variant of upscaling we lose some information about flow, but in return, we have simpler tools to study the important property of the process. As we show here, instead of the second rank tensor $\boldsymbol{\sigma}(\mathbf{x})$-the random media characteristic in local model, it is sufficient to use the first rank tensor, i.e., the vector - $\bar{\boldsymbol{\Pi}}(\mathbf{k})$ for the description the averaged model.

5. GLOBAL SYMMETRY

We continue the analysis of the averaged equations and assume that the random field $\boldsymbol{\sigma}(\mathbf{x})$ satisfies some symmetry conditions that are related to the structural properties of the field as a whole. We shall call this type of symmetry global.

ISOTROPY: Let the random conductivity tensor $\boldsymbol{\sigma}(\mathbf{x})$ be an isotropic field. In this case the imaginary vector $\bar{\boldsymbol{\Pi}}(\mathbf{k})$ in any orthogonal coordinate system is proportional to the vector $2\pi j\mathbf{k}$. It is invariant for any rotation and reflection on the coordinate planes $k_l = 0$ and the proportional coefficient depends on $|\mathbf{k}|$ only. We can write

$$\bar{\Pi}_l^{(i)}(\mathbf{k}) = -\bar{\Pi}_{*l}^{(i)}(|\mathbf{k}|)2\pi jk_l \tag{27}$$

where $\Pi_{*l}^{(i)}(|\mathbf{k}|)$ is a scalar and positive even function, such that $\bar{\Pi}_{*1}^{(i)} = \bar{\Pi}_{*2}^{(i)} = \bar{\Pi}_{*3}^{(i)} = \bar{\Pi}_{*}^{(i)}$.

Then

$$\bar{V}_l(\mathbf{k}) = -\mathrm{B}_{lm}^{(i)}(\mathbf{k})2\pi jk_m\bar{U}(\mathbf{k}) \ , \ \bar{\mathrm{B}}_{lm}^{(i)}(\mathbf{k}) = \Pi_{*}^{(i)}(|\mathbf{k}|)\delta_{lm} \tag{28}$$

and therefore, in *x*-space we have the relations:

$$V_l(\mathbf{x}) = -\int \mathrm{B}_{lm}^{(i)}(|\mathbf{x}-\mathbf{y}|)\frac{\partial U(\mathbf{y})}{\partial y_m}dy^3 \ \ , \ \ V_l(\mathbf{x}) = -\int \Pi_{*}^{(i)}(|\mathbf{x}-\mathbf{y}|)\frac{\partial U(\mathbf{y})}{\partial y_l}dy^3 \tag{29}$$

Here $\mathbf{B}^{(i)}(|\mathbf{x}|)$ is unique spherical tensor and $\Pi_{*}^{(i)}(|\mathbf{x}|)$is unique scalar function.

It is evident that the equation (29) is revesible and we can write

$$\bar{R}_{ml}^{(i)}(\mathbf{k})\bar{V}_l(\mathbf{k}) = -2\pi jk_m\bar{U}(\mathbf{k}) \ , \ \bar{R}_{ml}^{(i)}(\mathbf{k}) = \left[\Pi_{*}^{(i)}(|\mathbf{k}|)\right]\delta_{ml} \tag{30}$$

In x- space we have the non-local with unique kernel averaged condition of the momentum balance

$$\int R_{ml}^{(i)}\left(\left|\mathbf{x}-\mathbf{y}\right|\right)V_l\left(\mathbf{y}\right)dy^3=-\frac{\partial U\left(\mathbf{x}\right)}{\partial x_m} \tag{31}$$

Here the isotropic resistance tensor is $\mathbf{R}\left(\left|\mathbf{x}\right|\right)=T_F^{-1}\left[\bar{\mathbf{R}}\left(\left|\mathbf{k}\right|\right)\right]$.

ORTHOTROPY: If the field $\boldsymbol{\sigma}\left(\mathbf{x}\right)$ is orthotropic then there exists some orthogonal coordinate system such all the stochastic multipoint moments of the random field are invariant to the reflection in the planes $k_l=0$. In this case the components $\bar{\Pi}_l\left(\mathbf{k}\right)$ can be written as

$$\bar{\Pi}_l^{(o)}\left(\mathbf{k}\right)=-\bar{\Pi}_{*l}^{(o)}\left(\mathbf{k}\right)2\pi jk_l \tag{32}$$

where summation over l is not implied.

The functions $\bar{\Pi}_{*l}^{(o)}\left(\mathbf{k}\right)$ are positive and even of **k,** which depends on $\left|k_1\right|,\left|k_2\right|,\left|k_3\right|$. In an orthotropic system the averaged equations are in the forms:

$$\bar{V}_l\left(\mathbf{k}\right)=-\bar{\Pi}_{*l}^{(o)}\left(\mathbf{k}\right)2\pi jk_l\bar{U}\left(\mathbf{k}\right) \tag{33}$$

(no summation over *l!*)

$$\bar{V}_l\left(\mathbf{k}\right)=-\bar{\mathrm{B}}_{ml}^{(o)}\left(\mathbf{k}\right)2\pi jk_m\bar{U}\left(\mathbf{k}\right) \tag{34}$$

were the components tensor $\bar{\mathbf{B}}^{(o)}\left(\mathbf{k}\right)$ takes the form:

$$\bar{\mathrm{B}}_{ml}^{(o)}\left(\mathbf{k}\right)=\delta_{ml}\bar{\Pi}_{*m}^{(o)}\left(\mathbf{k}\right) \tag{35}$$

(no summation over m!)

which means that the tensor $\bar{\mathbf{B}}^{(o)}$ is diagonal.

In x-space in corresponding coordinate system we have non-local equations with unigue kernels

$$V_l\left(\mathbf{x}\right)=-\int\Pi_{*l}^{(o)}\left(\mathbf{x}-\mathbf{y}\right)\frac{\partial U\left(\mathbf{y}\right)}{\partial y_l}dy^3\ ,\ V_l\left(\mathbf{x}\right)=-\int B_{lm}^{(o)}\left(\mathbf{x}-\mathbf{y}\right)\frac{\partial U\left(\mathbf{y}\right)}{\partial y_m}dy^3 \tag{36}$$

Evidently the equation (35) is reversible and the averaged equation has the form:

$$\bar{R}_{ml}^{(o)}\left(\mathbf{k}\right)\bar{V}_l\left(\mathbf{k}\right)=-2\pi k_m\bar{U}\left(\mathbf{k}\right) \tag{37}$$

where $\bar{\mathbf{R}}^{(o)}\left(\mathbf{k}\right)=\left[\bar{\mathbf{B}}^{(o)}\left(\mathbf{k}\right)\right]^{-1}$ is the diagonal orthotropic tensor of resistance.

