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On Adding a Mean Structure to a Covariance Structure Model

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Abstract

The vast majority of structural equation models contain no mean structure, that is, the population means are estimated at the sample means and are then eliminated from modeling consideration. Generalized least squares methods are proposed to estimate potential mean structure parameters and to evaluate whether the given model can be successfully augmented with a mean structure. A simulation evaluates the performance of some alternative tests. A method that takes variability due to estimation of covariance structure parameters into account in the mean structure estimator, as well as in the weight matrix of the generalized least squares function, performs best. In small samples, the *F*-test and Yuan-Bentler adjusted χ^2 test perform best.

On Adding a Mean Structure to a Covariance Structure Model

In regression, path analysis, confirmatory factor analysis and general structural equation systems, it is possible to consider models whose parameters reproduce the covariance matrix of the variables, or, a more difficult challenge, to consider a model whose parameters reproduce not only the covariance matrix but also the means of the measured variables. Although structured means models were introduced into the literature about 30 years ago (e.g., Sörbom, 1974), extended technically in various ways (e.g., Meredith & Tisak, 1990), and have become popular in specialized contexts such as growth curve modeling (e.g., Duncan & Duncan, 1996) and multiple sample models, they are rarely invoked in single sample modeling practice. Certainly such models can be hard to fit, that is, they often are too restricted to accurately reproduce real data. Perhaps because of this, structural modeling analyses are typically limited to covariance structure analysis (e.g., Bentler & Dudgeon, 1996; Bollen, 1989; Hoyle, 1995). In this note, covariance structure analysis is proposed to be augmented with a post-modeling estimator of structured means and a test that can help a researcher evaluate whether it might make sense to consider adding a mean structure to the covariance structure model. For simplicity, the confirmatory factor analysis model is used for illustration.

The Factor Model

Consider the *k*-factor model for a random vector x of *p* observed variables

$$x = \mu + \Lambda\xi + \epsilon, \tag{1}$$

where μ is a constant vector, Λ is a *p x k* factor loading matrix, ξ is the vector of common factors, and ϵ is the vector of unique or error factors. In the standard application of confirmatory factor analysis, the common and unique factor means are taken to be zero, that is, $E(\xi)=0$ and $E(\epsilon)=0$. Then, taking expectations leads to

$$E(x) = \mu. \tag{2}$$

In general, the covariance structure of the data is given as

$$\Sigma = E(x - E(x))(x - E(x))' = \Lambda\Phi\Lambda' + \Psi, \tag{3}$$

where Σ is the covariance matrix of the observed variables x, Φ is the covariance matrix of the ξ, and Ψ is the covariance matrix of the ϵ. Although μ is a parameter of the model, in standard models it is not very interesting and is treated as a nuisance parameter. As a result, (2) is estimated at the sample mean $\bar{x}$ and is subsequently ignored. Modeling is based on the sample covariance matrix S. The free parameters of the structure (3) are estimated by some general method such as maximum likelihood to yield $\hat{\Lambda}$, $\hat{\Phi}$, and $\hat{\Psi}$. Conceptually, we may collect all the free parameters into a vector θ, and, when evaluating the significance of parameters, base this on the asymptotic distribution of the estimator $\hat{\theta}$, given by

$$n^{\frac{1}{2}}(\hat{\theta} - \theta) \xrightarrow{\mathcal{L}} \mathfrak{N}(0, \Omega). \tag{4}$$

With normal data and an asymptotically efficient parameter estimator, Ω is the inverse of the information matrix. Typically, a chi-square test also is used to evaluate the model structure (3). With normal data and maximum likelihood, this is the likelihood ratio test. Computations of these statistics completes the technical part of the usual covariance structure analysis. The meaning of the parameters and the model as a whole is the focus of subsequent evaluation by subject-matter specialists.

