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Abstract

Convexity In Contact Geometry And Reeb Dynamics

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Doctor of Philosophy in Mathematics

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Reeb flows are a rich, ubiquitous class of dynamical systems arising in symplectic geometry, which include billiard systems, many-body orbital systems, geodesic flows and many Hamiltonian flows. Convexity hypotheses play an important, albeit mysterious, role in the study of these flows. In this thesis, we discuss several new results in the study of convexity in symplectic geometry and Reeb dynamics.

In Chapter 1, we resolve a longstanding open problem on the intrinsic characterization of Reeb flows arising from Hamiltonian flows on the convex boundaries. Namely, we prove that dynamically convex Reeb flows, introduced by Hofer-Wysocki-Zehnder, are not all convex. Our proof uses a novel relation between Riemannian geometry and Reeb dynamics, and uses constructions of Abbondandolo-Bramham-Hryniewicz-Salomão.

In Chapter 2, we describe a powerful new framework for computationally modelling Reeb dynamics on the boundaries of convex polytopes. We apply this framework to provide new evidence and examples relating to the Viterbo conjecture, a major open problem in Reeb dynamics and quantitative symplectic geometry.

In Chapter 3, we study convex toric domains and toric surfaces. A longstanding conjecture in toric geometry states that the Gromov width is monotonic under inclusion of moment polytopes of closed toric varieties. We use methods from toric geometry and ECH to prove a generalization of this conjecture in dimension 4.

Contents

Introduction

A contact manifold (Y, ξ) is an odd dimensional manifold equipped with a hyperplane field $\xi \subset TY$, called the contact structure, that is the kernel of a 1-form α such that

$$\ker(d\alpha) \subset TY \text{ is rank } 1 \qquad \text{and} \qquad \alpha|_{\ker(d\alpha)} > 0$$

A 1-form satisfying this condition is called a contact form on (Y, ξ). Every contact form comes equipped with a natural Reeb vector field R, defined by

$$\alpha(R) = 1 \qquad \iota_R d\alpha = 0$$

The study of the dynamical properties of Reeb vector fields (e.g. the existence of closed orbits and their properties) is a topic of immense interest in contemporary symplectic geometry and dynamical systems. Indeed, many dynamical systems arising in physics can be interpreted in terms of Reeb dynamics. These include billiard systems, planetary systems and geodesic flows.

There are several natural notions of convexity that arise in contact geometry. For example, the boundary of any convex domain X in $\mathbb{C}^n$ containing 0 is equipped with a natural contact form. Contact forms arising in this way will be refered to (in this thesis) as convex. Convexity plays a prominent (albeit mysterious) role in contact geometry, and there are several significant conjectures about the Reeb flows of convex contact forms.

Outline

In this thesis, I discuss several new results in the study of convexity in contact geometry and Reeb dynamics, which I obtained during my tenure as a PhD student in collaboration with Oliver Edtmair, Michael Hutchings and Ben Wormleighton.

In **Chapter 1**, we recount joint work with Edtmair [**14**]. In that work, we settle a longstanding open problem regarding intrinsic characterizations of contact forms arising on the boundary of convex domains. Namely, we prove that dynamically convex contact forms, which have many of the Floer-theoretic properties of contact manifolds, are not all convex. In the course of the proof, we use a relation between extrinsic curvature and rotation to prove a novel new estimate on the Ruelle invariant of the Reeb flows on convex boundaries.

In **Chapter 2**, we discuss joint work with Hutchings [**15**]. In that work, we provide a new computational framework for the study of Reeb dynamics on convex contact forms, by developing the theory of singular Reeb dynamics on the boundaries of convex polytopes in $\mathbb{R}^4$. We also present the results of numerous experiments performed with these tools, including new examples of convex polytopes with interesting systolic properties.

In **Chapter 3**, we shift to the study of a different notion of convexity (and concavity) arising in the study of toric symplectic geometry. In joint work with Wormleighton [**16**], we apply methods developed to study the embedded contact homology of convex and concave toric domains to prove a number of new results about embeddings of toric domains into closed toric surfaces. In particular, we prove that the Gromov width of a toric surface is monotonic with respect to inclusion of the moment polytope.

Personal Acknowledgements

I would first like to thank my advisor Michael Hutchings for his mentorship throughout my 6 years of graduate school at UC Berkeley. Michael's friendly and supportive approach to advising was a big help. His no-nonsense approach to math has helped to keep me grounded, although admittedly my head is still in the clouds much of the time. He also provided much needed help and advice during some difficult professional and personal moments.

I have worked with many collaborators during my time at UC Berkeley. Mihai Munteanu was my first, and our collaboration grew out of friendship. I have also had the pleasure of working with Jordan Cotler, Xinshan Cui, Oliver Edtmair, Ben Wormleighton and my advisor Michael. Our work together makes up the bulk of this thesis. I thank all of them, and look forward to our future work together.

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Finally, and most importantly, I would like to thank my parents Natalie and Hector, and my sister Chloe, for their support. They have always been proud of my abilities in math and science, and have provided me with every resource that I could have asked for in education and in life. My mom is extremely hard working and driven, and she inspires me to reach for ever greater professional heights. My dad and sister bring artistry and eccentricity to the family, two traits that I have always valued and emmulated.

Mathematical Acknowledgements

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CHAPTER 1

3D Convex Contact Forms And The Ruelle Invariant

1. Introduction

Contact manifolds arise naturally as hypersurfaces in symplectic manifolds satisfying a certain stability condition. In fact, Weinstein introduced contact manifolds in **[79]** inspired by the following prototypical example of this phenomenon, due to Rabinowitz **[66]**.

Example 1.1. We say that a domain $X \subset \mathbb{R}^{2n}$ with smooth boundary Y is *star-shaped* if

$$0 \in \mathrm{int}(X) \qquad \text{and} \qquad \partial_r \text{ is transverse to } Y$$

Let ω and Z denote the standard symplectic form and Liouville vector field on $\mathbb{R}^{2n}$. That is

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i \qquad Z = \frac{1}{2}\sum_i x_i\partial_{x_i} + y_i\partial_{y_i} = \frac{1}{2}r\partial_r$$

Then the restriction $\lambda|_Y$ of the Liouville 1-form $\lambda = \iota_Z\omega$ is a contact form.

Example 1.2. The *standard* contact structure ξ on $S^{2n-1} \subset \mathbb{R}^{2n}$ is given by $\xi = \ker(\lambda|_{S^{2n-1}})$.

Every contact form on the standard contact sphere arises as the pullback of $\lambda|_Y$ via a diffeomorphism to some star-shaped boundary Y. Moreover, every star-shaped boundary Y admits such a map from the sphere. Thus, from the perspective of contact geometry, the study of star-shaped boundaries is equivalent to the study of contact forms on the standard contact sphere.

1.1. Convexity. In this part of this thesis, we are primarily interested in studying contact forms arising as boundaries of convex domains.

Definition 1.3. A contact form α on S^{2n-1} is *convex* if there is a convex star-shaped domain $X \subset \mathbb{R}^{2n}$ with boundary Y and a strict contactomorphism $(S^3, \alpha) \simeq (Y, \lambda|_Y)$.

In contrast to the star-shaped case, not every contact form on S^{2n-1} is convex, and the Reeb flows of convex contact forms possess many special dynamical properties, both proven and conjectural.

In **[77]**, Viterbo proposed a particularly remarkable systolic inequality for Reeb flows on convex boundaries. To state it, let (Y, α) be a closed contact manifold with contact form of dimension $2n-1$, and recall that the volume $\mathrm{vol}(Y, \alpha)$ and systolic ratio $\mathrm{sys}(Y, \alpha)$ are given by

$$\mathrm{vol}(Y,\alpha) = \int_Y \alpha \wedge d\alpha^{n-1} \qquad \text{and} \qquad \mathrm{sys}(Y,\alpha) = \frac{\min\{\text{period } T \text{ of an orbit}\}^n}{(n-1)!\,\mathrm{vol}(Y,\alpha)} \tag{1.1}$$

The weak Viterbo conjecture that originally appeared in **[77]** can be stated as follows.

Conjecture 1.4. *[77] Let α be a convex contact form on S^{2n-1}. Then the systolic ratio is bounded by* 1.

$$\mathrm{sys}(S^{2n-1}, \alpha) \leqslant 1$$

There is also a strong Viterbo conjecture (c.f. [**34**]), stating that all normalized symplectic capacities are equal on convex domains. For other special properties of convex domains, see [**37**,**77**].

Despite the plethora of distinctive properties that convex contact forms possess, a characterization of convexity entirely in terms of contact geometry has remained elusive.

Problem 1.5. Give an intrinsic characterization of convexity that does not reference a map to $\mathbb{R}^{2n}$.

1.2. Dynamical convexity. In the seminal paper [**37**], Hofer-Wysocki-Zehnder provided a candidate answer to Problem 1.5.

Definition 1.6 (Def. 3.6, [**37**]). A contact form α on S^3 is *dynamically convex* if the Conley-Zehnder index $\mathrm{CZ}(\gamma)$ of any closed Reeb orbit γ is greater than or equal to 3.

The Conley-Zehnder index of a Reeb orbit plays the role of the Morse index in symplectic field theory and other types of Floer homology (see §2.2 for a review). Thus, on a naive level, dynamical convexity may be viewed as a type of "Floer-theoretic" convexity. If X is a convex domain whose boundary Y has positive definite second fundamental form, then Y is dynamically convex [**37**, Thm 3.7]. Note that this condition is open and generic among convex boundaries.

In [**37**], Hofer-Wysocki-Zehnder proved that the Reeb flow of a dynamically convex contact form always admits a surface of section. In the decades since, dynamical convexity has been used as a key hypothesis in many significant works on Reeb dynamics and other topics in contact and symplectic geometry. See the papers of Hryniewicz [**39**], Zhou [**83**, **84**], Abreu-Macarini [**4**, **5**], Ginzburg-Gürel [**29**], Fraunfelder-Van Koert [**26**] and Hutchings-Nelson [**49**] for just a few examples. However, the following question has remained stubbornly open (c.f. [**26**, p. 5]).

Question 1.7. Is every dynamically convex contact form on S^3 also convex?

The recent paper [**1**] of Abbondandolo-Bramham-Hryniewicz-Salomão (ABHS) has suggested that the answer to Question 1.7 should be no. They construct dynamically convex contact forms on S^3 with systolic ratio close to 2. There is substantial evidence for the weak Viterbo conjecture (cf. [**15**]), and so these contact forms are likely *not* convex. However, this was not proven in [**1**].

Even more recently, Ginzburg-Macarini [**30**] addressed a version of Question 1.7 in higher dimensions that incorporates the assumption of symmetry under the antipod map $S^{2n-1} \to S^{2n-1}$. Their work did not address the general case of Question 1.7.

1.3. Main result. The main purpose of this part of this thesis is to resolve Question 1.7.

Theorem 1.8. *There exist dynamically convex contact forms α on S^3 that are not convex.*

Theorem 1.8 is an immediate application of Proposition 1.9 and 1.12, which we will now describe.

1.4. Ruelle bound. For our first result, recall that any closed contact 3-manifold (Y, ξ) with contact form α that satisfies $c_1(\xi) = 0$ and $H^1(Y;\mathbb{Z}) = 0$ has an associated *Ruelle invariant* [**69**]

$$\mathrm{Ru}(Y, \alpha) \in \mathbb{R}$$

Roughly speaking, the Ruelle invariant is the integral over Y of a time-averaged rotation number that measures the degree to which different Reeb trajectories twist counter-clockwise around each other (see §2.4 for a detailed review). Our result is stated most elegantly using the quantity

$$\mathrm{ru}(Y, \alpha) = \frac{\mathrm{Ru}(Y, \alpha)^2}{\mathrm{vol}(Y, \alpha)}$$

This *Ruelle ratio* is invariant under scaling of the contact form, unlike the Ruelle invariant itself.

In recent work [**47**] motivated by embedded contact homology, Hutchings investigated the Ruelle invariant of toric domains in $\mathbb{C}^2$. In that paper, the Ruelle invariant of the standard ellipsoid $E = E(a, b) \subset \mathbb{C}^2$ with symplectic radii $0 < a \leqslant b$ (see §3.1) was computed as

$$\mathrm{Ru}(E) = a + b \tag{1.2}$$

The systolic ratio and volume of E are well-known to be a/b and $ab/2$ respectively. This implies several constraints relating the systolic and Ruelle ratios. In particular, we have

$$\mathrm{ru}(E) = \frac{(\mathrm{sys}(E) + 1)^2}{\mathrm{sys}(E)} \quad \text{and thus} \quad 1 \leqslant \mathrm{ru}(E) \cdot \mathrm{sys}(E) = \frac{(a + b)^2}{b^2} \leqslant 4$$

Our first result may be viewed as a generalization of the estimate on the right to arbitrary convex contact forms on S^3.

Proposition 1.9 (Prop 3.1). *There are constants $C > c > 0$ such that, for any convex contact form α on S^3, the following inequality holds.*

$$c < \mathrm{ru}(S^3, \alpha) \cdot \mathrm{sys}(S^3, \alpha) < C \tag{1.3}$$

Note that a result of Viterbo [**77**, Thm 5.1] states that there exists a constant γ_2 such that $\mathrm{sys}(S^3, \alpha) \leqslant \gamma_2$ for any convex contact form. Thus, Proposition 1.9 also implies that

Corollary 1.10. *There is a constant $c > 0$ such that, for any convex contact form α on S^3, we have*

$$c < \mathrm{ru}(S^3, \alpha) \tag{1.4}$$

We have included a helpful visualization of Proposition 1.9 in the sys – ru plane in Figure 1.

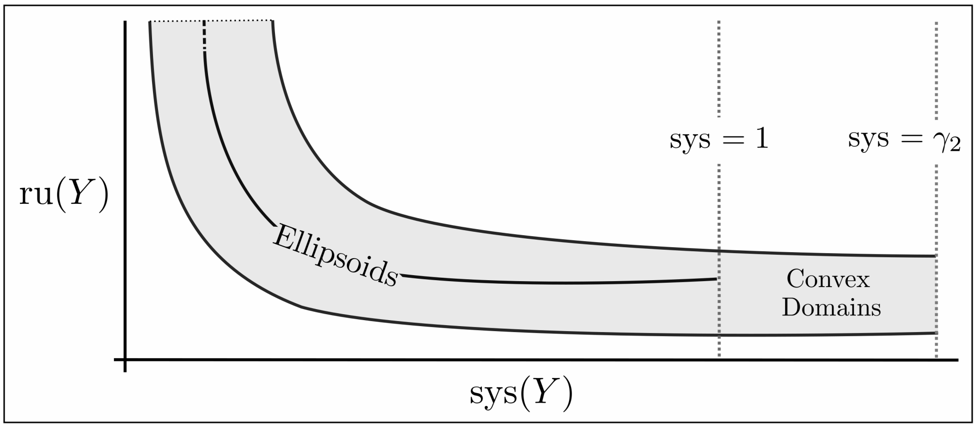
Let us explain the idea of the proof of Proposition 1.9. First, as explained above, the result holds for ellipsoids. By John's ellipsoid theorem, we can always sandwich a convex domain X between a standard ellipsoid and its scaling, after applying a linear symplectomorphism.

$$E(a, b) \subset X \subset 4 \cdot E(a, b)$$

Now note that the volume and minimum closed orbit length are monotonic under inclusion of convex domains. In particular, X satisfies

$$\frac{ab}{2} \leqslant \mathrm{vol}(X) \leqslant 2^8 \cdot \frac{ab}{2} \qquad \text{and} \qquad 2^{-8} \cdot \frac{a}{b} \leqslant \mathrm{sys}(Y) \leqslant 2^8 \cdot \frac{a}{b} \tag{1.5}$$

FIGURE 1. A plot of the region of the sys – ru plane containing convex contact forms, depicted in light red. The blue arc is the region occupied by ellipsoids, and the green lines represent the sys $= 1$ bound and the sys $= \gamma_2$ bound.



If the Ruelle invariant were also monotonic, then one could immediately acquire Proposition 1.9 from (1.5) and (1.2). Unfortunately, this is not evidently the case.

The resolution of this issue comes from a beautiful formula (Proposition 3.10) relating the second fundmantal form and local rotation of the Reeb flow on a contact hypersurface Y in $\mathbb{R}^4$. This is due originally to Ragazzo-Salomão [**67**], albeit in different language from this part of this thesis. Using this relation (§3.2), we derive estimates for the Ruelle invariant in terms of diameter, area and total mean curvature. By standard convexity theory (i.e. the theory of mixed volumes), these quantities are monotonic under inclusion of convex domains. This allows us to compare the Ruelle invariant of X to that of its sandwiching ellipsoids, and thus prove the result.

REMARK 1.11 (ENHANCING PROP 1.9). In future work, we plan to investigate optimal constants c and C for Proposition 1.9, and to generalize the result to higher dimensions.

1.5. A counterexample. In order to prove Theorem 1.8 using Proposition 1.9, we explicitly find a dynamically convex contact form that violates the estimate (1.4). This is the subject of our second new result.

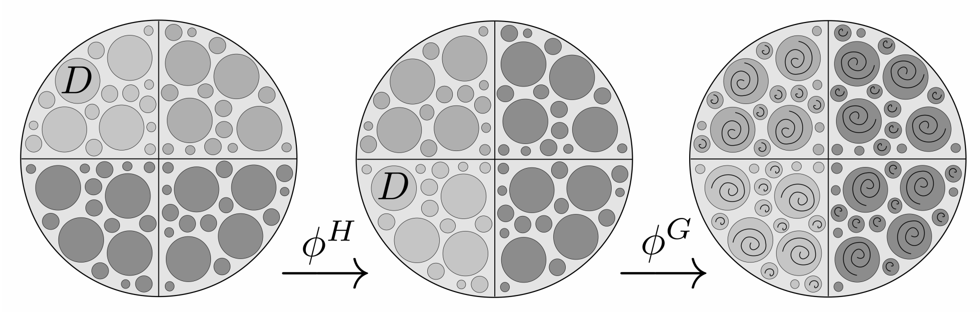
PROPOSITION 1.12 (PROP 4.1). *For every $\varepsilon > 0$, there is a dynamically convex contact form α on S^3 with*

$$\mathrm{vol}\big(S^3, \alpha\big) = 1 \qquad \mathrm{sys}(S^3, \alpha) \geqslant 1 - \varepsilon \qquad \mathrm{Ru}(S^3, \alpha) \leqslant \varepsilon$$

The construction of these examples follows the open book methods of Abbondandolo-Bramham-Hryniewicz-Salomão in [**1**]. Namely, we develop a detailed correspondence between the properties of a Hamiltonian disk map $\phi : \mathbb{D} \to \mathbb{D}$ and the properties of a contact form α on S^3 constructed using ϕ via the open book construction (see Proposition 4.10). This includes a new formula relating the Ruelle invariant of ϕ in the sense of [**69**] and the Ruelle invariant of (S^3, α).

We then construct a Hamiltonian disk map ϕ with all of the appropriate properties to produce a dynamically convex contact form on S^3 satisfying the conditions in Proposition 1.12. The map ϕ is acquired by composing two maps ϕ^H and ϕ^G. The map ϕ^H is a counter-clockwise rotation by angle $2\pi(1+1/n)$ for large n. The map ϕ^G is compactly supported on a disjoint union U of disks D, and rotates (most of) each disk D clockwise about its center by angle slightly less than 4π. See Figure 2 for an illustration of this map.

FIGURE 2. The map $\phi = \phi^G \circ \phi^H$ for $n = 4$. Here ϕ^H rotates $\mathbb{D}$ counter-clockwise by 45 degrees and ϕ^G twists each disk D by roughly 720 degrees clockwise.



Applying Proposition 4.10, we can show that the volume and Ruelle invariant of (S^3, α) are (up to negligible error) proportional to the following quantities.

$$\operatorname{vol}(S^3,\alpha) \sim \pi^2 - 2\sum_D \operatorname{area}(D)^2 \qquad \operatorname{Ru}(S^3,\alpha) \sim 2\pi - 2\sum_D \operatorname{area}(D)$$

By choosing U to fill most of $\mathbb{D}$ and choosing all of the disks in U to be very small, we can make the Ruelle invariant very small relative to the volume. This process preserves the minimal action of a closed orbit (up to a small error) and dynamical convexity, producing the desired example.

REMARK 1.13. Our examples *do not* coincide with the ABHS examples in [**1**]. However, we believe that improvements of Proposition 1.12 may make our analysis applicable to those examples.

Outline. This concludes the introduction **§1**. The rest of this part is organized as follows.

In **§2**, we cover basic preliminaries needed in later sections: the rotation number (§2.1), the Conley-Zehnder index (§2.2), invariants of Reeb orbits (§2.3) the Ruelle invariant (§2.4).

In **§3**, we prove Proposition 1.9. We start by discussing the curvature-rotation formula and some consequences (§3.2). We then derive a lower bound for a relevant curvature integral (§3.3). We conclude by proving the main bound (§3.4).

In **§4**, we prove Proposition 1.12. We first discuss general preliminaries on Hamiltonian disk maps (§4.1), open books (§4.2) and radial Hamiltonians (§4.3). We then construct a Hamiltonian flow on the disk (§4.4) before concluding with the main proof (§4.5).

2. Rotation numbers and Ruelle invariant

In this section, we review some preliminaries on rotation numbers, Conley-Zehnder indices and the Ruelle invariant, which we will need in later parts of this part.

2.1. Rotation number. Consider the universal cover $\widetilde{\mathrm{Sp}}(2)$ of the symplectic group $\mathrm{Sp}(2)$. We will view a group element Φ as a homotopy class of paths with fixed endpoints

$$\Phi : [0,1] \to \mathrm{Sp}(2) \quad \text{with} \quad \Phi(0) = \mathrm{Id}$$

Recall that a *quasimorphism* $q : G \to \mathbb{R}$ from a group G to the real line is a map such that there exists a $C > 0$ such that

$$|q(gh) - q(g) - q(h)| < C \qquad \text{for all } g, h \in G \tag{2.1}$$

A quasimorphism is *homogeneous* if $q(g^k) = k \cdot \sigma(g)$ for any $g \in G$. Finally, two quasimorphisms q and q' are called *equivalent* if the function $|q - q'|$ on G is bounded.

The universal cover of the symplectic group possesses a canonical homogeneous quasimorphism, due to the following result of Salamon-Simon [**72**].

Theorem 2.1 ([**72**], Thm 1). *There exists a unique homogeneous quasimorphism*

$$\rho : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$$

that restricts to the standard homomorphism $\rho : \widetilde{\mathrm{U}}(1) \to \mathbb{R}$ on the universal cover of the unitary group

$$\rho(\gamma) = L \qquad \textit{on the path } \gamma : [0,1] \to \mathrm{U}(1) \textit{ with } \gamma(t) = \exp(2\pi i L t) \tag{2.2}$$

Definition 2.2. The *rotation number* $\rho : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ is the quasimorphism in Theorem 2.1.

The rotation number is often characterized more explicitly in the literature as a lift of a map to the circle. More precisely, it is characterized as the unique lift

$$\widetilde{\sigma} : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R} \qquad \text{of} \qquad \sigma : \mathrm{Sp}(2) \to \mathbb{R}/\mathbb{Z} \qquad \text{such that} \qquad \widetilde{\sigma}(\mathrm{Id}) = 0 \tag{2.3}$$

where σ is defined as follows. Let $\Phi \in \mathrm{Sp}(2)$ have real eigenvalues λ, λ^{-1} and let $\Psi \in \mathrm{Sp}(2)$ have complex (unit) eigenvalues $\exp(\pm 2\pi i \theta)$ for $\theta \in (0, 1/2)$. Also fix an arbitrary $v \in \mathbb{R}^2 \backslash 0$. Then

$$\sigma(\Phi) = \begin{cases} 0 & \text{if } \lambda > 0 \\ 1/2 & \text{if } \lambda < 0 \end{cases} \quad \text{and} \quad \sigma(\Psi) = \begin{cases} \theta & \text{if } \langle iv, \Phi v \rangle > 0 \\ -\theta & \text{if } \langle iv, \Phi v \rangle < 0 \end{cases} \tag{2.4}$$

All of the elements of $\mathrm{Sp}(2)$ fall into one of the two categories above, and so σ is determined everywhere by (2.4).

Lemma 2.3. *The rotation number $\rho : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ is the unique lift of $\sigma : \mathrm{Sp}(2) \to \mathbb{R}/\mathbb{Z}$ with $\rho(\mathrm{Id}) = 0$.*

Proof. We verify the properties in Theorem 2.1. The lift $\widetilde{\sigma}$ is a quasimorphism by Lemmas 2.5 and 2.6 below. It is homogeneous since $\sigma(\Phi^k) = k \cdot \sigma(\Phi) \mod 1$, implying the same identity on the lift. Finally, if $\gamma : [0,1] \to \mathrm{Sp}(2)$ is given by $\gamma(t) = \exp(2\pi i L t)$ then

$$\sigma \circ \gamma : [0,1] \to \mathbb{R}/\mathbb{Z} \quad \text{is given by} \quad \sigma \circ \gamma(t) = Lt \mod 1 \in \mathbb{R}/\mathbb{Z}$$

This implies that the lift is $t \mapsto Lt$, so that $\widetilde{\sigma}(\gamma) = L$, and we have proven the needed criteria. □

We will also need to utilize several inhomogeneous versions of the rotation number depending on a choice of unit vector. These are defined a follows.

Definition 2.4. The *rotation number* $\rho_s : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ *relative to* $s \in S^1$ is the lift of the map

$$\sigma_s : \mathrm{Sp}(2) \to S^1 \qquad \Phi \mapsto |\Phi s|^{-1} \cdot \Phi s \in S^1 \subset \mathbb{R}^2$$

via the covering map $\mathbb{R} \to S^1 \subset \mathbb{C}$ given by $\theta \mapsto e^{2\pi i\theta} \cdot s$.

The rotation numbers relative to $s \in S^1$ and the lift of σ all agree up to a constant factor.

Lemma 2.5. *The maps* $\rho_s : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ *and the lift* $\widetilde{\sigma} : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ *of* σ *have bounded difference. More precisely, we have the following bounds.*

$$|\rho_s - \widetilde{\sigma}| \leqslant 1 \qquad \text{and} \qquad |\rho_s - \rho_t| \leqslant 1 \qquad \text{for any pair } s, t \in S^1 \tag{2.5}$$

Proof. First, assume that $\Phi : [0,1] \to \mathrm{Sp}(2)$ is a path such that $\Phi(t)$ has no negative real eigenvalues for any $t \in [0,1]$. Then

$$\widetilde{\sigma} \circ \Phi(t) \neq 1/2 \qquad \text{and} \qquad \sigma_s \circ \Phi(t) \neq -s \in S^1 \qquad \text{for any } s \in S^1 \text{ and } t \in [0,1]$$

It follows that the relevant lifts of $\sigma \circ \Phi$ and $\sigma_s \circ \Phi$ to maps $[0,1] \to \mathbb{R}$ remain in the interval $(-1/2, 1/2)$ for all t. Thus

$$\widetilde{\sigma}(\Phi) \in (-1/2, 1/2) \qquad \text{and} \qquad \rho_s(\Phi) \in (-1/2, 1/2)$$

This clearly implies (2.5). Since σ induces an isomorphism $\pi_1(\mathrm{Sp}(2)) \to \pi_1(S^1)$, we know that for any pair $\Phi, \Phi' \in \widetilde{\mathrm{Sp}}(2)$ lifting the same element of $\mathrm{Sp}(2)$, we have

$$\widetilde{\sigma}(\Phi) = \widetilde{\sigma}(\Phi') \qquad \text{implies} \qquad \Phi = \Phi'$$

In particular, the above analysis extends to any Φ with $\widetilde{\sigma}(\Phi) \in (-1/2, 1/2)$. In the general case, note that the path $\gamma : [0,1] \to S^1$ given by $\gamma(t) = \exp(\pi i \cdot kt)$ for an integer $k \in \mathbb{Z}$ satisfies

$$\widetilde{\sigma}(\gamma) = \rho_s(\gamma) = k/2 \qquad \widetilde{\sigma}(\Phi\gamma) = \widetilde{\sigma}(\Phi) + \widetilde{\sigma}(\gamma) \qquad \rho_s(\Phi\gamma) = \rho_s(\Phi) + \rho_s(\gamma)$$

Any path Ψ can be decomposed (up to homotopy) as $\Phi\gamma$ where γ is as above and $\Phi : [0,1] \to \mathrm{Sp}(2)$ is a path with $\widetilde{\sigma}(\Phi) \in (-1/2, 1/2)$. This reduces to the special case. □

This can be used to demonstrate that ρ_s is a quasimorphism. As noted in the proof of Lemma 2.3, this implies that $\widetilde{\sigma}$ is a quasimorphism as well.

Lemma 2.6. *The map* $\rho_s : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ *is a quasimorphism for any* $s \in S^1$*. In fact, we have*

$$|\rho_s(\Psi\Phi) - \rho_s(\Psi) - \rho_s(\Phi)| \leqslant 1 \qquad \text{for any} \qquad s \in S^1 \tag{2.6}$$

Proof. Let $\Phi : [0,1] \to \mathrm{Sp}(2)$ and $\Psi : [0,1] \to \mathrm{Sp}(2)$ be two elements of $\widetilde{\mathrm{Sp}}(2)$ viewed as paths in $\mathrm{Sp}(2)$. Consider the product $\Psi\Phi$ in the universal cover of $\mathrm{Sp}(2)$, represented by the path

$$\Phi(2t) \text{ for } t \in [0, 1/2] \qquad \text{and} \qquad \Psi(2t-1)\Phi(1) \text{ for } t \in [1/2, 1]$$

By examining the path $\sigma_s \circ \Psi\Phi : [0,1] \to S^1$ and the lift to $\mathbb{R}$, we deduce the following property.

$$\rho_s(\Psi\Phi) = \rho_{\Phi(s)}(\Psi) + \rho_s(\Phi) \tag{2.7}$$

Here $\Phi(s)$ is shorthand for the unit vector $\Phi_1(s)/|\Phi_1(s)|$. Applying Lemma 2.5, we have

$$|\rho_s(\Psi\Phi) - \rho_s(\Psi) - \rho_s(\Phi)| \leqslant |\rho_{\Phi(s)}(\Psi) - \rho_s(\Psi)| \leqslant 1$$

This proves the quasimorphism property. □

2.2. Conley-Zehnder index. Let $\mathrm{Sp}_\star(2) \subset \mathrm{Sp}(2)$ denote the subset of $\Phi \in \mathrm{Sp}(2)$ such that $\Phi - \mathrm{Id}$ is invertible. The *Conley-Zehnder index* is a continuous map

$$\mathrm{CZ} : \widetilde{\mathrm{Sp}}_\star(2) \to \mathbb{Z}$$

Here $\widetilde{\mathrm{Sp}}_\star(2)$ is the inverse image of $\mathrm{Sp}_\star(2)$ under $\pi : \widetilde{\mathrm{Sp}}(2) \to \mathrm{Sp}(2)$. The Conley-Zehnder index can be written using the rotation number as follows.

$$\mathrm{CZ}(\Phi) = \lfloor \rho(\Phi) \rfloor + \lceil \rho(\Phi) \rceil \tag{2.8}$$

There are several inequivalent ways to extend the Conley-Zehnder index to the entire symplectic group. We will follow [**37**, §3] and [**1**, §2.2], and use the following extension.

Convention 2.7. In this part of this thesis, the *Conley-Zehnder index* $\mathrm{CZ} : \widetilde{\mathrm{Sp}}(2) \to \mathbb{Z}$ will be the maximal lower semi-continuous extension of the ordinary Conley-Zehnder index.

The extension in Convention 2.7 can be bounded below in terms of the rotation number.

Lemma 2.8. *Let* $\Phi \in \widetilde{\mathrm{Sp}}(2)$. *Then*

$$\mathrm{CZ}(\Phi) \geqslant 2 \cdot \lceil \rho(\Phi) \rceil - 1 \tag{2.9}$$

Proof. For $\Phi \in \widetilde{\mathrm{Sp}}_\star(2)$, (2.9) is an immediate consequence of (2.8). In the other case, note that the maximal lower semicontinuous extension is defined by the property that

$$\mathrm{CZ}(\Phi) = \inf \lim_{i \to \infty} \mathrm{CZ}(\Phi_i) \qquad \text{for any } \Phi \notin \widetilde{\mathrm{Sp}}_\star(2)$$

Here the infimum is over all sequences $\Phi_i \in \widetilde{\mathrm{Sp}}_\star(2)$ with $\Phi_i \to \Phi$. Any $\Phi \notin \widetilde{\mathrm{Sp}}_\star(2)$ has eigenvalue 1, and so Lemma 2.3 implies that $\rho(\Phi) \in \mathbb{Z}$. Since ρ is continuous, we find that

$$\mathrm{CZ}(\Phi) = \inf \lim_{i \to \infty} \lfloor \rho(\Phi_i) \rfloor + \lceil \rho(\Phi_i) \rceil \geqslant \lfloor \rho(\Phi) - 1/2 \rfloor + \lceil \rho(\Phi) - 1/2 \rceil = 2 \cdot \lceil \rho(\Phi) \rceil - 1$$

This proves the lower bound in every case. □

2.3. Invariants of Reeb orbits. Let (Y, ξ) be a closed contact 3-manifold with $c_1(\xi) = 0$ and let α be a contact 1-form on Y.

Under this hypothesis on the Chern class, ξ is isomorphic as a symplectic vector-bundle to the trivial bundle $\mathbb{R}^2$. A *trivialization* τ of ξ is a bundle isomorphism

$$\tau : \xi \simeq \mathbb{R}^2 \qquad \text{denoted by} \qquad \tau(y) : \xi_y \simeq \mathbb{R}^2 \qquad \text{satisfying} \qquad \tau(y)^*\omega = d\alpha|_\xi$$

Two trivializations are *homotopic* if they are connected by a 1-parameter family of bundle isomorphisms. Given a trivialization τ, we may associate a *linearized* Reeb flow

$$\Phi_\tau : \mathbb{R} \times Y \to \mathrm{Sp}(2) \qquad \text{given by} \qquad \Phi_\tau(T, y) = \tau(\phi(T, y)) \circ d\phi(T, y) \circ \tau^{-1}(y) \tag{2.10}$$

Here $\phi : \mathbb{R} \times Y \to Y$ is the Reeb flow, i.e. the flow generated by the Reeb vector field R. The linearized flow lifts uniquely to a map

$$\widetilde{\Phi}_\tau : \mathbb{R} \times Y \to \widetilde{\mathrm{Sp}}(2) \qquad \text{with} \qquad \widetilde{\Phi}_\tau|_{0 \times Y} = \mathrm{Id} \in \widetilde{\mathrm{Sp}}(2)$$

We will refer to $\widetilde{\Phi}_\tau$ as the *lifted* linearized Reeb flow. Explicitly, it maps (y, T) to the homotopy class of the path $\Phi_\tau(\cdot, y)|_{[0,T]}$. Note that this lift satisfies the cocyle property

$$\widetilde{\Phi}_\tau(S + T, y) = \widetilde{\Phi}_\tau(T, \phi_S(y)) \cdot \widetilde{\Phi}_\tau(S, y) \tag{2.11}$$

DEFINITION 2.9. Let $\gamma : \mathbb{R}/L\mathbb{Z} \to Y$ be a closed Reeb orbit of Y. The *action* of γ is given by

$$\mathcal{A}(\gamma) = \int \gamma^*\alpha = L \tag{2.12}$$

Likewise, the *rotation number* and *Conley-Zehnder index* of γ with respect to τ are given by

$$\rho(\gamma,\tau) := \rho \circ \widetilde{\Phi}_\tau(L,y) \qquad \mathrm{CZ}(\gamma,\tau) := \mathrm{CZ}(\widetilde{\Phi}_\tau(L,y)) \qquad \text{where } y = \gamma(0) \tag{2.13}$$

These invariants depend only on the homotopy class of τ, and if $H^1(Y;\mathbb{Z}) = 0$ (e.g. if Y is the 3-sphere) there is a unique trivialization up to homotopy. In this case, we let

$$\rho(\gamma) := \rho(\gamma,\tau) \qquad \text{and} \qquad \mathrm{CZ}(\gamma) := \mathrm{CZ}(\gamma,\tau) \qquad \text{for any } \tau \tag{2.14}$$

In §4, we will need the following easy observation, which follows immediately from Lemma 2.8 and our way of defining CZ (see Convention 2.7).

LEMMA 2.10. *Let α be a contact form on S^3 with $\rho(\gamma) > 1$ for every closed Reeb orbit. Then α is dynamically convex.*

2.4. Ruelle invariant. Let (Y,ξ) be a closed contact 3-manifold with $c_1(\xi) = 0$ equipped with a contact form α and a homotopy class of trivialization $[\tau]$ of ξ. Here we discuss the *Ruelle invariant*

$$\mathrm{Ru}(Y,\alpha,[\tau]) \in \mathbb{R}$$

associated to the data of Y, α and $[\tau]$. This invariant was originally introduced by Ruelle in **[69]**

It will be helpful to describe a more general construction that subsumes that of the Ruelle invariant. For this purpose, we also fix a uniformly continuous quasimorphism

$$q : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$$

Pick a representative trivialization τ of $[\tau]$ and let $\widetilde{\Phi}_\tau : Y \times \mathbb{R} \to \widetilde{\mathrm{Sp}}(2)$ be the lifted linearized Reeb flow. We can associate a time-averaged version of q over the space Y, as follows.

PROPOSITION 2.11. *The* 1*-parameter family of functions $f_T : Y \to \mathbb{R}$ given by the formula*

$$f_T(y) := \frac{q \circ \widetilde{\Phi}_\tau(T,y)}{T} \tag{2.15}$$

converges in $L^1(Y;\mathbb{R})$ to a function $f(\alpha,q,\tau) : Y \to \mathbb{R}$ with the following properties.

(a) (Quasimorphism) If q and r are equivalent quasimorphisms, i.e. $|q - r|$ is bounded, then

$$f(\alpha,q,\tau) = f(\alpha,r,\tau)$$

(b) (Trivialization) If σ and τ are homotopic trivializations of ξ, then

$$f(\alpha,q,\sigma) = f(\alpha,q,\tau)$$

(c) (Contact Form) The integral $F(\alpha)$ of $f(\alpha,q,\tau)$ over Y is continuous in the C^2-topology on $\Omega^1(Y)$.

Proof. We prove the existence of the limit and the properties (a)-(c) separately.

Limit Exists. We apply Kingman's ergodic theorem [**52**]. Fix a constant $C > 0$ for the quasimorphism q satisfying (2.1). Let g_T denote the function on Y given by

$$g_T := Tf_T + C = q \circ \widetilde{\Phi}_\tau(-, T) + C$$

Note that g_T defines a sub-additive process, as described in [**52**, §1.3]. First, due to the cocycle property (2.11) we have

$$g_{S+T} = q \circ \widetilde{\Phi}_\tau(S+T, -) + C \leqslant q \circ \widetilde{\Phi}_\tau(S, -) + q \circ \widetilde{\Phi}_\tau(T, \phi_S(-)) + 2C = g_S + \phi_S^* g_T \tag{2.16}$$

We can analogously show that $g_{S+T} \geqslant g_S + \phi_S^* g_T - 2C$. In particular, if $T > 0$ is a sufficiently large time with $T = n + S$ and $S \in [0, 1]$, then

$$\int_Y g_T \cdot \alpha \wedge d\alpha \geqslant \sum_{k=0}^{n-1} \int_Y \phi_k^* g_1 \cdot \alpha \wedge d\alpha + \int_Y \phi_n^* g_S \cdot \alpha \wedge d\alpha - 2CT \geqslant -AT \tag{2.17}$$

Here A is any number larger than $2C$ and larger than the quantity

$$-\min\left\{\int_Y g_S \cdot \alpha \wedge d\alpha \ : \ S \in [0, 1]\right\}$$

Since g_T satsifies (2.16) and (2.17), we may apply Kingman's subadditive ergodic theorem [**52**, Thm 4] to conclude that there is a limiting function in L^1.

$$\frac{g_T}{T} \xrightarrow{L^1(Y;\mathbb{R})} f(\alpha, q, \tau) \in L^1(Y; \mathbb{R})$$

On the other hand, $\frac{g_T}{T}$ is Cauchy if and only if f_T is Cauchy, and they have the same limit, since

$$\|f_T - \frac{g_T}{T}\|_{L^1} \leqslant \frac{C}{T} \cdot \mathrm{vol}(Y, \alpha)$$

This proves that f_T converges in $L^1(Y; \mathbb{R})$ to $f(\alpha, q, \tau)$.

Quasimorphisms. Let q and r be equivalent quasimorphisms, and pick $C > 0$ such that $|q - r| < C$ everywhere. Then

$$\|\frac{q \circ \widetilde{\Phi}_\tau}{T} - \frac{r \circ \widetilde{\Phi}_\tau}{T}\|_{L^1} \leqslant \frac{C \cdot \mathrm{vol}(Y, \alpha)}{T}$$

Taking the limit as $T \to \infty$ shows that the limiting functions $f(\alpha, q, \tau)$ and $f(\alpha, r, \tau)$ are equal.

Trivializations. Let σ and τ be two trivializations of ξ in the homotopy class $[\tau]$. Then there is a transition map $\Psi : Y \to \mathrm{Sp}(2)$ given by

$$\Psi(y) : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \text{with} \qquad \Psi(y) = \tau(y) \cdot \sigma(y)^{-1}$$

The linearized flows of σ and τ are related via this transition map, by the following formula.

$$\Phi_\tau(T, y) = \Psi(\phi(T, y)) \cdot \Phi_\sigma(T, y) \cdot \Psi^{-1}(y)$$

The homotopy equivalence of σ and τ is equivalent to the fact that Ψ is null-homotopic, and in particular lifts to the universal cover of $\mathrm{Sp}(2)$. Thus we may write

$$\widetilde{\Phi}_\tau(T, y) = \widetilde{\Psi}(\phi(T, y)) \cdot \widetilde{\Phi}_\sigma(T, y) \cdot \widetilde{\Psi}^{-1}(y)$$

Here $\widetilde{\Psi} : Y \to \widetilde{\mathrm{Sp}}(2)$ is any lift of Ψ. The quasimorphism property of ρ now implies that

$$\|\frac{q \circ \widetilde{\Phi}_\sigma(T,y)}{T} - \frac{q \circ \widetilde{\Phi}_\tau(T,y)}{T}\|_{L^1} \leqslant \frac{2C + \sup|q \circ \widetilde{\Psi}| + \sup|q \circ \widetilde{\Psi}^{-1}|}{T} \cdot \mathrm{vol}(Y,\alpha)$$

Taking the limit as $T \to \infty$ shows that $f(\alpha, q, \sigma) = f(\alpha, q, \tau)$.

Contact Form. Fix a contact form α and an $\varepsilon > 0$. Since q is a quasimorphism, there exists a $C > 0$ depending only on q such that

$$|\rho \circ \widetilde{\Phi}_\tau(nT, y) - \sum_{k=0}^{n-1} \rho \circ \widetilde{\Phi}_\tau(T, \phi_T^k(y))| \leqslant Cn \qquad \text{for any } n, T > 0$$

We can divide by nT and rewrite this estimate in terms of f_T to see that

$$|f_{nT} - \frac{1}{n}\sum_{k=0}^{n-1} f_T \circ \phi_T^k| \leqslant \frac{C}{T} \qquad \text{for any } n, T > 0$$

integrate over Y and take the limit as $n \to \infty$ to acquire

$$|F(\alpha) - \int_Y f_T \cdot \alpha \wedge d\alpha| = \lim_{n\to\infty} |\int_Y (f_{nT} - f_T) \cdot \alpha \wedge d\alpha| \tag{2.18}$$

$$\leqslant \lim_{n\to\infty} |\int_Y (f_{nT} - \frac{1}{n}\sum_{k=0}^{n-1} f_T \circ \phi_T^k) \cdot \alpha \wedge d\alpha| \leqslant \frac{C \cdot \mathrm{vol}(Y,\alpha)}{T}$$

Next, fix a different contact form β. Let $\widetilde{\Psi}_\tau$ be the lifted linearized flow for β, and let

$$g_T : Y \to \mathbb{R} \quad \text{where} \quad g_T(y) = \frac{q \circ \widetilde{\Psi}_\tau(T, -)}{T}$$

Due to (2.18), we can fix a $T > 0$ such that, for all β sufficiently C^0-close to α, we have

$$|F(\alpha) - \int_Y f_T \cdot \alpha \wedge d\alpha| < \frac{\varepsilon}{3} \quad \text{and} \quad |F(\beta) - \int_Y g_T \cdot \beta \wedge d\beta| < \frac{2C\,\mathrm{vol}(Y,\alpha)}{T} < \frac{\varepsilon}{3} \tag{2.19}$$

Furthermore, we can choose β sufficiently C^2-close to α so that $\widetilde{\Phi}_\tau$ and $\widetilde{\Psi}_\tau$ are C^0-close on $Y \times [0,T]$ for any fixed $T > 0$. Thus, for β sufficiently close to α in C^3, we have

$$|\int_Y f_T \cdot \alpha \wedge d\alpha - \int_Y g_T \cdot \beta \wedge d\beta| \leqslant \|f_T - g_T\|_{C^0} \cdot \mathrm{vol}(Y,\alpha) + 2\|f_T\|_{C^0} \cdot |\mathrm{vol}(Y,\alpha) - \mathrm{vol}(Y,\beta)| \tag{2.20}$$

$$\leqslant \frac{c\|\widetilde{\Phi}_\tau - \widetilde{\Psi}_\tau\|_{C^0}}{T} \cdot \mathrm{vol}(Y,\alpha) + 2\|f_T\|_{C^0} \cdot |\mathrm{vol}(Y,\alpha) - \mathrm{vol}(Y,\beta)| < \frac{\varepsilon}{3}$$

Adding (2.19) and (2.20), we find that for β sufficiently C^2-close to α, we have $|F(\alpha) - F(\beta)| < \varepsilon$, which proves continuity.

This concludes the proof of the existence and properties of $f(\alpha, q, \tau)$, and of Proposition 2.11. □

Proposition 2.11 allows us to introduce the Ruelle invariant as an integral quantity, as follows.

Definition 2.12 (Ruelle Invariant). The *local rotation number* rot_τ of a closed contact manifold (Y, α) equipped with a (homotopy class of) trivialization τ is the following limit in L^1.

$$\text{(2.21)} \qquad \mathrm{rot}_\tau : Y \to \mathbb{R} \qquad \text{given by} \qquad \mathrm{rot}_\tau := \lim_{T\to\infty} \frac{\rho \circ \widetilde{\Phi}_\tau(T,-)}{T}$$

Similarly, the *Ruelle invariant* $\mathrm{Ru}(Y, \alpha, \tau)$ is the integral of the local rotation number over Y, i.e.

$$\text{(2.22)} \qquad \mathrm{Ru}(Y,\alpha,\tau) = \int_Y \mathrm{rot}_\tau \cdot \alpha \wedge d\alpha = \lim_{T\to\infty} \frac{1}{T} \int_Y \rho \circ \widetilde{\Phi}_\tau \cdot \alpha \wedge d\alpha$$

We will require an alternative expression for the Ruelle invariant in order to derive estimates later in this part.

The Reeb flow ϕ on Y preserves the contact structure, and so lifts to a flow on the total space of the contact structure ξ. Since this flow is fiberwise linear, it descends to the (oriented) projectivization $P\xi$. A trivialization τ determines an identification $P\xi \simeq Y \times \mathbb{R}/\mathbb{Z}$, and so a flow

$$\text{(2.23)} \qquad \bar{\Phi} : \mathbb{R} \times Y \times \mathbb{R}/\mathbb{Z} \to Y \times \mathbb{R}/\mathbb{Z} \qquad \text{generated by a vector field } \bar{R} \text{ on } Y \times \mathbb{R}/\mathbb{Z}$$

Let $\theta : Y \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ denote the tautological projection.