In x-space in corresponding coordinate system we have non-local with unique kernel averaged condition of the momentum balance

$$\int R_{ml}^{(o)}\left(\mathbf{x}-\mathbf{y}\right)V_l\left(\mathbf{y}\right)dy^3=-\frac{\partial U\left(\mathbf{x}\right)}{\partial x_m} \tag{38}$$

TRANSVERSAL ISOTROPY: In the case of transversal isotropy, it are invariants relative to the rotation around one axes of coordinate system, for example, k_3, and reflection on any plates $k_l = 0$, we have $\bar{\Pi}_{*1}^{(t)} = \bar{\Pi}_{*2}^{(t)} \neq \bar{\Pi}_{*3}^{(t)}$, where the scalar functions $\bar{\Pi}_{*l}^{(t)}(\mathbf{k})$ stays invariant over rotation and reflections. In this case of symmetry,

$$\bar{\Pi}_{l}^{(t)}(\mathbf{k}) = -\bar{\Pi}_{*l}^{(t)}(\mathbf{k}) 2\pi jk_l \tag{39}$$

(no summation over *l*!)

The function $\bar{\Pi}_{*l}^{(t)}(\mathbf{k})$ is positive and even w.r.t. **k** that depends on $(k_1^2 + k_2^2)^{1/2}$, $|k_1|$, $|k_2|$, and $|k_3|$. In the transversal isotropic system the averaged equations are:

$$\bar{V}_l(\mathbf{k}) = -\bar{\Pi}_{*l}^{(t)}(\mathbf{k}) 2\pi jk_l \bar{U}(\mathbf{k}) \tag{40}$$

(no summation over l!)

$$\bar{V}_l(\mathbf{k}) = -\bar{\mathrm{B}}_{lm}^{(t)}(\mathbf{k}) 2\pi jk_m \bar{U}(\mathbf{k}) \tag{41}$$

where the components of the tensor $\bar{\mathbf{B}}^{(t)}$ are:

$$\bar{\mathrm{B}}_{lm}^{(t)}(\mathbf{k}) = \delta_{lm} \bar{\Pi}_{*l}^{(t)}(\mathbf{k}) \tag{42}$$

It easy see that the tensor $\bar{\mathbf{B}}^{(t)}$ is diagonal and $\bar{\mathrm{B}}_{11}^{(t)}(\mathbf{k}) = \bar{\mathrm{B}}_{22}^{(t)}(\mathbf{k}) \neq \bar{\mathrm{B}}_{33}^{(t)}(\mathbf{k})$. In x-space in corresponding coordinate system we have non-local with unique kernels averaged equations

$$V_l(\mathbf{x}) = -\int \Pi_{*l}^{(t)}(\mathbf{x}-\mathbf{y}) \frac{\partial U(\mathbf{y})}{\partial y_l} dy^3 \quad , \; V_l(\mathbf{x}) = -\int B_{lm}^{(t)}(\mathbf{x}-\mathbf{y}) \frac{\partial U(\mathbf{y})}{\partial y_m} dy^3 \tag{43}$$

The averaged equation (40) is reversible and has the form

$$R_{ml}^{(t)}(\mathbf{k}) V_l(\mathbf{k}) = -2\pi k_m \bar{U}(\mathbf{k}) \tag{44}$$

where $\mathbf{R}^{(t)}(\mathbf{k}) = \left[\mathbf{B}^{(t)}(\mathbf{k})\right]^{-1}$ is the diagonal transversal tensor of resistance.

In x-space in corresponding coordinate system we have the non-local with unique kernel condition of the momentum balance

$$\int R_{ml}^{(t)}(\mathbf{x}-\mathbf{y}) V_l(\mathbf{y}) dy^3 = -\frac{\partial U(\mathbf{x})}{\partial y_m} \tag{45}$$

In summary, for any orthogonal coordinate systems in the case of isotropy the averaged equation is reversible and the tensors $\bar{\mathbf{B}}^{(i)}(k)$ and $\mathbf{R}^{(i)}(\mathbf{k})$ are spherical. In the case of transversal isotropy if the orthogonal coordinate system is oriented so that one of the axes, for example, k_3 coincides with the axis of rotation, and the other two are oriented arbitrarily, the averaged equation is reversible, the tensor $\overline{\mathbf{B}}^{(t)}(\mathbf{k})$ and $\mathbf{R}^{(t)}(\mathbf{k})$ are diagonal, and $\overline{\mathrm{B}}_{11}^{(t)}(\mathbf{k})=\overline{\mathrm{B}}_{22}^{(t)}(\mathbf{k})\neq\overline{\mathrm{B}}_{33}^{(t)}(\mathbf{k})$. Finally, in the case of orthotropy, if the axes of the orthogonal coordinate system are the orthotropy axes, the averaged equation, too, is reversible and the tensor $\overline{\mathbf{B}}^{(o)}(\mathbf{k})$ and $\mathbf{R}^{(o)}(\mathbf{k})$ are diagonal. However, it is well to bear in mind that in each of the studied cases of symmetry, the components of the tensors $\overline{\mathrm{B}}^{(\alpha)}(k)$, where $\alpha=i,t,o$, remain invariant related to the superscript α. Therefore, in all three basic cases of symmetry $(\alpha=i,t,o)$ with a suitable orientation of the coordinate axes, the averaged equation is

$$\overline{V}_l^{(\alpha)}(\mathbf{k})=-\overline{\mathrm{B}}_{lm}^{(\alpha)}(\mathbf{k})2\pi jk_m\overline{U}(\mathbf{k}) \tag{46}$$

where

$$\overline{\mathrm{B}}_{lm}^{(\alpha)}(\mathbf{k})=\delta_{lm}\overline{\Pi}_{*m}^{(\alpha)}(\mathbf{k})\ ,\ \overline{R}_{ml}^{(\alpha)}(\mathbf{k})=\delta_{ml}\left[\overline{\Pi}_{*m}^{(\alpha)}(\mathbf{k})\right]^{-1} \tag{47}$$

(no summations assumed in (43) over subscript m!)