The structured means model does not require the assumption that $E(\xi) = 0$, and replaces (2) with the structure

$$E(x) = \mu + \Lambda\mu_\xi, \tag{5}$$

where μ_ξ is a vector of factor means. In a single sample, this model is not identified, since there are p observed means on the left side of (5) but p plus k means on the right side of (5). Hence, to achieve identification, the convenient assumption $\mu = 0$ is made, so that

$$E(x) = \Lambda\mu_\xi. \tag{6}$$

This is the standard structured means modeling hypothesis. Under this hypothesis, the observed variable means are a linear combination of the factor means, with weights given by Λ. However, the parameters μ_ξ are not available when the covariance structure model (3) is estimated. Stated differently, when only (3) is modeled, the structured means hypothesis (6) cannot be evaluated. It will be seen that a small amount of added effort can provide information about (6).

The Distribution of a Residual

The problems of obtaining an efficient estimator of μ_ξ and also of obtaining a test statistic that can be used to evaluate the hypothesis (6) are closely related. We approach these problems using standard asymptotic statistical theory, based on the large sample distribution of $n^{\frac{1}{2}}(\overline{x} - \hat{\Lambda}\mu_\xi)$. First we decompose this expression into two components

$$n^{\frac{1}{2}}(\overline{x} - \hat{\Lambda}\mu_\xi) = n^{\frac{1}{2}}(\overline{x} - \Lambda\mu_\xi) - n^{\frac{1}{2}}(\hat{\Lambda}\mu_\xi - \Lambda\mu_\xi). \tag{7}$$

The distribution of the first part on the right is well known under the null hypothesis (6). It is just the distribution of the sample mean, that is,

$$n^{\frac{1}{2}}(\overline{x} - \Lambda\mu_\xi) \xrightarrow{\mathcal{L}} \mathfrak{N}(0, \Sigma), \tag{8}$$

which is asymptotically normal with covariance matrix Σ for an appropriately large sample of size n. The second part can be rewritten as

$$n^{\frac{1}{2}}(\hat{\Lambda}\mu_\xi - \Lambda\mu_\xi) = n^{\frac{1}{2}}\mu_\xi^* vec(\hat{\Lambda} - \Lambda)$$

where $\mu_\xi^* = (\mu_\xi' \otimes I_p)$ and the vec operator vectorizes the subsequent matrix. The asymptotic distribution of this expression depends on the distribution of the elements of Λ that are free parameters. If the vector of free parameters is denoted as λ, then covariance structure analysis under model (3) already has provided their asymptotic distribution as

$$n^{\frac{1}{2}}(\widehat{\lambda} - \lambda) \xrightarrow{\mathcal{L}} \mathfrak{N}(0, \Omega_{\lambda\lambda}),$$

where $\Omega_{\lambda\lambda}$ is the appropriate submatrix of Ω. Since $vec(\Lambda)$ contains the free parameters λ as well as some fixed or known elements that have no variance, the distribution of

$$n^{\frac{1}{2}} vec(\hat{\Lambda} - \Lambda) \xrightarrow{\mathcal{L}} \mathfrak{N}(0, \Omega_{\lambda\lambda}^*) \ ,$$

where $\Omega_{\lambda\lambda}^*$ is the matrix $\Omega_{\lambda\lambda}$ augmented by appropriate rows and columns of zeros corresponding to the fixed parameters in $vec(\Lambda)$. Thus the asymptotic distribution of the second right-hand part of (7) is given by

$$n^{\frac{1}{2}}\mu_\xi^* vec(\hat{\Lambda} - \Lambda) \xrightarrow{\mathcal{L}} \mathfrak{N}(0, \mu_\xi^* \Omega_{\lambda\lambda}^* \mu_\xi^{*\prime}). \tag{9}$$

The results in (8) and (9) can now be combined. In this paper it will be assumed that the mean parameters in (8) and the function of the factor loadings (9), derived from the covariance matrix, are asymptotically independent. This will occur in the important special case that the data are multivariate normally distributed. As a result,

$$n^{\frac{1}{2}}(\overline{x} \ - \ \hat{\Lambda}\mu_\xi) \xrightarrow{\mathcal{L}} \mathfrak{N}(0, \Upsilon) \tag{10}$$

where $\Upsilon = (\Sigma + \mu_\xi^* \Omega_{\lambda\lambda}^* \mu_\xi^{*\prime})$. This provides the basis for factor mean estimation and a test of the mean structure hypothesis.

