UNIVERSITY OF CALIFORNIA,
IRVINE

The Inverse Spectral Problem for Convex Planar Domains

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Amir B. Vig

Dissertation Committee:
Associate Professor Hamid Hezari, Chair
Professor Katya Krupchyk
Distinguished Professor Abel Klein

2020

DEDICATION

To the memories of Naqib Ahmad Khpulwaak, Deah Shaddy Barakat, Yusor Mohammad Abu-Salha, and Razan Mohammad Abu-Salha.

TABLE OF CONTENTS

LIST OF FIGURES

Page

ACKNOWLEDGMENTS

I would first like to thank my advisor Hamid Hezari for his mentorship, support and guidance while undertaking the projects in this dissertation. I would also like to thank Katya Krupchyk for teaching me a great deal about microlocal and semiclassical analysis in addition to being a constant source of inspiration and motivation throughout my time as a graduate student. I am also extremely grateful to Steve Zelditch, Jared Wunsch, Colin Guillarmou and Semyon Dyatlov for teaching me so much about microlocal analysis, spectral and scattering theory, dynamical systems, and geometric inverse problems in graduate minicourses around the world. I would also like to thank Jeremy Marzuola for first introducing me to partial differential equations and Jeff Streets, Xiangwen Zhang and Rick Schoen for teaching me so much geometry, which has broadened my mathematical horizons and opened many new doors. I am incredibly grateful to the Mathematical Sciences Research Institute for hosting me during the 2019 fall semester on microlocal analysis and to the many other conferences and summer schools from which I learned so much. I am eternally grateful to my friends Perry Kleinhenz, Thibault Lefeuvre, Francois Monard and James Upton for many inspiring mathematical conversations, friendship and support over the past several years. I am deeply appreciative of the tremendous amount of help and support I have received from Aubrey Rudd, Donna McConnell and Eugena Thompson in maneuvering both the mathematics department and university as a whole.

I would also like to thank Vadim Kaloshin and Alfonso Sorrentino for allowing the use of their images in Figure 3.1. I am grateful for the generous financial support I have received during graduate school. This includes: GAAN Fellowship (Fall 2014, Winter and Summer 2015), NSF grant DMS 1500703 (Summer 2016 under supervision of Katya Krupchyk), Faculty Mentorship Fellowship Honorable Mention (2016), Von Neumann Award for Outstanding Performance (2019), MSRI Program Associateship (Fall 2019), Simons Collaboration Grant

638398 (Fall 2019), Dissertation Fellowship (Fall 2019) and UCI Mathematics Department Fellowship (Spring 2020).

On a personal note, I would like to thank my parents, Dee Vig and Barbara DonCarlos for their unwavering love, support and encouragement throughout my life. I am also grateful to my brother Alex Vig for first introducing me to mathematics and piquing my interest in many interesting problems. I am forever indebted to the Waziri, Yunus and Olomi families for taking me in as one of their own and making me a part of their family during my time in California. I wouldn't be half the person I am today without the support of my closest friends Dawoud Waziri, Jasmine Afshar, Rahim Zalal, Zayd Yunus, Carl Santos, Sultan Latif, Walid Nasim, Madiha Bhatti and Abdullah Mohmand. Finally, I would like to thank Sarah Zamamiri for her everlasting love and support.

VITA

Amir B. Vig

EDUCATION

Doctor of Philosophy in Mathematics **2020**
University of California, Irvine *Irvine, California*

Master of Science in Mathematics **2016**
University of California, Irvine *Irvine, California*

Bachelor of Science in Mathematics **2014**
University of North Carolina at Chapel Hill *Chapel Hill, North Carolina*

RESEARCH EXPERIENCE

Graduate Research Assistant **2014–2020**
University of California, Irvine *Irvine, California*

TEACHING EXPERIENCE

Teaching Assistant **2014–2020**
University of California, Irvine *Irvine, California*

ABSTRACT OF THE DISSERTATION

The Inverse Spectral Problem for Convex Planar Domains

By

Amir B. Vig

Doctor of Philosophy in Mathematics

University of California, Irvine, 2020

Associate Professor Hamid Hezari, Chair

The purpose of this dissertation is to develop the spectral theory for bounded planar domains. A particularly effective method for understanding the Laplace spectrum is via the wave equation. Chapters 3-6 develop a general theory for explicitly solving the wave equation in terms of dynamical data associated to the billiard map on the underlying domain. In particular, we develop a microlocal Hadamard-Riesz type parametrix for the wave propagator near reflecting rays in bounded planar domains with smooth, strictly convex boundary. This parametrix then allows us to rederive an oscillatory integral representation for the wave trace appearing in [44] and compute its principal symbol explicitly in terms of geometric data associated to the billiard map. This results in new formulas for the wave invariants. The order of the principal symbol, which appears to be inconsistent in the works of [44] and [60], is also corrected. In those papers, the principal symbol was never actually computed and to our knowledge, this dissertation (in addition to the author's previous works [69] and [68]) contain the first explicit formulas for the principal symbol of the wave trace. The wave trace formulas we provide are localized near both simple lengths corresponding to nondegenerate periodic orbits and degenerate lengths associated to one parameter families of periodic orbits

tangent to a single rational caustic. Existence of a Hadamard-Riesz type parametrix for the wave propagator appears to be new in the literature. This is also based on the author's prior work [69] in the special case of elliptical domains and [68] for general convex billiard tables. It allows us to circumvent the symbol calculus in [12] and [25] when computing trace formulas, which are instead derived from our explicit parametrix and a rescaling argument via Hadamard's variational formula for the wave trace. These techniques also appear to be new in the literature. In Chapter 7, we investigate C^1 isospectral deformations of the ellipse with Robin boundary conditions, allowing both the Robin function and domain to deform simultaneously. We prove that if the deformations preserve the reflectional symmetries of the ellipse, then the first variation of both the domain and Robin function must vanish. If the deformation is in fact smooth, reparametrizing allows us to show that the first variation actually vanishes to infinite order. In particular, there exist no such analytic isospectral deformations. The key ingredients are a version of Hadamard's variational formula for variable Robin boundary conditions and an oscillatory integral representation of the wave trace variation which uses action angle coordinates for the billiard map. For the latter, we use the explicit parametrix for the wave propagator developed in the interior, microlocally near geodesic loops (see Chapter 5).

Chapter 1

Introduction

1.1 Background

The inverse spectral problem has a long history, dating back to Kac in 1966 [33], who asked the famous question "can one hear the shape of a drum?" Mathematically, this corresponds to uniquely determining a domain Ω from the spectrum of its Dirichlet, Neumann or Robin Laplacian. For bounded domains, the spectrum is purely discrete, consisting of eigenvalues λ_j^2 satisfying

$$\begin{cases} -\Delta u_j = \lambda_j^2 u_j & x \in \Omega \\ B u_j = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where u_j are smooth eigenfunctions on Ω and B is either the restriction operator (Dirichlet boundary conditions), normal differentiation (Neumann boundary conditions) or normal differentiation plus a prescribed function on $\partial\Omega$ (Robin boundary conditions). In [33], a positive answer to this question was obtained in the special case of a Euclidean ball by using heat invariants and the isoperimetric inequality. Almost immediately the original question of Kac, John Milnor found an example of 16 dimensional isospectral tori which were not isometric. Sunada later generalized this example by using an algebraic method, but the question for planar domains remained open until 1992, when distinct polygonal domains in $\mathbb{R}^2$ were found to be isospectral in [15]. The question is still widely open for convex and/or smooth domains, although there has been significant progress. For example, it is proved in [76] that analytic domains with a single isometric involution are spectrally determined assuming some additional generic dynamical constraints on the length spectrum. Melrose also showed that the set of isospectral planar domains is precompact in the C^∞ topology by a careful analysis of the heat invariants (see [47]). This was improved in [50], [51] and [52], where the authors proved genuine C^∞ compactness of the isospectral set. This result is based on the Polyakov formula for ζ-regularized determinants and applies to both bounded planar domains and closed surfaces. Recently, Hezari and Zelditch showed in [26] that ellipses of small eccentricity are spectrally determined amongst all smooth planar domains, which is the first positive result in such generality since Kac's original paper [33]. Thorough surveys of the inverse spectral problem are contained in [78], [74], [9] and [45].

A variety of approaches in local and global harmonic analysis have been taken to prove partial results in the direction of [33]. One particularly useful strategy is to use the wave group to deduce spectral information about the underlying geometric space, usually a Riemannian manifold. The motivation behind this approach stems from Duistermaat and Hörmander's propagation of singularities theorem, which says that singularities of solutions to the wave equation propagate along (possibly broken) bicharacteristics, which are lifts to $T^*(\mathbb{R} \times \Omega)$ of geodesic or billiard orbits. As linear waves can be superimposed, constructive interference is most pronounced along geodesics which are traversed infinitely often, i.e. periodic orbits. On the trace side, this is reflected in the Poisson relation:

$$\text{SingSupp Tr}e^{it\sqrt{-\Delta}} \subset \pm \text{LSP}(\Omega) \cup \{0\}, \tag{1.2}$$

where the lefthand side is the distributional trace of the half wave propagator (see Section 6) and the righthand side is the length spectrum of Ω (the closure of all lengths of periodic geodesic or billiard orbits together with $\{0\}$). This is due to [1] in the case of bounded planar domains. In particular, the formula (1.2) generalizes the Poisson summation formula on the torus $\mathbb{R}^n/\mathbb{Z}^n$ from elementary Fourier analysis (see [67], for example).

Asymptotic formulas near the singularities are given by the Selberg trace formula in the case of hyperbolic surfaces [64], the Duistermaat-Guillemin trace theorem for general smooth manifolds under a dynamical nondegeneracy condition [12], and a Poisson summation formula for strictly convex bounded planar domains due to Guillemin and Melrose [20]. However, since these trace formulas involve sums over all periodic orbits of a given length, it is theoretically possible that the contributions of distinct orbits having the same length could cancel out and the wave trace is actually smooth near a point in the length spectrum. We say that the length $L \in \mathbb{R}$ of a periodic orbit γ is simple if up to time reversal $(t \mapsto -t)$, γ is the unique periodic orbit of length L. Without length spectral simplicity, there is no way to deduce Laplace spectral information from the length spectrum alone. It is shown in [56] that generically, smooth convex domains have simple length spectrum associated to only nondegenerate periodic orbits. In that case, the following theorem holds:

Theorem 1.1.1 ([20], [57]). *Assume γ is a nondegenerate periodic billiard orbit in a bounded, strictly convex domain with smooth boundary and γ has length L which is simple. Then near L, the even wave trace has an asymptotic expansion*

$$Tr \cos t\sqrt{-\Delta} \sim \operatorname{Re}\left\{ a_\gamma (t - L + i0)^{-1} + \sum_{k=0}^{\infty} a_{\gamma k} (t - L + i0)^k \log(t - L + i0) \right\}, \qquad (1.3)$$

where the coefficients $a_{\gamma k}$ are the wave invariants associated to γ.

Calculations of the wave invariants associated to dynamically convenient orbits have proved extremely useful in the inverse spectral problem associated to (1.1). For example, the case of rotationally symmetric metrics on S^2 is analyzed in [72]. In [75], [77] and [24], the wave invariants associated to bouncing ball orbits are explicitly calculated using Feynman diagrams to analyze the stationary phase computation from which Balian-Bloch (wave trace) formulas are derived (see also [73]). Under mild dynamical conditions and some additional axial symmetry assumptions, these coefficients can be used to determine the Taylor series of a local boundary parametrization. In particular, this allows one to deduce that such domains are spectrally determined amongst a rich class of analytic, symmetric domains.

The wave invariants have also proved useful in variational inverse problems, going back to the seminal papers [17] and [18], where the authors proved spectral rigidity for closed manifolds with negative sectional curvature. This was recently generalized to Anosov surfaces in [53]. In the setting of bounded domains, it is proved in [25] that ellipses with Dirichlet/Neumann boundary conditions are infinitesimally spectrally rigid through smooth domains with the same symmetries. These results as well as spectral determination of the Robin function in [19] were generalized to Liouville billiard tables of classical type in [61] and [63]. They were also extended to the Robin setting in the author's previous work [69], where both the domain and Robin function were allowed to vary simultaneously. A recent breakthrough was obtained in [26], where the authors showed that ellipses of small eccentricity are spectrally determined. Thorough surveys of the inverse spectral problem are contained in [78], [74], [9] and [45].

This dissertation is partially inspired by [44], where the following theorem is proved:

Theorem 1.1.2 ([44]). *If Ω is a bounded and strictly convex planar region, there exists $N = N(\Omega)$ such that if $j > N$, then the contribution $\widehat{\sigma}_j$ of E_j to $\widehat{\sigma}_D$ is of the form*

$$\widehat{\sigma}_j = \frac{1}{2\pi} \int_0^\infty \int_0^L e^{i(t-\mu_j(s))\tau} a(s,\tau) ds d\tau, \tag{1.4}$$

where in terms of an arclength coordinate s *on* $\partial\Omega$*,*

$$\mu_j(s) = \Psi_j(s,s')\big|_{s=s'},$$

with $\psi(s,s') = L(\mathfrak{g})$*, with* $\mathfrak{g}$ *the length of a* j*-fold geodesic from* s *to* s' *and* a_j *is periodic in* s*, classical and elliptic of order zero, with principal part of the form* $e^{i\pi r_j/4}\alpha_j(s)$*,* $\alpha_j(s) > 0$*.*

Here, $E_j(t)$ is the cosine kernel associated to the parametrix for the j reflection wave operator constructed in [6], which is reviewed in Section 5.2. In particular, if L_j is a simple length corresponding to a nondegenerate periodic orbit, then Theorem 1.1.2 gives an asymptotic expansion for the localized wave trace. In fact, if Ω satisfies the noncoincidence condition (2.5), i.e. $|\partial\Omega|$ is not a limit point from below of the lengths of orbits of rotation number m/n for $m \geq 2$, then the trace in Theorem 1.1.2 is a spectral invariant. One of the main purposes of this dissertation is to explicitly calculate the principal symbol $a(s,\tau)$ of (1.4) in terms of geometric data associated to the billiard map, both in the case of simple lengths corresponding to nondegenerate periodic orbits and also for one parameter families of degenerate periodic orbits, all having the same length associated to a caustic of rational rotation number. It also corrects several errors in literature (including Theorem 1.1.2 above) on the order of $a \in S^{1/2}_{\text{cl}}(\partial\Omega)$.

Dual to the Laplace spectrum is the so called *length spectrum*, which is a discrete set of numbers containing the lengths of periodic orbits for the geodesic or billiard flow. The same inverse problem exists: can one determine a manifold up to isometry from its length spectrum? The answer is unfortunately negative, as was shown for the case of constant negative curvature in [70]. However, it is conjectured by Katok and Burns that the *marked length spectrum* does determine a smooth closed manifold up to isometry ([4]). Here, the marked length spectrum also encodes the homotopy classes of periodic geodesics. Marked length spectral rigidity was recently shown in [16] for Anosov manifolds. The relationship with the Laplace spectrum is contained in the Poisson relation, which tells us that the singularities of the wave trace are a subset of the length spectrum. Assymptotic formulas near the singularities are given by the Selberg trace formula for hyperbolic surfaces ([64]), the Duistermaat-Guillemin trace theorem ([12]) for general manifolds under a dynamical nondegeneracy condition, and a Poisson summation formula for strictly convex bounded planar domains due to Guillemin and Melrose ([20]). However, since these trace formulae involve sums over all periodic orbits of a given length, it is theoretically possible that the contributions of distinct orbits having the same length could cancel out and the wave trace is actually smooth near a point in the length spectrum. Hence, without length spectral simplicity, there is no way to deduce Laplace spectral information from the length spectrum alone.

While some results on spectral rigidity are known in the chaotic regime (see for example [17], [18] and [54]), very little is known about the completely integrable setting, in which the flow has a maximal number of conserved quantities (see Section 3). In the theory of dynamical systems, a famous conjecture of Birkhoff is that the only strictly convex planar domains with completely integrable billiards are ellipses. While this remains an open conjecture, much progress has been made in the local setting. It is shown in [34] that if a *rationally integrable* billiard table is sufficiently close to an ellipse, then it must be an ellipse. Rational integrability means that for each integer $q \geq 1$, the billiard map has invariant curves of rotation number $1/q$, consisting entirely of periodic points. Using Aubry-Mather theory, the authors then show that ellipses are length spectrally rigid (Corollary 14 of [34]).

The ellipse is smooth, convex and has completely integrable dynamics, which makes it an interesting object to study in the context of spectral theory. In fact:

Conjecture 1.1.3 (Melrose, [45])**.** *Ellipses are spectrally determined.*

In [10], the authors show that convex domains with $\mathbb{Z}_2$ axial symmetry which are sufficiently close to a circle in C^k are length spectrally rigid. These domains include ellipses of small eccentricity. It is shown in [56] that generically, convex domains have simple length spectrum and nondegenerate Poincaré map. Combining the results in [10] and [56], it then follows that $\mathbb{Z}_2$ symmetric domains close to a circle are Δ spectrally rigid amongst a generic class of symmetric domains. In [23], these results are extended to the Robin Laplacian, where the Robin function on the boundary is also allowed to deform through smooth functions with the same same $\mathbb{Z}_2$ symmetry. Our problem is similar in nature but considers ellipses of arbitrary eccentricity, which might not be close to a circle.

In the special case of an ellipse, the trace formulas developed in Chapters 6-7 are inspired by [25], [19], [20] and [55]. Guillemin and Melrose proved a version of the Poisson summation formula for bounded planar domains and then used this result in a subsequent article to show that for a fixed ellipse, a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric Robin function on the boundary is completely determined by the spectrum of the associated Laplacian. Hezari and Zelditch then proved infinitesimal spectral rigidity for Dirichlet/Neumann boundary conditions, while only letting the domain deform. In their proof, the authors used the symbol calculus in [12] to compute the trace of the wave kernel near periodic transversal reflecting rays. The problem addressed in this dissertation (see Theorem 2.2.1) allows both the domain and Robin function to deform simultaneously, which doesn't allow us to directly employ the results in [19] or [25].

Chapter 2

Statement of Results

2.1 General trace formulas

One of the key elements of this dissertation is to develop a Hadamard-Riesz type parametrix for the wave propagator in bounded planar domains with smooth, strictly convex boundary. This is done in Chapters 1-5. We then use this parametrix to produce asymptotic expansions for the distributional wave trace near isolated lengths in the length spectrum in Chapters 6-7. Let Ω be such a domain and denote by Δ the Dirichlet Laplacian on Ω. The even wave propagator $E(t)$ is defined to be the solution operator for the wave equation

$$\begin{cases} (\partial_t^2 - \Delta)E = 0 & (x \in \Omega) \\ E(0) = \mathrm{Id} & (x \in \Omega) \\ \partial_t E\big|_{t=0} = 0 & (x \in \Omega), \end{cases} \tag{2.1}$$

with Dirichlet boundary conditions. In spectral theoretic terms, we can write $E(t) = \cos t\sqrt{-\Delta}$, which is the even part of the half wave propagator $e^{it\sqrt{-\Delta}}$. In [6], it is shown that there exist Lagrangian distributions $E_j(t, x, y)$ such that microlocally away from the tangential rays, the Schwartz kernel of $E(t)$ is given by

$$E(t,x,y) = \sum_{j=-\infty}^{+\infty} E_j(t,x,y) + C^\infty(\mathbb{R}\times\Omega\times\Omega), \tag{2.2}$$

with E_j corresponding to a wave of j reflections at the boundary. The sum in (2.2) is locally finite in time. If we restrict our attention to waves which make j reflections and travel approximately once around the boundary, we have the following explicit parametrix for $E_j(t,x,y)$.

Theorem 2.1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth, strictly convex boundary. Then there exists $j_0 = j_0(\Omega) \in \mathbb{N}$ sufficiently large such that the following holds: for all $j \geq j_0$, there exists a tubular neighborhood U_j of the diagonal of the boundary $\Delta\partial\Omega \subset \Omega\times\Omega$ such that for all $(x,y) \in U_j$ and t less than but sufficiently close to $|\partial\Omega|$,*

$$E_j(t,x,y) = \sum_{\pm}\sum_{k=1}^{8}(-1)^j e^{\pm i\pi/4}\int_0^\infty e^{\pm i\tau(t-\Psi_j^k(x,y))} b_{j,k,\pm}(\tau,x,y)d\tau + C^\infty(\mathbb{R}\times U_j),$$

microlocally near geodesic loops of rotation number $1/j$. Here, $b_{j,k,\pm} \in S^{1/2}_{cl}(U_j\times\mathbb{R}^1)$ are classical elliptic symbols of order $1/2$ and the functions $\Psi_j^k(x,y)$ $(1 \leq k \leq 8)$ are lengths of the 8 billiard orbits connecting x to y in j reflections and approximately one rotation (see Theorem 3.3.1). The principal term in the asymptotic expansion for $b_{j,k,\pm}$ is given in boundary normal coordinates $x = (\mu,\varphi)$, $y = (\nu,\theta)$ (see Section 5.3) by

$$\frac{\pm|\tau|^{1/2}}{2} f^{1/2}(\mu,\varphi)\left|\frac{\partial\Psi_j^k}{\partial\mu}\frac{\partial\Psi_j^k}{\partial\theta}\frac{\partial^2\Psi_j^k}{\partial\nu\partial\varphi} + \frac{\partial\Psi_j^k}{\partial\varphi}\frac{\partial\Psi_j^k}{\partial\nu}\frac{\partial^2\Psi_j^k}{\partial\theta\partial\mu} - \frac{\partial\Psi_j^k}{\partial\mu}\frac{\partial\Psi_j^k}{\partial\nu}\frac{\partial^2\Psi_j^k}{\partial\theta\partial\varphi} - \frac{\partial\Psi_j^k}{\partial\varphi}\frac{\partial\Psi_j^k}{\partial\theta}\frac{\partial^2\Psi_j^k}{\partial\nu\partial\mu}\right|^{1/2},$$

where $f(\mu,\varphi)$ is a positive function which is smooth up to the boundary $(\mu = 0)$ and satisfies $f(0,\varphi) = 1$.

Theorem 2.1.1 bears a remarkable resemblance to Hadamard's parametrix for the wave propagator on boundaryless manifolds (see [22], [22]), where the phase functions $\pm\tau(t-\Psi_j^k(x,y))$ are replaced by $\pm\tau(t-r(x,y))$, with $r(x,y)$ the geodesic distance from x to y. In that case, for x,y near the diagonal, there is only one geodesic connecting x to y. In our setting, Theorem 3.3.1 shows that there are exactly 8 orbits connecting x to y in j reflections and approximately one rotation.

Remark 2.1.2. For $j \geq j_0(\Omega)$ and $x, y \in \partial\Omega$ sufficiently close to the diagonal, there exist only two orbits connecting x to y in j reflections and approximately one rotation. One is in the clockwise direction and the other is in the counterclockwise direction. In particular, when $x = y \in \partial\Omega$, there exists a unique geodesic loop based at x of rotation number $1/j$. Restricting the functions $\Psi_j^k(x,y)$ appearing in Theorem 2.1.1 to the diagonal of the boundary yield the *j-loop function*, which we denote by $\Psi_j(q,q)$.

As in the case of boundaryless manifolds, we can use the explicit parametrix in Theorem 2.1.1 to prove trace formulas. It is known that $E(t)$ has a well defined distributional trace

$$\operatorname{Tr}\cos t\sqrt{-\Delta} = \sum_{j=0}^{\infty} \cos t\lambda_j, \tag{2.3}$$

where λ_j^2 are the Dirichlet eigenvalues of $-\Delta$. The sum in (2.3) converges in the sense of tempered distributions and has singular support contained in the length spectrum

$$\mathrm{LSP}(\Omega) = \overline{\{\text{lengths of periodic billiard trajectories}\}} \cup \{0\},$$

together with $-\mathrm{LSP}(\Omega)$ (see Section 6). Each periodic billiard orbit in Ω can be classified according to its winding number m and the number of reflections n made at the boundary. Denote the collection of periodic orbits of this type by $\Gamma(m,n)$, normalized so that $m \leq n/2$. $\Gamma(m,n)$ is never empty by a theorem of Birkhoff [3]. The length spectrum can be decomposed accordingly as

$$\mathrm{LSP}(\Omega) = \bigcup_{m,n\in\mathbb{N}} \mathrm{length}(\Gamma(m,n)) \cup \mathbb{N}|\partial\Omega|, \tag{2.4}$$

where $|\partial\Omega|$ is the length of the boundary. Using the parametrix in Theorem 2.1.1, we can prove the following theorem.

Theorem 2.1.3. *Assume the conditions from Theorem 2.1.1 hold and Ω satisfies the non-coincidence condition:*

$$\textit{there exists } \varepsilon_0 > 0 \textit{ such that } \bigcup_{\substack{m\geq 2\\ n\geq 1}} \textit{length}(\Gamma(m,n)) \cap (|\partial\Omega| - \varepsilon_0, |\partial\Omega|) = \emptyset. \tag{2.5}$$

For $j \geq j_0$, define $t_j = \inf_{q\in\partial\Omega} \Psi_j(q,q)$ and $T_j = \sup_{q\in\partial\Omega} \Psi_j(q,q)$. Then on any sufficiently small neighborhood of $[t_j, T_j]$, $TrE(t)$ has the asymptotic expansion

$$(-1)^j \,\mathrm{Re}\left\{ e^{i\pi/4} \int_{\partial\Omega} \int_0^\infty e^{i\xi(t-\Psi_j(q,q))} a^j(q,\xi) d\xi dq \right\} + C^\infty(\mathbb{R}),$$

where $a^j(q,\xi)$ is a classical elliptic symbol of order $1/2$ with principal part given by

$$a_0^j(q,\xi) = 4\xi^{1/2} \sin\omega_{j,1}(q,q) \sin^{1/2}\omega_{j,2}(q,q) \left| \frac{\partial\omega_{j,1}}{\partial q'}(q,q) \right|^{1/2} X(q)\cdot N(q).$$

Here, $X(q)$ is the position vector to a boundary point q with respect to a fixed origin and $N(q)$ is the outward unit normal at q. The angles $\omega_{j,1}, \omega_{j,2}$ are the initial and final angles respectively of the unique billiard orbit $\gamma_j(q,q')$ which connects nearby boundary points q and q' in j reflections and approximately one counterclockwise rotation. The function $\Psi_j(q,q')$ is the length of $\gamma_j(q,q')$ and its restriction to the diagonal is the j-loop function.

Remark 2.1.4. As the position vector X is chosen with respect to an arbitrary interior point p, we can integrate out this symmetry over any measurable subset of the interior in the variable p to obtain a more invariant formula. In particular, we can integrate over open sets, curves and by a limiting argument, the boundary itself in order to obtain a smooth density on $\partial\Omega$. The noncoincidence condition (2.5) can be weakened and is in particular satisfied for ellipses (see [19]) and nearly circular domains (see [26]).

Theorem 2.1.3 provides an explicit formula for the principal term in the parametrix developed in [44]. In contrast to the methods employed in [44], the proof developed in the remainder of this dissertation uses Hadamard's variational formula for the wave trace and the explicit parametrix for the wave propagator appearing in Theorem 2.1.1. These techniques are of independent interest. Theorem 2.1.3 also provides clarity on a discrepancy in the literature regarding the order of the wave trace (cf. [60], [44]). Note that in Theorem 2.1.3, no assumptions are made on the nondegeneracy of orbits. If the length spectrum has high multiplicity, the wave trace is in general quite complicated. However, when periodic orbits come in a one parameter family corresponding to a caustic, we have the following:

Theorem 2.1.5. *Suppose Ω, $j \geq j_0$ are as in Theorem 2.1.3 and $\mathcal{C}$ is a caustic for Ω of rotation number $1/j$. If periodic orbits tangent to $\mathcal{C}$ have length L_j, then near $t = L_j$, $TrE(t)$ has the leading asymptotic*

$$c_j \operatorname{Re}\left\{e^{i\pi/4}(t - L_j - i0)^{-3/2}\right\},$$

where c_j is a wave invariant given by the formula

$$c_j = (-1)^{j+1} 4 \int_{\partial\Omega} \sin^{3/2} \omega_{j,1}(q,q) \left| \frac{\partial \omega_{j,1}}{\partial q'}(q,q) \right|^{1/2} X(q) \cdot N(q) dq.$$

In this case, $\omega_{j,1}(q,q) = \omega_{j,2}(q,q)$ is the measure of the angle of incidence for the unique periodic orbit of rotation number $1/j$ based at $q \in \partial\Omega$.

While KAM theory provides the existence of irrational caustics, it is shown in [35] that for a fixed j, the set of all smooth convex domains possessing a rational caustic of rotation number

$1/j$ is polynomially dense in the variable $1/j$ in the collection of all smooth domains, equipped with the C^∞ topology. Exponential density is also proven in the analytic category. As ellipses satisfy the noncoincidence condition (2.5) (see [19]) and are known to be completely integrable with confocal conic sections as caustics, we have the following corollary to Theorem 2.1.5.

Corollary 2.1.6. *For an ellipse Ω given by*

$$\Omega = \left\{ (x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\},$$

and $L_j \in LSP(\Omega)$ sufficiently close to $|\partial\Omega|$ corresponding to the length of billiard orbits of rotation number $1/j$, the wave invariants in Theorem 2.1.5 are given by

$$c_j = \int_0^{2\pi} \frac{(-1)^{j+1} 2ab \sin\omega_j \sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}\, d\varphi}{\sqrt{\cos\omega_j (a^2\sin^2\varphi + b^2\cos^2\varphi)(b^2 + (a^2 - b^2)\sin^2\varphi) G(\zeta_j) \sqrt{1 - k_{\zeta_j}^2 \sin\varphi}}}.$$

Here, $\zeta_j \in [0, b)$ is the parameter of the confocal ellipse

$$\mathcal{C}_{\zeta_j} = \left\{ (x, y) : \frac{x^2}{a^2 - \zeta_j^2} + \frac{y^2}{b^2 - \zeta_j^2} = 1 \right\},$$

to which the orbits of length L_j are tangent and k_{ζ_j} is given by

$$k_{\zeta_j}^2 = \frac{a^2 - b^2}{a^2 - {\zeta_j}^2}.$$

The parameter ζ_j depends on the rotation number $1/j$ and is defined implicitly by the equation

$$\frac{1}{j} = \frac{F(\arcsin \zeta_j / b; k_{\zeta_j})}{2K(\zeta_j)},$$

where $F(s;k)$ is the elliptic integral

$$\int_0^s \frac{d\tau}{\sqrt{1-k^2\sin^2\tau}}$$

and $K(\zeta_j) = F(\pi/2; k_{\zeta_j})$. The function $G(\zeta_j)$ is defined by

$$\frac{-k_{\zeta_j}^2}{(a^2-\zeta_j^2)}\int_0^{2\pi}\frac{\sin^2\tau d\tau}{(1-k_{\zeta_j}^2\sin^2\tau)^{3/2}} + (2j+2)\frac{d}{d\zeta^2}F(\arcsin\zeta/b; k_\zeta)\Big|_{\zeta=\zeta_j},$$

and $\omega_j = \omega_j(\varphi)$ are the angles of reflection for orbits tangent to $\mathcal{C}_{\zeta_j}$, given implicitly by the equation

$$\zeta_j^2 = \sin^2\omega_j(b^2 + (a^2-b^2)\sin^2\varphi).$$

Remark 2.1.7. Analogous formulas to those appearing in Theorem 2.1.3, Theorem 2.1.5 and Corollary 2.1.6 can also be proved for the Neumann and Robin wave traces. The formulas are less succinct but can be easily reproduced by following the proof of Theorem 2.1.3 and combining it with the Hadamard type variational formulas for Robin boundary conditions in the author's previous work [69].

2.2 Robin Spectral Rigidity of the Ellipse

In Chapter 7, we follow the author's previous work [69] in proving infinitesimal spectral rigidity of the ellipse with C^1 deformations of both the domain and Robin boundary conditions which preserve the symmetries of the ellipse. This means that the first variations of both the domain and Robin function vanish. To make this precise, we consider the eigenvalue problem for the Laplacian. Let Ω_0 be an ellipse and φ_ε a C^1 family of diffeomorphisms defined in a neighborhood of Ω_0 for $0 \le \varepsilon < \varepsilon_0$, such that $\varphi_0 = \mathrm{Id}$. Denote $\Omega_\varepsilon = \varphi_\varepsilon(\Omega_0)$ and let K_ε be a

C^1 family of smooth functions on $\partial\Omega_\varepsilon = \varphi_\varepsilon(\partial\Omega_0)$. The PDE we are interested in is

$$\begin{cases} -\Delta u_\varepsilon = \lambda^2(\varepsilon) u_\varepsilon, & x \in \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu} = K_\varepsilon u_\varepsilon, & x \in \partial\Omega_\varepsilon. \end{cases} \tag{2.6}$$

The equation (2.6) is said to have Robin boundary condition with Robin function K_ε. As $K_\varepsilon \in C^\infty(\partial\Omega_\varepsilon)$, it is shown in [66] that $-\Delta$ is self adjoint on $L^2(\Omega_\varepsilon)$ with densely defined domain $D = \{u \in H^2(\Omega_\varepsilon) : \partial_\nu u_\varepsilon = K_\varepsilon u_\varepsilon\}$. The analytic Fredholm theorem guarantees that (2.6) has nontrivial solutions only for a discrete collection of eigenvalues $\lambda_j^2 \in \mathbb{R}$, comprising the entire spectrum which we denote by Spec(Ω_ε). We may assume $\varphi_\varepsilon(x) = x + \rho_\varepsilon(x)\nu_x$ for $x \in \partial\Omega_0$, where ν_x is the outward unit normal to $\partial\Omega_0$ and ρ_ε is a smooth function on $\partial\Omega_0$. We also impose the restriction that ρ_ε and K_ε should be invariant under the symmetry group of the ellipse, which is generated by reflections through the coordinate axes and is isomorphic to the Klein four-group. We denote the Laplace operator with boundary conditions above by Δ_ε and prove the following result:

Theorem 2.2.1. *Let $\varphi_\varepsilon(\Omega_0) = \Omega_\varepsilon$ be a C^1 deformation of the ellipse through smooth domains with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetries and K_ε a C^1 family of Robin functions with the same symmetries. If the deformation is isospectral, i.e. Spec$(\Delta_\varepsilon) =$ Spec(Δ_0) for $0 \le \varepsilon < \varepsilon_0$, then $\dot{\rho} = \dot{K} = 0$.*

Here, we have written $\dot{\rho}(x) = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} \rho_\varepsilon(x)$ and $\dot{K}(x) = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} K_\varepsilon(x + \rho_\varepsilon(x)\nu_x)$ to denote the first variations, which are studied in more detail in Section 4.1. We also use the notation $\delta = \frac{d}{d\varepsilon}\big|_{\varepsilon=0}$. We then say the ellipse is infinitesimally spectrally rigid through domains and Robin boundary conditions with the symmetries of an ellipse. The prefix "infinitesimal" means that only the first variation vanishes. If it could be shown that no nontrivial such isospectral deformations exist, we would say the ellipse is spectrally rigid amongst such domains. As a quick corollary to Theorem 2.2.1, we have:

Corollary 2.2.2. *There are no nontrivial analytic isospectral deformations of the ellipse* *through $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric domains and Robin functions.*

This follows from a simple reparametrization argument which can be found in Section 3.2 of [25]. One of the main ingredients in our proof is a version of Hadamard's variational formula for variable Robin boundary conditions, which also appears to be new in the literature:

Theorem 2.2.3. *Let G_ε be the Green's kernel for the eigenvalue problem with Robin boundary conditions on Ω_ε. Then, for $x, y \in int(\Omega_0)$, $\delta G_\varepsilon(\lambda, x, y)$ is the distribution*

$$\int_{\partial\Omega_0} -\langle \nabla_2^T G_0(\lambda, x, q), \nabla_1^T G_0(\lambda, q, y)\rangle \dot{\rho} + (\lambda^2 \dot{\rho} + K_0^2 \dot{\rho} + K_0 \kappa \dot{\rho} + \dot{K}) G_0(\lambda, x, q) G_0(\lambda, q, y) dq.$$

Here, ∇_i^T denotes the tangential derivative in the ith spatial variable and dq is the natural line element on $\partial\Omega_0$ inherited from the flat metric on $\mathbb{R}^2$. The Green's kernel or Green's function is the Schwartz kernel of the resolvent $(-\Delta - \lambda^2)^{-1}$ for $\operatorname{Im} \lambda^2 > 0$. In Section 3, this result is extended to variational formulas for both the even wave trace $\operatorname{Tr} \cos(t\sqrt{-\Delta_\varepsilon})$ and simple eigenvalues. In particular, we prove the following in Section 7.3:

Theorem 2.2.4. *For the ellipse $\Omega_0 = \{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$, with boundary parametrized by $(x, y) = (a\cos\varphi, b\sin\varphi)$, there exists a $j_0 = j_0(\Omega)$ sufficiently large such that for all $j \geq j_0$, the variation of the wave trace near a length $L_j \in Lsp(\Omega_0)$ corresponding to a caustic of rotation number $1/j$ is given by*

$$\delta\, Tr \cos t\sqrt{-\Delta_\varepsilon} = c_j t \operatorname{Re}\left\{ e^{i\pi/4}(t - L_j - i0^+)^{-5/2} \right\} + L.O.T.,$$

The constants c_j are given by

$$c_j = \int_0^{2\pi} P(\lambda_j) Q(\varphi) \frac{\dot{\rho}(\varphi) d\varphi}{\sqrt{(b^2 + (a^2 - b^2)\sin^2\varphi) - \lambda_j^2}}.$$

Here, L.O.T. denotes lower order distributional terms and λ_j is the parameter of the confocal *ellipse*

$$\frac{x^2}{a^2-\lambda_j^2}+\frac{y^2}{b^2-\lambda_j^2}=1,$$

to which periodic orbits of length L_j are tangent. P and Q are certain nonzero analytic functions. Moreover, if $\dot{\rho}=0$, then the leading order term in the singularity expansion near L_j becomes

$$\delta\, Tr \cos t\sqrt{-\Delta_\varepsilon}=\widetilde{c}_j t\,\mathrm{Re}\left\{e^{i\pi/4}(t-L_j-i0^+)^{-1/2}\right\}+L.O.T.,$$

where the constants $\widetilde{c}_j$ are given by

$$\widetilde{c}_j=\int_0^{2\pi}\widetilde{P}(\lambda_j)\widetilde{Q}(\varphi)\frac{\dot{K}(\varphi)d\varphi}{\sqrt{(b^2+(a^2-b^2)\sin^2\varphi)-\lambda_j^2}}.$$

$\widetilde{P}$ and $\widetilde{Q}$ are again certain nonzero analytic functions.

Remark 2.2.5. In [19], it is shown that ellipses satisfy the noncoincidence condition 2.5, i.e. there exists a j_0 such that for all $j\geq j_0$, the length L_j of a periodic orbit with rotation numbers $1/j$ does not coincide with the length of any periodic orbit making more than 1 rotation. In particular, all periodic orbits of length L_j are tangent to a single confocal conic section. These orbits all have rotation number $1/j$, making j reflections at the boundary with winding number 1. In Section 7.3, we will send $j\to\infty$ and use the constants c_j and $\widetilde{c}_j$ to show that $\dot{\rho}$ and $\dot{K}$ vanish.

The idea of the proof of Theorem 2.2.1 is that if the deformation is isospectral, then the variation of the wave trace should also be zero. The Poisson relation, in this case due to Guillemin and Melrose ([19]), tells us that the singularities of the wave trace are contained in the length spectrum $\mathrm{Lsp}(\Omega_0)$, the set of lengths of periodic trajectories for the broken bicharacteristic (billiard) flow. Using microlocal analysis and in particular, Chazarain's parametrix, we can localize the wave kernel near the periodic transversal reflecting rays. From this, we obtain a singularity expansion for the wave trace variation, a Fourier integral distribution, near the length spectrum. We then cook up an oscillatory integral which microlocally approximates this distribution by using a special phase function associated to the billiard map. To do this, we actually construct an explicit parametrix for the microlocalized wave kernel near all orbits tangent to a confocal ellipse of rotation number $1/j$ (Theorem 5.4.3). In particular, this involves finding all orbits making approximately one rotation with a prescribed number of reflections which connect two points in an interior neighborhood of the diagonal of the boundary (Theorem 3.3.1). This is of independent interest in the theory of dynamical billiards.

In the special case of the ellipse, we can incorporate action angle coordinates for the billiard map, which allows us to convert the singularity expansion for the wave trace near $\mathrm{Lsp}(\Omega_0)$ into the product of a nonzero distribution and an elliptic integral as in Theorem 2.2.4. Since this expansion is valid near any isolated length sufficiently close to $|\partial\Omega|$, we can take a special sequence of caustics creeping closer and closer to the boundary. Following the ideas in [19], we send $j \to \infty$ and analyze the coefficients c_j and $\widetilde{c_j}$ in Theorem 2.2.4. These coefficients are analytic in the paramater λ^2 and since $\lambda_j^2 \to 0$ as $j \to \infty$, we see that they are actually flat at $\lambda = 0$. An application of the Stone-Weierstrass theorem then shows that $\dot{\rho} = 0$. Upon substituting $\dot{\rho} = 0$, we obtain a new singularity expansion for the subprincipal term with $\dot{K}$ only. The same tricks show $\dot{K} = 0$.