The equation (42) is reversible and for any α we have

$$\overline{R}_{ml}^{(\alpha)}(\mathbf{k})\overline{V}_l(\mathbf{k})=-2\pi jk_m\overline{U}(\mathbf{k}) \tag{48}$$

In x-space in corresponding coordinate system we have the non-local equations with unique kernels

$$V_l^{(\alpha)}(\mathbf{x})=-\int B_{lm}^{(\alpha)}(\mathbf{x}-\mathbf{y})\frac{\partial U(\mathbf{y})}{\partial y_m}dy^3\ ,\ \int R_{ml}^{(\alpha)}(\mathbf{x}-\mathbf{y})V_l(\mathbf{y})dy^3=-\frac{\partial U(\mathbf{x})}{\partial x_m} \tag{49}$$

Because $\overline{\Pi}_l^{(\alpha)}(\mathbf{k})$ are imaginary and odd functions of vector **k**, the components of diagonal tensor $\mathbf{B}^{(\alpha)}(\mathbf{k})$ are even and real functions. Now we write the component $\overline{\mathrm{B}}_{ll}^{(\alpha)}(\mathbf{k})$ in the following form:

$$\overline{\mathrm{B}}_{ll}^{(\alpha)}(\mathbf{k})=\overline{\mathrm{B}}_{ll}^{(\alpha)}(0)\overline{F}_{ll}^{(\alpha)}(\tilde{\mathbf{k}}) \tag{50}$$

Here $\overline{F}_{ll}^{(\alpha)}(\tilde{k}_1,\tilde{k}_2,\tilde{k}_3)$ is a dimensionless function of the dimensionless variables $\tilde{k}_l=\Delta_l k_l$, where Δ_l are linear scales of the random field $\boldsymbol{\sigma}(\mathbf{x})$, for example, the correlation scales. Assuming the existence of a *Taylor's* expansion of the function $\overline{F}_{ll}^{(\alpha)}(\tilde{\mathbf{k}})$ we can write:

$$\overline{\mathrm{B}}_{ll}^{(\alpha)}(k)=\overline{\mathrm{B}}_{ll}^{(\alpha)}(0)\sum_{n=n_1+n_2+n_3=0}^{\infty}\frac{1}{n!}\frac{\partial^n\overline{F}_{ll}^{(\alpha)}(0)}{\partial\tilde{k}_1\partial\tilde{k}_2\partial\tilde{k}_3}k_1^{n_1}k_2^{n_2}k_3^{n_3}\Delta_1^{n_1}\Delta_2^{n_2}\Delta_3^{n_3} \tag{51}$$

Substituting (51) into (46) and taking into account that all the odd derivatives of $\overline{F}_{ll}^{(\alpha)}(\tilde{\mathbf{k}})$ at $\tilde{\mathbf{k}}=0$ are zero, we can write the expansions for the mean velocity $V_l^{(\alpha)}(\mathbf{x})$ in x-space:

$$V_l^{(\alpha)}(\mathbf{x}) = -B_{ll}^{(\alpha)}(0) \sum_{n=n_1+n_2+n_3=0}^{\infty} \frac{(-1)^n \Delta_1^{2n_1} \Delta_2^{2n_2} \Delta_3^{2n_3}}{(2n)!(2\pi)^{2n}} \frac{\partial^{2n} \overline{F}_{ll}^{(\alpha)}(0)}{\partial \tilde{k}_1^{2n_1} \partial \tilde{k}_2^{2n_2} \partial \tilde{k}_3^{2n_3}} \frac{\partial^{2n+1} U(\mathbf{x})}{\partial x_l \partial x_1^{2n_1} \partial x_2^{2n_2} \partial x_3^{2n_3}} \quad (52)$$

$$V_l^{(\alpha)}(\mathbf{x}) = -\overline{\mathrm{B}}_{ll}^{(\alpha)}(0) \sum_{n=n_1+n_2+n_3=0}^{\infty} \frac{\Delta_1^{2n_1} \Delta_2^{2n_2} \Delta_3^{2n_3} \mathrm{I}_{ll,n}^{(\alpha)}(n_1, n_2, n_3)}{(2n)!} \frac{\partial^{2n+1} U(\mathbf{x})}{\partial x_l \partial x_1^{2n_1} \partial x_2^{2n_2} \partial x_3^{2n_3}} \quad (53)$$

where power moment of the dimensionless function $F_{ll}^{(\alpha)}(\tilde{\mathbf{y}}) = T_F^{-1}\left[\overline{F}_{ll}^{(\alpha)}(\tilde{\mathbf{k}})\right]$ of the dimensionless variables $\overline{y}_l = x_l / \Delta_l$ is $\mathrm{I}_{ll,n}^{(\alpha)}(n_1, n_2, n_3) = \int \tilde{y}_1^{2n_1} \tilde{y}_2^{2n_2} \tilde{y}_3^{2n_3} F_{ll}^{(\alpha)}(\tilde{\mathbf{y}}) d\tilde{y}^3$.

It is evident that from equation (50) we have $\overline{F}_{ll}^{(\alpha)}(0) = 1$ and then $\mathrm{I}_{ll,0}^{(\alpha)}(0,0,0) = 1$. The important question is: What is the behavior of the expansions (52) or (53) in the limiting case when $\Delta_l \to 0$ that corresponds to the theory of homogenization and the concept of effective conductivity (see *Bakhvalov and Panasenko*, 1984; *Zhikov et all*, 1993)? We should note that in the first terms of both expansions (n=0) the coefficients of the derivatives do not contain Δ_l explicitly. By setting some restrictions to the source density $f(\mathbf{x})$, the behavior the oldest derivatives of $U(\mathbf{x})$ can be sufficiently limited. All the other terms of these expansions tend to zero for $\Delta_l \to 0$. In this case we have the averaged equation:

$$V_l^{(\alpha)}(\mathbf{x}) = -\overline{\mathrm{B}}_{ll}^{(\alpha)}(0) \frac{\partial U(\mathbf{x})}{\partial x_l} \quad (54)$$