Factor Mean Estimator and Mean Structure Test

In order to obtain an estimator of μ_ξ, an obvious procedure is to minimize the generalized least squares function

$$n(\ \overline{x} \ - \hat{\Lambda}\mu_\xi)' W (\ \overline{x} \ - \hat{\Lambda}\mu_\xi),$$

using some logical choice for the weight matrix W. An optimal choice would be $W = \widehat{\Upsilon}^{-1}$, where $\widehat{\Upsilon}$ is a consistent estimator of Υ. However, Υ depends on the unknown μ_ξ. In order to make progress, we will obtain an initial consistent estimator of μ_ξ, and then use an optimal updating method to obtain a final fully efficient estimator. See, e.g., Bentler and Dijkstra (1985) for a discussion of two-step estimators in the context of structural modeling. In the first stage, the simplest procedure would be to use $W = I$, yielding the simple least squares estimator for μ_ξ as $(\hat{\Lambda}'\hat{\Lambda})^{-1}\hat{\Lambda}'\overline{x}$. A more efficient choice is to consider $W = \hat{\Sigma}^{-1}$, using the covariance structure estimator for Σ. This yields the quadratic form

$$T = n(\overline{x} - \hat{\Lambda}\mu_\xi)'\hat{\Sigma}^{-1}(\overline{x} - \hat{\Lambda}\mu_\xi), \tag{11}$$

which, when minimized with respect to μ_ξ, yields the estimator

$$\tilde{\mu}_\xi = (\hat{\Lambda}'\hat{\Sigma}^{-1}\hat{\Lambda})^{-1}\hat{\Lambda}'\hat{\Sigma}^{-1}\overline{x}. \tag{12}$$

This is a consistent estimator of the factor means, based only on the estimated covariance structure model and the sample means. It is interesting to note that if $\overline{x}$ were replaced by a particular score vector x in (12), the generalized least squares factor score estimator developed by Bentler and Yuan (1997) is obtained. However, in general $\tilde{\mu}_\xi$ is not fully efficient, since (11) is based on a misspecified weight matrix, that is, $\hat{\Sigma}^{-1}$ is not consistent for Υ^{-1}. However, in contrast to the least squares estimator, it will be efficient if the asymptotic covariance matrix in (9) is degenerate. This occurs, for example, when Λ contains no free parameters. While rare, this is not an unheard of modeling situation, since it will occur with specialized models such as the intraclass correlation model or standard growth curve models with fixed growth coefficients. In such a case, $\widehat{T}$ based on substituting (12) in (11) will have an asymptotic χ^2_{p-k} distribution, where k is the number of factors. In general, however, the distribution of $\widehat{T}$ can be represented by a mixture of chi-square variates (e.g., Satorra & Bentler, 1994), but this is difficult to deal with and can be circumvented.

In the second stage, a consistent estimator for Υ based on (12) is specified, and the updated final estimator $\widehat{\mu}_\xi$ is obtained. Specifically, $\widehat{\Sigma}$ and $\widehat{\Omega}^*_{\lambda\lambda}$ from the covariance structure analysis, along with $\tilde{\mu}_\xi$ from the first stage analysis, are used to obtain the consistent estimator

$$\hat{\Upsilon} = (\widehat{\Sigma} + \tilde{\mu}^*_\xi\widehat{\Omega}^*_{\lambda\lambda}\tilde{\mu}^{*\prime}_\xi), \tag{13}$$

where $\tilde{\mu}_\xi$ is used to generate $\tilde{\mu}^*_\xi$. This allows specification of the generalized least squares function

$$T_{\mu|\Sigma} = n(\overline{x} - \hat{\Lambda}\mu_\xi)'\hat{\Upsilon}^{-1}(\overline{x} - \hat{\Lambda}\mu_\xi), \tag{14}$$

which is minimized to yield the final factor score estimator

$$\widehat{\mu}_{\xi} = (\hat{\Lambda}'\hat{\Upsilon}^{-1}\hat{\Lambda})^{-1}\hat{\Lambda}'\hat{\Upsilon}^{-1}\overline{x}. \tag{15}$$