2.3 Schematic Outline

The proofs of Theorem 2.1.3, Theorem 2.1.5 and Corollary 2.1.6 use techniques from [69] and [68], which are reviewed throughout. The first step in the proof of Theorem 2.1.3 is to generalize the Hadamard-Riesz type parametrix for the wave propagator constructed in [69] for ellipses to arbitrary bounded domains with strictly convex, smooth boundary as in Theorem 2.1.1. This requires a dynamical classification (Theorem 3.3.1) of the cardinality and structure of all billiard orbits connecting interior points with a fixed number of reflections, analogous to Lemma 5.2 in [69]. Chapter 3 introduces language from dynamical systems necessary to describe the billiard (or broken bicharacteristic) flow, which later appears in the canonical relations of the wave propagator. The proof of Theorem 3.3.1 is relegated to Section 3.4, where it is broken up into several intermediate lemmas. This material is largely independent from the rest of the dissertation and is of separate interest from the perspective of dynamical billiards, irrespective of applications to spectral theory. In Section 3.2, we review the special properties of billiards in elliptical domains. An alternate proof of Theorem 3.3.1 for elliptical billiard tables is presented in Appendix B and uses Jacobi elliptic function theory.

Chapter 4 is then devoted to proving a version of Hadamard's variational formula with Robin boundary conditions (Theorem 2.2.3) and extending this to variational formulas for both the localized wave trace near lengths of periodic orbits (Theorem 4.1.7) and simple eigenvalues (Theorem 4.1.8). In Chapter 5, the length functionals corresponding to the orbits in Theorem 3.3.1 allow us to cook up explicit phase functions which parametrize the canonical relations for the wave propagator $e^{it\sqrt{-\Delta}}$, which is a Fourier integral operator microlocally away from the tangential rays. These canonical relations are discussed in detail in Section 5.2. This parametrix is microlocalized near approximate geodesic loops, which are particularly useful when integrating the wave kernel over the diagonal in the trace formulas which follow. Construction of an explicit parametrix for the wave propagator in the interior appears to be new in the literature and follows the author's prior works [69] in the special case of an ellipse and [68] for general convex domains. This construction is carried out in boundary normal coordinates and elliptical polar coordinates, both of which are used in Chapter 6 to prove concrete trace formulas.

In Section 6, we then use this parametrix to obtain a singularity expansion for a variation of the wave trace corresponding to radial dilation about an interior point, following the Hadamard type variational formulas in [25] and [69] which were reviewed in Chapter 4. The variational formulas are of interest because they allow us to restrict our study of the wave kernels to their behavior at the boundary. This technique for studying the wave trace at a fixed domain by variational methods also appears to be new in the literature. In Section 6.3, the honest principal symbol of the wave trace is computed in boundary normal coordinates by integrating its variation in the radial variable. This is essentially the fundamental theorem of calculus combined with some rescaling symmetries of the wave equation. In Section 6.3.1, it is shown that radially integrating subprincipal terms coming from the variational wave trace doesn't contribute to the principal symbol. As the order of the principal symbol computed in Section 6.3 appears to contradict other works in the literature, Appendix **??** provides an auxiliary confirmation via stationary phase that the order acheived in this dissertation is indeed correct. To our knowledge, these computations provide the first explicit formulas for the principal symbol of the wave trace associated to a convex billiard table.

Chapter 7 is devoted to proving Theorem 2.2.1, Theorem 2.2.2, Theorem 2.2.4 and Corollary 2.1.6. Section 7.2 analyzes the singularity coefficients of the variational trace formula in Theorem 7.1.4 by using Jacobi elliptic function theory. Section 7.3 converts the variational trace formula into an elliptic integral and then uses a method of Guillemin and Melrose to show that the first variation of both the Robin function and domain vanish, which completes the proof of Theorem 2.2.1.

Chapter 3

Billiards

3.1 Definitions

Before obtaining a singularity expansion for the wave trace, we first review the relevant background needed on billiards. This will be useful in our discussion of Chazarain's parametrix in Section 5.2. In this section, we denote by Ω a bounded strictly convex region in $\mathbb{R}^2$ with smooth boundary. This means that the curvature of $\partial\Omega$ is a strictly positive function. The billiard map is defined on the coball bundle of the boundary $B^*\partial\Omega = \{(q,\zeta) \in T^*\partial\Omega : |\zeta| < 1\}$, which can be identified with the inward part of the circle (cosphere) bundle $S^*_{\partial\Omega}\mathbb{R}^2$, via the natural orthogonal projection map. We can also identify $B^*\partial\Omega$ with $\mathbb{R}/\ell\mathbb{Z} \times (0,\pi)$, where $\ell = |\partial\Omega|$ is the length of the boundary. Define

$$
\begin{aligned}
t^1_\pm(y,\eta) &= \inf\{t > 0 : g^{\pm t}(y,\eta) \in \partial\Omega\}, \\
t^{-1}_\pm(y,\eta) &= \sup\{t < 0 : g^{\pm t}(y,\eta) \in \partial\Omega\},
\end{aligned}
$$

where π_1 is projection onto the first factor and $g^{\pm t}$ is the forwards $(+)$ or backwards $(-)$ geodesic flow on $\mathbb{R}^2$, corresponding to the Hamiltonian $H_\pm = \pm|\eta|$ (see Section 5.2). If

$(y,\zeta) \in B^*\partial\Omega$ is mapped to the inward pointing covector $(y,\eta) \in T^*_{\partial\Omega}\mathbb{R}^2$ under the inverse projection map, then we define

$$\beta^{\pm 1}(y,\eta) = g^{\widehat{t^1_{\pm}(y,\eta)}},$$

where a point $\widehat{(x,\xi)}$ is the reflection of ξ through the cotangent line $T^*_x\partial\Omega$. In other words, $\widehat{(x,\xi)}$ has the same footpoint and (co)tangential projection as (x,ξ), but reflected conormal component, so that it is again in the inward facing portion of the circle bundle. We call $\beta := \beta^{+1}$ the billiard map. It is well known that β preserves the natural symplectic form induced on $B^*(\partial\Omega)$ and is differentiable there, extending continuously up to the boundary. The maps $\beta^{\pm n}$ are defined via iteration and it is clear that $\beta^{-n} = (\beta^n)^{-1}$ for each $n \in \mathbb{Z}$. Associated to the billiard map is the billiard flow, or broken bicharacteristic flow, which we denote by Φ^t.

Geometrically, a billiard orbit corresponds to a union of line segments which are called links. A smooth closed curve $\mathcal{C}$ lying in Ω is called a caustic if any link drawn tangent to $\mathcal{C}$ remains tangent to $\mathcal{C}$ after an elastic reflection at the boundary of Ω. By elastic reflection, we mean that the angle of incidence equals the angle of reflection at an impact point on the boundary. We can map $\mathcal{C}$ onto the total phase space $B^*\partial\Omega$ to obtain a smooth closed curve which is invariant under β. If the dynamics are integrable, these invariant curves are precisely the Lagrangian tori which foliate phase space. A point P in $B^*\partial\Omega$ is called q-periodic ($q \geq 2$) if $\beta^q(P) = P$. We define the rotation number of a q-periodic orbit P by $\omega(P) = \frac{p}{q}$, where p is the winding number of the orbit generated by P, which we now define. We may consider the modified billiard map $\widetilde{\beta} = \Pi^*\beta$, where Π is the natural mapping from $\mathbb{R}/\ell\mathbb{Z} \times [0,\pi]$ to the closure of the coball bundle $\overline{B^*\partial\Omega}$. Pulling back by Π clearly preserves the notion of periodicity. There exists a unique lift $\widehat{\beta}$ of the map $\widetilde{\beta}$ to the closure of the universal cover $\mathbb{R} \times [0,\pi]$ which is continuous, ℓ periodic in first variable and satisfies $\widehat{\beta}(x,0) = (x,0)$. Given this normalization, for any point $(x, \theta) \in \mathbb{R}/\ell\mathbb{Z} \times [0, \pi]$ in a q periodic orbit of $\widetilde{\beta}$, we see that $\widehat{\beta}^q(x, \theta) = (x + p\ell, \theta)$ for some $p \in \mathbb{Z}$. We define this p to be the winding number of the orbit generated by $\Pi(x, \theta) \in \overline{B^*\partial\Omega}$. We see that even if a point $\Pi(x, \theta)$ generates an orbit which is not periodic in the full phase space but is such that $\pi_1(\widetilde{\beta}^q(x, \theta)) = x$ for some $q \in \mathbb{Z}$, we can still define a winding number in this case. Such orbits are called loops or geodesic loops. For a given periodic orbit, the winding number is independent of which point in the orbit is chosen, so we sometimes write $\omega(\gamma) = \omega(P)$ for any $P \in \gamma = \{P, \beta(P), \cdots, \beta^{q-1}(P)\}$. For deeper results and a more thorough introduction to dynamical billiards, we refer the reader to [65], [37], [60] and [62].

3.2 Elliptical Billiards

From here on, we let Ω be an ellipse with horizontal major axis, given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1.$$

The eccentricity of Ω is defined by $E = a^2 - b^2$.

Birkhoff conjectured that the only strictly convex integrable billiard tables are ellipses. Completely integrable means that there exists a foliation of the phase space by invariant submanifolds. In the context of Hamiltonian systems, this can be shown to be equivalent to the existence of a maximal number of Poisson commuting invariants, called first integrals. The (compact) energy level sets of regular values of these invariants can be shown to be diffeomorphic to tori and the leaves of a maximal such foliation are called Lagrangian tori. The Lagrangian tori are naturally parametrized by so called "angle coordinates" while the transversal directions in phase space are then parametrized by "action coordinates". It is well known that for each $Z \in (-E, 0) \cup (0, b]$, the confocal ellipse ($Z > 0$) or hyperbola

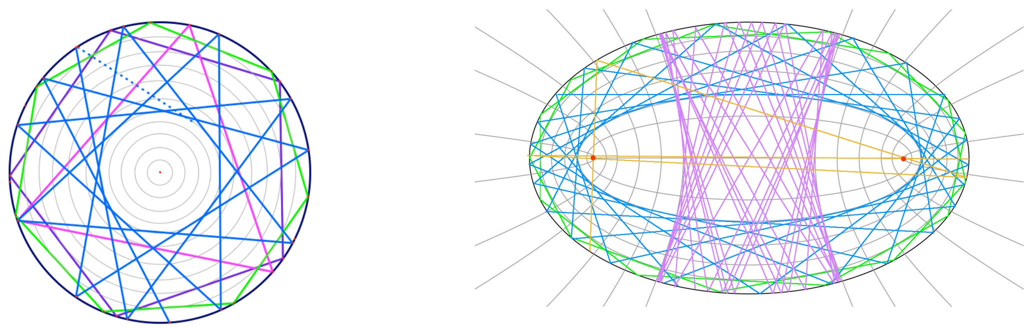

Figure 3.1: Billiards and caustics on the disk and ellipse. [1]

$(Z < 0)$ given by

$$\frac{x^2}{E+Z} + \frac{y^2}{Z} = 1,$$

is also a caustic (see Figure 3.1). A short proof of this can be found using elementary planar geometry in the appendix of [19]. For elliptical caustics, we follow the notation in [34] and [8] by setting $\lambda^2 = Z \geq 0$. In the context of [34], integrable is taken to mean that the union of all convex caustics has a non-empty interior in $\mathbb{R}^2$. Ellipses are both completely integrable and integrable in the sense of [34].

In 1822, Poncelet proved the following remarkable theorem:

Theorem 3.2.1 ([59], [58]). *Given an ellipse, if a primitive periodic billiard trajectory is tangent to a confocal conic section, then all orbits tangent to that caustic are also periodic, have the same periods, and have the same lengths.*

Hence, periodic orbits of a given length come in 1-parameter families. Poncelet's original theorem, as stated in his 1865 treatises [59] and [58], was much more general and concerned inscribing and circumscribing polygons about other confocal conic sections. There are several modern proofs of Poncelet's theorem. Because of this result, we can project confocal ellipses to invariant curves in the coball bundle $B^*\partial\Omega$. We will use $\omega(\mathcal{C})$ for the rotation

[1]Images courtesy of Vadim Kaloshin and Alfonso Sorrentino.

number of such an invariant curve. For ellipses, the rotation number of a periodic point is always the same as the rotation number of its corresponding invariant curve.

Birkhoff proved in [3] that for any smooth, strictly convex domain and any $\frac{p}{q} \in (0, \frac{1}{2}]$ in lowest terms, there exist at least two geometrically distinct periodic orbits with rotation number $\frac{p}{q}$. A strictly convex billiard table is said to be rationally integrable if for each $q \geq 3$, there exists a caustic consisting of periodic points of rotation number $\frac{1}{q}$. In particular, the rotation number of the invariant curve corresponding to such a caustic must be $\frac{1}{q}$. A third version of Birkhoff's conjecture is that ellipses are the only rationally integrable strictly convex billiard tables. We have seen that there exist many notions of integrability, yet Birkhoff's conjecture remains open for all of them. However, as mentioned in Section 1, a local version was proven in [34].

In [19], it is shown that periodic points of the billiard map for an ellipse are dense in phase space. Given a length $T \in \mathrm{Lsp}(\Omega)$, the associated fixed point set is denoted $F_T = \{(q, \zeta) \in B^*\partial\Omega : \Phi^T(q, \eta) = (q, \eta)\}$. In [19], the authors construct a special sequence of caustics converging to the boundary such that the associated fixed point submanifold in $B^*\partial\Omega$ has exactly two connected components, corresponding to forwards and backwards flow:

Proposition 3.2.2 ([19]). *Let $T_0 = |\partial\Omega|$ be the perimeter of the ellipse Ω. Then in every interval $(T_0 - \varepsilon, T_0)$, there exist infinitely many lengths $T \in Lsp(\Omega)$. For all but a finitely many such $T \in Lsp(\Omega)$, F_T is the union of two invariant curves which are mapped to each other by $(q, \zeta) \to (q, -\zeta)$.*

The time reversal map $(q, \zeta) \to (q, -\zeta)$ reverses the direction of a closed geodesic, which of course preserves its length and geometry. While the lengths T in Proposition 3.2.2 are not simple, it remains true as all periodic trajectories of length T are tangent to a single caustic. Proposition 3.2.2 will be crucial in evaluating our singularity expansion in Section

7.3 and allowing us to differentiate the constants c_j and $\widetilde{c_j}$ from Theorem 2.2.4 near $\lambda = 0$. That there is only one connected component up to symmetry will rule out any cancellation between terms in the variation of the wave trace.

3.3 8 Orbit Theorem

What will be crucial for us in later sections is a description of all orbits making a fixed number of reflections which connect interior points near the diagonal of the boundary in approximately one rotation. These orbits will allow us to cook up phase functions in Section 5.2 which parametrize the canonical relation of the wave propagator.

Theorem 3.3.1 (8 Orbit Theorem). *There exist $C_0 = C_0(\Omega) > 0$ and $j_0 = j_0(\Omega)$ sufficiently large such that for $j \geq j_0$ and any two points $x, y \in int(\Omega)$ which are C_0/j^4 close to the diagonal of the boundary, there exist precisely four distinct, broken geodesics of j reflections making approximately one counterclockwise rotation, emanating from x and terminating at y. Similarly, there exist four such orbits in the clockwise direction.*

The proof of Theorem 3.3.1 is based on several lemmas in Section 3.4 below and is inspired by the author's previous work in [69], where a similar construction is adapted to elliptical billiard tables. As in that paper, the proof actually provides more information. An independent proof in the special case of elliptical domains is contained in Appendix B. We now explain what is meant by approximately one rotation. Let $\xi \in S_x^*\Omega$ be one of the 4 covectors corresponding to the initial condition of a counterclockwise orbit described in Theorem 3.3.1. Denote by $\widehat{x} = \pi_1 g^{t_1^\dagger}(x, \xi)$ the first point of reflection at the boundary (π_1 is projection onto the first factor) and by $\widehat{y}$ the $(j+1)$st point of reflection at the boundary after the orbit reaches y. If x, y are $O(j^{-1})$ close to the diagonal of the boundary, then $|\widehat{x} - \widehat{y}| = O(j^{-1})$ (see Section 3.4). Also let ω be the angle of reflection made by the orbit at $\widehat{x}$ and note that $\widehat{x}, \widehat{y}$ and ω all depend implicitly on ξ. By approximately one counterclockwise rotation, we mean that for each of the initial covectors $\xi \in S_x^*\Omega$ of the 4 counterclockwise orbits provided by Theorem 3.3.1, we have

$$|\pi_1 \widehat{\beta}^j(\widehat{x}, \omega) - \widehat{y} - \ell| \leq \ell/100.$$

Here, $\ell = |\partial\Omega|$ and $\widehat{\beta}$ is the lift of the billiard map to the closure of the universal cover $\mathbb{R} \times [0, \pi]$ as described in Section 3.4. The choice of $\ell/100$ is somewhat arbitrary, but having ℓ in the numerator allows for scale invariance and finding the optimal constant is irrelevant for our purposes. The notion of approximately one clockwise rotation is defined analogously.

Definition 3.3.2. Of the four counterclockwise orbits emanating from x, two of them become tangent to a level curve of the distance function $d(z) = \text{dist}(z, \partial\Omega)$ before making a reflection at the boundary. We denote these orbits by T orbits (for tangency) and call their first links T links. The other two orbits make a reflection at the boundary before becoming tangent to a level curve of d and we call these N orbits (for nontangency); their first link is called an N link. Within either T or N category for the first link, the final link of one of the orbits reaches y before becoming tangent to a level curve (an N link) and the other has a point of tangency before reaching y (a T link). In this way, we obtain four types of counterlockwise orbits from x to y, which we denote by TT, TN, NT, and NN. The same classification also applies to the four clockwise orbits.

See Figure 3.2 for an example on the ellipse with $j = 4$. These configurations will be important in determining which limiting orbits give rise to geodesic loops of precisely j reflections as $(x, y) \to \Delta\partial\Omega$, where $\Delta : \partial\Omega \to \partial\Omega \times \partial\Omega$ is the diagonal embedding $x \mapsto (x, x)$.

Definition 3.3.3. For $1 \leq k \leq 8$, we set $\Psi_j^k(x, y)$ to be a branch of the length functional corresponding to one of the orbits in Theorem 3.3.1. It depends only on x, y, j and k. We use the convention that for a fixed number of reflections j, the indices $1 \leq k \leq 4$ correspond

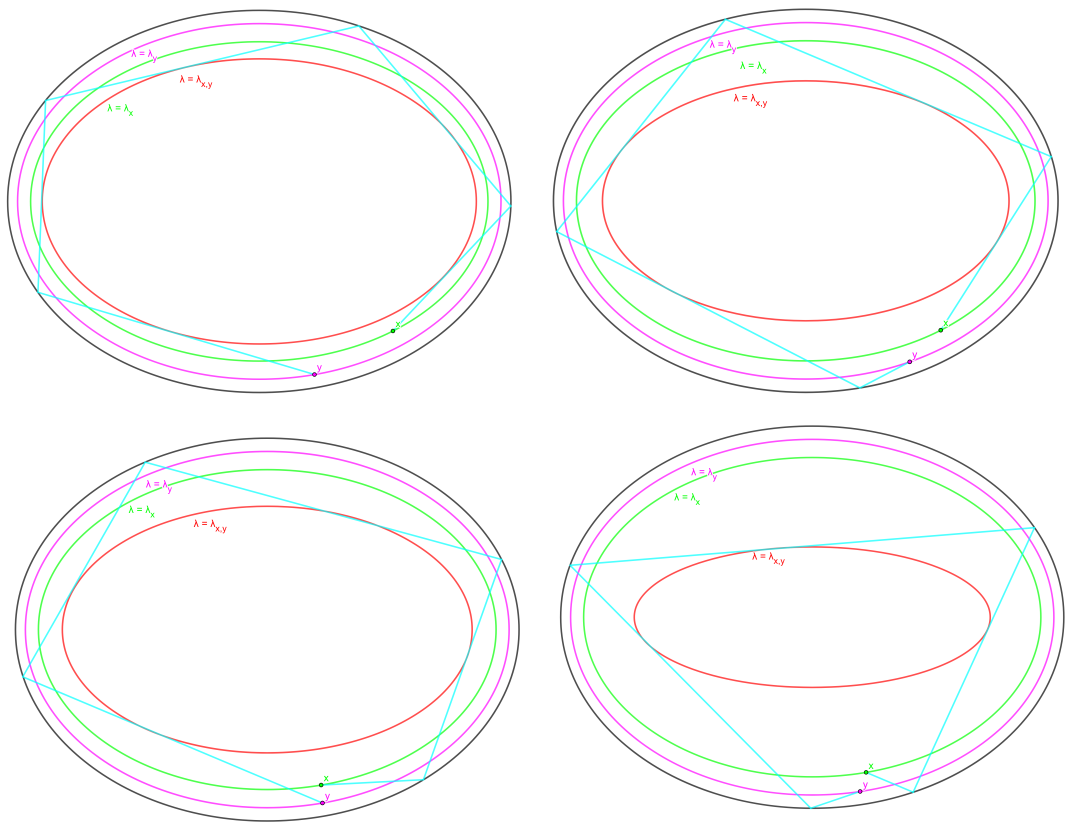

Figure 3.2: Counterclockwise orbit configurations TT, TN, NT, and NN corresponding to $j = 4$. The green and pink curves are the distance curves on which x and y respectively lie. The red curve is a caustic to which the billiard orbit is tangent in the completely integrable setting.

to the counterclockwise orbits TT, TN, NT, NN in that order and the indices $5 \leq k \leq 8$ correspond to their clockwise counterparts in the same order.

The author learned of a similar function in [44] (page 492), where its restriction to the boundary is defined. In such a case, i.e. if $x, y \in \partial\Omega$, it is stated in [44] and proved in [60] that only a single counterclockwise orbit of j reflections exists between the boundary points if they are sufficiently close and j is sufficiently large. The proof in Section 3.4 below also shows that as x and y approach the diagonal of the boundary from the interior of Ω, the corresponding orbits coalesce and converge to the orbits described in [44]. This in fact proves the claims made in [44] and provides a simpler alternative to the methods employed in [60]. The limiting orbits may have a different number of reflections, which is addressed in the proof of Lemma 6.2.1.

3.4 Proof of Theorem 3.3.1

We will construct conic subbundles $C^*_\pm(\Omega; j)$ of $T^*\Omega$ over a tubular neighborhood $\widetilde{U}_j$ of the boundary with the following property: for each $x_0 \in \widetilde{U}_j$, there exists a conic subset of $T^*_{x_0}\Omega\backslash 0$ such that all orbits emanating from x_0 which make j reflections and approximately one rotation in the sense of Section 3 have initial covector in $C^*_{x_0}(\Omega; j)$, the fiber of $\cup_\pm C^*_\pm(\Omega; j)$ at x_0. At each $x_0 \in \widetilde{U}_j$, there also exist distinguished covectors $\xi_0^\pm \in T^*_{x_0}(\Omega)\backslash 0$ which are tangent to distance curves foliating a neighborhood of the boundary. If $\widetilde{U}_j$ is chosen sufficiently small, the j reflection orbits emanating from $\xi_0^\pm$ will make less than a quarter rotation. By rotating ξ away from $\xi_0^\pm$ in either direction within the fiber of $C^*_\pm(\Omega; j)$ at x_0, we will show that the angle of reflection at the first impact point on the boundary increases monotonically. From the boundary, we can then take advantage of the twist property of the billiard map, and show that monotonicity of the incident angles causes orbits of a large number of reflections to wind around Ω. Two of the orbits from x to y will be obtained by perturbing ξ_0^+ in the counterclockwise direction and the other two will be obtained by perturbing ξ_0^+ in the clockwise direction. The four clockwise orbits will then be constructed in a similar manner, rotating ξ_0^- in both the clockwise and counterclockwise fiber directions. The arguments in this section will also provide the additional topological structure of the resulting orbits referenced in Definition 3.3.2, which can be seen in Figure 3.2.

3.4.1 Lazutkin coordinates

Recall that billiard map is defined on the coball bundle $B^*\partial\Omega$, which can be identified with the collection of inward facing covectors in the circle bundle $S^*_{\partial\Omega}\mathbb{R}^2$. Letting s denote the arclength parameter on $\partial\Omega$ and ϑ the angle an inward facing covector makes with the positively oriented boundary, we defined the modified billiard map $\widetilde{\beta}$ in arclength coordinates $(s, \vartheta) \in \mathbb{R}/\ell\mathbb{Z} \times (0, \pi)$. In this coordinate system, the modified billiard map is given by $\widetilde{\beta}(s_1, \vartheta_1) = (s_2, \vartheta_2)$, where

$$\begin{cases} s_2 = s_1 + a_1(s_1)\vartheta_1 + a_2(s_1)\vartheta_1^2 + a_3(s_1)\vartheta_1^3 + F(s_1, \vartheta_1)\vartheta_1^4 \\ \vartheta_2 = \vartheta_1 + b_2(s_1)\vartheta_1^2 + b_3(s_1)\vartheta_1^3 + G(s_1, \vartheta_1)\vartheta_1^4, \end{cases} \tag{3.1}$$

and a_i, b_i, F and G are smooth functions. This is a Taylor expansion in the angular variable near $\vartheta_1 = 0$. There are explicit formulas for the coefficients a_i, b_i. In particular, $a_1(s_1) = 2\rho(s_1)$ and $b_2 = -2/3\rho'(s_1)$, where $\rho(s_1)$ is the radius of curvature at s_1 (see [41]). In [41], the change of coordinates

$$x = \frac{\int_0^s \rho^{2/3}(t)dt}{\int_0^\ell \rho^{2/3}(t)dt}, \qquad \alpha = \frac{4\rho(s)\sin\vartheta/2}{\int_0^\ell \rho^{2/3}(t)dt} \tag{3.2}$$

was introduced near the circle corresponding $\vartheta = 0$. We call the coordinate system (x, α) in (3.2) Lazutkin coordinates. The advantage of this change of variables is that in these coordinates, the billiard map becomes a small perturbation of the translation map

$$\begin{pmatrix} x_1 \\ \alpha_1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} x_1 + \alpha_1 \\ \alpha_1 \end{pmatrix}, \tag{3.3}$$

which is completely integrable. The billiard map is given in Lazutkin coordinates by $(x_1, \alpha_1) \mapsto (x_2, \alpha_2)$, where

$$\begin{cases} x_2 = x_1 + \alpha_1 + \alpha_1^3 f(x_1, \alpha_1) \\ \alpha_2 = \alpha_1 + \alpha_1^4 g(x_1, \alpha_1), \end{cases} \tag{3.4}$$

for some smooth functions f, g. The original implementation of these coordinates was in [41], where it was shown as a consequence of the KAM theorem (see [39], [2], [48], [49]) that there exist an abundance of invariant tori for the billiard map near $\alpha = 0$. However, we will only use these coordinates to prove Theorem 3.3.1. Without loss of generality, we will often interchange the use of arclength and Lazutkin coordinates in the domain of the billiard map β.

3.4.2 Angles of reflection

Before estimating the billiard map and its iterates from the boundary, we need several preliminary estimates on the angles of reflection. In this section, we denote $\widehat{\beta}^k(x_1, \alpha_1) = (x_k, \alpha_k)$.

Lemma 3.4.1. *For each $c_1, c_2 > 0$, there exists $j_1 = j_1(\Omega, c_1, c_2) \in \mathbb{N}$ sufficiently large such that for $j \geq j_1$ and $c_1/j \leq \alpha_1 \leq c_2/j$, it follows that*

$$\frac{c_1}{(1 + c_1/j)^k j} \leq \alpha_k = \pi_2 \beta^k(x_1, \alpha_1) \leq \frac{c_2(1 + \frac{c_2}{j})^k}{j} \qquad (1 \leq k \leq j),$$

where π_2 is the projection onto the angular variable. In particular, $c_1 e^{-c_1}/j \le \pi_2\beta^k(x_1,\alpha_1) \le c_2 e^{c_2}/j$.

Proof. We use Lazutkin coordinates and proceed by induction. Suppose the claim is true for $1,\cdots,j-1$. Then

$$\begin{aligned}\pi_2\beta^j(x_1,\alpha_1) &= \pi_2\beta\circ\beta^{j-1}(x_1,\alpha_1) = \alpha_{j-1} + \alpha_{j-1}^4 g(x_1,\alpha_1)\\ &\le c_2(1+c_2/j)^{j-1}/j + (c_2(1+c_2/j)^{j-1}/j)^4 g(x_1,\alpha_1)\\ &= \frac{c_2(1+c_2/j)^{j-1}}{j}\left(1+\frac{c_2^3(1+c_2/j)^{3j-3}g(x_1,\alpha_1)}{j^3}\right).\end{aligned} \tag{3.5}$$

Choosing j_1 sufficiently large so that

$$\frac{c_2^3(1+c_2/j_0)^{3j_0-3}g(x_1,\alpha_1)}{j_0^3} \le \frac{c_2}{j_0}, \tag{3.6}$$

the upper bound follows. Similarly,

$$\begin{aligned}\pi_2\beta^j(x_1,\alpha_1) &= \pi_2\beta\circ\beta^{j-1}(x_1,\alpha_1) = \alpha_{j-1} + \alpha_{j-1}^4 g(x_1,\alpha_1)\\ &\ge \frac{c_1}{(1+c_1/j)^{j-1}j} + \left(\frac{c_1}{(1+c_1/j)^{j-1}j}\right)^4 g(x_1,\alpha_1)\\ &= \frac{c_1}{(1+c_1/j)^{j-1}j}\left(1+\frac{c_1^3}{(1+c_1/j)^{3j-3}j^3}g(x_1,\alpha_1)\right),\end{aligned} \tag{3.7}$$

which can be made greater than

$$\frac{c_1}{(1+c_1/j)^k j},$$

under the condition that

$$\frac{c_1}{j_1} + \frac{c_1^3 g(x_1,\alpha_1)(1+c_1/j_1)}{(1+c_1/j_1)^{3j_1-3}j_1^3} \ge 0.$$

This certainly holds for j_1 large enough, which concludes the proof. $\square$

Lemma 3.4.2. *There exist $C_1, C_2 > 0$ and $j_2 = j_2(\Omega) \in \mathbb{N}$ with the following property: for all $j \geq j_2$ and x, y which are $O(1/j)$ close to the diagonal of the boundary, any orbit γ of j reflections which emanates from x and terminates at y in approximately one rotation (in the sense of Section 3) has angles of reflection α_k in the range*

$$\frac{C_1}{j} \leq \alpha_k \leq \frac{C_2}{j} \qquad (1 \leq k \leq j).$$

Proof. Let $\widehat{x}, \widehat{y}$ denote the first and $(j+1)$st points of reflection on the boundary. Recall that by approximately one rotation, we mean that

$$|\widehat{\beta}^j(\widehat{x}, \alpha_1) - \widehat{y} - \ell| \leq \ell/100,$$

where $\widehat{\beta}$ is the lift to the universal cover and $\ell = |\partial\Omega|$. Each arc separated by moments of reflection x_p, x_{p+1} on the boundary has length $x_{p+1} - x_p$. As γ makes approximately one rotation, we see that there must exist one arc of length $x_{p+1} - x_p \geq (99/100)\ell/j$. Similarly, there must exist an arc of length $x_{m+1} - x_m \leq (101/100)\ell/j$. By Proposition 14.1 of [42], for each $1 \leq k \leq j$,

$$2\alpha_k \rho_{\min} \leq x_{k+1} - x_k \leq 2\alpha_k \rho_{\max}, \tag{3.8}$$

where $\rho_{\min}$ and $\rho_{\max}$ are the minimum and maximum radii of curvature respectively for $\partial\Omega$. In particular,

$$\frac{99\ell}{200\rho_{\max} j} \leq \alpha_p, \qquad \alpha_m \leq \frac{101\ell}{200\rho_{\min} j}. \tag{3.9}$$

We now apply Lemma 3.4.1 to $\beta^{\pm}$ at the points x_p and x_m to conclude that for all $1 \leq k \leq j$,

$$\frac{\exp\left(-\frac{99\ell}{200\rho_{\max}}\right)}{j} \leq \alpha_k \leq \frac{\exp\left(\frac{101\ell}{200\rho_{\min}}\right)}{j}. \tag{3.10}$$

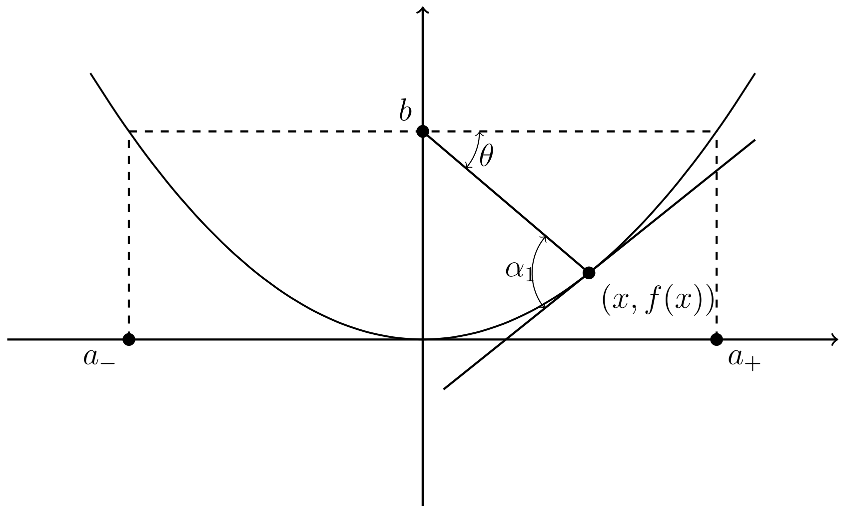

Figure 3.3: Angular derivative of the billiard flow.

Here we have used Lemma 3.4.1 applied to both the billiard map and its inverse, with the constants $c_1 = 99\ell/200\rho_{\max}$ and $c_2 = 101\ell/200\rho_{\min}$ which depend only on Ω. Hence, j_2 may be chosen uniformly for all orbits making approximately one rotation regardless of their initial positions. $\square$

We now investigate the angle of reflection at the first point of impact on the boundary. If $d(z) = \mathrm{dist}(z, \partial\Omega)$, then the level curves of d foliate a neighborhood of the boundary. For x_0 in the tubular neighborhood $\widetilde{U}_j$ (whose diameter remains to be specified), we define $\xi_0^{\pm} \in \pm T_{x_0}^* d^{-1}(d(x_0)) \cap S_{x_0}^*\Omega$ so that ξ_0^+ points in the counterclockwise direction and ξ_0^- points in the clockwise direction. Let $\xi \in S_{x_0}^*\Omega$ be identified with the corresponding point $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$, which denotes the clockwise angular parametrization of the fiber $S_{x_0}^*\Omega$, normalized so that ξ_0^+ and ξ_0^- correspond to 0 and π respectively. Denote by $\alpha_1 \in (0, \pi)$ the angle made between the billiard orbit emanating from (x_0, ξ) and the positively oriented cotangent line at the first point of reflection at the boundary. It is clear that $|\partial\xi/\partial\theta| = 1$.

Lemma 3.4.3. *There exist $c = c(\Omega) > 0$ and $j_3 = j_3(\Omega) \in \mathbb{N}$ such that if $j \geq j_3$ and $d(x_0) = O(1/j^2)$, then*

$$\frac{\partial \alpha_1}{\partial \theta} \geq c$$

for all initial conditions $(x_0, \xi(\theta))$ which generate a counterclockwise orbit of $j \geq j_0$ reflections, making approximately one rotation.

Proof. Let $x_0 \in \operatorname{int}(\Omega)$ be near the boundary and consider the point $x \in \partial\Omega$ which minimizes the Euclidean distance $|x_0 - x|$ among all boundary points. We then apply an affine change of coordinates $y \mapsto A(y - x)$ with $A \in SO(2)$ so that x is mapped to the origin and the positively oriented unit tangent vector at x is mapped to $(1, 0)$. The vector $x - x_0$ is perpendicular to both $T^*_x \partial\Omega$ and $T^*_{x_0} d^{-1}(|x - x_0|)$ (see Figure 3.3). Therefore, the point $A(x_0 - x)$ lies on the positive vertical axis and we denote it by $(0, b)$. In these coordinates, the boundary $\partial\Omega$ is locally parametrized by the graph of a smooth convex function f. Since $f'(0) = 0$, we see that $f(t) = \kappa t^2 + R(t)$, with κ denoting the curvature at the point corresponding to $(0, 0)$ and $R(t) = O(|t|^3)$. Dilating by a factor of κ^{-1}, we see that in rescaled coordinates, the function becomes $f(t) = t^2 + R(t)$, with $R(t)$ again of order $O(|t|^3)$.

In graph coordinates, α_1 becomes the angle between $(t, f(t)) - (0, b)$ and the negatively oriented cotangent line $T^*_{(t,f(t))}\partial\Omega$ as illustrated in Figure 3.3. This shows that

$$\cos\alpha_1 = \frac{(t, f(t) - b) \cdot (1, f'(t))}{\sqrt{t^2 + (f(t) - b)^2}\sqrt{1 + (f'(t))^2}} \geq 0, \tag{3.11}$$

since $0 \leq \alpha_1 \leq \pi/2$. Recall that $\xi_0^{\pm}$ are the unit covectors in the positive (+), resp. negative (-), cotangent line $T^*_{x_0} d^{-1}(|x - x_0|)$ corresponding to $\theta = 0$ and $\theta = \pi$ respectively. For now, we only consider counterclockwise orbits obtained by perturbing the orbit emanating from (x_0, ξ_0^+), as they correspond to $-\pi/2 \leq \theta \leq \pi/2$ in the statement of the lemma. Denote the points $a_- = \min\{f^{-1}(b)\}$ and $a_+ = \max\{f^{-1}(b)\}$. We first show that $d\alpha_1/dt < 0$ on $[0, a_+]$ so that the angle of incidence at the first impact point on the boundary is increasing as the initial covector of the trajectory winds clockwise in $S^*_{x_0}\Omega$ away from ξ_0^+, i.e. $d\alpha/dt < 0$ on $[0, a_+]$.

Noting that t and θ are negatively correlated, it suffices to show that the logarithmic t derivative of $\cos\alpha_1(t)$ is positive:

$$\frac{d}{dt}\log\cos\alpha_1(t) = \frac{(1+f''(f-b)+(f')^2)}{t+f'(f-b)} - \frac{t+f'(f-b)}{t^2+(f-b)^2} - \frac{f'f''}{1+(f')^2}. \tag{3.12}$$

Multiplying (3.12) through by a common denominator, which is positive for t, b small, we obtain

$$\begin{aligned} g(t,b) :=&(t+f'(f-b))(t^2+(f-b)^2)(1+(f')^2)\frac{d}{dt}\log\cos\alpha(x) \\ =&(1+f''(f-b)+(f')^2)(t^2+(f-b)^2)(1+(f')^2) \\ &-(t+f'(f-b))^2(1+(f')^2) - f'f''(t+f'(f-b))(t^2+(f-b)^2). \end{aligned} \tag{3.13}$$

To show that (3.13) is positive, we plug in our second order Taylor approximation for f and expand g to fourth order in a parabolic neighborhood of $t=0, b=0$:

$$g(t,b) = 3t^4 + 10t^2b^2 - 2b^3 + b^2 + O((t^2+b^2)^{5/2}). \tag{3.14}$$

As $f(t) = t^2 + O(t^3)$, we see that $a_\pm = \pm b^{1/2} + O(b^{3/2})$, so we choose $b \leq 1/j^2$ sufficiently small in order to make (3.14) positive in a parabolic neighborhood of origin. Examining the initial set up, choosing b small amounts to choosing x_0 close to the boundary, all of which can be done uniformly in x_0 and the curvature κ. Choosing a uniform b gives us a tubular neighborhood of the boundary $\widetilde{U}_j = \{x : 0 < d(x,\partial\Omega) < j^{-2}\}$, as mentioned in the begining of the section. We will later shrink $\widetilde{U}_j$ by a factor of j^{-2}, following Lemma 3.4.9.

While the lower bound in (3.14) is of order $b^2 = O(j^{-4})$, we in fact need uniform bounds for $\partial\alpha_1/\partial\theta$ which we now provide. The prefactor in (3.13) is nonvanishing for $t \neq 0, t \neq a_+$ and in particular, for all t corresponding to an orbit which makes approximately one rotation in j reflections. We first need to compare $\partial/\partial t$ and $\partial/\partial\theta$ in terms of b, which depends on j. In graph coordinates, θ becomes the angle of the first link from the horizontal axis (see Figure 3.3). Hence, $\theta = \arctan((b - f(t))/t)$ and

$$\frac{\partial \theta}{\partial t} = \frac{1}{1 + (b - f(t))^2/t^2} \left(-\frac{f'(t)}{t} - \frac{b - f(t)}{t^2} \right),$$

or equivalently,

$$\frac{\partial t}{\partial \theta} = -\left(\frac{t^2 + (b - f(t))^2}{tf'(t) + b - f(t)} \right), \tag{3.15}$$

where t is defined implicitly as a function of θ. Plugging (3.15) into (3.12) and using that (3.14) is bounded below by $b^2/2$ for b sufficiently small, we see that

$$\begin{aligned} \frac{\partial \alpha}{\partial \theta} &\geq \frac{b^2(t^2 + (b - f(t))^2)}{2(tf'(t) + b - f(t))(t + f'(t)(f(t) - b))(t^2 + (f(t) - b)^2)(1 + (f'(t))^2) \tan \alpha_1} \\ &= \frac{b^2}{2(tf'(t) + b - f(t))(t + f'(t)(f(t) - b))(1 + (f'(t))^2) \tan \alpha_1}. \end{aligned} \tag{3.16}$$

The denominator in (3.16) is nonvanishing and can be estimated above on $(0, a_+]$ by

$$c' tb \tan \alpha_1 + O(b^{5/2}). \tag{3.17}$$

for some $c' > 0$, independent of j. Observe that if $t \leq b$ for example, then the denominator is bounded above by $c'b^2$ and $\partial \alpha_1/\partial \theta \geq c''$, for some $c'' > 0$ which is also independent of j.