where $\overline{\mathrm{B}}_{ll}^{(\alpha)}(0)$ are the diagonal components of the effective conductivity tensor. Notice that according to the theory of homogenization the tensor of the effective conductivity exists and is constant in all *Euclidian* space R^3. This is true if, for any limited domain $Q \subset R^3$, the source density function $f(\mathbf{x})$ belongs to *Sobolev's* functional space $H^{-1}(Q)$, or for example belongs to the square integrable functional space $L^2(Q)$, that is embedded in $H^{-1}(Q)$ space. Furthermore, if in any orthogonal coordinate system the tensor of the local random conductivity is symmetric and positive definite, the tensor of the effective conductivity is also symmetric and positive definite, so-called elliptic (*Zhikov at all*, 1993). The principal part of the expansions (52) and (53) corresponds with the theory of homogenization limit and can be used for computing the effective conductivity. In the cases of symmetry: isotropic $(\alpha = i)$, transversal isotropy $(\alpha = t)$ and orthotropy $(\alpha = o)$ for appropriate coordinate systems, the averaged equation has the form:

$$\mathbf{V}^{(\alpha)}(\mathbf{x}) = -\mathbf{B}_*^{(\alpha)} \nabla U(\mathbf{x}), \quad \mathbf{R}_*^{(\alpha)} \mathbf{V}^{\alpha}(\mathbf{x}) = -\nabla U(\mathbf{x}) \quad (55)$$

where the diagonal tensors of the effective conductivity $\mathrm{B}_{*ml}{}^{(\alpha)} = \delta_{ml}\overline{\mathrm{B}}_{ll}^{(\alpha)}(0)$ and effective resistvity $R_{*lm}^{(\alpha)} = \delta_{lm}\left[B_{ll}{}^{(\alpha)}(0)\right]^{-1}$ are

$$\mathbf{B}_*^{(\alpha)} = \begin{pmatrix} \overline{\mathrm{B}}_{11}^{(\alpha)}(0) & 0 & 0 \\ 0 & \overline{\mathrm{B}}_{22}^{(\alpha)}(0) & 0 \\ 0 & 0 & \overline{\mathrm{B}}_{33}^{(\alpha)}(0) \end{pmatrix},\ \mathbf{R}_*^{(\alpha)} = \begin{pmatrix} \left[\overline{B}_{11}^{(\alpha)}(0)\right]^{-1} & 0 & 0 \\ 0 & \left[\overline{B}_{22}^{(\alpha)}(0)\right]^{-1} & 0 \\ 0 & 0 & \left[\overline{B}_{33}^{(\alpha)}(0)\right]^{-1} \end{pmatrix} \tag{56}$$

Obviously the principal axes for all tensors $\overline{\mathbf{B}}^{(\alpha)}(\mathbf{k})$, $\mathbf{B}_*^{(\alpha)}$ and $\mathbf{R}^{(\alpha)}(\mathbf{k}), \mathbf{R}_*^{(\alpha)}$ for any $\mathbf{k}$, for each α are identical to the respective coordinate axes.

Up to this point we have studied the fields with some symmetry in special orthogonal coordinate systems. If the orthogonal coordinate axes x_l' and k_l' are oriented arbitrarily and β_{lm} is the cosine of the angle between the axes x_l' and x_m, the effective conductivity tensor in the new coordinate system is $\mathbf{B}_*'^{(\alpha)} = \boldsymbol{\beta}\, \mathbf{B}_*^{(\alpha)} \boldsymbol{\beta}^{-1}$. This tensor is symmetric and positive definite (elliptic). The averaged equations in the arbitrary coordinate system x_l' has the forms $\mathbf{V}'^{(\alpha)}(\mathbf{x}') = -\mathbf{B}_*'^{(\alpha)}\nabla U(\mathbf{x}')$, $\mathbf{R}_*'^{(\alpha)}\mathbf{V}'^{(\alpha)}(\mathbf{x}) = -\nabla U(\mathbf{x})$. But what if the diagonal tensor $\mathbf{B}_*^{(\alpha)}$ is unknown? Or to put it more precisely, what if we know that there exists some symmetry but the orientation of the principal axes is not known nor the parameter α? In this case we return to equation (20), which is actual for any stochastically homogeneous positive defined random fields $\boldsymbol{\sigma}(\mathbf{x})$ and study the vector $\overline{\Pi}_l'(\mathbf{k}')$ again and its formal *Taylor's* expansion about $\mathbf{k}' = 0$:

$$\overline{\Pi}_l'(\mathbf{k}') = \sum_{n=n_1+n_2+n_3=0}^{\infty} \frac{1}{n!} \frac{\partial^n \overline{\Pi}_l'(0)}{\partial k_1'^{n_1} \partial k_2'^{n_2} \partial k_3'^{n_3}} k_1'^{n_1} k_2'^{n_2} k_3'^{n_3} \tag{57}$$

The component $\overline{\Pi}_l'(\mathbf{k}')$ is an odd function of $\mathbf{k}'$ and therefore at point $\mathbf{k}' = 0$ all even derivatives are zero. Thus

$$\overline{\Pi}_l'(\mathbf{k}') = \sum_{2n-1=n_1+n_2+n_3=1}^{\infty} \frac{1}{(2n-1)!} \frac{\partial^{2n-1} \overline{\Pi}_l'(0)}{\partial k_1'^{n_1} \partial k_2'^{n_2} \partial k_3'^{n_3}} k_1'^{n_1} k_2'^{n_2} k_3'^{n_3} \tag{58}$$

The linear part of this expansion on variable $\mathbf{k}'$ is:

$$\overline{\Pi}_l'^{1}(\mathbf{k}') = \frac{\partial \overline{\Pi}_l'(0)}{\partial k_m'} k_m' \tag{59}$$

Inserting (53) in (20) we can write the linear approximation on $\mathbf{k}'$ for $\overline{V}_l'(\mathbf{k}')$:

$$\bar{V}_l^1(\mathbf{k}') = -\left[-\frac{1}{2\pi j}\frac{\partial\bar{\Pi}_l'(0)}{\partial k_m'}\right] 2\pi j k_m' \bar{U}(\mathbf{k}) \tag{60}$$