Definition 2.13. The *rotation density* $\varrho_\tau : Y \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is the Lie derivative

$$\text{(2.24)} \qquad \varrho_\tau := \bar{R}(\theta)$$

Lemma 2.14. *The Ruelle invariant* $\mathrm{Ru}(Y, \alpha, \tau)$ *is written using the rotation density* ϱ_τ *as*

$$\mathrm{Ru}(Y,\alpha,\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \Big(\int_Y \bar{\Phi}_t^* \varrho_\tau(-,s) \cdot \alpha \wedge d\alpha \Big) dt \qquad \textit{for any fixed } s \in \mathbb{R}/\mathbb{Z}$$

Proof. By comparing Definition 2.4 with the formula (2.23), one may verify that

$$\sigma_s \circ \Phi_\tau(T,y) \quad \text{and} \quad \theta \circ \bar{\Phi}(T,y,s) - s \quad \text{are equal in } \mathbb{R}/\mathbb{Z}$$

Therefore, these formulas define a single map $\mathbb{R} \times Y \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, admitting a unique lift to a map $F : \mathbb{R} \times Y \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ that vanishes on $0 \times Y \times \mathbb{R}/\mathbb{Z}$. The first formula implies that

$$\text{(2.25)} \qquad F(T,y,s) = \rho_s \circ \widetilde{\Phi}_\tau(T,y)$$

On the other hand, let t be the $\mathbb{R}$-variable of F and $\theta \circ \bar{\Phi}$. Then the t-derivative of F is

$$\frac{dF}{dt}|_T = \frac{d}{dt}(\theta \circ \bar{\Phi})|_T = \bar{\Phi}_t^*(\mathcal{L}_{\bar{R}}(\theta))|_T = \bar{\Phi}_t^* \varrho_\tau$$

Integrating this identity and combining it with (2.25), we acquire the formula

$$\text{(2.26)} \qquad \rho_s \circ \widetilde{\Phi}_\tau(T,y) = F(T,y,s) = \int_0^T [\bar{\Phi}_t^* \varrho_\tau](y,s) \cdot dt$$

Now, since ρ_s and ρ are equivalent by Lemma 2.5, we can apply Proposition 2.11(a) to see that

$$\text{(2.27)} \qquad \mathrm{Ru}(Y,\alpha,\tau) = \lim_{T\to\infty} \int_Y \frac{\rho_s \circ \widetilde{\Phi}_\tau(T,-)}{T} \cdot \alpha \wedge d\alpha$$

We then apply (2.26) to see that the righthand side is given by

$$\lim_{T\to\infty}\frac{1}{T}\int_Y\int_0^T \bar{\Phi}_T^*\varrho(-,s)\cdot\alpha\wedge d\alpha = \lim_{T\to\infty}\frac{1}{T}\int_0^T\Big(\int_Y \bar{\Phi}_T^*\varrho(-,s)\cdot\alpha\wedge d\alpha\Big)dt \tag{2.28}$$

Combining the formulas (2.4) and (2.28) finishes the proof. □

3. Bounding the Ruelle invariant

Let $X \subset \mathbb{R}^4$ be a convex domain containing 0 in its interior, and let (Y,λ) be the contact boundary of X. In this section, we derive the following estimate for the Ruelle ratio.

PROPOSITION 3.1. *There exist positive constants c and C independent of Y such that*

$$c < \mathrm{ru}(Y,\lambda)\cdot\mathrm{sys}(Y,\lambda) < C$$

The proof follows the outline discussed in the introduction.

We begin (§3.1) with a review of the geometry of standard ellipsoids $E(a,b)$ in $\mathbb{C}^4$, including a variant of John's theorem (Corollary 3.6). We then present the key curvature-rotation formula (§3.2) and use it to bound the Ruelle invariant between two curvature integrals (Lemma 3.11). We then prove several bounds for one of these curvature integrals in terms of diameter, area and total mean curvature (§3.3). We collect this analysis together in the final proof (§3.4).

NOTATION 3.2. We will require the following notation throughout this section.

(a) g is the standard metric on $\mathbb{R}^4$ with connection ∇, and $\mathrm{dvol}_g = \frac{1}{2}\omega^2$ is the corresponding volume form. We also use $\langle u,v\rangle$ to denote the inner product of two vectors $u,v\in\mathbb{R}^4$.

(b) ν is the outward normal vector field to Y and ν^* is the dual 1-form with respect to g.

(c) σ is the restriction of g to Y and dvol_σ is the corresponding metric volume form. The volume form $\lambda\wedge d\lambda$ and dvol_σ are related (via the Liouville vector field Z of $\mathbb{R}^4$) by

$$\lambda\wedge d\lambda = \iota_Z(\frac{\omega^2}{2}|_Y) = \iota_Z(\mathrm{dvol}_g|_Y) = \iota_Z(\nu^*\wedge\mathrm{dvol}_\sigma) = \langle Z,\nu\rangle\,\mathrm{dvol}_\sigma \tag{3.1}$$

(d) S is the second fundamental form of Y, i.e. the bilinear form given on any $u,w\in TY$ by

$$S(u,w) := \langle\nabla_u\nu, w\rangle$$

(e) H is the mean curvature of Y. It is given by

$$H := \frac{1}{3}\,\mathrm{trace}\,S$$

3.1. Standard ellipsoids. Recall that a *standard ellipsoid* $E(a_1,\dots,a_n)\subset\mathbb{C}^n$ with parameters $a_i > 0$ for $i=1,\dots,n$ is defined as follows.

$$E(a_1,\dots,a_n) := \Big\{z=(z_i)\in\mathbb{C}^n \ :\ \sum_i\frac{\pi|z_i|^2}{a_i}\leqslant 1\Big\} \tag{3.2}$$

For example, $E(a) \subset \mathbb{C}$ is the disk of area a, and $E(a, \dots, a) \subset \mathbb{C}^n$ is the ball of radius $(a/\pi)^{1/2}$.

We beginn this section with a discussion of the Riemannian and symplectic geometry of standard ellipsoids in $\mathbb{C}^2$. All of the relevant geometric quantities for this section can be computed explicitly in this setting. Let us record the outcome of these calculations.

LEMMA 3.3 (ELLIPSOID QUANTITIES). *Let $E = E(a, b)$ be a standard ellipsoid with $0 < a < b$. Then*

(a) The diameter, surface area and volume of E are given by

$$\operatorname{diam}(E) = \frac{2}{\pi^{1/2}} \cdot b^{1/2} \qquad \operatorname{area}(\partial E) = \frac{4\pi^{1/2}}{3} \cdot \frac{b^2 a^{1/2} - b^{1/2} a^2}{b - a} \qquad \operatorname{vol}(E) = \frac{ab}{2}$$

(b) The total mean curvature of ∂E (i.e. the integral of the mean curvature over ∂E) is given by

$$\int_{\partial E} H \cdot \mathrm{dvol}_\sigma = \frac{2\pi}{3} \cdot \left(b + a + \frac{ab}{b - a} \cdot \log(b/a)\right)$$

(c) The minimum action of a closed orbit on ∂E and the systolic ratio of ∂E are given by

$$c(\partial E) = a \qquad \operatorname{sys}(\partial E) = \frac{a}{b}$$

(d) The Ruelle invariant of ∂E is given by

$$\operatorname{Ru}(\partial E) = a + b$$

The area, total mean curvature and volume are straightforward but tedious calculus computations, which we omit. The Ruelle invariant is computed in [**47**, Lem 2.1 and 2.2], while the minimum period of a closed orbit is computed in [**32**, §2.1].

Any convex boundary in $\mathbb{R}^{2n}$ can be sandwiched between a standard ellipsoid and a scaling of that ellipsoid by a factor of $2n$, after the application of an affine symplectomorphism. To see this, first recall the following well-known result of John.

THEOREM 3.4 (JOHN ELLIPSOID). *Let $X \subset \mathbb{R}^n$ be a convex domain. Then there exists an ellipsoid E centered at some $c \in X$ such that*

$$E \subset X \subset c + n(E - c)$$

Any ellipsoid E is carried to a standard ellipsoid $E(a, b)$ by some affine symplectomorphism T. Furthermore, note that we have the following elementary result, which can be demonstrated using a Moser argument.

LEMMA 3.5. *Let $\phi : (Y, \lambda) \to (Y', \lambda')$ be a diffeomorphism such that $\phi^*\lambda' = \lambda + df$. Then ϕ is isotopic to a strict contactomorphism.*

Since $\mathbb{R}^{2n}$ is contractible, $T^*\lambda = \lambda + df$ automatically on $\mathbb{R}^{2n}$. Thus, T carries any star-shaped hypersurface $Y = \partial X$ to a strictly contactomorphic $T(Y)$ by Lemma 3.5, and we conclude the following result.

COROLLARY 3.6. *Let $X \subset \mathbb{R}^{2n}$ be a convex domain with boundary Y. Then Y is strictly contactomorphic to the boundary ∂K of a convex domain K with $E(a_1, \dots, a_n) \subset K \subset 4 \cdot E(a_1, \dots, a_n)$.*

When a convex domain in $\mathbb{R}^4$ is squeezed between an ellipsoid and its scaling, we can estimate many important geometric quantities of X in terms of the ellipsoid itself.

Lemma 3.7. *Let $X \subset \mathbb{R}^4$ be a convex domain with smooth boundary Y such that*

$$E(a,b) \subset X \subset c \cdot E(a,b) \qquad \text{for some } b \geqslant a > 0 \text{ and } c \geqslant 0 \tag{3.3}$$

Then there is a constant $C > 0$ dependent only on c such that

$$b^{1/2} \leqslant \operatorname{diam}(X) \leqslant C \cdot b^{1/2} \qquad ba^{1/2} \leqslant \operatorname{area}(Y) \leqslant C \cdot ba^{1/2} \tag{3.4}$$

$$b \leqslant \int_Y H \cdot \operatorname{dvol}_\sigma \leqslant C \cdot b \qquad \frac{ab}{2} \leqslant \operatorname{vol}(X) \leqslant C \cdot ab \tag{3.5}$$

$$a \leqslant c(X) \leqslant C \cdot a \qquad C^{-1} \cdot \frac{a}{b} \leqslant \operatorname{sys}(Y) \leqslant C \cdot \frac{a}{b} \tag{3.6}$$

Remark 3.8. The optimal constants in the estimates (3.4)-(3.6) are not important to the arguments below. They could be explicitly computed in the following proof.

Proof. First, note that $c \cdot E(a,b)$ is also a standard ellipsoid. More precisely, we know that

$$c \cdot E(a,b) = E(c^2 \cdot a, c^2 \cdot b)$$

We now derive the desired estimates from Lemma 3.3 and the monotonicity of the relevant quantities under inclusion of convex domains.

The diameter $\operatorname{diam}(X)$ and volume $\operatorname{vol}(X)$ are monotonic with respect to inclusion of arbitrary open subsets, and so from Lemma 3.3(a) we acquire

$$b^{1/2} \leqslant \operatorname{diam}(X) \leqslant \frac{2c}{\pi^{1/2}} \cdot b^{1/2} \qquad \text{and} \qquad \frac{ab}{2} \leqslant \operatorname{vol}(X) \leqslant \frac{c^4}{2} \cdot ab$$

The surface area and total mean curvature are monotonic with respect to inclusion of convex domains, since

$$\int_Y H \operatorname{dvol}_\sigma = 4 \cdot V_2(X) \qquad \text{and} \qquad \operatorname{area}(Y) = 4 \cdot V_3(X)$$

Here $V_i(X)$ is the *ith cross-sectional measure* [**11**, §19.3], which is monotonic with respect to inclusions of convex domains by [**11**, p.138, Equation 13]. Furthermore, when $0 < a < b$ (and in the limit as $b \to a$), one may verify that

$$ba^{1/2} \leqslant \frac{b^2 a^{1/2} - b^{1/2} a^2}{b-a} \leqslant 3ba^{1/2} \qquad \text{and} \qquad b \leqslant b + a + \frac{ab}{b-a} \cdot \log(b/a) \leqslant 3b \tag{3.7}$$

Thus, by applying the monotonicity property, (3.7) and Lemma 3.3(a)-(b), we have

$$ba^{1/2} \leqslant \operatorname{area}(Y) \leqslant 3c^3 \cdot ba^{1/2} \qquad \text{and} \qquad b \leqslant \int_Y H \cdot \operatorname{dvol}_\sigma \leqslant 3c^2 \cdot b$$

Finally, the minimum orbit length $c(X)$ coincides with the 1st Hofer-Zehnder capacity $c_1^{HZ}(X)$ on convex domains, and is thus monotonic with respect to symplectic embeddings. Thus by Lemma 3.3(a) and (c), we have

$$a \leqslant c(X) \leqslant c^2 \cdot a \qquad \text{and} \qquad c^{-4} \cdot \frac{a}{b} \leqslant \frac{c(X)^2}{2\operatorname{vol}(X)} = \operatorname{sys}(Y) \leqslant c^4 \cdot \frac{a}{b}$$

This concludes the proof, after choosing C larger than the constants appearing above. $\square$

3.2. Curvature-rotation formula. Identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}^1$ via

$$\mathbb{R}^4 \ni (x_1, y_1, x_2, y_2) \mapsto x_1 + y_1 I + x_2 J + y_2 K \in \mathbb{H}^1$$

This equips $\mathbb{R}^4$ with a triple of complex structures.

$$I : T\mathbb{R}^4 \to T\mathbb{R}^4 \qquad J : T\mathbb{R}^4 \to T\mathbb{R}^4 \qquad K : T\mathbb{R}^4 \to T\mathbb{R}^4$$

We can utilize these structures to formulate an explicit representative of the standard homotopy class of trivialization $\tau : \xi \simeq \mathbb{R}^2$.

Definition 3.9. The *quaternion trivialization* $\tau : \xi \simeq Y \times \mathbb{C}$ is the symplectic trivialization given by

$$\tau : \xi \xrightarrow{\pi} Q \xrightarrow{q^{-1}} Y \times \mathbb{C}$$

Here $Q \subset TY$ is the symplectic sub-bundle $\operatorname{span}(J\nu, K\nu)$, $\pi : \xi \to Q$ is the projection map from ξ to Q along the Reeb direction, and $q : Y \times \mathbb{C} \to Q$ is the bundle map given on $z = a + ib$ by

$$q_p(z) := z \cdot J\nu_p = (a + Ib) \cdot J\nu_p \tag{3.8}$$

The key property of the quaternion trivialization is the following relation of the rotation density (see Definition 2.13) to extrinsic curvature, originally due to Ragazzo-Salomão (c.f. [**67**]).

Proposition 3.10 (Curvature-Rotation). *[**15**, Prop 4.7] Let $X \subset \mathbb{R}^4$ be a star-shaped domain with boundary Y transverse to the Liouville vector field Z of $\mathbb{R}^4$ and let τ be the quaternion trivialization. Then*

$$\varrho_\tau(y, s) = \frac{1}{2\pi \cdot \langle Z_y, \nu_y \rangle}(S(I\nu_y, I\nu_y) + S(s \cdot J\nu_y, s \cdot J\nu_y)) \tag{3.9}$$

As an easy consequence of (3.9), we have the following bound on the Ruelle invariant of Y.

Lemma 3.11. *The Ruelle invariant* $\operatorname{Ru}(Y)$ *is bounded by the following curvature integrals.*

$$\frac{1}{2\pi} \cdot \int_Y S(I\nu, I\nu) \, \mathrm{dvol}_\sigma \leqslant \operatorname{Ru}(Y) \leqslant \frac{3}{2\pi} \cdot \int_Y H \, \mathrm{dvol}_\sigma \tag{3.10}$$

Proof. By Lemma 2.14, we have the following integral formula for the Ruelle invariant.

$$\operatorname{Ru}(Y) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left(\int_Y [\bar{\Phi}_t^* \varrho_\tau](-, s) \cdot \lambda \wedge d\lambda \right) dt \tag{3.11}$$

By the curvature-rotation formula in Proposition 3.10, we can write the integrand as

$$[\bar{\Phi}_t^* \varrho_\tau](-, s) = \bar{\Phi}_t^* \Big(\frac{1}{\langle Z, \nu \rangle}(S(I\nu, I\nu) + S(s \cdot J\nu, s \cdot J\nu)) \Big) \tag{3.12}$$

To bound the righthand side of (3.12), note that $I\nu$, $s \cdot J\nu$ and $s \cdot K\nu$ form an orthonormal basis of TY with respect to the restricted metric $g|_Y$, so that

$$S(I\nu, I\nu) + S(s \cdot J\nu, s \cdot J\nu) + S(s \cdot K\nu, s \cdot K\nu) = \operatorname{trace}(S) = 3H$$

Furthermore, since Y is convex, the second fundamental form S is positive definite. Therefore by (3.12), we have the following lower and upper bound.

$$\bar{\Phi}_t^*\Big(\frac{S(I\nu, I\nu)}{\langle Z,\nu\rangle}\Big) \leqslant [\bar{\Phi}_t^*\varrho_\tau](-,s) \leqslant 3\cdot \bar{\Phi}_t^*\Big(\frac{H}{\langle Z,\nu\rangle}\Big) \tag{3.13}$$

To simplify the two sides of (3.13), let $F : Y\times S^1 \to \mathbb{R}$ be any map pulled back from a map $F : Y \to \mathbb{R}$. Since the flow $\bar{\Phi}_t$ on $Y\times S^1$ lifts the Reeb flow ϕ_t on Y, and ϕ_t preserves λ, we have

$$\bar{\Phi}_t^*\Big(\frac{F}{\langle Z,\nu\rangle}\Big)\cdot \lambda\wedge d\lambda = \phi_t^*\Big(\frac{F}{\langle Z,\nu\rangle}\Big)\cdot\lambda\wedge d\lambda = \phi_t^*\Big(F\cdot\frac{\lambda\wedge d\lambda}{\langle Z,\nu\rangle}\Big) = \phi_t^*\Big(F\cdot \mathrm{dvol}_\sigma\Big)$$

Since the integral of $\phi_t^*(F\cdot\mathrm{dvol}_\sigma)$ over Y is independent of t, we have

$$\frac{1}{T}\int_0^T\Big(\int_Y \bar{\Phi}_t^*\Big(\frac{F}{\langle Z,\nu\rangle}\Big)\cdot\lambda\wedge d\lambda\Big)dt = \frac{1}{T}\int_0^T\Big(\int_Y F\cdot\mathrm{dvol}_\sigma\Big)dt = \int_Y F\cdot\mathrm{dvol}_\sigma \tag{3.14}$$

By plugging in the estimate (3.13) to the integral formula (3.11) and applying (3.14) to the functions $S(I\nu, I\nu)$ and H on Y, we acquire the desired bound (3.10). □

3.3. Bounding curvature integrals. We now further simplify the lower bound of the Ruelle invariant in Lemma 3.11 by estimating (from below) the integral

$$\int_Y S(I\nu, I\nu)\cdot\mathrm{dvol}_\sigma$$

using the geometric quantities (e.g. area and diameter) appearing in §3.1. This will help us to leverage the sandwich estimates in Lemma 3.7 in the proof of the Ruelle invariant bound in §3.4.

Recall that $X\subset\mathbb{R}^4$ denotes a convex domain with smooth boundary Y. Let $\psi : \mathbb{R}\times Y\to Y$ be the flow by $I\nu$. Let S_T and H_T denote the time-averaged versions of $S(I\nu, I\nu)$ and H, respectively.

$$S_T := \frac{1}{T}\int_0^T S(I\nu, I\nu)\circ\psi_t dt \qquad H_T := \frac{1}{T}\int_0^T H\circ\psi_t dt \tag{3.15}$$

We will also need to consider a time-averaged acceleration function A_T on Y. Namely, let $\gamma : \mathbb{R}\to Y$ be a trajectory of $I\nu$ with $\gamma(0) = x$. Then we define

$$A_T := \frac{1}{T}\int_0^T |\nabla_{I\nu}I\nu|\circ\psi_t dt \qquad \text{or equivalently} \qquad A_T(x) = \frac{1}{T}\int_0^T |\ddot{\gamma}|dt \tag{3.16}$$

The first ingredient to the bounds in this section is the following estimate relating these three time-averaged functions.

Lemma 3.12. *For any $T > 0$, the functions A_T, H_T and S_T satisfy $A_T^2 \leqslant 3\cdot H_T\cdot S_T$ pointwise.*

Proof. In fact, the non-time-averaged version of this estimate holds. We will now show that

$$|\nabla_{I\nu}I\nu|^2 \leqslant 3H\cdot S(I\nu, I\nu) \tag{3.17}$$

To start, we need a formula for $\nabla_{I\nu}I\nu$ in terms of the second fundamental form, as follows.

$$\begin{aligned}\nabla_{I\nu}I\nu &= \langle\nu,\nabla_{I\nu}I\nu\rangle\nu + \langle I\nu,\nabla_{I\nu}I\nu\rangle I\nu + \langle J\nu,\nabla_{I\nu}I\nu\rangle J\nu + \langle K\nu,\nabla_{I\nu}I\nu\rangle K\nu \\ &= -\langle I\nu,\nabla_{I\nu}\nu\rangle\nu - \langle I^2\nu,\nabla_{I\nu}\nu\rangle I\nu - \langle IJ\nu,\nabla_{I\nu}\nu\rangle J\nu - \langle IK\nu,\nabla_{I\nu}\nu\rangle K\nu\end{aligned}$$

Applying the quaternionic relations $I^2 = -1$, $IJ = K$ and $IK = -J$, we can rewrite this as

$$-\langle I\nu, \nabla_{I\nu}\nu\rangle\nu + \langle \nu, \nabla_{I\nu}\nu\rangle I\nu - \langle K\nu, \nabla_{I\nu}\nu\rangle J\nu + \langle J\nu, \nabla_{I\nu}\nu\rangle K\nu$$

Finally, applying the definition of the second fundamental form we find that

$$\nabla_{I\nu} I\nu = -S(I\nu, I\nu)\nu - S(I\nu, K\nu)J\nu + S(I\nu, J\nu)K\nu$$

To estimate the righthand side, we note that $S(u,v)^2 \leqslant S(u,u)S(v,v)$ for any vectorfields u and v by Cauchy-Schwarz, since S is positive semi-definite. Thus we have

$$|\nabla_{I\nu} I\nu|^2 \leqslant S(I\nu, I\nu)^2 + S(I\nu, I\nu)S(J\nu, J\nu) + S(I\nu, I\nu)S(K\nu, K\nu) = 3H \cdot S(I\nu, I\nu)$$

This proves (3.17) and the desired estimate follows immediately by Cauchy-Schwarz.

$$A_T^2 = \Big(\frac{1}{T}\int_0^T |\nabla_{I\nu} I\nu| \circ \psi_t dt\Big)^2 \leqslant 3 \cdot \frac{1}{T}\int_Y H \circ \psi_t dt \cdot \frac{1}{T}\int_Y S(I\nu, I\nu) \circ \psi_t dt = 3H_T \cdot S_T \tag{3.18}$$

This concludes the proof of the lemma. □

As a consequence, we get the following estimate for the curvature integral of interest in terms of area, total mean curvature and the time-averaged acceleration A_T.

Lemma 3.13. *Let $\Sigma \subset Y$ be an open subset of Y and let $T > 0$. Then*

$$\int_Y S(I\nu, I\nu) \cdot \mathrm{dvol}_\sigma \geqslant \frac{\mathrm{area}(\Sigma)^2}{3 \cdot \int_Y H \,\mathrm{dvol}_\sigma} \cdot \min_\Sigma(A_T)^2 \tag{3.19}$$

Proof. We first note that $I\nu$ preserves the volume form dvol_σ, since

$$\mathcal{L}_{I\nu}(\mathrm{dvol}_\sigma) = d\iota_{I\nu}\,\mathrm{dvol}_\sigma = d\iota_R(\lambda \wedge d\lambda) = d^2\lambda = 0$$

Here R is the Reeb vector-field on Y. Thus, time-averaging leaves the integral over Y unchanged.

$$\int_Y H_T \,\mathrm{dvol}_\sigma = \int_Y H \,\mathrm{dvol}_\sigma \qquad \text{and} \qquad \int_Y S_T \,\mathrm{dvol}_\sigma = \int_Y S(I\nu, I\nu)\,\mathrm{dvol}_\sigma$$

We can thus integrate the estimate $A_T^2 \leqslant 3H_T \cdot S_T$ to see that

$$\min(A_T)^2 \cdot \mathrm{area}(\Sigma)^2 \leqslant \Big(\int_\Sigma A_T \cdot \mathrm{dvol}_\sigma\Big)^2 \leqslant \Big(\sqrt{3} \cdot \int_\Sigma H_T^{1/2} \cdot S_T^{1/2} \cdot \mathrm{dvol}_\sigma\Big)^2$$

$$\leqslant 3 \cdot \int_\Sigma H_T \cdot \mathrm{dvol}_\sigma \cdot \int_\Sigma S_T \cdot \mathrm{dvol}_\sigma \leqslant 3 \cdot \int_Y H \cdot \mathrm{dvol}_\sigma \cdot \int_Y S(I\nu, I\nu) \cdot \mathrm{dvol}_\sigma$$

After some rearrangement, this is the desired estimate. □

Every quantity on the righthand side of (3.19) can be controlled using the estimates in Lemma 3.7, with the exception of the term involving the time-averaged acceleration A_T. However, we can bound A_T in terms of $\mathrm{diam}(X)^{-1}$, using the following general fact about curves of unit speed.

Lemma 3.14. *Let $\gamma : [0,\infty) \to Y$ be a curve with $|\dot{\gamma}| = 1$ and let C satisfy $0 < C < 1$. Then*

$$\frac{1}{T}\int_0^T |\ddot{\gamma}| dt \geqslant \frac{C}{\mathrm{diam}(X)} \qquad \textit{for all } T \gg 0$$

Proof. Let T satisfy $T > CT + 2 \cdot \operatorname{diam}(Y)$. Then by Cauchy-Schwarz, we have

$$\operatorname{diam}(X) \int_0^T |\ddot{\gamma}| dt \geqslant \int_0^T |\gamma| \cdot |\ddot{\gamma}| dt \geqslant \int_0^T |\langle \ddot{\gamma}, \gamma \rangle| dt \geqslant \Big| \int_0^T \langle \ddot{\gamma}, \gamma \rangle dt \Big| \tag{3.20}$$

On the other hand, by integration by parts we acquire

$$\Big| \int_0^T \langle \ddot{\gamma}, \gamma \rangle dt \Big| \geqslant \Big| \int_0^T |\dot{\gamma}|^2 dt - \langle \gamma, \dot{\gamma} \rangle|_0^T \Big| \geqslant T - 2 \operatorname{diam}(X) \geqslant CT \tag{3.21}$$

Combining the estimates (3.20) and (3.21) yields the claimed bound. □

In particular, Lemma 3.14 implies that $A_T \geqslant C \cdot \operatorname{diam}(X)^{-1}$ for all $C < 1$ and sufficiently large T. Combining this with Lemma 3.13 and taking $C \to 1$, we acquire the following corollary.

Corollary 3.15. *Let $X \subset \mathbb{R}^4$ be a convex domain with smooth boundary Y. Then*

$$\int_Y S(I\nu, I\nu) \operatorname{dvol}_\sigma \geqslant \frac{\operatorname{area}(Y)^2}{3 \cdot \operatorname{diam}(X)^2 \cdot \int_Y H \operatorname{dvol}_\sigma} \tag{3.22}$$

We will use Corollary 3.15 in the proof of the main Ruelle invariant bound later in §3.4.

We will also need a less crude estimate on the time-averaged acceleration that uses the geometry of vector-field $I\nu$, but requires the hypothesis that X has small systolic ratio.

Lemma 3.16. *Suppose that X satisfies $E(a,b) \subset X \subset 4 \cdot E(a,b)$ and let $\Sigma \subset Y$ be the open subset*

$$\Sigma = Y \cap \mathbb{C} \times \operatorname{int}(E(b/2))$$

Then there is an $\varepsilon > 0$ and a $C > 0$ independent of a, b and X such that, if $a/b < \varepsilon$ and $T = b^{1/2}$, then

$$A_T \geqslant C \cdot a^{-1/2} \quad \text{on} \quad \Sigma \qquad \text{and} \qquad \operatorname{area}(\Sigma) \geqslant C \cdot \operatorname{area}(Y)$$

Proof. To bound A_T, the strategy is to show that the projection of $I\nu$ to the 2nd $\mathbb{C}$-factor is bounded along Σ by $(a/b)^{1/2}$. Thus, a length $T = b^{1/2}$ trajectory γ of $I\nu$ stays within a ball of diameter roughly $a^{1/2}$, and a variation of Lemma 3.14 implies the desired bound.

To bound area(Σ), the strategy is (essentially) to use the monotonicity of area under the inclusion $E(a,b) \subset X$ to reduce to the case of an ellipsoid. We can then use the estimates in Lemmas 3.3 and 3.7 to deduce the result.

Projection Bound. Let $\pi_j : \mathbb{R}^4 \simeq \mathbb{C}^2 \to \mathbb{C}$ denote the projections to each $\mathbb{C}$-factor for $i = 1, 2$. We begin by noting that there is an $A > 0$ independent of X, a and b such that

$$|\pi_2 \circ I\nu(x)| = |\pi_2 \circ \nu(x)| < A \cdot (a/b)^{1/2} \qquad \text{if} \qquad \pi_2(x) \in E(3b/4) \tag{3.23}$$

To deduce (3.23), assume that $x \in Y$ satisfies $\pi_2(x) \in E(3b/4)$ and that $\pi_2 \circ \nu(x) \neq 0$. Let $z \in 0 \times \partial E(b)$ be the unique vector such that $\pi_2(z - x)$ is a positive scaling of $\pi_2(\nu)$. Note that $z \in X$ since

$$0 \times E(b) \subset E(a,b) \subset X$$

Furthermore, since X is convex, we know that $\langle \nu(x), w - x \rangle \leqslant 0$ for any $w \in X$. Therefore

$$0 \geqslant \langle \nu(x), z - x \rangle = |\pi_2 \circ \nu(x))| \cdot |\pi_2(z - x)| + \langle \pi_1 \circ \nu(x), \pi_1(z - x) \rangle \tag{3.24}$$

Now note that since $\pi_2(x) \in E(3b/4)$ and $\pi_2(z) \in \partial E(b)$, we know that

$$|\pi_2(z-x)| \geqslant \frac{1-(3/4)^{1/2}}{\pi^{1/2}} \cdot b^{1/2}$$

Likewise, $\pi_1(X) \subset 4 \cdot E(a)$ so that $|\pi_1(z-x)| \leqslant 4a^{1/2}/\pi^{1/2}$. Finally, $|\pi_1 \circ \nu(x)| \leqslant |\nu(x)| = 1$. Thus, we can conclude that

$$|\pi_2 \circ \nu(x)| \leqslant \frac{|\pi_1 \circ \nu(x)| \cdot |\pi_1(z-x)|}{|\pi_2(z-x)|} \leqslant \frac{4}{1-(3/4)^{1/2}} \cdot (a/b)^{1/2}$$

Acceleration Bound. Now let $T = b^{1/2}$ and let $\gamma : [0,T] \to Y$ be a trajectory of $I\nu$ with $\gamma(0) \in \Sigma$. Since $\pi_2(\gamma(0)) \in E(b/2)$, we know that there is an interval $[0,S] \subset [0,T]$ where $\pi_2 \circ \gamma([0,S]) \subset E(3b/4)$. Thus, by (3.23), we know that for $t \in [0,S]$ we have

$$|\pi_2(\gamma(t)-\gamma(0))| \leqslant \int_0^t |\pi_2 \circ I\nu \circ \gamma| dt \leqslant A \cdot (a/b)^{1/2} \cdot t \leqslant A \cdot a^{1/2} \tag{3.25}$$

By picking $\varepsilon > 0$ small enough, we can ensure the following inequality.

$$Aa^{1/2} \leqslant (\frac{3b}{4\pi})^{1/2} - (\frac{b}{2\pi})^{1/2}$$

With this choice of ε, (3.25) implies that $\pi_2(\gamma(t)-\gamma(0)) \in E(3b/4)$ if $0 \leqslant t \leqslant T$. In fact, (3.25) implies that γ is inside of a ball, i.e.

$$\gamma(t) \in E(16a) \times E(\pi A^2 \cdot a) + p \subset B \cdot E(a,a) + p \quad \text{where} \quad p := 0 \times \pi_2(\gamma(0))$$

Here $B := (16+\pi A^2)^{1/2}$. The diameter of the ball $B \cdot E(a,a)$ is $2B \cdot (a/\pi)^{1/2}$. Therefore, by applying (3.20) and (3.21) we see that

$$\frac{2Ba^{1/2}}{\pi^{1/2}} \cdot A_T(x) = \frac{\operatorname{diam}(B \cdot E(a,a))}{T} \cdot \int_0^T |\ddot{\gamma}| dt \geqslant 1 - \frac{2\operatorname{diam}(B \cdot E(a,a))}{T} = 1 - \frac{4B}{\pi^{1/2}} \cdot (a/b)^{1/2}$$

We now choose $C > 0$ and $\varepsilon > 0$ independent of a, b and X, such that

$$A_T(x) \geqslant (\frac{\pi^{1/2}}{2B} - 2 \cdot (a/b)^{1/2}) \cdot a^{-1/2} \geqslant Ca^{-1/2} \quad \text{if} \quad a/b \leqslant \varepsilon$$

This proves the desired bound on time-averaged acceleration.

Area Bound. Let U denote the convex domain given by the intersection $X \cap (\mathbb{C} \times E(b/2))$. Note that we have the following inclusion.

$$E(a/2, b/2) \subset E(a,b) \cap (\mathbb{C} \times E(b/2)) \subset U$$

Furthermore, the boundary of U decomposes as follows.

$$\partial U = \Sigma \cup \Sigma' \qquad \text{where} \qquad \Sigma' := X \cap (\mathbb{C} \times \partial E(b/2))$$

Since $X \subset 4 \cdot E(a,b)$, we have $\Sigma' \subset R$ where R is the hypersurface

$$R := 4 \cdot E(a,b) \cap (\mathbb{C} \times \partial E(b/2)) = E(31a/2) \times \partial E(b/2)$$

Combining the above facts and applying the monotonicity of surface area under inclusion of convex domains, we find that

$$\operatorname{area}(\Sigma) = \operatorname{area}(\partial U) - \operatorname{area}(\Sigma') \geqslant \operatorname{area}(\partial E(a/2, b/2)) - \operatorname{area}(R)$$

By Lemma 3.7 and direct calculation, we compute the areas of $\partial E(a/2, b/2)$ and R to be

$$\operatorname{area}(\partial E(a/2,b/2)) \geqslant 2^{-3/2} \cdot ba^{1/2} \qquad \operatorname{area}(R) = \frac{31a}{2} \cdot (2\pi b)^{1/2} = 31 \cdot (\pi/2)^{1/2} \cdot (a/b)^{1/2} \cdot ba^{1/2}$$

Now let $B < 2^{-5/2}$ and choose $\varepsilon > 0$ small enough to that if $a/b < \varepsilon$ then

$$2^{-3/2} - 31 \cdot (\pi/2)^{1/2} \cdot (a/b)^{1/2} > B$$

By applying this inequality and the upper bound for area in Lemma 3.7, we find that for some $C > 0$ independent of X, a and b and an $\varepsilon > 0$ as above, we have

$$\operatorname{area}(\Sigma) \geqslant (2^{-3/2} - 31 \cdot (\pi/2)^{1/2} \cdot (a/b)^{1/2}) \cdot ba^{1/2} \geqslant C \cdot ba^{1/2} \geqslant \operatorname{area}(Y)$$

This yields the desired ara bound and concludes the proof of the lemma. $\square$

By plugging the bounds for A_T and $\operatorname{area}(\Sigma)$ from Lemma 3.16 into Lemma 3.13, we acquire the following variation of Corollary 3.15.

Corollary 3.17. *Let X be a convex domain with smooth boundary Y, such that $E(a,b) \subset X \subset 4 \cdot E(a,b)$. Then there exists a $C > 0$ and $\varepsilon > 0$ independent of X, a and b such that*

$$\int_Y S(I\nu, I\nu) \cdot \operatorname{dvol}_\sigma \geqslant C \cdot \frac{\operatorname{area}(Y)^2}{a \cdot \int_Y H \operatorname{dvol}_\sigma} \quad \text{if} \quad a/b < \varepsilon$$

3.4. Proof of main bound. We now combine the results of §3.1-3.3 to prove Proposition 3.1.

Proof. (Proposition 3.1) By Lemma 3.6, we may assume that X is sandwiched between standard ellipsoid $E(a,b)$ with $0 < a \leqslant b$ and a scaling.

$$E(a,b) \subset X \subset 4 \cdot E(a,b)$$

We begin by proving the lower bound, under this assumption. By Lemma 3.11, we have

$$\operatorname{Ru}(Y) \geqslant \frac{1}{2\pi} \cdot \int_Y S(I\nu, I\nu) \operatorname{dvol}_\sigma \tag{3.26}$$

By applying the lower bound in Corollary 3.15 and using the estimates for diameter, area, total curvature, volume and systolic ratio in Lemma 3.7, we see that for some $C > 0$ we have

$$\int_Y S(I\nu, I\nu) \cdot \operatorname{dvol}_\sigma \geqslant \frac{\operatorname{area}(Y)^2}{6\pi \cdot \operatorname{diam}(Y)^2 \cdot \int_Y H \operatorname{dvol}_\sigma} \geqslant C \cdot a \geqslant \operatorname{vol}(X)^{1/2} \cdot \operatorname{sys}(Y)^{1/2} \tag{3.27}$$

On the other hand, suppose that $\frac{a}{b} \ll 1$. Due to Lemma 3.7, this is equivalent to $\operatorname{sys}(Y) \ll 1$. By Corollary 3.17 and the estimates in Lemma 3.7, there are constants $A, B, C > 0$ with

$$\int_Y S(I\nu, I\nu) \operatorname{dvol}_\sigma \geqslant A \cdot \frac{\operatorname{area}(Y)^2}{a \cdot \int_Y H \operatorname{dvol}_\sigma} \geqslant B \cdot b \geqslant C \cdot \operatorname{vol}(X)^{1/2} \cdot \operatorname{sys}(Y)^{-1/2} \tag{3.28}$$

By assembling the estimate (3.26) with the two estimates (3.27) and (3.28), we deduce the following lower bound for some $C > 0$.

$$\operatorname{Ru}(Y) \geqslant C \cdot \operatorname{vol}(X)^{1/2} \cdot \operatorname{sys}(Y)^{-1/2} \tag{3.29}$$

After some rearrangement, this is the desired lower bound.

The second inequality is easier to show. By using the upper bound in Lemma 3.11 and the estimate for the mean curvature in Lemma 3.7, we see that for some $A, C > 0$ we have

$$\mathrm{Ru}(Y) \leqslant \int_Y H\,\mathrm{dvol}_\sigma \leqslant A \cdot b \leqslant C \cdot \mathrm{vol}(X)^{1/2} \cdot \mathrm{sys}(Y)^{-1/2} \tag{3.30}$$

This implies the desired upper bound, and concludes the proof. □

4. Non-convex, dynamically convex contact forms

In this section, we use the methods of [**1**] to construct a dynamically convex contact form with systolic ratio and volume close to 1, and arbitrarily small Ruelle invariant.

Proposition 4.1. *For every $\varepsilon > 0$, there exists a dynamically convex contact form α on S^3 with*

$$\mathrm{vol}(S^3, \alpha) = 1 \qquad \mathrm{sys}(S^3, \alpha) \geqslant 1 - \varepsilon \qquad \mathrm{Ru}(S^3, \alpha) \leqslant \varepsilon$$

4.1. Hamiltonian disk maps. We begin with some notation and preliminaries on Hamiltonian maps of the disk that we will need for the rest of the section.

Let $\mathbb{D} \subset \mathbb{R}^2$ denote the unit disk in the plane with ordinary coordinates (x, y) and radial coordinates (r, θ). We use λ and ω to denote the standard Liouville form and symplectic form.

$$\lambda := \frac{1}{2} r^2 d\theta = \frac{1}{2}(xdy - ydx) \qquad \text{and} \qquad \omega := rdr \wedge d\theta = dx \wedge dy$$

Let $\phi : [0, 1] \times \mathbb{D} \to \mathbb{D}$ be a the Hamiltonian flow (for $t \in [0, 1]$) generated by a time-dependent Hamiltonian on $\mathbb{D}$ vanishing on the boundary, i.e.

$$H : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R} \qquad \text{with} \qquad H|_{\partial\mathbb{D}} = 0$$

We let X_H denote the Hamiltonian vector field and adopt the convention that $\iota_{X_H}\omega = dH$. The differential of ϕ defines a map $\Phi : \mathbb{R} \times \mathbb{D} \to \mathrm{Sp}(2)$ with $\Phi|_{0\times\mathbb{D}} = \mathrm{Id}$, which lifts uniquely to a map

$$\widetilde{\Phi} : \mathbb{R} \times \mathbb{D} \to \widetilde{\mathrm{Sp}}(2) \qquad \text{satisfying} \qquad \widetilde{\Phi}(S + T, z) = \widetilde{\Phi}(T, \phi_S(z))\widetilde{\Phi}(S, z) \tag{4.1}$$

There are two key functions on $\mathbb{D}$ associated to the family of Hamiltonian diffeomorphisms ϕ. First, there is the action and the associated Calabi invariant.

Definition 4.2. The *action* $\sigma_\phi : \mathbb{D} \to \mathbb{R}$ and *Calabi invariant* $\mathrm{Cal}(\mathbb{D}, \phi) \in \mathbb{R}$ of ϕ are defined by

$$\sigma_\phi = \int_0^1 \phi_t^*(\iota_{X_H}\lambda + H) \cdot dt \qquad \text{and} \qquad \mathrm{Cal}(\mathbb{D}, \phi) = \int_{\mathbb{D}} \sigma \cdot \omega \tag{4.2}$$

The action measures the failure of ϕ to preserve λ, as captured by the following formula.

$$\phi_1^*\lambda - \lambda = d\sigma_\phi \tag{4.3}$$

Next, there is the rotation map and the associated Ruelle invariant. To discuss these quantities, we require the following lemma.

Lemma 4.3. *Let $\phi : [0,1] \times \mathbb{D} \to \mathbb{D}$ be the flow of a Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{D}$ with $\sigma_\phi > 0$. Then the sequences $r_n : \mathbb{D} \to \mathbb{R}$ and $s_n : \mathbb{D} \to \mathbb{R}$ given by*

$$r_n(z) := \frac{1}{n}\rho \circ \widetilde{\Phi}(n,z) \qquad \text{and} \qquad s_n(z) := \frac{1}{n}\sum_{k=0}^{n-1} \sigma_\phi \circ \phi^k(z)$$

converge in $L^1(\mathbb{D})$ to r_ϕ and s_ϕ, respectively. The map s_k^{-1} also converges to s_ϕ^{-1} in $L^1(\mathbb{D})$.

Proof. We apply Kingman's sub-additive ergodic theorem **[52]** to the map $g_n = r_n + C$ for sufficiently large $C > 0$. Applying (4.1) and the quasimorphism property of ρ, we find that

$$g_{m+n} \leqslant g_m + g_n \circ \phi^m$$

By Kingman's ergodic theorem, this implies that $\frac{g_n}{n}$ has a limit r_∞ in $L^1(\mathbb{D})$. Since $\|g_n - r_n\|_{L^1}$ is bounded, we acquire the same result for r_n.

By Birkhoff's ergodic theorem, s_n converges to a limit $s_\infty \in L^1(\mathbb{D})$. Note that for some $c > 0$, we have

$$c^{-1} \leqslant \sigma_\phi \leqslant c \quad \text{and therefore} \quad c^{-1} \leqslant s_n \leqslant c$$

Thus $s_\infty > 0$ pointwise almost everywhere and s_∞^{-1} is well-defined almost everywhere. Since $|s_n|^{-1} < c$, we can apply the dominated convergence theorem to conclude that s_∞^{-1} is integrable and $s_n^{-1} \to s_\infty^{-1}$ in L^1. A similar argument applies to r_n/s_n, which converges to r_∞/s_∞. □

Definition 4.4. The *rotation* $r_\phi : \mathbb{D} \to \mathbb{R}$ and *Ruelle invariant* $\mathrm{Ru}(\mathbb{D},\phi) \in \mathbb{R}$ of ϕ are defined by

$$r_\phi := \lim_{n\to\infty} r_n \qquad \text{and} \qquad \mathrm{Ru}(\mathbb{D},\phi) = \int_{\mathbb{D}} r_\phi \cdot \omega \tag{4.4}$$

Remark 4.5. Our Ruelle invariant $\mathrm{Ru}(\mathbb{D},\phi)$of a symplectomorphism of the disk agrees with the one introduced by Ruelle in **[69]**.

The action, rotation, Calabi invariant and rotation invariant depend only on the homotopy class of ϕ relative to the endpoints, or equivalently the element in the universal cover of $\mathrm{Ham}(\mathbb{D},\phi)$.

We conclude this review with a discussion of periodic points and their invariants.

Definition 4.6. A *periodic point* p of $\phi : \mathbb{D} \to \mathbb{D}$ is a point such that $\phi^k(p) = p$ for some $k \geqslant 1$. The period $\mathcal{L}(p)$, action $\mathcal{A}(p)$ and rotation number $\rho(p)$ of p are given, respectively, by

(4.5)

$$\mathcal{L}(p) := \min\{j > 0 | \phi^j(p) = p\} \qquad \mathcal{A}(p) = \sum_{i=0}^{\mathcal{L}(p)-1} \sigma_\phi \circ \phi^i(p) \qquad \rho(p) := \rho \circ \widetilde{\Phi}(\mathcal{L}(p), p)$$

Note that the rotation number can also be written as $\rho(p) = \mathcal{L}(p) \cdot r_\phi(p)$.

4.2. Open books of disk maps. We next review the construction of contact forms on S^3 from symplectomorphisms of the disk, using open books.

Construction 4.7. Let $H : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R}$ be a Hamiltonian with flow $\phi : [0,1] \times \mathbb{D} \to \mathbb{R}$ such that

(i) Near $\partial\mathbb{D}$, H is of the form $H(t,r,\theta) = C \cdot \pi(1-r^2)$ for some $C > 0$.

(ii) The action function σ_ϕ of the Hamiltonian is positive everywhere.

We now construct the *open book* contact form α on S^3 associated to $(\mathbb{D}, \phi)$. We proceed by producing two contact manifolds (U, α) and (V, β), then gluing them by a strict contactomorphism.