We now find a similar upper bound on $t(\theta)$ in terms of b, corresponding to the set of covectors $\xi(\theta)$ which produce orbits of j reflections and approximately one rotation. Let $b = rb_0$, where $b_0 = j^{-2}$ for $r \in (0, 1]$. Recall from Lemma 3.4.2 that there exist constants $C_1, C_2 > 0$ such that for all orbits making approximately one rotation and j reflections, we have $C_1/j \le \alpha_k \le C_2/j$ for each $1 \le k \le j$. Hence,

$$C_1 b_0^{1/2} \le \alpha_1 \le C_2 b_0^{1/2}. \tag{3.18}$$

Observe that equation (3.11) gives

$$\begin{aligned}\log\cos\alpha_1(t) &= \log(t + f'(t)(f(t)-b)) - 1/2\log(t^2 + (f(t)-b)^2)\\ &\quad - 1/2\log(1 + (f'(t))^2).\end{aligned} \tag{3.19}$$

Denote the terms in (3.19) above by A, B and C. Then,

$$\begin{aligned}A &= \log(t + (2t + O(t^2))(t^2 + O(t^3) - b))\\ &= \log t + \log(1 + 2t^2 - 2b + O(t^3) + O(b))\\ &= \log t + 2t^2 + O(t^3) + O(b).\end{aligned}$$

Similarly,

$$\begin{aligned}B &= -1/2\log(t^2 + t^4 - 2t^2 b + b^2 + O(t^5) + O(t^3 b))\\ &= -\log t - 1/2(t^2 - 2b + b^2/t^2 + O(t^3) + O(tb))\end{aligned}$$

and

$$C = -\log(1 + (2t + O(t^2))) = -4t^2 + O(t^3).$$

As $t = O(b^{1/2})$, we have

$$A + B + C = \frac{-b^2}{2t^2} + O(b).$$

On the other hand,

$$\begin{aligned} A + B + C &= \log\cos\alpha_1 = \log(1 - a_1^2/2 + O(\alpha_1^4)). \\ &= -\alpha_1^2/2 + O(b_0^{4/N}), \end{aligned}$$

which implies that

$$b^2/t^2 = r^2 b_0^2/t^2 \sim \alpha_1^2/2.$$

In particular, (3.18) implies

$$t \sim \frac{\sqrt{2} r b_0}{\alpha_1} \in \left[\frac{\sqrt{2} r b_0^{1/2}}{C_2}, \frac{\sqrt{2} r b_0^{1/2}}{C_1} \right]. \tag{3.20}$$

Note also that $\tan\alpha_1 \leq 2\alpha_1 \leq 2C_2 b_0^{1/2}$. Combining this with (3.20), we see that the denominator in equation (3.16) is bounded above by

$$c' t b \tan\alpha_1 + O(b^{5/2}) \leq c''' b (r b_0^{1-1/2}) b_0^{1/2} = c''' b^2, \tag{3.21}$$

for some $c''' > 0$ independent of j. This establishes the claim that $\partial\alpha/\partial\theta$ is uniformly bounded below on the set of initial covectors corresponding to orbits of j reflections and approximately one rotation. □

3.4.3 Monotonicity and the twist property

The significance of increasing the angle of reflection at the first impact point on the boundary is that after $j-1$ reflections at the boundary, the *jth* impact point winds around $\partial\Omega$ with approximate speed j. This is essentially the twist property of the billiard map (see [65] for a general definition of twist maps). The following lemma makes this notion quantitative.

Lemma 3.4.4. *There exists $j_4 = j_4(\Omega) \in \mathbb{N}$ such that if a trajectory makes $j \geq j_4$ reflections at the boundary and approximately one rotation, then the Jacobian of the iterated billiard map satisfies*

$$D\widehat{\beta}^j(x_1, \alpha_1) = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} + O(1/j)$$

in Lazutkin coordinates lifted to the universal cover. In particular, there exist constants $C_3, C_4, C_5, C_6 > 0$ depending only on Ω such that for any (x_1, α_1), we have

$$\begin{aligned} C_3 j &\leq \frac{\partial}{\partial \alpha_1} \pi_1 \widehat{\beta}^j(x_1, \alpha_1) \leq C_4 j \\ C_5 &\leq \frac{\partial}{\partial \alpha_1} \pi_2 \widehat{\beta}^j(x_1, \alpha_1) \leq C_6, \end{aligned}$$

where π_1 and π_2 denote projections onto the first and second components respectively.

Proof. In Lazutkin coordinates, $\beta(x_k, \alpha_k) = (x_k + \alpha_k + \alpha_k^3 f(x_k, \alpha_k), \alpha_k + \alpha_k^4 g(x_k, \alpha_k))$. Hence,

$$D\widehat{\beta}^j(x_1, \alpha_1) = \prod_{k=1}^{j} \begin{pmatrix} 1 + \alpha_k^3 \frac{\partial f}{\partial x}(x_k, \alpha_k) & 1 + 3\alpha_k^2 f(x_k, \alpha_k) + \alpha_k^3 \frac{\partial f}{\partial \alpha}(x_k, \alpha_k) \\ \theta_k^4 \frac{\partial g}{\partial x}(x_k, \alpha_k) & 1 + 4\alpha_k^3 g(x_k, \alpha_k) + \alpha_k^4 \frac{\partial g}{\partial \alpha}(x_k, \alpha_k) \end{pmatrix}, \tag{3.22}$$

where $(x_k, \alpha_k) = \widehat{\beta}^k(x_1, \alpha_1)$ and f, g are extended to be ℓ periodic in x on $\mathbb{R} \times [0, \pi]$. Each of the terms in the product can be decomposed into $(I_2 + B_k)$, where

$$B_k = \begin{pmatrix} \alpha_k^3 \frac{\partial f}{\partial x}(x_k, \alpha_k) & 1 + 3\alpha_k^2 f(x_k, \alpha_k) + \alpha_k^3 \frac{\partial f}{\partial \alpha}(x_k, \alpha_k) \\ \alpha_k^4 \frac{\partial g}{\partial x}(x_k, \alpha_k) & 4\alpha_k^3 g(x_k, \alpha_k) + \alpha_k^4 \frac{\partial g}{\partial \alpha}(x_k, \alpha_k) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + O(1/j^2). \tag{3.23}$$

Denote the constant matrix on the righthand side above by

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.24}$$

We have

$$\begin{aligned} D\widehat{\beta}^j = (I_2 + B_k)^j &= \sum_{k=0}^{j} \binom{j}{k} \begin{pmatrix} O(j^{-3}) & 1 + O(j^{-2}) \\ O(j^{-4}) & O(j^{-3}) \end{pmatrix}^k \\ &= I_2 + jN + O(1/j) + \sum_{k=2}^{j} \binom{j}{k} (N + O(1/j^2))^k. \end{aligned} \tag{3.25}$$

Noting that N is nilpotent of order 2 (i.e. $N^2 = 0$), we can estimate the sum above by

$$D\widehat{\beta}^j = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} + O(1/j) + \sum_{k=2}^{j} \binom{j}{k} O(j^{-2(k-1)}). \tag{3.26}$$

We want to show that the remainder term $\sum_{k=2}^{j} \binom{j}{k} O(j^{-2(k-1)})$ in (3.26) is small. Choosing j large so that $O(j^{-2(k-1)}) \leq 2^{-k}/j$,

$$\sum_{k \geq 2} O(j^{-2(k-1)}) \leq \frac{1}{j} \sum_{k=0}^{j} \binom{j}{k} 2^{-k} = \frac{1}{j}. \tag{3.27}$$

Noting that angular derivatives $\partial \pi_1 \widehat{\beta}^j / \partial \alpha_1$ and $\partial \pi_2 \widehat{\beta}^j / \partial \alpha_1$ are the $(1, 2)$ and $(2, 2)$ components of $D\widehat{\beta}^j$ respectively concludes the proof of the lemma. $\square$

Remark 3.4.5. As j increases, Lemma 3.4.4 shows that the Poincaré map associated to a periodic orbit of rotation number $1/j$ becomes more and more degenerate, corresponding to the accumulation of caustics at the boundary. In fact, there exist higher order Lazutkin coordinates for which the remainder above could be replaced by $O(j^{-N})$ for any $N \in \mathbb{N}$. However, this is not needed in the remainder of this dissertation.

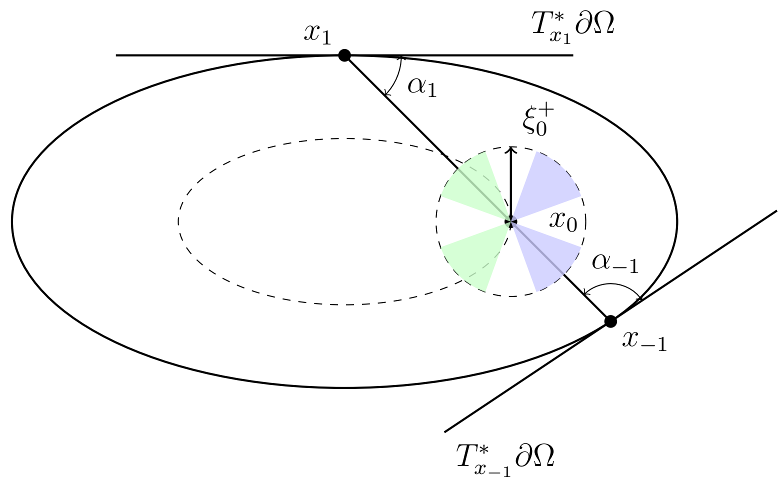

Figure 3.4: A counterclockwise perturbation of ξ_0^+. The green cones at x_0 are the length one covectors in $C^*_{x_0,+}(\Omega; j)$ and the blue cones at x_0 are the length one covectors in $C^*_{x_0,-}(\Omega; j)$.

The point x_j can be obtained in two ways. The first way is by flowing forwards to the boundary point $x_1 := \pi_1 g^{t_1^+}(x_0, \xi)$, iterating the billiard map j times and projecting onto the first component. The second way is by flowing backwards to the boundary point $x_{-1} := g^{t_1^-}(x_0, \xi)$ and iterating the billiard map $j+1$ times. If ξ is rotated in the clockwise direction away from ξ_0^+, it is convenient to use the angle α_1 at x_1, since it was shown in Lemma 3.4.3 that α_1 is increasing in θ. If ξ is rotated counterclockwise away from ξ_0^+, it is instead advantageous to use the point (x_{-1}, α_{-1}), since this again corresponds to a *clockwise* perturbation of ξ_0^-. In either regime $\theta < \theta_0^+$ or $\theta > \theta_0^+$, we can then take advantage of the monotonicity of x_j in $\alpha_{\pm 1}$ as ξ, or equivalently θ, varies. The next lemma shows that the point x_j is monotonically increasing for all orbits making approximately one rotation as $\xi(\theta)$ winds in either direction.

Lemma 3.4.6. *There exist $C_7 > 0$ and $j_5 = j_5(\Omega) \in \mathbb{N}$ such that for all $\theta \in (0, \pi/2)$ corresponding to a $j \geq j_5$ reflection orbit emanating from $(x_0, \xi(\theta))$ which makes approximately one rotation, we have $\partial x_j/\partial\theta \geq C_7 j$. Similarly, for $\theta \in (-\pi/2, 0)$ corresponding to an orbit of j reflections and approximately one rotation, we have $\partial x_j/\partial\theta \leq -C_7 j$.*

Proof. Denote $x_j = \pi_1 \beta^j(x_1, \alpha_1)$, where π_1 is the projection onto the first component. Then

x_j depends only on (x_0, ξ), or equivalently (x_0, θ) and can be written as the composition

$$x_j = \pi\beta^j(x_1(x_0, \theta), \alpha_1(x_0, \theta)).$$

As θ increases or decreases from 0, we have

$$\frac{\partial x_j}{\partial \theta} = \frac{\partial x_j}{\partial x_1}\frac{\partial x_1}{\partial \theta} + \frac{\partial x_j}{\partial \alpha}\frac{\partial \alpha}{\partial \theta}. \tag{3.28}$$

Recall from Lemma 3.4.4 that

$$C_3 j \leq \frac{\partial x_j}{\partial \alpha} \leq C_4 j, \tag{3.29}$$

for j large and positive constants C_3 and C_4. This derivative can be made arbitrarily large by choosing j accordingly. Also, Lemma 3.4.2 showed that all orbits making approximately one rotation with j reflections at the boundary have angles of reflection in the range $C_1/j \leq \alpha \leq C_2/j$ for positive constants C_1 and C_2. We can now use Lemma 3.4.3, which showed that $|\partial\alpha/\partial\theta| \geq c > 0$ independently of j for all θ producing an orbit of approximately one rotation in j reflections. Using Lazutkin coordinates, we also have that

$$\frac{\partial x_j}{\partial x_1}\frac{\partial x_1}{\partial \theta} = (1 + O(1/j))\frac{\partial x_1}{\partial \theta}. \tag{3.30}$$

In graph coordinates (see proof of Lemma 3.4.3), one can calculate that

$$\begin{aligned} \left|\frac{\partial x_1}{\partial \theta}\right| &= \left|\frac{\partial x_1}{\partial x}\frac{\partial x}{\partial \theta}\right| = \sqrt{1 + (f'(x))^2}\frac{xf'(x) + b - f}{x^2 + (b - f(x))^2} \\ &\leq 1 + O(b) \leq 2 \end{aligned} \tag{3.31}$$

which is bounded independently of j. Hence, the term (3.29) dominates in the expression (3.28) and x_j winds monotonically around $\partial\Omega$. □

At the endpoints $\theta = \pm\pi/2$, it is clear that the angle of reflection at $x_{\pm 1}$ is $\pi/2$ since the distance curves are parallel. Angles outside the range $\theta \in (-\pi/2, \pi/2)$ correspond to clockwise orbits and Lemma 3.4.2 showed that for an orbit making approximately one rotation and j reflections at the boundary, all angles of reflection satisfy $C_1/j \leq \alpha_k \leq C_2/j$, $(1 \leq k \leq j)$. As j reflection orbits emanating from ξ_0^+ (corresponding to $\theta = 0$) make only a quarter rotation, it is clear that the collection of covectors at x_0 whose trajectories make approximately one rotation in j reflections consists of two connected components in $S^*_{x_0}\Omega$.

Definition 3.4.7. The positive admissible cone $C^*_{x_0,+}(\Omega; j)$ at x_0 is defined to be the set of homogeneous extensions to $T^*_{x_0}\Omega$ of the two components in $S^*_{x_0}(\Omega)$ described above. The negative admissible cone $C^*_{x_0,-}(\Omega; j)$ at x_0 is defined by the same property for orbits making j reflections in the clockwise direction. Their union is denoted by $C^*_{x_0}(\Omega; j) = C^*_{x_0,+}(\Omega; j) \cup C^*_{x_0,-}(\Omega; j)$. See Figure 3.4 for an illustration of $C^*_{x_0}(\Omega; j)$.

3.4.4 Intersection points

We are finally ready to show that the intersection points of the last link with the distance curve of y wind around monotonically as we twist ξ in either direction. We first explain why the last link necessarily intersects $d^{-1}(\mathrm{dist}(y, \partial\Omega))$ twice.

Lemma 3.4.8. *There exists $j_6 = j_6(\Omega) \in \mathbb{N}$ such that for $j \geq j_6$ and any $N \geq 4$, the distance curve $\{x \in \Omega : dist(x, \partial\Omega) = j^{-N}\}$ is intersected exactly twice by any link emanating from the boundary at an angle greater than or equal to C_1/j, with C_1 the constant appearing in Lemma 3.4.2.*

Proof. For each point p in the distance curve $\{z \in \Omega : d(z, \partial\Omega) = j^{-N}\}$, the tangent line $T_p\{z \in \Omega : d(z, \partial\Omega) = j^{-N}\}$ intersects $\partial\Omega$ exactly twice by convexity. Recall that Lemma 3.4.2 gave $C_1/j \leq \alpha_k \leq C_2/j$ for each $1 \leq k \leq j$. We now show that the angles of reflection for links which are tangent to the distance curve $d^{-1}(j^{-N})$ are of order $O(j^{-N/2})$. In graph coordinates, $\partial\Omega$ is locally given by $(x, f(x))$ where $f(x) = \kappa x^2 + R(x)$, κ is the curvature of $\partial\Omega$ at the point corresponding to $(0,0)$ and $R(x) = O(x^3)$. By rescaling, we may assume that $\kappa = 1$. The distance curve $d^{-1}(j^{-N})$ can be parametrized in graph coordinates by

$$d^{-1}(j^{-N}) = \{(t, f(t)) + rN(t) : t \in \mathbb{R}\},$$

where $r = j^{-N}$ and

$$N(t) = \frac{(-f'(t), 1)}{\sqrt{1 + (f'(t))^2}}$$

is the unit normal to the graph at $(t, f(t))$. There exist precisely two lines through the origin which are tangent to $d^{-1}(j^{-N})$ in graph coordinates. The positive parameter t corresponding to a point of tangency satisfies

$$\frac{f(t) + r/(1 + (f'(t))^2)^{1/2}}{t - rf'(t)/(1 + (f'(t))^2)^{1/2}} = \frac{\partial_t(f(t) + r/(1 + (f'(t))^2)^{1/2})}{\partial_t(t - rf'(t)/(1 + (f'(t))^2)^{1/2})}, \tag{3.32}$$

where the lefthand side of (3.32) is the slope of a line connecting the origin to a point on $d^{-1}(r)$ and the righthand side is the slope of the tangent to $d^{-1}(r)$. Solving for t in terms of r in equation (3.32) yields $f'(t)t - f(t) = t^2 + O(t^3) = O(r)$. Plugging this in to the righthand side of (3.32), we see that the angle of tangency satisfies $\tan(\alpha_{\text{tangency}}) = O(r^{1/2})$ and hence $\alpha_{\text{tangency}} = O(j^{-2})$ if $N \geq 4$. The proof is complete by noting that the angles α_k coming from orbits making approximately one rotation and j reflections are bounded below by $C_1/j > O(j^{-4/2})$ for j sufficiently large.

□

Lemma 3.4.9. *Under the hypotheses of Lemma 3.4.8, denote by w_1 and w_2 the arclength coordinates on $d^{-1}(d(y))$ corresponding to the intersection points of $d^{-1}(d(y))$ and the last link of the j reflection orbit emanating from x. There exist $C_8 > 0$ and $j_7 = j_7(\Omega) \in \mathbb{N}$ such* *that if $j \geq j_7$ and x, y are $O(1/j^4)$ close the diagonal of the boundary, then $|\partial w_i/\partial\theta| \geq C_8 j$ ($i = 1, 2$) for all θ corresponding to $\xi(\theta)$ in the cone of admissible covectors at x_0.*

Proof. In graph coordinates, $\partial\Omega$ is again locally parametrized by $(t, f(t))$, where $f(t) = \kappa t^2 + R(t)$, κ is the curvature of $\partial\Omega$ at the point corresponding to $(0, 0)$ and $R(t) = O(t^3)$. By rescaling, we may assume that $\kappa = 1$. The distance curve $d^{-1}(d(y))$ on which y lies can be locally parametrized in graph coordinates by

$$d^{-1}(d(y)) = \{(t, f(t)) + rN(t)\}, \tag{3.33}$$

where $r = \mathrm{dist}(y, \partial\Omega) = O(j^{-4})$ and

$$N(t) = \frac{(-f'(t), 1)}{\sqrt{1 + (f'(t))^2}}$$

is the unit normal to the graph at $(t, f(t))$. The final link of the orbit connecting $x_j(\theta) \in \partial\Omega$ to $x_{j+1}(\theta) \in \partial\Omega$ can be parametrized by the line

$$Y = \tan(\alpha_j(\theta) + \beta_j(\theta))X + f(t_j(\theta)) - \tan(\alpha_j(\theta) + \beta_j(\theta))t_j(\theta), \tag{3.34}$$

where

$$\beta_j = \arccos\left(\frac{1}{\sqrt{1 + (f'(x_j))^2}}\right)$$

is the angle of the tangent to the graph with the horizontal axis and $t_j = t_j(\theta) \in \mathbb{R}$ is defined implicitly by the equation $(t_j, f(t_j)) = x_j \in \partial\Omega$. Noting that the Cartesian coordinates of the parametrization of $d^{-1}(d(y))$ satisfy

$$\begin{aligned} X &= t - \frac{rf'(t)}{\sqrt{1+(f'(t))^2}}, \\ Y &= f(t) + \frac{r}{\sqrt{1+(f'(t))^2}}, \end{aligned} \tag{3.35}$$

we plug these into equation (3.34) to obtain

$$\begin{aligned} \left(f(t) + \frac{r}{\sqrt{1+(f'(t))^2}}\right) &= \tan(\alpha_j(\theta)+\beta_j(\theta))\left(t - \frac{rf'(t)}{\sqrt{1+(f'(t))^2}}\right) \\ &\quad + f(t_j(\theta)) - \tan(\alpha_j(\theta)+\beta_j(\theta))t_j(\theta). \end{aligned} \tag{3.36}$$

By Lemma 3.4.8, there exist precisely two solutions $t = z_1(\theta), z_2(\theta)$ of equation (3.36) which correspond to the two intersection points of the last link with the distance curve $d^{-1}(d(y))$. Plugging $z_i(\theta)$ into equation (3.36), differentiating in θ and evaluating at θ_0 corresponding to $x_j(\xi(\theta_0))$ at the origin, we obtain

$$\begin{aligned} f'(z_i)z_i' - \frac{rf'(z_i)f''(z_i)z_i'}{(1+f(z_i)^2)^{3/2}} &= (\alpha_j' + \beta_j')(1+\tan^2\alpha_j)\left(z_i - \frac{rf'(z_i)}{\sqrt{1+(f'(z_i))^2}}\right) \\ &\quad + \tan\alpha_j\left(z_i' - \frac{rf''(z_i)z_i'}{\sqrt{1+(f'(z_i))^2}} + \frac{r(f'(z_i))^2f''(z_i)z_i'}{(1+(f'(z_i))^2)^{3/2}}\right) + 0 - \tan\alpha_j t_j'. \end{aligned} \tag{3.37}$$

We now collect all terms with z_i' in (3.37):

$$A_i z_i' = (\alpha_j' + \beta_j')(1+\tan^2\alpha_j)\left(z_i - \frac{rf'(z_i)}{\sqrt{1+(f'(z_i))^2}}\right) - \tan\alpha_j t_j', \tag{3.38}$$

where

$$A_i = f'(z_i) - \frac{rf'(z_i)f''(z_i)}{(1+f(z_i)^2)^{3/2}} - \tan\alpha_j\left(1 - \frac{rf''(z_i)}{\sqrt{1+(f'(z_i))^2}} + \frac{r(f'(z_i))^2f''(z_i)}{(1+(f'(z_i))^2)^{3/2}}\right).$$

Before dividing through equation (3.38) by A_i, we Taylor expand A_i in powers of $1/j$ to show that it is nonvanishing:

$$A_i = 2z_i - \tan\alpha_j + O(j^{-2}). \tag{3.39}$$

At this point, it is important to distinguish between z_1 and z_2. If $z_1 < z_2$, then insisting that y be $O(j^{-4})$ close to the boundary amounts to setting $z_1 = O(j^{-4})$ and $z_2 = t_{j+1} + O(j^{-4})$. Here, $t_{j+1} = t_{j+1}(\theta)$ is defined implicitly by the equation $(t_{j+1}, f(t_{j+1})) = x_{j+1} \in \partial\Omega$. When $i = 1$, $A_1 = -\alpha_j + O(j^{-4}) \leq -C_1/(2j)$ is nonvanishing for j sufficiently large. For $i = 2$, note that $\tan\alpha_j = f(z_2)/z_2 + O(j^{-4})$, which implies that $A_2 = z_2 + O(j^{-2})$ is again nonvanishing. Hence, we may divide through equation (3.38) by A_i. To estimate the righthand side of (3.38), we calculate that

$$\beta_j' = \frac{f'(t_j)}{\sqrt{1 - \frac{1}{1+(f'(t_j))^2}}} \frac{f''(t_j)t_j'}{(1 + (f(t_j)^2))^{3/2}}. \tag{3.40}$$

The first factor above is well defined by continuity at $t_j = 0$ and equals 1. Hence, $\beta_j' = 2t_j' + O(1/j)$. If $i = 1$, equation (3.38) becomes

$$(-\tan\alpha_j + O(j^{-4}))z_1' = -\tan\alpha_j t_j' + O(j^{-3}), \tag{3.41}$$

which implies that $z_1' = t_j' + O(j^{-2})$. When $i = 2$, equation (3.38) becomes

$$(z_2 + O(j^{-2}))z_2' = t_j' z_2 + O(1/j), \tag{3.42}$$

which again implies that $z_2' = t_j' + O(1/j)$. Recall that Lemma 3.4.6 showed x_j' is of order j in Lazutkin coordinates when $\xi(\theta) \in C^*_{x_0}(\Omega; j)$. To compare Lazutkin coordinates with t_j, note that in graph coordinates, the arclength parameter ds on $\partial\Omega$ is given by

$$ds = (1 + (f'(t))^2)^{1/2} dt.$$

As arclength coordinates and Lazutkin coordinates are comparable independently of j, we conclude that t_j' and hence z_i' are also of order j. We conclude the proof by noting that the arclength parameter ds' on $d^{-1}(d(y))$ in the parametrization (3.33) is given by

$$ds' = \left((1 + (f'(t))^2)^{1/2} + O(r)\right) dt,$$

which is also comparable to dt independently of j. □

3.4.5 Clockwise orbits

In Lemma 3.4.9, it was shown that there are precisely 4 counterclockwise orbits connecting $x = x_0$ to y in j reflections and approximately one rotation. The only constraint on x and y was that they were confined to a $O(j^{-4})$ neighborhood of the diagonal of the boundary. By reflecting Ω through the vertical axis, one obtains another smooth strictly convex domain and the reflections of x and y remain $O(j^{-4})$ close to the diagonal of the boundary. Hence, the same procedure produces exactly 4 counterclockwise orbits of j reflections from x to y making approximately one rotation in the reflected domain. Reflecting back through the vertical axis carries these 4 orbits to clockwise orbits in the original domain. This completes the proof of Theorem 3.3.1.

Remark 3.4.10. We can take $j_0 = j_0(\Omega) = \max\{j_i : 2 \leq i \leq 7\}$, with j_i as they appear in Lemmas 3.4.2, 3.4.3, 3.4.4, 3.4.6, 3.4.8, and 3.4.9. Similarly, we can choose a uniform constant C_0 as in the statement of Lemma 3.3.1. The tubular neighborhood U_j referenced in Theorem 2.1.1 can be taken to be $\{(x, y) \in \Omega \times \Omega : \text{dist}((x, y), \Delta\partial\Omega) \leq j^{-4}\}$ for $j \geq j_0$, where $\Delta : \partial\Omega \times \partial\Omega$ is the diagonal embedding $x \mapsto (x, x)$. For $(x, y) \in U_j$, both x and y satisfy the conditions of Lemma 3.4.9. The cone bundle $C_x^*(\Omega; j)$ is well defined whenever $\text{dist}(x, \partial\Omega) = O(j^{-4})$.

Remark 3.4.11. The proof of Theorem 3.3.1 in this section could actually be extended to a larger region of validity. In particular, the same methods allow us to prove the existence of 8 orbits of rotation number k/j for j sufficiently large in terms of k, connecting points in a comparable open neighborhood of the diagonal of the boundary. Additionally, we could also allow x and y to be further away from the diagonal as long as they are both sufficiently close to the boundary. However, we are only concerned with near diagonal terms in this dissertation for the purposes of deriving trace formulas.

Remark 3.4.12. If $x \in \partial\Omega$, then $\xi_0^{\pm}$ are tangent to the boundary and perturbing ξ away from ξ_0^+ in the clockwise direction is no longer well defined. Rather, it is equivalent to reflecting and then rotating ξ in the counterclockwise direction. Similarly, ξ_0^- cannot be rotated in the counter clockwise direction. Each of these restrictions reduces the number of j reflection orbits to y in approximately one rotation by 2. If additionally $y \in \partial\Omega$, the final link only makes one intersection with $d^{-1}(0) = \partial\Omega$ so there are only two orbits of j reflections connecting x to y in approximately one rotation. One is in the counterclockwise direction while the other is in the clockwise direction. In particular, when $x = y$, there is a *unique* geodesic loop (up to parametrization) of j reflections and exactly one rotation (i.e. rotation number $1/j$). The proof of these claims is much simpler, as it only requires Lemma 3.4.4.

Chapter 4

Variational Formulas

4.1 Variation of the Wave Trace

In this chapter, we derive variational formulas for the Green's function, simple eigenvalues and wave trace corresponding to the eigenvalue problem (2.6) with Robin boundary conditions. The PDE (2.6) has the weak formulation

$$\int_{\Omega_\varepsilon} \langle \nabla u_\varepsilon, \nabla \varphi_\varepsilon^{-1*} v\rangle - \lambda_j^2 u_\varepsilon \varphi_\varepsilon^{-1*} v\, dV = \int_{\Omega_\varepsilon} f_\varepsilon \varphi_\varepsilon^{-1*} v\, dV + \int_{\partial\Omega_\varepsilon} K_\varepsilon u_\varepsilon \varphi_\varepsilon^{-1*} v\, dq_\varepsilon, \tag{4.1}$$

for any $u_\varepsilon \in H^2(\Omega_\varepsilon)$ such that $\partial_\nu u_\varepsilon = K_\varepsilon u_\varepsilon$ (i.e. in the domain of $-\Delta$ with Robin boundary conditions), $v \in C^\infty(\Omega_0)$ and $f_\varepsilon \in \mathcal{E}'(\mathrm{int}(\Omega_\varepsilon))$ which is an inhomogeneous term for the PDE (2.6). Here, $dV = dx^1 \wedge dx^2$ is the volume form on $\mathbb{R}^2$ and dq_ε is the natural surface measure on $\partial\Omega_\varepsilon$ induced from the Euclidian metric. We refer to the the quantity

$$e(u, v) = \langle \nabla u, \nabla v\rangle - \lambda^2 uv$$

as the energy density.

4.1.1 Variational derivatives

Some care is needed to differentiate the expressions above. We begin by making precise our notion of first variation, following closely the presentation in [55].

Definition 4.1.1. If $u_\varepsilon \in C^1([0,\varepsilon_0], \mathcal{D}'(\Omega_\varepsilon))$ is a C^1 family of distributions, we write δu, δu_ε or $\dot{u}$ for the first variation of u_ε at $\varepsilon = 0$, as a distribution in Ω_0:

$$\delta u_\varepsilon = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} u_\varepsilon.$$

To simplify notation, for a single function u, we will oftentimes write $\delta u = \dot{u}$ and reserve the use of δ for preceding long formulas. If $\alpha \in C_0^\infty(\mathrm{int}\Omega_0)$ is a test function, then $\alpha \in C_0^\infty(\mathrm{int}\Omega_\varepsilon)$ for $\varepsilon \ll 1$ and we can define δu by

$$\delta u(\alpha) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} u_\varepsilon(\alpha),$$

i.e. the derivative of a function from $[0,\varepsilon_0)$ to $\mathbb{R}$. The issue with this definition is that if the supports of the distributions u_ε actually intersect $\partial\Omega_\varepsilon$, then the formula above only defines δu_ε in the interior of Ω_0 and not on the boundary even when u_ε is defined there. For instance, the Green's kernel is supported near the boundary in the setting of Robin boundary conditions.

To resolve this issue, we follow the ideas in [55], where some geometric heuristics motivate several precise definitions. The set of smooth domains in $\mathbb{R}^2$ is an infinite dimensional manifold Λ, on which the Lie group $\mathrm{Diff}(\mathbb{R}^2)$ acts. The Lie algebra of $\mathrm{Diff}(\mathbb{R}^2)$ is the space of smooth vector fields on $\mathbb{R}^2$. An initial domain Ω_0 and a curve φ_ε in $\mathrm{Diff}(\mathbb{R}^2)$ generate a curve in Λ, given by $\Omega_\varepsilon = \varphi_\varepsilon(\Omega_0)$. Hence, we can associate elements of the Lie algebra to tangent vectors at Ω_0. For any given deformation of Ω_0, we have an infinitesimal generator

$X = \frac{d\varphi_\varepsilon}{d\varepsilon}\big|_{\varepsilon=0}$. For a fixed $s \in \mathbb{R}$, we can associate to each $\Omega_0 \in \Lambda$ the fiber $H^s(\Omega_0)$, which is the L^2 based Sobolev space of order s. This defines a smooth vector bundle over Λ on which $\mathrm{Diff}(\mathbb{R}^2)$ acts via pullback (diffeomorphism invariance of the Sobolev spaces). These heuristics motivate the following definition:

Definition 4.1.2. For a curve $(u_\varepsilon, \Omega_\varepsilon)$ in the above vector bundle, we define the Lie derivative to be

$$\theta_X u_\varepsilon = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \varphi_\varepsilon^* u_\varepsilon.$$

We sometimes drop the ε subscript and write $\theta_X u_\varepsilon = \theta_X u$ for simplicity. The advantage of this definition is that it is well defined on the boundary of Ω_0 for $s > 1$, via the Sobolev embedding theorem $H^s \hookrightarrow C^0$.

Lemma 4.1.3. *If we suppose that $u_\varepsilon \in H^{s+1}(\Omega_\varepsilon)$ is supported away from the boundary and $\theta_X u \in H^s(\Omega_0)$, then δu_ε exists in $H^s(\Omega_0)$ and*

$$\theta_X u = \delta u_\varepsilon + X u_0.$$

Proof. The lemma follows from writing

$$\varphi_\varepsilon^* u_\varepsilon - u_0 = \varphi_\varepsilon^*(u_\varepsilon - u_0) + (\varphi_\varepsilon^* - 1)u_0,$$

dividing both sides by ε and sending $\varepsilon \to 0$. □

Remark 4.1.4. The formula in Lemma 4.1.3 is perfectly valid pointwise whenever $x \in \mathrm{int}\Omega_0$. Both θ_X and X are well defined operators on distributions supported near the boundary, so by setting $\delta u = \theta_X u - Xu$, we obtain an extension of Definition 4.1.1 for $s > 1$. From now on, we use X to denote the differential operator acting on distributions on the fixed domain Ω_0 and θ_X to denote the Lie derivative acting on distributions or differential forms on Ω_ε which may also depend on ε.

4.1.2 A general variational formula

We now derive a variational formula using the weak formulation (4.1). To obtain an integral equation on the fixed domain Ω_0, we pull back the energy density and apply the change of variables formula to (4.1):

$$\int_{\Omega_0} \varphi_\varepsilon^* e(u_\varepsilon, \varphi_\varepsilon^{-1*} v) \varphi_\varepsilon^* dV = \int_{\Omega_0} \varphi_\varepsilon^* f_\varepsilon v \varphi_\varepsilon^* dV + \int_{\partial\Omega_\varepsilon} K_\varepsilon u_\varepsilon \varphi_\varepsilon^{-1*} v dq_\varepsilon.$$

The pullback of the surface measure dq_ε in the last term on the right is more complicated, so for the time being, we leave it as is. We can rewrite this equation as

$$\int_{\Omega_0} e_\varepsilon(\varphi_\varepsilon^* u_\varepsilon, v) \varphi_\varepsilon^* dV = \int_{\Omega_0} \varphi_\varepsilon^* f_\varepsilon v dV + \int_{\partial\Omega_\varepsilon} K_\varepsilon u_\varepsilon \varphi_\varepsilon^{-1*} v dq_\varepsilon, \tag{4.2}$$

where

$$e_\varepsilon(u, v) = \varphi_\varepsilon^*(e(\varphi_\varepsilon^{-1*} u, \varphi_\varepsilon^{-1*} v))$$

is the conjugated energy density. While e_ε is a composition of operators, it is still of the form

$$e_\varepsilon = \sum_{|\alpha|,|\beta|\leq 2} c_{\alpha,\beta}(\varepsilon, x, y) D_x^\alpha D_y^\beta,$$

for some coefficients $c_{\alpha,\beta}$ depending smoothly on x, y and in a C^1 manner on ε. This will justify use of the product rule when computing ε derivatives in Lemma 4.1.5 below. Differentiating (4.2) in the parameter ε and setting $\varepsilon = 0$ yields

$$\begin{aligned} &\int_{\Omega_0} (e(\theta_X u, v) + (\theta_X e)(u_0, v)) dV + e(u_0, v)\theta_X dV \\ &\quad = \int_{\Omega_0} v\theta_X f dV + v f \theta_X dV + \delta \int_{\partial\Omega_\varepsilon} K_\varepsilon u_\varepsilon \varphi_\varepsilon^{-1*} v \, dq_\varepsilon, \end{aligned} \tag{4.3}$$

where the quantity

$$\theta_X e = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e_\varepsilon$$

is defined analogously to the formula in Definition 4.1.1.

Lemma 4.1.5. *For $u, v \in H^2(\Omega_0)$, differentiating the conjugated energy density e_ε yields*

$$(\theta_X e)(u, v) = Xe(u, v) - e(Xu, v) - e(u, Xv).$$

Proof. Consider the family of distributions $w_\varepsilon = e(\varphi_\varepsilon^{-1*}u, \varphi_\varepsilon^{-1*}v)$. Recalling that

$$e_\varepsilon(u, v) = \varphi_\varepsilon^* w_\varepsilon,$$

we see by Lemma 4.1.3 that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \varphi_\varepsilon^* w_\varepsilon = Xw_0 + \delta w_\varepsilon.$$

The first term is precisely $Xe(u, v)$ and the second term δw_ε is easily calculated by commuting the ε and x derivatives:

$$\begin{aligned}
\delta w_\varepsilon &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \langle \nabla \varphi_\varepsilon^{-1*}u, \nabla \varphi_\varepsilon^{-1*}v\rangle - \lambda^2 \varphi_\varepsilon^{-1*}u \varphi_\varepsilon^{-1*}v \\
&= \langle \nabla(-Xu), \nabla v\rangle + \langle \nabla u, \nabla(-Xv)\rangle - \lambda^2(-Xu)v - \lambda^2 u(-Xv) \\
&= -e(Xu, v) - e(u, Xv).
\end{aligned}$$

Commuting derivatives is always valid in the sense of distributions. In order to also restrict to $\varepsilon = 0$, note that $[\frac{d}{d\varepsilon}, \nabla_x]\varphi_\varepsilon^{-1*}u = 0$ in L^2 and this quantity only involves second derivatives of u, first order derivates of φ_ε^{-1} in ε, and x derivatives of φ_ε^{-1}. Here, $[\cdot, \cdot]$ denotes the commutator of two operators. As $u \in H^2$ and φ_ε is C^1 with respect to ε, $[\frac{d}{d\varepsilon}, \nabla_x]\varphi_\varepsilon^{-1*}u$ is a continuous family (in ε) of L_x^2 functions. Letting $\varepsilon \to 0$ then implies $[\delta, \nabla]\varphi_\varepsilon^{-1*}u = 0$. $\square$

Returning to the variation of (4.2), the Lie derivative of the volume form in equation (4.3) gives the divergence of X, which we would like to convert to a boundary integral, since X is only defined in a tubular neighborhood of $\partial\Omega_0$. Using again the formula $\theta_X = \delta + X$ and applying the divergence theorem to equation (4.3) above gives

$$\begin{aligned}\delta\int_{\Omega_0} e_\varepsilon(\varphi_\varepsilon^* u_\varepsilon, v)\varphi_\varepsilon^* dV &= \int_{\Omega_0} e(\dot{u}, v) - e(u_0, Xv)dV + \int_{\partial\Omega_0} e(u_0, v)X_\nu dq_0 \\ &= \int_{\Omega_0} v\dot{f} - (Xv)f dV + \int_{\partial\Omega_0} vfX_\nu dq_0 + \delta\int_{\partial\Omega_\varepsilon} K_\varepsilon u_\varepsilon \varphi_\varepsilon^{-1^*} v dq_\varepsilon.\end{aligned} \tag{4.4}$$

Here, $X_\nu = \langle X, \nu\rangle$ is the normal component of X. For the last term in (4.4), we can assume the perturbation is in the normal direction and parametrize the boundary by $\partial\Omega_0 \ni x \mapsto x + \rho_\varepsilon(x)\nu_x \in \partial\Omega_\varepsilon$, so that

$$\int_{\partial\Omega_\varepsilon} K_\varepsilon u_\varepsilon \varphi_\varepsilon^{-1*} v\, dq_\varepsilon = \int_{\partial\Omega_0} K_\varepsilon(x + \rho_\varepsilon(x)\nu_x)u_\varepsilon(x + \rho_\varepsilon(x)\nu_x)v(x)\, dq_\varepsilon. \tag{4.5}$$

We now recall a basic result from differential geometry:

Lemma 4.1.6. *The variation of surface measure is given by*

$$\delta dq_\varepsilon = \kappa\dot{\rho} dq_0,$$

where κ is the curvature of $\partial\Omega_0$.