By imposing some restrictions on the density $f(\mathbf{x})$ as discussed earlier, the oldest terms of the expansions of $\bar{V}_l(\mathbf{k}')$ vanish in the homogenization limit. In this case we have the averaged equation:

$$V_l'(\mathbf{x}') = -\mathrm{B}_{*lm}' \frac{\partial U(\mathbf{x}')}{\partial x_m'} \tag{61}$$

and in general case from (54) we find the real tensor of the effective conductivity, that is symmetric and positive definite

$$\mathrm{B}_{*lm}' = -\frac{1}{2\pi j}\frac{\partial\bar{\Pi}_l'(0)}{\partial k_m'} \tag{62}$$

Thus, if we know the components $\bar{\Pi}_l'(\mathbf{k}')$, we can find the effective conductivity tensor and by using the standard method we can find its real eigenvalues and the orthogonal eigenvectors. Transition to a new eigen orthogonal system that is associated with the eigenvectors and transformation of the tensor B_{*lm}' to the new coordinates lead to a diagonal tensor B_{*ml} , whose components are the eigenvalues for tensor B_{*lm}'. As we mentioned earlier, for each α in the new eigen coordinate system the tensor $\bar{\mathrm{B}}_{lm}(\mathbf{k})$ is diagonal with the following components:

$$\bar{\mathrm{B}}_{ll}(\mathbf{k}) = -\bar{\Pi}_l(\mathbf{k})/2\pi j k_l \ , \ \bar{B}_{lm}(\mathbf{k}) = 0 \text{, if } \ l \neq m \tag{63}$$

It is obvious that diagonal tensor $\bar{\mathbf{B}}(\mathbf{k})$ is unique and reversible.

6. ALTERNATIVE APPROACH

Majority of the works related to the present subject used a different approach. From the outset they wanted to find the relation between the averaged flow velocity field and the gradient of mean pressure (head). To examine the validity of this approach we return to the equation (20) again. To recast it to the form like the *Darcy's* Law we introduce some tensor $\bar{B}_{lm}(\mathbf{k})$ that satisfies the equation:

$$\bar{\Pi}_l(\mathbf{k}) = -\bar{B}_{lm}(\mathbf{k}) 2\pi j k_m \tag{64}$$

and after inserting (64) into (20) we have equation

$$\bar{V}_l(\mathbf{k}) = -\bar{B}_{lm}(\mathbf{k}) 2\pi j k_m \bar{U}(k) \tag{65}$$

In $\mathbf{x}$-space we have respectively

$$V_l(\mathbf{x}) = -\int B_{lm}(\mathbf{x}-\mathbf{y}) \frac{\partial U(\mathbf{y})}{\partial y_m} dy^3 \tag{66}$$

If we inserting (65) into (22), the condition of compatibility components vector $\bar{\mathbf{\Pi}}(\mathbf{k})$, we obtain condition of compatability for components of tensor $\bar{B}_{lm}(\mathbf{k})$

$$4\pi^2\bar{G}(\mathbf{k})k_l\bar{B}_{lm}(\mathbf{k})k_m = 1 \qquad (67)$$

It should be noted that last relationship we discussed that for any **k** the tensor $\bar{B}_{ml}(\mathbf{k})$ is elliptic is not proven, although the equation doesn't contradict it, because the factors – components k_l and k_m are non-arbitrary.

Assuming that tensor $\bar{\mathbf{B}}(\mathbf{k})$ is nonsingular we can rewrite the equation (64) in the form

$$\bar{\mathbf{R}}(\mathbf{k})\bar{\mathbf{\Pi}}(\mathbf{k}) = -2\pi j\mathbf{k} \ \ , \ \bar{\mathbf{R}}(\mathbf{k}) = \left[\bar{\mathbf{B}}(\mathbf{k})\right]^{-1} \qquad (68)$$

Multiplay the first equation from (68) by $\bar{U}(\mathbf{k})$ and taking into account (20) we can write the averaged equation

$$\bar{\mathbf{R}}(\mathbf{k})\bar{\mathbf{V}}(\mathbf{k}) = -2\pi j\mathbf{k}\bar{U}(\mathbf{k}) \qquad (69)$$

In $\mathbf{x}$-space we have respectively the conservative averaged equation of momentum balance

$$\int \mathbf{R}(\mathbf{x}-\mathbf{y})\mathbf{V}(\mathbf{x}-\mathbf{y})dy^3 = -\nabla U(\mathbf{x}) \qquad (70)$$

The definition of the Fourier transformation $\bar{\mathbf{B}}(\mathbf{k})$ with the system (64) or $\bar{\mathbf{R}}(\mathbf{k})$ with system (68) leads to three linear algebraic equations for each **k** and every of them contain three from nine unknown components $\bar{B}_{ml}(\mathbf{k})$ or $\bar{R}_{lm}(\mathbf{k})$ respectively. In the x-space this problem amounts to three differential equations with nine unknown nine function-components $B_{lm}(\mathbf{x})$ or three operator equations for unknown nine functions $R_{lm}(\mathbf{x})$.

$$\Pi_l(\mathbf{x}) = -\frac{\partial B_{lm}(\mathbf{x})}{\partial x_m} \quad , \ \int R_{lm}(\mathbf{x}-\mathbf{y})\Pi_l(\mathbf{y})dy^3 = -\frac{\partial}{\partial x_m} \qquad (71)$$

Both systems (64) and (68) and systems (71) are underdetermined and in general have unlimited sets of solutions.

It is well known that $\bar{\mathbf{B}}_g(\mathbf{k})$ -the general solution (all infinite set of solutions) for a singular non-uniform system of linear algebraic equations can be presented as a sum of any individual solution of system $\bar{\mathbf{B}}_0(\mathbf{k})$ and $\bar{\mathbf{B}}_*(\mathbf{k})$-which is any solution of the uniform system $\bar{\mathbf{B}}_*(\mathbf{k})\mathbf{k} = 0$ (The geometric sense of the uniform system is that all three vector-lines for the tensor $\bar{\mathbf{B}}_*(\mathbf{k})$ are orthogonal to the vector **k**). Namely for this reason as indicated by *Indelman and Abramovich*, 1994, the use in (23) any of general solutions

$\bar{\mathbf{B}}_g(\mathbf{k})$ with known vector $2\pi j\mathbf{k}\bar{U}(\mathbf{k})$, i.e., *Fourier*-transformation of $\nabla U(\mathbf{x})$, do not affect the computing $\bar{\mathbf{V}}(\mathbf{k})$.