The corresponding test statistic

$$\hat{T}_{\mu|\Sigma} = n(\overline{x} - \hat{\Lambda}\hat{\mu}_{\xi})'\hat{\Upsilon}^{-1}(\overline{x} - \hat{\Lambda}\hat{\mu}_{\xi}) \tag{16}$$

is, under the null hypotheses of a correctly specified covariance structure model (3) and the structured means model (5), asymptotically distributed as χ^2_{p-k}, where k is the number of factors. This can be shown using the usual minimum chi-square arguments (see, e.g., Ch. 23 of Ferguson, 1996). When $\hat{T}_{\mu|\Sigma}$ is large compared to degrees of freedom, the structured means hypothesis (5) can be rejected, but if the probability associated with the χ^2 is not too small, the structured means hypothesis is plausible. In that case, an investigator may consider the simultaneous estimation of the mean and covariance structure model to obtain an overall model test.

In principle, the estimator (15) could be used to update the estimated asymptotic covariance matrix (13), and the process (13)-(15) could be cycled through repeatedly until some convergence criterion is reached. Conceivably this might be helpful in very small samples, but, as noted for example by Bentler and Dijkstra (1985), asymptotically there is no advantage to doing so. Furthermore, at some point the computations become so heavy that one may as well directly and simultaneously estimate the covariance as well as mean structure. The whole point of the current development is to have a simple way to evaluate whether this might be worthwhile.

Alternative Test Statistics

The form of $T_{\mu|\Sigma}$ is suspiciously like that of a Hotelling T^2 variate, for which typical practice is to use the sample covariance matrix S in place of $\hat{\Sigma}$. Substituting this consistent estimator in (11) and (13), for example, is permissible and yields a closely related test statistic $T_{\mu|S}$ that has the same large sample properties as $T_{\mu|\Sigma}$. It also follows that it may be useful to evaluate the null hypothesis (5) using the transformed variate

$$\hat{F}_{\mu|\Sigma} = (n - p + k)\hat{T}_{\mu|\Sigma}/\{(p - k)(n - 1)\} \tag{17}$$

referred to the $F_{(p-k),(n-p+k)}$ distribution. In the context of covariance structure analysis, the F test performs very well (Yuan & Bentler, in press), but because of the lower degrees of freedom in this mean structure context, any advantage over evaluating $T_{\mu|\Sigma}$ using the χ^2_{p-k} distribution may be minimal. And of course $T_{\mu|S}$ may be used in (17) in place of $T_{\mu|\Sigma}$, yielding the comparable F statistic $F_{\mu|S}$.

Simulations

The four variable covariance matrix with means, given in Table 1, can be generated by a one-factor model as specified in (1), (3) and (5). The factor loading parameters for the variables in sequence are 1.0 (fixed), .6, .7, and .8, and all unique variances are 1. The factor mean is 3, and its variance is 1. Four sets of simulations were done with this model, each based on 500 repeated drawings of a given sample from a normal population having the parameters described in Table 1. The process was repeated at each of five sample sizes, 100, 200, 400, 800, and 1600.