Proof. A proof using normal coordinates can be found on page 6 of [7]. □

Hence, differentiating the entire boundary integral (4.5), we obtain

$$\begin{aligned}&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_{\partial\Omega_0} K_\varepsilon(x + \rho_\varepsilon(x)\nu_x)u_\varepsilon(x + \rho_\varepsilon(x)\nu_x)v(x)\, dq_\varepsilon \\ &\quad= \int_{\partial\Omega_0} \dot{K}u_0 v + K_0\theta_X uv + K_0 u_0 v\dot{\rho}\kappa dq_0.\end{aligned} \tag{4.6}$$

Actually, there are two ways to define $\dot{K}$ since it is currently ambiguous as to how to differentiate in x on the hypersurface $\partial\Omega_\varepsilon$. The first way involves extending K_ε radially to a function defined on a tubular neighborhood of $\partial\Omega_\varepsilon$, so that we may differentiate in x on an open subset of $\mathbb{R}^2$. The second way is to define $\dot{K}_\varepsilon(x) = d/d\varepsilon|_{\varepsilon=0} K_\varepsilon(x + \rho_\varepsilon \nu_x)$. While these two definitions differ pointwise, the integral formula remains the same and we adopt the second definition as it appears more naturally in the proof.

4.1.3 Variation of Green's kernel

Combining equations (4.4) and (4.6), we obtain the variational formula

$$\begin{aligned}
&\int_{\Omega_0} \langle \nabla \dot{u}, \nabla v\rangle - \lambda^2 \dot{u} v dV + \int_{\partial\Omega_0} (\langle \nabla u_0, \nabla v\rangle - \lambda^2 u_0 v) X_\nu dq_0 \\
&\quad = \int_{\Omega_0} \langle \nabla u_0, \nabla (Xv)\rangle - \lambda^2 u_0 X v - f X v + \dot{f} v dV \\
&\quad + \int_{\partial\Omega_0} f v X_\nu + \dot{K} u_0 v + K_0 \dot{u} v + K_0 (X u_0) v + K_0 u_0 v \dot{\rho} \kappa \, dq_0.
\end{aligned}$$

Recall that $\varphi_\varepsilon(x) = x + \rho_\varepsilon(x)\nu_x$ for $x \in \partial\Omega_0$, so that $X_\nu = \dot{\rho}$. Integrating by parts and collecting boundary terms, we see that

$$\begin{aligned}
&\int_{\Omega_0} \dot{u}(-\Delta - \lambda^2) v - (Xv)(-\Delta - \lambda^2) u_0 + f X v - \dot{f} v dV \\
&\qquad = \int_{\partial\Omega_0} - \dot{u} \nabla^\perp v - (\langle \nabla u_0, \nabla v\rangle - \lambda^2 u_0 v)\dot{\rho} + (Xv)\nabla^\perp u_0 \\
&\qquad\qquad + f v \dot{\rho} + \dot{K} u_0 v + K_0 \dot{u} v + K_0 (X u_0) v + K_0 u_0 v \dot{\rho} \kappa dq_0.
\end{aligned}$$

If u and v satisfy the PDE (2.6) with Robin boundary conditions, we have

$$\begin{aligned}
&\int_{\Omega_0} \dot{u} f_0 - \dot{f} v dV \\
&\quad = \int_{\partial\Omega_0} - \dot{u} K_0 v - (\langle \nabla u_0, \nabla v\rangle - \lambda^2 u_0 v)\dot{\rho} + (Xv) K_0 u_0 \\
&\qquad + f v \dot{\rho} + \dot{K} u_0 v + K_0 \dot{u} v + K_0 (X u_0) v + K_0 u_0 v \dot{\rho} \kappa dq_0 \\
&\qquad = \int_{\partial\Omega_0} -(\langle \nabla u_0, \nabla v\rangle - \lambda^2 u_0 v)\dot{\rho} + f v \dot{\rho} + 2K_0^2 u_0 v \dot{\rho} + \dot{K} u_0 v + K_0 u_0 v \dot{\rho} \kappa dq_0.
\end{aligned} \tag{4.7}$$

Noting that

$$\langle \nabla u_0, \nabla v\rangle = \langle \nabla^T u_0, \nabla^T v\rangle + \langle \nabla^\perp u_0, \nabla^\perp v\rangle = \langle \nabla^T u_0, \nabla^T v\rangle + K_0^2 u_0 v,$$

we can get rid of one of the $K_0^2 u_0 v \dot{\rho}$ terms in (4.7) in exchange for only using tangential derivatives.

To prove Theorem 2.2.3, we now fix $x_0 \in \operatorname{int}(\Omega_0)$ and denote by G_ε be the Greens function on Ω_ε. Formally, the Green's function is the Schwartz kernel of the resolvent $(-\Delta_\varepsilon - \lambda^2)^{-1}$. Setting $u_\varepsilon(x) = G_\varepsilon(\lambda, x, y)$, $v(x) = G_0(\lambda, x_0, x)$ and $f_\varepsilon(x) = \delta_{x_0}(x)$, we obtain

$$\begin{aligned}
&\delta G_\varepsilon(\lambda, x_0, y) = \\
&\qquad \int_{\partial\Omega_0} - \langle \nabla_2^T G_0(\lambda, x_0, q), \nabla_1^T G_0(\lambda, q, y)\rangle \dot{\rho} dq \\
&\qquad\quad + \int_{\partial\Omega_0} (\lambda^2 \dot{\rho} + K_0^2 \dot{\rho} + K_0 \kappa \dot{\rho} + \dot{K}) G_0(\lambda, x_0, q) G_0(\lambda, q, y) dq,
\end{aligned}$$

which is precisely Theorem 2.2.3 with x replaced by x_0. As $(\Delta_\varepsilon - \lambda^2) \in \Psi^2(\Omega_\varepsilon)$ is elliptic for $\operatorname{Im}\lambda$ positive, G_ε is a Lagrangian distribution with principal symbol in $S^{-2}_{1,0}$. Hence, G_ε is a family of distributions and δG_ε in particular, is a distribution of order -1 with wavefront set conormal to the diagonal $\{x = y\}$. As $x, y \in \operatorname{int}(\Omega_0)$ and $q \in \partial\Omega_0$ in Theorem 2.2.3, the points (x,q) and (q,y) are away from the diagonal, where the distribution is smooth. Hence, the tangential derivatives do not affect the smoothness or integrability. The distribution δG_ε can actually be extended up to the boundary using the method of layer potentials, although this is not needed in the remainder of this dissertation.

4.1.4 The wave trace and eigenvalues

We now want to find a formula for the variation of the distributional trace of the even wave propagator, $\cos(t\sqrt{-\Delta_\varepsilon})$, in terms of that of the Green's kernel. Recall that the wave propagator $e^{it\sqrt{-\Delta_\varepsilon}}$ has a distributional trace in the sense that

$$\int_{\mathbb{R}} e^{it\sqrt{-\Delta_\varepsilon}}\varphi(t)\,dt$$

is trace class for any Schwartz function φ. Its trace is

$$\sum_j \int_{\mathbb{R}} e^{it\lambda_j(\varepsilon)}\varphi(t)\,dt, \tag{4.8}$$

where $(\lambda_j^2(\varepsilon))_{j=1}^\infty$ are the eigenvalues of Δ_ε. The sum in (4.8) can be seen to be convergent via integration by parts combined with Weyl's law on the asymptotic distribution of eigenvalues. While Weyl's law is usually stated for Dirichlet or Neumann eigenvalues, the Robin and Neumman asymptotics actually agree up to leading order due to the fact that the boundary operators $\partial_\nu - K$ and ∂_ν of the Robin and Neumann Laplacians respectively have the same principal symbol. For example, see [71] or [32]. Taking real and imaginary parts, the analogous trace formulas hold for the even and odd wave kernels, which we denote by

$$E_R(t,x,y) = \cos t\sqrt{-\Delta_\varepsilon} \quad \text{and} \quad S_R(t,x,y) = \frac{\sin t\sqrt{-\Delta_\varepsilon}}{\sqrt{-\Delta_\varepsilon}},$$

respectively. The subscript R here is to denote the Robin boundary conditions. In this section, we shall prove:

Theorem 4.1.7. *The variation of the even wave trace is*

$$\delta\, Tr \cos t\sqrt{-\Delta_\varepsilon} = \int_{\partial\Omega_0} L_R^b(t,q,q)\dot{\rho} + \frac{t}{2} S_R(t,q,q)\dot{K}\, dq,$$

where we have defined

$$L_R^b(t,q,q') = \frac{t}{2}(-\nabla_1^T\nabla_2^T - \Delta_2 + K_0^2 + K_0\kappa)S_R.$$

Here, ∇_i^T is again the tangential derivative in the ith spacial variable and Δ_2 is the Euclidean Laplacian in the second spacial variable. The kernels are first differentiated in the interior using an extension of the tangential vector field and then restricted to the diagonal of the boundary. We will also prove:

Theorem 4.1.8. *If $\lambda_j^2(0)$ is a simple eigenvalue associated to the L^2 normalized eigenfunction Ψ_j, then*

$$\delta\lambda_j^2(\varepsilon) = \int_{\partial\Omega_0} |\nabla^T\Psi_j|^2\dot{\rho}\, dq - \int_{\partial\Omega_0} |\Psi_j|^2(\lambda_j^2\dot{\rho} + K_0^2\dot{\rho} + \dot{K} + K_0\kappa\dot{\rho})\, dq.$$

Both theorems are proved together by the same method:

Proof. Our derivation of the wave trace variation is based on Kato's variational formulas for sums of eigenvalues in [36]. One has to be careful, as an eigenvalue of higher multiplicity may not be C^1 in ε. Such eigenvalues can break off to become many different eigenvalues under deformation. However, if we denote by $m(\lambda_j^2)$ the multiplicity of $\lambda_j^2 = \lambda_j^2(0)$, we will see that the sum $\sum_1^{m(\lambda_j^2)} \lambda_{j,k}^2(\varepsilon)$ is in fact C^1 in ε. We actually prove a more general theorem: let g be holomorphic in a neighborhood of the eigenvalue λ_j^2 and denote the resolvent operator by

$\widetilde{R}_\varepsilon(z) = (-\Delta_\varepsilon - z)^{-1}$ for $z \notin \mathrm{Spec}(-\Delta_\varepsilon)$, with Schwartz kernel $\widetilde{G}_\varepsilon(z,x,y)$. We write $\widetilde{R}_\varepsilon$ and $\widetilde{G}_\varepsilon$ since the spectral parameter is z instead of z^2. Then, by the Cauchy integral formula, we have

$$\delta \sum_{k=1}^{m_j(\lambda_j^2)} g(\lambda_{j,k}^2(\varepsilon)) = \delta \mathrm{Tr} T_{g,\varepsilon}, \tag{4.9}$$

where

$$T_{g,\varepsilon} = \frac{-1}{2\pi i}\int_\gamma g(z)\widetilde{R}_\varepsilon(z)dz,$$

for γ a small, positively oriented circle enclosing only the λ_j^2 eigenvalue. When $\varepsilon = 0$, $T_{g,0}$ is g of the orthogonal projection onto the eigenspace of λ_j^2. As the eigenvalues do vary continuously in ε, for $\varepsilon \ll 1$, $T_{g,\varepsilon}$ is the total projector, i.e. g composed with the projection onto the direct sum of the eigenspaces of $\lambda_{j,k}^2(\varepsilon)$ for $1 \leq k \leq m(\lambda_j^2)$. $T_{g,\varepsilon}$ is in fact a C^1 family of operators in ε since the resolvent is. The trace can also be obtained by integrating $\widetilde{G}_\varepsilon$ over the diagonal, which combined with equation (4.9), gives

$$\delta \sum_{k=1}^{m_j(\lambda_j^2(0))} g(\lambda_{j,k}^2(\varepsilon)) = \frac{-1}{2\pi i}\int_\gamma g(z)\left(\delta \int_{\Omega_\varepsilon} \widetilde{G}_\varepsilon(z,x,x)dx\right)dz.$$

As in Section 4.1.2, we can pull back $\widetilde{G}_\varepsilon(z,x,x)dx$ via φ_ε^* to obtain an integral over a fixed domain, which we then differentiate. We have

$$\begin{aligned}\delta \sum_{k=1}^{m_j(\lambda_j^2(0))} g(\lambda_{j,k}^2(\varepsilon)) &= \frac{-1}{2\pi i}\int_\gamma g(z)\left\{\int_{\Omega_0} \delta\varphi_\varepsilon^*\widetilde{G}_\varepsilon(z,x,x)dx + \int_{\Omega_0} \widetilde{G}_0(z,x,x)\delta\varphi_\varepsilon^* dx\right\}dz \\ &= \frac{-1}{2\pi i}\int_\gamma g(z)\left\{\int_{\Omega_0} \theta_X\widetilde{G}_\varepsilon(z,x,x)dx + \int_{\Omega_0} \widetilde{G}_0(z,x,x)\mathrm{div}X dx\right\}dz,\end{aligned}$$

where the last line follows from the definition of θ_X and the standard fact from Riemannian geometry that the Lie derivative with respect to X of the volume form gives the divergence of X times the volume form, i.e. $\theta_X(dx) = \mathrm{div} X dx$. Using the formula for θ_X in Lemma 4.1.3 and the divergence theorem, we obtain

$$\frac{-1}{2\pi i}\int_\gamma g(z)\left\{\int_{\Omega_0}(\delta+X)\widetilde{G}_\varepsilon(z,x,x)dx - \int_{\Omega_0} X\widetilde{G}_0(z,x,x)dx + \int_{\partial\Omega_0} X_\nu \widetilde{G}_0(z,q,q)dq\right\}dz$$
$$= \frac{-1}{2\pi i}\int_\gamma g(z)\left\{\int_{\Omega_0}\delta\widetilde{G}_\varepsilon(z,x,x)dx + \int_{\partial\Omega_0}\dot{\rho}(q)\widetilde{G}_0(z,q,q)dq\right\}dz.$$

We now plug in our variational formula for the Green's kernel from Theorem 2.2.3 to see that

$$\begin{aligned}
\delta \sum_{k=1}^{m_j(\lambda_j^2(0))} g(\lambda_{j,k}^2(\varepsilon)) &= \frac{-1}{2\pi i}\int_\gamma g(z)\left\{\int_{\Omega_0}\delta\widetilde{G}_\varepsilon(z,x,x)dx + \int_{\partial\Omega_0}\dot{\rho}(q)\widetilde{G}_0(z,q,q)dq\right\}dz \\
&= \frac{-1}{2\pi i}\int_{\Omega_0}\int_{\partial\Omega_0}\int_\gamma g(z)(-\nabla_2^T\widetilde{G}_0(z,x,q)\cdot\nabla_1^T\widetilde{G}_0(z,q,x)\dot{\rho})\,dzdqdx \\
&+ \frac{-1}{2\pi i}\int_{\Omega_0}\int_{\partial\Omega_0}\int_\gamma g(z)(z\dot{\rho}+K_0^2\dot{\rho}+K_0\kappa\dot{\rho}+\dot{K})\widetilde{G}_0(z,x,q)\widetilde{G}_0(z,q,x)\,dzdqdx \\
&+ \frac{-1}{2\pi i}\int_\gamma\int_{\partial\Omega_0} g(z)\dot{\rho}(q)\widetilde{G}_0(z,q,q)dqdz.
\end{aligned}$$

Denote these three integrals by I_1, I_2 and I_3 and let $(\Psi_{j,k})_{k=1}^{m(\lambda_j^2)}$ be an orthonormal basis for the eigenspace corresponding to the eigenvalue λ_j^2. Then, via the Cauchy integral formula, we have

$$\begin{aligned}
I_1 &= \sum_1^{m(\lambda_j^2)}\int_\gamma\int_{\partial\Omega_0}\frac{g(z)}{(2\pi i)(z-\lambda_j^2)^2}\nabla^T\Psi_{j,k}(q)\cdot\nabla^T\Psi_{j,k}(q)\dot{\rho}(q)dqdz \\
&= g'(\lambda_j^2)\sum_{k=1}^{m(\lambda_j^2)}\int_{\partial\Omega_0}|\nabla^T\Psi_{j,k}(q)|^2\dot{\rho}dq.
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
I_2 =& - g(\lambda_j^2) \sum_{k=1}^{m(\lambda_j^2)} \int_{\partial\Omega_0} |\Psi_{j,k}(q)|^2 \dot{\rho}(q) dq \\
& - g'(\lambda_j^2) \int_{\partial\Omega_0} \left(\sum_{k=1}^{m(\lambda_j^2)} |\Psi_{j,k}(q)|^2 \right) \left(\lambda_j^2 \dot{\rho} + K_0^2 \dot{\rho} + K_0 \kappa \dot{\rho} + \dot{K} \right) dq \\
& = \int_{\partial\Omega_0} \left(\sum_{k=1}^{m(\lambda_j^2)} |\Psi_{j,k}(q)|^2 \right) \left(-g(\lambda_j^2)\dot{\rho} - g'(\lambda_j^2)(\lambda_j^2 \dot{\rho} + K_0^2 \dot{\rho} + K_0 \kappa \dot{\rho} + \dot{K}) \right) dq
\end{aligned}
$$

and

$$
I_3 = \sum_{k=1}^{m(\lambda_j^2)} \int_{\partial\Omega_0} g(\lambda_j^2) \dot{\rho}(q) |\Psi_{j,k}(q)|^2 dq.
$$

Combining these terms and noticing that I_3 cancels with one of the terms in I_2, we obtain

$$
\delta \sum_{k=1}^{m_j(\lambda_j^2)} g(\lambda_{j,k}^2(\varepsilon)) = g'(\lambda_j^2) \sum_{k=1}^{m_j(\lambda_j^2)} \int_{\partial\Omega_0} |\nabla^T \Psi_{j,k}|^2 \dot{\rho} - |\Psi_{j,k}|^2 (\lambda_j^2 \dot{\rho} + K_0^2 \dot{\rho} + K_0 \kappa \dot{\rho} + \dot{K}) dq. \tag{4.10}
$$

To compute the variation of the even wave trace in particular, set $g(z) = \cos(t\sqrt{z})$ in equation (4.10). Despite the square root, this is in fact an entire function since cosine is even. We have

$$
\begin{aligned}
& \delta \sum_{j=1}^{\infty} \sum_{k=1}^{m(\lambda_j^2)} \cos(t\lambda_{j,k}(\varepsilon)) = \sum_{j=1}^{\infty} \delta \sum_{k=1}^{m(\lambda_j^2)} \cos(t\lambda_{j,k}(\varepsilon)) \\
& = \sum_{j=1}^{\infty} \sum_{k=1}^{m(\lambda_j^2)} \frac{-t\sin(t\lambda_j)}{2\lambda_j} \int_{\partial\Omega_0} |\nabla^T \Psi_{j,k}|^2 \dot{\rho} - |\Psi_{j,k}|^2 (\lambda_j^2 \dot{\rho} + K_0^2 \dot{\rho} + K_0 \kappa \dot{\rho} + \dot{K}) dq.
\end{aligned}
$$

Writing this in terms of the wave kernels, we obtain

$$\delta \mathrm{Tr} \cos t\sqrt{-\Delta_\varepsilon} =$$
$$\int_{\partial\Omega_0} \frac{-t}{2}\nabla_1^T\nabla_2^T S_R(t,q,q)\dot{\rho} + \frac{t}{2}\dot{\rho}(-\Delta_2 + K_0^2 + K_0\kappa)S_R(t,q,q) + \frac{t}{2}S_R(t,q,q)\dot{K}\,dq. \tag{4.11}$$

Note that all but one of the terms above contain $\dot{\rho}$. Recalling that in the begining of the section, we defined

$$L_R^b(t,q,q') = \frac{t}{2}(-\nabla_1^T\nabla_2^T - \Delta_2 + K_0^2 + K_0\kappa)S_R$$

to be the coefficient of $\dot{\rho}$ in the expression (4.11), we obtain Theorem 4.1.7. Similarly, setting $g(z) = z$ in equation (4.9) easily yields Theorem 4.1.8 on the variation of simple eigenvalues. $\square$

Chapter 5

A Parametrix for the Wave Propagator

In this Chapter, we use microlocal analysis to obtain an oscillatory integral parametrix for the wave propagator in an open subset of $\mathbb{R} \times \Omega \times \Omega$ which contains the diagonal $\mathbb{R} \times \Delta(\partial\Omega)$. The wave kernel $e^{it\sqrt{-\Delta}}$ is actually *not* a Fourier integral operator (FIO) near the tangential rays (see [1], [46]), so we microlocalize the wave kernels near periodic transversal reflecting rays. We begin by reviewing FIOs and Chazarain's parametrix for the wave propagator in Sections 5.1 and 5.2. In Section 5.3, we then cook up oscillatory integrals for each term in Chazarain's parametrix, which microlocally approximate the wave propagator near the orbits described in Theorem 3.3.1. This approach is inspired by the formulas in [44] and will rely on the symbol calculus in Section 5.2.

5.1 Fourier integral operators (FIOs)

Let X and Y be open sets in $\mathbb{R}^{n_X}$ and $\mathbb{R}^{n_Y}$ respectively. If $a \in S^{\mu}_{1,0}(X \times \mathbb{R}^N)$ is a classical symbol of order μ and $\Theta \in C^\infty(X \times \mathbb{R}^N)$ is a nondegenerate phase function, then the linear operator

$$A(u) = \int_X \int_{\mathbb{R}^N} e^{i\Theta(x,\theta)} a(x,\theta) u(x)\, d\theta dx$$

is called a Lagrangian or Fourier integral distribution on X. Recall that a continuous linear operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ has an associated Schwartz kernel $K_A \in \mathcal{D}'(X \times Y)$. If K_A is given by a locally finite sum of Lagrangian distributions on $X \times Y$, then we say A is a Fourier integral operator (FIO). One can then show that the wavefront set of the kernel is contained in the image of the map $\iota_\Theta : (x,y) \mapsto (x, y, d_x\Theta, d_y\Theta)$ when restricted to the critical set $C_\Theta := \{d_\theta\Theta = 0\}$. The image of ι_Θ is in fact a conic Lagrangian submanifold $\Lambda_\Theta \subset T^*(X \times Y)$ and the map ι_Θ is a local diffeomorphism from C_Θ onto Λ_Θ. In this case, we say "Θ parametrizes Λ_Θ." The canonical relation or wavefront relation of A is defined by

$$WF'(A) = \{(x,\xi),(y,\eta) : (x,y,\xi,-\eta) \in WF(K_A)\} \subset T^*X \times T^*Y,$$

and describes how the operator A propagates singularities of distributions on which it acts. If ω_X and ω_Y denote the natural symplectic forms on T^*X and T^*Y respectively, one can more invariantly consider FIOs associated to general conic Lagranigan submanifolds $\Lambda \subset T^*X \times T^*Y$ (canonical relations), with respect to the symplectic form $\omega_X - \omega_Y$. The notion of a principal symbol for Fourier integral operators is more subtle than that for pseudodifferential operators. The principal symbol of A is a half density on Λ_Θ given in terms of the parametrization ι_Θ:

$$e = \iota_{\Theta *}(a_0 |dC_\Theta|^{1/2}), \tag{5.1}$$

where a_0 is the leading order term in the asymptotic expansion for a and $|dC_\Theta|^{1/2}$ is the half density associated with the Gelfand-Leray form on the level set $\{d_\theta \Theta = 0\}$. Here, we have ignored Maslov factors coming from the Keller-Maslov line bundle over Λ_Θ. These are nonzero factors $e^{i\sigma\pi/4}$ (σ is known as the Maslov index) which appear in front of the principal symbol as a result of the multiplicity of phase functions parametrizing the canonical relation Λ_Θ, possibly in different coordinate systems. While these factors allow the principal symbol to be defined in a more geometrically invariant way, we defer computation of the Maslov indices until Section 6.5. For a more thorough reference on the global theory of Lagrangian distributions, see [11], [28], [13], and [30]. The order of a Fourier integral operator is defined in such a way that when two Fourier integral operators' canonical relations intersect transversally, then the composition is again a Fourier integral operator and order of the composition is the sum of the orders:

$$\operatorname{order}(A) = m = \mu + \frac{1}{2}N - \frac{1}{4}(n_X + n_Y). \tag{5.2}$$

Recall that here, n_X and n_Y are the dimensions of X and Y respectively. In this case, we write $A \in I^m(X \times Y, \Lambda)$. This convention on orders also generalizes that of pseudodifferential operators, where $X = Y$ and $m = \mu$ coincides with the order of the corresponding symbol class. Sufficient conditions which guarantee that the composition exists are clean or transversal intersection of the two operators' canonical relations. In general, composition of Fourier integral operators and the associated symbol calculus is somewhat complicated, but we will not directly use the composition formula in what follows.

5.2 Chazarain's parametrix

The parametrix developed by Chazarain in [6] provides a microlocal description of the wave kernels near periodic transversal reflecting rays. The parametrices for

$$E(t) = \cos t\sqrt{-\Delta}, \qquad S(t) = \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}$$

are constructed in the ambient Euclidean space $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$. We only consider $S(t)$, as the formula for $E(t)$ is easily obtained from that of $S(t)$ by differentiating in t. We write $E(t,x,y)$ and $S(t,x,y)$ for the Schwartz kernels of $E(t)$ and $S(t)$ respectively. Following the work in [6], we can find a Lagrangian distribution

$$\widetilde{S}(t,x,y) = \sum_{j=-\infty}^{\infty} S_j(t,x,y), \qquad S_j \in I^{-5/4}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \Gamma_{\pm}^j), \tag{5.3}$$

which approximates $S(t)$ microlocally away from the tangential rays modulo a smooth kernel. We will describe the canonical relations $\Gamma_{\pm}^j$ momentarily and in particular, show that the sum in (5.3) is locally finite. We first explain what is meant by approximating $S(t)$ "microlocally away from the tangential rays." In general, two distributions $f, g \in \mathcal{D}'(\mathbb{R}^n)$ are said to agree microlocally near a closed cone $\Lambda_1 \subset T^*\mathbb{R}^n$ if $WF(u-v) \cap \Lambda_1 = \emptyset$. Similarly, using the language from Section 5.1, two operators $A, B : C^\infty(Y) \to C^\infty(X)$ are said to agree microlocally near a closed cone $\Lambda_2 \subset T^*X \times T^*Y$ if $WF'(A-B) \cap \Lambda_2 = \emptyset$. This second notion is what we will use to say that our parametrix approximates $S(t)$ microlocally near the canonical relations $\Gamma_{\pm}^j$.

The kernel of the wave propagator $S(t)$ satisfies

$$\begin{cases} S_{tt}(t) - \Delta S(t) = 0, & x \in \Omega, \\ S(t) = 0, & x \in \partial\Omega, \\ S(0) = 0, \\ S_t(0) = \delta(x - x_0), \end{cases}$$

for Dirichlet boundary condtions and

$$\begin{cases} S_{tt}(t) - \Delta S(t) = 0, & x \in \Omega, \\ (\partial_\nu - K)S(t) = 0, & x \in \partial\Omega, \\ S(0) = 0, \\ S_t(0) = \delta(x - x_0), \end{cases}$$

for Robin boundary conditions, where $x_0 \in \mathrm{int}\Omega$ and K a Robin function as in equation (2.6). Chazarain's construction begins with the solution of the homogeneous wave equation in the ambient Euclidean space:

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \mathbb{R}^2, \\ u(0, x) = 0, \\ u_t(0, x) = \delta(x - x_0). \end{cases}$$

In this case, we have an explicit representation for the fundamental solution of $\Box = \partial_t^2 - \Delta$, given by Kirchhoff's formula. If we restrict back to Ω, it is clear that the fundamental solution on $\mathbb{R}^2$ will in general not satisfy the Dirichlet/Robin boundary conditions. However, finite speed of propagation implies that it *does* satisfy the boundary condition for small time, since u vanishes identically in a neighborhood of $\partial\Omega$. If we let $d = d(x_0, \partial\Omega)$, then we obtain a solution u of the boundary value problem for $|t| < d$. The idea in [6] is to use the billiard flow and properties of FIOs to inductively extend the time interval on which the fundamental solution of the boundary value problem is defined.

To describe the canonical relations precisely, we first introduce some notation following the presentations in [6] and [25]. As the Euclidean wave operator $\Box_{\mathbb{R}^2}$ factors into $(\partial_t - \sqrt{\Delta})(\partial_t + \sqrt{-\Delta})$, there are two Hamiltonians corresponding to the symbol $\pm|\eta|$ of $\pm\sqrt{-\Delta}$. Let $H_\pm(y,\eta) = \pm|\eta|$ and g^t be the Hamiltonian flow, i.e. the flow map associated to the system

$$\begin{cases} \partial_t y = \frac{\partial H_\pm}{\partial \eta}, \\ \partial_t \eta = -\frac{\partial H_\pm}{\partial y}, \end{cases}$$

which is in fact just the reparametrized geodesic flow on $\mathbb{R}^2$. For $(y,\eta) \in T^*\Omega$ (or $T^*_{\partial\Omega}\mathbb{R}^2$ such that η is transversal to the boundary and inward pointing), recall that in Section 3 we defined

$$t^1_\pm = \inf_{t>0}\{t : \pi_1 g^t(y,\eta) \in \partial\Omega\},$$
$$t^{-1}_\pm = \sup_{t<0}\{t : \pi_1 g^t(y,\eta) \in \partial\Omega\},$$

where π_1 is projection onto the spatial variable. Note that $t^{\pm 1}_\pm$ depend on (y,η). We also have $t^1_\mp = -t^{-1}_\pm$. We then set

$$\lambda^1_\pm(y,\eta) = g^{\pm t^1_\pm}(y,\eta),$$
$$\lambda^{-1}_\pm(y,\eta) = g^{\pm t^{-1}_\pm}(y,\eta).$$

Also define $\widehat{\lambda^1_\pm(y,\eta)}$ to be the reflection of $\lambda^1_\pm(y,\eta)$ through the cotangent line at the boundary. In other words, $\widehat{\lambda^1_\pm(y,\eta)}$ and $\lambda^1_\pm(y,\eta)$ have the same cotangential components but opposite conormal components so that $\widehat{\lambda^1_\pm(y,\eta)}$ is inward pointing. The point $\widehat{\lambda^{-1}_\pm(y,\eta)}$ is defined analogously. We can then inductively define $t^j_\pm$ and $t^{-j}_\pm$ by the formulas

$$\begin{aligned} t^j_\pm &= \inf_{t>0}\{t : \pi_1 g^{\pm t}(\widehat{\lambda^{j-1}_\pm(y,\eta)}) \in \partial\Omega\}, \\ t^{-j}_\pm &= \sup_{t<0}\{t : \pi_1 g^{\pm t}(\widehat{\lambda^{-(j-1)}_\pm(y,\eta)}) \in \partial\Omega\}. \end{aligned}$$

The total travel time after j reflections is defined by

$$T^j_\pm = \begin{cases} \sum_{k=1}^{j} t^k_\pm & j > 0 \\ 0 & j = 0 \\ \sum_{k=j}^{-1} t^k_\pm & j < 0. \end{cases}$$

To study how the fundamental solution of $\Box_\Omega$ behaves at $\partial\Omega$ when we impose boundary condtions, we propagate the intial data by the free wave propagator on $\mathbb{R}^2$, restrict it to the boundary, reflect, and then propagate again. If such a construction is continued for $j \in \mathbb{Z}$ reflections at the boundary, it is shown in [6] that the FIOs S_j must have canonical relations

$$\Gamma^j_\pm = \begin{cases} (t, \tau, g^{\pm t}(y,\eta), y, \eta) : \tau = \pm|\eta| & j = 0, \\ (t, \tau, g^{\pm(t - T^j_\pm(y,\eta))}(\widehat{\lambda^j_\pm(y,\eta)}), y, \eta)) : \tau = \pm|\eta| & j \in \mathbb{Z}\backslash\{0\}. \end{cases}$$

Since $\widetilde{S}(t)$ is a microlocal parametrix, the canonical relation of the true solution operator $S(t)$ is also

$$\Gamma = \bigcup_{j\in\mathbb{Z},\pm} \Gamma^j_\pm.$$

Here, $j > 0$ and $j < 0$ correspond to reflections on the inside and outside of the boundary. In fact, there are four modes of propagation associated to the canonical relations $\Gamma^j_\pm$, corresponding to $\pm\tau \geq 0$ (forwards and backwards time) and $\pm j \geq 0$ (inside and outside bounces). See [25] for an explicit local model in the case of one reflection, where Ω is replaced by a half plane. We see that a point $(t, \tau, x, \xi, y, \eta)$ belongs to the canonical relation $\Gamma^j_\pm \subset T^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2)$ if the broken geodesic of j reflections emanating from (y, η) passes through (x, ξ). For a more thorough discussion of Chazarain's parametrix, see [6] and [20].

To be precise, Chazarain actually showed that there exists FIOs S_j such that the sum in (5.3) is a parametrix for the wave propagator $S(t)$ with canonical relation

$$\Gamma = \bigcup_{j \in \mathbb{Z}, \pm} \Gamma^j_\pm.$$

However, the canonical relations were never parametrized by explicit phase functions and the principal symbols of the operators S_j were not computed in [6]. For the remainder of this section, we concern ourselves with the task of explicitly computing them in terms of geometric and dynamical data associated to the billiard map.

5.3 Oscillatory integral representation

In this section, we cook up an oscillatory integral such that microlocally near the canonical relations $\Gamma^j_\pm$,

$$S_j(t, x, y) = \int_{-\infty}^{\infty} e^{i\Theta_j(t,\tau,x,y)} a_j(\tau, x, y)\, d\tau,$$

where S_j is the j^{th} term in Chazarain's parametrix corresponding to a wave with j reflections and $a_j \in S^\mu_{\text{cl}}$ is a classical symbol of order μ. We will only compute the principal symbol and use $L.O.T$ to denote lower order terms in the sense of Lagrangian distributions in what follows. Due to the presence of different Maslov factors on each branch of $\Gamma^j_\pm$ corresponding to $\pm\tau > 0$ (cf. Sections 5.1 and 6.5), it is actually more convenient to find operators

$$S_{j,\pm}(t, x, y) = \int_0^{\infty} e^{i\Theta_{j,\pm}(t,\tau,x,y)} a_{j,\pm}(\tau, x, y)\, d\tau, \tag{5.4}$$

so that $S_j = S_{j,+} + S_{j,-}$ and the phase functions associated to $S_{j,\pm}$ paramaterize Γ^j_+ and Γ^j_- individually.

We first make precise the notion of microlocalized FIOs. We would like to microlocalize $S(t)$ near periodic orbits of rotation number $1/j$. Oftentimes, it is required that such orbits be nondegenerate, in the sense that 1 is not an eigenvalue of the linearized Poincaré map. However, this assumption is not needed for our trace formulas, which work both for simple nondegenerate orbits as well as degenerate orbits coming in one parameter families as in the case of an ellipse (cf. Theorem 2.1.5 and Corollary 2.1.6). We recall that Ω is said to satisfy the concoincidence condition (2.5) if there exist no periodic orbits of rotation number m/n, $m \geq 2$ in a sufficiently small neighborhood of $|\partial\Omega|$. In this case, for j sufficiently large, periodic orbits of rotation number $1/j$ come in isolated families. This follows from the results in [44], which show that for j sufficiently large, no two orbits of distinct rotation numbers $1/j$ and $1/k$ can have the same length. It can be shown that periodic billiard orbits arise as critical points of the loop function $\Psi_j(q,q)$ (see [69], [26]). As in the statement of Theorem 2.1.3, denote $t_j = \inf_{q\in\partial\Omega}$ and $T_j = \sup_{q\in\partial\Omega}$. If Ω satisfies the noncoincidence condition 2.5, we can find a smooth cutoff function $\chi_1(t)$ which is identically equal to 1 on an open neighborhood of $[t_j, T_j]$ and vanishes in a neighborhood of all other $L \in \mathrm{Lsp}(\Omega)$. As noted in Section 5.2, each propagator S_j has canonical relations $\Gamma^j_\pm$. Denote by χ_2 a smooth cutoff function which is identically equal to 1 on $\Gamma^j = \cup_\pm \Gamma^j_\pm$, vanishing near the gliding rays $T^*\partial\Omega$ and conic in the fiber variables τ, ξ and η. Quantizing χ_2 gives a pseudodifferential operator which microlocalizes near the support of χ_2. For a reference, see Chapter 18 of [29]. We call such an operator a microlocal cutoff on $\Gamma^j_\pm$. The composition $\chi_1(t)\chi_2(t,x,y,D_t,D_x,D_y)S(t)$ is then smoothing away from the geodesic loops of rotation number $1/j$ and the trace of this composition is equal to the wave trace modulo C^∞ in an open neighborhood of $[t_j, T_j]$.

We now use Theorem 3.3.1 to find suitable phase functions $\Theta_{j,\pm}$ which parametrize $\Gamma^j_\pm$.

Define phase functions Θ_j^k by the formula

$$\Theta_{j,\pm}^k(t,\tau,x,y) = \pm\tau(t - \Psi_j^k(x,y)),$$

where Ψ_j^k are given in Definition 3.3.3. We then have,

Lemma 5.3.1. *The phase functions $\Theta_{j,\pm}^k(t,\tau,x,y)$ are smooth in an open neighborhood of the diagonal of the boundary and locally parametrize the canonical relations $\Gamma_\pm^j$. In particular, the fibers of both Γ_+^j and Γ_-^j lying over this neighborhood are unions of 8 connected components, which we denote by $\Gamma_\pm^{j,k}$ corresponding to $1 \leq k \leq 8$ as in Definition (3.3.3).*

Proof. The proof Lemma 5.3.1 is based on [69] and shows that the length functional is in fact a generating function for the billiard map. For any $x, y \in \Omega$ let

$$\begin{cases} L_{x,y} : \partial\Omega^j \to \mathbb{R}_+ \\ L_{x,y}(q_1, q_2, \cdots q_j) = |x - q_1| + \left\{\sum_{m=2}^j |q_m - q_{m-1}|\right\} + |q_j - y| \end{cases} \tag{5.5}$$

denote the length functional. We first show that billiard trajectories from x to y are in one to one correspondence with critical points of (5.5) with respect to $q \in \partial\Omega^j$. Let $g \in C^\infty(\mathbb{R}^2)$ be a defining function for $\partial\Omega$ and consider q as a variable in $\mathbb{R}^2 \times \cdots \times \mathbb{R}^2 = \mathbb{R}^{2j}$ rather than $\partial\Omega^j$. If q is a critical point of (5.5), then as in the method of Lagrange multipliers, by setting $x = q_0$ and $y = q_{j+1}$, we find that for $1 \leq m \leq j$, there exists $\lambda_m \in \mathbb{R}$ such that

$$\frac{\partial L_{x,y}}{\partial q_m} = \frac{q_m - q_{m-1}}{|q_m - q_{m-1}|} + \frac{q_m - q_{m+1}}{|q_m - q_{m+1}|} = \lambda_m \nabla_{q_m} g.$$

Since $\nabla_{q_m} g \perp \partial\Omega$, this implies that the two unit vectors in the formula for $\partial_{q_m} L_{x,y}$ have opposite tangential components, which is precisely the condition giving elastic collision at the boundary (angle of incidence equals angle of reflection). Similarly, if this condition is satisfied, then q is a critical point for (5.5).

We now consider the functions Ψ_j^k in Definition 3.3.3. We have

$$\Psi_j^k(x,y) = |x - q_1^k| + \left\{ \sum_{m=2}^{j} |q_m^k - q_{m-1}^k| \right\} + |q_j^k - y|, \tag{5.6}$$

where $q_m^k(x,y)$ is the mth impact point on the boundary for the billiard trajectory corresponding to Ψ_j^k. As opposed to the q_m in the length functional (5.5), q_m^k will in general have a nontrivial dependence on x and y. Differentiating (5.6) in x, we obtain

$$\begin{aligned}\frac{\partial \Psi_j^k}{\partial x_i} = \frac{x - q_1^k}{|x - q_1^k|} \cdot \frac{\partial}{\partial x_i}(x - q_1^k) + \left\{ \sum_{m=2}^{j} \frac{q_m^k - q_{m-1}^k}{|q_m^k - q_{m-1}^k|} \cdot \frac{\partial}{\partial x_i}(q_m^k - q_{m-1}^k) \right\} \\ + \frac{q_j^k - y}{|q_j^k - y|} \cdot \frac{\partial}{\partial x_i}(q_j^k - y).\end{aligned} \tag{5.7}$$

Since for each $x, y \in \Omega$, the path defined by (x, q^k, y) corresponds to a billiard trajectory, we see that all of the terms except the first telescope in (5.7). Hence,

$$d_x \Psi_j^k = \frac{x - q_1^k}{|x - q_1^k|}. \tag{5.8}$$

Similarly, differentiating (5.6) in y, we obtain

$$d_y \Psi_j^k = \frac{y - q_j^k}{|y - q_j^k|}. \tag{5.9}$$

Geometrically, these gradients are the incident and (reflected) outgoing unit directions of the billiard trajectories described in Theorem 3.3.1.