To construct U, we consider the contact form $dt + \lambda$ on $\mathbb{R} \times \mathbb{D}$. Due to the identity $d\sigma_\phi = \phi_1^*\lambda - \lambda$ in (4.3), the map f defined by

$$f : \mathbb{R} \times \mathbb{D} \to \mathbb{R} \times \mathbb{D} \qquad f(t,z) = (t - \sigma_\phi(z), \phi_1(z))$$

is a strict contactomorphism. Thus, we can form the manifold U as the following quotient space.

$$U = \mathbb{R} \times \mathbb{D}/\sim \qquad \text{defined by } (t,z) \sim f(t,z)$$

The contact form $dt + \lambda$ descends to a contact form α on U. Note that a fundamental domain of this quotient is given by

$$\Omega = \{(t,z) | 0 \leqslant t \leqslant \sigma_\phi(z)\}$$

To construct V, we choose a small $\varepsilon > 0$ and let

$$V := \mathbb{R}/\pi\mathbb{Z} \times \mathbb{D}(\varepsilon) \qquad \beta := (1-r^2)dt + \frac{C}{2}r^2 d\theta$$

Here $\mathbb{D}(\varepsilon) \subset \mathbb{C}$ is the disk of radius ε, t is the $\mathbb{R}/\pi\mathbb{Z}$ coordinate and (r,θ) are radial coordinates on $\mathbb{D}(\varepsilon)$. There is a strict contactomorphism Φ identifying subsets of U and V, given by

$$\Psi : V \backslash (\mathbb{R}/\pi\mathbb{Z} \times 0) \to U \qquad \text{with} \qquad \Psi(t,r,\theta) := (\frac{C}{2} \cdot \theta, \sqrt{1-r^2}, 2t - C\theta)$$

We now define $Y = \text{int}(U) \cup_\Psi V$ as the gluing of the interior of U and V via Φ, and α as the inherited contact form. Since ϕ is Hamiltonian isotopic to the identity, the resulting contact form (Y, α) is contactomorphic to standard contact S^3.

In order to relate various invariants associated to (S^3, α) and its Reeb orbits to corresponding structures for $(\mathbb{D}, \phi)$, we need to introduce a certain trivialization of ξ over U.

Construction 4.8. Let $(U, \xi|_U)$ be as in Construction 4.7. We let τ denote the continuous trivialization of $\xi|_U$ defined as follows. On the fundamental domain Ω, we let

(4.6)
$$\tau : \Omega \to \text{Hom}(\xi|_U, \mathbb{R}^2) \qquad \text{given by} \qquad \tau(t,z) := \exp(2\pi i t/\sigma_\phi(z)) \circ \Phi(t/\sigma_\phi(z), z) \circ \Pi_{\mathbb{D}}$$

Here $\Phi : [0,1] \times \mathbb{D} \to \mathbb{D}$ is the differential $d\phi$ of the flow $\phi : [0,1] \times \mathbb{D} \to \mathbb{D}$ and $\Pi_{\mathbb{D}} : \xi \to T\mathbb{D}$ denotes projection to the (canonically trivial) tangent bundle $T\mathbb{D}$ of $\mathbb{D}$. Note also that $\circ$ denotes composition of bundle maps.

To check that τ descends to a well-defined trivialization on U, we must check that it is compatible with the quotient map $f : \mathbb{R} \times \mathbb{D} \to \mathbb{R} \times \mathbb{D}$. Indeed, we have

$$\tau(\sigma_\phi(z), z) = \Phi(1,z) \circ \Pi_{\mathbb{D}} = \tau(0, \phi_1(z)) \circ df_{\sigma(z),z}$$

This precisely states that projection commutes with the isomorphism identifying tangent spaces in the quotient, so τ descends from Ω to U.

Lemma 4.9. *Let* $\tau : \xi|_U \to \mathbb{R}^2$ *be the trivialization in Construction 4.8. Then*

(a) *The restriction* $\tau|_K$ *of* τ *to any compact subset* $K \subset \mathrm{int}(U)$ *of the interior of* U *is the restriction of a global trivialization of* ξ *on* S^3.

(b) *The local rotation number* $\mathrm{rot}_\tau : U \to \mathbb{R}$ *of* $(U, \alpha|_U)$ *with respect to* τ *agres with the restriction of the local rotation number* $\mathrm{rot} : S^3 \to \mathbb{R}$ *of* (S^3, α) *with respect to the global trivialization.*

Proof. Let $V = \mathbb{R}/\pi\mathbb{Z} \times \mathbb{D}(\varepsilon)$ and Ψ be as in Construction 4.7. For any $\delta < \varepsilon$, we let $V(\delta) \subset V$ and $U(\delta) \subset U$ denote

$$V(\delta) := \mathbb{R}/\pi\mathbb{Z} \times D(\delta) \subset V \qquad \text{and} \qquad U(\delta) := \mathrm{int}(U)\backslash \mathrm{int}(\Psi(V(\delta)))$$

The sets $U(\delta)$ are an exhaustion of $\mathrm{int}(U)$ by compact, Reeb-invariant contact submanifolds.

To show (a), we assume that $K = U(\delta)$. The homotopy classes of trivializations $\mathcal{T}$ of ξ over $U(\delta)$ are in bijection with $H^1(U(\delta);\mathbb{Z}) \simeq \mathbb{Z}$. A map to $\mathbb{Z}$ classifying elements of $\mathcal{T}$ is given by

$$\mathcal{T} \to \mathbb{Z} \qquad \text{given by} \qquad \sigma \mapsto \mathrm{sl}(\gamma, \sigma)$$

Here $\mathrm{sl}(\gamma, \sigma)$ is the self-linking number (in the trivialization σ) of the following transverse knot.

$$\gamma : \mathbb{R}/2\pi\mathbb{Z} \to U(\delta) \qquad \gamma(\theta) = \Psi(0, \varepsilon, \theta) = (\frac{C\theta}{2}, \sqrt{1-\varepsilon^2}, -C\theta)$$

The knot γ bounds a Seifert disk $\Sigma = 0 \times \mathbb{D}(\varepsilon)$ in $V \subset S^3$. The foliation $\xi \cap \Sigma$ has a single positive elliptic singularity, so the self-linking number of the boundary γ with respect to the global trivialization is $\mathrm{sl}(\gamma) = -1$.

To compute $\mathrm{sl}(\gamma, \tau)$, we push γ into Σ along a collar neighborhood to acquire a nowhere zero section $\eta : \mathbb{R}/2\pi\mathbb{Z} \to \xi$ and then compose with τ to acquire a map $\tau \circ \eta : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2\backslash 0$. Up to isotopy through nowhere zero sections, we can compute that

$$\tau \circ \eta(\theta) = e^{i\theta} \in \mathbb{C} = \mathbb{R}^2$$

On the other hand, the self-linking number can be computed as the negative of the winding number of this map.

$$\mathrm{sl}(\gamma, \tau) = -\,\mathrm{wind}(\tau \circ \eta) = -1$$

This proves that τ agrees with the restriction of the global trivialization.

To show (b), note that since $U(\delta)$ is compact, we can choose a global trivialization of ξ on S^3

$$\sigma : \xi \simeq \mathbb{R}^2 \qquad \text{such that} \qquad \sigma|_{U(\delta)} = \tau|_{U(\delta)}$$

By Proposition 2.11(c), $\mathrm{rot}_\sigma = \mathrm{rot}$ on S^3 and so the local rotation numbers satisfy

$$\mathrm{rot}\,|_{U(\delta)} = \mathrm{rot}_\sigma\,|_{U(\delta)} = \mathrm{rot}_\tau\,|_{U(\delta)}$$

Since this holds for any δ, this shows (b) on all of $\mathrm{int}(U)$. Note that we assiduously avoided extending τ itself from $\mathrm{int}(U)$ to S^3 in this argument. □

Proposition 4.10 (Open Book). *Let* H *and* ϕ *be as in Construction 4.7. Then there exists a contact form* α *on* S^3 *with the following properties.*

(a) (Surface Of Section) There is an embedding $\iota : \mathbb{D} \to S^3$ such that $\iota(\mathbb{D})$ is a surface of section with return map ϕ_1 and first return time σ, and such that $\omega = \iota^* d\alpha$.

(b) (Volume) The volume of (S^3, α) is given by the Calabi invariant of $(\mathbb{D}, \phi)$, i.e.

$$\operatorname{vol}\big(S^3, \alpha\big) = \operatorname{Cal}(\mathbb{D}, \phi)$$

(c) (Ruelle) The Ruelle invariant of (S^3, α) is given by a shift of the Ruelle invariant of $(\mathbb{D}, \phi)$.

$$\operatorname{Ru}(S^3, \alpha) = \operatorname{Ru}(\mathbb{D}, \phi) + \pi$$

(d) (Binding) The binding $b = \iota(\partial\mathbb{D})$ is a Reeb orbit of action π and rotation number $1 + 1/C$.

(e) (Orbits) Every simple orbit $\gamma \subset S^3 \backslash b$ corresponds to a periodic point p of ϕ that satisfies

$$\operatorname{lk}(\gamma, b) = \mathcal{L}(p) \qquad \mathcal{A}(\gamma) = \mathcal{A}(p) \qquad \rho(\gamma) = \rho(p) + \mathcal{L}(p)$$

Proof. We prove each of these properties separately.

Surface Of Section. Define the inclusion $\iota : \mathbb{D} \to S^3$ as the following composition.

$$\iota : \mathbb{D} = 0 \times \mathbb{D} \to \mathbb{R} \times \mathbb{D} \xrightarrow{\pi} Y \simeq S^3$$

The surface $0 \times \mathbb{D}$ is transverse to the Reeb vector field ∂_t of $\mathbb{R} \times \mathbb{D}$ and intersects every flowline $\mathbb{R} \times z$. Also, $(\mathbb{R} \times z) \cap \Omega$ has action $\sigma_\phi(z)$ and ends on $(\sigma_\phi(z), z) \sim (0, \phi_1(z))$. Thus $\iota(\mathbb{D}) = \pi(0 \times \mathbb{D})$ is a surface of section with return time σ_ϕ and monodromy ϕ_1. Finally, note that

$$\iota^*(d\alpha) = d(dt + \lambda)|_{0 \times \mathbb{D}} = \omega$$

This verifies all of the properties of $\iota : \mathbb{D} \to Y \simeq S^3$ listed in (a).

Calabi Invariant. This property follows from a simple calculation of the volume using the fundamental domain Ω.

$$\operatorname{vol}(Y, \alpha) = \int_Y \alpha \wedge d\alpha = \int_\Omega dt \wedge d\lambda = \int_{\mathbb{D}} \sigma_\phi \cdot \omega = \operatorname{Cal}(\mathbb{D}, \phi)$$

Ruelle Invariant. Let $\operatorname{rot} : S^3 \to \mathbb{R}$ be the local rotation number of (S^3, α). By Lemma 4.9, the restriction of rot to the (open) fundamental domain $\Omega \subset S^3$ coincides with rot_τ. Since $S^3 \backslash \Omega$ is measure 0 in S^3, we thus have

$$\operatorname{Ru}(S^3, \alpha) = \int_{S^3} \operatorname{rot} \cdot \alpha \wedge d\alpha = \int_\Omega \operatorname{rot}_\tau \cdot dt \wedge \omega = \int_{\mathbb{D}} \iota^* \operatorname{rot}_\tau \cdot \sigma_\phi \omega \tag{4.7}$$

Here $\iota^* \operatorname{rot}_\tau$ denotes the pullback of rot_τ via the map $\iota : \mathbb{D} \to S^3$ from (a). We have used the Reeb invariance of rot_τ, i.e. the fact that $\operatorname{rot}_\tau(t, z) = \iota^* \operatorname{rot}_\tau(z)$.

To apply this alternative formula for $\operatorname{Ru}(S^3, \alpha)$, let T_k denote the kth positive time that the Reeb trajectory $\gamma : [0, \infty) \to S^3$ intersects the surface of section $\iota(\mathbb{D})$. Then

$$\iota^* \operatorname{rot}_\tau = \lim_{k \to \infty} \frac{\rho \circ \widetilde{\Phi}_\tau(T_k, -)}{T_k} = \lim_{k \to \infty} \frac{\rho \circ \widetilde{\Phi}(k, -) + k}{\sum_{i=0}^{k-1} \sigma_\phi \circ \phi^i} = \frac{r_\phi + 1}{s_\phi}$$

Here the maps r_ϕ and s_ϕ are the averaged rotation and action maps constructed in Lemma 4.3. By construction, these maps are invariant under pullback by ϕ. Thus

$$\int_{\mathbb{D}} \frac{r_\phi + 1}{s_\phi} \cdot \sigma_\phi \omega = \frac{1}{n} \sum_{k=0}^{n-1} \int_{\mathbb{D}} [\phi^k]^* \Big(\frac{r_\phi + 1}{s_\phi} \cdot \sigma_\phi \omega\Big) = \int_{\mathbb{D}} \frac{r_\phi + 1}{s_\phi} \cdot s_n \omega \quad \text{where} \quad s_n = \frac{1}{n} \sum_{k=0}^{n-1} \sigma_\phi \circ \phi^k$$

By Lemma 4.3, we know that $s_n \to s_\phi$ in $L^1(\mathbb{D})$. Thus, by combining the above formula in the $n \to \infty$ limit with (4.7), we acquire the desired property.

$$\mathrm{Ru}(S^3, \alpha) = \int_{\mathbb{D}} \frac{r_\phi + 1}{s_\phi} \cdot \sigma_\phi \cdot \omega = \int_{\mathbb{D}} \frac{r_\phi + 1}{s_\phi} \cdot s_\phi \cdot \omega = \int_{\mathbb{D}} (r_\phi + 1) \cdot \omega = \mathrm{Ru}(\mathbb{D}, \phi) + \pi$$

Binding. Let $b = \iota(\partial\mathbb{D})$ be the binding which coincides with $\mathbb{R}/\pi\mathbb{Z} \times 0$ in V. First note that the Reeb vector field is given on (V, β) by the following formula.

$$R_\beta = \partial_t + \frac{2}{C}\partial_\theta \tag{4.8}$$

Thus b is a Reeb orbit. Since b bounds a symplectic disk $\iota(\mathbb{D}) \subset S^3$ of area π, the action is π. To compute $\rho(b)$, note that there is a natural trivialization of $\xi|_V = \ker(\beta)$ given by

$$\nu : \xi|_V \subset TV \xrightarrow{\pi} T\mathbb{D}(\varepsilon) = \mathbb{R}^2$$

The Reeb flow $\phi : \mathbb{R} \times V \to V$ and the linearized Reeb flow $\Phi_\nu : \mathbb{R} \times V \to \mathrm{Sp}(2)$ with respect to ν can be calculated from (4.8), as follows.

$$\phi_t(s, z) = (s + t, e^{2it/C} \cdot z) \qquad \Phi_\nu(t, s, z) = e^{2it/C}$$

Thus the rotation number $\rho(b, \nu)$ of b in the trivialization ν is $1/C$. Finally, to compute the rotation number $\rho(b) = \rho(b, \tau)$ with respect to the global trivialization τ on ξ, we note that

$$\rho(b, \tau) - \rho(b, \nu) = \mu(\tau \circ \nu^{-1}|_b) = c_1(\xi|_{\iota(\mathbb{D})}, \tau) - c_1(\xi|_{\iota(\mathbb{D})}, \nu) = -c_1(\xi|_{\iota(\mathbb{D})}, \nu)$$

Here $\mu : \pi_1(\mathrm{Sp}(2)) \to \mathbb{Z}$ is the Maslov index and $c_1(\xi|_{\iota(\mathbb{D})}, -)$ is the relative Chern class of $\xi|_{\iota(\mathbb{D})}$ with respect to a given trivialization over $\iota(\partial\mathbb{D})$, which vanishes for τ.

On the other hand, the trivialization ν is specified by the section of $\xi|_{\iota(\mathbb{D})}$ given by pushing $\iota(\partial\mathbb{D})$ into $\iota(\mathbb{D})$ along a collar neighborhood. Thus, $-c_1(\xi|_{\iota(\mathbb{D})}, \nu)$ is precisely the self-linking number $\mathrm{sl}(b)$ of b. This number can be calculated as a signed count of singularities of the foliation $\xi \cap \iota(\mathbb{D})$, which has 1 elliptic singularity. Thus $\mathrm{sl}(b) = -1$ and $\rho(b) = 1 + 1/C$.

Orbits. An embedded closed orbit $\gamma : \mathbb{R}/L\mathbb{Z} \to Y$ of α that is disjoint from the binding b is equivalent to a closed orbit of $(U, \alpha|_U)$. The orbit γ intersects the surface of section $\iota(\mathbb{D})$ transversely at $n \geqslant 1$ times $T_0 = 0, T_1, \dots, T_n = L$. Let

$$p_k \in \mathbb{D} \qquad \text{be such that} \qquad \iota(p_k) = \gamma(T_k) \cap \iota(\mathbb{D})$$

Since $\iota(\mathbb{D})$ is a surface of section, we have $p_{i+1} = \phi(p_i)$ and since γ is closed, $p_n = p_0$. Thus $p = p_0$ is a periodic point of period

$$\mathcal{L}(p) = n = \iota_*[\mathbb{D}] \cdot [\gamma] = \mathrm{lk}(\gamma, b)$$

Next, note that on the interval $[T_i, T_{i+1}]$, γ restricts to a map $[T_i, T_{i+1}] \to \Omega$ given by $\gamma(t) = (t, \iota(p_i))$, from which it follows that

$$\mathcal{A}(\gamma) = \sum_{k=0}^{n-1} \int_{T_k}^{T_{k+1}} \gamma^*(dt + \alpha) = \sum_{k=0}^{n-1} \int_0^{\sigma(p_k)} dt = \sum_{k=0}^{n-1} \sigma \circ \phi^k(p) = \mathcal{A}(p)$$

Finally, due to Lemma 4.9 we may use the trivialization τ to compute the rotation number. For the purpose of abbreviation, we adopt the notation

$$y_i = \iota(p_i) = \gamma(T_i) \qquad L_i = T_{i+1} - T_i = \sigma_\phi(p_i)$$

Note that the lifted linearized Reeb flow with respect to τ at time L can be written as

$$\widetilde{\Phi}_\tau(L, \gamma(0)) = \widetilde{\Phi}_\tau(L_{n-1}, y_{n-1})\widetilde{\Phi}_\tau(L_{n-2}, y_{n-2}) \dots \widetilde{\Phi}_\tau(L_0, y_0) \tag{4.9}$$

The linearized Reeb flow $\widetilde{\Phi}_\tau(L_i, y_i)$ takes place along a trajectory connecting $(0, p_i)$ to $(\sigma_\phi(p_i), p_i)$ in the fundamental domain Ω. We may be directly compute from (4.6) that

$$\Phi_\tau(t, y_i) = \exp(2\pi i t/\sigma_\phi(p_i)) \circ \Phi(t/\sigma_\phi(z), p_i) \quad \text{and so} \quad \widetilde{\Phi}_\tau(L_i, y_i) = \widetilde{\Xi} \cdot \widetilde{\Phi}(1, p_i) \tag{4.10}$$

Here $\widetilde{\Xi}$ is the unique lift of $\mathrm{Id} \in \mathrm{Sp}(2)$ with $\rho(\widetilde{\Xi}) = 1$. This is a central element of $\widetilde{\mathrm{Sp}}(2)$, so combining (4.9) and (4.10) we have

$$\widetilde{\Phi}_\tau(L, \gamma(0)) = \widetilde{\Xi}^n \cdot \widetilde{\Phi}(1, \phi^{n-1}(p)) \cdot \widetilde{\Phi}(1, \phi^{n-2}(p)) \cdots \widetilde{\Phi}(1, p) = \widetilde{\Xi}^n \cdot \widetilde{\Phi}(n, p)$$

Since $\rho(\widetilde{\Xi} \cdot \widetilde{\Psi}) = 1 + \rho(\widetilde{\Psi})$ for any $\widetilde{\Psi} \in \widetilde{\mathrm{Sp}}(2)$, we can conclude that

$$\rho(\gamma) = \rho \circ \widetilde{\Phi}_\tau(L, \gamma(0)) = \rho \circ \widetilde{\Phi}(n, p) + n = \rho(p) + \mathcal{L}(p)$$

This completes the proof of (e), and the entire proposition. □

4.3. Radial Hamiltonians. A Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R}$ that is rotationally invariant will be called *radial*. In other words, H is radial if it can be written as

$$H(t, r, \theta) = h(t, r) \qquad \text{for a map} \qquad h : \mathbb{R}/\mathbb{Z} \times [0, 1] \to \mathbb{R}$$

We will require a few lemmas regarding radial Hamiltonians.

Lemma 4.11. *Let $H : \mathbb{D} \to \mathbb{R}$ be an autonomous, radial Hamiltonian with $H = h \circ r$. Then*

$$\sigma_\phi(r, \theta) = h(r) - \frac{1}{2} r h'(r) \qquad \textit{and} \qquad r_\phi(r, \theta) = -\frac{h'(r)}{2\pi r} \tag{4.11}$$

Proof. We calculate the Hamiltonian vector field X_H and the action function σ_ϕ as follows.

$$X_H = -\frac{h'}{r} \cdot \partial_\theta \qquad \text{and} \qquad \text{and} \qquad \sigma_\phi(r, \theta) = \int_0^1 \phi_t^*(-\frac{rh'(r)}{2} + h(r)) \cdot dt = h(r) - \frac{1}{2} r h'(r)$$

Here we use the fact that the Hamiltonian flow ϕ preserves any function of r. Next, we note that the differential $\Phi : \mathbb{R} \times \mathbb{D} \to \mathbb{D}$ of the flow ϕ is given by

$$\Phi(t, z)v = \exp(\frac{-h'}{r} \cdot it)v + \frac{it(rh'' - h')}{r^2} \cdot \exp(\frac{-h'}{r} \cdot it)z \cdot dr(v)$$

Note that if we use $s = iz/|z|$, then $dr(v) = 0$. Thus, if $\widetilde{\Phi} : \mathbb{R} \times \mathbb{D} \to \widetilde{\mathrm{Sp}}(2)$ denotes the lift of Φ, and ρ_s denotes the rotation number relative to s (see Definition 2.4) then

$$\Phi(t, z)s = \exp(\frac{-h'(r)}{r} \cdot it)s \qquad \text{and thus} \qquad \rho_s \circ \widetilde{\Phi}(T, z) = T \cdot \frac{-h'(r)}{2\pi r} \tag{4.12}$$

Since $\rho_s : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ and $\rho : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ are equivalent quasimorphisms (Lemma 2.5), we have

$$r_\phi = \lim_{T \to \infty} \frac{\rho \circ \widetilde{\Phi}(T, -)}{T} = \lim_{T \to \infty} \frac{\rho_s \circ \widetilde{\Phi}(T, -)}{T} = \frac{-h' \circ r}{2\pi r} \qquad \text{in } L^1(\mathbb{D})$$

This concludes the proof of the lemma. □

More generally, a Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R}$ is called *radial around* $p \in \mathbb{D}$ if H is invariant under rotation around p, i.e. if H can be written as

$$H(t, x, y) = h(t, r_p) \qquad \text{for a map} \qquad h : \mathbb{R}/\mathbb{Z} \times [0, 1] \to \mathbb{R}$$

Here $r_p : \mathbb{D} \to \mathbb{R}$ be the distance from p, i.e. $r_p(z) = |z - p|$.

Lemma 4.12. *Let $H : \mathbb{D} \to \mathbb{R}$ be an autonomous Hamiltonian that is radial around $p = (a, b) \in \mathbb{D}$, with $H = h \circ r_p$, in a neighborhood U of p. Then on U, we have*

$$\sigma_\phi = h(r_p) - \frac{1}{2} r_p h'(r_p) + u_p - \phi_1^* u_p \qquad \text{and} \qquad r_\phi = -\frac{h'(r_p)}{2\pi r_p} \tag{4.13}$$

Here the map $u_p : \mathbb{D} \to \mathbb{R}$ is given by $u_p(x, y) = (bx - ay)/2$.

Proof. Let λ_p be the radial Liouville form on $(\mathbb{D}, \omega)$ centered at p. That is, λ_p is given by

$$\lambda_p = \frac{1}{2}((x - a)dy - (y - b)dx) = \lambda + du_p$$

Let $\tau : \mathbb{D} \to \mathbb{R}$ be the function decribed in (4.13). Then by Lemma 4.11, we know that on U we have

$$d\tau = (\phi_1^* \lambda_p - \lambda_p) + (\phi_1^* du_p - u_p) = \phi_1^* \lambda - \lambda = d\sigma_\phi$$

Thus it suffices to check that $\sigma_\phi(p) = \tau(p)$. Since $rh'(p) = 0$ and $u_p(p) = u_p(\phi_1(p)) = 0$, we see that $\tau(p) = h(0) = H(p)$. On the other hand, $X_H(p) = 0$, we see that

$$\sigma_\phi(p) = \int_0^1 \phi_t^*(\lambda(X_H) + H)dt = \int_0^1 h(0)dt = \tau(p)$$

Thus $\sigma_\phi(p) = \tau(p)$. The formula for r_ϕ follows from identical arguments to Lemma 4.11. □

4.4. A special Hamiltonian map. We next construct a special Hamiltonian flow $\phi : [0, 1] \times \mathbb{D} \to \mathbb{D}$ whose corresponding contact form will provide our counterexample. We define ϕ as a product

$$\phi = \phi^H \bullet \phi^G$$

Here $\phi^G : [0, 1] \times \mathbb{D} \to \mathbb{D}$ and $\phi^H : [0, 1] \times \mathbb{D} \to \mathbb{D}$ are autonomous flows generated by G and H, and the product occurs in the universal cover of the group $\mathrm{Ham}(\mathbb{D}, \omega)$ of Hamiltonian diffeomorphisms of $(\mathbb{D}, \omega)$. We denote the Hamiltonian generating ϕ by

$$H\#G : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R}$$

To construct G and H, we must fix the following setup (which will be used for the rest of §4.4).

Setup 4.13. Fix an integer $n \geqslant 10$ and let $\mathbb{S}(n, k) \subset \mathbb{D}$ for $0 \leqslant k \leqslant n - 1$ be the sector of points with angle $2\pi k/n < \theta < 2\pi(k + 1)/n$.

Let $U \subset \mathbb{D}$ be a finite union of disjoint disks in $\mathbb{D}$ such that each of the component disks $D \subset U$ is contained in one of the sectors $\mathbb{S}(n, k)$ and such that for every $D \subset U$ the disk $e^{2\pi i/n} \cdot D$ is a component disk of U as well. Finally, let $\delta > 0$ be a constant that is smaller than the radius of each disk D, smaller than the distance between any two of the disks D and D', and smaller than the distance between D and the boundary of any of the sectors $\mathbb{S}(n, k)$.

For any subset $S \subset \mathbb{D}$, we use the notation

$$N(S) := \{z \in \mathbb{D} | |z - p| \leqslant \delta \text{ for some } p \in S\}$$

The neighborhoods $N(\partial D)$, $N(D)$, $N(U)$ and $N(\partial U)$ will be of particular importance.

We now introduce the two Hamiltonians H and G in some detail.

CONSTRUCTION 4.14. We let $H : \mathbb{D} \to \mathbb{R}$ denote the radial Hamiltonian given by the formula

$$H(r, \theta) := \frac{\pi(n+1)}{n} \cdot (1 - r^2) \tag{4.14}$$

The Hamiltonian vector field $X_H = \frac{2\pi(n+1)}{n} \cdot \partial_\theta$ and so the Hamiltonian flow is given by

$$\phi^H : \mathbb{R} \times \mathbb{D} \to \mathbb{D} \qquad \text{with} \qquad \phi^H(t, z) = \exp(\frac{2\pi(n+1)}{n} \cdot it) \cdot z \tag{4.15}$$

In particular, the time 1 flow is rotation by $\frac{2\pi}{n}$ and preserves the collection U.

CONSTRUCTION 4.15. We let $G : \mathbb{D} \to \mathbb{R}$ denote a Hamiltonian that is invariant under rotation by angle $2\pi/n$ and that vanishes away from $N(U)$. That is

$$G(z) = G(e^{2\pi i/n} \cdot z) \qquad \text{and} \qquad G|_{\mathbb{D}\backslash N(U)} = 0 \tag{4.16}$$

Furthermore, let D be a component disk of U that is centered at $p \in \mathbb{D}$ and with radius s. Then we also assume that G is radial about p in the neighborhood $N(D)$ of D, i.e.

$$G|_{N(D)} = g \circ r_p \qquad \text{for a function} \qquad g : [0, s + \delta] \to \mathbb{R} \tag{4.17}$$

Finally, we assume that the function g satisfies the following conditions.

$$g(r) = -\pi \cdot (2 - \delta) \cdot (s^2 - r^2) \qquad \text{if } r \leqslant s - \delta \tag{4.18}$$

$$g \leqslant 0 \qquad 0 \leqslant g' \leqslant 2\pi \cdot (2 - \delta) \cdot (s - \delta) \qquad \text{if } s - \delta \leqslant r \leqslant s + \delta \tag{4.19}$$

Note that (4.18) specifies G on the region $D \backslash N(\partial D)$ and (4.19) specifies G on the region $N(\partial D)$.

A crucial fact that we will use later without comment is that ϕ^G and ϕ^H commute as elements of the universal cover of $\mathrm{Ham}(\mathbb{D}, \omega)$. That is

$$\phi^G \bullet \phi^H = \phi^H \bullet \phi^G \qquad \text{and} \qquad G\#H = H\#G \quad \text{up to isotopy in } t \text{ relative to } 0, 1$$

The remainder of this section is devoted to calculating properties of the action, rotation and periodic points of the map ϕ.

LEMMA 4.16 (ACTION OF ϕ). *The action map $\sigma_\phi : \mathbb{D} \to \mathbb{R}$ and Calabi invariant* $\mathrm{Cal}(\mathbb{D}, \phi)$ *satisfy*

$$\sigma_\phi = \pi(1 + \frac{1}{n}) - 2 \sum_{D \subset U} \mathrm{area}(D) \cdot \chi_D + O(\delta) \qquad \textit{on} \quad \mathbb{D} \backslash N(\partial U) \tag{4.20}$$

$$\pi/2 \leqslant \sigma_\phi \leqslant 2\pi \qquad \textit{on all of} \quad \mathbb{D} \tag{4.21}$$

$$\mathrm{Cal}(\mathbb{D}, \phi) = \pi^2(1 + \frac{1}{n}) - 2 \sum_{D \subset U} \mathrm{area}(D)^2 + O(\delta) \tag{4.22}$$

Proof. Since ϕ^G and ϕ^H commute, we have $\sigma_G \circ \phi_1^H = \sigma_G$ and therefore

$$\sigma_\phi = \sigma_G \circ \phi_1^H + \sigma_H = \sigma_G + \sigma_H$$

Thus we must compute the action map of G and H. First, we note that H is radial by (4.14). Thus we apply Lemma 4.11 to see

$$\sigma_H = \pi(1 + \frac{1}{n}) \quad \text{on all of } \mathbb{D} \tag{4.23}$$

Next we compute the action map of G. Let D be a component disk of U centered at p and of radius s. We can apply Lemma 4.12 to see that

$$\sigma_G = -2\pi s^2 + \delta \cdot (-2\pi s^2) + (u_p - [\phi_1^G]^* u_p) = -2\,\mathrm{area}(D) + O(\delta) \quad \text{on } D \backslash N(\partial D)$$

Here the $u_p - [\phi_1^G]^* u_p$ is an $O(\delta)$ term because ϕ_1^G is a rotation of angle $\pi\delta$ on $D\backslash N(\partial D)$. Since $\sigma_G = 0$ outside of $N(D)$, we thus acquire the formula

$$\sigma_G = -2 \sum_{D \subset U} \mathrm{area}(D) \cdot \chi_D + O(\delta) \quad \text{on } \mathbb{D}\backslash N(\partial U) \tag{4.24}$$

Adding (4.23) and (4.24) yields the desired formula (4.20) and implies (4.21) away from $N(\partial U)$. On the neighborhood $N(\partial U)$, we have the formula

$$|\sigma_G| \leqslant |g(r_p) - \frac{1}{2} g'(r_p)| + O(\delta) \leqslant 4\pi s^2 + O(\delta) \leqslant \frac{\pi}{2} \qquad \text{on} \quad N(\partial U)$$

By adding this to the formula (4.23) for σ_H, we immediately acquire (4.21) on $N(\partial U)$. Finally, since $N(\partial U)$ has area $O(\delta)$, the Calabi invariant agrees with the integral of (4.20) over $\mathbb{D}\backslash N(\partial U)$ up to an $O(\delta)$ term. This proves (4.22). □

Lemma 4.17 (Rotation of ϕ). *The rotation map $r_\phi : \mathbb{D} \to \mathbb{R}$ and the Ruelle invariant* $\mathrm{Ru}(\mathbb{D}, \phi)$ *satisfy*

$$r_\phi = (1 + \frac{1}{n}) - 2 \sum_{D \subset U} \chi_D + O(\delta) \qquad \textit{on} \quad \mathbb{D}\backslash N(\partial U) \tag{4.25}$$

$$-1 + \frac{1}{n} + \delta \leqslant r_\phi \leqslant 1 + \frac{1}{n} \qquad \textit{on all of} \quad \mathbb{D} \tag{4.26}$$

$$\mathrm{Ru}(\mathbb{D}, \phi) = \pi(1 + \frac{1}{n}) - 2 \sum_{D \subset U} \mathrm{area}(D) + O(\delta) \tag{4.27}$$

Proof. In the universal cover of $\mathrm{Ham}(\mathbb{D}, \phi)$, the time k flow ϕ^k of $G\#H$ can be factored in terms of the time 1 flow $\phi^G : [0,1] \times \mathbb{D} \to \mathbb{D}$ of G and the time 1 flow $\phi^H : [0,1] \times \mathbb{D} \to \mathbb{D}$ of H, as follows.

$$\phi^k = (\phi^H \bullet \phi^G)^k = \phi^H \bullet \phi^G \bullet \phi^H \bullet \cdots \bullet \phi^H \bullet \phi^G$$

This factorization is inherited by the lifted differential $\widetilde{\Phi} : \mathbb{R} \times \mathbb{D} \to \widetilde{\mathrm{Sp}}(2)$ of $\phi : \mathbb{R} \times \mathbb{D} \to \mathbb{D}$ due to the cocycle property of $\widetilde{\Phi}$.

$$\widetilde{\Phi}(k,z) = \widetilde{\Phi}^H(1, \phi^G \circ \phi^{k-1}(z)) \bullet \widetilde{\Phi}^G(1, \phi^{k-1}(z)) \bullet \widetilde{\Phi}^H(1, \phi^G \circ \phi^{k-2}(z)) \bullet \cdots \bullet \widetilde{\Phi}^G(1, z) \tag{4.28}$$

To apply this, we note that the differential $\Phi^H : [0,1] \times \mathbb{D} \to \mathrm{Sp}(2)$ of the flow of H is given by

$$\Phi^H(t,z) = \exp(2\pi(1 + 1/n) \cdot it) \quad \text{for any } z \in \mathbb{D} \tag{4.29}$$

Likewise, the differential $\Phi^G : [0,1] \times \mathbb{D} \to \mathrm{Sp}(2)$ of the flow of G is given by the formula

(4.30)
$$\Phi^G(t,z) = \exp(-2(2-\delta)\pi \cdot it) \text{ if } z \in U\backslash N(\partial U) \qquad \text{and} \qquad \Phi^G(t,z) = \mathrm{Id} \text{ if } z \in \mathbb{D}\backslash N(D)$$

By combining (4.29) and (4.30) with the decomposition (4.28), we acquire the following formula.

$$\rho \circ \widetilde{\Phi}(k,z) = k \cdot \big(1 + \frac{1}{n} - 2\sum_{D \subset U} \chi_D(z) + O(\delta)\big) \qquad \text{if} \quad z \in \mathbb{D}\backslash N(\partial U) \tag{4.31}$$

By dividing (4.31) by k and taking the limit as $k \to \infty$, we acquire the first formula (4.25).

Next, we examine the rotation number in the region $N(\partial D)$. Fix a component disk $D \subset U$ centered at p and a point $z \in N(\partial D)$. Let $S \subset N(\partial D)$ be a circle centered at p with $z \in S$, and let $u \in T_z S$ be a unit tangent vector to S at z. Finally, let

$$S_i = \phi^i(S) \qquad z_i = \phi^i(z) \qquad w_i = \phi^G \circ \phi^i(z) \qquad u_i = \Phi(i,z)u \qquad v_i = \Phi^G(1,\phi^i(z))\Phi(i,z)u$$

Note that these points and vectors satisfy $z_i \in S_i$, $w_i \in S_i$, $u_i \in T_{z_i}S_i$ and $v_i \in T_{w_i}S_i$ for each i. By applying the decomposition (4.28) and the additivity property (2.7) of ρ_s, we see that

$$\rho_u(\widetilde{\Phi}(k,z)) = \sum_{i=0}^{k-1} \rho_{u_i}(\widetilde{\Phi}^G(1,z_i))) + \sum_{i=0}^{k-1} \rho_{v_i}(\widetilde{\Phi}^H(1,w_i)) \tag{4.32}$$

Since ϕ^H is just an orthogonal rotation, we can use (4.29) to immediately conclude that

$$\rho_{u_i}(\widetilde{\Phi}^G(1,z_i))) = 1 + \frac{1}{n} \tag{4.33}$$

On the other hand, since v_i is tangent to the circle S_i, we may use the formula (4.12) to see that

$$\rho_{v_i}(\widetilde{\Phi}^H(1,z_i))) = -\frac{g'(r_p(z))}{2\pi r_p(z)} \tag{4.34}$$

Here g is the function such that $G|_{N(D)} = g \circ r_p$. By our hypotheses, we know that

$$-2 + \delta \leqslant -\frac{(2-\delta)(s-\delta)}{s+\delta} \leqslant -\frac{g'(r_p(z))}{2\pi r_p(z)} \leqslant 0$$

By plugging in the formulas (4.32) and (4.33), we can estimate $\rho_u \circ \widetilde{\Phi}(k,z)$ as follows.

$$k \cdot (-1 + \frac{1}{n} + \delta) \leqslant \rho_u \circ \widetilde{\Phi}(k,z) \leqslant k \cdot (1 + \frac{1}{n})$$

We can therefore estimate r_ϕ. Since ρ_u and ρ are equivalent (Lemma 2.5) we find that

$$r_\phi(z) = \lim_{k\to\infty} \frac{\rho_u \circ \widetilde{\Phi}(k,z)}{k} \qquad \text{and thus} \qquad -1 + \frac{1}{n} + \delta \leqslant r_\phi(z) \leqslant 1 + \frac{1}{n}$$

Finally, since $N(\partial U)$ has area $O(\delta)$, the Ruelle invariant agrees with the integral of (4.25) over $\mathbb{D}\backslash N(\partial U)$ up to an $O(\delta)$ term. This proves (4.27). □

LEMMA 4.18 (PERIODIC POINTS OF ϕ). *The periodic points of $\phi : \mathbb{D} \to \mathbb{D}$ satisfy*

$$\mathcal{A}(p) \geqslant \pi \qquad \textit{and} \qquad \rho(p) + \mathcal{L}(p) > 1 \tag{4.35}$$

Proof. First, consider the center $c = 0 \in \mathbb{D}$, where $\phi = \phi^H$. This periodic point has period $\mathcal{L}(c) = 1$. Thus, due to Lemmas 4.16 and 4.17, the action and rotation number are given by

$$\mathcal{A}(c) = \sigma_\phi(c) = \pi(1 + \frac{1}{n}) \qquad \rho(c) = r_\phi(c) = 1 + \frac{1}{n}$$

Any other periodic point p of H has period $\mathcal{L}(p) \geqslant n$, since ϕ rotates the sector $\mathbb{S}(n,k)$ to the section $\mathbb{S}(n,k+1)$. Since $n \geqslant 2$ and $\sigma_\phi \geqslant \pi/2$ (by Lemma 4.16), the action of p is lower bounded, as follows.

$$\mathcal{A}(p) = \sum_{i=0}^{\mathcal{L}(p)-1} \sigma_\phi(\phi^i(p)) \geqslant \frac{\pi}{2} \cdot \mathcal{L}(p) \geqslant \pi$$

Likewise, we apply Lemma 4.17 to see that the rotation number of p is lower bounded as follows.

$$\rho(p) = \mathcal{L}(p) \cdot r_\phi(p) \geqslant \mathcal{L}(p) \cdot (-1 + \frac{1}{n} + \delta) \geqslant -\mathcal{L}(p) + 1 + \delta$$

In particular, the rotation number satisfies $\rho(p) + \mathcal{L}(p) > 1$. □

4.5. Main construction. We conclude this construction by proving Proposition 4.1. The result will be an easy consequence of Proposition 4.10 and the properties of the special flow ϕ of §4.4.

Proof. (Proposition 4.1) Let $\varepsilon > 0$. Choose an integer n, a union of disks $U \subset \mathbb{D}$ and a number $\delta > 0$, satisfying the properties of Setup 4.13. Additionally, choose a $\kappa > 0$ and suppose that the component disks $D \subset U$ satisfy

$$\pi - \kappa < \sum_{D \subset U} \text{area}(D) < \pi \qquad \text{and} \qquad \text{area}(D) \leqslant \pi\kappa \tag{4.36}$$

Let $\phi : [0,1] \times \mathbb{D} \to \mathbb{D}$ be the associated family of Hamiltonian diffeomorphisms from §4.4. By direct calculation and Lemma 4.16, we know that

$$G\#H = \pi(1 + \frac{1}{n}) \cdot (1 - r^2) \text{ near } \partial\mathbb{D} \qquad \text{and} \qquad \sigma_\phi > 0$$

Therefore we can associate a contact form α on S^3 to ϕ via Construction 4.7. We now show that (a scaling of) this contact form has all of the desired properties.

First, by Proposition 4.10(b) and Lemma 4.16, the volume of (S^3, α) is given by the formula

$$\text{vol}(S^3, \alpha) = \text{Cal}(\mathbb{D}, \phi) = \pi^2(1 + \frac{1}{n}) - 2\sum_{D \subset U} \text{area}(D)^2 + O(\delta)$$

Thus, by applying the inequalities in (4.36), we acquire the following estimates for the volume.

$$\pi^2(1 + \frac{1}{n}) + O(\delta) > \text{vol}(S^3, \alpha) > \pi^2(1 - 2\kappa) + O(\delta)$$

Next, by Proposition 4.10(c) and Lemma 4.17, the Ruelle invariant of (S^3, α) satisfies

$$\text{Ru}(S^3, \alpha) = \text{Ru}(\mathbb{D}, \phi) + \pi = \pi(2 + \frac{1}{n}) - 2\sum_{D \subset U} \text{area}(D) + O(\delta)$$

Again, we can then use the inequalities in (4.36) to acquire estimates for the Ruelle invariant.

$$\frac{\pi}{n} + 2\kappa + O(\delta) > \text{Ru}(S^3, \alpha) > \frac{\pi}{n} + O(\delta)$$

Last, by Proposition 4.10(d) the binding $b = \iota(\partial\mathbb{D})$ in S^3 has action and rotation number given by

$$\mathcal{A}(b) = \pi \qquad \rho(b) = 1 + \frac{1}{1+1/n} > 1$$

Due to Proposition 4.10(e) and Lemma 4.18, every periodic orbit of (S^3, α) other than b satisfies

$$\mathcal{A}(\gamma) \geqslant \pi \qquad \rho(\gamma) > 1$$

In particular, α is a dynamically convex contact form. To conclude the proof, we now note that by choosing δ and κ sufficiently small, and choosing n sufficiently large, we can guarantee that

$$\frac{\mathrm{Ru}(S^3,\alpha)}{\mathrm{vol}(S^3,\alpha)^{1/2}} \leqslant \frac{\pi/n + 2\kappa + O(\delta)}{\pi(1-2\kappa+O(\delta))^{1/2}} < \varepsilon \qquad \text{and}$$

$$\mathrm{sys}(Y,\alpha) = \frac{\min\{\mathcal{A}(\gamma)|\gamma \text{ is an orbit of } \alpha\}^2}{\mathrm{vol}(S^3,\alpha)} \geqslant \frac{\pi^2}{\pi^2(1+1/n+O(\delta))} > 1-\varepsilon$$

By scaling α so that $\mathrm{vol}(Y,\alpha) = 1$, we arrive at a contact form satisfying all of the properties of Proposition 4.1. This finishes the proof and the main construction of this section. □

CHAPTER 2

Computing Reeb Dynamics On 4d Convex Polytopes

1. Introduction and main results

This part of this paper is about computational methods for testing Viterbo's conjecture and related conjectures, via combinatorial Reeb dynamics.

1.1. Review of Viterbo's conjecture. We first recall two different versions of Viterbo's conjecture. Consider $\mathbb{R}^{2n} = \mathbb{C}^n$ with coordinates $z_i = x_i + \sqrt{-1}y_i$ for $i = 1, \ldots, n$. Define the standard Liouville form

$$\lambda_0 = \frac{1}{2}\sum_{i=1}^{n}(x_i\,dy_i - y_i\,dx_i)\,.$$

Let X be a compact domain in $\mathbb{R}^{2n}$ with smooth boundary Y. Assume that X is "star-shaped", by which we mean that Y is transverse to the radial vector field. Then the 1-form $\lambda = \lambda_0|_Y$ is a contact form on Y. Associated to λ are the contact structure $\xi = \operatorname{Ker}(\lambda) \subset TY$ and the Reeb vector field R on Y, characterized by $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$. A *Reeb orbit* is a periodic orbit of R, i.e. a map $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ for some $T > 0$ such that $\gamma'(t) = R(\gamma(t))$, modulo reparametrization. The *symplectic action* of a Reeb orbit γ, denoted by $\mathcal{A}(\gamma)$, is the period of γ, or equivalently

$$\mathcal{A}(\gamma) = \int_{\mathbb{R}/T\mathbb{Z}} \gamma^*\lambda_0. \tag{1.1}$$

Reeb orbits on Y always exist. This was first proved by Rabinowitz [**66**] and is a special case of the Weinstein conjecture; see [**43**] for a survey. We are interested here in the minimal period of a Reeb orbit on Y, which we denote by $\mathcal{A}_{\min}(X) \in (0, \infty)$, and its relation to the volume of X. For this purpose, define the *systolic ratio*

$$\operatorname{sys}(X) = \frac{\mathcal{A}_{\min}(X)^n}{n!\operatorname{vol}(X)}.$$

The exponent ensures that the systolic ratio of X is invariant under scaling of X; and the constant factor is chosen so that if X is a ball then $\operatorname{sys}(X) = 1$.

Conjecture 1.1 (weak Viterbo conjecture). *Let $X \subset \mathbb{R}^{2n}$ be a compact convex domain with smooth boundary such that $0 \in \operatorname{int}(X)$. Then $\operatorname{sys}(X) \leqslant 1$.*

Conjecture 1.1 asserts that among compact convex domains with the same volume, $\mathcal{A}_{\min}$ is largest for a ball. Although the role of the convexity hypothesis is somewhat mysterious, some hypothesis beyond the star-shaped condition is necessary: it is shown in [**3**] that there exist star-shaped domains in $\mathbb{R}^4$ with arbitrarily large systolic ratio[1].

[1]It is further shown in [**1**] that there are star-shaped domains in $\mathbb{R}^4$ which are *dynamically convex* (meaning that every Reeb orbit on the boundary has rotation number greater than 1, see Proposition 1.9(a) below) and have systolic ratio $2 - \varepsilon$ for $\varepsilon > 0$ arbitrarily small.

One motivation for studying Conjecture 1.1 is that it implies the Mahler conjecture in convex geometry [**7**].

To put Conjecture 1.1 in more context, recall[2] that a *symplectic capacity* is a function c mapping some class of $2n$-dimensional symplectic manifolds to $[0, \infty]$, such that:

- (Monotonicity) If there exists a symplectic embedding $\varphi : (X, \omega) \to (X', \omega')$, then $c(X, \omega) \leqslant c(X', \omega')$.
- (Conformality) If $r > 0$ then $c(X, r\omega) = rc(X, \omega)$.

Of course we can regard (open) domains in $\mathbb{R}^{2n}$ as symplectic manifolds with the restriction of the standard symplectic form $\omega = \sum_{i=1}^{n} dx_i\, dy_i$. Conformality for a domain $X \subset \mathbb{R}^{2n}$ means that $c(rX) = r^2 c(X)$.