Simulation 1. This study evaluates the two-stage estimation process that culminates in the factor mean estimator (15) and the test statistic (16). Since there are four sample means, and there is one factor mean, the test statistic should be distributed as a chi-square variate with 3 degrees of freedom. Results of the simulation are given in Table 2. Each row gives the results for samples of a given sample size. The columns give the number of rejections of the null hypothesis at α=.05 across the 500 replications, the mean and standard deviations of the statistic $\hat{T}_{\mu|\Sigma}$, and then the corresponding information based on the F-test $\hat{F}_{\mu|\Sigma}$. The number of rejections should be approximately 25 to show nominal performance. There is a slight tendency to overreject the structured means hypothesis at the smallest sample size, but asymptotic performance certainly can be seen with 800 observations. Since the mean of a χ^2 variate is the degrees of freedom, the sample means of $\hat{T}_{\mu|\Sigma}$ should be approximately 3, while the standard deviation should be $\sqrt{6}$=2.449. The means as well as standard deviations across the 500 replications are somewhat too large at smallest sample sizes, but performance on these measures is also good with 800 or more observations.

At the smallest sample size, the performance of the *F*-test is superior to that of the χ^2 variate in terms of number of rejections, but at the larger sample sizes it is virtually identical. Improved small sample performance by the *F*-test is similar to that observed for covariance structure model tests by Yuan and Bentler (in press). More specifically, the mean and variance of an *F*-variate are given by

$$E(F_{n_1,n_2}) = \frac{n_2}{n_2-2} \quad \text{and } Var(F_{n_1,n_2}) = \frac{2n_2^2(n_1+n_2-2)}{n_1(n_2-2)^2(n_2-4)},$$

respectively. For sample sizes 100, 200, 400, 800, and 1600, these formulas yield means and standard deviations of $F_{3,n-3}$, respectively as:

$$E(F) = 1.021,\ 1.010,\ 1.005,\ 1.003,\ 1.001$$
$$Std(F) = .856,\ .835,\ .825,\ .821,\ .819.$$

Comparing these theoretical values to the empirical results shown in the right part of Table 2, at the smaller sample sizes both the empirical means and standard deviations are a little bit larger than those of the theoretical *F*-distribution. Similar performance for an *F*-test was found in a different modeling context by Yuan and Bentler (in press).

Simulation 2. Although the simulation results of Table 2 verify that our χ^2 approach seems to be asymptotically valid, in small samples the test means and standard deviations are too large. Such a bias to a test statistic has been observed for the asymptotically distribution free covariance structure test statistic. Yuan and Bentler (1997) proposed a simple correction to this test statistic that works substantially better than the uncorrected statistic in realistic sized samples. In the current context, this requires computing the modified χ^2statistic

$$\hat{T}^*_{\mu|\Sigma} = \hat{T}_{\mu|\Sigma}/\{1 + \hat{T}_{\mu|\Sigma}/n\}, \qquad (18)$$

and referring it to the reference χ^2_{p-k} distribution. It will be seen that as $n \to \infty$, this becomes equivalent to (16), i.e., it is a small sample correction. As can be seen from Table 3, at the small sample sizes, this modified statistic improves on the original statistic in terms of number of rejections, while at the larger sample sizes it performs the same. It also has means and standard deviations that are closer to those expected from the reference χ^2 distribution. In fact, the test (18) performs equivalently to the *F*-test in the smallest samples (compare to Table 2).

Simulation 3. In order to get some idea of the importance of taking into account the variability in $\hat{\Lambda}$ to the accurate estimation of μ_ξ and testing a mean structure, the estimator $\tilde{\mu}_\xi$ given in (12) and the associated test statistic $\hat{T}$ derived from (11) were studied under identical conditions as in the previous studies. Perhaps if the variability given by $\tilde{\mu}^*_\xi \hat{\Omega}^*_{\lambda\lambda} \tilde{\mu}^{*\prime}_\xi$ is relatively small, it could be ignored without much worry. The results, given in Table 4, are dramatic. At all sample sizes, the true null hypothesis is overwhelmingly -- almost always -- rejected. There is no evidence that χ^2 and *F* reference distributions can be used to describe the empirical behavior of these test statistics.