We now consider the maps

$$\iota_{\Theta_{j,\pm}^k} : (t, \tau, x, y) \mapsto (t, \tau, x, d_x \Theta_j^k, y, -d_y \Theta_j^k) = (t, \tau, x, -\tau d_x \Psi_j^k, y, \tau d_y \Psi_j^k) \tag{5.10}$$

on the critical set $C_{\Theta^k_{j,\pm}} = \{t - \Psi^k_j = 0\}$. Inserting formulas (5.8) and (5.9) into (5.10) and comparing with the canonical graphs

$$\Gamma^j_\pm = \begin{cases} (t, \tau, g^{\pm t}(y, \eta), y, \eta) : \tau = \pm|\eta| & j = 0, \\ (t, \tau, g^{\pm(t - T^j_\pm(y,\eta))}\widehat{\lambda_j(y, \eta)}, y, \eta) : \tau = \pm|\eta| & j \in \mathbb{Z}\backslash\{0\} \end{cases}$$

from Section 5.2, we see that $\iota_{\Theta^k_{j,\pm}} : C_{\Theta^k_{j,\pm}} \to \Gamma^j_\pm$ is a local diffeomorphism. Since $1 \leq k \leq 8$, it follows that both Γ^j_+ and Γ^j_- are the unions of 8 connected components. □

For a fixed j, the multiple components $\Gamma^{j,k}_\pm$ of the canonical relation indexed by k can be seen to intersect over the boundary. As our parametrix for the propagators described in [6] are in fact modified by microlocal cutoffs supported away from the tangential rays in $S^*\partial\Omega$, we may assume $\Gamma^{j,k}_\pm$ are smooth nonintersecting Lagrangian submanifolds. It would be interesting to study mapping properties of the operators S^k_j as the canonical relations degenerate near the glancing set in future work. We now want to derive an explicit oscillatory integral representation (5.4) for $S(t)$ in a specific coordinate system adapted to Ω. We first need to better understand the forwards and backwards symbols on Γ.

Recall that the Hadamard type variational formula for the wave trace in Theorem 4.1.7 involved the integral of wave kernels over the diagonal of the boundary. To understand the principal symbol first in the interior, we study how the propagator reflects at the boundary. In particular, we want to study the canonical relations $\Gamma^j_\pm$ restricted to the fibers over the boundary, which we now describe. Denote by

$$A^0_\pm = \{(0, \tau, y, \eta, y, \eta) : \tau = \pm|\eta|\}$$

the fibers corresponding to zero reflections. If we flow out from $A^0_\pm$ by the Hamiltonian flow $\psi^{\pm t}$ of $H = \tau \pm |\xi|$, we obtain $\Gamma^0_\pm$. Note that $\psi^{\pm t}$ consists of geodesics lifted to $T^*(\mathbb{R} \times \Omega)$.

Consider the following subsets of $\Gamma^0_\pm$:

$$
\begin{aligned}
A^1_\pm &= \{(t, \tau, \psi^{\pm t}(y,\eta), y, \eta) : t > 0, \psi^t(y,\eta) \in T^*_{\partial\Omega}\mathbb{R}^2, \tau = \pm|\eta|\}, \\
A^{-1}_\pm &= \{(t, \tau, \psi^{\pm t}(y,\eta), y, \eta) : t < 0, \psi^t(y,\eta) \in T^*_{\partial\Omega}\mathbb{R}^2, \tau = \pm|\eta|\}.
\end{aligned}
$$

If $\widehat{\ }$ denotes reflection in the left factor, we have

$$
\Gamma^1_\pm = \bigcup_{t\in\mathbb{R}} \widehat{A^1_\pm}, \qquad \Gamma^{-1}_\pm = \bigcup_{t\in\mathbb{R}} \widehat{A^{-1}_\pm}.
$$

We define $A^j_\pm$ and $\widehat{A^j_\pm}$ similarly and note that these are precisely the fibers of the canonical relations $\Gamma^j_\pm$ lying over the boundary. In the next section, we will compute the principal symbol of the wave propagator in coordinates on the critical set. Therefore, we first need to better understand the forwards and backwards symbols on Γ.

Proposition 5.3.2. *Let $e_\pm$ denote the principal symbol of $S(t)$ on $\Gamma = \bigcup_{j\in\mathbb{Z},\pm} \Gamma^j_\pm$. Then, we have*

$$
\begin{aligned}
e_\pm &= \frac{(-1)^j}{2\tau i}|dt \wedge dy \wedge d\eta|^{1/2} \qquad \textit{(Dirichlet)} \\
e_\pm &= \frac{1}{2\tau i}|dt \wedge dy \wedge d\eta|^{1/2} \qquad \textit{(Neumann/Robin)}
\end{aligned}
$$

Furthermore, the principal symbol for the wave propagator with $K = 0$ (Neumann boundary conditions) coincides with $e_\pm$ (Robin boundary conditions).

Proof. As the proof for Dirichlet boundary conditions can be found in [25], let us only treat the case of Robin/Neumann boundary conditions following [69]. As in [6] and [25], denote by σ_0 the symbol of the restriction to $t = 0$, σ_r the symbol of the boundary restriction operator r, and σ_B the symbol of $rN - K \in \Psi^1$. Here, N is an extension of the unit normal vector field on $\partial\Omega$ to a tubular neighborhood of the boundary acting as a differential operator and

K is again a Robin function. We have the following implications:

$$
\begin{aligned}
\Box S = 0 &\implies \mathcal{L}_H e_\pm = 0, \\
S\big|_{t=0} = 0 &\implies \sigma_0 \circ e_+ + \sigma_0 \circ e_- = 0, \\
\frac{d}{dt} S\big|_{t=0} = Id &\implies \tau\sigma_0 \circ e_+ - \tau\sigma_0 \circ e_- = \frac{1}{i}\sigma_{Id}, \\
(rN - rK)S = 0 &\implies \sigma_B \circ e_\pm = \sigma_B \circ e_\pm\big|_{A^j_\pm} + \sigma_B \circ e_\pm\big|_{\widehat{A}^j_\pm} = 0.
\end{aligned}
\tag{5.11}
$$

The first assertion in (5.11) follows from Theorem 5.3.1 of [13] and the remaining formulas are clear. At the boundary, the symbol σ_B is given by

$$\langle \lambda(y,\eta), \nu_y \rangle \sigma_r$$

on $\Gamma_{\partial\Omega} \circ A^j_\pm$ and

$$\langle \widehat{\lambda(y,\eta)}, \nu_y \rangle \sigma_r = -\langle \lambda(y,\eta), \nu_y \rangle \sigma_r$$

on $\Gamma_{\partial\Omega} \circ \widehat{A}^j_\pm$. The symbol of rK doesnt appear since multiplication by K is a ΨDO of order 0 while, $N \in \Psi^1$. Hence, the fourth equation in (5.11) implies that on the boundary, we have

$$\langle \lambda(y,\eta), \nu_y \rangle \sigma_r \circ e_\pm(\lambda(y,\eta)) - \langle \lambda(y,\eta), \nu_y \rangle \sigma_r \circ e_\pm(\widehat{\lambda(y,\eta)}) = 0. \tag{5.12}$$

Note that in equation (5.12), both sides involve the composition $\sigma_r \circ e_\pm$. The formula for the principal symbol of the composition of FIOs is quite complicated, but is discussed more thoroughly in [20], [25], [12] and [13]. Since $\langle \lambda(y,\eta), \nu_y \rangle$ is nonvanishing, equation (5.12) tells us that the direct and reflected symbols coincide on the boundary. Multiplying the second equation in (5.11) by τ and adding/subtracting it to the third equation gives

$$\sigma_0 \circ e_\pm = \frac{1}{2\tau i}\sigma_{Id}$$

in the interior. It is elementary to see that σ_{Id} is the canonical half density $|dt \wedge dy \wedge d\eta|^{1/2}$.

The first equation in (5.11) implies that the symbol is invariant under geodesic flow, so the claim follows on $\Gamma^1_\pm$. In fact, since we have already noted that the direct and reflected symbols coincide over the boundary, the fact that $\Gamma^j_\pm$ is the flowout of $\widehat{A}^j_\pm$ then implies that the claim extends to all $\Gamma^j_\pm$. $\square$

Remark 5.3.3. Chazarain's parametrix actually computes the full symbol by solving successive transport equations and Borel summing the terms. The full proof and explicit computation of the symbol can be found in the original French paper [6]. The actual solution operator could be obtained from $S(t)$ by adding correction terms via Duhamel's principle. However, we only need the principal symbol in our calculation. In [20], a more general situation is treated in which both $M = \Omega \times \{t = 0\}$ and $M' = \partial\Omega \times \mathbb{R}$ are nonglancing, noncharacteristic hypersurfaces for the wave propagator. A Fourier integral operator is then constructed iteratively to solve the localized hyperbolic pseudodifferential equation.

As Proposition 5.3.2 gives $e_\pm = \frac{1}{2\tau i}|dt \wedge dy \wedge d\eta|^{1/2}$ and we now know that the phase functions $\Theta^k_{j,\pm}(t, \tau, x, y) = \pm\tau(t - \Psi^k_j(x, y))$ parametrize connected components of $\Gamma^j_\pm$ lying over an open neighborhood of the diagonal of the boundary, we can compute the principal term in the asymptotic expansion of $a_{j,\pm}$ appearing in the expression (5.4). Since there are 16 phases for each j, corresponding to $1 \leq k \leq 8$ and $\pm\tau > 0$, (5.4) should actually be a sum of 16 oscillatory integrals:

$$S_j(t, x, y) = \sum_{\pm} \sum_{k=1}^{8} \int_0^\infty e^{i\Theta^k_{j,\pm}(t,\tau,x,y)} a_{j,k,\pm}(\tau, x, y) d\tau. \tag{5.13}$$

Recalling formula (5.1) for the principal symbol of a Fourier integral operator, we must compute the Gelfand-Leray form on the critical set $d_\tau \Theta^k_{j,\pm} = 0$. The Leray measure coming from the Gelfand-Leray form is coordinate invariant and it is ultimately more convenient to utilize different coordinate systems for general calculations and those specific to the ellipse.

5.4 Computations in boundary normal coordinates

For computations in general convex domains, it is convenient to introduce boundary normal coordinates which we now describe following the presentation in [43]. Fix a point $p \in \partial\Omega$. For each $q \in \partial\Omega$ near p, we denote by $\gamma_q(\mu)$ the unit speed geodesic with initial condition conormal to $\partial\Omega$ at q and inward pointing. Now denote by φ the boundary coordinate which parametrizes $\partial\Omega$ with respect to arclength, such that p is given by $\varphi = 0$. The coordinate φ can be extended smoothly inside Ω so that it is constant along $\gamma_q(\mu)$ for every fixed q near p. For $\varepsilon > 0$ sufficiently small, (μ, φ) is then a smooth coordinate system in an ε neighborhood of $p \in \overline{\Omega}$. In these coordinates, the Euclidean metric is locally given by the warped product

$$g_{\text{Eucl.}} = d\mu^2 + f(\mu, \varphi) d\varphi^2,$$

where f is a locally defined function which is smooth up to the boundary. In particular, f satisfies $f(0, \varphi) = 1$. This coordinate system is convenient near the boundary because conformal multiples the vector fields $\partial/\partial\mu$ and $\partial/\partial\varphi$ extend the orthogonal and and tangential gradients respectively in a tubular neighborhood of the boundary. As the canonical relations $\Gamma_{\pm}^{j}$ involve both x and y variables, we change x variables to (μ, φ) and y variables to (ν, θ) according to the procedure described above. In [69], elliptical polar coordinates were instead used of boundary normal coordinates to achieve a similar decomposition into normal and tangential vector fields.

Without loss of generality, we use boundary normal coordinates instead of Euclidean coordinates in the domain of Ψ_j^k from here on. We now compute the Gelfand-Leray form in boundary normal coordinates.

Lemma 5.4.1. *The canonical relation of each operator in* (5.13) *is parametrized in boundary* *normal coordinates by*

$$\Gamma_{\pm}^{j,k} = \{(t,\tau,\mu,\varphi,\xi,\nu,\theta,\eta) = (\Psi_j^k,\tau,\mu,\varphi,\mp\tau d_{\mu,\varphi}\Psi_j^k,\nu,\theta,\pm\tau d_{\nu,\theta}\Psi_j^k)\}.$$

The Gelfand-Leray form on $C_{\Theta_{j,\pm}^k}$ is given by

$$dC_{\Theta_{j,\pm}^k} = \mp f(\mu,\varphi)f(\nu,\theta)d\tau\wedge d\mu\wedge d\varphi\wedge d\nu\wedge d\theta.$$

Proof. The first claim follows directly from Lemma 5.3.1 and the observation that

$$d_\tau\Theta_{j,\pm}^k = \pm(t-\Psi_j^k) = 0$$

on the critical set $C_{\Theta_{j,\pm}^k}$. From this, it is clear that $(\tau,\mu,\varphi,\nu,\theta)$ form a smooth coordinate system on $C_{\Theta_{j,\pm}^k}$. The Gelfand-Leray form is uniquely defined on $C_{\Theta_{j,\pm}^k}$ by the condition

$$d\left(d_\tau\Theta_{j,\pm}^k\right)\wedge dC_{\Theta_{j,\pm}^k} = f(\mu,\varphi)f(\nu,\theta)dt\wedge d\tau\wedge d(\mu,\varphi)\wedge d(\nu,\theta), \tag{5.14}$$

where the righthand side of (5.14) coincides with the Euclidian volume form on $\mathbb{R}_t\times\mathbb{R}_\tau\times\Omega\times\Omega$. Hence,

$$dC_{\Theta_{j,\pm}^k} = \mp f(\mu,\varphi)f(\nu,\theta)d\tau\wedge d\mu\wedge d\varphi\wedge d\nu\wedge d\theta,$$

as claimed. □

We now change variables and use Lemma 5.4.1 to compute the principal symbol of (5.13).

Dropping the j, k indices on Ψ_j^k in place of differentiation, we have

$$
\begin{aligned}
dt &= \Psi_\mu d\mu + \Psi_\varphi d\varphi + \Psi_\nu d\nu + \Psi_\theta d\theta,\\
dy &= f(\nu,\theta) d\nu \wedge d\theta,\\
d\eta_1 &= \Psi_\nu d\tau + \tau(\Psi_{\mu\nu} d\mu + \Psi_{\nu\varphi} d\varphi + \Psi_{\nu\nu} d\nu + \Psi_{\nu\theta} d\theta),\\
d\eta_2 &= \Psi_\theta d\tau + \tau(\Psi_{\theta\mu} d\mu + \Psi_{\theta\varphi} d\varphi + \Psi_{\theta\nu} d\nu + \Psi_{\theta\theta} d\theta)
\end{aligned}
$$

on the canonical relation $\Gamma_\pm^{j,k}$. Wedging all these terms together, we see that

$$
\begin{aligned}
dt \wedge dy \wedge d\eta =& \tau(f(\nu,\theta)(\Psi_\mu \Psi_\theta \Psi_{\nu\varphi} + \Psi_\varphi \Psi_\nu \Psi_{\theta\mu}\\
&- \Psi_\mu \Psi_\nu \Psi_{\theta\varphi} - \Psi_\varphi \Psi_\theta \Psi_{\nu\mu})) d\tau \wedge d\mu \wedge d\varphi \wedge d\nu \wedge d\theta.
\end{aligned} \tag{5.15}
$$

Definition 5.4.2. Denote by $A_j^k(\mu,\varphi,\nu,\theta)$ the functions

$$
f(\mu,\varphi)(\Psi_\mu \Psi_\theta \Psi_{\nu\varphi} + \Psi_\varphi \Psi_\nu \Psi_{\theta\mu} - \Psi_\mu \Psi_\nu \Psi_{\theta\varphi} - \Psi_\varphi \Psi_\theta \Psi_{\nu\mu}),
$$

where each $\Psi = \Psi_j^k$ depends implicitly on j, k, μ, φ, ν and θ.

On each of the canonical relations $\Gamma_\pm^{j,k}$ (cf. Lemma 5.3.1), equation (5.15) implies that

$$
e_\pm = (-1)^j \frac{|dt \wedge dy \wedge d\eta|^{1/2}}{2\tau i} = (-1)^j \frac{\pm 1}{2|\tau|^{1/2} i} |A_j^k(\mu,\varphi,\nu,\theta) dC_{\Theta_{j,\pm}^k}|^{1/2} \tag{5.16}
$$

for Dirichlet boundary conditions and

$$
e_\pm = \frac{|dt \wedge dy \wedge d\eta|^{1/2}}{2\tau i} = \frac{\pm 1}{2|\tau|^{1/2} i} |A_j^k(\mu,\varphi,\nu,\theta) dC_{\Theta_{j,\pm}^k}|^{1/2} \tag{5.17}
$$

for Neumann boundary conditions. As a result, we conclude the following description of the operators S_j.

Theorem 5.4.3. *Microlocally near $\Gamma_\pm^j$, the following oscillatory integral is a parametrix for*

$S_j(t,x,y)$ with Dirichlet boundary conditions in an open neighborhood of $\Delta\partial\Omega \subset \Omega \times \Omega$:

$$S_j(t) = (-1)^j \sum_{\pm} \sum_{k=1}^{8} \int_0^\infty e^{\pm i\tau(t-\Psi_j^k)} \frac{\pm 1}{2|\tau|^{1/2} i} |A_j^k|^{1/2} d\tau + L.O.T.$$

Here, L.O.T. denotes lower order Lagrangian distributions, using the convention (5.2).

Definition 5.4.4. We define the operators $S_{j,\pm}^k(t)$ appearing in Theorem 5.4.3 by

$$S_{j,\pm}^k(t) = (-1)^j \sum_{\pm} \sum_{k=1}^{8} \int_0^\infty e^{\pm i\tau(t-\Psi_j^k)} \frac{\pm 1}{2|\tau|^{1/2} i} |A_j^k|^{1/2} d\tau.$$

Together with Theorem 3.3.1 and the choice of j_0, U_j in Remark 3.4.10, we can immediately derive Theorem 2.1.1 from Theorem 5.4.3. To see this, note that

$$E(t) = \frac{d}{dt} S(t).$$

On a symbolic level, this implies that the symbol of $E(t)$ is $\tau e_\pm$, with $e_\pm$ given by Proposition 5.3.2. This explains the order of $b_{j,k,\pm} \in S^{1/2}(U_j \times \mathbb{R}^1)$ appearing in Theorem 2.1.1. Each $(\pm)$ branch of the propagators $S_{j,\pm}^k$ should be multiplied by Maslov factors $e^{\pm i\pi\sigma_j/4}$. We will compute $\sigma_j = 1$ in Section 6.5, which completes the proof of Theorem 2.1.1. In the next chapter, we will use this explicit oscillatory integral representation to compute the wave trace.

5.5 Computations in elliptical polar coordinates

Let us now denote by S_R the wave operator for the ellipse

$$\left\{ (x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$$

appearing in Theorem 2.2.1, together with corresponding Robin boundary conditions. We want to derive an explicit formula for the principal symbol $e_\pm$ of S_R in coordinates. Referring to formula (5.1) for the principal symbol of an FIO, we easily see that in our setting, $dC_{\Theta^k_{j,\pm}} = \mp d\tau \wedge dx \wedge dy$. In Proposition 5.3.2, we calculated that $e_\pm = \frac{1}{2\tau i}|dt \wedge dy \wedge d\eta|^{1/2}$. Since we now know that the phase functions $\Theta^k_{j,\pm,\pm}(t,\tau,x,y) = \pm\tau(t - \Psi^k_j(x,y))$ locally parametrize the connected components of $\Gamma^j_\pm$, we now want to calculate a_0 (as in Section 5.1) by introducing a conformal change of coordinates which is suitable to computing the symbol $e_\pm$ for the ellipse:

Definition 5.5.1. Elliptical polar coordinates are defined on $\mathbb{R}^2 \backslash [-c, c]$ by the equations:

$$x_1 = c \cosh \mu \cos \varphi, \qquad x_2 = c \sinh \mu \sin \varphi.$$

Here, $(\mu, \varphi) \in (0, \infty) \times \mathbb{R}/(2\pi\mathbb{Z})$ and $c = \sqrt{a^2 - b^2}$ is the semifocal distance, i.e. the distance between the origin and a focal point of the ellipse

$$\Omega = \left\{ (x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.$$

It is easy to check that the Euclidean metric in these coordinates is a conformal multiple of $d\mu d\varphi$:

$$dx_1 dx_2 = c^2(\cosh^2 \mu - \cos^2 \varphi) d\mu d\varphi.$$

In particular, the vector fields $\partial/\partial\mu$ and $\partial/\partial\varphi$ are orthogonal at each point. For a fixed $\mu > 0$, the φ coordinate parametrizes a confocal ellipse of eccentricity $1/\cosh(\mu)$. When projected onto the phase space $B^*\partial\Omega$, these curves are precisely the invariant Lagrangian tori for the billiard map. In particular, fixing

$$\mu = \mu_0 = \cosh^{-1}\left(\frac{a}{\sqrt{a^2 - b^2}}\right)$$

gives a parametrization of the boundary $\partial\Omega$. Similarly, for a fixed $\varphi > 0$, the μ coordinate parametrizes a branch of a confocal hyperbola. The reason we use these coordinates is because up to a conformal factor, tangential differentiation on the ellipse becomes $\partial/\partial\varphi$ while differentiating in the normal direction becomes $\partial/\partial\mu$. Since the wave kernel S_R is a function of both the x and y variables, we use elliptic coordinates for y as well:

$$y_1 = c\cosh\nu\cos\theta, \qquad y_2 = c\sinh\nu\sin\theta.$$

Note that the Leray form is coordinate independent. Hence, without loss of generality, we compute that in elliptical coordinates,

$$\begin{aligned} dC_{\Theta^k_{j,\pm}} &= \mp d\tau \wedge dx \wedge dy \\ &= \mp a^4(\cosh^2\mu - \cos^2\varphi)(\cosh^2\nu - \cos^2\theta) d\tau \wedge d\mu \wedge d\varphi \wedge d\nu \wedge d\theta. \end{aligned}$$

On the critical set, we have

$$(t, \tau, \mu, \varphi, \xi, \nu, \theta, \eta) = (\Psi^k_j, \tau, \mu, \varphi, -\tau d_{\mu,\varphi}\Psi^k_j, \nu, \theta, \tau d_{\nu,\theta}\Psi^k_j).$$

From now on, we drop the the j, k subscripts and write $\Psi^k_j = \Psi$ so that we may use subscripts to denote derivatives. We have

$$\begin{aligned} dt &= \Psi_\mu d\mu + \Psi_\varphi d\varphi + \Psi_\nu d\nu + \Psi_\theta d\theta, \\ dy &= a^2(\cosh^2\nu - \cos^2\theta) d\nu \wedge d\theta, \\ d\eta_1 &= \Psi_\nu d\tau + \tau(\Psi_{\mu\nu} d\mu + \Psi_{\nu\varphi} d\varphi + \Psi_{\nu\nu} d\nu + \Psi_{\nu\theta} d\theta), \\ d\eta_2 &= \Psi_\theta d\tau + \tau(\Psi_{\theta\mu} d\mu + \Psi_{\theta\varphi} d\varphi + \Psi_{\theta\nu} d\nu + \Psi_{\theta\theta} d\theta). \end{aligned}$$

Wedging all these terms, we find that

$$\begin{aligned} dt \wedge dy \wedge d\eta =& \tau a^2(\cosh^2\nu - \cos^2\theta)(\Psi_\mu\Psi_\theta\Psi_{\nu\varphi} + \Psi_\varphi\Psi_\nu\Psi_{\theta\mu} \\ & - \Psi_\mu\Psi_\nu\Psi_{\theta\varphi} - \Psi_\varphi\Psi_\theta\Psi_{\nu\mu}) d\tau \wedge d\mu \wedge d\varphi \wedge d\nu \wedge d\theta. \end{aligned}$$

Keeping in mind that all Ψ terms depend on j and k, we denote the above factor by

$$A_j^k(\mu,\varphi,\nu,\theta) = a^2(\cosh^2\nu - \cos^2\theta)(\Psi_\mu\Psi_\theta\Psi_{\nu\varphi} + \Psi_\varphi\Psi_\nu\Psi_{\theta\mu} - \Psi_\mu\Psi_\nu\Psi_{\theta\varphi} - \Psi_\varphi\Psi_\theta\Psi_{\nu\mu}). \tag{5.18}$$

It will be clear from context which coordinate system we use when referring to the functions A_j^k. On each of the canonical relations $\Gamma_\pm^{j,k}$ (see Lemma 5.3.1), we have

$$e_\pm = \frac{|dt \wedge dy \wedge d\eta|^{1/2}}{2\tau i} = \frac{\pm 1}{2|\tau|^{1/2} i} \left| A_j^k d\tau \wedge d\mu \wedge d\varphi \wedge d\nu \wedge d\theta \right|^{1/2}.$$

As a result, we have proved:

Theorem 5.5.2. *Microlocally near $\Gamma_\pm^j$, the following oscillatory integral is a parametrix for S_j in an open neighborhood of $\Delta\partial\Omega \subset \Omega \times \Omega$:*

$$S_j(t,\mu,\varphi,\nu,\theta) = \sum_{\pm,k=1}^{8} \int_0^\infty e^{\pm i\tau(t-\Psi_j^k(x,y))} \frac{\pm 1}{2|\tau|^{1/2} i} |A_j^k(\mu,\varphi,\nu,\theta)|^{1/2} d\tau + L.O.T.$$

We denote the operators in this sum by $S_{j,\pm}^k$ and and also define $S_j^k = S_{j,+}^k + S_{j,-}^k$.

Chapter 6

Dirichlet Wave Trace Formulas

In this section, we use the parametrix in Theorem 2.1.1, or equivalently Theorem 5.4.3, to compute an integral formula for the wave trace corresponding to Dirichlet boundary conditions in a general convex domain. Formally, the wave trace is the Fourier transform of the spectral measure $\sum_j \delta(\lambda - \lambda_j)$, where $\{\lambda_j^2\}$ are the Dirichlet (resp. Neumann or Robin) eigenvalues of $-\Delta$ on Ω. This is a distribution of the form

$$\sum_j e^{it\lambda_j}, \tag{6.1}$$

which can be seen to be weakly convergent by Weyl's law on the asymptotic distribution of Laplace eigenvalues (see [32], [71]). The connection between this distribution and the wave equation lies in the fact that (6.1) is actually the trace of $e^{it\sqrt{-\Delta}}$, the propagator associated to the half wave operator $(\partial_t - i\sqrt{-\Delta})$. Since such a unitary operator is not trace class, we mean that for any Schwartz function φ, the regularized operators

$$\int_{-\infty}^{\infty} \varphi(t) e^{it\sqrt{-\Delta}} dt$$

are of trace class and have trace

$$\sum_j \int_{-\infty}^{\infty} \varphi(t) e^{it\lambda_j} dt.$$

The same holds for the even and odd wave operators

$$E(t) = \cos t\sqrt{-\Delta}, \qquad S(t) = \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}, \tag{6.2}$$

and we consider these as they appear more naturally in Chazarain's parametrix (cf. Section 5.2) and $E(t)$ solves the initial boundary value problem (2.1).

Recall that $\chi_1(t)$ is a cutoff function near the lengths $[t_j, T_j]$ of geodesic loops and Ω is assumed to satisfy the noncoincidence condition (2.5). Modulo Maslov factors and a smooth error term, we have

$$\chi_1(t)\mathrm{Tr}E = \int_\Omega E_j(t, x, x) dx, \tag{6.3}$$

where $E_j = \frac{\partial}{\partial t} S_j$ and S_j is the j bounce wave appearing in Chazarain's parametrix (5.3).

As our parametrix in Theorem 5.4.3 is only valid near the boundary, we want to write the wave trace as an integral over the boundary. There are two ways to do this. The first involves writing $E_j = \Delta B_j$ for a specially chosen distribution B_j so that formally, we may integrate by parts to obtain an integral of ∇B_j in terms of E_j over the boundary. The disadvantage of this approach is nonlocality of Δ^{-1}, which makes it difficult to compute B_j explicitly. The second way is to use Hadamard type variational formulas for the wave trace, which were derived in [25] and [69]. As shown in those papers, taking the variation of the wave trace automatically includes an integration by parts which puts the integral (6.3) on the boundary. The advantage of this approach is that the formulas are quite explicit.

6.1 Rescaling techniques

Our idea is to rescale the domain Ω by dilating about an interior point. Setting $\Omega_r = r\Omega$ for $0 \leq r \leq 1$, the domain shrinks to a point when $r = 0$. Denote by Δ_r the Laplacian on Ω_r with Dirichlet boundary conditions. Formally, one then has

$$\int_{0^+}^{1} \frac{\partial}{\partial r} \mathrm{Tr} \cos t\sqrt{-\Delta_r} dr = \mathrm{Tr} \cos t\sqrt{-\Delta_1} - \lim_{\varepsilon \to 0} \mathrm{Tr} \cos t\sqrt{-\Delta_\varepsilon}. \tag{6.4}$$

Here, the integral bounds $(0^+, 1]$ on the lefthand side of (6.4) correspond to a distributional limit, where the domain of integration is taken to be $(\varepsilon, 1]$ as $\varepsilon \to 0^+$. Equation (6.4) is essentially the fundamental theorem of calculus. The quantity $\partial_r \cos t\sqrt{-\Delta_{\Omega_r}}$ is given explicitly by variational formulas for the wave trace in [69] and [25]. As $r \to 0^+$, the eigenvalues of $-\Delta_r$ increase rapidly and monotonically to $+\infty$. This contributes highly oscillatory terms to the wave trace (6.1), which tend weakly to zero with r. We will in fact only scale down to $r = r_0 > 0$ rather than $r = 0$, for r_0 sufficiently small at the expense of smooth errors. This is explained in detail in Section 6.3.1. In this section, we derive more explicit integral formulas for the lefthand side of equation (6.4). In Section 6.3, we will then justify these computations in more detail and show that they indeed give the trace of $\cos t\sqrt{-\Delta_1}$.

As Δ is translation invariant, we may assume without loss of generality that the origin O is an interior point of Ω. For each $r \in (0, 1]$, consider the dilated domain $\Omega_r = r\Omega$ with respect to O. Define wave propagators on the rescaled domains Ω_r by

$$E_r(t) = \cos t\sqrt{-\Delta_r} \qquad\qquad S_r(t) = \frac{\sin t\sqrt{-\Delta_r}}{\sqrt{-\Delta_r}}.$$

The formulas in [25] and [69] show that

$$\partial_r \operatorname{Tr} \cos t\sqrt{-\Delta_r} = \begin{cases} \frac{t}{2}\int_{\partial\Omega_r} \nabla_1^{\perp}\nabla_2^{\perp} S(t,q,q)\dot{\rho}\, dq & \text{Dirichlet BC} \\ \frac{-t}{2}\int_{\partial\Omega_r} (\nabla_1^T\nabla_2^T + \Delta_2) S(t,q,q)\dot{\rho}\, dq & \text{Neumann BC,} \end{cases} \tag{6.5}$$

where $\nabla_{1,2}^{\perp}$ are the unit orthogonal gradients on $\partial\Omega_r$ and $\dot{\rho}$ is the normal component of the variational vector field ∂_r, which will be computed in Section 6.3. Note that the integrals in (6.5) are over the diagonal of the boundary. We now discuss how to calculate these integrals more precisely.

6.2 Wave propagators near the boundary

As we are localizing the wave trace near lengths of geodesic loops with j reflections, it turns out we only need to insert a select few of the operators from Theorem 5.4.3 into the formula (6.5).

Lemma 6.2.1. *Modulo Maslov factors and lower order distributions, the radially differentiated even wave trace localized near $[t_j, T_j]$ is given by*

$$\sum_{\pm}\int_{\partial\Omega}\frac{t}{2}\nabla_1^{\perp}\nabla_2^{\perp}(S^1_{j-1,\pm} + S^5_{j-1,\pm} + S^2_{j,\pm} + S^3_{j,\pm} \\ + S^6_{j,\pm} + S^7_{j,\pm} + S^4_{j+1,\pm} + S^8_{j+1,\pm})(t,q,q)\dot{\rho}\, dq$$

in the Dirichlet case and

$$\sum_{\pm}\int_{\partial\Omega}\frac{t}{2}(-\nabla_1^T\nabla_2^T - \Delta_2)(S^1_{j-1,\pm} + S^5_{j-1,\pm} + S^2_{j,\pm} + S^3_{j,\pm} \\ + S^6_{j,\pm} + S^7_{j,\pm} + S^4_{j+1,\pm} + S^8_{j+1,\pm})(t,q,q)\dot{\rho}\, dq$$

in the Neumann case.

For completeness, we repeat the proof derived in [69] as it contains substantial geometric insight.

Proof. For the localized wave trace, we only need to consider orbits which contribute to the singularities in $[t_j, T_j]$. Recall that for positive time, Theorem 3.3.1 gives 8 orbits connecting x to y in j reflections and approximately one rotation. These orbits coalesce into geodesic loops as $(x, y) \to \Delta\partial\Omega$. However, as the orbits coalesce within various configurations, not all of the limiting orbits will have j reflections. As Ω satisfies the noncoincidence condition (2.5), only the limiting geodesic loops of rotation number $1/j$ will contribute to the wave trace near $[t_j, T_j]$. See Figure 3.2 for visualizing the geometric arguments which follow. As $(x, y) \to \Delta\partial\Omega$, the two orbits of j reflections in TT configuration ($k = 1, 5$) converge geometrically to a loop of $j + 1$ reflections. The additional vertex appears at the boundary point where x and y coalesce. Similarly, the two NN orbits of j reflections ($k = 4, 8$) can be seen to converge to loops of $j - 1$ reflections. In this case, the first and last moments of reflection at the boundary converge to a single impact point. The four orbits of j reflections in TN ($k = 2, 6$) and NT ($k = 3, 7$) configurations preserve exactly j reflections in the limit. Hence, when $x, y \in \text{int}\Omega$ converge to $\Delta\partial\Omega$, only 4 of the 8 orbits contribute to geodesic loops of j reflections. However, in the same limit, two additional TT orbits of $j - 1$ reflections converge to a loop of $j - 1 + 1 = j$ reflections. Similarly, two NN orbits of $j + 1$ reflections converge to a loop of $j + 1 - 1 = j$ reflections. Any other orbit from x to y with strictly less than $j - 1$ or strictly more than $j + 1$ reflections at the boundary cannot converge to a loop of j reflections. As we have localized the wave trace near the isolated set of lengths $[t_j, T_j]$ and Ω satisfies the noncoincidence condition (2.5), only the $4 + 2 + 2 = 8$ orbits which converge geometrically to a loop of exactly j reflections will contribute to singularities here. All additional orbits contribute smooth errors to the wave trace in a small neighborhood of $[t_j, T_j]$.

It should also be clarified that although the parametrices $S_j(t,x,y)$ are constructed in the interior, we can in fact extend them continuously to the diagonal of the boundary and this extension coincides with that of the true propagator $S(t,x,x)$ ($x \in \partial\Omega$) modulo lower order terms. Both propagators agree up to lower order Lagrangian distributions in the interior, microlocally near the canonical relations $\Gamma_{\pm}^{j}$. The explicit oscillatory integral representation for each $S_j(t,x,y)$ in fact shows that they extend continuously up to the boundary and its diagonal, since the functions $\Psi_j^k(x,y)$ do. The true wave kernel $S(t,x,y)$ also extends continuously up to the boundary as a family of distributions. To see this, note that

$$S(t,x,y) = \sum_j \frac{\sin(t\lambda_j)}{\lambda_j}\psi_j(x)\overline{\psi_j(y)},$$

where $(\psi_j)_{j=1}^{\infty}$ is an L^2 orthonormal basis of Dirichlet or Neumann eignenfunctions corresponding to eigenvalues $(\lambda_j^2)_{j=1}^{\infty}$. Multiplying by a test function $\varphi(t)$ and integrating by parts $4k$ times, we see that

$$\int_{-\infty}^{\infty} S(t,x,y)\varphi(t)dt = \int_{-\infty}^{\infty}\sum_j \frac{\sin(t\lambda_j)}{\lambda_j^{4k+1}}\psi_j(x)\overline{\psi_j(y)}\partial_t^{4k}\varphi(t)dt. \tag{6.6}$$

Combining Weyl's law on the asymptotic growth of λ_j (see [71], [32]) and Hörmander's L^∞ bounds for eigenfunctions (see [27]), we see that the integrand in (6.6) can be made absolutely convergent for k sufficiently large. An application of the dominated convergence theorem then shows that (6.6) is actually smooth in x, y, so $S(t,x,y)$ has a smooth extension to the diagonal of the boundary as a distribution in t. In particular, both distributions agree up to lower order terms microlocally near the fibers of $\Gamma_{\pm}^{j}$ lying over diagonal of the boundary, as was required for the variational trace formula. □

Definition 6.2.2. As shown in the proof of Lemma 6.2.1 above, for each j, there exist 8 limiting trajectories which converge geometrically to geodesic loops of exactly j reflections.

We denote the set of these trajectories by $\mathcal{G}_j(x, y)$ and say that $\gamma_{m,k} \in \mathcal{G}_j$ if $\gamma_{m,k}$ makes $m = j-1, j$ or $j+1$ reflections at the boundary and corresponds to the length functional Ψ_m^k. The length functionals Ψ_j^2, Ψ_j^3, Ψ_j^6, Ψ_j^7, Ψ_{j+1}^4, Ψ_{j+1}^8, Ψ_{j-1}^1 and Ψ_{j-1}^5 corresponding to orbits in $\mathcal{G}_j$ coincide for $x = y \in \partial\Omega$ on the diagonal. We denote their common value by $\Psi_j(x, x)$, which is the j-loop function..

As we obtained a rather explicit formula for $S_j(t, x, y)$ in Theorem 5.4.3, it now remains to differentiate the kernels $S_m^k(t, x, y)$ and substitue them into Lemma 6.2.1. Using our oscillatory integral representation for $S_m^k(t, x, y)$ in Theorem 5.4.3, we find that microlocally near $\Gamma_\pm^{m,k}$ and $t \in [t_j, T_j]$, modulo lower order terms in an open neighborhood of the diagonal of the boundary, we have

$$\nabla_1^\perp \nabla_2^\perp S_{m,\pm}^k = (-1)^{m+1} \sum_\pm \int_0^\infty e^{\pm i\tau(t-\Psi_m^k)} \frac{\pm(\nabla_1^\perp \Psi_m^k)(\nabla_2^\perp \Psi_m^k)|\tau|^{3/2}}{2i} |A_m^k|^{1/2} d\tau \tag{6.7}$$

for Dirichlet boundary conditions. The formula for Neumann boundary conditions is less succinct and from here on, we only study Dirichlet boundary conditions. We have only written the terms coming from $\nabla_{1,2}^\perp$ falling on the exponential in equation (6.7), as all other terms don't contribute positive powers of τ and can be regarded as lower order in the singularity expansion. The operators $\nabla_{1,2}^\perp$ in the integrand of (6.7) are conformal multiples of the vector fields $\frac{\partial}{\partial\mu}$ and $\frac{\partial}{\partial\nu}$ coming from boundary normal coordinates.

As equation (6.5) tells us that the variation of the wave trace is given by integrating the differentiated sine kernels over the diagonal of the boundary, we want to understand the restriction of (6.7) to the boundary. We already noted that the j-loop function is well defined for j sufficiently large. The differentiated kernels in equation (6.7) also have factors of A_j^k and $\nabla_{1,2}^\perp \Psi_j^k$ in the integrand. We now discuss how to extend these derivatives of Ψ_j^k to the diagonal of the boundary in a similar manner. In the proof of Lemma 5.3.1 (cf. [69]), the x

and y gradients of the functions Ψ_j^k are shown to be:

$$d_x \Psi_j^k = \frac{x - q_1^k}{|x - q_1^k|}, \quad d_y \Psi_j^k = \frac{y - q_j^k}{|y - q_j^k|}. \tag{6.8}$$

Geometrically, these are the incident and reflected outgoing unit directions of the corresponding billiard trajectories at x and y. The expression $\nabla_1^\perp \Psi_j^k$ in (6.7) can easily be seen to be $\pm \sin \omega_{j,1}^k$, where $\omega_{j,1}^k$ is the angle made between the initial link of the billiard trajectory and the oriented tangent line to the level set of the distance function on which x lies (as in the foliation used in the proof of Theorem 3.3.1). We use the positively oriented tangent line for $1 \leq k \leq 4$ and the negatively oriented tangent line for $5 \leq k \leq 8$. The sign $\pm$ depends on the TT, TN, NT or NN configuration of the corresponding orbit. Similarly, $\nabla_2^\perp \Psi_j^k = \pm \sin \omega_{j,2}^k$, where $\omega_{j,2}^k$ is the angle made between the final link of the billiard trajectory and the positively ($1 \leq k \leq 4$) or negatively ($5 \leq k \leq 8$) oriented tangent line to the distance curve on which y lies. As $x, y \to \Delta \partial \Omega$, the absolute value of the angles associated to trajectories in the $\mathcal{G}_j$ converge to the initial and final angles of reflection of the unique limiting geodesic loop. We are careful to point out that only the absolute values of the angles converge, since for example, the final angles of reflection at y associated to orbits in TN and NN configurations actually converge to the *negative* of the final angle in the limiting trajectory. This phenomenon will be important for avoiding cancellation in the equation (6.19) of Section 6.3.1, where we add together all wave kernels and integrate them over the diagonal to obtain a trace formula. All limiting loops are automatically in TT configuration.