Following the usual convention in symplectic geometry, for $r > 0$ define the ball

$$B(r) = \left\{ z \in \mathbb{C}^n \;\middle|\; \pi |z|^2 \leqslant r \right\}$$

and the cylinder

$$Z(r) = \left\{ z \in \mathbb{C}^n \;\middle|\; \pi |z_1|^2 \leqslant r \right\}.$$

We say that a symplectic capacity c is *normalized* if it is defined at least for all compact convex domains in $\mathbb{R}^{2n}$ and if

$$c(B(r)) = c(Z(r)) = r.$$

An example of a normalized symplectic capacity is the *Gromov width* c_{Gr}, where $c_{\mathrm{Gr}}(X, \omega)$ is defined to be the supremum over r such that there exists a symplectic embedding $B(r) \to (X, \omega)$. It is immediate from the definition that c_{Gr} is monotone and conformal. Since symplectomorphisms preserve volume, we have $c_{\mathrm{Gr}}(B(r)) = r$; and the Gromov nonsqueezing theorem asserts that $c_{\mathrm{Gr}}(Z(r)) = r$.

Another example of a normalized symplectic capacity is the *Ekeland-Hofer-Zehnder capacity*, denoted by c_{EHZ}. If X is a compact convex domain with smooth boundary such that $0 \in \operatorname{int}(X)$, then

$$c_{\mathrm{EHZ}}(X) = \mathcal{A}_{\min}(X). \tag{1.2}$$

This is explained in [**8**, Thm. 2.2], combining results from [**23**, **38**].

Any symplectic capacity which is defined for compact convex domains in $\mathbb{R}^{2n}$ with smooth boundary is a C^0 continuous function of the domain (i.e., continuous with respect to the Hausdorff distance between compact sets), and thus extends uniquely to a C^0 continuous function of all compact convex sets in $\mathbb{R}^{2n}$.

Conjecture 1.2 (strong Viterbo conjecture[3]). *All normalized symplectic capacities agree on compact convex sets in* $\mathbb{R}^{2n}$.

Conjecture 1.2 implies Conjecture 1.1, because if Conjecture 1.2 holds, and if X is a compact convex domain with smooth boundary and $0 \in \operatorname{int}(X)$, then

$$\mathcal{A}_{\min}(X)^n = c_{\mathrm{EHZ}}(X)^n = c_{\mathrm{Gr}}(X)^n \leqslant n! \operatorname{vol}(X).$$

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[2]The precise definition of "symplectic capacity" varies in the literature. For an older but extensive survey of symplectic capacities see [**18**].

[3]The original version of Viterbo's conjecture from [**77**] asserts that a normalized symplectic capacity, restricted to convex sets in $\mathbb{R}^{2n}$ of a given volume, takes its maximum on a ball. (This follows from what we are calling the "strong Viterbo conjecture" and implies what we are calling the "weak Viterbo conjecture".) Viterbo further conjectured that the maximum is achieved only if the interior of the convex set is symplectomorphic to an open ball; cf. Question 1.21 below.

Here the second equality holds by Conjecture 1.2; and the inequality on the right holds because if there exists a symplectic embedding $B(r) \to X$, then $r^n/n! = \mathrm{vol}(B(r)) \leqslant \mathrm{vol}(X)$.

There are also interesting families of non-normalized symplectic capacities. For example, there are the Ekeland-Hofer capacities defined in **[24]**; more recently, and conjecturally equivalently, positive S^1-equivariant symplectic homology was used in **[33]** to define a symplectic capacity $c_k^{S^1}$ for each integer $k \geqslant 1$. Each equivariant capacity $c_k^{S^1}(X)$ is the symplectic action of some Reeb orbit, which when X is generic (so that λ is nondegenerate) has Conley-Zehnder index $n - 1 + 2k$ (see §1.3 below). Some other symplectic capacities give the total action of a finite set of Reeb orbits, such as the ECH capacities in the four-dimensional case **[44]**, or the symplectic capacities defined by Siegel using rational symplectic field theory **[70]**.

Conjectures 1.1 and 1.2 are known for some special examples such as S^1-invariant convex domains **[35]**, but they have not been well tested more generally. To test Conjecture 1.1, and as a first step towards computing other symplectic capacities and testing conjectures about them, we need good methods for computing Reeb orbits, their actions, and their Conley-Zehnder indices. The plan is to understand Reeb orbits on a smooth convex domain in terms of "combinatorial Reeb orbits" on convex polytopes approximating the domain.

1.2. Combinatorial Reeb orbits. Let X be any compact convex set in $\mathbb{R}^{2n}$ with $0 \in \mathrm{int}(X)$, and let $y \in \partial X$. The *tangent cone*, which we denote by T_y^+X, is the closure of the set of vectors v such $y + \varepsilon v \in X$ for some $\varepsilon > 0$. For example, if ∂X is smooth at y, then T_y^+X is a closed half-space whose boundary is the usual tangent space $T_y\partial X$.

Also define the *positive normal cone*

$$N_y^+X = \left\{v \in \mathbb{R}^{2n} \;\middle|\; \langle x - y, v\rangle \leqslant 0 \;\; \forall x \in X\right\}.$$

If ∂X is smooth at y, then N_y^+X is a one-dimensional ray and consists of the outward pointing normal vectors to ∂X at y.

Finally, define the *Reeb cone*

$$R_y^+X = T_y^+X \cap \mathbf{i}N_y^+X$$

where $\mathbf{i}$ denotes the standard complex structure on $\mathbb{C}^n = \mathbb{R}^{2n}$. If ∂X is smooth near y, then R_y^+X is the ray consisting of nonnegative multiples of the Reeb vector field on ∂X at y. Indeed, in this case we can write

$$T_y\partial X = \left\{v \in \mathbb{R}^{2n} \;\middle|\; \langle \nu, v\rangle = 0\right\}$$

where ν is the outward unit normal vector to ∂X at y; and the Reeb vector field at y is given by

$$R_y = 2\frac{\mathbf{i}\nu}{\langle \nu, y\rangle}. \tag{1.3}$$

Suppose now that X is a convex polytope (i.e. a compact set given by the intersection of a finite set of closed half-spaces) in $\mathbb{R}^{2n}$ with $0 \in \mathrm{int}(X)$. Our convention is that a *k-face* of X is a k-dimensional subset $F \subset \partial X$ which is the interior of the intersection with ∂X of some set of the hyperplanes defining X. For a given k-face F, the tangent cone T_y^+X,

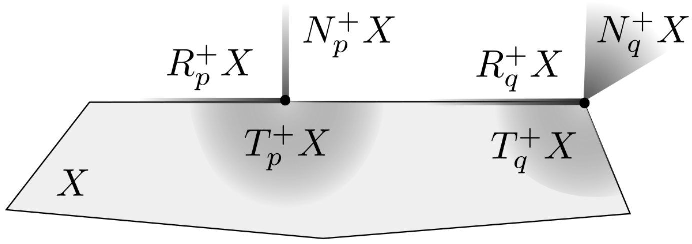


FIGURE 1. We depict the tangent, normal and Reeb cones for two points $p, q \in X$ in a polytope $X \subset \mathbb{R}^2$.

the positive normal cone N_y^+X, and the Reeb cone R_y^+X are the same for all $y \in F$. Thus we can denote these cones by T_F^+X, N_F^+X, and R_F^+X respectively.

We will usually restrict attention to polytopes of the following type:

DEFINITION 1.3. A *symplectic polytope* in $\mathbb{R}^4$ is a convex polytope X in $\mathbb{R}^4$ such that $0 \in \operatorname{int}(X)$ and no 2-face of X is Lagrangian, i.e., the standard symplectic form $\omega_0 = \sum_{i=1}^2 dx_i\, dy_i$ restricts to a nonzero 2-form on each 2-face.

Symplectic polytopes are generic, in the sense that in the space of polytopes in $\mathbb{R}^4$ with a given number of 3-faces, the set of non-symplectic polytopes is a proper subvariety.

PROPOSITION 1.4. *(proved in §3.2) If X is a symplectic polytope in $\mathbb{R}^4$, then the Reeb cone R_F^+X is one-dimensional for each face F.*

DEFINITION 1.5. Let X be a symplectic polytope in $\mathbb{R}^4$. A *combinatorial Reeb orbit* for X is a finite sequence $\gamma = (\Gamma_1, \ldots, \Gamma_k)$ of oriented line segments in ∂X, modulo cyclic permutations, such that for each $i = 1, \ldots, k$:

- The final endpoint of Γ_i agrees with the initial endpoint of $\Gamma_{i+1 \bmod k}$.
- There is a face F of X such that $\operatorname{int}(\Gamma_i) \subset F$, the endpoints of Γ_i are on the boundary of (the closure of) F, and Γ_i points in the direction of R_F^+X.

The *combinatorial symplectic action* of a combinatorial Reeb orbit as above is defined by

$$\mathcal{A}_{\mathrm{comb}}(\gamma) = \sum_{i=1}^{k} \int_{\Gamma_i} \lambda_0.$$

To give a better idea of what combinatorial Reeb orbits look like, we have the following lemma.

LEMMA 1.6. *(proved in §3.3) Let X be a symplectic polytope in $\mathbb{R}^4$. Then the Reeb cones of the faces of X satisfy the following:*

- *If E is a 3-face, then R_E^+X consists of all nonnegative multiples of the Reeb vector field on E.*
- *If F is a 2-face, then R_F^+X points into a 3-face E adjacent to F, and agrees with R_E^+X.*
- *If L is a 1-face, then one of the following possibilities holds:*
  - *R_L^+X points into a 3-face E adjacent to L and agrees with R_E^+X. In this case we say that L is a* good *1-face.*
  - *R_L^+X is tangent to L, and does not agree with R_E^+X for any of the 3-faces E adjacent to L. In this case we say that L is a* bad *1-face.*

- *If P is a 0-face, then R_P^+X points into a 3-face E or bad 1-face L adjacent to F and agrees with R_E^+X or R_L^+X respectively.*

Remark 1.7. The reason we assume that X has no Lagrangian 2-faces in Definition 1.3 is that if F is a Lagrangian 2-face, then R_F^+X is two-dimensional and tangent to F. In fact, $\partial R_F^+X = R_{E_1}^+X \cup R_{E_2}^+X$ where E_1 and E_2 are the two 3-faces adjacent to F. In this case we do not have a well-posed "combinatorial Reeb flow" on ∂X.

Definition 1.8. A combinatorial Reeb orbit as above is:

- *Type 1* if it does not intersect the 1-skeleton of X;
- *Type 2* if it intersects the 1-skeleton of X, but only in finitely many points which are some of the endpoints of the line segments Γ_i;
- *Type 3* if it contains a bad 1-face.

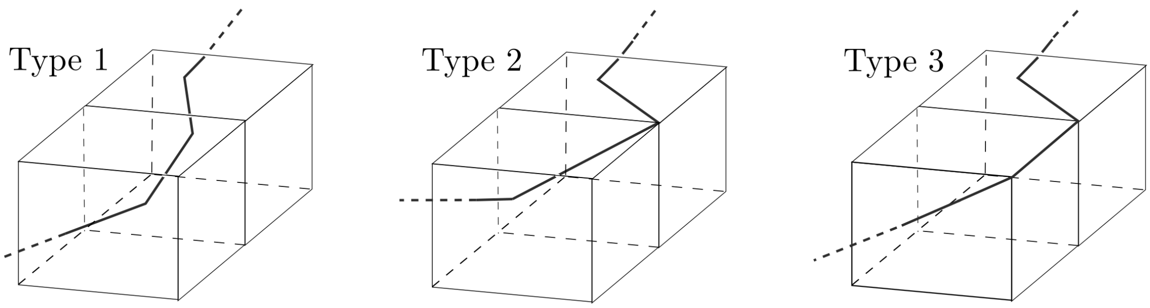


Figure 2. We depict sub-trajectories of the three types of orbits, in red. Each cube above represents a 3-face of a hypothetical 4-polytope.

It follows from the definitions that each combinatorial Reeb orbit is of one of the above three types. Type 1 Reeb orbits are the most important for our computations. We expect that Type 2 combinatorial Reeb orbits do not exist for generic polytopes; see Conjecture 1.24 below. Type 3 combinatorial Reeb orbits generally cannot be eliminated by perturbing the polytope; but we will see in Theorem 1.11(iii) below that they do not contribute to the symplectic capacities that we are interested in. See Remark 5.8 for some intuition for this.

1.3. Rotation numbers and the Conley-Zehnder index. Let X be a compact star-shaped domain in $\mathbb{R}^4$ with smooth boundary Y. Let $\Phi_t : Y \to Y$ denote the time t flow of the Reeb vector field R. The derivative of Φ_t preserves the contact form λ, and thus for each $y \in Y$ defines a map

$$d\Phi_t : \xi_y \longrightarrow \xi_{\Phi_t(y)}$$

which is symplectic with respect to $d\lambda$.

We say that a Reeb orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ is *nondegenerate* if the "linearized return map"

$$d\Phi_T : \xi_{\gamma(0)} \longrightarrow \xi_{\gamma(0)} \tag{1.4}$$

does not have 1 as an eigenvalue. The contact form λ is called nondegenerate if all Reeb orbits are nondegenerate.

Now fix a symplectic trivialization $\tau : \xi \to Y \times \mathbb{R}^2$. If γ is a Reeb orbit as above, then the trivialization τ allows us to regard the map (1.4) as an element of $\mathrm{Sp}(2)$. Moreover, the family of maps

$$\left\{\mathbb{R}^2 \xrightarrow{\tau^{-1}} \xi_{\gamma(0)} \xrightarrow{d\Phi_t} \xi_{\gamma(t)} \xrightarrow{\tau} \mathbb{R}^2\right\}_{t\in[0,T]}$$

defines a path in $\mathrm{Sp}(2)$ from the identity to the map (1.4). As we review in Appendix 7, this path has a well-defined *rotation number*, which we denote by

$$\rho(\gamma) \in \mathbb{R}.$$

This rotation number does not depend on the choice of global trivialization τ.

If γ is nondegenerate (which holds automatically when $\rho(\gamma)$ is not an integer), then the *Conley-Zehnder index* of γ is defined by

$$\mathrm{CZ}(\gamma) = \lfloor\rho(\gamma)\rfloor + \lceil\rho(\gamma)\rceil \in \mathbb{Z}. \tag{1.5}$$

PROPOSITION 1.9. *Let X be a compact convex domain in $\mathbb{R}^4$ with smooth boundary Y and with $0 \in \mathrm{int}(X)$. Then:*

(a) Every Reeb orbit γ in Y has $\rho(\gamma) > 1$. In particular, if γ is nondegenerate then $\mathrm{CZ}(\gamma) \geqslant 3$.

(b) There exists a Reeb orbit γ which is action minimizing, i.e. $\mathcal{A}(\gamma) = \mathcal{A}_{\min}(X)$, with

$$\rho(\gamma) \leqslant 2.$$

If γ is also nondegenerate then the inequality is strict, so that $\mathrm{CZ}(\gamma) = 3$.

PROOF. (a) was proved by Hofer-Wysocki-Zehnder **[37]**.

(b) follows from the construction of the Ekeland-Hofer-Zehnder capacity and an index calculation of Hu-Long **[41]**. In fact, it was recently shown by Abbondandolo-Kang **[2]** and Irie **[50]** that $c_{\mathrm{EHZ}}(X)$ agrees with a capacity defined from symplectic homology, which by construction is the action of some Reeb orbit γ with $\rho(\gamma) \leqslant 2$, with equality only if γ is degenerate. □

Suppose now that X is a symplectic polytope in $\mathbb{R}^4$. As we explain in Definition 2.23, each Type 1 combinatorial Reeb orbit γ has a well-defined *combinatorial rotation number*, which we denote by $\rho_{\mathrm{comb}}(\gamma) \in \mathbb{R}$. There is also a combinatorial notion of nondegeneracy for γ, which automatically holds when $\rho_{\mathrm{comb}}(\gamma) \notin \mathbb{Z}$. When γ is a nondegenerate Type 1 combinatorial Reeb orbit, we can then define its *combinatorial Conley-Zehnder index* by analogy with (1.5) as

$$\mathrm{CZ}_{\mathrm{comb}}(\gamma) = \lfloor\rho_{\mathrm{comb}}(\gamma)\rfloor + \lceil\rho_{\mathrm{comb}}(\gamma)\rceil. \tag{1.6}$$

The combinatorial rotation number and combinatorial Conley-Zehnder index of a Type 2 combinatorial Reeb orbit are not defined; and although we do not need this, it would be natural to define the combinatorial rotation number and combinatorial Conley-Zehnder index of a Type 3 combinatorial Reeb orbit to be $+\infty$.

1.4. Smooth-combinatorial correspondence. Let X be a convex polytope in $\mathbb{R}^{2n}$. If $\varepsilon > 0$, define the *ε-smoothing* of X by

$$X_\varepsilon = \left\{ z \in \mathbb{R}^{2n} \;\middle|\; \operatorname{dist}(z, X) \leqslant \varepsilon \right\}. \tag{1.7}$$

The domain X_ε is convex and has C^1-smooth boundary. The boundary is C^∞ smooth except along strata arising from the boundaries of the faces of X; see §5.1 for a detailed description.

Our main results are the following two theorems, giving a correspondence between combinatorial Reeb dynamics on a symplectic polytope in $\mathbb{R}^4$, and ordinary Reeb dynamics on ε-smoothings of the polytope.

There is a slight technical issue here: since ∂X_ε is only C^1 smooth, the Reeb vector field on ∂X_ε is only C^0, so that for a Reeb orbit γ, the linearized Reeb flow (1.4) might not be defined. If γ is transverse to the strata where ∂X_ε is not C^∞ (which is presumably true for all γ if X and ε are generic), then the Reeb flow in a neighborhood of γ has a well-defined linearization; we call such orbits *linearizable*. It turns out that a non-linearizable Reeb orbit γ on ∂X_ε still has a well-defined rotation number $\rho(\gamma)$, defined in §5.4.

The following theorem describes how combinatorial Reeb orbits give rise to Reeb orbits on smoothings. See Lemma 6.1 for a more precise statement.

Theorem 1.10. *(proved in §6.1) Let X be a symplectic polytope in $\mathbb{R}^4$, and let γ be a nondegenerate Type 1 combinatorial Reeb orbit for X. Then for all $\varepsilon > 0$ sufficiently small, there is a distinguished Reeb orbit γ_ε on ∂X_ε such that:*

- *(i)* γ_ε *converges in* C^0 *to* γ *as* $\varepsilon \to 0$.
- *(ii)* $\lim_{\varepsilon\to 0} \mathcal{A}(\gamma_\varepsilon) = \mathcal{A}_{\mathrm{comb}}(\gamma)$.
- *(iii)* γ_ε *is linearizable and nondegenerate,* $\rho(\gamma_\varepsilon) = \rho_{\mathrm{comb}}(\gamma)$, *and* $\mathrm{CZ}(\gamma_\varepsilon) = \mathrm{CZ}_{\mathrm{comb}}(\gamma)$.

The following theorem describes how Reeb orbits on smoothings give rise to combinatorial Reeb orbits.

Theorem 1.11. *(proved in §6.2) Let X be a symplectic polytope in $\mathbb{R}^4$. Then there are constants $c_F > 0$ for each 0-, 1-, or 2-face F of X with the following property.*

Let $\{(\varepsilon_i, \gamma_i)\}_{i=1,\ldots}$ be a sequence of pairs such that $\varepsilon_i > 0$; γ_i is a Reeb orbit on $\partial X_{\varepsilon_i}$; and $\varepsilon_i \to 0$ as $i \to \infty$. Suppose that $\rho(\gamma_i) < R$ where R does not depend on i. Then after passing to a subsequence, there is a combinatorial Reeb orbit γ for X such that:

- *(i)* γ_i *converges in* C^0 *to* γ *as* $i \to \infty$.
- *(ii)* $\lim_{i\to\infty} \mathcal{A}(\gamma_i) = \mathcal{A}_{\mathrm{comb}}(\gamma)$.
- *(iii)* γ *is either Type 1 or Type 2.*
- *(iv) If γ is Type 1, then for i sufficiently large, γ_i is linearizable and $\rho(\gamma_i) = \rho_{\mathrm{comb}}(\gamma)$. If γ is also nondegenerate, then for i sufficiently large, γ_i is nondegenerate and* $\mathrm{CZ}(\gamma_i) = \mathrm{CZ}_{\mathrm{comb}}(\gamma)$.
- *(v) Let $F_1, \ldots, F_k$ denote the faces containing the endpoints of the segments of the combinatorial Reeb orbit γ. Then*

$$\sum_{i=1}^{k} c_{F_i} \leqslant R. \tag{1.8}$$

Remark 1.12. One can compute explicit constants c_F – see §6.2 for details – and the resulting bound (1.8) is crucial in enabling finite computations. For example, combinatorial Reeb orbits with a given action bound could have arbitrarily many segments winding in a "helix" around a bad 1-face. However the bound (1.8) ensures that combinatorial Reeb orbits with too many segments will not arise as limits of sequences of smooth Reeb orbits with bounded rotation number.

Theorem 1.11 allows one to compute the EHZ capacity of a four-dimensional polytope as follows:

Corollary 1.13. *Let X be a symplectic polytope in $\mathbb{R}^4$. Then*

$$c_{\mathrm{EHZ}}(X) = \min\{\mathcal{A}_{\mathrm{comb}}(\gamma)\} \tag{1.9}$$

where the minimum is over combinatorial Reeb orbits γ with $\sum_i c_{F_i} \leqslant 2$ which are either Type 1 with $\rho_{\mathrm{comb}}(\gamma) \leqslant 2$ or Type 2.

Remark 1.14. If the coordinates of the vertices of X are rational, then the combinatorial action of every combinatorial Reeb orbit is rational. It follows from Theorem 1.11 that in this case, $c_{\mathrm{EHZ}}(X)$, as well as the other symplectic capacities mentioned in §1.1 determined by actions of Reeb orbits, are all rational.

To explain why Corollary 1.13 follows from Theorem 1.11, we need to recall a result of Künzle [**53**] as explained by Artstein-Avidan and Ostrover [**8**].

Definition 1.15. If X is any compact convex set in $\mathbb{R}^{2n}$ with $0 \in \mathrm{int}(X)$, a *generalized Reeb orbit* for X is a map $\gamma : \mathbb{R}/T\mathbb{Z} \to \partial X$ for some $T > 0$ such that γ is continuous and has left and right derivatives at every point, which agree for almost every t, and the left and right derivatives at t are in $R^+_{\gamma(t)}X$. If γ is a generalized Reeb orbit, define its symplectic action by (1.1).

Proposition 1.16. *[**8**, Prop. 2.7] If X is a compact convex set in $\mathbb{R}^{2n}$ with $0 \in \mathrm{int}(X)$, then*

$$c_{\mathrm{EHZ}}(X) = \min\{\mathcal{A}(\gamma)\}$$

where the minimum is taken over all generalized Reeb orbits.

Proof of Corollary 1.13. Pick a sequence of positive numbers ε_i with $\lim_{i\to\infty} \varepsilon_i = 0$. For each i, by equation (1.2), we can find a Reeb orbit γ_i on $\partial X_{\varepsilon_i}$ with $\mathcal{A}(\gamma_i) = c_{\mathrm{EHZ}}(X_{\varepsilon_i})$. By Proposition 1.9(b), we can assume that $\rho(\gamma_i) \leqslant 2$. By Theorem 1.11, it follows that after passing to a subsequence, there is a combinatorial Reeb orbit γ for X, satisying the conditions in Corollary 1.13, such that

$$\mathcal{A}_{\mathrm{comb}}(\gamma) = \lim_{i\to\infty} \mathcal{A}(\gamma_i) = \lim_{k\to\infty} c_{\mathrm{EHZ}}(X_{\varepsilon_i}) = c_{\mathrm{EHZ}}(X).$$

Here the last equality holds by the C^0 continuity of c_{EHZ}. We conclude that

$$c_{\mathrm{EHZ}}(X) \geqslant \min\{\mathcal{A}_{\mathrm{comb}}(\gamma)\}$$

where the minimum is over combinatorial Reeb orbits γ satisfying the conditions in Corollary 1.13.

The reverse inequality follows from Proposition 1.16, because by Definitions 1.5 and 1.15, every combinatorial Reeb orbit is a generalized Reeb orbit. (For a symplectic polytope in $\mathbb{R}^4$, a "generalized Reeb orbit" is equivalent to a generalization of a "combinatorial Reeb orbit" in which there may be infinitely many line segments.) ☐

REMARK 1.17. Haim-Kislev [**36**, Thm. 1.1] gives a different formula for c_{EHZ} of a convex polytope, which is valid in $\mathbb{R}^{2n}$ for all n. That formula implies that in the minimum (1.9), we can also assume that γ has at most one segment in each 3-face.

1.5. Experiments testing Viterbo's conjecture. If X is a convex polytope in $\mathbb{R}^{2n}$, define its systolic ratio by

$$\operatorname{sys}(X) = \frac{c_{\mathrm{EHZ}}(X)^n}{n! \operatorname{vol}(X)}.$$

Note that c_{EHZ} is translation invariant, so we can make this definition without assuming that $0 \in \operatorname{int}(X)$.

Since every compact convex domain in $\mathbb{R}^{2n}$ can be C^0 approximated by convex polytopes, it follows that the weak version of Viterbo's conjecture, namely Conjecture 1.1, is true if and only if every convex polytope X has systolic ratio $\operatorname{sys}(X) \leqslant 1$. The combinatorial formula for the systolic ratio given by Corollary 1.13 allows us to test this conjecture by computer when $n = 2$. In particular, we ran optimization algorithms over the space of k-vertex convex polytopes in $\mathbb{R}^4$ to find local maxima of the systolic ratio[4]. In the results below, when listing the vertices of specific polytopes, we use Lagrangian coordinates (x_1, x_2, y_1, y_2).

5-vertex polytopes (4-simplices). Experimentally[5], every 4-simplex X has systolic ratio

$$\operatorname{sys}(X) \leqslant 3/4.$$

The apparent maximum of 3/4 is achieved by the "standard simplex" with vertices

$$(0,0,0,0), (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1).$$

REMARK 1.18. Corollary 1.13 does not directly apply to (a translate of) this polytope because it has some Lagrangian 2-faces. For examples like these, we find numerically that a slight perturbation of the polytope to a symplectic polytope (to which Corollary 1.13 does apply) has systolic ratio very close to the claimed value. One can compute the systolic ratio of a polytope with Lagrangian 2-faces rigorously using a generalization of Corollary 1.13. For the particular example above, one can also compute the systolic ratio by hand using [**36**, Thm. 1.1].

We have found families of other examples of 4-simplices with systolic ratio 3/4, including some with no Lagrangian 2-faces. An example is the simplex with vertices

$$(0,0,0,0), (1,-1/3,0,0), (0,-1/3,1,0), (-2/3,-1,2/3,0), (0,0,0,1).$$

6-vertex polytopes. We found families of 6-vertex polytopes with systolic ratio equal to 1. An example is the polytope with vertices

$$(0,0,0,0), (1,0,0,0), (0,0,1,0), (0,0,0,1), (0,-1,1,0), (-1,-1,0,1).$$

This was somewhat of a surprise, since previously the simplest known polytope with systolic ratio equal to 1 was a Hanner polytope, the Lagrangian product of a square and a diamond, with 16 vertices. (This polytope corresponds to the equality case of Mahler's

[4]This is a somewhat involved process; convergence to a local maximum becomes very slow once one is close. It helps to mod out the space of polytopes by the 15-dimensional symmetry group generated by translations, linear symplectomorphisms, and scaling. To find exact local maxima, one can look at symplectic invariants, such as areas of 2-faces, and guess what these are converging to.

[5]Perhaps this could be proved analytically using the formula in [**36**, Thm. 1.1].

conjecture in two dimensions under the reduction from the Mahler conjecture to the weak Viterbo conjecture in [**7**].)

7-vertex polytopes. We also found families of 7-vertex polytopes with systolic ratio 1. One example has vertices

$$(0,0,0,0), (1,0,0,0), (0,0,1,0), (0,0,0,1),$$
$$(1/3,-2/3,2/3,0), (-1,-1,0,1/2), (0,0,1/3,-1/3).$$

Presumably there exist k-vertex polytopes in $\mathbb{R}^4$ with systolic ratio equal to 1 for every $k \geqslant 6$.

The 24-cell. We also found a special example of a polytope with systolic ratio 1: a rotation of the 24-cell (one of the six regular polytopes in four dimensions). See §2.4 for details.

We have heavily searched the spaces of polytopes with 7 or fewer vertices and have not found any counterexamples to Viterbo's conjecture. For polytopes with 8 vertices, our computer program starts becoming slower (sometimes taking minutes per polytope on a standard laptop instead of seconds), and we have not yet searched as extensively.

Towards a proof of the weak Viterbo conjecture? Let X be a star-shaped domain in $\mathbb{R}^4$ with smooth boundary Y. Following [**3**], we say that X is *Zoll* if every point on Y is contained in a Reeb orbit with minimal action. Note that:

(a) If X is strictly convex and a local maximizer for the systolic ratio of convex domains in the C^0 topology, then X is Zoll.

(b) If X is Zoll, then X has systolic ratio $\operatorname{sys}(X) = 1$.

Part (a) holds because if X is strictly convex and if $y \in Y$ is not on an action mimizing Reeb orbit, then one can shave some volume off of X near y without creating any new Reeb orbits of small action. Part (b) holds by a topological argument going back to [**78**]. Of course, these observations are not enough to prove Conjecture 1.1, since we do not know that the systolic ratio for convex domains takes a maximum, let alone on a strictly convex domain. But this does suggest the following strategy for proving Conjecture 1.1 via convex polytopes.

Definition 1.19. Let X be a convex polytope in $\mathbb{R}^4$ with $0 \in \operatorname{int}(X)$. We say that X is *combinatorially Zoll* if there is an open dense subset U of ∂X such that every point in U is contained in a combinatorial Reeb orbit (avoiding any Lagrangian 2-faces of X) with combinatorial action equal to $c_{\mathrm{EHZ}}(X)$.

We have checked by hand that the above examples of polytopes with systolic ratio equal to 1 are combinatorially Zoll. This suggests:

Conjecture 1.20. *Let X be a convex polytope in $\mathbb{R}^4$ with $0 \in \operatorname{int}(X)$. Then:*

(a) If X is combinatorially Zoll, then $\operatorname{sys}(X) = 1$.

(b) If k is sufficiently large ($k \geqslant 6$ might suffice) and if X maximizes systolic ratio over convex polytopes with $\leqslant k$ vertices, then X is combinatorially Zoll.

Part (a) of this conjecture can probably be proved following the argument in the smooth case. Part (b) might be much harder. But both parts of the conjecture together would imply the weak Viterbo conjecture (using a compactness argument to show that for each k the systolic ratio takes a maximum on the space of convex polytopes with $\leqslant k$ vertices).

QUESTION 1.21. If a convex polytope X in $\mathbb{R}^4$ is combinatorially Zoll, then is $\operatorname{int}(X)$ symplectomorphic to an open ball?

1.6. Experiments testing other conjectures. One can also use Theorems 1.10 and 1.11 to test conjectures about Reeb orbits that do not have minimal action. For example, if X is a convex domain with smooth boundary and $0 \in \operatorname{int}(X)$ such that $\lambda_0|_{\partial X}$ is nondegenerate, and if k is a positive integer, define

$$\mathcal{A}_k(X) = \min\{\mathcal{A}(\gamma) \mid \mathrm{CZ}(\gamma) = 2k+1\}, \tag{1.10}$$

where the minimum is over Reeb orbits γ on ∂X. In particular $\mathcal{A}_1(X) = \mathcal{A}_{\min}(X)$ by Proposition 1.9(b).

CONJECTURE 1.22. *For X as above we have $\mathcal{A}_2(X) \leqslant 2\mathcal{A}_1(X)$.*

This conjecture has nontrivial content when every action-minimizing Reeb orbit has rotation number at least 3/2. (If an action-minimizing Reeb orbit has rotation number less than 3/2, then its double cover has Conley-Zehnder index 5 and thus verifies the conjectured inequality.) To explain how to test this, we need the following definitions.

DEFINITION 1.23. Let X be a symplectic polytope in $\mathbb{R}^4$. Let $L > 0$. We say that X is *L-nondegenerate* if:

- X does not have any Type 2 combinatorial Reeb orbit γ with $\mathcal{A}_{\mathrm{comb}}(\gamma) \leqslant L$.
- Every Type 1 combinatorial Reeb orbit γ with $\mathcal{A}_{\mathrm{comb}}(\gamma) \leqslant L$ is nondegenerate, see Definition 2.23.

It follows from Theorem 1.11 that if a symplectic polytope X is L-nondegenerate, then for all $\varepsilon > 0$ sufficiently small, all Reeb orbits on ∂X_ε with action less than L are nondegenerate.

CONJECTURE 1.24. *For any integer k and any real number L, the set of L-nondegenerate symplectic polytopes with k vertices is dense in the set of all k-vertex convex polytopes containing 0, topologized as an open subset of $\mathbb{R}^{4k}$.*

DEFINITION 1.25. Let k be a positive integer and let X be a symplectic polytope in $\mathbb{R}^4$. Suppose that X is L-nondegenerate and has a combinatorial Reeb orbit γ with $\mathcal{A}(\gamma) < L$ and $\mathrm{CZ}_{\mathrm{comb}}(\gamma) = 2k+1$. By analogy with (1.10), define

$$\mathcal{A}_k^{\mathrm{comb}}(X) = \min\left\{\mathcal{A}_{\mathrm{comb}}(\gamma) \mid \mathrm{CZ}_{\mathrm{comb}}(\gamma) = 2k+1\right\}$$

where the minimum is over combinatorial Reeb orbits γ with combinatorial action less than L.

Conjecture 1.22 is now equivalent[6] to the following:

CONJECTURE 1.26. *Let X be a symplectic polytope in $\mathbb{R}^4$. Assume that $\mathcal{A}_1^{\mathrm{comb}}(X)$ and $\mathcal{A}_2^{\mathrm{comb}}(X)$ are defined. Then*

$$\mathcal{A}_2^{\mathrm{comb}}(X) \leqslant 2\mathcal{A}_1^{\mathrm{comb}}(X).$$

[6]More precisely, by Theorem 1.10, if X is a polytope as above for which $\mathcal{A}_1^{\mathrm{comb}}(X)$ and $\mathcal{A}_2^{\mathrm{comb}}(X)$ are defined, and if $\mathcal{A}_2^{\mathrm{comb}}(X) > 2\mathcal{A}_1^{\mathrm{comb}}(X)$, then Conjecture 1.22 fails for (nondegenerate C^∞ perturbations of) ε-smoothings of X for ε sufficiently small. Thus Conjecture 1.22 implies Conjecture 1.26. If Conjecture 1.24 is true, then one can conversely show, by approximating smooth domains by L-nondegenerate symplectic polytopes, that Conjecture 1.26 implies Conjecture 1.22.

One can use Theorems 1.10 and 1.11 to compute $\mathcal{A}_k^{\mathrm{comb}}(X)$. One can then test Conjecture 1.26 by using optimization algorithms to try to maximize the ratio $\mathcal{A}_2^{\mathrm{comb}}(X)/(2\mathcal{A}_1^{\mathrm{comb}}(X))$. So far we have not found any example where this ratio is greater than 1.

1.7. The rest of this part. In §2, we investigate Type 1 combinatorial Reeb orbits in detail, we define the combinatorial rotation number, and we work out the example of the 24-cell. In §3, we establish foundational facts about the combinatorial Reeb flow on a symplectic polytope. In §4 we review a symplectic trivialization of the contact structure on a star-shaped hypersurface in $\mathbb{R}^4$ defined using the quaternions. We explain a key curvature identity due to Hryniewicz and Salomão which implies that in the convex case, the rotation number of a Reeb trajectory increases monotonically as it evolves. In §5 we study the Reeb flow on a smoothing of a polytope. In §6 we use this work to prove the smooth-combinatorial correspondence of Theorems 1.10 and 1.11. In the appendix, we review basic facts about rotation numbers that we need throughout.

2. Type 1 combinatorial Reeb orbits

Let X be a symplectic polytope in $\mathbb{R}^4$. In this section we give what amounts to an algorithm for finding the Type 1 combinatorial Reeb orbits and their combinatorial symplectic actions, see Proposition 2.14. (Our actual computer implementation uses various optimizations not discussed here.) We also define combinatorial rotation numbers and work out the example of the 24-cell.

2.1. Symplectic flow graphs. We start by defining "symplectic flow graphs", which keep track of the combinatorics needed to find Type 1 Reeb orbits.

Definition 2.1. A *linear domain* is an intersection of a finite number of open or closed half-spaces in an affine space, or an affine space itself.

Definition 2.2. The *tangent space* TA of a linear domain A is the tangent space T_xA for any $x \in A$; the tangent spaces for different x are canonically isomorphic to each other via translations.

Definition 2.3. Let A and B be linear domains. An *affine map* $\phi : A \to B$ is the restriction of an affine map between affine spaces containing A and B. Such a map induces a map on tangent spaces which we denote by $T\phi : TA \to TB$.

Definition 2.4. Let A and B be linear domains. A *linear flow* from A to B is a triple $\Phi = (D, \phi, f)$ consisting of:

- the *domain of definition*: a linear domain $D \subset A$.
- the *flow map*: an affine map $\phi : D \to B$.
- the *action function*: an affine function $f : D \to \mathbb{R}$.

We sometimes write $\Phi : A \to B$. In the examples of interest for us, ϕ is injective, and $f \geqslant 0$.

Definition 2.5. Let $\Phi = (D, \phi, f)$ be a linear flow from A to B and let $\Psi = (E, \psi, g)$ be a linear flow from B to C. Their *composition* is the linear flow $\Psi \circ \Phi : A \to C$ defined by

$$\Psi \circ \Phi = (\phi^{-1}(E), \psi \circ \phi, f + g \circ \phi).$$

Remark 2.6. Composition of linear flows is associative, and there is an identity linear flow $\iota_A : A \to A$ given by $\iota_A = (A, \mathrm{id}_A, 0)$. If $\Phi_i = (D_i, \phi_i, f_i)$ is a linear flow from A_{i-1} to A_i for $i = 1, \ldots, k$, and if $\Phi = (D, \phi, f)$ is the composition $\Phi_k \circ \cdots \circ \Phi_1$, then for $x \in D$, we have

$$f(x) = \sum_{i=1}^{k} f_i((\phi_{i-1} \circ \cdots \circ \phi_1)(x)). \tag{2.1}$$

Definition 2.7. A *linear flow graph* G is a triple $G = (\Gamma, A, \Phi)$ consisting of:

- A directed graph Γ with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$.
- For each vertex v of Γ, an open linear domain A_v.
- For each edge e of Γ from u to v, a linear flow $\Phi_e = (D_e, \phi_e, f_e) : A_u \to A_v$.

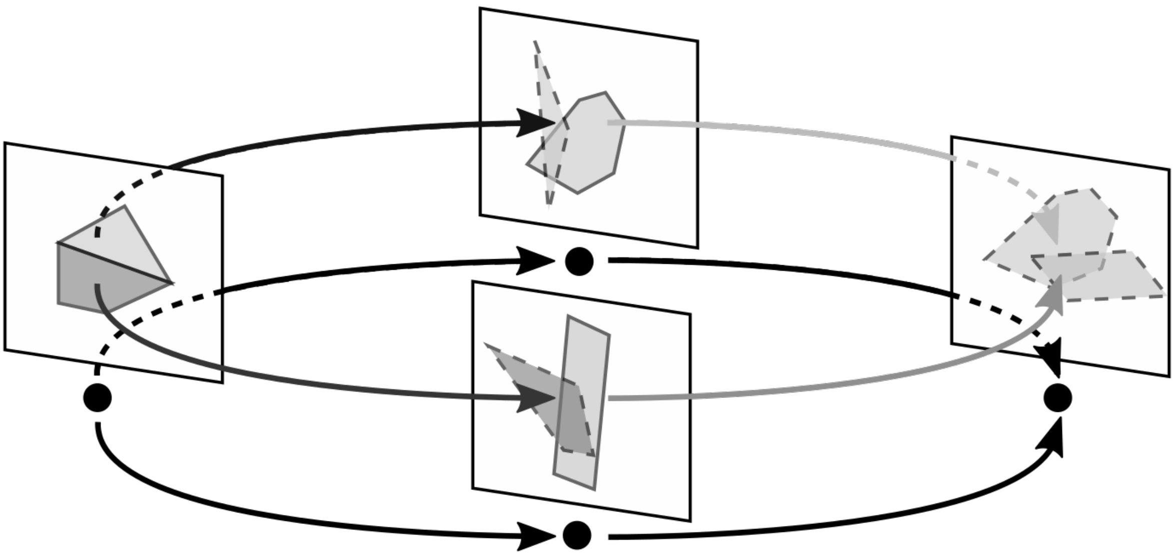


Figure 3. An example of a flow graph with 4 nodes and 4 edges. The linear domains and flows are depicted above their corresponding nodes and edges.

Let $G = (\Gamma, A, \Phi)$ be a linear flow graph. If $p = e_1 \ldots e_k$ is a path in Γ from u to v, we define an associated linear flow

$$\Phi_p = (D_p, \phi_p, f_p) : A_u \longrightarrow A_v$$

by

$$\Phi_p = \Phi_{e_k} \circ \cdots \circ \Phi_{e_1}.$$

Definition 2.8. A *trajectory* γ of G is a pair $\gamma = (p, x)$, where p is a path in Γ and $x \in D_p$.

Definition 2.9. A *periodic orbit* of G is an equivalence class of trajectories $\gamma = (p, x)$ where p is a cycle in Γ and x is a fixed point of ϕ_p, i.e. $\phi_p(x) = x$. Two such trajectories $\gamma = (p, x)$ and $\eta = (q, y)$ are equivalent if there are paths r and s in Γ such that $p = rs$, $q = sr$, and $\phi_r(x) = y$. We often abuse notation and denote the periodic orbit by $\gamma = (p, x)$, instead of by the equivalence class thereof.

Definition 2.10. The *action* of a periodic orbit $\gamma = (p, x)$ is defined by $f(\gamma) = f_p(x)$.

Definition 2.11. A periodic orbit $\gamma = (p, x)$, where p is a cycle based at u, is *degenerate* if the induced map on tangent spaces $T\phi_p : TD_u \to TD_u$ has 1 as an eigenvalue. Otherwise we say that γ is *nondegenerate*.

Definition 2.12. An $2n$-dimensional *symplectic flow graph* G is a quadruple $G = (\Gamma, A, \omega, \Phi)$ where:

- (Γ, A, Φ) is a linear flow graph in which each linear domain A_v has dimension $2n$.
- ω assigns to each vertex v of Γ a linear symplectic form ω_v on TA_v.

We require that if e is an edge from u to v, then $\phi_e^* \omega_v = \omega_u$.

2.2. The symplectic flow graph of a 4d symplectic polytope.

Definition 2.13. Let X be a symplectic polytope in $\mathbb{R}^4$. We associate to X the two-dimensional symplectic flow graph $G(X) = (\Gamma, A, \omega, \Phi)$ defined as follows:

- The vertex set of Γ is the set of 2-faces of X. The linear domain associated to a vertex is simply the corresponding 2-face, regarded as a linear domain in $\mathbb{R}^4$. If F is a 2-face, then the symplectic form ω_F on TF is the restriction of the standard symplectic form ω_0 on $\mathbb{R}^4$.
- If F_1 and F_2 are 2-faces, then there is an edge e in Γ from F_1 to F_2 if and only if there is a 3-face E adjacent to F_1 and F_2, and a trajectory of the Reeb vector field R_E on E from some point in F_1 to some point in F_2. In this case, the linear flow

$$\Phi_e = (D_e, \phi_e, f_e) : F_1 \longrightarrow F_2$$

  is defined as follows:
  - The domain D_e is the set of $x \in F_1$ such that there exists a trajectory of R_E from x to some point $y \in F_2$.
  - For x as above, $\phi_e(x) = y$, and $f_e(x)$ is the time it takes to flow along the vector field R_E from x to y, or equivalently the integral of λ_0 along the line segment from x to y.

In the above definition, note that ϕ_e and f_e are affine, because the vector field R_E on E is constant by equation (1.3). A simple calculation as in [**37**, Eq. (5.10)] shows that the map ϕ_e is symplectic.

Proposition 2.14. *Let X be a symplectic polytope in $\mathbb{R}^4$. Then there is a canonical bijection*

$$\{\text{periodic orbits of } G(X)\} \longleftrightarrow \{\text{Type } 1 \text{ combinatorial Reeb orbits of } X\}.$$

If (p, x) is a periodic orbit of $G(X)$, and if γ is the corresponding combinatorial Reeb orbit, then

$$f(p, x) = \mathcal{A}_{\text{comb}}(\gamma). \tag{2.2}$$

Proof. Suppose $(p = e_1 \cdots e_k, x)$ is a periodic orbit of $G(X)$. Let E_i denote the 3-face of X associated to e_i. There is then a combinatorial Reeb orbit $\gamma = (L_1, \ldots, L_k)$, where L_i is the line segment in E_i from $\phi_{e-1} \circ \cdots \circ \phi_{e_1}(x)$ to $\phi_{e_i} \circ \cdots \circ \phi_{e_1}(x)$. It follows from Definitions 1.5 and 2.13 that this construction defines a bijection from periodic orbits of $G(X)$ to combinatorial Reeb orbits of X. The identification of actions (2.2) follows from equation (2.1). □

By Proposition 2.14, to find the Type 1 Reeb orbits[7] of X, one can compute the symplectic flow graph $G(X) = (\Gamma, A, \omega, \Phi)$, enumerate the cycles in the graph Γ, and for each cycle p, compute the fixed points of the map ϕ_p in the domain D_p. In order to avoid searching for arbitrarily long cycles in the graph Γ in the cases of interest, we now need to discuss combinatorial rotation numbers.

2.3. Combinatorial rotation numbers.

Definition 2.15. A *trivialization* of a $2n$-dimensional symplectic flow graph $G = (\Gamma, A, \omega, \Phi)$ is a pair $(\tau, \widetilde{\phi})$ consisting of:

- For each vertex u of Γ, an isomorphism of symplectic vector spaces
$$\tau_u : (TA_u, \omega_u) \xrightarrow{\simeq} (\mathbb{R}^{2n}, \omega_0).$$
- For each edge e in Γ from u to v, a lift $\widetilde{\phi}_{e,\tau} \in \widetilde{\mathrm{Sp}}(2n)$ of the symplectic matrix
$$\tau_v \circ T\phi_e \circ \tau_u^{-1} \in \mathrm{Sp}(2n).$$

Here ω_0 denotes the standard symplectic form on $\mathbb{R}^{2n}$, and $\widetilde{\mathrm{Sp}}(2n)$ denotes the universal cover of the symplectic group $\mathrm{Sp}(2n)$. We sometimes abuse notation and denote the trivialization $(\tau, \widetilde{\phi})$ simply by τ.

If $p = e_1 \ldots e_n$ is a path in Γ from u to v, we define
$$\widetilde{\phi}_{p,\tau} = \widetilde{\phi}_{e_n,\tau} \circ \cdots \circ \widetilde{\phi}_{e_1,\tau} \in \widetilde{\mathrm{Sp}}(2n).$$

Definition 2.16. Let $G = (\Gamma, A, \omega, \Phi)$ be a 2-dimensional symplectic flow graph, let τ be a trivialization of G, and let p be a path in Γ. Define the *rotation number* of p with respect to τ by
$$\rho_\tau(p) = \rho(\widetilde{\phi}_{p,\tau}) \in \mathbb{R},$$
where the right hand side is the rotation number on $\widetilde{\mathrm{Sp}}(2)$ reviewed in Appendix 7.