When the tensors $\bar{\mathbf{B}}_g(\mathbf{k})$ are not singular, we can obtain the tensors $\bar{\mathbf{R}}_g(\mathbf{k})=\left[\bar{\mathbf{B}}_g(\mathbf{k})\right]^{-1}$-the general solutions of (68), which are not unique. In this case $\mathbf{R}_g(\mathbf{k})$ can be written as a sum of any individual solution $\bar{\mathbf{R}}_0(\mathbf{k})$ and $\bar{\mathbf{R}}_*(\mathbf{k})$ - any solution of uniform system $\bar{\mathbf{R}}_*(\mathbf{k})\bar{\mathbf{\Pi}}(\mathbf{k})=0$. (The geometric sense of the uniform system is that all vector-lines of the tensor $\bar{\mathbf{R}}_*(\mathbf{k})$ are orthogonal to the vector $\bar{\mathbf{\Pi}}(\mathbf{k})$. If we select any individual solution $\bar{\mathbf{B}}_0(\mathbf{k})$, we can find $\bar{\mathbf{R}}_0(\mathbf{k})=\left[\bar{\mathbf{B}}_0(\mathbf{k})\right]^{-1}$. The corresponding tensor $\bar{\mathbf{R}}_*(\mathbf{k})=-\bar{\mathbf{R}}_0(\mathbf{k})\bar{\mathbf{B}}_*(\mathbf{k})\left[\bar{\mathbf{B}}_0(\mathbf{k})+\bar{\mathbf{B}}_*(\mathbf{k})\right]^{-1}$ does not affect the computation of $2\pi j\mathbf{k}U(\mathbf{k})$ from equation (69) or $\nabla U(\mathbf{x})$ from equation (70).

On the other hand, in the case of global symmetry as shown above, the number of unknown functions reduces to three or less and it is possible to find a unique solution, the diagonal tensor $\bar{\mathbf{B}}(\mathbf{k})$. Even if there is no reason to believe that the types of the global symmetry discussed above exist, the fact remains that if the stochastically homogeneous field of the local random conductivity tensor $\sigma_{ml}(\mathbf{x})$ is symmetric and elliptic, the tensor of the effective conductivity $\bar{\mathrm{B}}_{ml}(0)$ is symmetric and elliptic as well. In this general case the eigen orthogonal coordinate system exists in which the tensor $\bar{\mathbf{B}}(0)$ is diagonal.

In general case it would appear reasonable to find in eigen coordinate system special solutions of system (64) and the uniform system as diagonal tensors for all $\mathbf{k}$, that are simple and convenient for matching and identification. Then we select solutions $\tilde{\bar{\mathbf{B}}}(\mathbf{k})=0$ for uniform system . In this case we find diagonal tensors $\bar{\mathbf{B}}(\mathbf{k})$ and $\bar{\mathbf{R}}(\mathbf{k})=\left[\bar{\mathbf{B}}(\mathbf{k})\right]^{-1}$

$$\bar{B}_{ll}(\mathbf{k})=-\bar{\Pi}_l(\mathbf{k})/2\pi jk_l\ ,\ \bar{B}_{lm}(\mathbf{k})=0\ \text{ if }\ l\neq m$$

$$\bar{R}_{ll}(\mathbf{k})=-2\pi jk_l/\bar{\Pi}_l(\mathbf{k})\ ,\bar{R}_{lm}(\mathbf{k})=0\ \text{ if }\ l\neq m \tag{72 }$$

Note that if we use any orthogonal coordinate system k_l'' that is different from the eigen system, and write:

$$\bar{\mathrm{B}}_{ll}(\mathbf{k}'')=-\bar{\Pi}_l(\mathbf{k}'')/2\pi jk_l''\quad ,\ \bar{\mathrm{B}}_{lm}(\mathbf{k}'')=0\ \textit{if}\ \ l\neq m \tag{73}$$

which is also a exact solution of the system defined by (64), we can see that the limit of $\bar{\mathrm{B}}_{ll}(\mathbf{k}'')$ does not exist when $\mathbf{k}''\to 0$. In fact, inserting the linear part of expansion $\bar{\Pi}_l(\mathbf{k}'')$ in the form of (53) into (61), we have:

$$\overline{\mathrm{B}}_{ll}\left(\mathbf{k}''\right)=\frac{1}{2\pi j}\frac{\partial\overline{\Pi}_l\left(\mathbf{k}''\right)}{\partial k''_m}\frac{k''_m}{k''_l} \tag{74}$$

(Here, the summation over m is implied!) Because $\frac{\partial\overline{\Pi}_l\left(\mathbf{k}''\right)}{\partial k''_m}\neq 0$ for $l\neq m$ and because the $\mathbf{k}''_l$ coordinate system is not eigen, as $\mathbf{k}''\to 0$ the $\lim\overline{\mathrm{B}}_{ll}\left(\mathbf{k}''\right)$ does not exist.

Therefore in (72) we have solutions, which are exact, continuous and reversible with the formulated constrains.

For finding the continuous exact and reversible tensor- solution of system (64) in any orthogonal coordinate system it should be from (62) compute the tensor of effective conductivity and find it eigenvalue and eigenvectors and the $\boldsymbol{\beta}$ -matrix of transition from original coordinate system to eigen system. Then we need find the diagonal tensor of effective conductivity $\overline{\mathbf{B}}_*\left(0\right)$ in eigen system and from system (72) find components for diagonal tensor $\overline{\mathbf{B}}\left(\mathbf{k}\right)$. Now it remains to return to initial coordinate system using the matrix $\boldsymbol{\beta}^{-1}$.

As we showed in section 5 this method in case of global symmetry lead to exact, unique and reversible solution. The difference lies in the fact that in case of global symmetry the components of diagonal tensor dependent from invariants, which related to type of symmetry. It is evident that finding the solutions $\overline{\mathbf{B}}\left(\mathbf{k}\right)$ and $\overline{\mathbf{R}}\left(\mathbf{k}\right)$ we not assume that for any $\mathbf{k}$ these tensors by $\mathbf{k}\neq 0$ must be elliptic.

