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ABSTRACT

We discuss the s-channel unitarity constraint of the multiperipheral bootstrap models. We show that, if the low subenergy part of the kernel is weak, it is possible to have self-consistent Regge poles. On the other hand, if the low subenergy part of the kernel is strong, we can never have self-consistent Regge poles. In this case the most important J-plane singularity is a cut with branch point at 1 and the π-π scattering amplitude goes to infinity slower than $s/\log s$ at high energy.

I. INTRODUCTION

It is a well-known fact that although the general idea of the singularities in the complex angular momentum plane is very useful in the description of high-energy phenomena, the simple pure Regge-pole models fail in many respects,[1] e.g., the polarization effects. In terms of the singularities in the complex angular momentum plane, these may be interpreted as being due to the presence of cuts. Many cut models have been proposed.[1] In most of them a "Born term" was chosen and the s-channel unitarity condition was imposed. Although the usual practice is to choose pure Regge poles as "Born term," the main criticism for such models is the choice of "Born term."

The multiperipheral models[2,3] (MPM) are very useful for describing general features of high-energy collisions. The models predict Regge behavior for elastic and total cross sections, a constant elasticity, a $\log s$ behavior for multiplicity, and a small average transverse momentum for the secondary particles produced in high-energy collisions. The MPM, in its simplest form, regards the kernel of the integral equation as given, not constraint by the self-consistency condition. In this version the output Regge pole can have intercept greater than 1 when the strength of the input kernel is very large. This violates the Froissart bound[4] on the total cross section at high energy. A more refined version of the MPM imposes the self-consistency condition.[5] It has been shown by Chew and Snider[6] that in this version the s-channel unitarity constraint is built in. In other words, if a solution is found to a multiperipheral bootstrap equation (MPBE), it is no longer necessary to consider s-channel absorption as a subsequent correction.

The purpose of this paper is to explore the s-channel unitarity constraint of multiperipheral bootstrap models (MPBM). In particular, we show explicitly from the eigenvalue equation of the integral equation that we can have either a leading Regge pole with intercept $\alpha(0) < 1$, or a cut with branch point $\alpha(0) = 1$. Therefore the MPBM can provide a consistent framework to discuss the Regge cut. We also find that, if the input low subenergy kernel is weak, we have $\alpha(0) < 1$. But if the input low subenergy kernel is strong we will have a Regge cut with branch point at $\alpha(0) = 1$ as the leading singularity, a connection which has been shown in ϕ^3 type field theory by Chang and Yan.[7]

In Sec. II, we first present the argument of Chew and Snider.[6] We then restrict ourselves to the forward scattering problem. The eigenvalue equation for the MPBE is written down explicitly using the trace approximation. Section III discusses the existence of solution, its relation to the strength of the low-energy kernel, and the nature of discontinuity across the cut when the leading singularity is a cut in complex angular momentum plane. We make some relevant remarks in Sec. IV.

II. MULTIPERIPHERAL BOOTSTRAP MODEL AND S-CHANNEL UNITARITY

We consider here the oldest of MPM--that of Amati-Bertocchi-Fubini-Stanghellini-Tonin[2] (ABFST). According to ABFST the absorptive part $F(p,q;\, k)$ of the π-π scattering amplitude satisfies the integral equation

$$F(p,q;\, k) = F_1(p,q;\, k) + \frac{1}{8\pi^4} \int \frac{d^4q'\; F_1(p,-q';\, k)\, F(q',q;\, k)}{[(q'+k)^2 - m_\pi^2][(q'-k)^2 - m_\pi^2]}. \tag{2.1}$$

The integral equation, together with the kinematics, is represented graphically in Fig. 1.