Suppose now that X is a symplectic polytope in $\mathbb{R}^4$. We now define a canonical trivialization τ of the symplectic flow graph $G(X)$ which has the useful property that if (p, x) is a periodic orbit of $G(X)$, and if γ is the corresponding combinatorial Reeb orbit on X from Proposition 2.14, then the rotation number $\rho_\tau(p)$ is the limit of the rotation numbers of Reeb orbits on smoothings of X that converge to γ.

Fix matrices $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathrm{SO}(4)$ which represent the quaternion algebra, such that $\mathbf{i}$ is the standard almost complex structure. It follows from the formula $\omega_0(V, W) = \langle \mathbf{i}V, W \rangle$, together with the quaternion relations, that the matrices $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are symplectic. In examples below, in the coordinates x_1, x_2, y_1, y_2, we use the choice
$$\mathbf{i} = \begin{pmatrix} & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}.$$

[7] When testing Viterbo's conjecture and related conjectures, although all Type 1 orbits of X are detected by the flow graph $G(X)$, in view of Corollary 1.13 we must also account for Type 2 orbits. One can do this by either (1) extending $G(X)$ to a flow graph that includes the lower-dimensional faces of X or (2) working with a flow graph $G(X)$ whose linear domains A_F are the closures of the 2-faces, rather than 2-faces themselves. We use the first strategy in our computer program.

Definition 2.17. Let X be a symplectic polytope in $\mathbb{R}^4$. We define the *quaternionic trivialization* $(\tau, \widetilde{\phi})$ of the symplectic flow graph $G(X)$ as follows.

- Let F be a 2-face of X. We define the isomorphism

$$\tau_F : TF \xrightarrow{\simeq} \mathbb{R}^2$$

as follows. By Lemma 1.6, there is a unique 3-face E adjacent to F such that the Reeb cone R_F^+ consists of the nonnegative multiples of the Reeb vector field R_E, and the latter points into E from F. Let ν denote the outward unit normal vector to E. If $V \in TF$, define

$$\tau_F(V) = (\langle V, \mathbf{j}\nu\rangle, \langle V, \mathbf{k}\nu\rangle). \tag{2.3}$$

- If e is an edge from F_1 to F_2, define $\widetilde{\phi}_{e,\tau} \in \widetilde{\mathrm{Sp}}(2)$ to be the unique lift of the symplectic matrix

$$\tau_{F_2} \circ T\phi_e \circ \tau_{F_1}^{-1} \in \mathrm{Sp}(2) \tag{2.4}$$

that has rotation number in the interval $(-1/2, 1/2]$.

The following lemma verifies that this is a legitimate trivialization.

Lemma 2.18. *Let X be a symplectic polytope in $\mathbb{R}^4$. If F is a 2-face of X, then the linear map τ_F in* (2.3) *is an isomorphism of symplectic vector spaces.*

Proof. Let E and ν be as in the definition of τ_F. Then $\{\mathbf{i}\nu, \mathbf{j}\nu, \mathbf{k}\nu\}$ is an orthonormal basis for TE. We have $\omega_0(\mathbf{i}\nu, \mathbf{j}\nu) = \omega_0(\mathbf{i}\nu, \mathbf{k}\nu) = 0$ and $\omega_0(\mathbf{j}\nu, \mathbf{k}\nu) = 1$. If V and W are any two vectors in $TF \subset TE$, then expanding them in this basis, we find that $\omega_0(V, W) = \omega_0(\tau_F(V), \tau_F(W))$. □

Remark 2.19. An alternate convention for the quaternionic trivialization would be to define an isomorphism

$$\tau'_F : TF \xrightarrow{\simeq} \mathbb{R}^2$$

as follows. Let E' be the other 3-face adjacent to F (so that the Reeb vector field $R_{E'}$ points out of E along F), and let ν' denote the outward unit normal vector to E'. Define

$$\tau'_F(V) = (\langle V, \mathbf{j}\nu'\rangle, \langle V, \mathbf{k}\nu'\rangle).$$

This is also an isomorphism of symplectic vector spaces by the same argument as in Lemma 2.18.

Definition 2.20. If X is a symplectic polytope in $\mathbb{R}^4$ and F is a 2-face of X, define the *transition matrix*

$$\psi_F = \tau_F \circ (\tau'_F)^{-1} \in \mathrm{Sp}(2).$$

Lemma 2.21. *If X is a symplectic polytope in $\mathbb{R}^4$ and F is a 2-face of X, then the transition matrix ψ_F is positive elliptic (see Definition 7.7).*

Proof. We compute that

$$(\tau'_F)^{-1} = \left(\mathbf{j}\nu' - \frac{\langle \mathbf{j}\nu', \nu\rangle}{\langle \mathbf{i}\nu', \nu\rangle}\mathbf{i}\nu', \mathbf{k}\nu' - \frac{\langle \mathbf{k}\nu', \nu\rangle}{\langle \mathbf{i}\nu', \nu\rangle}\mathbf{i}\nu'\right). \tag{2.5}$$

To simplify notation, write $a_1 = \langle v', v\rangle$, $a_2 = \langle \mathbf{i}v', v\rangle$, $a_3 = \langle \mathbf{j}v', v\rangle$, and $a_4 = \langle \mathbf{k}v', v\rangle$. It then follows from (2.3) and (2.5) that

$$\psi_F = \frac{1}{a_2}\begin{pmatrix} a_1a_2 - a_3a_4 & -a_2^2 - a_4^2 \\ a_2^2 + a_3^2 & a_1a_2 + a_3a_4 \end{pmatrix}$$

Then $\operatorname{Tr}(\psi_F) = 2\langle v', v\rangle \in (-2, 2)$, so ψ_F is elliptic. Moreover $a_2 > 0$ by Lemma 3.9 below, so ψ_F is positive elliptic. $\square$

Corollary 2.22. *If E is a 3-face of X, if F_1 and F_2 are 2-faces of X, and if there is a trajectory of the Reeb vector field on E from some point in F_1 to some point in F_2, then $\widetilde{\phi}_{e,\tau}$ has rotation number in the interval $(0, 1/2)$.*

Proof. It follows from the definitions that the map (2.4) agrees with the transition matrix ψ_{F_2}. By Lemma 2.21, this matrix is positive elliptic. It then follows from Lemma 7.8 that its mod $\mathbb{Z}$ rotation number is in the interval $(0, 1/2)$. $\square$

Definition 2.23. Let X be a symplectic polytope in $\mathbb{R}^4$. Let γ be a Type 1 combinatorial Reeb orbit for X.

- We define the *combinatorial rotation number* of γ by

  $$\rho_{\mathrm{comb}}(\gamma) = \rho_\tau(p),$$

  where (p, x) is the periodic orbit of $G(X)$ corresponding to γ in Proposition 2.14, and τ is the quaternionic trivialization of X.
- We say that γ is *nondegenerate* if the periodic orbit (p, x) is nondegenerate as in Definition 2.11. In this case we define the *combinatorial Conley-Zehnder index* of γ by equation (1.6).

Remark 2.24. By Corollary 2.22, the combinatorial rotation number is the rotation number of a product of elements of $\widetilde{\mathrm{Sp}}(2)$ each with rotation number in the interval $(0, 1/2)$. A formula for computing the rotation number of such a product is given by Proposition 7.9.

2.4. Example: the 24-cell. We now compute the symplectic flow graph $G(X) = (\Gamma, A, \omega, \Phi)$ and the quaternionic trivialization τ for the example where X is the 24-cell with vertices

$$(\pm 1, 0, 0, 0), (0, \pm 1, 0, 0), (0, 0, \pm 1, 0), (0, 0, 0, \pm 1), (\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2).$$

The polytope X has 24 three-faces, each of which is an octahedron. The 3-faces are contained in the hyperplaces

$$\pm x_1 \pm x_2 = 1,\ \pm x_1 \pm y_1 = 1,\ \pm x_1 \pm y_2 = 1,\ \pm x_2 \pm y_1 = 1,\ \pm x_2 \pm y_2 = 1,\ \pm y_1 \pm y_2 = 1.$$

There are 96 two-faces, each of which is a triangle; thus the graph Γ has 96 vertices. It follows from the calculations below that none of the 2-faces is Lagrangian, so that X is a symplectic polytope.

To understand the edges of the graph Γ, consider for example the 3-face E contained in the hyperplane $x_1 + y_1 = 1$. The vertices of this 3-face are

$$(1, 0, 0, 0), (1/2, \pm 1/2, 1/2, \pm 1/2), (0, 0, 1, 0).$$

The unit normal vector to this face is

$$\nu = \frac{1}{\sqrt{2}}(1,0,1,0).$$

The Reeb vector field on E is

$$R_E = 2\left(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right).$$

Thus the Reeb flow on E flows from the vertex $(1,0,0,0)$ to the vertex $(0,0,1,0)$ in time $1/2$. Each of the four 2-faces of E adjacent to $(1,0,0,0)$ flows to one of the four 2-faces of E adjacent to $(0,0,1,0)$, by an affine linear isomorphism.

For example, let F_1 be the 2-face with vertices $(1,0,0,0)$, $(1/2,1/2,1/2,\pm 1/2)$, and let F_2 be the 2-face with vertices $(0,0,1,0)$, $(1/2,1/2,1/2,\pm 1/2)$. Then F_1 flows to F_2, so there is an edge e in the graph Γ from F_1 to F_2. More explicitly, we can parametrize F_1 as

$$\left(1-\frac{t_1+t_2}{2}, \frac{t_1+t_2}{2}, \frac{t_1+t_2}{2}, \frac{t_1-t_2}{2}\right), \quad t_1,t_2>0,\ t_1+t_2<1,$$

and we can parametrize F_2 as

$$\left(\frac{t_1+t_2}{2}, \frac{t_1+t_2}{2}, 1-\frac{t_1+t_2}{2}, \frac{t_1-t_2}{2}\right), \quad t_1,t_2>0,\ t_1+t_2<1.$$

With respect to these parametrizations, the flow map ϕ_e is simply

$$\phi_e(t_1,t_2) = (t_1,t_2).$$

The domain D_e of ϕ_e is all of F_1, and the action function is

$$f_e(t_1,t_2) = \frac{1-t_1-t_2}{2}.$$

It turns out that for every other 3-face E', there is a linear symplectomorphism A of $\mathbb{R}^4$ such that $AX = X$ and $AE = E'$. In fact, we can take A to be right multiplication by an appropriate unit quaternion. It follows from this symplectic symmetry that the Reeb flow on each 3-face behaves analogously. Putting these Reeb flows together, one finds that the graph Γ consists of 8 disjoint 12-cycles. (This example is highly non-generic!) Further calculations show that for each 12-cycle p, the map ϕ_p is the identity, so that every point in the interior of a 2-face is on a Type 1 combinatorial Reeb orbit. Moreover, the action of each such orbit is equal to 2. In particular, X is "combinatorially Zoll" in the sense of Definition 1.19. Also, the volume of X is 2, so X has systolic ratio 1.

To see how the quaternionic trivialization works, let us compute $\widetilde{\phi}_{e,\tau}$ for the edge e above. For the 2-face F_1 above, the isomorphism τ_{F_1} is given in terms of the unit normal vector ν to E. We compute that

$$\mathbf{j}\nu = \frac{1}{\sqrt{2}}(0,1,0,-1), \qquad \mathbf{k}\nu = \frac{1}{\sqrt{2}}(0,1,0,1).$$

It follows that in terms of the basis $(\partial_{t_1}, \partial_{t_2})$ for TF_1, we have

$$\tau_{F_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For the 2-face F_2 above, the isomorphism τ_{F_2} is given in terms of the unit normal vector to the *other* 3-face adjacent to F_2. This other 3-face is in the hyperplane $x_2 + y_1 = 1$ and so has unit normal vector

$$\nu' = \frac{1}{\sqrt{2}}(0, 1, 1, 0).$$

We then similarly compute that in terms of the basis $(\partial_{t_1}, \partial_{t_2})$ for TF_2, we have

$$\tau_{F_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

Therefore the matrix (2.4) for the edge e is

$$\tau_{F_2} \circ T\phi_e \circ \tau_{F_1}^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

This matrix is positive elliptic and has eigenvalues $e^{\pm i\pi/3}$. It follows that its lift $\widetilde{\phi}_{e,\tau}$ in $\widetilde{\mathrm{Sp}}(2)$ has rotation number $1/6$.

For one of the other three edges associated to E, the matrix (2.4) is the same as above, and for the other two edges associated to E, the matrix is $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, whose lift also has rotation number $1/6$. It then follows from the quaternionic symmetry of X mentioned earlier that for every edge e' of the graph Γ, the lift $\widetilde{\phi}_{e',\tau}$ is one of the above two matrices with rotation number $1/6$. One can further check that for each 12-cycle in the graph, one obtains just one of the above two matrices repeated 12 times, so each corresponding Type 1 combinatorial Reeb orbit has rotation number equal to 2.

3. Reeb dynamics on symplectic polytopes

The goal of this section is to Proposition 1.4 and Lemma 1.6, describing the Reeb dynamics on the boundary of a symplectic polytope in $\mathbb{R}^4$.

3.1. Preliminaries on tangent and normal cones. We now prove some lemmas about tangent and normal cones which we will need; see §1.2 for the definitions.

Recall that if C is a cone in $\mathbb{R}^m$, its *polar dual* is defined by

$$C^o = \{y \in \mathbb{R}^m \mid \langle x, y\rangle \leqslant 0 \ \forall x \in X\}.$$

LEMMA 3.1. *Let X be a convex set in $\mathbb{R}^m$ and let $y \in \partial X$. Then*

$$N_y^+ X = (T_y^+ X)^o, \qquad T_y^+ X = (N_y^+ X)^o.$$

PROOF. If C is a closed cone then $(C^o)^o = C$, so it suffices to prove that $N_y^+ X = (T_y^+ X)^o$.

To show that $N_y^+ X \subset (T_y^+ X)^\circ$, let $v \in N_y^+ X$ and $w \in T_y^+ X$; we need to show that $\langle v, w\rangle \leqslant 0$. By the definition of $T_y^+ X$, there exist a sequence of vectors $\{w_i\}$ and a sequence of positive real numbers $\{\varepsilon_i\}$ such that $y + \varepsilon_i w_i \in X$ for each i and $\lim_{i\to\infty} w_i = w$. By the definition of $N_y^+ X$ we have $\langle v, w_i\rangle \leqslant 0$, and so $\langle v, w\rangle \leqslant 0$.

To prove the reverse inclusion, if $v \in (T_x^+ X)^o$, then for any $x \in X$ we have $x - y \in T_y^+ X$, so $\langle v, x - y\rangle \leqslant 0$. It follows that $v \in N_y^+ X$. □

If X is a convex polytope in $\mathbb{R}^m$ and if E is an $(m-1)$-face of X, let ν_E denote the outward unit normal vector to E.

Lemma 3.2. *Let X be a convex polytope in $\mathbb{R}^m$ and let F be a face of X. Let $E_1, \ldots, E_k$ denote the $(m-1)$-faces whose closures contain F. Then*

$$T_F^+ X = \{w \in \mathbb{R}^m \mid \langle w, \nu_{E_i} \rangle \leqslant 0 \;\; \forall i = 1, \ldots, k\}, \tag{3.1}$$

$$N_F^+ X = \operatorname{Cone}(\nu_{E_1}, \ldots, \nu_{E_k}). \tag{3.2}$$

Proof. Let $y \in F$, and let B be a small ball around y. Then $B \cap X = \cap_i (B \cap H_i)$ where $\{H_i\}$ is the set of all defining half-spaces for X whose boundaries contain F. The boundaries of the half-spaces H_i are the hyperplanes that contain the $(m-1)$-faces $E_1, \ldots, E_k$. It follows that $B \cap X$ is the set of $x \in B$ such that $\langle x - y, \nu_{E_i} \rangle \leqslant 0$ for each $i = 1, \ldots, k$. Equation (3.1) follows. Taking polar duals and using Lemma 3.1 then proves (3.2). □

Lemma 3.3. *Let X be a convex polytope in $\mathbb{R}^m$ and let F be a face of X. Let $v \in N_F^+ X \setminus \{0\}$ and let $w \in T_F^+ X \setminus \{0\}$. Then $\langle v, w \rangle = 0$ if and only if there is a face E of X with $F \subset \overline{E}$ such that $v \in N_E^+ X$ and $w \in T_F^+ \overline{E}$.*

Here if $E \neq F$ then $T_F^+ \overline{E}$ denotes the tangent cone of the polytope $\overline{E}$ at the face F of $\overline{E}$; if $E = F$, then we interpret $T_F^+ \overline{E} = TF$.

Proof of Lemma 3.3. As in Lemma 3.2, let $E_1, \ldots, E_k$ denote the $(m-1)$-faces adjacent to F.

($\Rightarrow$) By the definitions of $N_F^+ X$ and $T_F^+ X$, if $v \in N_F^+ X$ and $w \in T_F^+ X$ then $\langle v, w \rangle \leqslant 0$. Assume also that v and w are both nonzero and $\langle v, w \rangle = 0$. Then we must have $v \in \partial N_F^+ X$ and $w \in \partial T_F^+ X$; otherwise we could perturb v or w to make the inner product positive, which would be a contradiction.

Since $w \in \partial T_F^+ X$, it follows from (3.1) that $\langle w, \nu_{E_i} \rangle = 0$ for some i. By renumbering we can arrange that $\langle w, \nu_{E_i} \rangle = 0$ if and only if $i \leqslant l$ where $1 \leqslant l \leqslant k$. Let $E = \cap_{i=1}^{l} E_i$. Then E is a face of X adjacent to F, and $w \in T_F^+ \overline{E}$.

We now want to show that $v \in N_E^+ X$. By (3.2), we can write $v = \sum_{i=1}^{k} a_i \nu_{E_i}$ with $a_i \geqslant 0$. Since $\langle v, w \rangle = 0$ and $\langle w, \nu_{E_i} \rangle = 0$ for $i \leqslant l$ and $\langle w, \nu_{E_i} \rangle < 0$ for $i > l$, we must have $a_i = 0$ for $i > l$. Thus $v \in \operatorname{Cone}(\nu_{E_1}, \ldots, \nu_{E_l})$, so by (3.2) again, $v \in N_F^+ X$.

($\Leftarrow$) Assume that there is a face E adjacent to X such that $v \in N_E^+ X$ and $w \in T_F^+ \overline{E}$. We can renumber so that $E = \cap_{i=1}^{l} E_i$ where $1 \leqslant l \leqslant k$. Then $v \in \operatorname{Cone}(\nu_{E_1}, \ldots, \nu_{E_l})$, and $\langle w, \nu_{E_i} \rangle = 0$ for $i \leqslant l$, so $\langle v, w \rangle = 0$. □

3.2. The combinatorial Reeb flow is locally well-posed. We now prove Proposition 1.4, asserting that the "combinatorial Reeb flow" on the boundary of a symplectic polytope in $\mathbb{R}^4$ is locally well-posed. This is a consequence of the following two lemmas:

Lemma 3.4. *Let X be a convex polytope in $\mathbb{R}^4$, and let F be a face of X. Then the Reeb cone*

$$R_F^+ X = \mathbf{i} N_F^+ X \cap T_F^+ X$$

has dimension at least 1*.*

Note that there is no need to assume that $0 \in \operatorname{int}(X)$ in the above lemma, because the Reeb cone is invariant under translation of X.

Lemma 3.5. *Let X be a symplectic polytope in $\mathbb{R}^4$ and let F be a face of X. Then the Reeb cone $R_F^+ X$ has dimension at most* 1*.*

PROOF OF LEMMA 3.4. The proof has four steps.

Step 1. We need to show that there exists a unit vector in R_F^+X. We first rephrase this statement in a way that can be studied topologically.

Define

$$B = \left\{(v,w) \in N_F^+X \times T_F^+X \;\middle|\; \|v\| = \|w\| = 1,\ \langle v,w\rangle = 0\right\}.$$

Define a fiber bundle $\pi : Z \to B$ with fiber S^2 by setting

$$Z_{(v,w)} = \left\{u \in \mathbb{R}^4 \;\middle|\; \|u\| = 1,\ \langle u,v\rangle = 0\right\}.$$

Define two sections

$$s_0, s_1 : B \longrightarrow Z$$

by

$$\begin{aligned} s_0(v,w) &= \mathbf{i}v, \\ s_1(v,w) &= w. \end{aligned}$$

To show that there exists a unit vector in R_F^+X, we need to show that there exists a point $(v,w) \in B$ with $s_0(v,w) = s_1(v,w)$.

Step 2. Let

$$B_0 = \left\{w \in \partial T_F^+X \;\middle|\; \|w\| = 1\right\}.$$

The space B_0 is the set of unit vectors on the boundary of a nondegenerate cone, and thus is homeomorphic to S^2. Recall from the proof of Lemma 3.3 that if $(v,w) \in B$ then $w \in B_0$. We now show that the projection $B \to B_0$ sending $(v,w) \mapsto w$ is a homotopy equivalence.

To do so, observe that by Lemma 3.3, we have

$$B = \bigcup_{F \subset E} \left\{v \in N_E^+X \mid \|v\| = 1\right\} \times \left\{w \in T_F^+\overline{E} \;\middle|\; \|w\| = 1\right\}. \tag{3.3}$$

If F is a 3-face, then in the union (3.3), we only have $E = F$; there is a unique unit vector $v \in N_E^+X$, and so the projection $B \to B_0$ is a homeomorphism.

If F is a 2-face, then in (3.3), E can be either F itself, or one of the two three-faces adjacent to F, call them E_1 and E_2. The contribution from $E = F$ is a cylinder, while the contributions from $E = E_1$ and E_2 are disks which are glued to the cylinder along its boundary. The projection $B \to B_0$ collapses the cylinder to a circle, which again is a homotopy equivalence.

If F is a 1-face, with k adjacent 3-faces, then the contribution to (3.3) from $E = F$ consists of two disjoint closed k-gons. Each 2-face E adjacent to F contributes a square with opposite edges glued to one edge of each k-gon. Each 3-face E adjacent to F contributes a bigon filling in the gap between two consecutive squares. The projection $B \to B_0$ collapses each k-gon to a point and each bigon to an interval, which again is a homotopy equivalence.

Finally, suppose that F is a 0-face. Then $E = F$ makes no contribution to (3.3), since $TF = \{0\}$ contains no unit vectors. Now B_0 has a cell decomposition consisting of a k-cell for each $(k+1)$-face adjacent to F. The space B is obtained from B_0 by thickening each 0-cell to a closed polygon, and thickening each 1-cell to a square. Again, this is a homotopy equivalence.

Step 3. The S^2-bundle $Z \to B$ is trivial. To see this, observe that Z is the pullback of a bundle over $N_F^+X \backslash \{0\}$, whose fiber over v is the set of unit vectors orthogonal to v. Since

$N_F^+ X \backslash \{0\}$ is contractible, the latter bundle is trivial, and thus so is Z. In particular, the bundle Z has two homotopy classes of trivialization, which differ only in the orientation of the fiber. We now show that, using a trivialization to regard s_0 and s_1 as maps $B \to S^2$, the mod 2 degrees of these maps are given by $\deg(s_0) = 0$ and $\deg(s_1) = 1$.

It follows from the triviality of the bundle Z that $\deg(s_0) = 0$.

To prove that $\deg(s_1) = 1$, we need to pick an explicit trivialization of Z. To do so, fix a vector $v_0 \in \operatorname{int}(T_F^+ X)$. Let S denote the set of unit vectors in the orthogonal complement $v_0^\perp$. Let $P : \mathbb{R}^4 \to v_0^\perp$ denote the orthogonal projection. We then have a trivialization

$$Z \xrightarrow{\simeq} B \times S$$

sending

$$((v, w), u) \longmapsto ((v, w), Pu/\|Pu\|).$$

Note here that for every $(v, w) \in B$, the restriction of P to $v^\perp$ is an isomorphism, because otherwise v would be orthogonal to v_0, but in fact we have $\langle v, v_0 \rangle < 0$.

With respect to this trivialization, the section s_1 is a map $B \to S$ which is the composition of the projection $B \to B_0$ with the map $B_0 \to S$ sending

$$w \longmapsto Pw/\|Pw\|.$$

The former map is a homotopy equivalence by Step 2, and the latter map is a homeomorphism because v_0 is not parallel to any vector in $\partial T_F^+ X$. Thus $\deg(s_1) = 1$.

Step 4. We now complete the proof of the lemma. Suppose to get a contradiction that there does not exist a point $p \in B$ with $s_0(p) = s_1(p)$. It follows, using a trivialization of Z to regard s_0 and s_1 as maps $B \to S^2$, that s_1 is homotopic to the composition of s_0 with the antipodal map. Then $\deg(s_1) = -\deg(s_0)$. This contradicts Step 3. □

Remark 3.6. It might be possible to generalize Lemma 3.4 to show that if X is any convex set in $\mathbb{R}^{2n}$ with nonempty interior and if $z \in \partial X$, then the Reeb cone $R_z^+ X$ is at least one dimensional.

We now prepare for the proof of Lemma 3.5.

Lemma 3.7. *Let X be a convex polytope in $\mathbb{R}^{2n}$. Then for every face F of X, there exists a face E with $F \subset \overline{E}$ such that*

$$R_F^+ X \subset T_F^+ \bar{E}.$$

Proof. Let $\{E_i\}_{i=1}^N$ denote the set of faces whose closures contain F. By Lemma 3.3, we have

$$R_F^+ X \subset \bigcup_{i=1}^N T_F^+ \bar{E}_i. \tag{3.4}$$

Let V denote the subspace of $\mathbb{R}^{2n}$ spanned by $R_F^+ X$. Note that since the latter set is a cone, it has a nonempty interior in V. We claim now that $V \subset TE_i$ for some i. If not, then $V \cap TE_i$ is a proper subspace of V for each i. But by (3.4), we have

$$R_F^+ X = \left(\cup_i T_F^+ \bar{E}_i\right) \cap R_F^+ X \subset (\cup_i TE_i) \cap V.$$

This is a contradiction, since the left hand side has a nonempty interior in V, while the right hand side is a union of proper subspaces of V.

Since $V \subset TE_i$, it follows that $R_F^+ X \subset T_F^+ \bar{E}_i$, because by (3.4) again,

$$\begin{aligned} R_F^+ X &= R_F^+ X \cap V = R_F^+ X \cap TE_i \\ &\subset TE_i \cap \left(\bigcup_j T_F^+ \bar{E}_j \right) = T_F \bar{E}_i, \end{aligned}$$

□

Lemma 3.8. *Let X be a convex polytope in $\mathbb{R}^{2n}$, and let F be a face of X. Let $v \in R_F^+ X$. Suppose that $v \in \operatorname{int}(T_F^+ \overline{E})$ for some $(2n-1)$-face E whose closure contains F. Then v is a positive multiple of $\mathbf{i}\nu_E$.*

Proof. Let $E = E_1, \ldots, E_N$ denote the $(2n-1)$-faces whose closures contain F, and let ν_i denote the outward unit normal vector to E. Since $v \in \operatorname{int}(T_F^+ \overline{E})$, we have $\langle v, \nu_1 \rangle = 0$ and $\langle v, \nu_i \rangle < 0$ for $i > 1$. Since $-\mathbf{i}v \in N_F^+ X$, it follows from Lemma 3.2 that we can write

$$-\mathbf{i}v = \sum_{i=1}^{N} a_i \nu_i$$

with $a_i \geqslant 0$. Since $\langle v, \mathbf{i}v \rangle = 0$, we conclude that $a_i = 0$ for $i > 1$. Thus $-\mathbf{i}v = a_1 \nu_1$, and $a_1 > 0$. □

Proof of Lemma 3.5. Suppose v_0, v_1 are distinct unit vectors in $R_F^+ X$. By Lemma 3.7, there is a 3-face E such that v_0 and v_1 are both in $T_F^+ \bar{E}$. In particular, v_1 and v_2 are linearly independent.

Since v_0 and v_1 are both in the cone $R_F^+ X$, it follows that if $t \in [0,1]$ then the affine linear combination $(1-t)v_0 + tv_1$ is also in this cone. Since v_0 and v_1 are linearly independent, these affine linear combinations cannot be in the interior of $T_F^+ \overline{E}$, or else this would contradict the projective uniqueness in Lemma 3.8. Consequently v_0 and v_1 are both contained in $T_F^+ \overline{E'}$ for some 2-face E' on the boundary of $\overline{E}$.

We now have

$$\omega(v_0, v_1) = \langle v_0, -\mathbf{i}v_1 \rangle \leqslant 0,$$

where the inequality holds since $v_0 \in T_F^+ X$ and $-\mathbf{i}v_1 \in N_F^+ X$. By a symmetric calculation, $\omega(v_1, v_0) \leqslant 0$. It follows that $\omega(v_0, v_1) = 0$. Since v_0 and v_1 are linearly independent vectors in TE', this contradicts the hypothesis that $\omega|_{TE'}$ is nondegenerate. □

3.3. Description of the Reeb cone. We now prove Lemma 1.6, describing the possibilities for the Reeb cone of a face of a symplectic polytope in $\mathbb{R}^4$.

Lemma 3.9. *Let X be a convex polytope in $\mathbb{R}^4$ and let F be a 2-face of X. Let E_1 and E_2 denote the 3-faces adjacent to F, and let ν_i denote the outward unit normal vector to E_i.*

(a) If $\langle \mathbf{i}\nu_1, \nu_2 \rangle < 0$, then every nonzero vector w in the Reeb cone $R_{E_1}^+$ points into E_1 from F, that is $w \in \operatorname{int}(T_F^+ \overline{E_1})$.

(b) If $\langle \mathbf{i}\nu_1, \nu_2 \rangle > 0$, then every nonzero vector w in the Reeb cone $R_{E_1}^+$ points out of E_1 from F, that is $w \in \operatorname{int}(-T_F^+ \overline{E_1})$.

(c) If $\langle \mathbf{i}\nu_1, \nu_2 \rangle = 0$, then F is Lagrangian.

Proof. Let η denote the unit normal vector to F in $T\overline{E_1}$ pointing into E_1. The vector η must be a linear combination of ν_1 and ν_2 (since it is normal to F), it must be orthogonal to ν_1 (since it is tangent to E_1), and it must have negative inner product with ν_2 (since it points into E_1). It follows that

$$\eta = \frac{-\nu_2 + \langle \nu_1, \nu_2 \rangle \nu_1}{\| - \nu_2 + \langle \nu_1, \nu_2 \rangle \nu_1 \|}. \tag{3.5}$$

The vector w points into E_1 if and only if $\langle \eta, w \rangle > 0$, and the vector w points out of E_1 if and only if $\langle \eta, w \rangle < 0$. For w in the Reeb cone of E_1, we know that w is a positive multiple of $\mathbf{i}\nu_1$. By equation (3.5), we have

$$\langle \eta, \mathbf{i}\nu_1 \rangle = \frac{-\langle \mathbf{i}\nu_1, \nu_2 \rangle}{\| - \nu_2 + \langle \nu_1, \nu_2 \rangle \nu_1 \|}.$$

Thus if $\langle \mathbf{i}\nu_1, \nu_2 \rangle$ is nonzero, then it has opposite sign from $\langle \eta, w \rangle$. This proves (a) and (b).

If $\langle \mathbf{i}\nu_1, \nu_2 \rangle = 0$, then $\omega(\mathbf{i}\nu_1, \mathbf{i}\nu_2) = 0$, but $\mathbf{i}\nu_1$ and $\mathbf{i}\nu_2$ are linearly independent tangent vectors to F, so F is Lagrangian. This proves (c). □

Lemma 3.10. *Let X be a convex polytope in $\mathbb{R}^4$ and let F be a 2-face of X. If $TF \cap R_F^+ X \neq \{0\}$, then F is Lagrangian.*

Proof. If $w \in TF \cap R_F^+ X$, then for any other vector $u \in TF$, we have

$$\omega(w, u) = \langle \mathbf{i}w, u \rangle = 0$$

since $-\mathbf{i}w \in N_F^+ X$. If we also have $w \neq 0$, then it follows that F is Lagrangian. □

Proof of Lemma 1.6. If F is a 3-face, then by the definition of the Reeb cone, $R_F^+ X$ consists of all nonnegative multiples of $\mathbf{i}\nu_F$; and $\mathbf{i}\nu_F$ is a positive multiple of the Reeb vector field on F by equation (1.3).

Suppose now that F is a k-face with $k < 3$, and that w is a nonzero vector in the Reeb cone $R_F^+ X$. Applying Lemma 3.3 to $v = -\mathbf{i}w$ and w, we deduce that there is a face E of X with $F \subset \overline{E}$ such that $-\mathbf{i}w \in N_E^+ X$ and $w \in T_F^+ \overline{E}$. In particular,

$$w \in TE \cap R_E^+ X. \tag{3.6}$$

By Lemma 3.10 and our hypothesis that X is a symplectic polytope, E is not a 2-face.

If F is a 2-face, we conclude that w is in the Reeb cone $R_E^+ X$ for one of the 3-faces E adjacent to F. By Lemma 3.9, w must point into E.

If F is a 1-face, then E is either a 3-face adjacent to F, or F itself. In the case when $E = F$, the vector w cannot be in the Reeb cone of any 3-face F_3 adjacent to F. The reason is that if F_2 is one of the two 2-faces with $F \subset \overline{F_2} \subset \overline{F_3}$, then by Lemma 3.9, the Reeb cone of F_3 is not tangent to F_2, so it certainly cannot be tangent to F.

If F is a 0-face, then E is adjacent to F and is either a 3-face or a 1-face. If E is a 1-face, then it is a bad 1-face by (3.6). □

4. The quaternionic trivialization

In this section let $Y \subset \mathbb{R}^4$ be a smooth star-shaped hypersurface with the contact form $\lambda = \lambda_0|_Y$ and contact structure $\xi = \mathrm{Ker}(\lambda)$. We now define a special trivialization τ of the contact structure ξ, and we prove a key property of this trivialization.

4.1. Definition of the quaternionic trivialization. The following definition is a smooth analogue of Definition 2.17.

DEFINITION 4.1. Define the *quaternionic trivialization*

$$\tau : \xi \xrightarrow{\simeq} Y \times \mathbb{R}^2 \tag{4.1}$$

as follows. If $y \in Y$ and $V \in T_yY$, let ν denote the outward unit normal to Y at y, and define

$$\tau(V) = (y, \langle V, \mathbf{j}\nu\rangle, \langle V, \mathbf{k}\nu\rangle).$$

By abuse of notation, for fixed $y \in Y$ we write $\tau : \xi_y \xrightarrow{\simeq} \mathbb{R}^2$ to denote the restriction of (4.1) to ξ_y followed by projection to $\mathbb{R}^2$.

From now on we always use the quaternionic trivialization τ for smooth star-shaped hypersurfaces in $\mathbb{R}^4$.

LEMMA 4.2. *The quaternionic trivialization τ is a symplectic trivialization of ξ.*

PROOF. Same calculation as the proof of Lemma 2.18(a). □

REMARK 4.3. The inverse

$$\tau^{-1} : Y \times \mathbb{R}^2 \xrightarrow{\simeq} \xi$$

is described as follows. Recall from (1.3) that the Reeb vector field at y is a positive multiple of $\mathbf{i}\nu$. Then $\tau^{-1}(y,(1,0))$ is obtained by projecting $\mathbf{j}\nu$ to ξ_y along the Reeb vector field, while $\tau^{-1}(y,(0,1))$ is obtained by projecting $\mathbf{k}\nu$ to ξ_y along the Reeb vector field.

4.2. Linearized Reeb flow. We now make some definitions which we will need in order to bound the rotation numbers of Reeb orbits and Reeb trajectories.

DEFINITION 4.4. If $y \in Y$ and $t \geqslant 0$, define the *linearized Reeb flow* $\phi(y,t) \in \mathrm{Sp}(2)$ to be the composition

$$\mathbb{R}^2 \xrightarrow{\tau^{-1}} \xi_y \xrightarrow{d\Phi_t} \xi_{\Phi_t(y)} \xrightarrow{\tau} \mathbb{R}^2 \tag{4.2}$$

where $\Phi_t : Y \to Y$ denotes the time t flow of the Reeb vector field, and τ is the quaternionic trivialization. Define the *lifted linearized Reeb flow* $\widetilde{\phi}(y,t) \in \widetilde{\mathrm{Sp}}(2)$ to be the arc

$$\widetilde{\phi}(y,t) = \{\phi(y,s)\}_{s\in[0,t]}. \tag{4.3}$$

Note that we have the composition property

$$\widetilde{\phi}(y, t_2 + t_1) = \widetilde{\phi}(\phi_{t_1}(y), t_2) \circ \widetilde{\phi}(y, t_1).$$

Next, let $\mathbb{P}\xi$ denote the "projectivized" contact structure

$$\mathbb{P}\xi = (\xi \backslash Z)/\sim$$

where Z denotes the zero section, and two vectors are declared equivalent if they differ by multiplication by a positive scalar. Thus $\mathbb{P}\xi$ is an S^1-bundle over Y. The Reeb vector field R on Y canonically lifts, via the linearized Reeb flow, to a vector field $\widetilde{R}$ on $\mathbb{P}\xi$.

The quaternionic trivialization τ defines a diffeomorphism

$$\overline{\tau} : \mathbb{P}\xi \xrightarrow{\simeq} Y \times S^1.$$

Let
$$\sigma : \mathbb{P}\xi \longrightarrow S^1$$
denote the composition of $\overline{\tau}$ with the projection $Y \times S^1 \to S^1$.

Definition 4.5. Define the *rotation rate*
$$r : \mathbb{P}\xi \longrightarrow \mathbb{R}$$
to be the derivative of σ with respect to the lifted linearized Reeb flow,
$$r = \widetilde{R}\sigma.$$
Define the *minimum rotation rate*
$$r_{\min} : Y \longrightarrow \mathbb{R}$$
by
$$r_{\min}(y) = \min_{\widetilde{y} \in \mathbb{P}\xi_y} r(\widetilde{y}).$$

It follows from (7.6) and (7.7) that we have the following lower bound on the rotation number of the lifted linearized flow of a Reeb trajectory.

Lemma 4.6. *Let y be a smooth star-shaped hypersurface in $\mathbb{R}^4$, let $y \in Y$, and let $t \geqslant 0$. Then*
$$\rho(\widetilde{\phi}(y,t)) \geqslant \int_0^t r_{\min}(\Phi_s(y))ds.$$

4.3. The curvature identity. We now prove a key identity which relates the linearized Reeb flow, with respect to the quaternionic trivialization τ, to the curvature of Y. This identity (in different notation) is due to U. Hryniewicz and P. Salomão [40]. Below, let $S : TY \otimes TY \to \mathbb{R}$ denote the second fundamental form defined by
$$S(u,w) = \langle \nabla_u \nu, w \rangle,$$
where ν denotes the outward unit normal vector to Y, and ∇ denotes the trivial connection on the restriction of $T\mathbb{R}^4$ to Y. Also write $S(u) = S(u,u)$.

Proposition 4.7. *Let Y be a smooth star-shaped hypersurface in $\mathbb{R}^4$, let $y \in Y$, let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, and write $\sigma = \theta/2\pi \in \mathbb{R}/\mathbb{Z}$. Then at the point $\overline{\tau}^{-1}(y,\sigma) \in \mathbb{P}\xi$, we have*
$$\widetilde{R}\sigma = \frac{1}{\pi\langle \nu, y\rangle}\left(S(\mathbf{i}\nu) + S(\cos(\theta)\mathbf{j}\nu + \sin(\theta)\mathbf{k}\nu)\right). \tag{4.4}$$

Proof. It follows from the definitions that
$$\begin{aligned} 2\pi\widetilde{R}\sigma =& \langle \mathcal{L}_R((\cos\theta)\mathbf{j}\nu + (\sin\theta)\mathbf{k}\nu), (\sin\theta)\mathbf{j}\nu - (\cos\theta)\mathbf{k}\nu\rangle \\ =& -(\cos^2\theta)\langle \mathcal{L}_R\mathbf{j}\nu, \mathbf{k}\nu\rangle + (\sin^2\theta)\langle \mathcal{L}_R\mathbf{k}\nu, \mathbf{j}\nu\rangle \\ &+ (\sin\theta\cos\theta)(\langle \mathcal{L}_R\mathbf{j}\nu, \mathbf{j}\nu\rangle - \langle \mathcal{L}_R\mathbf{k}\nu, \mathbf{k}\nu\rangle). \end{aligned} \tag{4.5}$$
We compute
$$\begin{aligned} \langle \mathcal{L}_R\mathbf{j}\nu, \mathbf{k}\nu\rangle &= \langle \nabla_R\mathbf{j}\nu - \nabla_{\mathbf{j}\nu}R, \mathbf{k}\nu\rangle \\ &= \frac{2}{\langle \nu, y\rangle}\left(\langle \nabla_{\mathbf{i}\nu}\mathbf{j}\nu, \mathbf{k}\nu\rangle - \langle \nabla_{\mathbf{j}\nu}\mathbf{i}\nu, \mathbf{k}\nu\rangle\right) \\ &= \frac{2}{\langle \nu, y\rangle}\left(-\langle \nabla_{\mathbf{i}\nu}\nu, \mathbf{i}\nu\rangle - \langle \nabla_{\mathbf{j}\nu}\nu, \mathbf{j}\nu\rangle\right) \\ &= \frac{2}{\langle \nu, y\rangle}\left(-S(\mathbf{i}\nu) - S(\mathbf{j}\nu)\right). \end{aligned} \tag{4.6}$$

Here in the second to third lines we have used the fact that multiplication on the left by a constant unit quaternion is an isometry. Similar calculations show that

$$\langle \mathcal{L}_R \mathbf{k}\nu, \mathbf{j}\nu \rangle = \frac{2}{\langle \nu, y \rangle} \left(S(\mathbf{i}\nu) + S(\mathbf{k}\nu) \right), \tag{4.7}$$

$$\langle \mathcal{L}_R \mathbf{j}\nu, \mathbf{j}\nu \rangle = -\langle \mathcal{L}_R \mathbf{k}\nu, \mathbf{k}\nu \rangle = \frac{2}{\langle \nu, y \rangle} S(\mathbf{j}\nu, \mathbf{k}\nu). \tag{4.8}$$

Plugging (4.6), (4.7) and (4.8) into (4.5) proves the curvature identity (4.5). □

Remark 4.8. Since the second fundamental form is positive definite when Y is strictly convex, and positive semidefinite when Y is convex, by Lemma 4.6 we obtain the following corollary: *If Y is a convex star-shaped hypersurface in $\mathbb{R}^4$ then $\widetilde{R}\sigma \geqslant 0$ everywhere, so $\widetilde{\phi}(y,t)$ has nonnegative rotation number for all $y \in Y$ and $t \geqslant 0$. If Y is a strictly convex star-shaped hypersurface in $\mathbb{R}^4$ then $\widetilde{R}\sigma > 0$ everywhere, so $\widetilde{\phi}(y,t)$ has positive rotation number for all $y \in Y$ and $t > 0$.*

5. Reeb dynamics on smoothings of polytopes

In §5.1 and §5.2 we study the Reeb flow on the boundary of a smoothing of a symplectic polytope in $\mathbb{R}^4$. In §5.3 and §5.4 we explain some more technical issues arising from the fact that the smoothing is only C^1, and in particular how to make sense of the "rotation number" of Reeb trajectories. In §5.5 we derive important lower bounds on this rotation number.

5.1. Smoothings of polytopes. If $X \subset \mathbb{R}^m$ is a compact convex set and $\varepsilon > 0$, define the ε-smoothing X_ε of X by equation (1.7). Observe that X_ε is convex. Denote its boundary by $Y_\varepsilon = \partial X_\varepsilon$. We now describe Y_ε more explicitly, in a way which mostly does not depend on ε. We first have:

Lemma 5.1. *If X is a compact convex set then*

$$Y_\varepsilon = \{ y \in \mathbb{R}^m \mid \operatorname{dist}(y, X) = \varepsilon \}.$$

Proof. The left hand side is contained in the right hand side because distance to X is a continuous function on $\mathbb{R}^m$. The reverse inclusion holds because given $y \in \mathbb{R}^m$ with $\operatorname{dist}(y,X) = \varepsilon$, since X is compact and convex, there is a unique point $x \in X$ which is closest to y. By convexity again, X is contained in the closed half-space $\{z \in \mathbb{R}^m \mid \langle z, y - x \rangle \leqslant 0\}$. It follows that $\operatorname{dist}(t(y-x), X) = \varepsilon t$ for $t > 0$, so that $y \in \partial X_\varepsilon$. □

Definition 5.2. If $X \subset \mathbb{R}^m$ is a compact convex set, define the "blown-up boundary"

$$Y_0 = \left\{ (y, v) \mid y \in \partial X,\ v \in N_y^+ X,\ |v| = 1 \right\} \subset \partial X \times S^{m-1}.$$

We then have the following lemma, which is proved by similar arguments to Lemma 5.1:

Lemma 5.3. *Let $X \subset \mathbb{R}^m$ be a compact convex set and let $\varepsilon > 0$. Then:*

(a) There is a homeomorphism

$$Y_0 \xrightarrow{\simeq} Y_\varepsilon$$

sending $(y, v) \mapsto y + \varepsilon v$.

(b) *The inverse homeomorphism sends $y \mapsto (x, \varepsilon^{-1}(y - x))$ where x is the unique closest point in X to y.*

(c) *For $y \in Y_\varepsilon$, if x is the closest point in X to y, then the positive normal cone $N_y^+ X_\varepsilon$ is the ray consisting of nonnegative multiples of $y - x$.*

Suppose now that $X \subset \mathbb{R}^m$ is a convex polytope and $\varepsilon > 0$.

Definition 5.4. If F is a face of X, define the *ε-smoothed face*

$$F_\varepsilon = \{x \in Y_\varepsilon \mid \operatorname{dist}(x, F) = \varepsilon\}.$$

By Lemma 5.3, we have

$$Y_\varepsilon = \bigsqcup_F F_\varepsilon$$

and

$$F_\varepsilon = F + \{v \in N_F^+ X \mid |v| = \varepsilon\}.$$

In particular, it follows that Y_ε is a C^1 smooth hypersurface, and it is C^∞ except along strata[8] of the form $\partial F + \{v \in N_F^+ X \mid |v| = \varepsilon\}$.

5.2. The Reeb flow on a smoothed symplectic polytope. Suppose now that X is a symplectic polytope in $\mathbb{R}^4$ and $\varepsilon > 0$. As noted above, $Y_\varepsilon = \partial X_\varepsilon$ is a C^1 convex hypersurface, and as such it has a well-defined C^0 Reeb vector field, which is smooth except along the strata of Y_ε arising from the boundaries of the faces of X. We now investigate the Reeb flow on Y_ε in more detail, as well as the lifted linearized Reeb flow $\widetilde{\phi}$ from Definition 4.4.

General remarks. By Lemma 5.3, a point in Y_ε lives in an ε-smoothed face F_ε for a unique face F of X, and thus has the form $y + \varepsilon v$ where $y \in F$ and $v \in N_F^+ X$ is a unit vector. By equation (1.3) and Lemma 5.3(c), the Reeb vector field at $y + \varepsilon v$ is given by

$$R_{y+\varepsilon v} = \frac{2\mathbf{i}v}{\langle v, y\rangle + \varepsilon}. \tag{5.1}$$

Lemma 5.5. *The Reeb vector field* (5.1) *on the ε-smoothed face F_ε, regarded as a map $F_\varepsilon \to \mathbb{R}^4$, depends only $v \in N_F^+ X$ and not on the choice of $y \in F$.*

Proof. This follows from equation (5.1), because for fixed $v \in N_F^+ X$ and for two points $y, y' \in F$, by the definition of positive normal cone we have $\langle v, y - y'\rangle = 0$. □

Smoothed 3-faces. The Reeb flow on a smoothed 3-face is very simple.