Now let us compare the approaches discussed above and estimate their adequacy and utility for describing the averaged flow and studying appropriate direct and inverse problems. We shall call the approach presented in the present paper and the alternative approach with symbols P and A, respectively.

1. In x-space the approach P leads to the equation that relates the mean velocity field with the mean pressure (head) field. But the approach A relates the mean velocity field with the gradient of mean pressure (head).
2. These equations are non-local and contain convolutions. The kernels are a vector-operator $\boldsymbol{\Pi}\left(\mathbf{x}\right)$ in P and a tensor-operator $\overline{\mathbf{B}}\left(\mathbf{x}\right)$ in A.
3. Describing averaged flow in Fourier-space k leads to linear algebraic equations that contain the vector $\overline{\boldsymbol{\Pi}}\left(\mathbf{k}\right)$ in P and tensor $\overline{\mathbf{B}}\left(\mathbf{k}\right)$ in A.
4. Under P-approach for each conductivity field there exists a unique vector-operator $\overline{\boldsymbol{\Pi}}\left(\mathbf{x}\right)$ or a vector $\overline{\boldsymbol{\Pi}}\left(\mathbf{k}\right)$.

5. Under A-approach for each conductivity field an unlimited set of operators $\mathbf{B}(\mathbf{x})$ or tensors $\bar{\mathbf{B}}(\mathbf{k})$ exists. Any of them can be used to compute the exact mean velocity vector. Actually by this operation one utilizes the P-approach.
6. In P-approach to exactly and uniquely solve the inverse problem for finding the vector $\bar{\mathbf{\Pi}}(\mathbf{k})$, it is sufficient to know a pair of fields, $\bar{U}(\mathbf{k})$ and $\bar{\mathbf{V}}(\mathbf{k})$, which are consistent with the density $\bar{f}(\mathbf{k})$.
7. In A-approach the inverse problem for finding tensor $\bar{\mathbf{B}}(\mathbf{k})$ in general is ill-posed.

We again return to the averaged basic system (16)-(19) but now we eliminate the function $\bar{f}(\mathbf{k})$ from equations (16) and (17). In this case we have the equation

$$2\pi j\bar{G}(\mathbf{k})k_l\bar{V}_l(k)=\bar{U}(\mathbf{k}) \tag{75}$$

Multiplying the equation (75) by $(-2\pi jk_m)$ for $m=1,2,3$ we can write

$$\bar{\bar{R}}_{lm}(\mathbf{k})\bar{V}_l(\mathbf{k})=-2\pi jk_m\bar{U}(\mathbf{k}) \tag{76}$$

where the symmetric tensor $\bar{\bar{R}}_{lm}(\mathbf{k})$ is

$$\bar{\bar{R}}_{lm}(\mathbf{k})=4\pi^2k_lk_m\bar{G}(\mathbf{k}) \tag{77}$$

It is easy to see that for any $\mathbf{k}$ the tensor is singular because the determinant $\left\|\bar{\bar{R}}_{lm}(\mathbf{k})\right\|=0$ and the reciprocal tensor $\bar{\bar{\mathbf{R}}}^{-1}(\mathbf{k})$ does not exist. Generally speaking, the system (76) is not reversible.

If we know the velocity field $\bar{\mathbf{V}}(\mathbf{k})$, we can use the system (76) for computing $-2\pi jk_m\bar{U}(\mathbf{k})$.

In x-space we have the conservative non-local equation of moment balance

$$\int\bar{\bar{R}}_{lm}(\mathbf{x}-\mathbf{y})\bar{V}_l(\mathbf{y})dy^3=-\frac{\partial U(\mathbf{x})}{\partial x_m},\ \bar{\bar{R}}_{lm}(\mathbf{x})=-\frac{\partial^2G(\mathbf{x})}{\partial x_l\partial x_m} \tag{78}$$

We can use the system (78) for computing $\nabla U(\mathbf{x})$. It is evident that for this action a knowledge of the *Green's* function $G(\mathbf{x})$ is enough. On the contrary, in general it is impossible to find the filed $\mathbf{V}(\mathbf{x})$ from the system in (78). This exact result demonstrates again that A-approach have serious contradictions.

7. NON-STEADY TRANSIENT FLOW WITH SOURCES

Let us consider the stochastic system of equations in a three-dimensional unbounded domain:

$$\frac{\partial m(\mathbf{x},t)}{\partial \ t} + \frac{\partial v_l(\mathbf{x},t)}{\partial \ x_l} = f(\mathbf{x},t) \tag{79}$$

$$m(\mathbf{x},t) = \alpha(\mathbf{x}) u(\mathbf{x},t) \tag{80}$$

$$\mathbf{r}(\mathbf{x})\mathbf{v}(\mathbf{x}) = -\nabla u(\mathbf{x},t) \tag{81}$$

$$u(\mathbf{x},t_0) = 0 \tag{82}$$

Here scalar function $\alpha(\mathbf{x})$ and tensor of $\mathbf{r}(\mathbf{x})$ are statistically homogeneous random fields of the storage capacity and resistence , respectively, and $u(\mathbf{x},t)$ is the pressure. We introduce the random Green's function $g(\mathbf{x},t,\mathbf{y},\tau)$, which is the solution to the system (79)-(82) for $f(\mathbf{x},t) = \delta(\mathbf{x}-\mathbf{y})\delta(t-\tau)$. Let us introduce in the same way:

$$G(\mathbf{x}-\mathbf{y},t-\tau) = \langle g(\mathbf{x},t,\mathbf{y},\tau)\rangle \tag{83}$$

$$N(\mathbf{x}-\mathbf{y},t-\tau) = \langle \alpha(\mathbf{x}) g(\mathbf{x},t,\mathbf{y},\tau)\rangle \tag{84}$$

$$\Gamma_l(\mathbf{x}-\mathbf{y},t-\tau) = -\left\langle \sigma_{lj}(\mathbf{x}) \frac{\partial g(\mathbf{x},t,\mathbf{y},\tau)}{\partial x_j} \right\rangle \tag{85}$$