Lemma 6.2.3. *On the diagonal of the boundary, the factors $A_{j-1}^1, A_{j-1}^5 A_j^2, A_j^3, A_j^6, A_j^7, A_{j+1}^4$ and A_{j+1}^8 corresponding to orbits in $\mathcal{G}_j$ coincide up to a sign. We denote their common (absolute) value by $|A_j|$, which at a point $(0, \varphi, 0, \varphi)$ in boundary normal coordinates satisfies*

$$|A_j(0, \varphi, 0, \varphi)| = \frac{1}{\sin \omega_{j,2}} \left| \frac{\partial \omega_{j,1}}{\partial \theta} \right|.$$

Here, $\omega_{j,1}$ and $\omega_{j,2}$ are the initial and final angles of incidence respectively made by the unique *counterclockwise parametrized geodesic loop of rotation number $1/j$ based at $(0, \varphi, 0, \varphi) \in \Delta\partial\Omega$.*

Proof. Let (m, k) denote an admissible pair of indices corresponding to an orbit $\gamma_{m,k} \in \mathcal{G}_j$. Recall the notation in the proof of Lemma 5.3.1 (cf. [69]), where we described a billiard trajectory by the point $(x, q, y) \in \Omega \times \partial\Omega^m \times \Omega$. Let us first assume $1 \leq k \leq 4$. If $x \in \Omega$, we denote the positive angle between $q_1 - x$ and the positively oriented tangent line to the leaf of the foliation by distance curves on which x lies by $\omega_{j,1}^k$ (cf. Section 3.4). Similarly, if $y \in \Omega$, let us also denote the positive angle between $y - q_j$ and the positively oriented tangent line to the distance curve on which y lies by $\omega_{j,2}^k$. Note that $\omega_{j,1}^k$ and $\omega_{j,2}^k$ depend on x, y, j and k. Using the warped product structure of boundary normal coordinates $(\mu, \varphi, \nu, \theta)$, we have

$$\begin{aligned} \nabla_x^T &= (f)^{-1/2}\frac{\partial}{\partial\varphi}, & \nabla_y^T &= (f)^{-1/2}\frac{\partial}{\partial\theta}, \\ \nabla_x^\perp &= \frac{\partial}{\partial\mu}, & \nabla_y^\perp &= \frac{\partial}{\partial\nu}. \end{aligned} \tag{6.9}$$

Equation (6.8), which was derived from the proof of Lemma 5.3.1 in [69], then shows that

$$\nabla_x^\perp \Psi_j^k(x,y) = \begin{cases} -\sin\omega_{j,1}^k, & k = 1,2 \\ \sin\omega_{j,1}^k, & k = 3,4, \end{cases} \qquad \nabla_y^\perp \Psi_j^k(x,y) = \begin{cases} -\sin\omega_{j,2}^k, & k = 1,3 \\ \sin\omega_{j,2}^k, & k = 2,4, \end{cases} \tag{6.10}$$

and

$$\nabla_x^T(x,y) = -\cos\omega_{j,1}^k, \quad (1 \leq k \leq 4) \qquad \nabla_y^T = \cos\omega_{j,2}^k, \quad (1 \leq k \leq 4). \tag{6.11}$$

Combining (6.9), (6.10), and (6.11), we obtain

$$\nabla_{(\mu,\varphi)}\Psi_j^k(\mu,\varphi,\nu,\theta) = \begin{cases} (-\sin\omega_{j,1}^k, -f^{1/2}(\mu,\varphi)\cos\omega_{j,1}^k), & k=1 \\ (-\sin\omega_{j,1}^k, -f^{1/2}(\mu,\varphi)\cos\omega_{j,1}^k), & k=2 \\ (\sin\omega_{j,1}^k, -f^{1/2}(\mu,\varphi)\cos\omega_{j,1}^k), & k=3 \\ (\sin\omega_{j,1}^k, -f^{1/2}(\mu,\varphi)\cos\omega_{j,1}^k), & k=4 \end{cases}$$
$$\nabla_{(\nu,\theta)}\Psi_j^k(\mu,\varphi,\nu,\theta) = \begin{cases} (-\sin\omega_{j,2}^k, f^{1/2}(\nu,\theta)\cos\omega_{j,2}^k), & k=1 \\ (\sin\omega_{j,2}^k, f^{1/2}(\nu,\theta)\cos\omega_{j,2}^k), & k=2 \\ (-\sin\omega_{j,2}^k, f^{1/2}(\nu,\theta)\cos\omega_{j,2}^k), & k=3 \\ (\sin\omega_{j,2}^k, f^{1/2}(\nu,\theta)\cos\omega_{j,2}^k), & k=4. \end{cases} \tag{6.12}$$

Using (6.12) to calculate the (μ,φ,ν,θ) Hessian of Ψ_j^k, we find

$$\frac{\partial^2\Psi_j^k}{\partial\theta\partial\varphi}(\mu,\varphi,\nu,\theta) = f^{1/2}(\mu,\varphi)\sin\omega_{j,1}^k\frac{\partial\omega_{j,1}^k}{\partial\theta}, \qquad (1\le k\le 4)$$
$$\frac{\partial^2\Psi_j^k}{\partial\nu\partial\varphi}(\mu,\varphi,\nu,\theta) = f^{1/2}(\mu,\varphi)\sin\omega_{j,1}^k\frac{\partial\omega_{j,1}^k}{\partial\nu}, \qquad (1\le k\le 4)$$
$$\frac{\partial^2\Psi_j^k}{\partial\theta\partial\mu}(\mu,\varphi,\nu,\theta) = \begin{cases} -\cos\omega_{j,1}^k\frac{\partial\omega_{j,1}^k}{\partial\theta}, & k=1,2 \\ \cos\omega_{j,1}^k\frac{\partial\omega_{j,1}^k}{\partial\theta}, & k=3,4 \end{cases} \tag{6.13}$$
$$\frac{\partial^2\Psi_j^k}{\partial\nu\partial\mu}(\mu,\varphi,\nu,\theta) = \begin{cases} -\cos\omega_{j,1}^k\frac{\partial\omega_{j,1}^k}{\partial\nu}, & k=1,2 \\ \cos\omega_{j,1}^k\frac{\partial\omega_{j,1}^k}{\partial\nu}, & k=3,4. \end{cases}$$

Inserting (6.13) into the expression (5.15) for A_j^k in all possible configurations $(1\le k\le 4)$, we find that on the boundary,

$$A_m^k(0,\varphi,0,\theta) = \begin{cases} -\cos\omega_{j,2}^k \frac{\partial\omega_{j,1}^k}{\partial\nu} - \sin\omega_{j,2}^k \frac{\partial\omega_{j,1}^k}{\partial\theta}, & k=1 \\ -\cos\omega_{j,2}^k \frac{\partial\omega_{j,1}^k}{\partial\nu} + \sin\omega_{j,2}^k \frac{\partial\omega_{j,1}^k}{\partial\theta}, & k=2 \\ +\cos\omega_{j,2}^k \frac{\partial\omega_{j,1}^k}{\partial\nu} + \sin\omega_{j,2}^k \frac{\partial\omega_{j,1}^k}{\partial\theta}, & k=3 \\ +\cos\omega_{j,2}^k \frac{\partial\omega_{j,1}^k}{\partial\nu} - \sin\omega_{j,2}^k \frac{\partial\omega_{j,1}^k}{\partial\theta}, & k=4. \end{cases} \tag{6.14}$$

Before evaluating this expression on the diagonal of the boundary, we differentiate $\omega_{j,1}^k$ in the direction $L = (y-q)/|y-q|$ of the last link to see that

$$\begin{aligned} 0 &= \nabla_L \omega_{j,1}^k \\ &= \begin{cases} \cos\omega_{j,2}^k \nabla_y^T \omega_{j,1}^k - \sin\omega_{j,2}^k \nabla_y^\perp \omega_{j,1}^k, & k=1,3 \\ \cos\omega_{j,2}^k \nabla_y^T \omega_{j,1}^k + \sin\omega_{j,2}^k \nabla_y^\perp \omega_{j,1}^k, & k=2,4 \end{cases} \\ &= \begin{cases} \frac{1}{f^{1/2}} \cos\omega_{j,2}^k \frac{\partial\omega_{j,1}^k}{\partial\theta} - \sin\omega_{j,2}^k \frac{\partial\omega_{j,1}^k}{\partial\nu}, & k=1,3 \\ \frac{1}{f^{1/2}} \cos\omega_{j,2}^k \frac{\partial\omega_{j,1}^k}{\partial\theta} + \sin\omega_{j,2}^k \frac{\partial\omega_{j,1}^k}{\partial\nu}, & k=2,4. \end{cases} \end{aligned}$$

This implies that

$$\sqrt{f(\nu,\theta)} \frac{\partial\omega_{j,1}^k}{\partial\nu} = \begin{cases} +\cot\omega_{j,2}^k \frac{\partial\omega_{j,1}^k}{\partial\theta}, & k=1,3 \\ -\cot\omega_{j,2}^k \frac{\partial\omega_{j,1}^k}{\partial\theta}, & k=2,4. \end{cases} \tag{6.15}$$

Inserting the formula (6.15) into (6.14) and evaluating on $\partial\Omega$, we find that

$$|A_m^k(0,\varphi,0,\theta)| = \left(\frac{\cos^2\omega_{j,2}^k}{\sin\omega_{j,2}^k} + \frac{\sin^2\omega_{j,2}^k}{\sin\omega_{j,2}^k} \right) \left| \frac{\partial\omega_{j,1}^k}{\partial\theta} \right| = \frac{1}{\sin\omega_{j,2}^k} \left| \frac{\partial\omega_{j,1}^k}{\partial\theta} \right|.$$

Note that for $1 \le k \le 4$ and x, y on the boundary, both $\omega_{j,1}^k$ and $\omega_{j,2}^k$ are independent of k. They coincide with the initial and final angles of the unique orbit connecting x to y in

j reflections and approximately one counterclockwise rotation. As the orbits corresponding to $5 \leq k \leq 8$ can be viewed as counterclockwise orbits in the reflected domain, we instead defined $\omega_{j,1}^k$ and $\omega_{j,2}^k$ to be the angles made between the initial and final links and the *negatively* oriented tangent lines to the distance curves on which x and y respectively lie. With this convention, we see by symmetry that the roles of $\omega_{j,1}^k$ and $\omega_{j,2}^k$ are interchanged:

$$\nabla_{(\mu,\varphi)}\Psi_j^k(\mu,\varphi,\nu,\theta) = \begin{cases} (-\sin\omega_{j,1}^k, f^{1/2}(\mu,\varphi)\cos\omega_{j,1}^k), & k=5 \\ (-\sin\omega_{j,1}^k, f^{1/2}(\mu,\varphi)\cos\omega_{j,1}^k), & k=6 \\ (\sin\omega_{j,1}^k, f^{1/2}(\mu,\varphi)\cos\omega_{j,1}^k), & k=7 \\ (\sin\omega_{j,1}^k, f^{1/2}(\mu,\varphi)\cos\omega_{j,1}^k), & k=8 \end{cases}$$

$$\nabla_{(\nu,\theta)}\Psi_j^k(\mu,\varphi,\nu,\theta) = \begin{cases} (-\sin\omega_{j,2}^k, -f^{1/2}(\nu,\theta)\cos\omega_{j,2}^k), & k=5 \\ (\sin\omega_{j,2}^k, -f^{1/2}(\nu,\theta)\cos\omega_{j,2}^k), & k=6 \\ (-\sin\omega_{j,2}^k, -f^{1/2}(\nu,\theta)\cos\omega_{j,2}^k), & k=7 \\ (\sin\omega_{j,2}^k, -f^{1/2}(\nu,\theta)\cos\omega_{j,2}^k), & k=8. \end{cases} \tag{6.16}$$

Replacing (6.12) by (6.16), parallel computations to those above then show that for $5 \leq k \leq 8$,

$$|A_m^k(0,\varphi,0,\varphi)| = \frac{1}{\sin\omega_{j,2}^k}\left|\frac{\partial\omega_{j,1}}{\partial\theta}\right|.$$

Note that for $5 \leq k \leq 8$, $\omega_{j,1}^{k-4} - \omega_{j,2}^k \to 0$ as $x, y \to \partial\Omega$. Similarly, $\omega_{j,2}^{k-4} - \omega_{j,1}^k \to 0$ as $x, y \to \partial\Omega$. Hence,

$$|A_m^k(0,\varphi,0,\varphi)| = \frac{1}{\sin\omega_{j,1}}\left|\frac{\partial\omega_{j,2}}{\partial\varphi}\right|, \tag{6.17}$$

for $5 \leq k \leq 8$. Note that the righthand side of (6.17) involves angles at boundary points and does not depend on k. Also observe that

$$\begin{aligned}\frac{\partial^2 \Psi_j^k}{\partial\varphi\partial\theta} &= \pm \sin\omega_{j,2}^k \frac{\partial \omega_{j,2}^k}{\partial\varphi} \\ \frac{\partial^2 \Psi_j^k}{\partial\theta\partial\varphi} &= \pm \sin\omega_{j,1}^k \frac{\partial \omega_{j,1}^k}{\partial\theta}.\end{aligned} \tag{6.18}$$

Setting the two equations in (6.18) equal implies that

$$\frac{1}{\sin\omega_{j,1}}\left|\frac{\partial\omega_{j,2}}{\partial\varphi}\right| = \frac{1}{\sin\omega_{j,2}}\left|\frac{\partial\omega_{j,2}}{\partial\varphi}\right|,$$

which combined with (6.17), concludes the proof of the lemma. □

Remark 6.2.4. In [69], complete integrability of the ellipse actually implies that $\omega_{j,1} = \omega_{j,2}$ for elliptical billiards i.e. every geodesic loop is in fact a periodic orbit. In that case, the angular derivative $\partial\omega/\partial\theta$ appearing in Lemma 6.2.3 was explicitly calculated using action angle coordinates and Jacobi elliptic functions (see Chapter 7 below). In general, it may be difficult to compute $\partial\omega_1/\partial\theta$ explicitly despite its relatively simple geometric interpretation.

6.3 Principal symbol computation

In this section, we refine the integral formulas obtained in Section 6 and use them to compute the principal symbol of the wave trace. We also compute Maslov factors on each branch of the wave propagator and provide estimates on the subprincipal terms of the parametrix described in Theorem 5.4.3. These estimates ensure that radially integrating subprincipal terms in the variational wave trace does not contribute to the leading order asymptotics in equation (6.4). Combining these estimates with formulas in Section 6 then allows us to complete the proofs of Theorem 2.1.3, Theorem 2.1.5 and Corollary 2.1.6.

We begin by evaluating the quantity $\dot{\rho}$, coming from the variational formula (6.5). Let

$X(s) = (x(s), y(s))$ be an arclength parametrization of $\partial\Omega$. Then, $rX(s) = (rx(s), ry(s))$ parametrizes $\partial\Omega_r$ and coincides with a multiple of the variational vector field $\partial/\partial r$. The unit normal to $\partial\Omega_r$ at (rx, ry) is given by $N(s) = (-y'(s), x'(s))$, which is independent of r. Hence, $\dot{\rho}(s) = X \cdot N$. Plugging this into (6.5) together with (6.7), we see that locally near $[t_j, T_j]$, modulo Maslov factors and lower order terms, $\partial_r \mathrm{Tr} \cos t\sqrt{-\Delta_{\Omega_r}}$ is given by

$$\sum_{\gamma_{m,k} \in \mathcal{G}_j, \pm} \frac{\pm t(-1)^{m+1}}{4i} \int_{\partial\Omega_r} \int_0^\infty \Delta^*(e^{\pm i\tau(t - \Psi_m^k)} |\tau|^{3/2} (\nabla_1^\perp \Psi_m^k)(\nabla_2^\perp \Psi_m^k) |A_m^k|^{1/2}) X \cdot N d\tau ds_r$$

(for Dirichlet boundary conditions). Recall that here, Δ^* is pullback under the diagonal embedding $\Delta : \partial\Omega \to \partial\Omega \times \partial\Omega$ so that all terms depending on both x and y in integrand above are evaluated at the diagonal of the boundary $\{x = y = s \in \partial\Omega\}$. Note that the functions Ψ_m^k, A_m^k as well as the operators $\nabla_i^\perp$ and the measure ds_r all depend on r. It is easily verified that Ψ_m^k and ds_r are homogeneous of degree one in r, $\nabla_{1,2}^\perp \Psi_m^k$ is homogeneous of degree 0, and A_m^k is homogeneous of degree -1. In particular, the orthogonal gradients restricted to the diagonal of the boundary are still given by

$$\nabla_i^\perp \Psi_j^k(s, s) = \begin{cases} \pm \sin \omega_{j,1} & i = 1 \\ \pm \sin \omega_{j,2} & i = 2, \end{cases}$$

where $\omega_{j,1}$ and $\omega_{j,2}$ are the angles made with the initial and final links respectively of $\gamma_j(s)$ and the positively oriented tangent line $T_s\partial\Omega$. These quantities are clearly invariant under dilation, which is in fact a conformal transformation and the $\pm$ signs only depend on whether the orbits emanating from s originated from orbits in the TT, TN, NT, or NN configuration.

6.3.1 Integrating the principal symbol

Recall that χ_1 was a smooth cutoff identically equal to 1 near $[t_j, T_j]$ as discussed in Section 5.3. Adding up the various contributions from orbits in $\mathcal{G}_j$, we find that locally near $[t_j, T_j]$, modulo Maslov factors and lower order terms, we have

$$\int_{0^+}^{1} \chi_1(t/r)\partial_r \mathrm{Tr} \cos t\sqrt{-\Delta_r} dr =$$
$$\sum_{\pm} \frac{\pm 2t(-1)^j}{i} \int_{0^+}^{1} \int_{0}^{|\partial\Omega_1|} \int_{0}^{\infty} \Delta^*(e^{\pm i\tau(t-r\Psi_j)}|\tau|^{3/2} r^{1/2} \sin\omega_1 \sin\omega_2 |A_j|^{1/2} X \cdot N) d\tau ds_1 dr. \tag{6.19}$$

In order to apply the formal manipulation suggested by equation (6.4) in Section 6, we need to understand how intertwining localization, integration and radial differentiation in (6.19) combine to produce the desired trace formula. In equation (6.19), we have also assumed that integrating the principal symbol of the radially differentiated wave trace localized near $[rt_j, rT_j]$ in fact gives us the principal symbol of the honest wave trace near $[t_j, T_j]$. This relies on a structural characterization of the subprincipal terms in our parametrix for S_j (see Lemma 6.4.1 below). The latter is carried out in Lemma 6.3.1 of Section 6.4 and we first address the localization problem.

Lemma 6.3.1. *To obtain the principal symbol of the wave trace localized near lengths in $[t_j, T_j]$ of a given periodic orbit, it suffices to integrate the variational formula localized near $[rt_j, rT_j]$ from r_0 to 1 rather than 0^+ to 1, where $r_0 = r_0(j)$ is chosen sufficiently small in terms of j. The remainder*

$$\int_0^{r_0} \partial_r(\chi_1(t/r)\, Tr \cos t\sqrt{-\Delta_r}) dr \tag{6.20}$$

is in fact smooth for t near $[t_j, T_j]$.

Proof. The localized wave trace near $[t_j, T_j]$ is given by

$$\chi_1(t)\mathrm{Tr}\cos t\sqrt{\Delta_1}. \tag{6.21}$$

All geodesic loops of rotation number $1/j$ have length in the interval $[t_j, T_j]$, where $t_j = \inf_{\omega(\gamma)=1/j} \mathrm{length}(\gamma)$ and $T_j = \sup_{\omega(\gamma)=1/j} \mathrm{length}(\gamma)$ (recall that ω is the function which outputs the rotation number of a given periodic orbit). The work in [44] shows that there exist a $j_0 \in \mathbb{N}$ such that for $j, k > j_0$, the intervals $[t_j, T_j]$ and $[t_k, T_k]$ are disjoint whenever $j \neq k$. Moreover, we also have the asymptotic $T_j - t_j = O(j^{-\infty})$. As the Euclidean Laplacian scales by r^{-2} under the dilation $x \to rx$, the wave trace also scales by

$$\mathrm{Tr}\cos t\sqrt{-\Delta_{\Omega_r}} = \sum_j \cos t\lambda_j/r, \tag{6.22}$$

where $(\lambda_j^2)_j$ are the Dirichlet eigenvalues of Δ_{Ω_1}. It is clear that the rescaled trace (6.22) is a smooth family of distributions in the weak topology. Hence, for $0 < r_0 < 1$ to be chosen sufficiently small later, we have

$$\int_{r_0}^{1} \partial_r\left(\chi_1(t/r)\mathrm{Tr}\cos t\sqrt{-\Delta_r}\right)dr = \chi_1(t)\mathrm{Tr}\cos t\sqrt{-\Delta_1} - \chi_1(t/r_0)\mathrm{Tr}\cos t\sqrt{-\Delta_{r_0}}. \tag{6.23}$$

Note that $[t_j, T_j] \subset [0, |\partial\Omega|)$, so if $t > 2r_0|\partial\Omega|$, then $\chi_1(t/r_0) = 0$. Hence, we choose $0 < \delta < t_j/(2|\partial\Omega|)$.

However, in contrast to (6.23), the quantity integrated in (6.19) was localized near $[rt_j, rT_j]$ only *after* taking the variation of the wave trace. We see that the integrand in equation (6.23) splits into

$$\partial_r\left(\mathrm{Tr}\cos t\sqrt{-\Delta_r}\chi_j(t/r)\right) = \chi_j(t/r)\partial_r\mathrm{Tr}\cos t\sqrt{-\Delta_r} - r^{-2}\chi_j'(t/r)\mathrm{Tr}\cos t\sqrt{-\Delta_r}. \tag{6.24}$$

As χ_j was identically equal to 1 on $[t_j, T_j]$, we see that $\chi_j'(\cdot/r)$ vanishes on $[rt_j, rT_j]$. Hence, $\mathrm{Supp}\chi_j'(\cdot/r) \cap \mathrm{SingSupp}\ \mathrm{Tr}\cos\cdot\sqrt{-\Delta_r} = \emptyset$ and integrating the second term on the righthand side of (6.24) in r contributes only smooth errors in t to the wave trace near $[t_j, T_j]$. □

Remark 6.3.2. For any Schwartz function φ on $\mathbb{R}$, we have

$$\mathrm{Tr}\cos t\sqrt{-\Delta_\varepsilon}(\varphi) = \sum \int_{\mathbb{R}} \varphi(t)\cos t\lambda_j/r. \tag{6.25}$$

Integrating by parts $4k$ times in the formula (6.25), we have

$$\mathrm{Tr}\cos t\sqrt{-\Delta_\varepsilon}(\varphi) = \sum_j \left(\frac{r}{\lambda_j}\right)^{4k} \int_{\mathbb{R}} \varphi^{(4k)}(t)\cos t\lambda_j/r. \tag{6.26}$$

For k sufficiently large, (6.26) can be seen to be absolutely convergent by Weyl's law and an application of the dominated convergence theorem then shows that

$$\lim_{\varepsilon\to 0^+} \mathrm{Tr}\cos t\sqrt{-\Delta_\varepsilon} = 0$$

in the sense of distributions. In the Neumann case, $\lambda_0 = 0$, which implies by the same logic that

$$\lim_{\varepsilon\to 0^+} \mathrm{Tr}\cos t\sqrt{-\Delta_\varepsilon} = 1.$$

This only contributes a smooth error to the trace formula in (2.1.3). However, the proof of Lemma 6.3.1 above actually resolves the issues near $r = 0$ at the expense of smooth errors. Nonetheless, it seemed conceptually more clear to motivate the variational approach by anticipating that the highly oscillatory terms near $r = 0$ would vanish in the limit.

6.3.2 Proof of Theorem 2.1.3

We now make the change of variables $\xi = r\tau$, which turns equation (6.19) into

$$\sum_{\pm} \frac{(-1)^j 2t}{\pm i} \int_{r_0}^{1} \int_0^{|\partial\Omega_1|} \int_0^{\infty} \Delta^*(e^{\pm i\xi t/r} e^{-i\xi\Psi_j} |\xi|^{3/2} \sin\omega_{j,1} \sin\omega_{j,2} |A_m^k|^{1/2} X \cdot N) d\xi ds_1 \frac{dr}{r^2}. \tag{6.27}$$

Separating out the oscillatory dr integral, we see that

$$\int_{r_0}^{1} e^{\pm i\xi t/r} r^{-2} dr = \frac{e^{\pm i\xi t} - e^{\pm i\xi t/r_0}}{\pm i\xi t}.$$

This simplifies (6.27) to

$$\begin{aligned} \sum_{\pm} \pm 2(-1)^{j+1} \int_0^{|\partial\Omega_1|} \int_0^{\infty} \Delta^*(e^{\pm i\xi(t-\Psi_j)} |\xi|^{1/2} \sin\omega_1 \sin\omega_2 |A_j|^{1/2} X \cdot N) d\xi ds \\ \mp 2(-1)^{j+1} \int_0^{|\partial\Omega_1|} \int_0^{\infty} \Delta^*(e^{\pm i\xi(t/r_0-\Psi_j)} |\xi|^{1/2} \sin\omega_1 \sin\omega_2 |A_j|^{1/2} X \cdot N) d\xi ds. \end{aligned} \tag{6.28}$$

We now apply the nonstationary phase lemma to the second term in (6.28); the critical set is given by $\{t = r_0\Psi_j(s,s)\}$ and since all terms are localized by χ_1 near $[t_j, T_j]$, our choice of r_0 in Lemma 6.3.1 shows that $r_0\Psi_j(s,s) \notin [t_j, T_j]$ for all $s \in \partial\Omega$. Hence, this second term is actually smooth. Equation (6.28) is exactly the quantity appearing in Theorem 2.1.3 without Maslov factors. In Section 6.5, it is explained why the real part of the integral is taken in Theorem 2.1.3, ultimately owing to conjugacy of the Maslov factors on the two branches of Γ^j. In Section 6.4 below, it is verified that radially integrating the principal symbol of the variational wave trace indeed gives the principal symbol of the honest wave trace (cf. remarks preceding Lemma 6.3.1).

6.4 Lower order terms

As mentioned in the remarks following equation (6.19), we have to justify that integrating the principal symbol of $\partial_r \mathrm{Tr}\cos t\sqrt{-\Delta_{\Omega_r}}$ in the variable r gives the principal symbol of $\mathrm{Tr}\cos t\sqrt{-\Delta_{\Omega_1}}$. The parametrix

$$S(t) = \sum_j S_j(t)$$

described in [6] approximates the wave kernel modulo C^∞ and Theorem 5.4.3 gives us an oscillatory integral expansion for each S_j in terms of the operators S_m^k modulo lower order terms. In addition to the principal symbol computed in Lemma 5.3.2, S_m^k in fact has a complete parametrix on each dilated domain Ω_r:

$$S_{m,\pm}^k(t,x,y;r) = \int_0^{+\infty} e^{\pm i\tau(t-\Psi_m^k(x,y;r))} a_{m,k,\pm}(t,\tau,x,y;r)d\tau, \tag{6.29}$$

where $a_{m,k,\pm} \in S_{\mathrm{cl}}^{1/2}(\mathbb{R}\times\mathbb{R}\Omega\times\Omega)$ is a classical elliptic symbol of order 1/2, implicitly depending on r. The full symbol has an asymptotic expansion in negative half integer powers of τ:

$$a_{m,k,\pm}(t,\tau,x,y;r) \sim \sum_{p=0}^{\infty} a_{m,k,p,\pm}(t,x,y;r)|\tau|^{1/2-p}. \tag{6.30}$$

We already know that $\Psi_j^k(x,y;r) = r\Psi_j^k(x/r,y/r;1)$ is homogeneous of degree one in r. The potential problem is that integrating lower order terms $a_{m,k,p,\pm}$ in the radial variable r, for $p \geq 1$ could actually produce large powers of τ which contribute to the principal symbol of $\mathrm{Tr}\cos t\sqrt{-\Delta_1}$. We now show that this is not the case, by taking advantage of certain homogeneities for the dilated kernels and symmetries in the subprincipal terms on the critical set.

Lemma 6.4.1. *The full symbol* (6.30) *can be made independent of t for x, y in the open region where the parametrix 5.4.3 is valid. Moreover, each term in the asymptotic expansion* *contributes $O(|\tau|^{-p-1})$ to the full symbol of the wave trace as $\tau \to \pm\infty$.*

Proof. On the canonical relation $\Gamma_{\pm}^{m,k}$, we have $t = \Psi_m^k$. Hence, substituting $\widetilde{a}_{m,k,p,\pm}(x, y; r) := a_{m,k,p,\pm}(\Psi_j^k(x, y), x, y; r)$ in (6.30) defines the kernel of an operator which differs from $S(t)$ by a smoothing operator microlocally near $\Gamma_{\pm}^{m,k}$. Note also that the wave kernel has the following homogeneity property:

$$S(t, x, y; r) = r^{-1} S(t/r, x/r, y/r; 1) \qquad x, y \in \Omega_r. \tag{6.31}$$

This can be seen by computing

$$(\partial_t^2 - \Delta_r) r S(t/r, x/r, y/r; 1) = r^{-1}(\partial_s^2 - \Delta_1) S(s, z, w; r)\Big|_{\substack{s=t/r \\ z=x/r \\ w=y/r}} = 0,$$

for $x, y \in \Omega_r$. As both kernels in (6.31) solve the homogeneous wave equation with the same initial conditions

$$\begin{aligned} S(0, x, y; r) &= r^{-1} S(0, x/r, y/r; 1) = 0, \\ \frac{\partial S}{\partial t}(0, x, y; r) &= r^{-1} \frac{\partial}{\partial t} S(t/r, x/r, y/r; 1)\Big|_{t=0} = \delta(x - y), \end{aligned} \tag{6.32}$$

uniqueness for the wave equation then implies the asserted homogeneity. Upon equating the phase functions via a change of variables, we obtain the symbolic equation

$$a_{m,k,p,\pm}(t, rx, ry; r) = r^{1/2-p} a_{m,k,p,\pm}(t/r, x, y; 1) = r^{1/2-p} \widetilde{a}_{m,k,p,\pm}(x, y; 1). \tag{6.33}$$

The integrand appearing in the variation of the wave trace (6.5) is $\nabla_1^{\perp} \nabla_2^{\perp} S_r(t, q, q)$. We should actually replace the gradients with the correctly scaled ones $r^{-1} \nabla_i^{\perp}$, which are the differential operators corresponding to unit length vector fields for each r. We calculate that

$$\begin{aligned}&\left(r^{-2}\nabla_1^{\perp}\nabla_2^{\perp}S_r\right)(t,rs,rs)\\&=\sum_{p\geq 0,\pm}\int_0^{\infty}e^{\pm i\tau(t-r\Psi_m^k)}(\nabla_1^{\perp}\Psi_m^k)(\nabla_2^{\perp}\Psi_m^k)a_{m,k,p,\pm}(t,rs,rs;r)|\tau|^{3/2-p}d\tau.\end{aligned}\tag{6.34}$$

Changing variables $\xi = r\tau$ and using the homogeneity in equation (6.31) in addition to that of $\nabla\Psi_m^k$, we see that integrating (6.34) in r gives

$$\begin{aligned}&\int_{r_0}^1\frac{\partial}{\partial r}\operatorname{Tr}\cos t\sqrt{-\Delta_r}dr\\&=\sum_{p\geq 0,\pm}\int_0^{|\partial\Omega|}\int_{r_0}^1\int_0^{\infty}e^{\pm i\xi(t/r-\Psi_m^k)}(\nabla_1^{\perp}\Psi_m^k)(\nabla_2^{\perp}\Psi_m^k)\widetilde{a}_{m,k,p,\pm}(s,s;1)|\xi|^{3/2-p}\frac{d\xi}{r^2}drds.\end{aligned}\tag{6.35}$$

Proceeding exactly as in equations (6.27) and (6.28) from the previous section, we conclude the proof of the lemma. □

Remark 6.4.2. While Lemma 6.4.1 shows that integrating the subprincipal terms a_p in the radial variable, for $p \geq 1$, does not contribute large powers of τ, it is still possible that smooth remainders blow up near $r = 0$. However, replacing 0^+ by r_0 as in Lemma 6.3.1 resolves this issue.

6.5 Maslov factors

To explicitly compute the Maslov factors $\sigma_j^{\pm}$ on $\Gamma_{\pm}^j$, we use an argument due to Keller ([38]), following the presentation in [14]. The free wave propagator $U(t) = e^{-it\sqrt{-\Delta}}$ on $\mathbb{R}^2$ has an integral kernel given by

$$U(t,x,y) = (2\pi)^{-2}\int_{\mathbb{R}^2_\xi}e^{i(\langle x-y,\xi\rangle-|\xi|t)}d\xi|dx\wedge dy|^{1/2},\tag{6.36}$$

considered as a distributional half density (cf. Section 5.1). By changing variables and applying the method of stationary phase, it is shown in [14] that the principal symbol of $U(t)$ on $N^*\{|x-y|=t\} = \Gamma^0_\pm$ is

$$e^{-i\pi/4}\left(\frac{\tau}{t}\right)^{1/2}|d\tau \wedge dx \wedge dy|^{1/2}. \tag{6.37}$$

Hence, the Maslov factors on $\Gamma^0_\pm$ are given by $\sigma_0^\pm = \pm 1$. The arguments in [20], which are in turn based on the construction in [6], then show that after a reflection at the boundary, the Maslov factors remain unchanged. Hence, $\sigma_j^\pm = \pm 1$ for all $j \in \mathbb{Z}$. Both of the propagators corresponding to Γ^j_+ and Γ^j_- contribute to the wave trace singularity near $[t_j, T_j]$, owing to the two branches of propagation:

$$S(t) = \frac{e^{it\sqrt{-\Delta}} - e^{-it\sqrt{-\Delta}}}{2i\sqrt{-\Delta}}. \tag{6.38}$$

Hence, we multiply each $\pm$ branch of the Lagrangian distributions $S_{j,\pm}$ by $e^{\pm i\pi/4}$, which explains the real parts taken in Theorem 2.1.3, Theorem 2.1.5 and Corollary 2.1.6. Noting that the principal symbol of $\sqrt{-\Delta}$ is $|\xi|$ and $\tau = \pm|\eta| = \pm|\xi|$ on the canonical relation $\Gamma^0_\pm$, equations (6.37) and (6.38) also show that the principal symbol of $S(t)$ on Γ^0 has the correct order.

6.6 Proofs of Theorem 2.1.5 and Corollary 2.1.6

We now show how Theorem 2.1.5 and Corollary 2.1.6 follow directly from Theorem 2.1.3. Recall from Section 3 that a caustic is a smooth curve $\mathcal{C}$ in Ω such that every link tangent to $\mathcal{C}$ remains tangent to $\mathcal{C}$ after a reflection at the boundary. It is well known (see [37], [65]) that if one periodic orbit is tangent to a caustic $\mathcal{C}$, then every orbit tangent to $\mathcal{C}$ is in fact periodic, with the same period (number of bounces) and winding number. In this case, we say $\mathcal{C}$ is a rational caustic and use $\omega(\mathcal{C})$ to denote the (rational) rotation number of any orbit γ tangent to $\mathcal{C}$. Hence, rational caustics correspond to highly degenerate periodic orbits in the sense that they are not isolated and 1 is an eigenvalue of the Poincaré map. Let L_j be the length of a periodic orbit which is tangent to a caustic $\mathcal{C}_j$, making j reflections at the boundary and one rotation. As all orbits tangent to $\mathcal{C}$ are periodic orbits, each $q \in \partial\Omega$ is a critical point of the loop function, i.e. $\partial_q \Psi_j(q, q) = 0$. In this case, the length function $\Psi_j(q, q)$ appearing in the phase of $\operatorname{Tr} \cos t\sqrt{-\Delta}$ (as in Theorem 2.1.3) is the constant function L_j, i.e. every j reflection loop is in fact a periodic orbit of length L_j. Assuming the noncoincidence condition (2.5) on Ω, all periodic orbits of length L_j arise in this way. Tangency to $\mathcal{C}$ also implies that the angles ω_1 and ω_2 in the amplitude are equal. Hence, the wave trace in Theorem 2.1.3 is given by

$$\operatorname{Re}\left\{(-1)^j e^{-i\pi/4} 4 \int_{\partial\Omega} \int_0^\infty e^{i\xi(t-L_j)} |\xi|^{1/2} \sin^{3/2}\omega_1 \left|\frac{\partial \omega_1}{\partial \theta}\right|^{1/2} X \cdot N d\xi dq\right\}. \tag{6.39}$$

As the dq and $d\xi$ integrals can be separated, we obtain the Fourier transform of the homogeneous distribution

$$\chi_+^{3/2}(\xi) = \begin{cases} \xi^{3/2} & \xi \geq 0 \\ 0 & \xi < 0 \end{cases}$$

One can define χ_+^a similarly as an L^1_{loc} function for $\operatorname{Re} a > -1$ and these distributions fact be analytically continued to a larger region of $a \in \mathbb{C}$. It is shown in [31] (Chapter 7) that the Fourier transform of χ_+^a (with dual variable t) is given by $e^{-i\pi(a+1)/2}(t - i0)^{-a-1}$. The proof of Theorem 2.1.5 is concluded by evaluating the Fourier transform in equation (6.39) at the point $t - L_j$.

In the case of an ellipse, the billiard flow is known to be completely integrable and each confocal ellipse is in fact a caustic. Moreover, Poncelet's Theorem (see [59], [58]) implies that all periodic orbits tangent to a given confocal ellipse have the same length. Hence, all periodic orbits in the ellipse correspond to rational caustics, so Theorem 2.1.5 applies. Calculations from the author's previous work using action-angle coordinates and Jacobi elliptic function theory allow for the explicit computation of $\partial\omega/\partial\theta$ appearing in the integrand of (6.39) (see Section 5.5 of [69] or Section 7.3 below). As the boundary of an ellipse

$$\left\{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}$$

is easily parametrized by $(a\cos\varphi, b\sin\varphi)$ for $\varphi \in [0, 2\pi)$, the quantity $X(q) \cdot N(q)dq$ can be explicitly calculated. Combining these observations with Theorem 2.1.5, we obtain the formula appearing in Corollary 2.1.6.

Chapter 7

Robin Spectral Rigidity of the Ellipse

7.1 Modified wave propagators near the boundary of an ellipse

In this section, we use microlocal analysis to obtain a singularity expansion for the variation of the wave trace near the length spectrum. Using Theorem 4.1.7, we can rewrite

$$\delta \mathrm{Tr} \cos t\sqrt{-\Delta_\varepsilon} = \pi_* \Delta^* r_1 r_2 (L_R^b \dot{\rho} + \frac{t}{2} S_R \dot{K}), \tag{7.1}$$

where r_1, r_2 are the boundary restriction operators, $\Delta : \partial\Omega \to \partial\Omega \times \partial\Omega$ is the diagonal embedding, π_* is integration over the fibers, and the modified propagator L_R^b is given by

$$L_R^b = \frac{t}{2}(-\nabla_1^T \nabla_2^T - \Delta_2 + K_0^2 + K_0 \kappa) S_R.$$

This manipulation of notation is just to illustrate how one can decompose the wave trace variation into the composition of simpler Fourier integral operators. However, the wave kernels L_R^b and S_R in our formula are *not* FIOs near the glancing rays emanating from

$S^*\partial\Omega$. In [25], the authors microlocalize the wave kernels near periodic nonglancing orbits and calculate the principal symbol of the composition (7.1) for Dirichlet and Neumann boundary conditions using the symbol calculus in [12]. In contrast to the methods employed in [25], we instead take a more direct approach which avoids an application of the symbol calculus in [12].

Recall that according to Theorem 4.1.7, the variation of the wave trace is given by

$$\delta \mathrm{Tr} \cos t\sqrt{-\Delta_\varepsilon} = \int_{\partial\Omega} L_R^b(t,q,q)\dot{\rho} + \frac{t}{2} S_R(t,q,q)\dot{K}\, dq.$$

The highest order terms come only from the differentiated sine kernels, which implies that

$$\delta \mathrm{Tr} \cos t\sqrt{-\Delta_\varepsilon} = \int_{\partial\Omega} \frac{t}{2}((-\nabla_1^T\nabla_2^T - \Delta_2)S_R)(t,q,q)\dot{\rho}\, dq + L.O.T.$$

In fact, we can discard even more terms in the singularity expansion:

Lemma 7.1.1. *Modulo Maslov factors and distributions of lower order, the variation of the localized (even) wave trace near an isolated length L_j is given by*

$$\sum_{\pm} \int_{\partial\Omega} \frac{t}{2}((-\nabla_1^T\nabla_2^T - \Delta_2)(S^1_{j-1,\pm} + S^5_{j-1,\pm} + S^2_{j,\pm} + S^3_{j,\pm}$$
$$+ S^6_{j,\pm} + S^7_{j,\pm} + S^4_{j+1,\pm} + S^8_{j+1,\pm}))(t,q,q)\dot{\rho}\, dq$$

The proof of Lemma 7.1.1 follows exactly the same argument as that of Lemma 6.2.1. As in that case, we again group together all relevant orbits and their lengths:

Definition 7.1.2. As shown in the proof of Lemma 6.2.1, for each j, there exist 8 limiting trajectories which converge geometrically to periodic orbits of exactly j reflections. We denote the set of these trajectories by $\mathcal{G}_j(x,y)$ and say that $\gamma_{m,k} \in \mathcal{G}_j$ if $\gamma_{m,k}$ makes $m = j-1, j$ or $j+1$ reflections at the boundary and corresponds to the length functional Ψ^k_m. By the results in [44], [19] and [60], the length functionals $\Psi_j^2, \Psi_j^3, \Psi_j^6, \Psi_j^7, \Psi_{j+1}^4, \Psi_{j+1}^8, \Psi_{j-1}^1$ and Ψ_{j-1}^5 corresponding to orbits in $\mathcal{G}_j$ actually coincide for $x, y \in \partial\Omega$ near the diagonal. We denote their common value by Ψ_j.