Lemma 5.6. *Let $X \subset \mathbb{R}^4$ be a symplectic polytope, let $\varepsilon > 0$, and let E be a 3-face of X with outward unit normal vector ν.*

(a) *The Reeb vector field on E_ε, regarded as a map $E_\varepsilon \to \mathbb{R}^4$, agrees with the Reeb vector field on E, up to rescaling by a positive constant which limits to 1 as $\varepsilon \to 0$.*

(b) *If $\gamma : [0, t] \to E_\varepsilon$ is a Reeb trajectory, then $\widetilde{\phi}(\gamma(0), t) = 1 \in \widetilde{\mathrm{Sp}}(2)$.*

(c) *If $y \in \partial E$, then at the point $y + \varepsilon\nu \in Y_\varepsilon$, the Reeb vector field on Y_ε is not tangent to ∂E_ε.*

[8]We do not also need to mention strata of the form $F + \partial\{v \in N_F^+ X \mid |v| = \varepsilon\}$, because any point in $\partial N_F^+ X$ is contained in $N_E^+ X$ where E is a face with $F \subset \partial E$.

Proof. (a) This follows from equation (5.1).

(b) For $s \in [0,t]$, the Reeb flow $\Phi_s : Y_\varepsilon \to Y_\varepsilon$ is a translation on a neighborhood of $\gamma(0)$. Consequently the linearized Reeb flow $d\Phi_s : \xi_{\gamma(0)} \to \xi_{\gamma(s)}$ is the identity, if we regard $\xi_{\gamma(0)}$ and $\xi_{\gamma(s)}$ as (identical) two-dimensional subspaces of $\mathbb{R}^4$. The quaternionic trivialization $\tau : \mathbb{R}^2 \to \xi_{\gamma(s)}$ likewise does not depend on $s \in [0,t]$. Consequently $\phi(y,s) = 1$ for all $s \in [0,t]$. Thus $\widetilde{\phi}(y,t)$ is the constant path at the identity in $\mathrm{Sp}(2)$.

(c) It is equivalent to show that the Reeb vector field on E at y is not tangent to ∂E. If the Reeb vector field on E at y is tangent to ∂E, then it is tangent to some 2-face $F \subset \partial E$. By Lemma 3.10, the face 2-face F is Lagrangian, contradicting our hypothesis that the polytope X is symplectic. □

Smoothed 2-faces. Let F be a 2-face. Let E_1 and E_2 be the 3-faces adjacent to F. By Lemma 1.6, we can choose these so that R_{E_2} points out of F; and a similar argument shows that then R_{E_1} points into F. Let ν_1 and ν_2 denote the outward unit normal vectors to E_1 and E_2 respectively. By Lemma 3.2, the normal cone N_F^+ consists of nonnegative linear combinations of ν_1 and ν_2. Let $\{v, w\}$ be an orthonormal basis for $F^\perp$, such that the orientation given by (v,w) agrees with the orientation given by (ν_1,ν_2). For $i = 1,2$ we can write $\nu_i = (\cos\theta_i)v + (\sin\theta_i)w$ where $0 < \theta_2 - \theta_1 < \pi$. We then have a homeomorphism

$$
\begin{aligned}
F \times [\theta_1,\theta_2] &\xrightarrow{\simeq} F_\varepsilon, \\
(y,\theta) &\longmapsto y + \varepsilon((\cos\theta)v + (\sin\theta)w).
\end{aligned}
\tag{5.2}
$$

In the coordinates (y,θ), the Reeb vector field R on F_ε depends only on θ by Lemma 5.5, and has positive ∂_θ coordinate for both $\theta = \theta_1$ and $\theta = \theta_2$ by our choice of labeling of E_1 and E_2. By equation (5.1), Lemma 3.10, and our hypothesis that the polytope X is symplectic, the ∂_θ component of the Reeb vector field is positive on all of F_ε.

Let $U_{F,\varepsilon} \subset F$ denote the set of $y \in F$ such that the Reeb flow on Y_ε starting at $(y,\theta_1) \in F_\varepsilon$ stays in F_ε until reaching a point in $F \times \{\theta_2\}$, which we denote by $(\phi_{F,\varepsilon}(y),\theta_2)$. Thus we have a well-defined "flow map" $\phi_{F,\varepsilon} : U_{F,\varepsilon} \to F$.

Lemma 5.7. *Let F be a two-face of a symplectic polytope $X \subset \mathbb{R}^4$. Then:*

- *(a) The flow map $\phi_{F,\varepsilon} : U_{F,\varepsilon} \to F$ above is translation by a vector $V_{F,\varepsilon} \in TF$.*
- *(b) $|V_{F,\varepsilon}| = O(\varepsilon)$ and $\lim_{\varepsilon\to 0} U_{F,\varepsilon} = F$.*
- *(c) Let $y \in U_{F,\varepsilon}$ and let t be the Reeb flow time on F_ε from $y + \varepsilon\nu_1$ to $\phi_{F,\varepsilon}(y) + \varepsilon\nu_2$. Then $\phi(y,t) \in \mathrm{Sp}(2)$ agrees with the transition matrix ψ_F in Definition 2.20, and $\widetilde{\phi}(y,t) \in \widetilde{\mathrm{Sp}}(2)$ is the unique lift of ψ_F with rotation number in the interval $(0,1/2)$.*

Proof. (a) If $y, y' \in U_{F,\varepsilon}$, then it follows from the translation invariance in Lemma 5.5 that $\phi_{F,\varepsilon}(y) - y = \phi_{F,\varepsilon}(y') - y'$, so $\phi_{F,\varepsilon}$ is a translation.

(b) It follows from equation (5.1) that for each v, the Reeb vector field $R_{y+\varepsilon v}$, regarded as a vector in $\mathbb{R}^4$, has a well-defined limit as $\varepsilon \to 0$, which by Lemma 3.10 is not tangent to F. Since ∂_θ, regarded as a vector in $\mathbb{R}^4$, has length ε, it follows that the flow time of the Reeb vector field on F_ε from $F \times \{\theta_1\}$ to $F \times \{\theta_2\}$ is $O(\varepsilon)$. Consequently the translation vector $V_{F,\varepsilon}$ has length $O(\varepsilon)$, and the complement $F \setminus U_{F,\varepsilon}$ of the domain of the flow map is contained within distance $O(\varepsilon)$ of ∂F.

(c) Write $y_1 = y + \varepsilon v_1$ and $y_2 = \phi_{F,\varepsilon}(y) + \varepsilon v_2$. By part (a) and the translation invariance in Lemma 5.5, the time t Reeb flow Φ_t on Y_ε restricted to $U_{F,\varepsilon} + \varepsilon v_1$ is a translation in $\mathbb{R}^4$. Hence the derivative of Φ_t on the full tangent space of Y_ε, namely

$$d\Phi_t : T_{y_1}Y_\varepsilon \longrightarrow T_{y_2}Y_\varepsilon,$$

restricts to the identity on TF. We now have a commutative diagram

$$\begin{tikzcd}
\xi_{y_1} \arrow[r] \arrow[d, "d\Phi_t"'] & TF \arrow[r, "\tau'_F"] \arrow[d, "1"'] & \mathbb{R}^2 \arrow[d, "\psi_F"] \\
\xi_{y_2} \arrow[r] & TF \arrow[r, "\tau_F"] & \mathbb{R}^2.
\end{tikzcd}$$

Here the upper left horizontal arrow is projection along the Reeb vector field in $T_{y_1}Y_\varepsilon$, and the lower left horizontal arrow is projection along the Reeb vector field in $T_{y_2}Y_\varepsilon$. The right horizontal arrows were defined in Definition 2.17 and Remark 2.19. The left square commutes because $d\Phi_t$ preserves the Reeb vector field. The right square commutes by Definition 2.20. The composition of the arrows in the top row is the quaternionic trivialization τ on ξ_{y_1}, and the composition of the arrows in the bottom row is the quaternionic trivialization τ on ξ_{y_2}. Going around the outside of the diagram then shows that $\phi(y,t) = \psi_F$.

To determine the lift $\widetilde{\phi}(y,t)$, note that this is actually defined for, and depends continuously on, any $\varepsilon > 0$ and any pair of hyperplanes E_1 and E_2 that do not contain the origin and that intersect in a non-Lagrangian 2-plane F. Thus we can denote this lift by $\widetilde{\phi}(E_1,E_2,\varepsilon) \in \widetilde{\mathrm{Sp}}(2)$. Now fixing E_1, F, and ε, we can interpolate from E_1 and E_2 via a 1-parameter family of hyperplanes $\{E_s\}_{s\in[1,2]}$ such that $0 \notin E_s$ and $E_1 \cap E_s = F$ for $1 < s \leqslant 2$. The rotation number $\rho : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ then gives us a continuous map

$$\begin{aligned} f : (1,2] &\longrightarrow \mathbb{R}, \\ s &\longmapsto \rho\left(\widetilde{\phi}(E_1,E_s,\varepsilon)\right) \end{aligned}$$

We have $\lim_{\tau\searrow 1} \widetilde{\phi}(E_1,E_s,\varepsilon) = 1$, so $\lim_{s\searrow 1} f(s) = 0$. On the other hand, for each $s \in (1,2]$, the fractional part of $f(s)$ is in the interval $(0,1/2)$ by Lemma 2.21. It follows by continuity that $f(s) \in (0,1/2)$ for all $s \in (1,2]$. Thus $f(2) \in (0,1/2)$, which is what we wanted to prove. □

Smoothed 1-faces. The Reeb flow on a smoothed 1-face is more complicated, but we will not need to analyze this in detail. We just remark that one can see the difference between good and bad 1-faces in the Reeb dynamics on their smoothings. Namely:

Remark 5.8. If L is a bad 1-face, then by definition, there is a unique unit vector $v \in N_L^+X$ such that $\mathbf{i}v$ is tangent to L. The line segment $L + \varepsilon v \subset L_\varepsilon$ is then a Reeb trajectory. On the complement of this line in L_ε, the Reeb vector field spirals around the line, with the number of times that it spirals around going to infinity as $\varepsilon \to 0$. This gives some intuition why Type 3 combinatorial Reeb orbits do not correspond to limits of sequences of Reeb orbits on smoothings with bounded rotation number.

By contrast, if L is a good 1-face, then the Reeb vector field on L_ε always has a nonzero component in the N_L^+X direction.

Smoothed 0-faces. If P is a 0-face, then by Lemma 5.3, P_ε is identified with a domain in S^3. By equation (5.1), the Reeb vector field on this domain agrees, up to reparametrization, with the standard Reeb vector field on the unit sphere in $\mathbb{R}^4$.

5.3. Non-smooth strata. We now investigate in more detail how Reeb trajectories on Y_ε intersect the strata where Y_ε is not C^∞.

Let Σ denote the subset of Y_ε where Y_ε is not locally C^∞. By the discussion at the end of §5.1, we can write

$$\Sigma = \Sigma_1 \sqcup \Sigma_2 \sqcup \Sigma_3$$

where:

- Σ_1 is the disjoint union of sets

$$P + \{v \in N_L^+ X \mid |v| = \varepsilon\} \tag{5.3}$$

  where P is a vertex of X, and L is a 1-face adjacent to P.
- Σ_2 is the disjoint union of sets

$$L + \{v \in N_F^+ X \mid |v| = \varepsilon\} \tag{5.4}$$

  where L is a 1-face, and F is a 2-face adjacent to L.
- Σ_3 is the disjoint union of sets

$$F + \varepsilon \nu$$

  where F is a 2-face, and ν is the outward unit normal vector to one of the two 3-faces E adjacent to F.

Lemma 5.9. *Let $X \subset \mathbb{R}^4$ be a symplectic polytope, let $\varepsilon > 0$, and let $\gamma : [a, b] \to Y_\varepsilon$ be a Reeb trajectory. Then there exist a nonnegative integer k and real numbers $a \leqslant t_1 < t_2 < \cdots < t_k \leqslant b$ with the following properties:*

(a) $\gamma(t_i) \in \Sigma$ for each i.

(b) For each $i = 0, \ldots, k$, one of the following possibilities holds:

(i) γ maps (t_i, t_{i+1}) to $Y_\varepsilon \setminus \Sigma$. (Here we interpret $t_0 = a$ and $t_{k+1} = b$.)

(ii) γ maps (t_i, t_{i+1}) to a Reeb trajectory in a component of Σ_1. (Each component of Σ_1 contains at most one Reeb trajectory of positive length.)

(iii) γ maps (t_i, t_{i+1}) to a Reeb trajectory in a component of Σ_2. (This can only happen when the corresponding 2-face F is complex linear, and in this case the component of Σ_2 is foliated by Reeb trajectories.)

Proof. We need to show that a Reeb trajectory intersects Σ in isolated points, or in Reeb trajectories of the types described in (ii) and (iii).

We have seen in §5.2 that the Reeb vector field is transverse to all of Σ_3. Thus the Reeb trajectory γ intersects Σ_3 only in isolated points.

Next let us consider the Reeb vector field on a component of Σ_2 of the form (5.4). As in §5.2, let E_1 and E_2 denote the 3-faces adjacent to F, with outward unit normal vectors ν_1 and ν_2 respectively. The smoothing F_ε is parametrized by (5.2). This parametrization extends by the same formula to a parametrization of $\overline{F_\varepsilon}$ by $\overline{F} \times [\theta_1, \theta_2]$. The latter parametrization includes the component (5.4) of Σ_2 as the restriction to $L \times [\theta_1, \theta_2]$. By equation (5.1), at the point corresponding to (y, θ) in (5.2), the Reeb vector is given by

$$R = \frac{2}{\langle (\cos\theta)v + (\sin\theta)w, y\rangle + \varepsilon}\mathbf{i}((\cos\theta)v + (\sin\theta)w). \tag{5.5}$$

This vector is tangent to the component (5.4) if and only if the orthogonal projection of $\mathbf{i}((\cos\theta)v + (\sin\theta)w)$ to F is parallel to L.

If the projections of $\mathbf{i}v$ and $\mathbf{i}w$ to F are not parallel, then this tangency will only happen for isolated values of θ, and since the Reeb vector field on $\overline{F_\varepsilon}$ always has a positive ∂_θ component, a Reeb trajectory will only intersect the component (5.4) in isolated points.

If on the other hand the projections of $\mathbf{i}v$ and $\mathbf{i}w$ to F are parallel, then there is a nontrivial linear combination of $\mathbf{i}v$ and $\mathbf{i}w$ whose projection to F is zero. This means that there is a nonzero vector ν perpendicular to F such that $\mathbf{i}\nu$ is also perpendicular to F. This means that $F^\perp$ is complex linear, and thus F is also complex linear. Then $\mathbf{i}v$ and $\mathbf{i}w$ are both perpendicular to F, so in the parametrization (5.2), the Reeb vector field vector field (5.5) is a just a positive multiple of ∂_θ.

The conclusion is that a Reeb trajectory will intersect each component (5.4) of Σ_2 either in isolated points, or (when F is complex linear) in Reeb trajectories which, in the parametrization (5.2), start on $L \times \{\theta_1\}$ and end on $L \times \{\theta_2\}$, keeping the L component constant.

Finally we consider the Reeb vector field on a component (5.3) of Σ_1. The set of vectors v that arise in (5.3) is a domain D in the intersection of the sphere $|v| = \varepsilon$ with the hyperplane $L^\perp$. As we have seen at the end of §5.2, the Reeb vector field on Y_ε at a point in (5.3) agrees, up to scaling, with the standard Reeb vector field on the sphere $|v| = \varepsilon$, whose Reeb orbits are Hopf circles. There is a unique Hopf circle C contained entirely in $L^\perp$. All other Hopf circles intersect $L^\perp$ transversely. Thus any Reeb trajectory in Y_ε intersects the component (5.3) in isolated points and/or the arc corresponding to $C \cap D$, if the latter intersection is nonempty. □

5.4. Rotation number of Reeb trajectories. Suppose $\gamma : [a, b] \to Y_\varepsilon$ is a Reeb trajectory. Let $D \subset Y_\varepsilon$ be a disk through $\gamma(a)$ tranverse to γ, and let $D' \subset Y_\varepsilon$ be a disk through $\gamma(b)$ transverse to γ. We can identify D with a neighborhood of 0 in $\xi_{\gamma(a)}$, and D' with a neighborhood of 0 in $\xi_{\gamma(b)}$, via orthogonal projection in $\mathbb{R}^4$. If D is small enough, then there is a well-defined map continuous map $\phi : D \to D'$ with $\phi(\gamma(a)) = \gamma(b)$, such that for each $x \in D$, there is a unique Reeb trajectory near γ starting at x and ending at $\phi(x)$.

Lemma 5.10. *Let X be a symplectic polytope in $\mathbb{R}^4$, let $\varepsilon > 0$, and let $\gamma : [a, b] \to Y_\varepsilon$ be a Reeb trajectory. Then there is a unique (independent of the choice of D and D') homeomorphism*

$$P_\gamma : \xi_{\gamma(a)} \longrightarrow \xi_{\gamma(b)}$$

such that:

(a)

$$\lim_{x\to 0} \frac{\phi(x) - P_\gamma(x)}{\|x\|} = 0. \tag{5.6}$$

(b) P_γ is linear along rays, i.e. if $x \in \xi_{\gamma(a)}$ and $c > 0$ then $P_\gamma(cx) = cP_\gamma(x)$.

This map P_γ has the following additional properties:

(c) If γ does not include any arcs as in Lemma 5.9(ii)-(iii), and in particular if γ does not intersect any smoothed 0-face or smoothed 1-face, then P_γ is linear.

(d) For $t \in (a, b)$ we have the composition property

$$P_\gamma = P_{\gamma|_{[t,b]}} \circ P_{\gamma|_{[a,t]}}.$$

(e) For $t \in [a, b]$, the homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$ given by the composition

$$\mathbb{R}^2 \xrightarrow{\tau^{-1}} \xi_{\gamma(a)} \xrightarrow{P_{\gamma|_{[a,b]}}} \xi_{\gamma(t)} \xrightarrow{\tau} \mathbb{R}^2$$

is a continuous, piecewise smooth function of t.

Proof. Uniqueness of the homeomorphism P_γ follows from properties (a) and (b). Independence of the choice of D and D' follows from properties (a) and (b) together with continuity of the Reeb vector field. Assuming existence of the homeomorphism P_γ, the composition property (d) follows from uniqueness.

We now need to prove existence of the homeomorphism satisfying properties (a), (b), (c), and (e). Let $a \leqslant t_1 < t_2 < \cdots < t_k \leqslant b$ be the subdivision of the inteveral $[a, b]$ given by Lemma 5.9. For $i = 0, \ldots, k$, let γ_i denote the restriction of γ to $[t_i, t_{i+1}]$, where we interpret $t_0 = a$ and $t_k = b$. It is enough to prove existence of a homeomorphism

$$P_{\gamma_i} : \xi_{\gamma(t_i)} \longrightarrow \xi_{\gamma(t_{i+1})}$$

with the required properties for each i. The desired homeomorphism P_γ is then given by the composition $P_k \cdots P_0$.

For case (i) in Lemma 5.9, a homeomorphism P_{γ_i} with properties (a), (b), and (e) is given by the usual linearized return map on the smooth hypersurface $Y_\varepsilon \backslash \Sigma$ from $t_i + \delta$ to $t_{i+1} - \delta$, in the limit as $\delta \to 0$. Since P_{γ_i} is linear, we also obtain property (c).

For case (ii) or (iii) in Lemma 5.9, the existence of P_{γ_i} with the desired properties follows from the fact that γ_i is on a smooth hypersurface separating two regions of Y_ε, on each of which the Reeb vector field is C^∞. □

Remark 5.11. In case (ii) or (iii) above, the description of the Reeb flow in §5.2 allows us to write down the map P_{γ_i} quite explicitly. Namely, for a suitable trivialization, P_{γ_i} is given by the flow for some positive time of a continuous, piecewise smooth vector field V on $\mathbb{R}^2$, which is the derivative of a shear on one half of $\mathbb{R}^2$, and which is the derivative of a rotation or the identity on the other half of $\mathbb{R}^2$. For case (ii), the vector field has the form

$$V(x, y) = \begin{cases} -y\partial_x, & x \geqslant 0, \\ x\partial_y - y\partial_x, & x \leqslant 0. \end{cases} \tag{5.7}$$

For case (iii), the vector field has the form

$$V(x, y) = \begin{cases} x\partial_y, & x \geqslant 0, \\ 0, & x \leqslant 0. \end{cases} \tag{5.8}$$

Since the map $P_\gamma : \xi_{\gamma(a)} \to \xi_{\gamma(b)}$ sends rays to rays, it induces a well-defined map $\mathbb{P}\xi_{\gamma(a)} \to \mathbb{P}\xi_{\gamma(b)}$. It follows from Lemma 5.10(c),(d) and equations (5.7) and (5.8) that the latter map is C^1. Similarly to (4.2), we obtain a C^1 diffeomorphism of S^1 given by the composition

$$S^1 \xrightarrow{\tau^{-1}} \mathbb{P}\xi_{\gamma(a)} \xrightarrow{P_\gamma} \mathbb{P}\xi_{\gamma(b)} \xrightarrow{\tau} S^1.$$

Stealing the notation from Definition 4.4, let us denote this map by $\phi(y,t)$ where $y = \gamma(a)$ and $t = b - a$. By analogy with (4.3), we define

$$\widetilde{\phi}(y,t) = \{\phi(y,s)\}_{s\in[0,t]} \in \widetilde{\mathrm{Diff}}(S^1).$$

This then has a well-defined rotation number, see Appendix 7, which we denote by

$$\rho(\gamma) = \rho(\widetilde{\phi}(y,t)) \in \mathbb{R}.$$

5.5. Lower bounds on the rotation number. We now prove the following lower bound on the rotation number.

Lemma 5.12. *Let X be a symplectic polytope in $\mathbb{R}^4$. Then there exists a constant $C > 0$, depending only on X, such that if $\varepsilon > 0$ is small, then the following holds. Let $\gamma : [a,b] \to Y_\varepsilon$ be a Reeb trajectory, and assume that if $t \in (a,b)$ and E is a 3-face then $\gamma(t) \notin E_\varepsilon$. Then*

$$\rho(\gamma) \geqslant C\varepsilon^{-1}(b-a).$$

Proof. Define a function

$$r_\varepsilon^{\min} : Y_\varepsilon \longrightarrow \mathbb{R}$$

as follows. A point Y_ε can by uniquely written as $y + \varepsilon v$ where $y \in Y$ and v is a unit vector in N_y^+X. Then define

$$r_\varepsilon^{\min}(y+\varepsilon v) = \min_{\theta\in\mathbb{R}/2\pi\mathbb{Z}} \frac{1}{\pi(\langle v,y\rangle + \varepsilon)}(S(\mathbf{i}v) + S(\cos(\theta)\mathbf{j}v + \sin(\theta)\mathbf{k}v)). \tag{5.9}$$

Here $S : TY_\varepsilon \to \mathbb{R}$ is the single-argument version of the second fundamental form, which is well-defined, even though along the non-smooth strata of Y_ε there is no corresponding bilinear form.

More explicitly, $T_{y+\varepsilon v}Y_\varepsilon$, regarded as a subspace of $\mathbb{R}^4$, does not depend on ε. A tangent vector $V \in T_{y+\varepsilon v}Y_\varepsilon$ can be uniquely decomposed as

$$V = V_T + V_N \tag{5.10}$$

where $V_T \in T_y\partial X$ is tangent to a face F such that $y \in \overline{F}$ and $v \in N_F^+X$, and $V_N \in T_vN_y^+X$ is perpendicular to v. We then have

$$S(V) = \varepsilon^{-1}|V_N|^2. \tag{5.11}$$

Lemma 4.6 and Proposition 4.7 carry over to the present situation to show that

$$\rho(\gamma) \geqslant \int_a^b r_\varepsilon^{\min}(\gamma(s))ds. \tag{5.12}$$

In (5.9), by compactness, there is a uniform upper bound on $\langle v,y\rangle$ for $y \in \partial X$ and $v \in N_y^+X$ a unit vector. Thus by (5.11) and (5.12), to complete the proof of the lemma, it is enough to show that there is a constant $C > 0$ such that

$$|(\mathbf{i}v)_N|^2 + |(\cos(\theta)\mathbf{j}v + \sin(\theta)\mathbf{k}v)_N|^2 \geqslant C \tag{5.13}$$

whenever $y \in \partial X$, $v \in N_y^+X$ is a unit vector, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, and $y + \varepsilon v$ is not in the closure of E_ε where E is a 3-face. To prove this, it is enough to show that for each k-face F with $k < 3$, there is a uniform positive lower bound on the left hand side of (5.13) for all $y \in F$, all unit vectors v in N_F^+X that are not normal to a 3-face adjacent to F, and all θ.

If $k = 2$, then we have a positive lower bound on $|(\mathbf{i}v)_N|^2$ by the discussion of smoothed 2-faces in §5.2.

If $k = 1$, denote the 1-face F by L. If v is on the boundary of N_L^+X, then we have a positive lower bound on $|(\mathbf{i}v)_N|^2$ as in the case $k = 2$ above. Suppose now that v is in the interior of N_L^+X. We have a positive lower bound on $|(\mathbf{i}v)_N|^2$ when $\mathbf{i}v_N$ is away from the Reeb cone of L. This is sufficient when L is a good 1-face. If L is a bad 1-face, then we have to consider the case where $\mathbf{i}v$ is on or near the Reeb cone R_L^+X. If $\mathbf{i}v$ is in the Reeb cone, then all vectors in $V \in T_{y+\varepsilon v}Y_\varepsilon$ that are not in the real span of the Reeb cone R_L^+X have $V_N \neq 0$. Since the vectors $\cos(\theta)\mathbf{j}v + \sin(\theta)\mathbf{k}v$ are all unit length and orthogonal to $\mathbf{i}v$, we get a positive lower bound on $|(\cos(\theta)\mathbf{j}v + \sin(\theta)\mathbf{k}v)_N|^2$ for all θ when $\mathbf{i}v$ is on or near the Reeb cone.

Suppose now that $k = 0$. If v is on the boundary of N_L^+X, then the desired lower bound follows as in the cases $k = 1$ and $k = 2$ above. If v is in the interior of N_F^+X, then we have $|(\mathbf{i}v)_N|^2 = 1$. □

We now deduce a related rotation number bound. Let $\gamma : [a,b] \to Y_\varepsilon$ be a Reeb trajectory. By Lemma 5.3, we can write

$$\gamma(t) = y(t) + \varepsilon v(t)$$

where $y(t) \in \partial X$ and $v(t)$ is a unit vector in $N_{y(t)}^+X$ for each t.

Lemma 5.13. *Let X be a symplectic polytope in $\mathbb{R}^4$. Then there exists a constant $C > 0$, depending only on X, such that if $\varepsilon > 0$ is small and $\gamma : [a,b] \to Y_\varepsilon$ is a Reeb trajectory as above, then*

$$\rho(\gamma) \geqslant C \int_a^b |v'(s)| ds.$$

Proof. By Lemma 5.12, it is enough to show that there is a constant C such that

$$|v'(s)| \leqslant C\varepsilon^{-1}.$$

To prove this last statement, observe that by equation (5.1), in the notation (5.10) we have

$$v'(s) = \frac{2\varepsilon^{-1}}{\langle v(s), y(s)\rangle + \varepsilon}(\mathbf{i}v(s))_N.$$

Thus

$$|v'(s)| \leqslant \frac{2\varepsilon^{-1}}{\langle v(s), y(s)\rangle + \varepsilon}.$$

If $y \in \partial X$ and $v \in N_y^+X$ is a unit vector, then $\langle v, y\rangle > 0$ because X is convex and $0 \in \operatorname{int}(X)$. By compactness, there is then a uniform lower bound on $\langle v, y\rangle$ for all such pairs (y, v). □

6. The smooth-combinatorial correspondence

We now prove Theorems 1.10 and 1.11.

6.1. From combinatorial to smooth Reeb orbits. We first prove Theorem 1.10. In fact we will prove a slightly more precise statement in Lemma 6.1 below.

Let X be a symplectic polytope in $\mathbb{R}^4$ and let $\gamma = (L_1, \ldots, L_k)$ be a Type 1 combinatorial Reeb orbit. This means that there are 3-faces $E_1, \ldots, E_k$ and 2-faces $F_1, \ldots, F_k$ such that F_i is adjacent to E_{i-1} and E_i, and L_i is an oriented line segment in E_i from a point in F_i to a point in F_{i+1} which is parallel to the Reeb vector field on E_i. Here the subscripts $i-1$ and $i+1$ are understood to be mod k. Below we will regard γ as a piecewise smooth parametrized loop $\gamma : \mathbb{R}/T\mathbb{Z} \to X$, where $T = \mathcal{A}_{\mathrm{comb}}(\gamma)$, which traverses the successive line segments L_i as Reeb trajectories.

Lemma 6.1. *Let X be a symplectic polytope in $\mathbb{R}^4$, and let $\gamma = (L_1, \ldots, L_k)$ be a nondegenerate Type 1 combinatorial Reeb orbit. Then there exists $\delta > 0$ such that for all $\varepsilon > 0$ sufficiently small:*

(a) There is a unique Reeb orbit γ_ε on the smoothed boundary Y_ε such that

$$|\gamma_\varepsilon - \gamma|_{C^0} < \delta.$$

(b) γ_ε converges in C^0 to γ as $\varepsilon \to 0$.

(c) γ_ε does not intersect F_ε where F is a 0-face or 1-face.

(d) γ_ε is linearizable, i.e. has a well-defined linearized return map.

(e) $\mathcal{A}(\gamma_\varepsilon) - \mathcal{A}_{\mathrm{comb}}(\gamma) = O(\varepsilon)$.

(f) γ_ε is nondegenerate, $\rho(\gamma_\varepsilon) = \rho_{\mathrm{comb}}(\gamma)$, and $\mathrm{CZ}(\gamma_\varepsilon) = \mathrm{CZ}_{\mathrm{comb}}(\gamma)$.

Proof. *Setup.* For $i = 1, \ldots, k$, let p_i denote the initial point of the segment L_i. Using the notation E_i, F_i above, let D_i denote the set of points $y \in F_i$ such that Reeb flow along E_i starting at y reaches a point in F_{i+1}, which we denote by $\phi_i(y)$. Thus we have a well-defined affine linear map

$$\phi_i : D_i \longrightarrow F_{i+1}.$$

and by definition $\phi_i(p_i) = p_{i+1}$. In particular, the composition

$$\phi_k \circ \cdots \circ \phi_1 : F_1 \longrightarrow F_1$$

is an affine linear map defined in a neighborhood of p_1 sending p_1 to itself. For $V \in TF_1$ small, this composition sends

$$p_1 + V \longmapsto p_1 + AV,$$

where A is a linear map $TF_1 \to TF_1$. Since the combinatorial Reeb orbit γ is assumed nondegenerate, the linear map A does not have 1 as an eigenvalue.

By Lemma 5.7(a), the Reeb flow along the smoothed 2-face $(F_i)_\varepsilon$ induces a well-defined map

$$\phi_{F_i,\varepsilon} : U_{F_i,\varepsilon} \longrightarrow F_i \tag{6.1}$$

which is translation by a vector $V_{F_i,\varepsilon}$.

Proof of (a). If $\varepsilon > 0$ is sufficiently small, then p_i is in the domain $U_{F_i,\varepsilon}$ for each i, and Reeb orbits on Y_ε that are C^0 close to γ correspond to fixed points of the composition

$$\phi_{F_1,\varepsilon} \circ \phi_k \circ \cdots \circ \phi_2 \circ \phi_{F_2,\varepsilon} \circ \phi_1 : F_1 \longrightarrow F_1. \tag{6.2}$$

It follows from the above that for $V \in TF_1$ small, the composition (6.2) sends

$$p_1 + V \longmapsto p_1 + AV + W_\varepsilon \tag{6.3}$$

where $W_\varepsilon \in TF_1$ has length $O(\varepsilon)$. Since the linear map $A - 1$ is invertible, the affine linear map (6.3) has a unique fixed point $p_1 + V$ for some $V \in TF_1$. If ε is sufficiently small, this fixed point will also be in the domain of the composition (6.2), and thus will correspond to the desired Reeb orbit γ_ε.

Proof of (b). This holds because for the above fixed point, V has length $O(\varepsilon)$.

Proof of (c). The Reeb orbit γ_ε does not intersect F_ε where F is a 0-face or 1-face, by the definition of the domain of the map (6.1).

Proof of (d). This follows from Lemma 5.10(c).

Proof of (e). The symplectic action of the Reeb orbit γ_ε is the sum of its flow times over the smoothed 2-faces $(F_i)_\varepsilon$, plus the sum of its flow times over the smoothed 3-faces $(E_i)_\varepsilon$. The former sum is $O(\varepsilon)$ as explained in the proof of Lemma 5.7(b). The latter sum is $(1 + O(\varepsilon))$ times the sum of the corresponding flow times over the 3-faces E_i, and the latter differs from $\mathcal{A}_{\mathrm{comb}}(\gamma)$ by $O(\varepsilon)$, because the fixed point of (6.3) has distance $O(\varepsilon)$ from p_1.

Proof of (f). Let T_ε denote the period of γ_ε, and let y_ε be a point on the image of γ_ε in E_k. If F is a 2-face, let $\widetilde{\psi}_F \in \widetilde{\mathrm{Sp}}(2)$ denote the lift of the transition matrix ψ_F in Definition 2.20 with rotation number in the interval $(0, 1/2)$. By Lemmas 5.6(b) and 5.7(c), the lifted return map $\widetilde{\phi}(y_\varepsilon, T_\varepsilon)$ is given by

$$\widetilde{\phi}(y_\varepsilon, T_\varepsilon) = \widetilde{\psi}_{F_k} \circ \cdots \circ \widetilde{\psi}_{F_1}. \tag{6.4}$$

Nondegeneracy of the combinatorial Reeb orbit γ means that the projection

$$\phi(y_\varepsilon, T_\varepsilon) = \psi_{F_k} \circ \cdots \circ \psi_{F_1} \in \mathrm{Sp}(2)$$

does not have 1 as an eigenvalue, so γ_ε is nondegenerate. Moreover, it follows from (6.4) and the definition of combinatorial rotation number in Definition 2.23 that $\rho_{\mathrm{comb}}(\gamma) = \rho(\gamma_\varepsilon)$. This implies that $\mathrm{CZ}_{\mathrm{comb}}(\gamma) = \mathrm{CZ}(\gamma_\varepsilon)$. □

6.2. From smooth to combinatorial Reeb orbits.

Proof of Theorem 1.11. We proceed in four steps.

Step 1. We claim that for each i, the Reeb orbit γ_i can be expressed as a concatenation of a finite number, k_i, of arcs such that:

(a) Each endpoint of an arc maps to the boundary of E_{ε_i} where E is a 3-face.

(b) For each arc, either:
   (i) There is a 3-face E such that the interior of the arc maps to E_{ε_i}, or
   (ii) No point in the interior of the arc maps to E_{ε_i} where E is a 3-face.

The above decomposition follows from parts (a) and (b)(i) of Lemma 5.9, because the boundary of E_{ε_i} where E is a 3-face is contained in the singular set Σ. (Note that the decomposition into arcs in Lemma 5.9 is a subdivision of the above decomposition into arcs. Moreover, if $k_i > 1$, then k_i is even and the arcs alternate between types (i) and (ii).)

Step 2. We claim now that there is a constant $C > 0$, not depending on i, such that if $\gamma : [a.b] \to Y_{\varepsilon_i}$ is an arc of type (ii) above, then if we write $\gamma(t) = y(t) + \varepsilon_i v(t)$ for $y(t) \in \partial X$ and $v(t) \in N^+_{y(t)}X$ a unit vector, then we have

$$\int_a^b |v'(s)ds| \geqslant C. \tag{6.5}$$

To see this, note that by (a) above, there are 3-faces E and E' such that $\gamma(a) \in \overline{E_{\varepsilon_i}}$ and $\gamma(b) \in \overline{E'_{\varepsilon_i}}$. Then $v(a) = \nu_E$, where ν_E denotes the outward unit normal vector to E, and likewise $v(b) = \nu_{E'}$. If $E \neq E'$, then the integral in (6.5) is bounded from below by the distance in S^3 between ν_E and $\nu_{E'}$, and this distance has a uniform positive lower bound because X has only finitely many 3-faces, each with distinct outward unit normal vectors.

We now consider the case where $E = E'$. The proof of Lemma 5.13 shows that there is a neighborhood U of ν_E in S^3, and a constant $C > 0$, such that for any point $y + \varepsilon_i v \in Y_{\varepsilon_i} \setminus E_{\varepsilon_i}$ with $v \in U$, with respect to the decomposition (5.10), we have $|(\mathbf{i}v)_N|^2 \geqslant C$. By shrinking the the neighborhood U, we can replace this last inequalty with $\langle (\mathbf{i}v)_N, \nu_E \rangle > 0$. Since $v'(t)$ is a positive multiple of $(\mathbf{i}v(t))_N$, it follows that the path $[a, b] \to S^3$ sending $t \mapsto v(t)$ must initially exit the neighborhood U before returning to ν_E. So in this case, we can take the constant C in (6.5) to be twice the distance in S^3 from ν_E to ∂U.

Step 3. We now show that we can pass to a subsequence so that the sequence of Reeb orbits γ_i on Y_{ε_i} converges in C^0 to a Type 1 or Type 2 combinatorial Reeb orbit γ for X.

By Lemma 5.12 and our hypothesis that $\rho(\gamma_i) < R$, we must have $k_i > 1$ when i is sufficiently large. Then, by Lemma 5.13 and Step 2, there is an i-independent upper bound on k_i. We can then pass to a subsequence such that k_i is equal to an even constant k.

By compactness, we can pass to a further subsequence such that the endpoints of the k arcs from Step 1 for γ_i converge to k points in the 2-skeleton of X. By Lemma 5.6, the $k/2$ arcs of type (i) converge to Reeb trajectories on 3-faces of X. On the other hand, by Lemma 5.12, for each arc of type (ii), the length of its parametrizing interval converges to 0. A compactness argument also shows that there is an upper bound on the length of the Reeb vector field on Y_{ε_i}. It follows that each arc of type (ii) is converging in C^0 to a point. Then γ_i converges in C^0 to a Type 1 or Type 2 combinatorial Reeb orbit consisting of the line segments on 3-faces given by the limits of the $k/2$ arcs of type (i).

Step 4. To complete the proof, we now prove that the subsequence and limiting orbit constructed above satisfy all of the requirements (i)-(v) of the theorem.

We have proved assertions (i) and (iii). Assertion (ii) follows from the proof of Lemma 6.1(e). Assertion (iv) follows from the proof of Lemma 6.1(d),(f). Assertion (v) follows from Lemma 5.13 and Step 2. (To get explicit constants C_F, one only needs to consider the case $E \neq E'$ in Step 2.) □

7. Appendix: Rotation numbers

Let $\widetilde{\mathrm{Sp}}(2)$ denote the universal cover of the group $\mathrm{Sp}(2)$ of 2×2 real symplectic matrices. Let $\mathrm{Diff}(S^1)$ denote the group of orientation-preserving C^1 diffeomorphisms[9] of $S^1 = \mathbb{R}/\mathbb{Z}$, and let $\widetilde{\mathrm{Diff}}(S^1)$ denote its universal cover. In this appendix, we review two invariants of elements of $\widetilde{\mathrm{Sp}}(2)$, and more generally $\widetilde{\mathrm{Diff}}(S^1)$: the rotation number ρ and the "minimum rotation number" r. The former is a standard notion in dynamics and is a key ingredient in Theorem 1.11; and we use the latter to bound the former. We also explain how to use rotation numbers to efficiently compute certain products in $\widetilde{\mathrm{Sp}}(2)$, which is needed for our algorithms.

7.1. Rotation numbers of circle diffeomorphisms. We can identify the universal cover $\widetilde{\mathrm{Diff}}(S^1)$ with the group of C^1 diffeomorphisms $\Phi : \mathbb{R} \to \mathbb{R}$ which are $\mathbb{Z}$-equivariant in the sense that $\Phi(t + 1) = \Phi(t) + 1$ for all $t \in \mathbb{R}$. Such a diffeomorphism of $\mathbb{R}$ descends

[9]For the most part we could work more generally with orientation-preserving homeomorphisms.

to an orientation-preserving diffeomorphism of S^1, and this defines the covering map $\widetilde{\mathrm{Diff}}(S^1) \to \mathrm{Diff}(S^1)$.

Definition 7.1. Given $\sigma \in S^1$, we define the *rotation number with respect to* σ, denoted by

$$r_\sigma : \widetilde{\mathrm{Diff}}(S^1) \longrightarrow \mathbb{R},$$

as follows. Let Φ be a $\mathbb{Z}$-equivariant diffeomorphism of $\mathbb{R}$ as above. Let $t \in \mathbb{R}$ be a lift of $\sigma \in \mathbb{R}/\mathbb{Z}$. We then define

$$r_\sigma(\Phi) = \Phi(t) - t. \tag{7.1}$$

Definition 7.2. Given $\Phi \in \widetilde{\mathrm{Diff}}(S^1)$, we define the *rotation number*

$$\rho(\Phi) = \lim_{n\to\infty} \frac{r_\sigma(\Phi^n)}{n} \in \mathbb{R} \tag{7.2}$$

where $\sigma \in S^1$. This limit does not depend on the choice of σ. Equivalently,

$$\rho(\Phi) = \lim_{n\to\infty} \frac{\Phi^n(t) - t}{n} \tag{7.3}$$

where $t \in \mathbb{R}$.

Note that we have the $\mathbb{Z}$-equivariance property

$$\rho(\Phi + 1) = \rho(\Phi) + 1. \tag{7.4}$$

We can bound the rotation number as follows.

Definition 7.3. We define the *minimum rotation number* $r : \widetilde{\mathrm{Diff}}(S^1) \to \mathbb{R}$ by

$$r\left(\Phi\right) = \min_{\sigma\in S^1} r_\sigma\left(\Phi\right). \tag{7.5}$$

Alternatively, if $\Phi \in \widetilde{\mathrm{Diff}}(S^1)$ is presented as a piecewise smooth path $\{\phi_t\}_{t\in[0,1]}$ in $\mathrm{Diff}(S^1)$ with $\phi_0 = \mathrm{id}_{S^1}$, then

$$r(\Phi) = \min_{\sigma\in S^1} \int_0^1 \frac{d}{ds}\phi_s(\sigma)ds.$$

In particular, it follows that

$$r(\Phi) \geqslant \int_0^1 \min_{\sigma\in S^1}\left(\frac{d}{ds}\phi_s(\sigma)\right) ds. \tag{7.6}$$

It follows from the definitions that

$$\rho(\Phi) \geqslant r(\Phi). \tag{7.7}$$

7.2. A partial order.

Definition 7.4. We define a partial order $\geqslant$ on $\widetilde{\mathrm{Diff}}(S^1)$ as follows:

$$\Phi \geqslant \Psi \text{ if and only if } r_s(\Phi) \geqslant r_s(\Psi) \text{ for all } s \in S^1. \tag{7.8}$$

Equivalently, $\Phi(t) \geqslant \Psi(t)$ for all $t \in \mathbb{R}$.

Lemma 7.5. *The partial order $\geqslant$ on $\widetilde{\mathrm{Diff}}(S^1)$ is left and right invariant.*

Proof. Let $\Phi, \Psi, \Theta \in \widetilde{\mathrm{Diff}}(S^1)$, and suppose that $\Phi \geqslant \Psi$, i.e.

$$\Phi(t) \geqslant \Psi(t) \tag{7.9}$$

for every $t \in \mathbb{R}$. We need to show that $\Phi\Theta \geqslant \Psi\Theta$ and $\Theta\Phi \geqslant \Theta\Psi$.

Since $\Theta : \mathbb{R} \to \mathbb{R}$ is an orientation preserving diffeomorphism, it preserves the order on $\mathbb{R}$, so it follows from (7.9) that

$$\Theta(\Phi(t)) \geqslant \Theta(\Psi(t))$$

for every $t \in \mathbb{R}$, so $\Theta\Phi \geqslant \Theta\Psi$.

On the other hand, replacing t by $\Theta(t)$ in the inequality (7.9), we deduce that

$$\Phi(\Theta(t)) \geqslant \Psi(\Theta(t))$$

for every $t \in \mathbb{R}$, so $\Phi\Theta \geqslant \Psi\Theta$. □

Lemma 7.6. *If $\Phi, \Psi \in \widetilde{Diff}(S^1)$ and $\Phi \geqslant \Psi$, then $\rho(\Phi) \geqslant \rho(\Psi)$.*

Proof. By (7.3), it is enough to show that given $t \in \mathbb{R}$, we have $\Phi^n(t) \geqslant \Psi^n(t)$ for each positive integer n. This follows by induction on n, using the fact that Φ preserves the order on $\mathbb{R}$. □

7.3. Rotation numbers of symplectic matrices. There is a natural homomorphism $\mathrm{Sp}(2) \to \mathrm{Diff}(S^1)$, sending a symplectic linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ to its action on the set of positive rays (identified with $\mathbb{R}/\mathbb{Z}$ by the map sending $t \in \mathbb{R}/\mathbb{Z}$ to the ray through $e^{2\pi i t}$). This lifts to a canonical homomorphism $\widetilde{\mathrm{Sp}}(2) \to \widetilde{\mathrm{Diff}}(S^1)$. Under this homomorphism, the invariants r_s, r, and ρ defined above pull back to functions $\widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$, which by abuse of notation we denote using the same symbols.

We can describe the rotation number $\rho : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ more explicitly in terms of the following classification of elements of the symplectic group $\mathrm{Sp}(2)$.

Definition 7.7. Let $A \in \mathrm{Sp}(2)$. We say that A is

- *positive hyperbolic* if $\mathrm{Tr}(A) > 2$ and *negative hyperbolic* if $\mathrm{Tr}(A) < -2$.
- a *positive shear* if $\mathrm{Tr}(A) = 2$ and a *negative shear* if $\mathrm{Tr}(A) = -2$.
- *positive elliptic* if $-2 < \mathrm{Tr}(A) < 2$ and $\det(()[v, Av]) > 0$ for all $v \in \mathbb{R}^2 \backslash \{0\}$.
- *negative elliptic* if $-2 < \mathrm{Tr}(A) < 2$ and $\det(()[v, Av]) < 0$ for all $v \in \mathbb{R}^2 \backslash \{0\}$.

By the equivariance property (7.4), the rotation number $\rho : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ descends to a "mod $\mathbb{Z}$ rotation number" $\bar{\rho} : \mathrm{Sp}(2) \to \mathbb{R}/\mathbb{Z}$.

Lemma 7.8. *The mod $\mathbb{Z}$ rotation number $\bar{\rho} : Sp(2) \to \mathbb{R}/\mathbb{Z}$ can be computed as follows:*

$$\bar{\rho}(A) = \begin{cases} 0 & \textit{if} \quad A \textit{ is positive hyperbolic or a positive shear,} \\ \frac{1}{2} & \textit{if} \quad A \textit{ is negative hyperbolic or a negative shear,} \\ \theta & \textit{if} \quad A \textit{ is positive elliptic with eigenvalues } e^{\pm 2\pi i \theta} \textit{ for } \theta \in (0, \frac{1}{2}), \\ -\theta & \textit{if} \quad A \textit{ is negative elliptic with eigenvalues } e^{\pm 2\pi i \theta} \textit{ for } \theta \in (0, \frac{1}{2}). \end{cases}$$

Proof. In the first two cases, A has 1 or -1 as an eigenvalue. This means that there exists $s \in S^1$ which is fixed or sent to its antipode, and one can use this s in the definition (7.2).