It was assumed in the original ABFST model that F_1 is the low-energy (resonant) part of the π-π elastic amplitude. The modern version[8] of the MPM recognizes the importance of dividing the kernel F_1 into two pieces, one corresponding to the low subenergy part, the other corresponding to the high subenergy "tail." The important concepts of diffractive dissociation and multiperipheralism have been unified in this way.[9] It was also realized that it is fruitful to impose the bootstrap condition on the Pomeranchuk trajectory by requiring the input pole used in the high subenergy part of the kernel to be consistent with the leading output pole.[5] Such kinds of MPM are designated as multiperipheral bootstrap models (MPBM). This has been used to explain the schizophrenic behavior of the Pomeranchuk trajectory.[5]

In the original ABFST model it is clear that the output pole can have intercept greater than 1, if the low subenergy kernel is strong enough. This indicates that in the original version, the s-channel unitarity constraint has not been taken into account. Chew and Snider[6] showed that if the kernel is separated into a low subenergy part plus a high subenergy tail and by requiring the bootstrap condition on the Pomeranchuk trajectory, the s-channel unitarity constraint is built into the model. Therefore, once the solution of the multiperipheral bootstrap equation (MPBE) has been found (if a solution exists), the subsequent s-channel absorption correction will no longer be necessary.

We reproduce here the argument of Chew and Snider.[6] The function F_1 is assumed to have the form (see Fig. 2)

$$F_1(s,t) = F_1^R(s,t)\,\theta(s^* - s) + \frac{1}{16\pi^2 \lambda^{\frac{1}{2}}(s,m_\pi^2,m_\pi^2)} \iint \frac{d\tau_1 d\tau_2\, \theta\left\{-\lambda(t,\tau_1,\tau_2) + \frac{t\tau_1\tau_2}{\frac{s}{4} - m_\pi^2}\right\}}{\left\{-\lambda(t,\tau_1,\tau_2) + \frac{t\tau_1\tau_2}{\frac{s}{4} - m_\pi^2}\right\}^{\frac{1}{2}}} \times T^*(s,\tau_2)\,T(s,\tau_1)\,\theta(s - s^*), \qquad (2.2)$$

where T is the π-π scattering amplitude and

$$\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc\,. \qquad (2.3)$$

When we do the s-channel partial-wave projection of Eq. (2.1), both terms on the right-hand side will give rise to positive contributions. It is also clear that at the high-energy limit F_1 is projected to $|f_\ell|^2$ because the first term in Eq. (2.2) vanishes in this limit. From Eq. (2.1), we find

$$\mathrm{Im}\, f_\ell = |f_\ell|^2 + R_\ell\,, \qquad (2.4)$$

where $R_\ell \geq 0$ is the projection of the second term of Eq. (2.1) and corresponds to the multiparticle states. Here our normalization is

$$f_\ell \equiv \frac{S_\ell - 1}{2i}\,. \qquad (2.5)$$

Evidently the s-channel unitarity constraint is satisfied in the model. This, together with the analytic property of F_1 in t, is sufficient to establish the unitarity bound of Froissart on the total cross section in the high s limit.

We show in the following that this unitarity constraint on the total cross section can also be obtained from the eigenvalue equation of the MPBE. To establish this it suffices to restrict ourselves to the forward scattering case. For simplicity we neglect the complication due to the isospin property. We use the trace approximation to solve the integral equation. We will comment on this approximation in Sec. IV.

From now on our notation follows closely that of CRS.[8] We begin with the diagonalized integral equation for forward scattering,

$$F^\lambda(\tau,\tau') = F_1^\lambda(\tau,\tau') + \int_{-\infty}^{0} d\tau''\, F^\lambda(\tau,\tau'')\, K^\lambda(\tau'',\tau')\,, \qquad (2.6)$$

where $F^\lambda(\tau,\tau')$ is defined by

$$F^\lambda(\tau,\tau') \equiv \int_{4m_\pi^2}^{\infty} ds\, \frac{e^{-(\lambda+1)\eta(s,\tau,\tau')}}{\lambda + 1}\, F^+(s,\tau,\tau')\,, \qquad (2.7)$$

with

$$\cosh \eta(s,\tau,\tau') \equiv \frac{s - \tau - \tau'}{2(\tau\tau')^{\frac{1}{2}}} . \tag{2.8}$$