As we obtained a rather explicit formula for S_j in Theorem 5.5.2, it now remains to differentiate the kernels S_j^k and substitute them into Lemma 7.1.1. Using our oscillatory integral representation for $S_{j,\pm}^k$ in Theorem 5.5.2, we find that microlocally near $\Gamma_\pm^{j,k}$ and $t = L_j$,

$$(-\nabla_1^T \nabla_2^T - \Delta_2) S_{j,\pm}^k = \frac{\mp t}{2} \int_0^\infty e^{\pm i\tau(t - \Psi_j^k(x,y))} \frac{(-i\tau)^2 |\nabla_2^\perp \Psi_j^k|^2 |A_j^k|^{1/2}}{2|\tau|^{1/2} i} \dot{\rho}\, d\tau + L.O.T. \quad (7.2)$$

in an open neighborhood of the diagonal of the boundary. We have only written the terms coming from $-\nabla^T \nabla^T - \Delta$ acting on the phase function in equation (7.2), as all other terms don't contribute positive powers of τ and can be regarded as lower order in the singularity expansion. The operator $\nabla^\perp$ in the integrand of (7.2) is a conformal multiple of the vector field $\frac{\partial}{\partial \nu}$ coming from elliptical polar coordinates, which gives an extension of the normal vector field to a neighborhood of the boundary. As Theorem 4.1.7 tells us that the variation of the wave trace is given by integrating the kernels L_R^b and S_R over the diagonal of the boundary, we want to understand the restriction of (7.2) to the boundary. In Definition 7.1.2, we noted that the length of the unique orbit connecting two boundary points with j reflections is well defined. When $x = y = q \in \partial\Omega$, $\Psi_j(q, q)$ gives precisely the length L_j of a periodic orbit with j reflections emanating from and terminating at q. By Poncelet's Theorem (3.2.1), $\Psi_j(q, q)$ is actually equal to the constant function L_j, which simplifies the phase in equation (7.2). The differentiated kernels in equation (7.2) also have factors of A_j^k and $\nabla_2^\perp \Psi_j^k$ in the integrand. We now discuss how to extend these derivatives of Ψ_j^k to the boundary in a manner analogous to that of Definition 7.1.2.

We have already computed the x, y gradient of the functions Ψ_j^k in equations (5.8) and

(5.9) in the proof of Lemma 5.3.1:

$$d_x \Psi_j^k = \frac{x - q_1^k}{|x - q_1^k|}, \quad d_y \Psi_j^k = \frac{y - q_j^k}{|y - q_j^k|}.$$

Geometrically, these are the incident and reflected outgoing unit directions of the corresponding billiard trajectories at x and y. The expression $|\nabla_y^{\perp} \Psi_j^k|^2$ in (7.2) can easily be seen to be $\sin^2 \omega_j^k$, where ω_j^k is the angle made between the terminal link of the billiard trajectory and the positively oriented tangent line to the confocal ellipse on which y lies. As $x, y \to \partial\Omega$, the absolute value of these angles associated to trajectories in the $\mathcal{G}_j$ converge to the terminal angle of the unique limiting orbit connecting boundary points in [44]. We are careful to point out that only the absolute values of the angles converge, since the angles associated to orbits in TN and NN configurations actually converge to *minus* the angle of incidence of the limiting trajectory. All of the limiting orbits which connect boundary points in [44] are automatically in TT configuration.

In order to understand the factor A_j^k, we must compute the Hessian of Ψ_j^k and its restriction to the boundary. Recall that A_j^k is given by equation (5.18) in elliptical polar coordinates. We can further simplify that expression to obtain the following:

Lemma 7.1.3. *On the boundary, all of the factors $A_{j-1}^1, A_{j-1}^5 A_j^2, A_j^3, A_j^6, A_j^7, A_{j+1}^4$ and A_j^8 corresponding to orbits in $\mathcal{G}_j$ coincide up to sign. We denote their common (absolute) value by $|A_j|$, which on $\Delta\partial\Omega$ in particular, satisfies the equation*

$$|A_j(\mu_0, \theta, \mu_0, \theta)| = \left| \frac{g^5(\mu_0, \theta)}{\sin \omega} \frac{\partial \omega}{\partial \theta} \right|.$$

Here, ω is the angle of incidence of the unique periodic orbit with rotation number $1/j$ based at $(a \cos\theta, b \sin\theta) \in \partial\Omega$ and $g(\mu_0, \theta) = \left(a^2(\cosh^2 \mu_0 - \cos^2 \theta)\right)^{1/2}$ is the inverse of the conformal factor in elliptical coordinates.

The proof of Lemma 7.1.3 follows exactly the same as that of Lemma 6.2.3 except that elliptical polar coordinates now carry the conformal factor g instead of f appearing in the warped product boundary normal coordinates. Substituting the formula for A_j in Lemma 7.1.3 into (7.2) and performing the integral (7.2) in τ, we obtain:

Corollary 7.1.4. *The variation of the wave trace localized near $t = L_j$ is*

$$4t\,\mathrm{Re}\left\{e^{i\pi/4}(t - L_j - i0^+)^{-5/2}\right\}\int_{\partial\Omega} |\nabla^\perp \Psi_j|^2 |A_j(q,q)|^{1/2}\dot{\rho}(q)\,dq + L.O.T.$$

Proof. For a fixed j, the parametrices corresponding to Γ_+^j and Γ_-^j are first multiplied by Maslov factors of the form $e^{\pm i\sigma\pi/4}$. This is due to the multiplicity of phase functions parametrizing the canonical relation of the wave propagator, as briefly described in Section 5.1. It is well known that the Maslov factors on the two branches Γ_-^j and Γ_+^j of the canonical relation are conjugate to one another, owing to the two modes of propagation:

$$\sin t\sqrt{-\Delta} = \frac{e^{it\sqrt{-\Delta}} - e^{-it\sqrt{-\Delta}}}{2it\sqrt{-\Delta}}.$$

While in principle, the Maslov indices might depend on j, it is shown in [20] that the Maslov indices in Chazarain's parametrix remain unchanged after a reflection at the boundary. The Maslov indices $\pm\sigma$ were in fact be explicitly computed in Section 6.5 in Chapter 6. The contributions of the wave kernels corresponding to Γ_+^j and Γ_-^j are then added and the result follows from a limiting argument for the Fourier transform of the homogeneous distribution $\tau^{3/2}\mathbb{1}_{(0,\infty)}$, as can be found in Chapter 7 of [31]. Since the 8 terms in (7.2) corresponding to orbits in $\mathcal{G}_j$ coincide on the boundary, we multiply the final integral by a factor of 8. $\square$

Since both the kernels on $\Gamma_\pm^j$ contribute to the wave trace singularity near L_j, we sum together the contributions of S_j^- and S_j^+ which explains the real parts in Theorem 2.2.4 and Corollary 7.1.4.

7.2 Calculating $\frac{\partial \omega}{\partial \theta}$

To calculate the angular derivative in Lemma 7.1.3, we will first relate it to the billiard map and then utilize some special dynamical properties of the ellipse in action angle coordinates. Recall that the billiard map takes place on the coball bundle $B^*\partial\Omega$, which is diffeomorphic to the inward facing portion of the circle bundle with footpoints on the boundary and can be parametrized by coordinates $(\varphi, \omega) \in S^1 \times S^1$ (although we only consider the nontangential, inward pointing directions corresponding to $0 < \omega < \pi$). Between any two points $x, y \in \partial\Omega$, the results of [44] show that there exists a unique broken geodesic of j reflections emanating from x and terminating at y. This geodesic makes an initial angle of ω (depending on both x and y) with the tangent line $T_x\partial\Omega$. Setting $x = y$ above gives the angle $\omega(x)$ corresponding to a periodic orbit, which we considered in Lemma 7.1.3. Letting $\varphi, \theta \in S^1$ be the angular variables which parametrize x and y respectively in elliptical polar coordinates, we need to calculate the quantity $\frac{\partial \omega}{\partial \theta}$, evaluated on the diagonal $\{\varphi = \theta\}$. Consider the map

$$\begin{cases} B_j : S^1 \times (0, \pi) \to S^1, \\ B_j(\varphi, \omega(\varphi, \theta)) = \pi_1 \circ \beta^{j+1}(\varphi, \omega) = \theta, \end{cases} \tag{7.3}$$

where π_1 is the projection onto the first factor. Fixing φ and differentiating both sides in θ gives

$$\frac{\partial B_j}{\partial \omega}(x, \omega(\theta)) \frac{\partial \omega}{\partial \theta} = 1.$$

Hence, we have

$$\frac{\partial \omega}{\partial \theta} = \frac{1}{\frac{\partial B_j}{\partial \omega}(x, \omega(\theta))}. \tag{7.4}$$

We will use formula (7.4) to calculate $\frac{\partial\omega}{\partial\theta}$. Recall that the linearized Poincaré map of the iterated billiard map β^{j+1} at a periodic point (φ,ω) is given by

$$P_{j+1}(\varphi,\omega) = \begin{pmatrix} \frac{\partial\beta_1^{j+1}}{\partial\varphi} & \frac{\partial\beta_1^{j+1}}{\partial\omega} \\ \frac{\partial\beta_2^{j+1}}{\partial\varphi} & \frac{\partial\beta_2^{j+1}}{\partial\omega} \end{pmatrix}, \tag{7.5}$$

where β_1^{j+1} and β_2^{j+1} are the first and second components (in elliptical coordinates) of β^{j+1}. We are precisely interested in the $(1,2)$ entry of this matrix.

To evaluate this quantity, we will use action angle coordinates for elliptical billiards, which we now describe, following the presentation in [34], [8] and [5]. We begin by developing some basic elliptic function theory.

Elliptic functions and elliptic integrals were first studied in the context of computing the arclength of an ellipse. It is therefore no surprise that these same objects appear naturally in the study of elliptical billiards. Formally, an elliptic function is given by a doubly periodic, meromorphic function on the complex plane. One way to obtain elliptic functions is by inverting elliptic integrals:

Definition 7.2.1. An incomplete elliptic integral of the first kind is an integral of the form

$$F(\varphi,k) = \int_0^{\varphi} \frac{d\tau}{\sqrt{1-k^2\sin^2\tau}}.$$

The quantity φ is referred to as the amplitude and k is the modulus. A complete elliptic integral of the first kind is given by fixing $\varphi = \pi/2$:

$$K(k) = F(\pi/2,k).$$

Note that for a fixed k, $F(\varphi,k)$ is an increasing function of φ. The amplitude function

$\mathrm{am}(s;k)$ is obtained by inverting F in the variable φ:

$$s = \int_0^{\mathrm{am}(s;k)} \frac{d\tau}{\sqrt{1 - k^2 \sin^2 \tau}}. \tag{7.6}$$

Definition 7.2.2. The Jacobi elliptic functions are defined by

$$\begin{aligned} \mathrm{cn}(s;k) &= \cos(\mathrm{am}(s;k)), \\ \mathrm{sn}(s;k) &= \sin(\mathrm{am}(s;k)). \end{aligned}$$

These are elliptic functions with periods $4K(k)$ and $4iK(k')$, where $k \in (0,1)$ is called the modulus and $k' = \sqrt{1-k^2}$ is the complimentary modulus. The reason these elliptic functions are useful is that they provide coordinates on phase space in which the billiard map becomes a simple translation. To the confocal ellipse

$$C_\lambda = \left\{ (x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} = 1 \right\},$$

we associate the following parameters:

$$\begin{aligned} k_\lambda^2 &= \frac{a^2 - b^2}{a^2 - \lambda^2}, \\ \delta_\lambda &= 2F(\arcsin(\lambda/b); k_\lambda) = 2\int_0^{\arcsin(\lambda/b)} \frac{d\tau}{\sqrt{1 - k_\lambda^2 \sin^2 \tau}}. \end{aligned} \tag{7.7}$$

Let us also denote the $4K(k_\lambda)$ periodic boundary parametrization associated to the caustic C_λ by $q_\lambda(s) = (-a\mathrm{sn}(s;k_\lambda), b\mathrm{cn}(s;k_\lambda))$ (a reflection about the y axis of the parametrization considered in [8]). It is proven in [5] that for all $s \in \mathbb{R}$ the line segment connecting $q_\lambda(s)$ and $q_\lambda(s + \delta_\lambda)$ is tangent to the caustic C_λ. In other words, (s, λ) are precisely the action-angle coordinates from Section 3 and in these coordinates, the billiard map is given by a linear rotation along the invariant tori, which are the projections of C_λ onto $B^*\partial\Omega$.

To relate these action angle coordinates to the elliptical polar coordinates in which we calculated A_j (see Lemma 7.1.3), note that

$$(x_1, x_2) = (a\cos\varphi, b\sin\varphi) = (-a\mathrm{sn}(t_\varphi; k_\lambda), b\mathrm{cn}(t_\varphi; k_\lambda)),$$

which implies

$$\varphi = \mathrm{am}(t_\varphi; k_\lambda) + \frac{\pi}{2}. \tag{7.8}$$

Here, t_φ is defined implicitly by the equations (7.6) and (7.8). Similarly, we find that

$$B_j(\varphi, \omega) = \pi/2 + \mathrm{am}(t_\varphi + (j+1)\delta_\lambda; k_\lambda).$$

Differentiating B_j in θ, we obtain

$$\frac{\partial B_j}{\partial \omega}(\varphi, \omega(\theta))\frac{\partial \omega}{\partial \theta} = 1 \implies \frac{\partial \omega}{\partial \theta} = \frac{1}{\frac{\partial B_j}{\partial \omega}(\varphi, \omega(\theta))}.$$

By the chain rule, we see that

$$\frac{\partial B_j}{\partial \omega} = \frac{\partial \mathrm{am}}{\partial s}\left(\frac{\partial t_\varphi}{\partial \lambda^2}\frac{\partial \lambda^2}{\partial \omega} + (j+1)\frac{d\delta_\lambda}{d\lambda^2}\frac{\partial \lambda^2}{\partial \omega}\right) + \frac{\partial \mathrm{am}}{\partial k_\lambda^2}\frac{dk_\lambda^2}{d\lambda^2}\frac{\partial \lambda^2}{\partial \omega}. \tag{7.9}$$

We can factor out $\frac{\partial \lambda^2}{\partial \omega}$ from (7.9) and calculate each of the individual terms explicitly. Using the implicit function theorem, we find that

$$\frac{\partial \mathrm{am}}{\partial s} = \sqrt{1 - k_\lambda^2 \mathrm{sn}^2(s; k_\lambda)},$$

evaluated at the point $s = t_\varphi + (j+1)\delta_\lambda$. We now find the derivative of t_φ with respect to λ. Since x is fixed, we know that the argument $\varphi = \mathrm{am}(t_\varphi; k_\lambda)$ must also be fixed.

Differentiating in λ under the integral, we see that

$$\frac{\partial t_\varphi}{\partial \lambda^2} = \frac{k_\lambda^2}{(a^2-\lambda^2)} \int_0^{\mathrm{am}(t_\varphi;k_\lambda)} \frac{\sin^2\tau d\tau}{(1-k_\lambda^2\sin^2\tau)^{3/2}}.$$

Using the formula (7.7) for δ_λ, we also calculate that

$$\frac{d\delta_\lambda}{d\lambda^2} = \frac{1}{b\lambda\sqrt{1-\frac{k_\lambda^2\lambda^2}{b^2}}} \frac{1}{\sqrt{1-\lambda^2/b^2}} + \frac{2k_\lambda^2}{a^2-\lambda^2} \int_0^{\arcsin(\lambda/b)} \frac{\sin^2\tau d\tau}{(1-k_\lambda^2\sin^2\tau)^{3/2}}. \tag{7.10}$$

For the last term in (7.9), write

$$s = \int_0^{\mathrm{am}(s;k_\lambda)} \frac{d\tau}{\sqrt{1-k_\lambda^2\sin^2\tau}}$$

and differentiate both sides in the variable k_λ^2. We find that

$$\frac{\partial \mathrm{am}}{\partial k_\lambda^2} = -\sqrt{1-k_\lambda^2\mathrm{sn}^2(s;k_\lambda)} \int_0^{\mathrm{am}(s;k_\lambda)} \frac{\sin^2\tau d\tau}{(1-k_\lambda^2\sin^2\tau)^{3/2}},$$

where s is evaluated at $t_x+(j+1)\delta_\lambda$. It is also easy to see that

$$\frac{dk_\lambda^2}{d\lambda^2} = \frac{k_\lambda^2}{(a^2-\lambda^2)}.$$

At the critical λ corresponding to the angle $\omega(x)$ generating a periodic orbit at x, we have $\mathrm{am}(t_\varphi+(j+1)\delta_\lambda) = \mathrm{am}(t_\varphi)+2\pi$. Using this, we calculate that

$$\frac{\partial B_j}{\partial \omega} = \frac{\partial \lambda^2}{\partial \omega}\sqrt{1-k_\lambda^2\mathrm{sn}^2(t_\varphi;k_\lambda)} \left(\frac{-k_\lambda^2}{(a^2-\lambda^2)} \int_0^{2\pi} \frac{\sin^2\tau d\tau}{(1-k_\lambda^2\sin^2\tau)^{3/2}} + (j+1)\frac{d\delta_\lambda}{d\lambda^2} \right). \tag{7.11}$$

We have not simplified the expression in parentheses in (7.11), since what will ultimately be important is that this term depends only on λ and not on φ or ω. Let us denote this factor by

$$G(\lambda) = \frac{-k_\lambda^2}{(a^2 - \lambda^2)} \int_0^{2\pi} \frac{\sin^2 \tau d\tau}{(1 - k_\lambda^2 \sin^2 \tau)^{3/2}} + (j+1)\frac{d\delta_\lambda}{d\lambda^2}. \tag{7.12}$$

The term $\partial \lambda^2 / \partial \omega$ is more difficult to compute and will rely on a geometric lemma, which we now present. As the ellipse is foliated by caustics, for each $(x, \omega) \in S^*_{\partial\Omega}(\mathbb{R}^2)$, there exists a unique λ such that the line segments of the billiard flow are always tangent to the confocal ellipse/hyperbola of parameter λ. The following lemma expresses this relationship between the confocal caustic and angle of incidence:

Lemma 7.2.3. *The billiard ray emanating from $(a\cos\varphi, b\sin\varphi)$ at angle ω is tangent to the elliptical caustic C_λ, where the relationship between λ and ω is given by*

$$\lambda^2 = \sin^2(\omega)\left(b^2 + (a^2 - b^2)\sin^2(\varphi)\right).$$

Proof. For simplicity, consider the complexified parametrization of the ellipse given by $\gamma(\varphi) = a\cos\varphi + ib\sin\varphi$. The tangent line at $\gamma(\varphi)$ is then parametrized by

$$(a\cos\varphi + ib\sin\varphi) + t(-a\sin\varphi + ib\cos\varphi).$$

Rotating this line counterclockwise by the angle ω, we see that the billiard ray is parametrized by

$$L_\omega(t, \varphi) = (a\cos\varphi + ib\sin\varphi) + te^{i\omega}(-a\sin\varphi + ib\cos\varphi).$$

Taking real and imaginary parts, we find that

$$\begin{aligned}
\operatorname{Re} L_\omega(t, \varphi) &= a\cos\varphi + t(-a\sin\varphi\cos\omega - b\cos\varphi\sin\omega), \\
\operatorname{Im} L_\omega(t, \varphi) &= b\sin\varphi + t(b\cos\varphi\cos\omega - a\sin\varphi\sin\omega).
\end{aligned}$$

For a given φ, ω, there exist infinitely many caustics which intersect the line L_ω. However, only one such caustic intersects L_ω at a single point of tangency. To find the parameter of this caustic, we look for a solution of the equation

$$\frac{\mathrm{Re}(L_\omega(t,\varphi))^2}{a^2-\lambda^2}+\frac{\mathrm{Im}(L_\omega(t,\varphi))^2}{b^2-\lambda^2}=1.$$

This is a quadratic equation in the variable t and a caustic corresponding to a point of tangency will give rise to a repeated root. Thus, to find λ^2, we set the discriminant of this equation equal to zero. For convenience let us put

$$\begin{aligned}
A &= -\, a\sin\varphi\cos\omega - b\cos\varphi\sin\omega,\\
B &= \quad b\cos\varphi\cos\omega - a\sin\varphi\sin\omega.
\end{aligned}$$

If we set the discriminant equal to zero, we obtain

$$\left(\frac{aA\cos\varphi}{a^2-\lambda^2}+\frac{bB\sin\varphi}{b^2-\lambda^2}\right)^2=\left(\frac{A^2}{a^2-\lambda^2}+\frac{B^2}{b^2-\lambda^2}\right)\left(\frac{a^2\cos^2\varphi}{a^2-\lambda^2}+\frac{b^2\sin^2\varphi}{b^2-\lambda^2}-1\right).$$

After some obvious cancellations and simplifications, multiplying both sides by $(a^2-\lambda^2)(b^2-\lambda^2)$ gives

$$\lambda^2=\frac{(Ab\cos\varphi+Ba\sin\varphi)^2}{A^2+B^2}.$$

Two simple computations show that

$$A^2+B^2=a^2\sin^2\varphi+b^2\cos^2\varphi$$

and

$$Ab\cos\varphi+Bb\sin\varphi=-\sin(\omega)(a^2\sin^2\varphi+b^2\cos^2\varphi).$$

Plugging these into the above equation for λ^2 completes the proof of the lemma.

□

Differentiating the formula in Lemma 7.2.3 by ω gives

$$\frac{\partial \lambda^2}{\partial \omega} = 2 \sin \omega \cos \omega (b^2 + (a^2 - b^2) \sin^2 \varphi), \tag{7.13}$$

which is the last term in (7.9) we needed to calculate.

Remark 7.2.4. In [20], a different formula is derived in the usual circular polar coordinates with angular parameter α. In this case,

$$\lambda^2 = M(\alpha)(1 - \cos(2\omega)),$$

where

$$M(\alpha) = \frac{a^2 + b^2}{2} - \frac{a^2 b^2}{a^2 + b^2 - (a^2 - b^2)\cos(2\alpha)}.$$

The relationship between α and the elliptical angular coordinate φ is given by

$$\alpha = \arctan\left(\frac{b}{a}\tan\varphi\right).$$

7.3 Converting the singularity into an elliptic integral

7.3.1 Proof of $\dot{\rho} = 0$

It now remains to convert the singularity expansion in Corollary 7.1.4 into an elliptic integral, following the ideas in [19]. Recall that according to Corollary 7.1.4, the variation of the localized wave trace is given by

$$\delta \mathrm{Tr} \cos t\sqrt{-\Delta_\varepsilon} = 4t \,\mathrm{Re}\left\{ e^{\sigma i \pi/4}(t - L_j - i0^+)^{-5/2} \right\} \int_{\partial\Omega_0} |\nabla^\perp \Psi_j|^2 |A_j(q,q)|^{1/2} \dot{\rho}(q)\, dq + L.O.T.$$

The computations in Section 7.2 combined with Lemma 7.1.3 lead us to the formula

$$|A_j| = \left|\frac{g^5}{\sin\omega}\frac{\partial\omega}{\partial\theta}\right| = \left|\frac{g^5}{\sin\omega}\frac{1}{\frac{\partial\lambda^2}{\partial\omega}G(\lambda)\sqrt{1-k_\lambda^2\mathrm{sn}^2(t_\varphi)}}\right|, \tag{7.14}$$

on the diagonal of the boundary, where $G(\lambda)$ is given by (7.12) and $\frac{\partial\lambda^2}{\partial\omega}$ is given by (7.13). The formulas (7.12) and (7.10) show that G is in fact a nonvanishing analytic function of λ. Putting this all together, we see that the principal term in the variation of the wave trace in Corollary 7.1.4 is given by the product of the distribution

$$4t\,\mathrm{Re}\left\{e^{i\pi/4}(t-L_j-i0^+)^{-5/2}\right\}$$

and the factor

$$c_j = \int_{\partial\Omega}\frac{\sin(\omega)|g(\mu_0,\varphi)|^{5/2}}{\left|2G(\lambda)\cos(\omega)(b^2+(a^2-b^2)\sin^2\varphi)\sqrt{1-k_\lambda^2\mathrm{sn}^2(t_\varphi;k_\lambda)}\right|^{1/2}}\dot{\rho}(\varphi)\,dq(\varphi). \tag{7.15}$$

Since we are evaluating on the diagonal of the boundary, Poncelet's Theorem (3.2.1) guarantees that the parameter λ corresponding to a j-periodic geodesic is in fact independent of φ. Hence, the $G(\lambda)$ factor can be pulled outside of the integral in (7.15). However, both ω, and t_φ in the integrand depend on φ, so it remains to compute this dependency and parametrize the boundary explicitly. Recall that Lemma 7.2.3 gives us

$$\lambda^2 = \sin^2(\omega)\left(b^2+(a^2-b^2)\sin^2(\varphi)\right).$$

Since $\omega \notin \{0, \pi/2, \pi\}$ for orbits which are tangent to a confocal ellipse, this equation determines the trigonometric terms appearing in the integrand of (7.15) up to a sign:

$$\begin{aligned} \sin\omega &= \sqrt{\frac{\lambda^2}{b^2 + (a^2 - b^2)\sin^2(\varphi)}}, \\ \cos\omega &= \pm\sqrt{1 - \frac{\lambda^2}{b^2 + (a^2 - b^2)\sin^2(\varphi)}}. \end{aligned} \tag{7.16}$$

To simplify notation, set $C(\varphi) = \left(b^2 + (a^2 - b^2)\sin^2(\varphi)\right)$. Recalling that in elliptic coordinates, if we fix

$$\mu_0 = \cosh^{-1}\left(\frac{a}{\sqrt{a^2 - b^2}}\right),$$

then we obtain a parametrization of $\partial\Omega_0$ in terms of $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, given by

$$\gamma(\varphi) = (a\cos\varphi, b\sin\varphi).$$

In these coordinates, the line element on $\partial\Omega_0$ is

$$dq(\varphi) = \sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}\, d\varphi.$$

Recalling that $\cos\varphi = -\mathrm{sn}(t_\varphi; k_\lambda)$ due to our convention (7.8) on the phase shift, a simple computation using formula (7.16) then shows that

$$\cos\omega\sqrt{1 - k_\lambda^2 \mathrm{sn}^2(t_\varphi; k_\lambda)} = \frac{1}{|C(\varphi)(a^2 - \lambda^2)|^{1/2}}(C(\varphi) - \lambda^2),$$

which simplifies the integrand in (7.15) to

$$\left|\frac{\lambda^2 g(\mu_0, \varphi)^5}{2G(\lambda)}\right|^{1/2} \left|\frac{a^2 - \lambda^2}{C(\varphi)}\right|^{1/4} \frac{\gamma^* \dot{\rho}}{\sqrt{C(\varphi) - \lambda^2}}. \tag{7.17}$$

Reinserting this into the integrand in (7.15), we obtain

$$\int_0^{2\pi} \left|\frac{\lambda^2 g(\mu_0,\varphi)^5(a^2\cos^2\varphi + b^2\sin^2\varphi)}{2G(\lambda)}\right|^{1/2} \left|\frac{a^2-\lambda^2}{C(\varphi)}\right|^{1/4} \frac{\gamma^*\dot{\varphi}d\varphi}{\sqrt{C(\varphi)-\lambda^2}}, \tag{7.18}$$

from which the first formula in Theorem 2.2.4 follows.

We now set the integral in equation (7.18) equal to zero. Since λ is nonzero and independent of φ, we can divide out the separated terms depending only on λ from both sides and what is left will be a nonzero analytic function $F(\varphi)$ multiplied by $\dot{\rho}(\varphi)(C(\varphi)-\lambda^2)^{-1/2}$:

$$\int_0^{2\pi} \frac{F(\varphi)\gamma^*\dot{\rho}(\varphi)d\varphi}{\sqrt{C(\varphi)-\lambda^2}} = 0. \tag{7.19}$$

Here, we have written

$$F(\varphi) = \left|g(\mu_0,\varphi)^5(a^2\cos^2\varphi + b^2\sin^2\varphi)\right|^{1/2} |C(\varphi)|^{-1/4}.$$

Using the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry condition on $\dot{\rho}$, the equation (7.19) reduces to

$$\int_0^{\pi/2} \frac{F(\varphi)\gamma^*\dot{\rho}(\varphi)d\varphi}{\sqrt{C(\varphi)-\lambda^2}} = 0. \tag{7.20}$$

We now recall that since the singularity expansion is localized near an isolated length L_j, we have $\lambda = \lambda_j$ and $\lambda_j \to 0$ as $j \to \infty$. The expression (7.20) is actually analytic in the parameter λ^2 and vanishes at each λ_j^2. Since it has an accumulation point of zeros, it is actually flat at $\lambda = 0$. Differentiating k times under the integral (7.20) in the parameter λ^2 and evaluating at $\lambda = 0$, we see that

$$\int_0^{\pi/2} \frac{F(\varphi)\gamma^*\dot{\rho}(\varphi)d\varphi}{|C(\varphi)|^{1/2+k}} = 0.$$

It is clear that the functions $|C(\varphi)|^{-k}$ form a subalgebra of $C(S^1)$. Since we have restricted the domain to $(0, \pi/2)$, this subalgebra also separates points, and hence by the Stone-Weierstrass theorem,

$$\frac{F(\varphi)\gamma^* \dot{\rho}(\varphi)}{|C(\varphi)|^{1/2}} \equiv 0. \tag{7.21}$$

Since F and C are nonvanishing, equation (7.21) implies that $\dot{\rho} = 0$.

7.3.2 Proof of $\dot{K} = 0$

We now return to the first variation of the Robin function K. According to Theorem 4.1.7,

$$\delta \mathrm{Tr} \cos t\sqrt{-\Delta_\varepsilon} = \int_{\partial\Omega} L_R^b(t, q, q)\dot{\rho} + \frac{t}{2} S_R(t, q, q)\dot{K}\, dq.$$

Since we have just shown that $\dot{\rho} = 0$ in Section 7.3.1, the variational trace formula in Theorem 4.1.7 becomes

$$\delta \mathrm{Tr} \cos t\sqrt{-\Delta_\varepsilon} = \int_{\partial\Omega} \frac{t}{2} S_R(t, q, q)\dot{K}\, dq. \tag{7.22}$$

In Section 5.5, we cooked up an oscillatory integral representation for S_j microlocally near the canonical relations $\Gamma_\pm^j$ lying over an open neighborhood of the diagonal of the boundary (Theorem 5.5.2). Without the $L_R^b\dot{\rho}$ terms, we do not differentiate the sine kernel in the integrand of (7.22). Again following the formulas for homogeneous distributions in [31], we see that near $t = L_j$,

$$\delta \mathrm{Tr} \cos t\sqrt{-\Delta_\varepsilon} = -8t\, \mathrm{Re}\left\{e^{i\pi/4}(t - T - i0^+)^{-1/2}\right\} \int_{\partial\Omega} |A_j^k|^{1/2} dq,$$

where the minus sign is due to the appearance of the same Maslov index $\sigma = 1$ appearing in the principal term. Plugging in the formula (7.14) for A_j gives the second formula in Theorem 2.2.4. Similar computations to those in Section 7.3.1 lead us to the equation

$$\int_0^{\pi/2} \left| \frac{g^5(\mu_0, \varphi)\sqrt{C(\varphi)}\sqrt{a^2 - \lambda_j^2}}{2G(\lambda_j)\lambda_j^2} \right|^{1/2} \frac{\dot{K}(\varphi) d\varphi}{\sqrt{C(\varphi) - \lambda_j^2}} = 0.$$

Discarding the separated terms depending only on λ_j and Taylor expandng at $\lambda = 0$ as before, we see via the Stone-Weierstrass theorem that $\dot{K} = 0$, which concludes the proof of Theorem 2.2.1.

Remark 7.3.1. In [25], the principal symbol computation for Neumann boundary conditions follows closely the work of [12]. The principal symbols for the Neumann and Robin wave propagators agree and the computations in [25] yield

$$\delta \mathrm{Tr} \cos(t\sqrt{-\Delta}) \sim \frac{t}{2} \mathrm{Re} \left\{ \left(\sum_{\Gamma \subset F_T} C_\Gamma \int_\Gamma \dot{\rho} \gamma_1 \, d\mu_L \right) (t - T + i0^+)^{-5/2} \right\},$$

where the sum is over connected components Γ of the fixed point set F_T and $\gamma_1 \in C(B^*\partial\Omega)$ is given by $\gamma_1(q, \zeta) = \sqrt{1 - |\zeta|^2}$. The coefficients C_Γ are nonzero Maslov factors coming from the stationary phase computation in [25] and μ_L is the Leray measure on $\Gamma \subset B^*\partial\Omega$, which is computed in [19]. In our computations, all of these factors are explicit.

Bibliography

[1] K. G. Andersson and R. B. Melrose. The propagation of singularities along gliding rays. *Invent. Math.*, 41(3):197–232, 1977.

[2] V. I. Arnold. Proof of a theorem of A. N. Kolmogorov on the preservation of conditionally periodic motions under a small perturbation of the Hamiltonian. *Uspehi Mat. Nauk*, 18(5 (113)):13–40, 1963.

[3] G. D. Birkhoff. *Dynamical systems.* With an addendum by Jurgen Moser. American Mathematical Society Colloquium Publications, Vol. IX. American Mathematical Society, Providence, R.I., 1966.

[4] K. Burns and A. Katok. Manifolds with nonpositive curvature. *Ergodic Theory Dynam. Systems*, 5(2):307–317, 1985.

[5] S.-J. Chang and R. Friedberg. Elliptical billiards and Poncelet's theorem. *Journal of Mathematical Physics*, 29:1537–1550, July 1988.

[6] J. Chazarain. Paramétrix du problème mixte pour l'équation des ondes à l'intérieur d'un domaine convexe pour les bicaractéristiques. pages 165–181. Astérisque, No. 34–35, 1976.

[7] T. H. Colding and W. P. Minicozzi, II. *A course in minimal surfaces*, volume 121 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2011.

[8] J. Damasceno, M. J. Dias Carneiro, and R. Ramírez-Ros. The billiard inside an ellipse deformed by the curvature flow. *Proc. Amer. Math. Soc.*, 145(2):705–719, 2017.

[9] K. Datchev and H. Hezari. Inverse problems in spectral geometry. In *Inverse problems and applications: inside out. II*, volume 60 of *Math. Sci. Res. Inst. Publ.*, pages 455–485. Cambridge Univ. Press, Cambridge, 2013.

[10] J. de Simoi, V. Kaloshin, and Q. Wei. Dynamical spectral rigidity among $\mathbb{Z}_2$-symmetric strictly convex domains close to a circle (appendix b coauthored with h. hezari). *Ann. of Math. (2)*, 186(1):277–314, 2017. Appendix B coauthored with H. Hezari.

[11] J. J. Duistermaat. *Fourier integral operators*, volume 130 of *Progress in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, 1996.

[12] J. J. Duistermaat and V. W. Guillemin. The spectrum of positive elliptic operators and periodic bicharacteristics. *Invent. Math.*, 29(1):39–79, 1975.

[13] J. J. Duistermaat and L. Hörmander. Fourier integral operators. II. *Acta Math.*, 128(3-4):183–269, 1972.

[14] G. A. Ford, A. Hassell, and L. Hillairet. Wave propagation on Euclidean surfaces with conical singularities. I: Geometric diffraction. *J. Spectr. Theory*, 8(2):605–667, 2018.

[15] C. Gordon, D. L. Webb, and S. Wolpert. One cannot hear the shape of a drum. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):134–138, 1992.

[16] C. Guillarmou and T. Lefeuvre. The marked length spectrum of Anosov manifolds. *ArXiv e-prints*, June 2018.

[17] V. Guillemin and D. Kazhdan. Some inverse spectral results for negatively curved 2-manifolds. *Topology*, 19(3):301–312, 1980.

[18] V. Guillemin and D. Kazhdan. Some inverse spectral results for negatively curved n-manifolds. In *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, pages 153–180. Amer. Math. Soc., Providence, R.I., 1980.

[19] V. Guillemin and R. Melrose. An inverse spectral result for elliptical regions in $\mathbf{R}^2$. *Adv. in Math.*, 32(2):128–148, 1979.

[20] V. Guillemin and R. Melrose. The Poisson summation formula for manifolds with boundary. *Adv. in Math.*, 32(3):204–232, 1979.

[21] V. Guillemin and R. Melrose. A cohomological invariant of discrete dynamical systems. In *E. B. Christoffel (Aachen/Monschau, 1979)*, pages 672–679. Birkhäuser, Basel-Boston, Mass., 1981.

[22] J. Hadamard. *Lectures on Cauchy's problem in linear partial differential equations.* Dover Publications, New York, 1953.

[23] H. Hezari. Robin spectral rigidity of nearly circular domains with a reflectional symmetry. *Comm. Partial Differential Equations*, 42(9):1343–1358, 2017.

[24] H. Hezari and S. Zelditch. Inverse spectral problem for analytic $(\mathbb{Z}/2\mathbb{Z})^n$-symmetric domains in $\mathbb{R}^n$. *Geom. Funct. Anal.*, 20(1):160–191, 2010.

[25] H. Hezari and S. Zelditch. C^∞ spectral rigidity of the ellipse. *Anal. PDE*, 5(5):1105–1132, 2012.

[26] H. Hezari and S. Zelditch. One can hear the shape of ellipses of small eccentricity. *arXiv e-prints*, page arXiv:1907.03882, Jul 2019.

[27] L. Hörmander. The spectral function of an elliptic operator. *Acta Math.*, 121:193–218, 1968.

[28] L. Hörmander. Fourier integral operators. I. *Acta Math.*, 127(1-2):79–183, 1971.

[29] L. Hörmander. *The analysis of linear partial differential operators. III*, volume 274 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985. Pseudodifferential operators.

[30] L. Hörmander. *The analysis of linear partial differential operators. IV*, volume 275 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985. Fourier integral operators.

[31] L. Hörmander. *The analysis of linear partial differential operators. I*. Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[32] V. Ivrii. 100 years of Weyl's law. *Bull. Math. Sci.*, 6(3):379–452, 2016.

[33] M. Kac. Can one hear the shape of a drum? *Amer. Math. Monthly*, 73(4, part II):1–23, 1966.

[34] V. Kaloshin and A. Sorrentino. On the local Birkhoff conjecture for convex billiards. *Ann. of Math. (2)*, 188(1):315–380, 2018.

[35] V. Kaloshin and K. Zhang. Density of convex billiards with rational caustics. *Nonlinearity*, 31(11):5214–5234, 2018.

[36] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[37] A. B. Katok. Billiard table as a playground for a mathematician. In *Surveys in modern mathematics*, volume 321 of *London Math. Soc. Lecture Note Ser.*, pages 216–242. Cambridge Univ. Press, Cambridge, 2005.

[38] J. B. Keller. A geometrical theory of diffraction. In *Calculus of variations and its applications. Proceedings of Symposia in Applied Mathematics, Vol. 8*, pages 27–52. For the American Mathematical Society: McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958.

[39] A. N. Kolmogorov. On conservation of conditionally periodic motions for a small change in Hamilton's function. *Dokl. Akad. Nauk SSSR (N.S.)*, 98:527–530, 1954.

[40] V. V. Kozlov and D. V. Treshchëv. *Billiards*, volume 89 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1991. A genetic introduction to the dynamics of systems with impacts, Translated from the Russian by J. R. Schulenberger.

[41] V. F. Lazutkin. Existence of caustics for the billiard problem in a convex domain. *Izv. Akad. Nauk SSSR Ser. Mat.*, 37:186–216, 1973.

[42] V. F. Lazutkin. *KAM theory and semiclassical approximations to eigenfunctions*, volume 24 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1993. With an addendum by A. I. Shnirelman.

[43] J. M. Lee and G. Uhlmann. Determining anisotropic real-analytic conductivities by boundary measurements. *Comm. Pure Appl. Math.*, 42(8):1097–1112, 1989.

[44] S. Marvizi and R. Melrose. Spectral invariants of convex planar regions. *J. Differential Geom.*, 17(3):475–502, 1982.

[45] R. Melrose. The inverse spectral problem for planar domains. In *Instructional Workshop on Analysis and Geometry, Part I (Canberra, 1995)*, volume 34 of *Proc. Centre Math. Appl. Austral. Nat. Univ.*, pages 137–160. Austral. Nat. Univ., Canberra, 1996.

[46] R. Melrose and M. Taylor. Boundary problems for wave equations with grazing and gliding rays no citation. Unpublished lecture notes.

[47] R. B. Melrose. Isospectral sets of drumheads are compact in c^∞. 2007.

[48] J. Moser. On invariant curves of area-preserving mappings of an annulus. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, 1962:1–20, 1962.

[49] J. Moser. Remark on the paper: "On invariant curves of area-preserving mappings of an annulus" [Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1962**, 1–20; MR0147741 (26 #5255)]. *Regul. Chaotic Dyn.*, 6(3):337–338, 2001.

[50] B. Osgood, R. Phillips, and P. Sarnak. Compact isospectral sets of plane domains. *Proc. Nat. Acad. Sci. U.S.A.*, 85(15):5359–5361, 1988.

[51] B. Osgood, R. Phillips, and P. Sarnak. Compact isospectral sets of surfaces. *J. Funct. Anal.*, 80(1):212–234, 1988.

[52] B. Osgood, R. Phillips, and P. Sarnak. Extremals of determinants of Laplacians. *J. Funct. Anal.*, 80(1):148–211, 1988.

[53] G. P. Paternain, M. Salo, and G. Uhlmann. Spectral rigidity and invariant distributions on Anosov surfaces. *J. Differential Geom.*, 98(1):147–181, 2014.

[54] G. P. Paternain, M. Salo, and G. Uhlmann. Tensor tomography: progress and challenges. *Chin. Ann. Math. Ser. B*, 35(3):399–428, 2014.