In the third case, A is conjugate to rotation by $2\pi\theta$. One can then lift A to an element of $\widetilde{\mathrm{Sp}}(2)$ whose image in $\widetilde{\mathrm{Diff}}(S^1)$ is a $\mathbb{Z}$-equivariant diffeomorphism $\Phi : \mathbb{R} \to \mathbb{R}$ such that $|\Phi^n(t) - t - n\theta| < 1$ for each $t \in \mathbb{R}$. It then follows from (7.3) that $\rho(\Phi) = \theta$. The last case is analogous. □

7.4. Computing products in $\widetilde{\mathbf{Sp}}(2)$. Observe that $\widetilde{\mathrm{Sp}}(2)$ can be identified with the set of pairs (A, r), where $A \in \mathrm{Sp}(2)$ and $r \in \mathbb{R}$ is a lift of $\overline{\rho}(A) \in \mathbb{R}/\mathbb{Z}$. The identification sends a lift $\widetilde{A}$ to the pair $(A, \rho(\widetilde{A}))$.

For computational purposes, we can keep track of the lifts of A using less information, which is useful when for example we do not want to compute $\overline{\rho}(A)$ exactly. Namely, we can identify a lift $\widetilde{A}$ with a pair (A, r), where r is either an integer (when A has positive eigenvalues), an open interval $(n, n+1/2)$ for some integer n (when A is positive elliptic), a half-integer (when A has negative eigenvalues), or an open interval $(n-1/2, n)$ (when A is negative elliptic).

The following proposition allows us to compute products in the group $\widetilde{\mathrm{Sp}}(2)$ in terms of the above data, in the cases that we need (see Remark 2.24).

Proposition 7.9. *Let $\widetilde{A}, \widetilde{B} \in \widetilde{Sp}(2)$. Suppose that $\rho(\widetilde{A}) \in (0, 1/2)$. Then*

$$\rho(\widetilde{B}) \leqslant \rho(\widetilde{A}\widetilde{B}) \leqslant \rho(\widetilde{B}) + \frac{1}{2}.$$

To apply this proposition, if for example $\widetilde{B}$ is described by the pair $(B, (m, m+1/2))$, then it follows that $\widetilde{A}\widetilde{B}$ is described by either $(AB, (m, m+1/2))$, $(AB, m+1/2)$, or $(AB, (m+1/2, m+1))$. To decide which of these three possibilities holds, by Lemma 7.8 it is enough to check whether AB is positive elliptic, has negative eigenvalues, or is negative elliptic.

Proof of Proposition 7.9. Let Φ and Ψ denote the elements of $\widetilde{\mathrm{Diff}}(S^1)$ determined by $\widetilde{A}$ and $\widetilde{B}$ respectively. Let $\Theta : \mathbb{R} \to \mathbb{R}$ denote translation by 1/2. By Lemma 7.8, $\widetilde{A}$ projects to a positive elliptic element of Sp(2). It follows that with respect to the partial order on $\widetilde{\mathrm{Diff}}(S^1)$, we have

$$\mathrm{id}_{\mathbb{R}} \leqslant \Phi \leqslant \Theta.$$

By Lemma 7.5, we can multiply on the right by Ψ to obtain

$$\Psi \leqslant \Phi\Psi \leqslant \Theta\Psi.$$

Using Lemma 7.6, we deduce that

$$\rho(\Psi) \leqslant \rho(\Phi\Psi) \leqslant \rho(\Theta\Psi).$$

Since Ψ comes from a linear map, it commutes with Θ, so we have

$$\rho(\Theta\Psi) = \rho(\Psi) + \frac{1}{2}.$$

Combining the above two lines completes the proof. □

CHAPTER 3

ECH Embedding Obstructions For Rational Surfaces

1. Introduction

A symplectic embedding of symplectic manifolds $(X, \omega) \to (X', \omega')$ of the same dimension is a smooth embedding $\varphi : X \to X'$ that intertwines the symplectic form, i.e. $\varphi^*\omega' = \omega$. The study of symplectic embeddings has been a major topic in symplectic geometry ever since Gromov proved his eponymous non-squeezing theorem, stating that

$$B^{2n}(r) \text{ symplectically embeds into } B^2(R) \times \mathbb{C}^{n-1} \qquad \Longleftrightarrow \qquad r \leqslant R$$

Symplectic capacities provide the primary tool for obstructing symplectic embeddings. Roughly speaking, a symplectic capacity c is a numerical invariant associated to a symplectic manifold (usually in a restricted class, e.g. exact) such that $c(X) \leqslant c(X')$ whenever X symplectically embeds into X'. The most famous example is the *Gromov width* of X, defined by

$$c_G(X) := \sup\{\pi r^2 \ : \ B(r) \text{ symplectically embeds into } X\} \tag{1.1}$$

Capacities like c_G have been used to great effect to provide complete solutions to many symplectic embedding problems.

One family of capacities that have been applied with particular success in dimension 4 are the *ECH capacities* c_k^{ECH} (one for each integer $k \geqslant 1$) introduced by Hutchings in [**44**]. These capacities are defined using embedded contact homology (or ECH for short), a version of Floer homology for contact 3-manifolds with a deep connection to Seiberg-Witten theory. They also provide sharp embedding obstructions for several 4-dimensional symplectic embedding problems, such as ellipsoids into ellipsoids [**59**] and (more generally) of concave toric domains into convex toric domains [**20**]. This part of this thesis is about symplectic embedding obstructions derived using ECH.

1.1. ECH capacities via algebraic geometry. Our present story begins with the work of Wormleighton (the second author of this part of this thesis) in [**80**], which we now review in some detail.

Recall that a *toric domain* X_Ω is the inverse image $\mu^{-1}(\Omega)$ of a compact subset $\Omega \subset [0, \infty)^2$ with open interior under the standard moment map on $\mathbb{C}^2$.

$$\mu : \mathbb{C}^2 \to \mathbb{R}^2 \qquad (z_1, z_2) \mapsto (\pi|z_1|^2, \pi|z_2|^2)$$

The region Ω is called the *moment image*. A toric domain X_Ω is *convex* if $\Omega = K \cap [0, \infty)^2$ where $K \subset \mathbb{R}^2$ is a convex set and $0 \in K$. Likewise, X_Ω is *concave* if $\Omega = C \cap [0, \infty)^2$ where $\mathbb{R}^2 \backslash C$ is convex and $0 \in C$. Finally, a *rational* toric domain is a convex toric domain where Ω is the convex hull of finitely many rational points in $[0, \infty)^2$.

The ECH capacities of toric domains have been studied extensively (c.f. [**17**,**20**,**45**,**46**]). For rational toric domains, the ECH capacities can be combinatorially computed using the moment polytope Ω, and these computations bear a remarkable resemblance to calculations arising in the algebraic geometry of $\mathbb{Q}$-line bundles over toric surfaces. This observation was first leveraged (for ellipsoids) in the work of Cristofaro-Gardiner–Kleinman [**22**]. In [**80**], Wormleighton formalized it as a theorem.

To state this theorem we observe that, given a moment polytope Ω, there is in addition to X_Ω, an associated projective algebraic surface Y_Ω described by the inner normal fan of Ω. This surface can be singular, and may alternately be viewed as a toric, symplectic orbifold with moment polytope Ω. It comes equipped with a canonical ample $\mathbb{R}$-divisor A_Ω on Y_Ω.

Theorem 1.1 ([**80**, Thm. 1.5]). *Let X_Ω be a rational toric domain and (Y_Ω, A_Ω) be the corresponding polarized toric surface. Then*

$$c_k^{\mathrm{ECH}}(X_\Omega) = \inf_{D \in \mathrm{nef}(Y_\Omega)_{\mathbb{Q}}} \{D \cdot A_\Omega : h^0(D) \geqslant k+1\} \tag{1.2}$$

Here the infimum is over all nef $\mathbb{Q}$-divisors in Y_Ω. For the more symplectically minded reader, a nef divisor may be thought of as a homology class that is represented by a disconnected J-curve, and which has non-negative intersection with any other J-curve. For example, in $\mathbb{P}^2$ this is every non-negative multiple of the hyperplane class $[\mathbb{P}^1]$, while in $\mathbb{P}^1 \times \mathbb{P}^1$ this is every non-negative combination of $[\mathbb{P}^1 \times \mathrm{pt}]$ and $[\mathrm{pt} \times \mathbb{P}^1]$.

Theorem 1.1 allows one to leverage the computational tools developed for toric geometry to perform calculations, and implies a number of nice results about the asymptotics of the ECH capacities as $k \to \infty$. See [**80**] for more results.

1.2. Geometric explanation. The proof of Theorem 1.1 in [**80**] is largely combinatorial, and amounts to checking that the two quantities agree using previously known explicit formulas. Thus, it is natural to wonder if there is some deeper geometric phenomenon at play. We now sketch a heuristic argument suggesting that this is indeed the case.

To start, given a moment polytope Ω, we observe that the surface with divisor (Y_Ω, A_Ω) and domain X_Ω are related. Indeed, the interior X_Ω° of X_Ω and the complement $Y_\Omega \backslash A_\Omega$ are equivariantly symplectomorphic and one can write down a "collapsing map" $\pi : \partial X_\Omega \to A_\Omega$ whose fibers are generically circles. If Y_Ω is smooth, we can (roughly speaking) write

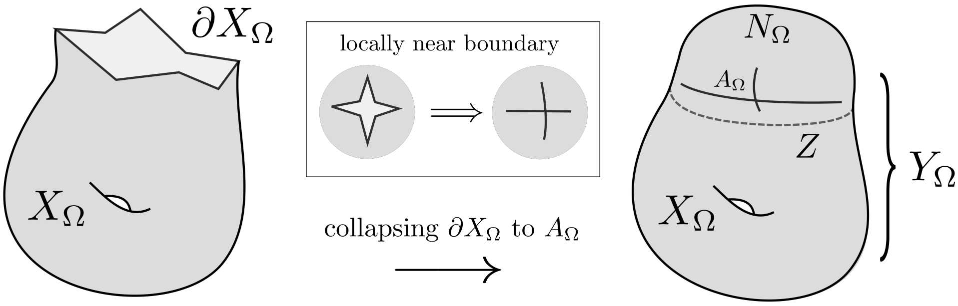
$$Y_\Omega = X_\Omega \cup_Z N_\Omega \tag{1.3}$$

where N_Ω is a very thin neighborhood of A_Ω and Z is the boundary of N_Ω. Thus we have the following picture.

Now we return to a discussion of capacities. Dissecting the construction of c_k^{ECH}, we find that the 1st ECH capacity of X_Ω is (again, roughly speaking) computed as the minimum area of certain disconnected holomorphic curves u in $\widehat{Z} = \mathbb{R} \times Z$ satisfying some conditions. First, each component C of u is embedded, cylindrical at $\pm\infty$ and comes with an integer weight $n_C \in \mathbb{Z}_+$. Second, u must pass through a point $p \in \widehat{Z}$ (fixed for all u). The kth ECH capacity of X_Ω is given by sequences u_i of k such curves with matching ends at $\pm\infty$.

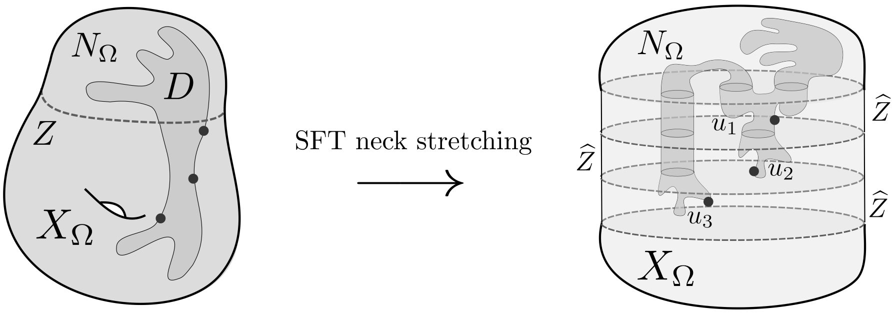
One way that sequences u_i of this form arise naturally is by neck stretching Y_Ω along the hypersurface Z. Namely, a disconnected curve $D \subset Y_\Omega$ with embedded components that is equipped with integer weights on its components and that passes through k

FIGURE 1. The relationship between Y_Ω and X_Ω.



generic points in Y_Ω will (if it survives the stretching process) produce a sequence u_i as above.

FIGURE 2. Neck stretching divisors to acquire ECH curves.



The curve D is essentially an effective, integral Weil divisor. If D passes through k points, then we expect the moduli of divisors $\mathcal{M}_D$ in the class of D to satisfy $\dim(\mathcal{M}_D) \geqslant 2k$. Furthermore, the area of D in Y_Ω is given by $A_\Omega \cdot D$ since A_Ω is Poincare dual to the Kahler form on Y_Ω.

The above discussion leads us to expect an inequality of the following form, which strongly resembles one direction of the equality (1.2).

$$c_k^{\mathrm{ECH}}(X_\Omega) \leqslant \min\{A_\Omega \cdot D \mid \text{effective divisors } D \text{ with } \dim(\mathcal{M}_D) \geqslant 2k + \text{more (?)})\}$$

Note that, in the above discussion, we did not reference the fact that X_Ω and Y_Ω arose via toric geometry or that $X_\Omega = Y_\Omega \backslash A_\Omega$. In fact, the entire argument seems sensible if (Y, A) is an arbitrary projective surface with ample divisor and $X \subset Y$ is an embedded exact symplectic sub-domain.

REMARK 1.2. A more precise perspective on the curve D in Y_Ω is that it arises in the moduli space count used to define the *Gromov-Taubes invariant* of a symplectic 4-manifold [**58**, **73**]. This neck stretching phenomenon is, morally speaking, the reason that ECH is the Floer theory categorifying the Gromov-Taubes invariants.

In practice, this fact is formalized using the isomorphism of ECH with a variant of Seiberg-Witten-Floer homology [**76**], and the equivalent of the Gromov-Taubes invariants with the Seiberg-Witten invariants [**75**]. In order to make the discussion of this section (§1.2) rigorous, we will make use of these equivalences via a result of Hutchings (see Theorem 2.14 in §2.6).

1.3. Main results. We are now ready to state the main theorem of this part of this thesis, which formalizes the discussion of §1.2. First we recall the notion of algebraic capacity from [**80–82**].

Definition 1.3 (Definition 3.2). The *kth algebraic capacity* $c_k^{\text{alg}}(Y, A)$ of a rational projective surface Y with ample $\mathbb{R}$-divisor A is

$$c_k^{\text{alg}}(Y, A) := \inf_{D \in \text{Nef}(Y)_{\mathbb{Z}}} \{D \cdot A : \chi(D) \geqslant k + \chi(\mathcal{O}_Y)\}$$

Here $\text{Nef}(Y)_{\mathbb{Z}}$ denotes the set of nef $\mathbb{Z}$-divisors on Y.

Recall that a *star-shaped domain* $X \subset \mathbb{C}^2$ is a codimension 0 sub-manifold with boundary possessing a point $p \in X$ with the property that any other point $q \in X$ is connected to p by a line segment in X. We do not require X to have smooth boundary.

Theorem 1.4. *(Theorem 3.5) Let $X \to Y$ be a symplectic embedding of a star-shaped domain X into a smooth rational projective surface (Y, ω_A) with a ample $\mathbb{R}$-divisor A with $[\omega_A] = PD[A]$. Then*

$$c_k^{\text{ECH}}(X) \leqslant c_k^{\text{alg}}(Y, A) \tag{$\star$}$$

Remark 1.5. Methods of algebraic geometry have been applied extensively to symplectic embedding problems for rational and toric surfaces, and our result is just one more perspective on this story. We refer the reader to the work of McDuff [**60**], McDuff-Polterovich [**65**], Anjos-Lalonde-Pinsonnault [**6**], Casals-Vianna [**13**] and Christofaro-Gardiner-Holm-Mandini-Pires [**21**] for just a few examples. Likewise, rationality is a key assumption in many embedding results (even those that use purely symplectic methods). See, for example, the work of Buse-Hind [**12**] and Opshtein [**64**]. Note that our references here are not at all exhaustive.

Remark 1.6. The formula ($\star$) provides a new computational tool for studying the ECH capacities of star shaped domains living within divisor complements. Indeed, the nef cones of surfaces are very well studied and many structural results exist which may be brought to bear while studying c^{ECH} via Theorem 3.5. Furthermore, the nef cone is often polyhedral, and thus methods from convex optimization can be utilised to compute c^{alg}. We hope to explore the combinatoral and computational implications of ($\star$) in future work.

Although we were originally motivated to prove Theorem 1.4 in order to study non-toric surfaces, many interesting implications appear even in the toric setting. In particular, [**80**, Thm. 1.5] implies that the inequality in Theorem 1.4 is an equality for certain divisor complements, and this is key to our applications. We will now discuss the three results on symplectic embeddings into smooth toric surfaces that we will prove.

For our first application, we prove that these obstructions are sharp for embeddings of concave toric domains into toric surfaces.

Theorem 1.7. *(Theorem 4.13) Let X_Δ be a concave toric domain with interior $X_\Delta^\circ \subset X_\Delta$, and let (Y_Ω, A_Ω) be a smooth toric surface. Then*

$$X_\Delta^\circ \textit{ symplectically embeds into } Y_\Omega \quad \Longleftrightarrow \quad c_k^{\text{ECH}}(X_\Delta) \leqslant c_k^{\text{alg}}(Y_\Omega, A_\Omega)$$

This result uses a similar result of Christofaro-Gardiner in [**20**], for embeddings of concave domains into convex domains. Theorem 1.7 essentially shows that the extra freedom provided by gluing the divisor A_Ω into X_Ω° makes no difference for embeddings of concave domains.

For our next application, we prove the following result that includes a folk conjecture about the Gromov width. Let Ξ be the moment polygon of a concave toric domain and define the Ξ-*width* by

$$c_\Xi(X) := \sup\{r : X_{r\Xi} \text{ symplectically embeds in } X\}$$

When Ξ is the triangle with vertices $(0,0),(1,0),(0,1)$ the Ξ-width c_Ξ is just the Gromov width c_G.

Theorem 1.8. *(Corollary 4.14 + Corollary 4.15) Let Ξ be the moment polygon of a concave toric domain. Suppose $\Omega \subset \Delta$ is an inclusion of moment polytopes of smooth toric projective surfaces. Then*

$$c_\Xi(Y_\Omega) \leqslant c_\Xi(Y_\Delta)$$

In particular, the Gromov widths satisfy

$$c_G(Y_\Omega) \leqslant c_G(Y_\Delta)$$

In fact, we prove Theorem 1.8 (and Theorem 1.7) for all projective surfaces (even singular ones) that possess a smooth fixed point. Note that any smooth symplectic toric 4-manifold is a smooth projective toric surface (c.f. [**55**]) so Theorem 1.8 may be stated in those terms as well.

Remark 1.9. There have been previous results (c.f. [**21**, Thm. 1.2]) indicating that a ball (or more generally, ellipsoid) embeds into a toric domain if and only if it embeds into the corresponding toric surface. These results are related to Theorem 1.8, and can actually be used to recover some cases. See §4.4 for more discussion.

Finally, we prove an estimate of the Gromov width of a toric surface in terms of the lattice width of its moment polygon. This result is [**9**, Conjecture 3.12].

Definition 1.10. The *lattice width* $w(\Omega)$ of a moment polytope is defined by

$$w(\Omega) := \min_{l \in \mathbb{Z}^n \setminus 0} \Big(\max_{p,q \in \Omega} \langle l, p - q \rangle \Big)$$

Theorem 1.11. *(Corollary 4.19) Let Ω be a moment polygon with a smooth vertex. Then*

$$c_G(Y_\Omega) \leqslant w(\Omega)$$

In particular, this holds when Ω is Delzant or, equivalently, when the toric surface Y_Ω is smooth.

Theorem 1.11 follows from Theorem 1.8 and a rigorous version of a heuristic argument from [**9**].

Remark 1.12. The assumption that the moment polytope has a smooth vertex in Theorems 1.7, 1.8 and 1.11 is an technical assumption that may be removable with different methods.

1.4. Future directions. There are a number of interesting research directions along the lines of [**80**] and this part of this thesis that are worth exploring. We will comment on these now.

First, Theorem 1.1 in [**80**] gives an equality for the ECH capacities, and it is natural to ask when Theorem 3.5 can be upgraded to an equality as well. Here is a guess along those lines.

Conjecture 1.13 (ECH of divisor complements). *Let (Y, ω_A) be a rational projective surface with an ample $\mathbb{R}$-divisor A such that* $\operatorname{sing}(Y) \subseteq \operatorname{supp}(A)$ *and suppose* $Y \setminus \operatorname{supp}(A)$ *is deformation-equivalent to a ball. Then,*

$$c_k^{\mathrm{ECH}}(Y \setminus \operatorname{supp}(A)) = c_k^{\mathrm{alg}}(Y, A)$$

Note that $Y \backslash A$ can still be viewed as the interior of a star shaped domain with corners X. Proving Conjecture 1.13 would require either a clever argument for packing X or a very delicate understanding of the ECH and Reeb dynamics of smoothings of X.

Beyond the ECH capacities, there are finer obstructions defined (by Hutchings in [**46**]) for embeddings of convex toric domains into other convex toric domains. These invariants are still poorly understood. The hope is that they could help solve some of the more obstinate embedding problems, such as the problem of embedding polydisks into ellipsoids.

Question 1.14. Let Δ and Ω be rational moment polytopes. Is there a framework for treating the obstructions of [**46**] to embeddings $X_\Delta \to X_\Omega$ in terms of the algebraic geometry of Y_Δ and Y_Ω?

Finally, our proof of Theorem 1.8 for the Gromov width requires only a family of capacities that provide sharp obstructions for embeddings of the ball into convex toric domains, and an extension of these invariants to closed toric surfaces satisfying a set of axioms (see Proposition 4.9). It is interesting to ask if the proof of Theorem 1.8 can be ported to higher dimensions using another family of holomorphic curve based capacities, such as the S^1-equivariant symplectic homology capacities of Gutt-Hutchings [**32**] or the rational SFT capacities of Siegel [**71**].

Outline. This concludes **§1**, the introduction. The rest of this part of this thesis is organized as follows.

In **§2**, we cover preliminaries in Seiberg-Witten theory (2.1) and embedded contact homology (2.5). We then prove an important estimate of the ECH capacities of a star-shaped domain in terms of a minimum area over Seiberg-Witten non-zero classes. We should note that this is where the "neck stretching" part of the argument is made formal.

In **§3**, we discuss the algebraic capacities in earnest (§3.1). We then prove Theorem 1.4 using the results of §2 and methods from algebraic geometry (§3.2).

In **§4**, we discuss the applications to toric surfaces. We start with a review of toric surfaces (4.1) and toric domains (4.2). We then show that the algebraic capacities of a (possibly singular) surface satisfy a set of nice axioms (4.3). Finally, we apply the axioms to prove Theorems 1.7-1.11.

2. ECH capacities and Seiberg–Witten theory

In this section, we review some aspects of Seiberg–Witten theory (§2.1) and embedded contact homology (§2.5). Our goal is to prove an estimate for the ECH capacities in terms of the Seiberg–Witten invariants in §2.6.

2.1. Seiberg–Witten invariants. The Seiberg–Witten invariant of a closed 4-manifold X with $b^+(X) \geqslant 1$ and a spin-c structure $\mathfrak{s}$ is an integral smooth invariant denoted by

$$\mathrm{SW}_X(\mathfrak{s}) \in \mathbb{Z}$$

A symplectic manifold X has a canonical spin-c structure $\mathfrak{s}_X$. Since spin-c structures on X are a torsor for $H_2(X;\mathbb{Z})$, SW_X in the symplectic setting can be viewed as map

$$\mathrm{SW}_X : H_2(X;\mathbb{Z}) \to \mathbb{Z} \qquad A \mapsto \mathrm{SW}_X(A) := \mathrm{SW}_X(\mathfrak{s}_X + A) \tag{2.1}$$

In later sections (e.g. §3.2), we will often refer to the set of mod 2 Seiberg-Witten non-zero classes

$$\mathrm{SW}(X) := \{A \in H^2(X;\mathbb{Z}) \ : \ \mathrm{SW}_X(A) = 1 \mod 2\} \subset H^2(X;\mathbb{Z}) \tag{2.2}$$

In this section, we discusss several properties of the these invariants that we will apply in later sections. See [**57**,**61**] for a more detailed review.

Let us briefly recall the construction of SW_X for X symplectic and spin-c structure $\mathfrak{s} = \mathfrak{s}_X + A$. Choose a metric g and a self-dual 2-form μ. Given this data, we can consider the Seiberg-Witten equations for a pair (a,ψ) of a spin-c connection a on $\mathfrak{s}$ and a spinor $\psi \in \Gamma(S^+)$.

$$D_a\psi = 0 \qquad F_a^+ + \sigma(\psi) = \mu \tag{2.3}$$

Proposition 2.1. *For generic (g,μ), the moduli space $\mathcal{M}(A)$ of pairs (a,ψ) modulo a natural $C^\infty(X;S^1)$ action is a closed manifold of dimension*

$$\mathrm{I}(A) = c_1(X)\cdot A + A^2 \tag{2.4}$$

The Seiberg-Witten invariant $\mathrm{SW}_X(A)$ is acquired by integrating a certain natural top-dimensional cohomology class over $\mathcal{M}(A)$. It is independent of the choice of (g,μ) if $b^+(X) \geqslant 2$.

2.2. Wall–crossing. When $b^+(X) = 1$, two different Seiberg-Witten invariants arise depending on the choice of (g,μ). More precisely, we have invariants

$$\mathrm{SW}_X(A) \quad \text{if } (g,\mu) \text{ satisfies } \quad \frac{i}{2\pi}\int_Y \mu \wedge \tau_g > [\omega]\cdot A \tag{2.5}$$

$$\mathrm{SW}^-_X(A) \quad \text{if } (g,\mu) \text{ satisfies } \quad \frac{i}{2\pi}\int_Y \mu \wedge \tau_g < [\omega]\cdot A \tag{2.6}$$

Here τ_g is the unique self-dual, g-harmonic 2-form satisfying $[\tau_g] = [\omega]$ in $H^2(Y)$, and if g is compatible with ω then $\tau_g = \omega$. We refer to the space of data satisfying as (2.5) as the *symplectic* chamber and the other space of data as the *non-symplectic* chamber (cf. [**74**, p. 463]).

The two invariants SW_X and SW^-_X are related by a well-known wall-crossing formula. Here is a simple version of this formula that we will use momentarily (cf. Li-Liu [**56**, Prop 1.1]).

Theorem 2.2 (Wall Crossing). *Let X be a closed symplectic 4-manifold with $b_1(X) = 0$ and $b^+(X) = 1$, and let $A \in H_2(X)$ satisfy $I(A) \geqslant 0$. Then*

$$\mathrm{SW}_X(A) = \mathrm{SW}_X^-(A) \pm 1 \tag{2.7}$$

2.3. Gromov-Taubes. There is a deep alternate formulation of the Seiberg-Witten invariants using J-holomorphic curves due primarily to Taubes [**74**,**75**], who introduced the *Gromov-Taubes invariants*

$$\mathrm{Gr}_X : H_2(X) \to \mathbb{Z} \qquad A \mapsto \mathrm{Gr}_X(A)$$

Given a choice of compatible complex structure J on X, $\mathrm{Gr}_X(A)$ is a signed count of points in a certain 0-dimensional moduli space $\mathcal{M}_A(J)$ of disconnected J-curves C in homology class A that pass through $k = I(A)/2$ generic points of X. Of course, J must be chosen so that $\mathcal{M}_A(J)$ is transversely cut out in an appropriate sense.

Theorem 2.3 (Taubes). *The Seiberg-Witten and Gromov-Taubes invariants agree, i.e.* $\mathrm{SW}_X = \mathrm{Gr}_X$.

Theorem 2.3 is extremely powerful and has a number of surprising consequences. For example, we have the following effectiveness result.

Proposition 2.4 (Effective Classes). *Let Y be a smooth projective surface. Then every Seiberg-Witten non-zero class is effective, i.e.* $\mathrm{SW}(Y) \subseteq \overline{\mathrm{NE}}(Y)$.

Proof. Let J be the projective complex structure on Y. Any J-holomorphic map $u : \Sigma \to Y$ from a closed (possibly disconnected) Riemann surface Σ is, of course, algebraic. If $A \in H_2(Y)$ is non-zero and non-effective, then no such curve can exist. In particular, the Gromov-Taubes moduli space $\mathcal{M}_A(J)$ is empty (and thus transverse), so $\mathrm{SW}_Y(A) = \mathrm{Gr}_Y(A) = 0$.

There is a slight technical point when $A = 0$. In this case, the empty curve is counted as the unique J-curve of homology class 0, so $\mathrm{Gr}_Y(0) = 1$. This covers the statement in that case. □

2.4. SW for rational surfaces. We can use Proposition 2.2 and the wall-crossing formula in Theorem 2.2 to compute the mod 2 Seiberg-Witten invariants of a rational surface. This calculation is key to §3.

Proposition 2.5 (Rational Surfaces). *Let Y be a smooth rational surface. Then*

$$\mathrm{SW}(Y) = \{A \in \overline{\mathrm{NE}}(Y) \ : \ I(A) \geqslant 0\}$$

The proof is a direct generalization of the calculation for $\mathbb{P}^2$, and requires the following lemma.

Lemma 2.6. *Every smooth, rational, projective surface X admits a psc (positive scalar curvature) metric.*

Proof. Every minimal rational surface M has a psc metric [**54**, Thm 1], e.g. $\mathbb{P}^2$. In particular, $\bar{\mathbb{P}}^2$ also has a psc metric. Thus by the work of Gromov-Lawson [**31**, Thm 1], the connect sum $X = M\#k\bar{\mathbb{P}}^2$ has a psc metric for any $k \geqslant 0$. This covers all rational surfaces. □

Proof. (Proposition 2.5) Let Y be the smooth rational surface above with Kahler form ω. Every rational projective surface Y satisfies $b^+(Y) = 1$ and $b_1(Y) = 0$.

By Proposition 2.4, every Seiberg-Witten non-zero class A is effective. Furthermore, every such class A must satisfy $I(A)$ since the moduli space $\mathcal{M}(A)$ must have dimension $I(A) \geqslant 0$. Thus

$$\mathrm{SW}(Y) \subseteq \{A \in \overline{\mathrm{NE}}(Y) \ : \ I(A) \geqslant 0\}$$

We thus must prove inclusion in the other direction.

Thus let $A \in H_2(Y)$ be an effective class with $I(A) \geqslant 0$. Let (g, μ) be a pair of a psc metric and a C^0-small self-dual 1-form μ. The pair is in the non-symplectic chamber. Indeed, $[\omega] \cdot A > 0$ since ω is ample and A is effective, while $\int \mu \wedge \tau_g \simeq 0$. The μ-perturbed Seiberg-Witten moduli space is empty since g has psc [**63**, Cor 2.2.6 and Cor 2.2.18]. Thus the Seiberg-Witten invariant $\mathrm{SW}^-(A)$ in this chamber vanishes. By the wall-crossing formula of Theorem 2.2, we thus conclude that

$$\mathrm{SW}^+_Y(A) = \mathrm{SW}^-_Y(A) \pm 1 = 1 \mod 2$$

This concludes the proof. □

Here are a few examples of the above calculation for specific rational surfaces.

Example 2.7 (Projective Plane). The homology $H_2(\mathbb{P}^2)$ is generated by the hyperplane class H and the effective classes are $\overline{\mathrm{NE}}(\mathbb{P}^2) = \mathrm{Cone}(H)$. Furthermore, the anti-canonical is $-K = 3H$ so

$$I(kH) = kH \cdot 3H + kH \cdot H = (k^2 + 3k)$$

Thus $I(A) \geqslant 0$ for any effective class and so by Proposition 2.5, $\mathrm{SW}(\mathbb{P}^2) = \mathrm{Cone}(H)$.

Example 2.8 (Line Times Line). The effective cone $\overline{\mathrm{NE}}(\mathbb{P}^1 \times \mathbb{P}^1)$ is generated by the two classes $D_1 = [\mathbb{P}^1 \times p]$ and $D_2 = [p \times \mathbb{P}^1]$. These intersect as follows.

$$D_1 \cdot D_1 = D_2 \cdot D_2 = 0 \qquad D_1 \cdot D_2 = 1$$

The anti-canonical divisor $-K$ is $2D_1 + 2D_2$. This is an ample class, so again $I(A) \geqslant 0$ for any A and we acquire $\mathrm{SW}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathrm{Cone}(D_1, D_2)$.

2.5. Embedded contact homology. Here we review embedded contact homology as a symplectic field theory, as presented in [**42**] (also see [**45**]).

Definition 2.9. A *contact 3-manifold* (Y, ξ) is a 3-manifold Y with a 2-plane bundle $\xi \subset TY$ that is the kernel $\xi = \ker(\alpha)$ of a contact form. A *contact form* α is a 1-form satisfying

$$\alpha \wedge d\alpha > 0 \quad \text{everywhere}$$

The *Reeb vector-field* R of α is the unique vector-field satisfying $\alpha(R) = 1$ and $d\alpha(R, \cdot) = 0$, and a *Reeb orbit* is a closed orbit (modulo reparametrization) of R.

The *embedded contact homology*, or ECH for short, of a closed contact 3-manifold (Y, ξ) is a $\mathbb{Z}/2$-graded $\mathbb{Z}/2$-module denoted by

$$\mathrm{ECH}(Y, \xi) = \bigoplus_{[\Gamma] \in H_1(Y;\mathbb{Z})} \mathrm{ECH}(Y, \xi; [\Gamma])$$

The ECH group comes equipped with a degree -2 *U-map* and a distinguished *empty set* class.

$$U : \mathrm{ECH}(Y, \xi; [\Gamma]) \to \mathrm{ECH}(Y, \xi; [\Gamma]) \qquad [\varnothing] \in \mathrm{ECH}(Y, \xi; [0])$$

The $\mathbb{Z}/2$ grading on $\mathrm{ECH}(Y,\xi;[0])$ can be canonically enhanced to a $\mathbb{Z}/2m$-grading where $[\varnothing]$ has grading 0 and m is defined by

$$m := \min\{\langle c_1(\xi);[\Sigma]\rangle \;:\; [\Sigma] \in H_2(Y;\mathbb{Z})\}$$

The simplest example of ECH groups are those of the 3-sphere.

Proposition 2.10. *(c.f. [45]) The embedded contact homology* $\mathrm{ECH}(S^3,\xi)$ *of the* 3*-sphere is given by*

$$\mathrm{ECH}(S^3,\xi) = \mathbb{Z}/2[U^{-1}]$$

as a $\mathbb{Z}/2[U]$*-module, where* $|U^{-1}| = 2$ *and* U *acts in the obvious way.*

Given a choice of contact form α for (Y,ξ), one can enhance the ECH groups of Y to a family of *filtered ECH groups* $\mathrm{ECH}^L(Y,\alpha;[\Gamma])$ parametrized by $L \in [0,\infty)$ equipped with natural maps

(2.8)

$$\iota_L^K : \mathrm{ECH}^L(Y,\alpha;[\Gamma]) \to \mathrm{ECH}^K(Y,\alpha;[\Gamma]) \quad \text{and} \quad \iota_L : \mathrm{ECH}^L(Y,\alpha;[\Gamma]) \to \mathrm{ECH}(Y,\xi;[\Gamma])$$

Each filtered ECH group comes equipped with a U-map and empty set class, and these structures are compatible with the maps (2.8).

$$U^L : \mathrm{ECH}^L(Y,\alpha;[\Gamma]) \to \mathrm{ECH}^L(Y,\alpha;[\Gamma]) \qquad [\varnothing]^L \in \mathrm{ECH}^L(Y,\xi;[0])$$

Furthermore, the inclusions ι_L^K respect composition and the ordinary ECH is the colimit of the filtered ECH groups via the maps ι_L.

We can give a simple definition of the ECH capacities in terms of the formal structure of ECH described above.

Definition 2.11. The *k-th ECH capacity* $c_k(Y,\alpha)$ of a closed contact 3-manifold is defined by

$$c_k(Y,\alpha) = \min\Big\{L \;:\; [\varnothing] = U^k \circ \iota_L(\sigma) \text{ for } \sigma \in \mathrm{ECH}^L(Y,\alpha;[0])\Big\}$$

The *k-th ECH capacity* $c_k(X,\lambda)$ of a Liouville domain (X,λ) is the k-th ECH capacity of its boundary $(\partial X, \lambda|_{\partial X})$ as a contact manifold.

The ECH capacities are (non-normalized) capacities on the category of Liouville domains.

Proposition 2.12. *The ECH capacities* $c_k(\cdot)$ *satisfy the following axioms.*

(a) (Inclusion) If $X \to X'$ *is a symplectic embedding of Liouville domains, then* $c_k(X,\lambda) \leqslant c_k(X',\lambda')$.

(b) (Scaling) If (X,λ) *is a Liouville domain then* $c_k(X, C\cdot\lambda) = C \cdot c_k(X,\lambda)$ *for any constant* $C > 0$.

The ECH groups are the homology of an ECH chain group $\mathrm{ECC}(Y,\alpha,J)$ depending on a choice of non-degenerate[1] contact form α and a complex structure J on the symplectization of Y satisfying certain conditions. The chain group is freely generated over $\mathbb{Z}/2$ by *orbit sets*

$$\Gamma = \{(\gamma_i, m_i)\}_{i=1}^k \qquad \gamma_i \text{ is an embedded Reeb orbit and } m_i \in \mathbb{Z}_+$$

[1]A non-degenerate contact form is one where the differential of the Poincare return map along any orbit has no 1-eigenvalues.

satisfying the condition that $m_i = 1$ if the orbit γ_i is hyperbolic. Given an element x of $\mathrm{ECC}(Y, \alpha, J)$ and an orbit set Γ, we denote the Γ-coefficient of x by $\langle x, \Gamma \rangle$. The differential

$$\partial : \mathrm{ECC}(Y, \alpha, J) \to \mathrm{ECC}(Y, \alpha, J)$$

is defined by a holomorphic curve count. More precisely, if $\Gamma_+ = \{(\gamma_i, m_i)\}_1^k$ and $\Gamma_- = \{(\eta_i, n_i)\}_1^l$ are admissible orbit sets, then the Γ_--coefficient of $\partial\Gamma_+$ is given by

$$\langle \partial\Gamma_+, \Gamma_- \rangle = \#\mathcal{M}_1(Y, J)/\mathbb{R}$$

Here $\#\mathcal{M}_1(Y, J)/\mathbb{R}$ is a count of (possibly disconnected) holomorphic curves in the symplectization of Y that have ECH index 1 with positive ends at Γ_+ and negative ends at Γ_-. The *ECH index* $I(C)$ of a homology class in $C \in H_2(Y, \Gamma_+ \cup \Gamma_-)$ is defined by

$$I(C) = \langle c_\tau(\xi), C \rangle + Q_\tau(C, C) + \sum_{i=1}^{k} \sum_{j=1}^{m_i} \mathrm{CZ}_\tau(\gamma_j^i) - \sum_{i=1}^{l} \sum_{j=1}^{n_i} \mathrm{CZ}_\tau(\eta_j^i) \tag{2.9}$$

Here $c_\tau(\xi)$ is the relative 1st Chern class, $Q_\tau(C, C)$ is the relative intersection form and CZ_τ is the Conley-Zehnder index (all relative to a trivialization τ of the contact structure).

Embedded contact homology has a vaguely TQFT-like structure, whereby certain types of cobordisms between contact manifolds induce maps on the (filtered) ECH groups.

Definition 2.13. A *strong symplectic cobordism* X between contact manifolds $Y_\pm$ with contact form, denoted by

$$(X, \omega) : (Y_+, \alpha_+) \to (Y_-, \alpha_-)$$

is a symplectic manifold (X, ω) with oriented boundary $\partial X = Y_+ - Y_-$ such that $\omega|_{Y_\pm} = \pm d\alpha_\pm$. The *area class* $[\omega, \alpha_\pm] \in H^2(X, \partial X)$ of (X, ω) is the class of the relative de Rham cycle

$$(\omega, \alpha_+ - \alpha_-) \in \Omega^2(X) \oplus \Omega^1(Y_+ \cup -Y_-)$$

We use $[\Sigma] : [\Gamma_+] \to [\Gamma_-]$ to denote a relative class in $H_2(X, \partial X)$ whose image under the boundary map $\partial : H_2(X, \partial X) \to H_1(\partial X)$ is given by

$$[\Gamma_+] \oplus [\Gamma_-] \in H_2(Y_+) \oplus H_2(Y_-) \simeq H(\partial X)$$

For convenience, we use $\rho[\Sigma] \to \mathbb{R}$ denote the pairing of $[\Sigma]$ with the area class $[\omega, \alpha_\pm]$. Explicitly, we have the formula

$$\rho[\Sigma] = \int_\Sigma \omega - \int_{\partial_+\Sigma} \alpha_+ + \int_{\partial_-\Sigma} \alpha_-$$

With the above notation, we can state the following result of Hutchings regarding the functoriality of ECH with respect to strong symplectic cobordisms.

Theorem 2.14 (Hutchings [**42**]). *A strong symplectic cobordism $X : Y_+ \to Y_-$ and let $[\Sigma] : [\Gamma_+] \to [\Gamma_-]$ be a class in $H_2(X, \partial X)$. Then there is a canonical, ungraded map*

$$\mathrm{ECH}^L(X; [\Sigma]) : \mathrm{ECH}^L(Y_+, \alpha_+; [\Gamma_+]) \to \mathrm{ECH}^{L+\rho[\Sigma]}(Y_-, \alpha_-; [\Gamma_-]) \tag{2.10}$$

$$\mathrm{ECH}(X; [\Sigma]) : \mathrm{ECH}(Y_+, \xi_+; [\Gamma_+]) \to \mathrm{ECH}(Y_-, \xi_-; [\Gamma_-]) \tag{2.11}$$

These maps are compatible with composition, and satisfy the following axioms.

(a) *(Curve Counting) There exists a chain map* Φ *inducing* $\mathrm{ECH}(X;[\Sigma])$*, of the form*

$$\Phi^L : ECC_L(Y_+, \alpha_+; [\Gamma_+]) \to ECC^{L+\rho[\Sigma]}(Y_-, \alpha_-; [\Gamma_-])$$

that "counts curves" in the following sense: if $\Gamma_\pm$ *are orbit sets in* $Y_\pm$ *such that*

$$\langle \Phi_A(\Gamma_+), \Gamma_- \rangle = 1$$

then there is a holomorphic current[2] C *of ECH index* 1 *asymptotic at* $\pm\infty$ *to* $\Gamma_\pm$*.*

(b) *(Filtration) The maps commute with the inclusion maps* ι_L^K *and* ι_L*, e.g.*

$$\begin{array}{ccc} \mathrm{ECH}^L(Y_+, \alpha_+; [\Gamma_+]) & \xrightarrow{\mathrm{ECH}^L(X;[\Sigma])} & \mathrm{ECH}^{L+\rho[\Sigma]}(Y_-, \alpha_-; [\Gamma_-]) \\ \downarrow & & \downarrow \\ \mathrm{ECH}^K(Y_+, \alpha_+; [\Gamma_+]) & \xrightarrow{\mathrm{ECH}^K(X;[\Sigma])} & \mathrm{ECH}^{K+\rho[\Sigma]}(Y_-, \alpha_-; [\Gamma_-]) \end{array}$$

(c) *(U-Map) The maps commute with the* U*-maps, e.g.*

$$U^{L+\rho[\Sigma]} \circ \mathrm{ECH}^L(X;[\Sigma]) = \mathrm{ECH}^L(X;[\Sigma]) \circ U^L$$

(d) *(Seiberg–Witten) Let* (P, ξ) *be a contact* 3*-manifold. Consider a pair of strong symplectic cobordisms and their composition, denoted by*

$$N : \varnothing \to P \qquad X : P \to \varnothing \quad \text{and} \quad Y = N \cup_Z X : \varnothing \to \varnothing$$

Fix homology classes $[A] \in H_2(N, Z)$ *and* $[B] \in H_2(X, Z)$ *with* $\partial[A] = \partial[B]$*. Then*

$$\mathrm{ECH}(X, [B]) \circ U^k \circ \mathrm{ECH}(N, [A]) = \sum_{[C] \in S} \mathrm{SW}_Y([C])$$

Here $S \subset H_2(X)$ *is shorthand for the set of homology classes satisfying*

$$[C] \cap N = [A] \qquad [C] \cap X = [B] \quad \text{and} \quad I([C]) = 2k$$

The analogue functoriality result for exact symplectic cobordisms is well established [**48**, Theorem 1.9] and has been used extensively, e.g. to define the ECH capacities [**44**]. Non-exact cobordisms and the foundations provided by [**42**] have been used to define Gromov-Taubes invariants for non-symplectic manifolds [**27**, **28**].

Remark 2.15 (Proof of Theorem 2.14). Since a detailed treatment of Theorem 2.14 has yet to appear in the literature outside of [**42**], we include a brief discussion of its proof. It is similar to the exact case in [**48**] with some small modifications.

The basic strategy of [**48**] and [**42**] is to establish a filtered version of the Taubes isomorphism between filtered ECH and an energy filtered version of Kronheimer-Mrowka's monopole Floer homology (MFH) groups [**48**, §3]. Cobordism maps on filtered ECH can then be *defined* so that they intertwine the analogous maps on filtered MFH via these isomorphisms [**48**, §5.1]. Theorem 2.14(c)-(d) follow more or less immediately from this strategy [**48**, Cor 5.3].

The proof in [**48**, §6] of the analogue of Theorem 2.14(a)-(b) uses a well-known argument for producing instantons counted in MFH cobordism maps from holomorphic curves [**48**, §6.2] and an SFT/Gromov compactness argument [**48**, §6.4]. In the non-exact setting, the required compactness can be guaranteed by only considering cobordism maps $\mathrm{MFH}(X, \mathfrak{s})$ in MFH induced by a symplectic cobordism (X, ω) equipped with a

[2]This is a formal positive integer combination of embedded holomorphic curves.

specific spin-c structure $\mathfrak{s}$, determined by a fixed relative homology class $[\Sigma] : [\Gamma_+] \to [\Gamma_-]$ with $\mathfrak{s} = \mathfrak{s}_\omega + PD[\Sigma]$. The energy of the instantons and holomorphic curves involved in $\mathrm{MFH}(X, \mathfrak{s})$ obey a uniform bound in terms of the actions of the ends $\Gamma_\pm$ and $\rho[\Sigma]$. In particular, the moduli space of curves admits a compactification in the SFT topology and the arguments of [**48**, §6] can be slightly modified to handle this case.

2.6. From ECH to SW. We now conclude the section by applying the formal structure of the ECH groups in §2.5 to estimate for the ECH capacities of a star-shaped domain embedded into closed symplectic manifolds.

Proposition 2.16. *Let $(X, \lambda) \subset \mathbb{R}^4$ be a star-shaped domain with restricted Liouville form λ and let (Y, ω) be a closed symplectic 4-manifold. Fix an embedding*

$$\iota : (X, d\lambda) \to (Y, \omega)$$

Then the ECH capacities of X satisfy

$$c_k(X) \leqslant \inf_{[\Sigma] \in \mathrm{SW}(Y)} \{\langle \omega, [\Sigma] \rangle \ : \ I([\Sigma]) \geqslant 2k\} \tag{2.12}$$

Remark 2.17. This result is based on the proofs in [**42**, §2.2].