A similar formula defines the inhomogeneous term $F_1^{\lambda}(\tau,\tau')$. The functions $F^+(s,\tau,\tau')$ and $F_1(s,\tau,\tau')$ is normalized such that

$$\begin{aligned} F^+(s,-m_\pi^2,-m_\pi^2) &= \lambda^{\frac{1}{2}}(s,m_\pi^2,m_\pi^2)\, \sigma^{\pi\pi}_{tot}(s), \\ F_1(s,-m_\pi^2,-m_\pi^2) &= \lambda^{\frac{1}{2}}(s,m_\pi^2,m_\pi^2)\, \sigma^{\pi\pi}_{e\ell}(s). \end{aligned} \tag{2.9}$$

The projected kernel is given by

$$K^{\lambda}(\tau'',\tau') = \frac{1}{(\tau'' - m_\pi^2)^2} \int_{4m_\pi^2}^{\infty} ds\, C(s)\, \frac{e^{-(\lambda+1)\eta(s,\tau'',\tau')}}{\lambda + 1}, \tag{2.10}$$

where

$$C(s) = \lambda^{\frac{1}{2}}(s,m_\pi^2,m_\pi^2)\, \sigma^{\pi\pi}_{e\ell}(s). \tag{2.11}$$

The inversion formula of Eq. (2.7) is

$$2(\tau\tau')^{\frac{1}{2}} F^+(s,\tau,\tau') = \frac{1}{2\pi i} \int d\lambda(\lambda + 1)\, \frac{e^{(\lambda+1)\eta}}{\sinh \eta}\, F^{\lambda}(\tau,\tau'), \tag{2.12}$$

where the contour of integration is from $-i\infty$ to $i\infty$, and passing to the right of all the singularities of $F^{\lambda}(\tau,\tau')$ in the complex λ-plane.

The trace approximation amounts to replacing $D(\lambda)$ by $1 - \text{Trace } K^{\lambda}$, which has been evaluated by CRS explicitly. The result is

$$D(\lambda) = 1 - \frac{1}{\lambda(\lambda + 1)} \left\{ \frac{2}{\lambda + 2} R + \left(\frac{\Delta}{s_0}\right)^{\lambda} C\, e^{b(\lambda-\beta)} \times \int_{(\lambda-\beta)(b+\ell n\frac{s^*}{s_0})}^{\infty} \frac{e^{-x}}{x}\, dx \right\}, \tag{2.13}$$

where

$$\beta = 2\alpha^{in}(0) - 1 , \tag{2.14}$$

$$b = \frac{a}{2(\alpha^{in})'(0)} > 0, \qquad C > 0, \tag{2.15}$$

$$R = \frac{1}{16\pi^3} \int_{4m_\pi^2}^{s^*} \frac{ds}{s}\, \lambda^{\frac{1}{2}}(s,m_\pi^2,m_\pi^2)\, \sigma^{\pi\pi}_{e\ell}(s) > 0. \tag{2.16}$$

Here $\alpha^{in}(0)$ is the input Pomeranchuk intercept and a is the constant in the Regge residue $e^{a\tau}$, Δ is the cutoff in the τ integration and s_0 is some mass scale smaller than s^*. In CRS no attempt was made to incorporate the bootstrap condition. The requirement of self-consistency can be taken care of easily. We have only to make the replacement of $\alpha^{in}(0)$ in Eq. (2.14) by λ. Or in other words, we replace $\lambda - \beta$ by $\lambda - (2\lambda - 1) = 1 - \lambda$.