[55] J. Peetre. On Hadamard's variational formula. *J. Differential Equations*, 36(3):335–346, 1980.

[56] V. M. Petkov and L. N. Stoyanov. *Geometry of reflecting rays and inverse spectral problems*. Pure and Applied Mathematics (New York). John Wiley & Sons, Ltd., Chichester, 1992.

[57] V. M. Petkov and L. N. Stoyanov. *Geometry of the generalized geodesic flow and inverse spectral problems.* John Wiley & Sons, Ltd., Chichester, second edition, 2017.

[58] J.-V. Poncelet. *Traité des propriétés projectives des figures. Tome I.* Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Sceaux, 1995. Reprint of the second (1865) edition.

[59] J.-V. Poncelet. *Traité des propriétés projectives des figures. Tome II.* Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Sceaux, 1995. Reprint of the second (1866) edition.

[60] G. Popov. Invariants of the length spectrum and spectral invariants of planar convex domains. *Comm. Math. Phys.*, 161(2):335–364, 1994.

[61] G. Popov and P. Topalov. Liouville billiard tables and an inverse spectral result. *Ergodic Theory Dynam. Systems*, 23(1):225–248, 2003.

[62] G. Popov and P. Topalov. On the integral geometry of Liouville billiard tables. *Comm. Math. Phys.*, 303(3):721–759, 2011.

[63] G. Popov and P. Topalov. Invariants of isospectral deformations and spectral rigidity. *Comm. Partial Differential Equations*, 37(3):369–446, 2012.

[64] A. Selberg. Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Indian Math. Soc. (N.S.)*, 20:47–87, 1956.

[65] S. Tabachnikov. *Geometry and billiards*, volume 30 of *Student Mathematical Library.* American Mathematical Society, Providence, RI; Mathematics Advanced Study Semesters, University Park, PA, 2005.

[66] M. E. Taylor. *Partial differential equations II. Qualitative studies of linear equations*, volume 116 of *Applied Mathematical Sciences.* Springer, New York, second edition, 2011.

[67] A. Uribe. Trace formulae. In *First Summer School in Analysis and Mathematical Physics (Cuernavaca Morelos, 1998)*, volume 260 of *Contemp. Math.*, pages 61–90. Amer. Math. Soc., Providence, RI, 2000.

[68] A. Vig. The Wave Trace and Birkhoff Billiards. *arXiv e-prints*, page arXiv:1910.06441, Oct. 2019.

[69] A. Vig. Robin spectral rigidity of the ellipse. *The Journal of Geometric Analysis*, 2020.

[70] M.-F. Vignéras. Variétés riemanniennes isospectrales et non isométriques. *Ann. of Math. (2)*, 112(1):21–32, 1980.

[71] E. M. E. Zayed. Short-time asymptotics of the heat kernel on bounded domain with piecewise smooth boundary conditions and its applications to an ideal gas. *Acta Mathematicae Applicatae Sinica, English Series*, 20(2):215, Jun 2004.

[72] S. Zelditch. The inverse spectral problem for surfaces of revolution. *J. Differential Geom.*, 49(2):207–264, 1998.

[73] S. Zelditch. Spectral determination of analytic bi-axisymmetric plane domains. *Geom. Funct. Anal.*, 10(3):628–677, 2000.

[74] S. Zelditch. The inverse spectral problem. In *Surveys in differential geometry. Vol. IX*, volume 9 of *Surv. Differ. Geom.*, pages 401–467. Int. Press, Somerville, MA, 2004. With an appendix by Johannes Sjöstrand and Maciej Zworski.

[75] S. Zelditch. Inverse spectral problem for analytic domains. I. Balian-Bloch trace formula. *Comm. Math. Phys.*, 248(2):357–407, 2004.

[76] S. Zelditch. Inverse spectral problem for analytic domains. II. $\mathbb{Z}_2$-symmetric domains. *Ann. of Math. (2)*, 170(1):205–269, 2009.

[77] S. Zelditch. Inverse spectral problem for analytic domains. II. $\mathbb{Z}_2$-symmetric domains. *Ann. of Math. (2)*, 170(1):205–269, 2009.

[78] S. Zelditch. Survey on the inverse spectral problem. *ICCM Not.*, 2(2):1–20, 2014.

Appendix A

An Auxiliary Check on the Order of S_j^k

A.1 An auxiliary check on the order of S_j^k

Given a discrepancy in the works [44] and [60], we provide an additional check on the order of a_0^j in Theorem 2.1.3. Assume Ω satisfies the noncoincidence condition (2.5) and let $\rho \in \mathcal{S}(\mathbb{R})$ be a test function such that $\text{Supp}\rho \subset [t_j - \varepsilon, T_j + \varepsilon]$, where ε is sufficiently small to ensure $\text{Supp}\rho \cap \text{LSP}(\Omega) = [t_j, T_j]$. Let $L_j \in [t_j, T_j]$ denote the length of a periodic orbit of rotation number $1/j$. We will compute the quantity

$$\mathcal{F}(\rho(t)\text{Tr}\cos t\sqrt{-\Delta}) = \int e^{-i\lambda t}\rho(t)\text{Tr}\cos t\sqrt{-\Delta}dt \tag{A.1}$$

in two different ways. Recalling our parametrix for $\cos t\sqrt{-\Delta}$, we see that (A.1) is given by

$$\sum_{\pm}\int_{\mathbb{R}_t}\int_0^\infty\int_{\partial\Omega} e^{\pm i\tau(t-\Psi_j(q,q))-\lambda t}\rho(t)\tau^m a^j(q)dqd\tau dt, \tag{A.2}$$

where m is the purported order of $a^j(q)$. Changing variables by $\xi = \tau/\lambda$, we see that (A.2) becomes

$$\sum_{\pm} \int_{\mathbb{R}_t} \int_0^\infty \int_{\partial\Omega} e^{\pm i\lambda\xi(t-\Psi_j(q,q))-\lambda t} \rho(t) \lambda^{m+1} \xi^m a_0(q) dq d\xi dt. \tag{A.3}$$

To understand the λ asymptotics of this oscillatory integral, we apply the method of stationary phase. First assume L_j is a simple nondegenerate length. On the critical set, $d_{t,\xi,q}(\xi(t - \Psi_j(q,q)) - t) = 0$, which implies

$$\begin{cases} t = \Psi_j(q,q) \\ \xi = 1 \\ d_q\psi_j(q,q) = 0. \end{cases}$$

Hence, (A.1) is given by

$$(2\pi)^{3/2}\lambda^{m-1/2} \sum_{q: d_q\Psi_j(q,q)=0} \frac{e^{i\pi/4 \mathrm{sgn}\ \mathrm{Hess}(\Psi_j(q,q))}\rho(\Psi_j(q,q))a_0(q)}{|\partial_q^2(\Psi_j(q,q))|^{1/2}} + o(\lambda^{m-1/2}). \tag{A.4}$$

Recall that periodic orbits of rotation number $1/j$ arise from critical points of the j-loop function, i.e. $d_q\Psi_j(q,q) = 0$ implies that the geodesic loop of j reflections based at q is actually a periodic orbit. Since L_j was assumed to be simple, corresponding to a unique nondegenerate orbit, there are precisely j such boundary points q and the sum in (A.4) is actually finite. It is shown in Theorem 3 of [40] that nondegeneracy of γ also implies $\partial_q^2(\Psi_j(q,q)) \neq 0$. Now, by the formulas in [20], we know that for a simple length L_j corresponding to a nondegenerate periodic orbit of j reflections, the leading asymptotic of the wave trace modulo Maslov factors is given by

$$\sum \cos t\lambda_j = \frac{L_j}{|\det(I - P_\gamma)|^{1/2}}(t - L_j + i0)^{-1} \mod L^1_{loc}(\mathbb{R}), \tag{A.5}$$

where P_γ is the Poincare map associated to the unique periodic orbit γ of length L_j. Formulas in [31] tell us that the Fourier transform of the righthand side of (A.5) is a constant multiple of the Heaviside function. Comparing degrees of homogeneity in (A.4) and (A.5) immediately implies that $m = 1/2$.

If there are infinitely many critical points of $\Psi_j(q,q)$ in the phase of (A.4), the analysis is more subtle. For example, Poncelet's theorem for elliptical billiards actually implies that periodic orbits of a fixed length and rotation number come in one parameter families. For j sufficiently large, every boundary point is the base point for a unique periodic orbit tangent to a single confocal ellipse, making j reflections and a single rotation. The lengths of these orbits are independent of the base point, which implies $d_q\Psi_j(q,q)$ vanishes identically. In this case, $\Psi_j(q,q) = L_j$ and applying stationary phase to (A.3) yields

$$2\pi\lambda^m e^{i\lambda L_j}\rho(L_j)\int_{\partial\Omega} a_0(q)dq + o\left(\lambda^m\right). \tag{A.6}$$

While the expansion (A.5) is no longer valid for high length spectral multiplicity, the formulas in [25] and [69] show that the *variation* of the wave trace near such a period in the length spectrum is a distribution of the form $d_j \operatorname{Re}(t - L_j + i0)^{-5/2}$ for some constant d_j. Formally, the wave trace has one higher degree of regularity than its variation, which suggests that the wave trace is of the form $c_j(t-L_j+i0)^{-3/2}$, in agreement with Theorem 2.1.5 and Corollary 2.1.6. Comparing this with the asymptotics in (A.6) and formulas in [31] for the Fourier transform of homogeneous distributions, we see again that $m = -1-(-3/2) = 1/2$. The order of a_0^j was also confirmed in the recent paper [26].

Appendix B

The 8 Orbit Theorem in Elliptical Polar Coordinates

B.1 Proof of Theorem 3.3.1

This section is dedicated to an alternate proof of Theorem 3.3.1 in the special case of an elliptical billiard table. Fix $j \in \mathbb{Z}$ large and choose a corresponding tubular neighborhood U of $\partial\Omega$ with the following property: *for any $x \in U$, the j-fold broken geodesic emanating from x which is tangent to the confocal ellipse on which x lies makes less than a quarter rotation.* By this we mean that if $x = (a \cosh \mu_x \cos \varphi_x, b \sinh \mu_x \sin \varphi_x)$, then the $j+1$ impact point at the boundary has angular component in the interval $\varphi_x \leq \varphi_{x,j} \leq \varphi_x + \pi/2$. This is certainly possible since the billiard flow is continuous on the closure of its phase space and if x is on the boundary, the corresponding orbit is stationary. Denote by C_{λ_x} the confocal ellipse on which x lies. We will perturb this orbit by holding x fixed and increasing the parameter λ^2

of the confocal caustic

$$C_\lambda = \left\{ z \in \mathbb{R}^2 : \frac{z_1^2}{a^2 - \lambda^2} + \frac{z_2^2}{b^2 - \lambda^2} = 1 \right\},$$

to which the orbit is tangent. This can be done by rotating the initial covector of the trajectory slightly in any direction within $S_x^*(\mathbb{R}^2)$ so that it makes a nonzero angle with the tangent line to C_{λ_x} at x. As λ^2 increases, the associated angle also increases, the confocal ellipses shrink and heuristically, the j-fold broken geodesic begins to rotate more and more around Ω. This is precisely the twist property of the billiard map: for a fixed point x in the base, the straight line in phase space obtained by letting the angular component vary becomes twisted under iteration of the billiard map. If y is another point which is sufficiently close to x, we claim that there exist 8 angles in $S_x^*(\mathbb{R}^2)$ such that the last link of the corresponding billiard trajectory intersects y.

Four of these orbits will be oriented in the counterclockwise direction and four in the clockwise direction. Of the four counterclockwise orbits, two of them will correspond to rotating the initial covector in $S_x^*(\mathbb{R}^2)$ in the counterclockwise direction and two will result from a rotation in the opposite direction. We show that in elliptical polar coordinates, the angular components of both intersection points of the last link (after j reflections) with the confocal ellipse on which y lies are increasing as we rotate the initial covector within either direction in $S_x^*(\mathbb{R}^2)$. Hence, both intersection points will wind around the confocal ellipse until they eventually coincide with y. The clockwise orbits will then be constructed by a simple reflection argument.

B.1.1 Notation

Let λ_x and λ_y denote the parameters of the confocal ellipses on which x and y lie respectively. If $\lambda_x > \lambda_y$, then it is clear that the last link of any billiard emanating from x will intersect C_{λ_y} exactly twice. If $\lambda_y > \lambda_x$ but x and y are sufficiently close, then λ_x and λ_y are also close, so we can arrange that the billiard emanating from x which is tangent to C_{λ_y} makes less than a half rotation. Hence, the orbits making approximately one full rotation will necessarily have final links which intersect C_{λ_y} twice. We only consider such x, y, which lie in an open neighborhood of $\Delta\partial\Omega \subset \Omega\times\Omega$, where $\Delta : \partial\Omega \to \partial\Omega\times\partial\Omega$ is the diagonal embedding.

The aim is to prove that the angular components of both intersection points of the last link with the caustic C_{λ_y} are increasing. To do this, we will consider a variant of the map B_j defined in Section 5.5. Recall that we defined

$$\begin{cases} B_j : S^1 \times (0,\pi) \to S^1, \\ B_j(\varphi, \omega(\varphi,\theta)) = \pi_1 \circ \beta^{j+1}(\varphi,\omega) = \theta, \end{cases}$$

where β is the billiard map on the coball bundle $B^*(\partial\Omega)$ of the boundary. Since the coball bundle of the boundary is diffeomorphic to the collection of inward facing covectors in $S^*_{\partial\Omega}(\mathbb{R}^2)$ and the latter is more geometrically natural, we lose no generality by considering B_j or β as a map on $S^*_{\partial\Omega}(\mathbb{R}^2)$. We've assumed that $x, y \in \text{int}\Omega$, so we must first flow to the boundary in order to study the billiard map. Let $\alpha \in S^1$ and denote by $(x_1(\alpha), \omega_1(\alpha))$ the point obtained by evolving (x,α) under the forward billiard flow for $t^1_+(x,\alpha)$ units of time and then reflecting at the boundary. Recall from Section 3 that we defined

$$t^1_\pm(y,\alpha) = \inf\{t > 0 : g^{\pm t}(y,\alpha) \in \partial\Omega\}.$$

To the point $(x_1(\alpha), \omega_1(\alpha)) \in S^*_{\partial\Omega}(\mathbb{R}^2)$ which is now fixed, we may apply B_{j-1} to obtain the angular component of the jth impact point. Denote the corresponding boundary point by

$$x_j(\alpha) = (a \cos B_{j-1}(x_1(\alpha), \omega_1(\alpha)), b \sin B_{j-1}(x_1(\alpha), \omega_1(\alpha))).$$

Denote by α_x the angle which the positively oriented tangent line $T_x C_{\lambda_x}$ makes with the positive x axis. If α is sufficiently close to α_x, the first j iterates under billiard map will also make less than a quarter rotation by our original set up. Our strategy is to let α vary within an open cone of directions which parametrizes all possible counterclockwise orbits emanating from x, and show that precisely 4 of these angles result in orbits which reach y after j reflections, making approximately one rotation.

For each α, there exists a unique λ such that the corresponding orbit is tangent to the caustic C_λ. For example, λ_x corresponds to α_x.

Definition B.1.1. Let U^*_x be the set of $\alpha \in S^*_x(\mathbb{R}^2)$ such that the corresponding orbits avoid the region between the focal points. Then, denote the homogeneous extension of U^*_x to $T^*_x\mathbb{R}^2$ by C^*_x, which is precisely the fiber cone at x of admissible initial covectors we consider.

Remark B.1.2. If x varies smoothly in Ω, so do the associated fibers and in this way, we obtain a smooth cone bundle, which we denote by $C^*(\Omega) \subset T^*(\Omega)$, over a tubular neighborhood of $\partial\Omega$.

B.1.2 Derivatives of β

If α increases, so does the parameter λ of the confocal caustic, i.e. $d\lambda/d\alpha > 0$ for $\alpha > \alpha_x$. Similarly, as α decreases, the parameter λ also increases, as can be seen by considering the backwards (clockwise) orbit. Hence, λ_x is a local minimum for α near α_x. By the implicit function theorem, it is clear that λ is a smooth function of α as long as the corresponding forwards and backwards orbits do not enter between the focal points. Let us first consider the case in which α increases. For $\alpha > \alpha_x$, there is a one to one correspondence between α and λ. In this case, it is geometrically clear that the angular component of x_1 is also increasing in α and $\partial x_1/\partial\alpha$ is bounded independently of j, λ, ω.

To obtain a large lower bound on the speed at which the two intersection points wind around C_{λ_y}, we first find a corresponding lower bound on

$$\frac{\partial}{\partial\lambda} B_j(x_1(\alpha(\lambda)), \omega_1(\alpha(\lambda))) = \frac{\partial B_j}{\partial x_1}\frac{\partial x_1}{\partial\alpha}\frac{\partial\alpha}{\partial\lambda} + \frac{\partial B_j}{\partial\omega}\frac{\partial\omega}{\partial\alpha}\frac{\partial\alpha}{\partial\lambda}. \tag{B.1}$$

By a slight abuse of notation, we have systematically confused x_1 with its angular component $\theta(x_1)$, since they are in one to one correspondence modulo factors of 2π. We begin with a simple lemma which will be used throughout the section.

Lemma B.1.3. *Let $x, y \in \Omega$ be $O(1/j)$ close to the diagonal of the boundary and γ be an orbit of j reflections which is tangent to the caustic C_λ and connects x to y. Also denote by ω_k $(1 \leq k \leq j)$ the angle of reflection made at the kth impact point on the boundary. Then $\lambda = O(1/j)$ and $\omega_k = O(1/j)$ for all $1 \leq k \leq j$.*

Proof. Recall that by γ making approximately one rotation we mean that $|\widehat{\beta}(\widehat{x}, \omega_1) - \widehat{y} - \ell| < \ell/100$ (see the remarks following Theorem 3.3.1). If in action angle coordinates (cf. Section 7.2), $\widehat{x}$ is given by $q_\lambda(s) = (-a\mathrm{sn}(s, k_\lambda), b\mathrm{cn}(s, k_\lambda))$ for some $s \in \mathbb{R}$, we have that the link

$$q_\lambda(s+\delta_\lambda) - q_\lambda(s) = (-a\mathrm{sn}(s+\delta_\lambda, k_\lambda), b\mathrm{cn}(s+\delta_\lambda, k_\lambda)) - (-a\mathrm{sn}(s, k_\lambda), b\mathrm{cn}(s, k_\lambda)),$$

is tangent to the confocal ellipse C_λ. Lifting to the universal cover $\mathbb{R} \times (0, \pi)$ as in Section 3 and setting $\widetilde{s} \in \mathbb{R}/\ell\mathbb{Z}$ to be the arclength parameter corresponding to s, we have $\pi_1\widehat{\beta}(\widetilde{s}, \omega_1) =$

$\widetilde{s} + \delta_\lambda$ where π_1 is projection onto the first factor. Similarly,

$$\pi_1 \widehat{\beta}^j(\widetilde{s}, \omega_1) = \widetilde{s} + j\delta_\lambda. \tag{B.2}$$

If y is given by $(-a\mathrm{sn}(t, k_\lambda), b\mathrm{cn}(t, k_\lambda))$ in action angle coordinates and $\widetilde{t}$ is the associated arclength parameter, then x being $O(1/j)$ close to y and the boundary is equivalent to requiring $|\widetilde{s} - \widetilde{t}| = O(1/j)$. Then (B.2) implies that

$$|\pi_1 \widehat{\beta}^j(\widetilde{s}, \omega_1) - \widehat{y} - \ell| = |\widetilde{s} + j\delta_\lambda - \widetilde{t} - \ell + O(1/j)| \leq \ell/100. \tag{B.3}$$

Hence, $\delta_\lambda = \ell/j + O(j^{-2})$. Note that

$$2 \arcsin \lambda/b \leq \delta_\lambda = 2 \int_0^{\arcsin(\lambda/b)} \frac{d\tau}{\sqrt{1 - k_\lambda^2 \sin^2 \tau}} \leq 2\sqrt{1 - \lambda^2 k_\lambda^2/b} \arcsin \lambda/b. \tag{B.4}$$

As $\sqrt{1 - \lambda^2 k_\lambda^2/b} = 1 + O(\lambda)$ and $\arcsin \lambda/b = \lambda/b + O(\lambda^2)$, $\delta_\lambda = O(1/j)$ immediately implies that $\lambda = O(1/j)$. The relationship between ω_1 and λ is given by Lemma 7.2.3: $\lambda^2 = \sin^2 \omega_1 (b^2 + (a^2 - b^2) \sin^2 \widehat{x})$. As the coefficient of $\sin^2 \omega_1$ is bounded above by a and below by b, λ and ω_1 are of the same order near zero. Furthermore, each link of γ is tangent to C_λ for a fixed $\lambda = O(1/j)$ and hence, the same logic also implies that $\omega_k = O(1/j)$ for each $1 \leq k \leq j$. This concludes the proof of the lemma. $\square$

We are now ready to estimate (B.1).

Lemma B.1.4. *For j large, we have*

$$\frac{\partial}{\partial \lambda} B_j(x_1(\alpha(\lambda)), \omega_1(\alpha(\lambda))) \gtrsim j$$

Proof. In Section 5.5, we found a rather explicit expression for

$$\frac{\partial B_j}{\partial \omega} = \frac{\partial \text{am}}{\partial s}\left(\frac{\partial t_\varphi}{\partial \lambda^2}\frac{\partial \lambda^2}{\partial \omega} + (j+1)\frac{d\delta_\lambda}{d\lambda^2}\frac{\partial \lambda^2}{\partial \omega}\right) + \frac{\partial \text{am}}{\partial k_\lambda^2}\frac{dk_\lambda^2}{d\lambda^2}\frac{\partial \lambda^2}{\partial \omega}, \tag{B.5}$$

where $x_1 \in \partial\Omega$ is fixed and ω is evaluated at the critical angle $\omega(x_1)$ corresponding to a periodic orbit. Using the formulas in Section 5.5, we can actually let x_1 vary in α and evaluate at any ω; in particular, we can evaluate at $\omega = \omega_1$ corresponding to the angle of reflection at the first impact point on the boundary. Let us first examine the term in (B.5) with a coefficient of $(j+1)$. We see that

$$\frac{\partial \text{am}}{\partial s} = \sqrt{1 - k_\lambda^2 \text{sn}^2(t_\varphi + (j+1)\delta_\lambda; k_\lambda)}, \tag{B.6}$$

$$\frac{d\delta_\lambda}{d\lambda^2} = \frac{1}{b\lambda\sqrt{1-\frac{k_\lambda^2\lambda^2}{b^2}}}\frac{1}{\sqrt{1-\lambda^2/b^2}} + \frac{2k_\lambda^2}{a^2-\lambda^2}\int_0^{\arcsin(\lambda/b)} \frac{\sin^2\tau d\tau}{(1-k_\lambda^2\sin^2\tau)^{3/2}}, \tag{B.7}$$

$$\frac{\partial \lambda^2}{\partial \omega} = 2\sin\omega\cos\omega(b^2 + (a^2-b^2)\sin^2(\varphi)). \tag{B.8}$$

If j is sufficiently large, we can ensure that $\lambda = O(1/j)$ is in turn small. Hence, (B.6) can bounded below by $(1-b^2/a^2)/2$ and recalling Lemma 7.2.3, the product of $1/\lambda$ and $\sin\omega$ coming from equations (B.7) and (B.8) can be estimated below by $1/a$. For $\lambda \ll 1$, all of the remaining terms can easily be bounded below by a positive constant depending only on a and b. Hence, we have

$$(j+1)\frac{\partial \text{am}}{\partial s}\frac{d\delta_\lambda}{d\lambda^2}\frac{\partial \lambda^2}{\partial \omega} \gtrsim j.$$

We now consider the first term $\frac{\partial \text{am}}{\partial s}\frac{\partial t_\varphi}{\partial \lambda^2}\frac{\partial \lambda^2}{\partial \omega}$ in (B.5). The first and third factors of this product are clearly bounded above and below, independently of ω, λ, and j near $\lambda = 0$ by the same arguments as before. Recall from Section 5.5 that

$$\frac{\partial t_\varphi}{\partial \lambda^2} = \frac{k_\lambda^2}{(a^2-\lambda^2)}\int_0^{\text{am}(t_\varphi;k_\lambda)} \frac{\sin^2\tau d\tau}{(1-k_\lambda^2\sin^2\tau)^{3/2}},$$

which is also clearly bounded in magnitude by a positive constant independent of ω, λ and

j.

In a similar manner, we would like to estimate the final term $\frac{\partial \mathrm{am}}{\partial k_\lambda^2}\frac{dk_\lambda^2}{d\lambda^2}\frac{\partial \lambda^2}{\partial \omega}$ in (B.5). Recall from Section 5.5 that

$$\frac{\partial \mathrm{am}}{\partial k_\lambda^2} = -\sqrt{1-k_\lambda^2 \mathrm{sn}^2(s;k_\lambda)}\int_0^{\mathrm{am}(s;k_\lambda)} \frac{\sin^2\tau d\tau}{(1-k_\lambda^2\sin^2\tau)^{3/2}},$$
$$\frac{dk_\lambda^2}{d\lambda^2} = \frac{k_\lambda^2}{(a^2-\lambda^2)}.$$

Both terms can be bounded independently of ω, λ and j. Combining this with the earlier bound for (B.8), we see that

$$\frac{\partial B_j}{\partial \omega} \gtrsim j. \tag{B.9}$$

We also need to estimate the remaining terms in (B.1). In particular, we must bound

$$\frac{\partial \omega}{\partial \alpha}\frac{\partial \alpha}{\partial \lambda} = \frac{\partial \omega}{\partial \lambda}$$

from below by a positive constant and

$$\frac{\partial B_j}{\partial x_1}\frac{\partial x_1}{\partial \alpha}\frac{\partial \alpha}{\partial \lambda} = \frac{\partial B_j}{\partial x_1}\frac{\partial x_1}{\partial \lambda}$$

in magnitude. Lemma 7.2.3 tells us that if $0 \leq \omega \leq \pi/3$, then

$$\frac{\partial \lambda}{\partial \omega} = \frac{1}{\cos\omega}\frac{1}{\sqrt{b^2+(a^2-b^2)\sin^2\varphi}} \leq \frac{2}{b},$$

so that

$$\frac{\partial \omega}{\partial \lambda} \geq \frac{b}{2}. \tag{B.10}$$

It is also geometrically clear that

$$\left|\frac{\partial x_1}{\partial \alpha}\frac{\partial \alpha}{\partial \lambda}\right| = \left|\frac{\partial x_1}{\partial \lambda}\right| \lesssim 1. \tag{B.11}$$

By writing out B_j in action angle coordinates, we see that

$$\frac{\partial B_j}{\partial x_1} = \frac{\partial \mathrm{am}}{\partial s}(t_\varphi + (j+1)\delta_\lambda; k_\lambda)\frac{\partial t_\varphi}{\partial \varphi}, \tag{B.12}$$

where t_φ is defined implicitly by the equation $\varphi = \mathrm{am}(t_\varphi; k_\lambda) + \pi/2$. Hence, by the implicit function theorem and our previous bound for (B.6), $\frac{\partial B_j}{\partial x_1}$ is both nonnegative and bounded independently of ω, λ and j. Combining (B.9), (B.10), (B.11) and (B.12), we see that

$$\frac{\partial B_j}{\partial \lambda}(x_1(\alpha(\lambda)), \omega(\lambda)) \gtrsim j.$$

□

B.1.3 Intersection points

We are now ready to prove that the intersection points with the last link wind monotonically around the caustic C_{λ_y} on which y lies as we increase λ. In tandem with Lemma B.1.4, we will also need information about how the final angle ω_j at the jth reflection depends on λ.

Lemma B.1.5. *If $\omega_j = \pi_2\beta^j(x_1, \omega_1)$ denotes the angle of reflection at the jth impact point on the boundary, then $\partial\omega_j/\partial\lambda = O(1)$, i.e. it's derivative bounded independently of j.*

Proof. Recall Lemma 7.2.3, which gave

$$\lambda^2 = \sin^2\omega_j(b^2 + (a^2 - b^2)\sin^2 x_j).$$

Differentiating this as in the proof of Lemma B.1.4 but now using that x_j depends on λ, we see that

$$2\lambda = 2\frac{\partial \omega_j}{\partial \lambda}\sin\omega_j\cos\omega_j(b^2+(a^2-b^2)\sin^2 x_j) + 2\sin^2\omega_j(a^2-b^2)\sin x_j\cos x_j\frac{\partial x_j}{\partial \lambda}. \tag{B.13}$$

Substituting again the formula in Lemma 7.2.3 into equation (B.13), we find that

$$\frac{\partial \omega_j}{\partial \lambda} = \frac{(b^2+(a^2-b^2)\sin^2 x_j)^{1/2} - \sin\omega_j(a^2-b^2)\sin x_j\cos x_j\frac{\partial x_j}{\partial \lambda}}{(b^2+(a^2-b^2)\sin^2 x_j)\cos\omega_j}. \tag{B.14}$$

Lemma B.1.3 tells us that $\lambda, \omega_j = O(1/j)$ and hence $\cos\omega_j = 1 + O(1/j), \sin\omega_j = O(1/j)$. This implies that the denominator of (B.14) is bounded below by a positive constant for j larger than some fixed j_0 depending only on Ω. The only unbounded term in the numerator is $\partial x_j/\partial\lambda = \partial B_j/\partial\lambda \gtrsim j$, as was shown in Lemma B.1.4. However, an examination of the proof of Lemma B.1.4 actually shows that $\partial x_j/\partial\lambda = O(1/j)$ and hence, $\sin\omega_j\frac{\partial x_j}{\partial\lambda} = O(1)$, which implies that (B.14) is in fact bounded. $\square$

With Lemmas B.1.3, B.1.4 and B.1.5, we can prove half of Theorem 3.3.1:

Lemma B.1.6. *For j sufficiently large, the angular components in elliptical polar coordinates of both intersection points φ_j^1 and φ_j^2 of the final link with the caustic C_{λ_y} are monotonically increasing in λ with approximate speed j.*

Proof. Recall from the proof of Lemma 7.2.3 that the billiard ray emanating from x_j at angle ω_j is parametrized by

$$L(t) = (a\cos x_j + ib\sin x_j) + te^{i\omega_j}(-a\sin x_j + ib\cos x_j). \tag{B.15}$$

Taking real and imaginary parts, we find that

$$\operatorname{Re} L_{\omega_j}(t) = a\cos x_j + t(-a\sin x_j\cos\omega_j - b\cos x_j\sin\omega_j),$$
$$\operatorname{Im} L_{\alpha_j}(t) = b\sin x_j + t(b\cos x_j\cos\omega_j - a\sin x_j\sin\omega_j).$$

For simplicity, denote by A the coefficient of t in (B.15). Converting the line L from parametric form, we find that

$$\frac{z - a\cos x_j}{-a\sin x_j\cos\omega_j - b\cos x_j\sin\omega_j} = \frac{w - b\sin x_j}{b\cos x_j\cos\omega_j - a\sin x_j\sin\omega_j}. \tag{B.16}$$

We know that L has exactly two intersection points with

$$C_{\lambda_y} = \left\{(z,w)\in\mathbb{R}^2 : \frac{z^2}{a^2-\lambda_y^2} + \frac{w^2}{b^2-\lambda_y^2} = 1\right\},$$

the caustic on which y lies. These correspond to two different values of t in equation (B.15). At either intersection point, the values of x, y in (B.16) are constrained to be of the form $(c\cosh\mu_j\cos\varphi_j, c\sinh\mu_j\sin\varphi_j)$, where $c = \sqrt{a^2-b^2}$, $\mu_j = \cosh^{-1}((a^2-\lambda_y^2)^{1/2}/c)$ and $\varphi_j \in [0, 2\pi)$ is the angular parameter in elliptical polar coordinates for the caustic C_{λ_y}. For simplicity, denote

$$c_j = c\cosh\mu_j = (a^2-\lambda_y^2)^{1/2}, \quad s_j = c\sinh\mu_j = (b^2-\lambda_y^2).$$

To begin, assume that φ_j is either the first intersection point or the second intersection point. Only at the end of the proof will we need to distinguish between the two cases. We want to show precisely that $\varphi_j' = \frac{\partial}{\partial\lambda}\varphi_j > 0$ so that the intersection points wind around C_{λ_y}. Solving for w in equation (B.16) and substituting $z = c_j\cos\varphi_j, w = s_j\sin\varphi_j$, we see that

$$s_j\sin\varphi_j = (c_j\cos\varphi_j - a\cos x_j)\frac{\operatorname{Im} A}{\operatorname{Re} A} + b\sin x_j. \tag{B.17}$$

Differentiating (B.17) and collecting terms, we have

$$D_1\varphi_j' = D_2 x_j' + D_3\omega_j', \tag{B.18}$$

where

$$D_1 = (s_j \cos\varphi_j \operatorname{Re} A + c_j \sin\varphi_j \operatorname{Im} A)$$
$$D_2 = (-s_j \sin\varphi_j f + a \sin x_j \operatorname{Im} A + c_j \cos\varphi_j g - a\cos x_j g + b\cos x_j \operatorname{Re} A + b\sin x_j f),$$
$$D_3 = (-s_j \sin\varphi_j \operatorname{Im} A + c_j \cos\varphi_j \operatorname{Re} A - a\cos x_j \operatorname{Re} A + b\sin x_j \operatorname{Im} A),$$

and

$$f = (-a\cos\omega_j \cos x_j + b\sin\omega_j \sin x_j),$$
$$g = (-a\sin\omega_j \cos x_j - b\cos\omega_j \sin x_j).$$

As both intersection points are within $O(1/j)$ of x_j, we have also that $|x_j - \varphi_j| = O(1/j)$ and $D_3 = O(1/j)$. However, setting $\omega_j = 0$, $\varphi_j = x_j$, $c_j = a$, and $s_j = b$, we see that D_1, D_2 and D_3 all vanish. Instead we Taylor expand each coefficient to second order in $1/j$. Recall that we are free to choose y as close to the boundary as we want, so $\lambda_y = O(j^{-N})$, $c_j = a + O(j^{-N})$, and $s_j = b + O(j^{-N})$ for any $N \in \mathbb{N}$. Expanding D_1 and simplifying, we see that

$$D_1 = ab\cos\omega_j \sin(\varphi_j - x_j) - \sin\omega_j(a^2\sin^2 x_j + b^2\cos^2 x_j) + O(j^{-2})). \tag{B.19}$$

Similarly, we obtain

$$D_2 = ab\cos\omega_j \sin(\varphi_j - x_j) - \sin\omega_j(a^2\sin^2 x_j + b^2\cos^2 x_j) + O(j^{-2}). \tag{B.20}$$

and

$$D_3 = a^2 \sin x_j(\cos x_j - \cos\varphi_j) + b^2 \cos x_j(\sin x_j - \sin\varphi_j). \tag{B.21}$$

Hence, $D_1 = D_2 + O(j^{-2})$ and $D_3 = O(1/j)$. As $x_j' \gtrsim j$ by Lemma B.1.4, we are done as long as we can show that D_1, D_2 don't vanish to first order in $1/j$ so that we may divide through by them in equation (B.18). If φ_j is the first intersection point, then $\varphi_j = x_j + O(j^{-N})$ for any N so modulo $O(j^{-2})$ terms, D_1 becomes $-\omega_j(a^2 \sin^2 x_j\varphi + b^2\cos^2 x_j) \leq -c\omega_j \leq -c'/j$ for some constants $c, c' > 0$ depending only on the curvature of $\partial\Omega$. Hence, we may assume φ_j is the second intersection point, in which case choosing y sufficiently close to $\partial\Omega$ amounts to setting $\varphi_j = x_{j+1} + O(j^{-N})$ for any desired N.

Simplifying further, we see that modulo terms of order $O(j^{-2})$,

$$D_1 = ab(x_{j+1} - x_j) - \omega_j(a^2\sin^2 x_j + b^2\cos^2 x_j). \tag{B.22}$$

We want to show that the positivity of the first term in (B.22) outweighs the second term. Recall from Section 7.2 that $x_{j+1} - x_j = \mathrm{am}(s + \delta_\lambda, k_\lambda) - \mathrm{am}(s, k_\lambda)$ for some $s \in \mathbb{R}$ and λ (not λ_y) corresponding to the caustic to which the orbit is tangent. Hence,

$$(\mathrm{am}(s+\delta_\lambda, k_\lambda) - \mathrm{am}(s, k_\lambda))\frac{1}{\sqrt{1-\lambda^2 k_\lambda^2/b}} \geq \int_{\mathrm{am}(s,k_\lambda)}^{\mathrm{am}(s+\delta_\lambda,k_\lambda)} \frac{d\tau}{\sqrt{1-k_\lambda^2\sin^2\tau}} = \delta_\lambda, \tag{B.23}$$

which implies that

$$\begin{aligned}(x_{j+1} - x_j) &\geq \delta_\lambda\sqrt{1-\lambda^2k_\lambda^2/b} = 2\sqrt{1-\lambda^2k_\lambda^2/b}\int_0^{\arcsin(\lambda/b)} \frac{d\tau}{\sqrt{1-k_\lambda^2\sin^2\tau}} \\ &\geq 2\sqrt{1-\lambda^2k_\lambda^2/b}\arcsin\lambda/b.\end{aligned} \tag{B.24}$$

As $\lambda = O(1/j)$ for j large, $\arcsin\lambda/b = \lambda/b + O(j^{-2})$ and $(1-\lambda^2k_\lambda^2/b)^{1/2} = 1 + O(\lambda) =$

$1 + O(1/j)$. Hence, modulo terms of order $O(j^{-2})$, we have

$$ab(x_{j+1} - x_j) \geq 2a\lambda. \tag{B.25}$$

Now recall Lemma 7.2.3, which gives

$$\lambda^2 = \sin^2(\omega_j)\left(b^2 + (a^2 - b^2)\sin^2(x_j)\right).$$

Near $\omega_j = 0, \lambda = 0$, this tells us that

$$\omega_j = \lambda\left(b^2 + (a^2 - b^2)\sin^2(x_j)\right)^{-1/2} + O(j^{-2}).$$

The term in D_1 with a $\sin\omega_j$ can hence be estimated modulo $O(j^{-2})$ by

$$\omega_j(a^2\sin^2 x_j + b^2\cos^2 x_j) = \lambda\frac{a^2\sin^2 x_j + b^2\cos^2 x_j}{\sqrt{b^2 + (a^2 - b^2)\sin^2 x_j}} + O(j^{-2}). \tag{B.26}$$

The coefficient of λ in (B.26) is positive and maximized at critical points corresponding to $\sin x_j = 0, \cos x_j = 0$ or

$$b^2\cos^2 x_j = (a^2 - 2b^2)\sin^2 x_j + 2b^2. \tag{B.27}$$

If $\sin x_j = 0$, then $\alpha_j = \lambda/b + O(j^{-2})$ and the leading order coefficient of ω_j in the formula for D_1 is b^2. Hence, (B.25) implies that

$$ab(x_{j+1} - x_j) \geq 2a\lambda > b\lambda = b^2\omega_j$$

and $D_1 > 0$ is nonvanishing. If $\cos x_j = 0$, then $\omega_j = \lambda/a + O(j^{-2})$ and the leading order coefficient of ω_j in the formula for D_1 is a^2. Hence, (B.25) again implies that

$$ab(x_{j+1} - x_j) \geq 2a\lambda > a\lambda = a^2\omega_j$$

and $D_1 > 0$ is nonvanishing. In the third case, solving (B.27) results in

$$(a^2 - b^2)\sin^2 x_j + b^2 = 0,$$

which impossible unless $b^2 = 0$. □

In this way, we obtain two orbits from x to y by letting α increase within the $\alpha \geq \alpha_x$ regime. Dynamically, these orbits can be characterized by having a point of tangency to a confocal ellipse before a moment of reflection at the boundary in the forwards direction.

We now consider the regime in which $\alpha \leq \alpha_x$. There is a different one to one correspondence between α and λ but many of the equations above remain completely valid. In particular (B.9) and (B.10) are unchanged. Hence, by essentially the same bounds as in Lemma B.1.4 before, if $\varphi_j^{(1)}$ and $\varphi_j^{(2)}$ denote the angular components of the intersection points in elliptical polar coordinates, we have

$$\frac{\partial \varphi_j^{(1)}}{\partial \lambda} \gtrsim j, \qquad \frac{\partial \varphi_j^{(2)}}{\partial \lambda} \gtrsim j$$

in the $\alpha \leq \alpha_x$ regime. This provides two additional orbits, which are dynamically characterized by having a reflection at the boundary before becoming tangent to a confocal ellipse.

To obtain the four *clockwise* orbits, note that we can first apply the isometry $\mathbb{R}^2 \ni (z, w) \mapsto (-z, w)$, obtain four counterclockwise orbits as above, and then reflect back. In the last link of any orbit, the first point of intersection with C_{λ_y} is reached before a point of tangency with a confocal ellipse while the second intersection point is reached after a point of tangency. These characterizations are important in Section 5.5 for understanding four different types of orbit configurations and determining which types of limiting orbits give rise to periodic orbits of precisely j reflections as (x, y) approach the diagonal of the boundary.

Remark B.1.7. For sufficiently large j and x, y both lying *on the boundary* near the diagonal, the existence of a single such geodesic for general smooth, strictly convex domains was proven in [21], [44] and [60]. The eight orbits in the statement of Theorem 3.3.1 can be seen to collapse into the orbits described in [44] as (x, y) approach the diagonal of the boundary from any direction. However, there may be a different number of reflections in the limiting orbit (see the proof of Lemma 6.2.1).