Proof. Let (Z, α) be the contact boundary of (X, λ) and let $[\Sigma] \in H_2(Y)$ be any $\mathbb{Z}$-homology class satisfying the constraints laid out in (2.12).

$$\mathrm{SW}_Y([\Sigma]) = 1 \mod 2 \ \text{ and } \ I([\Sigma]) \geqslant 2k$$

It suffices to demonstrate the following inequality for any such $[\Sigma]$.

$$c_k(X) \leqslant A := \langle \omega, [\Sigma] \rangle$$

Since $c_k(X) \leqslant c_j(X)$ for $j = I([\Sigma])/2$, we can assume that $k = j = I([\Sigma])/2$. Furthermore, it is equivalent to show that for all $\varepsilon > 0$ sufficiently small, there exists a class

$$\eta \in \mathrm{ECH}^{A+\varepsilon}(Z, \xi; [0]) \quad \text{with} \quad U^k \iota_A \eta = [\varnothing] \in \mathrm{ECH}(Z, \xi; [0]) \tag{2.13}$$

To find an η that satisfies (2.13), we consider the splitting of Y into X (or rather, the image $\iota(X)$) and $N = Y \backslash X$. If we denote the contact boundary of X by (Z, ξ), we can interpret this as pair of strong symplectic cobordisms

$$N : \varnothing \to Z \qquad X : Z \to \varnothing$$

Since X is diffeomorphic to a 4-ball, the pair of maps

$$H_2(Y) \xrightarrow{-\cap X} H_2(X, \partial X) \quad \text{and} \quad H_2(Y) \xrightarrow{-\cap P} H_2(P, \partial P)$$

are, respectively, the 0 map and an isomorphism. Let $[S] = [\Sigma] \cap X$ be the intersection of $[\Sigma]$ with X. Note that we have

$$A = \langle [\omega], [\Sigma] \rangle = \rho[S] + \rho[0] = \rho[S]$$

Now we let $\varepsilon > 0$ be small and arbitrary, and define the desired class η by

$$\eta = \mathrm{ECH}^A(P; [S])[\varnothing] \in \mathrm{ECH}^{A+\varepsilon}(Z, \xi; [0]) \quad \text{where} \quad [\varnothing] \in \mathrm{ECH}^{\varepsilon}(\varnothing; [0]) \simeq \mathbb{Z}/2[\varnothing]$$

We would like to show that $U^k \iota_{A+\varepsilon} \eta = [\varnothing]$. To start, pick a chain map lifting the ECH cobordism map as in Thm. 2.14(a). That is,

$$\Phi : \mathbb{Z}/2 \to \mathrm{ECC}^{\varepsilon + A}(Z, \alpha; [0]) \quad \text{with} \quad [\Phi(\varnothing)] = \eta$$

If Γ_- is any orbit set such that $\langle \Phi(\varnothing), \Gamma_- \rangle = 1$, then by Theorem 2.14(a) we know that there is a holomorphic current C of ECH index 0 with empty positive boundary and negative boundary Γ_-. If we let $C' \subset Z$ be a surface with positive boundary Γ_-, so that $|\Gamma_-| = I(C')$, then by the additivity of the ECH index we have

$$2k = I([\Sigma]) = I(C) + I(C') = I(C') = |\Gamma_-|$$

Thus we know that η is homogenous of grading $2k$, and so $U^k \circ \iota_{A+\varepsilon}(\eta)$ is grading 0. In particular, by Proposition 2.10, we have

$$U^k \iota_{A+\varepsilon} \eta \in ECH_0(Z, \xi; [0]) \simeq ECH_0(S^3; [0]) = \mathbb{Z}/2[\varnothing]$$

On the other hand, by Theorem 2.14(b) and (d), we know that

$$\mathrm{ECH}(X; [0]) \circ U^k \circ \iota_{A+\varepsilon} \eta = \mathrm{ECH}^A(X; [0]) \circ U^k \circ \mathrm{ECH}(X; [S])[\varnothing] = c$$

Here $c \in \mathbb{Z}/2$ is the sum over $[C]$ with $[C] \cap X = [0]$ and $[C] \cap P = [S]$ of $\mathrm{SW}_Y([C])$ mod 2. Since $[\Sigma]$ is the unique such class and $\mathrm{SW}_Y([\Sigma]) = 1 \mod 2$, we find that $c = 1$. Thus, $U^k \iota_{A+\varepsilon} \eta$ is non-zero and we must have

$$U^k \iota_{A+\varepsilon} \eta = [\varnothing]$$

This proves that for every ε, there is a class $\eta \in ECH^{A+\varepsilon}(Z, \xi; [0])$ satisfying (2.13), and thus concludes the proof. □

Remark 2.18. The proof of Proposition 2.16 generalizes immediately to Liouville domains (X, λ) that satisfy the following criteria.

(a) The map $H_2(\partial X) \xrightarrow{\iota_*} H_2(X)$ is 0.
(b) The contact manifold $(\partial X, \xi)$ has torsion chern class, i.e. $c_1(\xi) = 0 \in H^2(\partial X; \mathbb{Q})$.
(c) The empty set $[\varnothing]$ is the unique class of ECH grading 0 in the image of the U-map.

The conclusion of Proposition 2.16 must be appropriately modified so that (2.12) is a minimum over all classes $[\Sigma]$ such that $[\Sigma] \cap X = 0$. In practice, the most difficult criterion to verify is (c). This holds, for instance, when $[\varnothing]$ is the unique ECH index 0 class. It is also believed to hold for circle bundles over a 2-sphere (c.f. the unpublished thesis of Ferris **[25]** and the forthcoming work of Nelson–Weiler **[62]**).

3. Algebraic capacities and birational geometry

We now construct of the algebraic capacities (§3.1) and prove Theorem 3.5 (§3.2).

Conventions 3.1. In this section, all surfaces will be projective normal algebraic surfaces over the complex numbers, not necessarily smooth, unless otherwise specified.

Let $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. We work in the Néron–Severi group $\mathrm{NS}(Y) \subseteq H^2(Y, \mathbb{Z})$ of Weil $\mathbb{Z}$-divisors regarded up to algebraic equivalence. We denote $\mathrm{NS}(Y)_{\mathbb{K}} := \mathrm{NS}(Y) \otimes_{\mathbb{Z}} \mathbb{K}$. We say that a $\mathbb{Z}$-divisor D on a surface Y is $\mathbb{Q}$-*Cartier* if some integer multiple of D is Cartier; that is, the sheaf $\mathcal{O}(D)$ is a line bundle. Y is said to be $\mathbb{Q}$-*factorial* if every Weil $\mathbb{Z}$-divisor on Y is $\mathbb{Q}$-Cartier. Every toric surface is $\mathbb{Q}$-factorial. A $\mathbb{Q}$-Cartier $\mathbb{R}$-divisor D on Y is *nef* if $D \cdot C \geqslant 0$ for all curves $C \subseteq Y$. Denote by $\mathrm{Nef}(Y)_{\mathbb{K}}$ the classes in $\mathrm{NS}(Y)_{\mathbb{K}}$ corresponding to nef divisors.

3.1. Construction of algebraic capacities. Let Y be a $\mathbb{Q}$-factorial projective surface and let A be an ample $\mathbb{R}$-divisor on Y. We recall the optimisation problems of [**80–82**] that are designed to emulate ECH capacities in the context of algebraic geometry.

Definition 3.2 ([**80**, §4.5] or [**81**, Def. 2.2]). The *kth algebraic capacity* of (Y, A) are given by

$$c_k^{\mathrm{alg}}(Y, A) := \inf_{D \in \mathrm{Nef}(Y)_{\mathbb{Z}}} \{D \cdot A : \chi(D) \geqslant k + \chi(\mathcal{O}_Y)\} \tag{3.1}$$

Remark 3.3. Note that it follows from Kleiman's criterion for nef-ness that this infimum in (3.1) is always achieved.

The *index* of a $\mathbb{Z}$-divisor D on Y is given by $I(D) := D \cdot (D - K_Y)$. When Y is smooth or has at worst canonical singularities [**68**] we have Noether's formula

$$\chi(D) = \chi(\mathcal{O}_Y) + \frac{1}{2} I(D) \tag{3.2}$$

Furthermore, if ω_A is the Kahler class induced by A via the embedding into $\mathbb{P}H^0(k\mathcal{O}(A))$ for $k \gg 0$ (which is defined because A is ample) we may write

$$D \cdot A = \langle \omega_A, D \rangle = \int_D \omega_A \tag{3.3}$$

In these cases, we can alternatively write the algebraic capacities as

$$c_k^{\mathrm{alg}}(Y, A) = \inf_{D \in \mathrm{Nef}(Y)_{\mathbb{Z}}} \{\langle \omega_A, D \rangle \ : \ I(D) \geqslant 2k\} \tag{3.4}$$

which is very similar to the upper bound for c_k^{ECH} in Proposition 2.16.

3.2. Relating ECH capacities and algebraic capacities. We seek to prove the following result.

Theorem 3.4. *Suppose Y is a smooth rational surface, and let A be an ample $\mathbb{R}$-divisor on Y. Then*

$$\inf_{D \in \mathrm{SW}(Y)} \{D \cdot A : I(D) \geqslant 2k\} = \inf_{D \in \mathrm{Nef}(Y)_{\mathbb{Z}}} \{D \cdot A : I(D) \geqslant 2k\} =: c_k^{\mathrm{alg}}(Y, A)$$

By combining Proposition 2.16, the formula (3.4) and Theorem 3.4, we immediately acquire the main result, which we state again for completeness.

Theorem 3.5. *Suppose that $X \to Y$ is a symplectic embedding of a star-shaped domain X into a smooth rational projective surface Y with a ample $\mathbb{R}$-divisor A and symplectic form ω_A satisfying $[\omega_A] = PD[A]$. Then*

$$c_k^{\mathrm{ECH}}(X) \leqslant c_k^{\mathrm{alg}}(Y, A) \tag{$\star$}$$

Remark 3.6. We only require an upper bound of the Seiberg-Witten quantity by c_k^{alg} for the purposes of this part of this thesis. However, Theorem 3.4 is satisfying because it demonstrates that the algebraic capacities are (as obstructions) just as sensitive as the Seiberg-Witten theoretic quantities.

We treat the case of smooth rational surfaces using the Minimal Model Program. To begin, recall that the nef cone is contained within the effective cone, i.e.

$$\mathrm{Nef}(Y)_{\mathbb{Z}} \subseteq \overline{\mathrm{NE}}(Y)$$

We calculated the Seiberg-Witten theory of a rational surface in Proposition 2.5. That calculation implies the inequality of Theorem 3.4

$$\inf_{D \in \mathrm{SW}(Y)} \{D \cdot A : I(D) \geqslant 2k\} = \inf_{D \in \overline{\mathrm{NE}}(Y)} \{D \cdot A : I(D) \geqslant 2k\}$$

This immediately implies that we have an inequality in one direction.

$$\inf_{D \in \mathrm{SW}(Y)} \{D \cdot A : I(D) \geqslant 2k\} \leqslant \inf_{D \in \mathrm{Nef}(Y)_{\mathbb{Z}}} \{D \cdot A : I(D) \geqslant 2k\} \tag{3.5}$$

For the converse inequality, we will show for that each Seiberg–Witten nonzero divisor there is a nef divisor that is 'preferable' from the perspective of the optimisation problems above. For this purpose, we adopt the following terminology.

Definition 3.7. Let Y be a $\mathbb{Q}$-factorial surface. We say that a Weil $\mathbb{Q}$-divisor D_0 is

(a) *index-preferable* to another Weil $\mathbb{Q}$-divisor D if $I(D_0) \geqslant I(D)$ and

(b) *area-preferable* D if $D_0 \cdot A \leqslant D \cdot A$ for all ample $\mathbb{R}$-divisors A on Y.

A Weil $\mathbb{Q}$-divisor D_0 that is both area- and index-preferable will simply be called *preferable*. Note that D_0 is area-preferable to D if and only if $D - D_0$ is effective.

To construct preferable divisors in general we will use the *isoparametric transform* IP_Y of [**10**]. This takes an effective divisor D to a new divisor $\mathrm{IP}_Y(D)$ given by

$$\mathrm{IP}_Y(D) := D - \sum_{D \cdot D_i < 0} \left\lceil \frac{D \cdot D_i}{D_i^2} \right\rceil D_i \tag{3.6}$$

Here the sum is over prime divisors D_i with $D \cdot D_i < 0$ and, in particular, $\mathrm{IP}_Y(D) = D$ if D is nef. We denote by $\mathrm{IP}^n_Y(D)$ the result of iterating IP^n_Y n times. In [**10**], the following result is proven.

Theorem 3.8 ([**10**, Thm. 1.1 + 1.2]). *For any effective divisor D on a smooth surface Y we have*

$$h^0(D) = h^0(\mathrm{IP}_Y(D))$$

Then for all sufficiently large $n \gg 0$, we have $\mathrm{IP}^n_Y(D) = \mathrm{IP}^\infty_Y(D)$ for some nef $\mathrm{IP}^\infty_Y(D) \in \mathrm{Nef}(Y)_{\mathbb{Z}}$.

We will need to know what IP_Y does to area and index. For area, the answer is quite simple.

Lemma 3.9. *Let D be effective and A be ample. Then $A \cdot \mathrm{IP}_Y(D) \leqslant A \cdot D$.*

Proof. If D_i is a prime divisor with $D_i \cdot D < 0$ and D is effective, then $D_i^2 < 0$. Thus the coefficients of the sum in (3.6) are positive. Since A is ample, $A \cdot D_i < 0$. These two facts imply the result. $\square$

The answer for the index is more complicated. For this, we need the following lemma.

Lemma 3.10. *Let Y be a smooth surface with D an effective divisor on Y. Suppose $C_1, \ldots, C_n$ is a collection of curves intersecting D negatively. Then either one of the C_i is a (-1)-curve or*

$$I(D') \geqslant I(D) \quad \textit{where} \quad D' = D - \sum_{i=1}^{n} \left\lceil \frac{D \cdot C_i}{C_i^2} \right\rceil C_i$$

In particular, $I(\mathrm{IP}_Y(D)) \geqslant I(D)$ if no (-1)-curve intersects D negatively.

Proof. Suppose $n = 1$ so that there is only one curve C. If $C^2 = -1$ we are done, so let $C^2 = -r$ for $r \geqslant 2$. Let $D \cdot C = -\ell$ so that

$$D' = D - \left\lceil \frac{\ell}{r} \right\rceil C =: D - mC$$

Let $\pi \colon Y \to \overline{Y}$ be the contraction of C to the singular surface $\overline{Y}$. We can compute

$$\begin{aligned} I(D') &= (D - mC) \cdot (D - mC - K_Y) \\ &= I(D) - 2mD \cdot C + (-mC) \cdot (-mC - K_Y) \\ &= I(D) + 2m\ell + (-mC) \cdot (-mC - \pi^* K_{\overline{Y}} - \frac{2-r}{r} C) \\ &= I(D) + 2m\ell - m^2 r - (2-r)m \end{aligned}$$

Now observe that $1 > m - \frac{\ell}{r} \geqslant 0$ by definition and so $\ell + r > rm$. Furthermore, $r \geqslant 2$ and $m \geqslant 1$. Using these facts, we can compute the following lower bound.

$$2m\ell - mr(m + \frac{2-r}{r}) > 2m\ell - (\ell + r)(m + \frac{2-r}{r})$$

$$= m\ell + \ell \cdot \frac{r-2}{r} - mr + r - 2 \geqslant m\ell - r(m-1) - 2 > (m-1)\ell - 2 \geqslant -2$$

In particular, $I(D') > I(D) - 2$. However $I(\cdot)$ is even and so we must have $I(D') \geqslant I(D)$.

Now induct on the number of curves. Suppose the formula holds for a set of n curves meeting an effective divisor negatively. Suppose curves $C_1, \ldots, C_n, C$ intersect D negatively. If any of the curves is a (-1)-curve then we are done. Assume not. Notate

$$D \cdot C = -\ell, \quad C^2 = -r, \quad \left\lceil \frac{D \cdot C}{C^2} \right\rceil = m \ \text{ and } \ F = \sum_{i=1}^{n-1} m_i C_i$$

so that $D' = D - F - mC$. Compute

$$\begin{aligned} I(D - F - mC) = & \\ &= I(D - F) + 2mF \cdot C - 2mD \cdot C + I(-mC) \\ &\geqslant I(D - F) + 2m\ell - mr(m + \frac{2-r}{r}) \\ &> I(D - F) + (m-1)\ell - 2 \\ &\geqslant I(D - F) - 2 \end{aligned}$$

where we used that $F \cdot C \geqslant 0$ since F is effective and supported away from C. By inductive assumption $I(D - F) \geqslant I(D)$ and so we have $I(D') > I(D) - 2$. Since $I(\cdot)$ is even we can conclude that $I(D') \geqslant I(D)$ as desired. ◻

Proof of Thm. 3.4. We simply need to show that for any divisor in $\mathrm{SW}(Y)$, there exists a preferable nef divisor. In other words, we must construct a map

$$\mathcal{N}_Y\colon \mathrm{SW}(Y) \to \mathrm{Nef}(Y)_{\mathbb{Z}}$$

taking a Seiberg–Witten nonzero divisor to a preferable nef $\mathbb{Z}$-divisor. We now construct these maps by induction on the number of blow ups necessary to make Y from a minimal surface.

For minimal rational surfaces the existence of an $\mathcal{N}_Y$ is clear. In the cases of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, we have $\mathrm{SW}(Y) = \mathrm{Nef}(Y)_{\mathbb{Z}}$. Hirzebruch surfaces, on the other hand, have no (-1)-curves. Thus we can set $\mathcal{N}_Y(D) = \mathrm{IP}_Y^n(D)$ for $n \gg 0$. Lemmas 3.10 and 3.9 imply that the result is preferable.

Now assume that such a function exists for all rational surfaces expressible as $b-1$ blowups of a minimal rational surface. Let Y be a surface expressed as b blowups of a minimal rational surface, and for any (-1)-curve $E \subseteq Y$ denote the contraction by $\pi_E\colon Y \to \overline{Y}_E$. We define $\mathcal{N}_Y(D)$ by the following procedure.

(a) If $D \cdot C \geqslant 0$ for all curves $C \subseteq Y$ then D is nef and we define $\mathcal{N}_Y(D) = D$.

(b) If $D \cdot E \leqslant 0$ for some (-1)-curve E, write $D = \pi_E^*\overline{D} + mE$ for some $\overline{D} \in \mathrm{SW}(\overline{Y}_E)$ and for some $m \geqslant 0$. The inductive hypothesis implies that there exists a nef $\mathbb{Z}$-divisor $\overline{D}_0$ preferable to $\overline{D}$. Define $\mathcal{N}(D) = \pi_E^*\overline{D}_0$.

(c) If $D \cdot E > 0$ for all (-1)-curves E on Y but $D \cdot C < 0$ for some $(-r)$-curve C with $r \geqslant 2$, recursively apply (a)-(c) to $\mathrm{IP}_Y(D)$ instead of D and define $\mathcal{N}_Y(D)$ as the result.

This procedure terminates: if $\mathrm{IP}_Y^n(D)$ eventually intersects a (-1)-curve negatively then (b) outputs a nef divisor. If $\mathrm{IP}_Y^n(D)$ does not intersect a (-1)-curve nonpositively for any n then after a finite number of steps we reach $\mathrm{IP}_Y^\infty(D) \in \mathrm{Nef}(Y)_{\mathbb{Z}}$ by Theorem 3.8, which is returned by (a). Note that the application of Theorem 3.8 is valid by Proposition 2.4.

We claim that $\mathcal{N}_Y(D)$ is nef and preferable to D. Indeed, all three steps (a)-(c) only improve the area and index constraints. This claim is trivial for (a) and follows from Lemmas 3.9 and 3.10 for (c). (b) produces a preferable nef $\mathbb{Z}$-divisor since $\pi_E^*\overline{D}$ is preferable to $D = \pi_E^*\overline{D} + mE$ from direct calculation (noting that $m \geqslant 0$), and then $\pi_E^*\overline{D}_0$ is nef and preferable to $\pi_E^*\overline{D}$ since $\overline{D}_0$ is preferable to $\overline{D}$. □

4. Toric Surfaces

We now apply Theorem 3.5 to the study of embeddings into projective toric surfaces. We begin with a review of toric surfaces (§4.1) and toric domains (§4.2). We then demonstrate that the algebraic capacities on toric surfaces are uniquely characterized by a set of axioms (§4.3). Finally, we discuss the main applications: obstructing embeddings of concave toric domains into toric surfaces, and monotonicity of the Gromov width under inclusion of moment polygons (§4.4).

4.1. Toric varieties. We start with a brief review of toric varieties. Our main reference is **[19]**.

DEFINITION 4.1. A *(projective normal) toric variety* Y of dimension n over $\mathbb{C}$ is a projective normal variety with a $(\mathbb{C}^\times)^n$-action acting faithfully and transitively on a Zarisiki open subset of Y.

Every toric variety Y can be described (uniquely, up to isomorphism) by either a *fan* $\Sigma \subset \mathbb{R}^n$ [**19**, Def 3.1.2 and Cor 3.1.8] or a *moment polytope* $\Omega \subset \mathbb{R}^n$ [**19**, Def 2.3.14]. A fan Σ for Y can be recovered from a moment polytope Ω for Y by passing to the *inner normal fan* $\Sigma(\Omega)$ of Ω [**19**, Prop 3.1.6]. We will focus on the polytope perspective, since it will be more important in this part of this thesis.

DEFINITION 4.2. A *moment polytope* $\Omega \subset \mathbb{R}^n$ is a convex polytope with rational vertices and open interior. We denote the corresponding toric variety by Y_Ω.

Note that given a scalar $S > 0$ in $\mathbb{Q}$ or an affine map $T : \mathbb{Z}^2 \to \mathbb{Z}^2$, we can scale Ω to $S\Omega$ or apply T to acquire $T\Omega$. There are naturally induced isomorphisms of varieties $Y_{S\Omega} \simeq Y_\Omega$ and $Y_{T\Omega} \simeq Y_\Omega$.

DEFINITION 4.3. A *smooth vertex* $v \in \Omega$ of a moment polytope is a vertex such that there exists a neighborhood $U \subset \mathbb{R}^n_{\geqslant 0}$ of 0, a neighborhood $V \subset \Omega$ of v, a scaling S and a $\mathbb{Z}$-affine isomorphism T such that $ST(U) = V$ and $ST(v) = 0$. Otherwise a vertex is *singular*.

On a projective toric variety, each face $F \subset \Omega$ determines a $\mathbb{Q}$-Cartier divisor D_F. Every torus invariant divisor is in the span of these divisors D_F, and every divisor class is represented by a torus-invariant divisor [**19**, 4.1.3]. Furthermore, every moment polytope Ω for a toric variety Y_Ω is associated to a natural divisor A_Ω given as a combination of these face divisors.

DEFINITION 4.4. The *associated divisor* A_Ω of the moment polytope Ω is defined as

$$A_\Omega = \sum_F a_F D_F$$

Here for each face $F \subset \Omega$, we define $u_F \in \mathbb{Z}^n$ and $a_F \in \mathbb{Q}$ by the following conditions.

$$\langle u_F, x \rangle = -a_F \text{ for } x \in F \qquad u_F \text{ is primitive in } \mathbb{Z}^n\text{, inward to } \Omega \text{ and normal to } F$$

Note that the equation $\langle u_F, x \rangle = -a_F$ defines a hyperplane that we denote by Π_F.

LEMMA 4.5. *The associated divisor A_Ω of a moment polytope Ω has the following properties.*

(a) (Ample) A_Ω is an ample divisor, and so defines an projective embedding to projective space.

$$|kD_\Omega| : Y_\Omega \to \mathbb{P}H^0(Y_\Omega, kA_\Omega) \quad \textit{for } k \gg 0 \tag{4.1}$$

(b) (Translation/Scaling) Let $T \in \mathrm{GL}_n(\mathbb{Z})$, $V \in \mathbb{Z}^n$ and $S \in \mathbb{Q}$. Then

$$D_{T\Omega} = D_\Omega \qquad D_{\Omega+V} = D_\Omega + P_V \qquad D_{S\Omega} = S \cdot D_\Omega$$

Here P_V is a principle divisor depending on V.

PROOF. For (a), see [**19**, Prop 6.1.10]. For (b), see [**19**, §4.2, Ex 4.2.5(a)] for the translation property. The scaling and linear map properties follow from Definition 4.4. ◻

More generally, any $\mathbb{T}^n$-equivariant $\mathbb{Q}$-divisor $D = \sum a_F D_F$ is associated to a half-space arrangement consisting of the half-spaces H_F and a (possibly empty) polytope P_F given by

$$H_F = \{x \in \mathbb{R} : \langle u_F, x\rangle \geqslant -a_F\} \qquad P_F = \cap_F \Pi_F$$

The dimension of the space of sections $h^0(D)$ is given by the number of lattice points $|P_F \cap \mathbb{Z}^n|$ [**19**, §7.1, p. 322]. A divisor is ample if and only if $\partial H_F \cap P_D$ is an open subset of ∂H_F for each F, and nef if $\partial H_F \cap P_D$ is non-empty for each F.

We are primarily interested in *toric surfaces* , i.e. projective toric varieties of complex dimension 2. In this case, the embedding (4.1) gives Y the structure of a symplectic orbifold by restriction of the Kahler form on $\mathbb{P}^N$. Every toric surface is an orbifold [**19**, Thm. 3.1.19] since every two-dimensional fan is simplicial (dually, every polygon is simple).

4.2. Toric domains. We next review the theory of toric domains. Let ω_{std} denote the standard symplectic form on $\mathbb{C}^n$ and let μ denote the standard moment map

$$\mu : \mathbb{C}^n \to \mathbb{R}^n_{\geqslant 0} \qquad (z_1, \ldots, z_n) \mapsto (\pi|z_1|^2, \ldots, \pi|z_n|^2)$$

Definition 4.6. A *toric domain* (X_Ω, ω) is the inverse image $\mu^{-1}(\Omega)$ of a closed subset $\Omega \subset [0,\infty)^2$ with open interior, equipped with the symplectic form $\omega_{\text{std}}|_{X_\Omega}$ and moment map $\mu|_{X_\Omega}$.

A toric domain X_Ω is *convex* if $\Omega = C \cap [0,\infty)^n$ where $C \subset \mathbb{C}^n$ is a convex and contains 0 in its interior, *concave* if the compliment $\mathbb{R}^2_+\backslash\Omega$ is convex in $\mathbb{C}^n$ and *free* if Ω is convex and contained in $\mathbb{R}^n_+ \subset \mathbb{R}^n_{\geqslant 0}$ (i.e. disjoint from the coordinate axes). Finally, Ω is *rational* if it is a moment polytope in the sense of Definition 4.2 (i.e. a polytope with rational vertices).

A fundamental fact in this part of this thesis is that a convex rational domain X can be compactified to toric surfaces Y by collapsing the boundary ∂Y so that it becomes the associated ample divisor A of Y. More precisely, we have the following result.

Lemma 4.7. *Let Ω be a rational, convex domain polytope with toric variety (Y_Ω, A_Ω) and toric surface X_Ω. Then there is a $\mathbb{T}^n$-equivariant symplectomorphism*

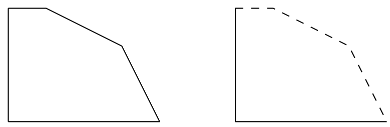
$$Y_\Omega \backslash \operatorname{supp}(A_\Omega) \simeq X^\circ_\Omega$$

Proof. Let $\mu : Y_\Omega \to \mathbb{R}^n_{\geqslant 0}$ and $\nu : X_\Omega \to \mathbb{R}^n_{\geqslant 0}$ denote the moment maps of Y_Ω and X_Ω. Define Ω° to be the complement $\Omega\backslash(\partial\Omega \cap \mathbb{R}^n_+)$. First note that Ω° is the moment image of both $Y_\Omega\backslash\text{supp}(A_\Omega)$ under μ and X°_Ω under ν. For X°_Ω this is clear, and true for any convex domain.

For Y_Ω, write the associated ample divisor as $A_\Omega = \sum_F a_F \cdot D_F$. By examination of Definition 4.4, we see that $a_F = 0$ if and only if F is on a plane passing through 0. Since $\Omega = K \cap \mathbb{R}^n_{\geqslant 0}$ for some convex K, we know that Ω intersects each coordinate hyperplane $H_i = \{x \in \mathbb{R}^n | x_i = 0\}$ along a single face F_i and every other face F_i is not contained in a plane containing the origin (essentially by convexity). Thus $a_{F_i} = 0$ for each i and $a_F \neq 0$ for every other face F. Thus $Y_\Omega\backslash\text{supp}(A_\Omega) = \mu^{-1}(\Omega^\circ)$.

Now that we have shown that X°_Ω and $Y_\Omega\backslash\text{supp}(A_\Omega)$ have the same moment images, we just apply an open version of Delzant's theorem, e.g. the result of Kershon-Lerman

FIGURE 3. Moment polytopes for Y_Ω and $Y_\Omega \backslash D_\Omega$



[51, Thm 1.3]. Note that, in that result, there is a homological obstruction $\mathfrak{o}$ to the equivalence of two spaces with the same moment image

$$\mathfrak{o} \in H^2(X_\Omega^\circ; R) = H^2(Y_\Omega \backslash \operatorname{supp}(A_\Omega); R)$$

for some abelian group R. This obstruction necessarily vanishes since X_Ω° is contractible. $\square$

Note that (essentially by definition) a moment polytope $\Omega \subset \mathbb{R}^n$ is equivalent to a convex, rational polytope $\mathbb{R}^n_{\geqslant 0}$ by scalings and $\mathrm{GL}_n(\mathbb{Z})$-affine maps if and only if Ω has a smooth vertex.

EXAMPLE 4.8. Considering ellipsoids $X_\Omega = E(a, b)$ and the corresponding toric varieties $\mathbb{P}(1, a, b)$, we recover the (well-)known compactifications

$$\mathbb{P}^2 \backslash H = B(1)^\circ \text{ and } \mathbb{P}(1, a, b) \backslash H = E(a, b)^\circ$$

where $H = \mathcal{O}(1)$ is a hyperplane section in each variety respectively.

4.3. Axioms of c^{alg} for toric surfaces. This section is devoted to proving that the algebraic capacities of toric surfaces satisfy a set of important formal properties.

THEOREM 4.9. *Let Y_Ω be a projective toric surface with moment polytope Ω and associated ample divisor A_Ω. Then the kth algebraic capacity satisfies the following axioms.*

(a) (Scaling/Affine Maps) If $S > 0$ is a constant and $T : \mathbb{Z}^2 \to \mathbb{Z}^2$ is an affine isomorphism, then

$$c_k^{\mathrm{alg}}(Y_{S\Omega}, A_{S\Omega}) = S \cdot c_k^{\mathrm{alg}}(Y_\Omega, A_\Omega) \quad \text{and} \quad c_k^{\mathrm{alg}}(Y_{T\Omega}, A_{T\Omega}) = c_k^{\mathrm{alg}}(Y_\Omega, A_\Omega)$$

(b) (Inclusion) If $\Omega \subset \Delta$ is an inclusion of moment polytopes, then

$$c_k^{\mathrm{alg}}(Y_\Omega, A_\Omega) \leqslant c_k^{\mathrm{alg}}(Y_\Delta, A_\Delta)$$

(c) (Blow Up) If $\pi : Y_{\tilde{\Omega}} \to Y_\Omega$ is a birational toric morphism with one exceptional fiber E and associated ample divisor $A_{\tilde{\Omega}} = \pi^* A_\Omega - \varepsilon E$ for $\varepsilon > 0$ small, then

$$c_k^{\mathrm{alg}}(Y_{\tilde{\Omega}}, A_{\tilde{\Omega}}) \leqslant c_k^{\mathrm{alg}}(Y_\Omega, A_\Omega)$$

(d) (Embeddings) If $X \subset \mathbb{R}^4$ be a star-shaped domain that symplecically embed into Y_Ω, then

$$c_k^{\mathrm{ECH}}(X) \leqslant c_k^{\mathrm{alg}}(Y_\Omega, A_\Omega)$$

(e) (Domains) If Ω is a (convex or free) domain polytope and X_Ω is the associated toric domain, then

$$c_k^{\mathrm{ECH}}(X_\Omega) = c_k^{\mathrm{alg}}(Y_\Omega, A_\Omega)$$

Furthermore, axioms (a)-(e) uniquely characterize the algebraic capacities c_k^{alg} on toric surfaces.

Proof. We will need some of these properties to prove the others, so we must proceed in a particular order. We first prove (a), (c) and (e) which are mutually independent. We then apply these properties to acquire (b) and apply Theorem 3.5 to acquire (d).

(a) - Scaling/Affine Maps. First, note that a toric domain transforms as $Y_{S\Omega} = Y_\Omega$ and the divisor transforms as $A_{S\Omega} = S \cdot A_\Omega$. So the scaling axiom follows from Definition 3.2.

Next, we must show invariance if T is either linear or a translation. If $T \in \mathrm{GL}_2(\mathbb{Z})$ is linear, then T is an automorphism on the Lie algebra $\mathbb{R}^2 \simeq \mathfrak{t}^2$ of $\mathbb{T}^2$ induced by a group automorphism of $\mathbb{T}^2$. Thus (Y_Ω, A_Ω) and $(Y_{T\Omega}, A_{T\Omega})$ are identical after pulling back by this automorphism, and the algebraic capacities must agree. If T is a translation then $Y_\Omega = Y_{T\Omega}$ and $A_\Omega = A_{T\Omega} + R$ where R is a principle divisor determined by T. On the other hand, $A_\Omega \cdot D$ for a divisor D depends only on the divisor class of A_Ω, and so invariance follows from Definition 3.2.

(c) - Blow Up. Let D be a nef $\mathbb{Q}$-divisor on Y that achieves the optimum defining $c_k^{\mathrm{alg}}(Y, A)$, i.e.

$$c_k^{\mathrm{alg}}(Y, A) = D \cdot A \text{ and } \chi(D) \geqslant k + 1$$

Consider the proper transform $\pi^* D$ of D on $\widetilde{Y}$, which is nef. This has $\chi(\pi^* D) = \chi(D) \geqslant k + 1$. Therefore, the algebraic capacities satisfy

$$c_k^{\mathrm{alg}}(Y_{\widetilde{\Omega}}, A_{\widetilde{\Omega}}) \leqslant \pi^* D \cdot A_{\widetilde{\Omega}} = \pi^* D \cdot (\pi^* A_\Omega - \varepsilon E) = D \cdot A_\Omega = c_k^{\mathrm{alg}}(Y_\Omega, A_\Omega)$$

(e) - Domains. This is simply a restatement of Theorem 4.15 and Theorem 4.18 of [**80**], which state that if Ω is is a convex domain polytope or a convex free polytope, then

$$c_k^{\mathrm{ECH}}(X_\Omega) = \inf_{D \in \mathrm{nef}(Y_\Omega)_{\mathbb{Q}}} \{D \cdot A_\Omega : h^0(D) \geqslant k + 1\} \tag{4.2}$$

This result is phrased in terms of $\mathbb{Q}$-divisors, and also uses global sections instead of the Euler characteristic. However, since Y_Ω is toric we have Demazure vanishing.

Lemma 4.10 ([**19**, Thm. 9.3.5.]). *Suppose Y is a toric surface and D is a nef $\mathbb{Q}$-divisor. Then*

$$h^p(D) = 0 \text{ for all } p > 0$$

Thus $h^0(D) = \chi(D)$. Moreover, we have the following Lemma (see [**81**, Lem. 2.1]).

Lemma 4.11. *Let D be a nef $\mathbb{Q}$-divisor on Y_Ω. Then there exists a nef $\mathbb{Z}$-divisor with*

$$h^0(D') = h^0(D) \qquad A_\Omega \cdot D' \leqslant A_\Omega \cdot D$$

Proof. Without loss of generality assume D is a torus-invariant divisor and let $D = \sum a_F D_F$. Consider the round-down of D, defined by

$$\lfloor D \rfloor := \sum \lfloor a_F \rfloor D_F$$

which is a $\mathbb{Z}$-divisor with $P_D \cap \mathbb{Z}^n = P_{\lfloor D \rfloor} \cap \mathbb{Z}^n$. The difference $D - \lfloor D \rfloor$ is effective and so $\lfloor D \rfloor \cdot A \leqslant D \cdot A$. Unfortunately, $\lfloor D \rfloor$ may not be nef.

To fix this, we modify $\lfloor D \rfloor$ to a nef divisor D' by translating some of the hyperplanes $H_F = \{x | \langle u_F, x \rangle \geqslant -\lfloor a_F \rfloor\}$ (see §4.1) for $\lfloor D \rfloor$ inwards if necessary. (Here we are using the nef criterion discussed in §4.1.) This is equivalent to subtracting some integer multiple of the prime divisor D_F and hence only reduces the area. We must also translate each hyperplane only until it meets a lattice point in $P_{\lfloor D \rfloor}$ for $\lfloor D \rfloor$, so that

$h^0(D') = h^0(\lfloor D \rfloor)$. Note that every lattice point in $\mathbb{Z}^n$ is in one of the translates of H_F, for each F, so we can always perform this translation process while ensuring that $P_{D'} \cap \mathbb{Z}^n = P_D \cap \mathbb{Z}^n = P_{\lfloor D \rfloor} \cap \mathbb{Z}^n$. In particular, $h^0(D') = h^0(D)$. □

Lemmas 4.10 and 4.11 together imply that the following two infima are equal.

$$\inf_{D \in \mathrm{nef}(Y_\Omega)_{\mathbb{Q}}} \{D \cdot A_\Omega : h^0(D) \geqslant k+1\} = \inf_{D \in \mathrm{nef}(Y_\Omega)_{\mathbb{Z}}} \{D \cdot A_\Omega : \chi(D) \geqslant k + \chi(\mathcal{O}_Y)\}$$

In view of (4.2) and Definition 3.2, we conclude that $c_k^{\mathrm{ECH}}(X_\Omega) = c_k^{\mathrm{alg}}(Y_\Omega, A_\Omega)$.

(b) - Inclusion. Let $\Omega \subset \Delta$ be an inclusion of moment polytopes. By the application of an affine transformation $T : \mathbb{Z}^2 \to \mathbb{Z}^2$ to both Ω and Δ, we may assume that Ω and Δ are in $(0,\infty)^2 \subset \mathbb{R}^2$, and thus are convex free polytopes. By (e) and the fact that $X_\Omega \subset X_\Delta$, we have

$$c_k^{\mathrm{alg}}(Y_\Omega, A_\Omega) = c_k^{\mathrm{ECH}}(X_\Omega) \leqslant c_k^{\mathrm{ECH}}(X_\Delta) = c_k^{\mathrm{alg}}(Y_\Delta, A_\Delta)$$

(d) - Embeddings. Let $X \to Y_\Omega$ be a symplectic embedding of a star-shaped domain. If Y_Ω has no singularities (i.e. no singular fixed points), this is simply Theorem 3.5. Otherwise, since X is a smooth and compact, its image misses the singular fixed points. Thus we can take a toric resolution $\pi : Y_{\widetilde{\Omega}} \to Y_\Omega$, where $\widetilde{\Omega}$ is acqurired from Ω by cutting off small triangles from the singular corners. For sufficiently small cuts, $Y_{\widetilde{\Omega}}$ inherits an embedding $X \to Y_{\widetilde{\Omega}}$ and thus we have

$$c_k^{\mathrm{ECH}}(X) \leqslant c_k^{\mathrm{alg}}(Y_{\widetilde{\Omega}}, A_{\widetilde{\Omega}}) \leqslant c_k^{\mathrm{alg}}(Y_\Omega, A_\Omega)$$

Here we apply either the blow up axiom (c) or the inclusion axiom (b).

Uniqueness. Finally, to argue that these axioms uniquely determine c_k^{alg}, let d_k^{alg} be another family of numerical invariants satisfying axioms (a)-(e). The blow up and inclusion axioms imply that c_k^{alg} and d_k^{alg} agree if and only if they agree on all polytopes Ω such that Y_Ω is non-singular. Any such polytope is equivalent to a domain polytope by scaling and affind transformation, so by (a) we merely need to check those polytopes. Then (e) implies that the invariants must agree for those polytopes. □

Remark 4.12. Theorem 3.5 and the blow up property (c) can be used together to give an indendent proof of the upper bound of the ECH capacities by the algebraic capacities in Theorem 4.15 of **[80]**. However, we are not aware of a proof that establishes a lower bound which is not essentially equivalent to the one provided in **[80]**. A fundamentally different proof could potentially shed light on an approach to Conjecture 1.13.

4.4. Embeddings to toric surfaces. We now prove the main applications of this part, which are easy consequences of the axioms in Theorem 4.9. We start by showing that the algebraic capacities are complete obstructions for embeddings of the interiors of concave toric domains into a toric surfaces, in terms of c^{ECH} and c^{alg}.

Theorem 4.13. *Let X_Δ be a concave toric domain and let (Y_Ω, A_Ω) be a projective toric surface with a smooth fixed point. Then*

$$X_\Delta^\circ \textit{ symplectically embeds into } Y_\Omega \quad \Longleftrightarrow \quad c_k^{\mathrm{ECH}}(X_\Delta) \leqslant c_k^{\mathrm{alg}}(Y_\Omega, A_\Omega)$$

Proof. Suppose that $X_\Delta^\circ \to Y_\Omega$ is a symplectic embedding, and let X_i be an exhaustion of X_Δ° by star-shaped domains. Then

$$c_k^{\mathrm{ECH}}(X_\Delta) = \lim_{i\to\infty} c_k^{\mathrm{ECH}}(X_i) \leqslant c_k^{\mathrm{alg}}(Y_\Omega, A_\Omega)$$

On the other hand, suppose that $c_k^{\mathrm{ECH}}(X_\Delta) \leqslant c_k^{\mathrm{alg}}(Y_\Omega, A_\Omega)$. Since Y_Ω has a torus fixed point, we can scale by an $S > 0$ and apply an affine map $T : \mathbb{Z}^2 \to \mathbb{Z}^2$ so that $TS(\Omega)$ is a convex domain polygon for convex toric domain $X_{TS(\Omega)}$. Applying axioms (a) and (e) of Theorem 4.9, we acquire

$$c_k^{\mathrm{ECH}}(X_{S\Delta}) \leqslant c_k^{\mathrm{alg}}(Y_{TS(\Omega)}, A_{TS(\Omega)}) = c_k^{\mathrm{ECH}}(X_{TS(\Omega)})$$

Now we apply a well-known result [**20**, Thm. 1.2] of Cristofaro-Gardiner stating that a concave toric domain $X_{S\Delta}$ embeds into a convex toric domain $X_{TS(\Omega)}$ if and only if the ECH capacities of $X_{S\Delta}$ are bounded by those of $X_{TS(\Omega)}$. Thus we acquire a symplectic embedding

$$X_{S\Delta}^\circ \to X_{TS(\Omega)}^\circ \subset Y_{TS(\Omega)} \simeq Y_{S\Omega}$$

Since scaling the moment image merely scales the symplectic form accordingly, we thus acquire a symplectic embedding $X_\Delta^\circ \to Y_\Omega$. □

Corollary 4.14. *Let $\Omega \subset \Delta$ be an inclusion of moment polygons, each of which has a smooth vertex. Then the Gromov widths satisfy*

$$c_G(Y_\Omega) \leqslant c_G(Y_\Delta)$$

In particular, c_G is monotonic with respect to inclusions of the moment polytope for smooth toric surfaces.

Proof. Let $B(r) \to Y_\Omega$ be a symplectic embedding of a closed ball of symplectic radius r. Then by the embedding axiom and inclusion axiom in Theorem 4.9, we have

$$c_k^{\mathrm{ECH}}(B(r)) \leqslant c_k^{\mathrm{alg}}(Y_\Omega, A_\Omega) \leqslant c_k^{\mathrm{alg}}(Y_\Delta, A_\Delta)$$

Thus by Theorem 4.13, we have an embedding $B^\circ(r) \to Y_\Delta$ of the open ball of symplectic radius r, so $r \leqslant c_G(Y_\Delta)$. Taking the sup over all such embeddings $B(r) \to Y_\Omega$ yields $c_G(Y_\Omega) \leqslant c_G(Y_\Delta)$. □

In fact, we can prove a more general result than Corollary 4.14. Namely, given a moment image Ξ for a concave toric domain and a symplectic manifold Y, define the *Ξ-width* $c_\Xi(Y)$ by

$$c_\Xi(Y) := \sup\{r \ : \ X_{r\Xi} \text{ embeds symplectically into } Y\}$$

Then by the same argument as in Corollary 4.14, we have the following result.

Corollary 4.15. *Let $\Omega \subset \Delta$ be an inclusion of moment polygons, each of which has a smooth vertex. Then*

$$c_\Xi(Y_\Omega) \leqslant c_\Xi(Y_\Delta)$$

Remark 4.16. It seems that one can also execute the proof of Corollary 4.15 using only the fact that a ball $B(r)$ embeds into X_Ω if and only if it embeds into Y_Ω (see [**21**, Thm 1.2]) and the inclusion axiom (b) of Theorem 4.9. However, this would not cover any singular surfaces, and furthermore the stronger Corollary 4.15 requires the results of this part of this thesis.

A consequence of Theorem 4.13 is that the Ξ-width of a convex toric domain X_Ω where Ω has rational slopes agrees with the Ξ-width of the toric surface Y_Ω.

Corollary 4.17. *Suppose Ω is a convex domain with rational slopes. Then*

$$c_\Xi(X_\Omega) = c_\Xi(Y_\Omega)$$

4.5. Gromov width and lattice width. We use Corollary 4.14 to provide a combinatorial upper bound for the Gromov width of a toric surface as conjectured in **[9]**. We recall the definition of the lattice width.

Definition 4.18. The *lattice width* $w(\Omega)$ of a moment polytope is defined by

$$w(\Omega) := \min_{l \in \mathbb{Z}^n \setminus 0} \left(\max_{p,q \in \Omega} \langle l, p - q \rangle \right)$$

Corollary 4.19. *Let Ω be a moment polygon with a smooth vertex. Then $c_G(X_\Omega) \leqslant w(\Omega)$.*

Proof. We implement the heuristic argument in [**9**, Rmk 3.13] rigorously. Let $l \in \mathbb{Z}^2 \setminus 0$ be the vector such that

$$w(\Omega) = \sup_{p,q \in \Omega} |\langle l, p - q \rangle|$$

We can choose an element $A \in \mathrm{GL}_2(\mathbb{Z})$ such that $(A^T)^{-1}(l) = e = (1,0)$ is the x-basis vector. This implies that

$$\langle e, A(p-q) \rangle = \langle (A^{-1})^T l, A(p-q) \rangle = \langle l, p-q \rangle = w(\Omega) = w(A\Omega)$$

Thus the lattice width of $A\Omega$ is achieved in the direction of e. We can thus fit $A\Omega$ in a rectangle R of width $a_1 = w(\Omega)$ and very large height $a_2 \gg a_1$. Since $A\Omega \subset R$, we apply Corollary 4.19 to acquire the inequality

$$c_G(Y_\Omega) = c_G(Y_{A\Omega}) \leqslant c_G(Y_R)$$

On the other hand, $Y_R \simeq \mathbb{P}^1(a_1) \times \mathbb{P}^1(a_2)$ and since $a_2 \gg a_1$, we have that $c_G(Y_R) = a_1 = w(\Omega)$. $\square$

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