Therefore, the eigenvalue equation of the MPBE is, in the trace approximation,

$$D(\lambda) = 1 - \frac{1}{\lambda(\lambda + 1)} \left\{ \frac{2R}{\lambda + 2} + \left(\frac{\Delta}{s_0}\right)^{\lambda} C\, e^{b(1-\lambda)} \int_{(1-\lambda)(b+\ell n\frac{s^*}{s_0})}^{\infty} \times \frac{e^{-x}}{x}\, dx \right\} = 0. \tag{2.17}$$

III. EXISTENCE OF SOLUTION TO THE EIGENVALUE EQUATION

In this section we want to investigate under what condition can we have zero of $D(\lambda)$, i.e., self-consistent Regge pole. For this purpose, we concentrate on Eq. (2.17). Let us define α_0 $(\alpha_0 > 0)$ by

$$1 - \frac{2R}{\alpha_0(\alpha_0 + 1)(\alpha_0 + 2)} = 0. \tag{3.1}$$

One might call this α_0 the intercept of the "bare" Regge pole, since this is the intercept of the output pole if the high-energy tail is left out. From Eq. (3.1) and Eq. (2.16), it is clear that α_0 increases when the low subenergy part of the kernel increases. We separate our discussions into two different cases according to $\alpha_0 < 1$ or $\alpha_0 \geqslant 1$.

(i) $\alpha_0 < 1$. Since

$$1 - \frac{2R}{\lambda(\lambda + 1)(\lambda + 2)} > 0 \qquad \text{for } \alpha_0 < \lambda < 1,$$

it is possible to have solutions for $D(\lambda) = 0$ in $\alpha_0 < \lambda < 1$. This is the case when C is small (weak Pomeranchuk part). Notice that we cannot have a solution with $\lambda \geqslant 1$. This is due to the fact that when $\lambda \geqslant 1$ the second term inside the bracket in Eq. (2.17) will develop an imaginary part. The self-consistent solution in this case has been studied by Chew and Snider.[5]

(ii) $\alpha_0 \geqslant 1$. In this case no matter what the value of C is, there will be no solution of Eq. (2.17) at all. For in the region $0 < \lambda < 1$, we have

$$1 - \frac{2R}{\lambda(\lambda + 1)(\lambda + 2)} < 1 - \frac{2R}{\alpha_0(\alpha_0 + 1)(\alpha_0 + 2)} = 0$$

and

$$-\frac{1}{\lambda(\lambda + 1)}\left(\frac{\Delta}{s_0}\right)^{\lambda} C\, e^{b(1-\lambda)} \int_{(1-\lambda)(b+\ell n\frac{s^*}{s_0})}^{\infty} dx\, \frac{e^{-x}}{x} < 0.$$

We cannot have a solution with $\lambda \geqslant 1$ either, because in this region the second term inside the bracket of Eq. (2.17) will develop an imaginary part. Thus we have established the fact that it is impossible to have Regge pole as solution in the case $\alpha_0 \geqslant 1$. We conclude that the leading singularity has to be Regge cut. We can determine the branch point of the Regge cut and study the discontinuity of F^λ across the cut near the branch point. It is convenient for this purpose to define $\bar{F}^\lambda$ by

$$\bar{F}^\lambda(\tau,\tau') = \frac{1}{(\tau - m_\pi^2)} F^\lambda(\tau,\tau') \frac{1}{(\tau' - m_\pi^2)}. \tag{3.2}$$

Notice that the singularity structure in the λ-plane is the same for F^λ and $\bar{F}^\lambda$. Our integral equation can be written formally as

$$\bar{F}^\lambda = \bar{F}_1^{\,\lambda} + \bar{F}^\lambda \bar{F}_1^{\,\lambda},$$

or

$$\bar{F}^\lambda = \frac{\bar{F}_1^{\,\lambda}}{1 - \bar{F}_1^{\,\lambda}} = -1 + \frac{1}{1 - \bar{F}_1^{\,\lambda}}. \tag{3.3}$$