We consider T_{FL} and T_{FL}^{-1} - the direct and inverse *Fourier-Laplace* transforms and use the following designations:

$$\bar{G}(\mathbf{k},\mu) = T_{FL}G, \quad \bar{N}(\mathbf{k},\mu) = T_{FL}N, \quad \bar{\Gamma}_l(\mathbf{k},\mu) = T_{FL}\Gamma_l \tag{86}$$

and introduce the following scalar function and vector:

$$\bar{S}(k,\mu) = \bar{N}(k,\mu)\bar{G}^{-1}(k,\mu), \quad \bar{\Pi}_l(k,\mu) = \bar{\Gamma}_l(k,\mu)\bar{G}^{-1}(k,\mu) \tag{87}$$

It easy to show that $\mu\bar{N}(\mathbf{k},\mu) + 2\pi i k_l \bar{\Pi}_l(\mathbf{k},\mu) = \bar{G}^{-1}(\mathbf{k},\mu)$. Thus, the averaged system is

$$\frac{\partial M(\mathbf{x},t)}{\partial t} + \frac{\partial V_l(\mathbf{x},t)}{\partial x_l} = f(\mathbf{x},t) \tag{88}$$

$$M(\mathbf{x},t) = \int\int_{t_0}^{t} S(\mathbf{x}-\mathbf{y}, \ t-\tau) U(\mathbf{y},\tau) \ dy^3 d\tau \tag{89}$$

$$V_l(\mathbf{x},t) = -\int\int_{t_0}^{t} \Pi_l(\mathbf{x}-\mathbf{y},t-\tau) U(\mathbf{y},\tau) dy^3 \, d\tau \tag{90}$$

$$U(\mathbf{x},t_0)\ =0 \tag{91}$$

here $U(\mathbf{x},t)=\langle u(\mathbf{x},t)\rangle$, $V_l(\mathbf{x},t)=\langle v_l(\mathbf{x},t)\rangle$, $M(\mathbf{x},t)=\langle m(\mathbf{x},t)\rangle$, $S(\mathbf{x},t)=T_{FL}^{-1}\,\bar{S}(\mathbf{k},\mu)$, and $\mathbf{\Pi}(\mathbf{x},t)=T_{FL}^{-1}\,\bar{\mathbf{\Pi}}(\mathbf{k},\mu)$.

8. NON-REACTIVE SOLUTE TRANSPORT

We consider a stochastic system of equations in a three dimensional unbounded domain:

$$\frac{\partial a(\mathbf{x},t)}{\partial t}+\frac{\partial q_l(\mathbf{x},t)}{\partial x_l}\ =\ f(\mathbf{x},t) \tag{92}$$

$$a(\mathbf{x},t)=\ \theta(\mathbf{x})c(\mathbf{x},t) \tag{93}$$

$$q_l(\mathbf{x},t)=v_l(\mathbf{x},t)c(\mathbf{x},t)-D_{lj}\frac{\partial c(\mathbf{x},t)}{\partial x_j} \tag{94}$$

$$c(\mathbf{x},t_0)=0 \tag{95}$$

Here $c(\mathbf{x},t),\theta(\mathbf{x}),\mathbf{q}(\mathbf{x},t),\mathbf{D}$ are the concentration of solute, random porosity, solute flux and non-random dispersion tensor respectively.

We introduce the random Green's function $g_c(\mathbf{x},t,\mathbf{y},\tau)$ which is the solution to the system (92)-(95) for $f(\mathbf{x},t)=\delta(\mathbf{x}-\mathbf{y})\delta(t-\tau)$, and relations: $G_c(\mathbf{x}-\mathbf{y},t-\tau)=\langle g_c\rangle$,

$$H(\mathbf{x}-\mathbf{y},t-\tau)=\langle\theta(\mathbf{x})g_c\rangle,\ P_l(\mathbf{x}-\mathbf{y},t-\tau)=\langle v_l(\mathbf{x})g_c\rangle \tag{96}$$

$$\bar{G}_c(\mathbf{k},\mu)=T_{FL}G_c,\ \bar{H}(\mathbf{k},\mu)=T_{FL}H,\ \bar{P}_l(\mathbf{k},\mu)=T_{FL}P_l,\ \bar{\tilde{\theta}}(\mathbf{k},\mu)=\bar{H}\bar{G}^{-1},\ \bar{W}_l(\mathbf{k},\mu)=\bar{P}_l\bar{G}^{-1}$$

It easy to show that $\mu\bar{\tilde{\theta}}+2\pi ik_l\bar{W}_l=\bar{G}^{-1}$. Thus, the averaged system is:

$$\frac{\partial A(\mathbf{x},t)}{\partial t}+\frac{\partial Q_l(\mathbf{x},t)}{\partial x_l}=f(\mathbf{x},t) \tag{97}$$

$$A(\mathbf{x},t)=\int\int_{t_0}^{t}\tilde{\theta}(\mathbf{x}-\mathbf{y},t-\tau)C(\mathbf{y},\tau)d\,y^3d\,\tau \tag{98}$$

$$Q_l(\mathbf{x},t)=\int\int_{t_0}^{t}W_l(\mathbf{x}-\mathbf{y},t-\tau)C(\mathbf{y},\tau)dy^3d\,\tau\ \ -D_{lj}\frac{\partial\ \ C(x,t)}{\partial x_j} \tag{99}$$

$$C(\mathbf{x},t_0)=0 \tag{100}$$

Here

$$C(\mathbf{x},t)=\langle c(\mathbf{x},t)\rangle,\ Q_l(\mathbf{x},t)=\langle q_l(\mathbf{x},t)\rangle,\ \tilde{\theta}(\mathbf{x},t)=T_{FL}^{-1}\bar{\tilde{\theta}}(\mathbf{k},\mu),\ W_l(\mathbf{x},t)=T_{FL}^{-1}\bar{W}_l(\mathbf{k},\mu) \tag{101}$$

9. SUMMARY

We have described a general form of the exactly averaged equations of flow and transport in a stochastically homogeneous unbounded field with sources. We examined the validity of the averaged descriptions for the given fields and the generalized law for non-local models. A variant of the generalization for a given field with a unique kernel-vector and in some cases with a unique kernel-tensor was presented. We discussed the problem of uniqueness and the properties of the non-local averaged equations for three types of global symmetry: isotropic, transversal isotropic, and orthotropic. We analyzed the structure of non-local equations in the general case of stochastically homogeneous fields.

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