It is easy to see from Eq. (3.3) that the singularity structure of $\bar{F}^\lambda$ is the same as $1/(1 - \bar{F}_1^{\,\lambda})$ or that of $1/[D(\lambda)]$. Assuming the branch point of $\bar{F}^\lambda$ is at $\lambda = \lambda_0$, we can calculate the high subenergy part of $\bar{F}_1^{\,\lambda}$. It is readily seen that the function $\bar{F}_1^{\,\lambda}$ is analytic for Re $\lambda > \beta = 2\lambda_0 - 1$ and has a branch point at $\lambda = 2\lambda_0 - 1$. Also, we have just established that the singularity structure of $\bar{F}^\lambda$ is the same as $1/[D(\lambda)]$, which implies the coincidence of the branch point. That is,

$$\lambda_0 = 2\lambda_0 - 1, \qquad \text{or} \quad \lambda_0 = 1. \tag{3.4}$$

Thus in the case of $\alpha_0 \geqslant 1$ we have Regge cut with branch point at $\lambda = 1$ as the leading singularity in the complex angular momentum plane. As we have shown before, we cannot have $D(\lambda) = 0$. That implies that $\bar{F}^\lambda$ cannot be infinite as $\lambda \to 1$. Therefore, the discontinuity of $\bar{F}^\lambda$ across the cut should be finite at $\lambda = 1$. Starting with this result we can use the connection between the solution F and the high-energy tail in F_1 to argue that this discontinuity has to be zero at $\lambda = 1$. The reason is as follows. Since the discontinuity is finite as $\lambda \to 1$, we find that the high subenergy tail of $F_1{}^\lambda$ is of the form

$$\frac{1}{\lambda + 1} \int_{-\Delta}^{0} dt \, \frac{(-t)^{\lambda+1}}{(t - m_\pi{}^2)^2} \, C \int_{s^*}^{\infty} ds \, \frac{s^{-(\lambda+1)}\left(\frac{s}{s_0}\right)^\beta}{b + \ell n\left(\frac{s}{s_0}\right)} \, \frac{1}{\left|\ell n\left(\frac{s}{s_0}\right) + d\right|^2} . \tag{3.5}$$

The only difference between Eq. (3.5) and Eq. (VI.9) of CRS is the presence of the factor $1/|\ell n(s/s_0) + d|^2$, which is a reflection of the fact that now it is the cut and not the Regge pole that controls the high-energy behavior. It can be readily proved that

$$\int_{s^*}^{\infty} ds \, \frac{s^{-(\lambda+1)}\left(\frac{s}{s_0}\right)^\beta}{b + \ell n\left(\frac{s}{s_0}\right)} \, \frac{1}{\left|\ell n\left(\frac{s}{s_0}\right) + d\right|^2}$$

is analytic for Re $\lambda > \beta = 2\lambda_0 - 1 = 1$ and has a cut with branch point at $\lambda = \beta = 1$. We can calculate the discontinuity across the cut at $\lambda = 1$. The result is zero. This implies that the discontinuity of F^λ across the cut at $\lambda = 1$ is zero. Or, stated in terms of the high-energy behavior, F goes to infinity slower than s/log s as s approaches infinity.

Summarizing our results, we find that if the "bare" Regge pole has intercept $\alpha_0 < 1$, we may have self-consistent Regge pole as our solution. However, if the "bare" Regge pole has intercept $\alpha_0 \geqslant 1$, we can never have Regge pole as solution of MPBE. In this case (if the solution of MPBE exists) the leading singularity will be a cut with branch point at $\lambda = 1$. We have shown from the eigenvalue equation that the solution of MPBE respects the Froissart bound on the total cross section at high energy. We have also established the connection between the nature of the leading J-plane singularity and the strength of the low subenergy kernel as was first pointed out by Chang and Yan[7] in a ϕ^3 field theory.

IV. DISCUSSION

In this section we make some remarks concerning the generality of our results.

1. In CRS the signature factors have been ignored in the high subenergy tail of the kernel as well as the high-energy tail of the inhomogeneous term. These can be taken into account easily. The only change is replacing C by $C|i - \cot(\pi\lambda/2)|^2$.

2. Although we reach our conclusion starting from the ABFST pion-pole dominance model and using the trace approximation, the results we find are expected to be true for general MPM. In fact, we can reach exactly the same conclusion if we use the multi-Regge formalism developed by Ciafaloni, DeTar, and Misheloff.[10] In that case the eigenvalue equation of MPBE has been written down by Chew and Snider. The starting point in our investigation will then be the equation

$$1 - a - x_p = \epsilon_p \int_0^\infty d\xi \, \frac{e^{-\xi}}{x_p + \frac{b_p}{\gamma_p}\xi} \; . \qquad (4.1)$$

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FOOTNOTES AND REFERENCES

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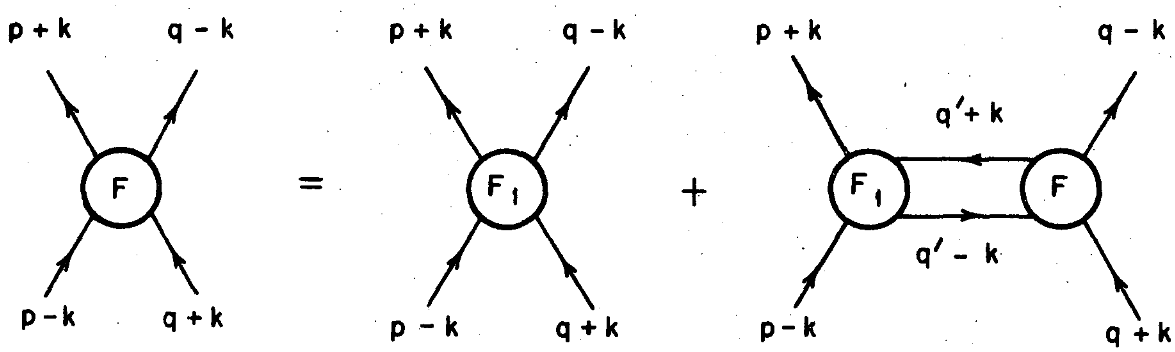
FIGURE CAPTIONS

Fig. 1. The kinematics of the multiperipheral integral equation.

Fig. 2. The splitting of the function F_1.



$t = 4k^2$,
$s = (p+q)^2$

Fig. 1

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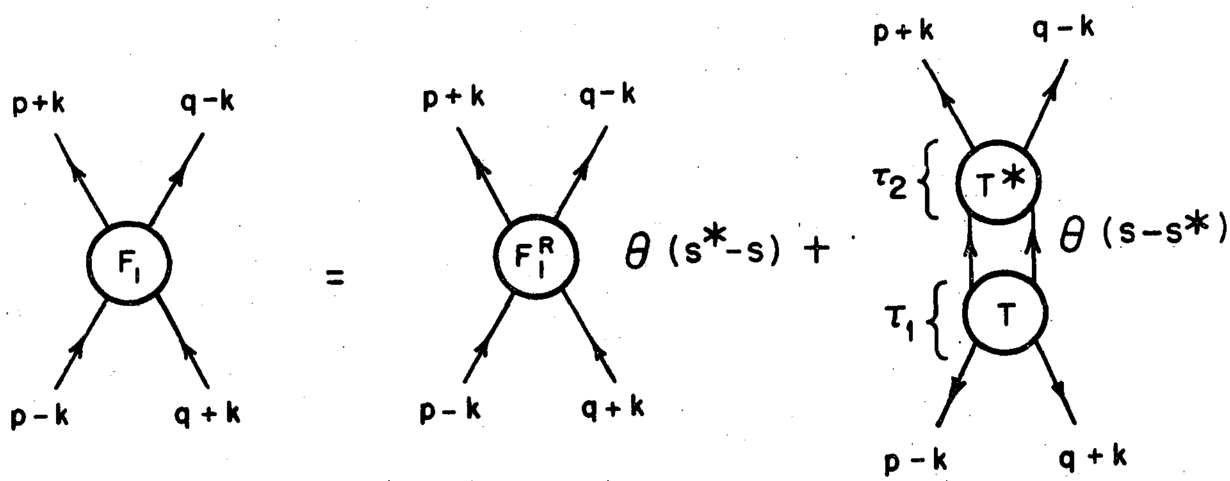

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Fig. 2