Simulation 4. The results of Simulation 3 could be due to use of an inappropriate weight matrix *W* or else a poor choice of estimator of μ_ξ. In order to shed some light on these alternatives, the simulation was repeated with use of the estimator $\tilde{\mu}_\xi$ given in (12), applied to the generalized least squares function (14). Thus the weight matrix is asymptotically correct, based on (13), but the factor mean estimator is the first stage estimator that does not use an asymptotically correct weight matrix. The results are given in Table 5. The true structured means model is rejected about 40% of the time at all sample sizes when using both χ^2 and *F* reference distributions. While this is an improvement over that of Simulation 3, it still provides a radical contrast to the two-stage approach given in Simulations 1 and 2. Evidently, both the factor mean estimator and the quadratic form test statistic must be based on the consistent estimator $\hat{\Upsilon}$ of Υ given in (13) to obtain approximately nominal performance.

Discussion

Although developed in the context of the confirmatory factor analysis model, with the specific factor score hypothesis (5), the proposed two-stage approach to evaluating potential mean structures is obviously general. The Bentler-Weeks model (e.g., Bentler, 1995) describes the relation among variables ν using the relation $\nu = B\nu + \Gamma\xi$, where B and Γ are coefficient matrices, ξ is a vector of independent variables, and ν contains ξ as well as dependent variables η. The corresponding covariance structure is given by

$$\Sigma = G(I - B)^{-1}\Gamma\Phi\Gamma'(I - B)^{-1\prime}G', \tag{19}$$

where G is a known matrix selecting observed variables x from the variables ν and Φ is the covariance matrix of the ξ variables. The corresponding mean structure can be taken as

$$E(x) = G(I - B)^{-1}\Gamma\mu_{\xi}. \tag{20}$$

Clearly, the theory developed above is directly applicable to such generalized models. The covariance structure model is estimated using (19) rather than (3), and the mean structure hypothesis is given by (20) rather than (5). As can be seen from the latter comparison, the expression $G(I - B)^{-1}\Gamma$ has to replace Λ in the technical development, with the asymptotic covariances of free elements of B and Γ being used instead of those of Λ when obtaining the asymptotically correct estimator $\hat{\Upsilon}$. Λ is similarly replaced in (15) and (16) when obtaining the factor means and test statistics.

The simulations verified the critical nature that the proposed consistent estimator $\hat{\Upsilon}$ of Υ plays in these analyses. It will be obvious that the factor mean estimator (15) immediately implies a new factor score estimator $\hat{\xi} = (\hat{\Lambda}'\hat{\Upsilon}^{-1}\hat{\Lambda})^{-1}\hat{\Lambda}'\hat{\Upsilon}^{-1}x$. A more classical estimator, analogous to (12), was given by Bentler and Yuan (1997) as $\hat{\xi} = (\hat{\Lambda}'\hat{\Sigma}^{-1}\hat{\Lambda})^{-1}\hat{\Lambda}'\hat{\Sigma}^{-1}x$; this is equivalent to the well-known Bartlett estimator under standard regularity conditions. However, the Bartlett, regression, and related factor score estimators all have been derived under the assumption that Λ is known. In our application, this is clearly a dangerous assumption, and it is quite likely also to be problematic in factor score estimation. That is, it also may be important in factor score estimation to take into account sampling variability due to the fact that Λ is not known and needs to be estimated. This is a topic we will discuss in detail elsewhere.

While our simulations with a small structured means factor model showed that our proposed new tests seem to perform reasonably well under the null hypothesis, further thorough simulation studies are needed to evaluate power, to find the limitations of the proposed methodology, as well as to provide better evidence on the relative performance of the alternative χ^2 and F tests. Finally, it should be noted EQS 6 computes the two-stage statistics described here. The implementation is quite simple, as was shown, and could be incorporated into any structural modeling program.

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Table 1
Population Covariances and Means for Simulation

Covariance Matrix			
2.00			
.60	1.36		
.70	.42	1.49	
.80	.48	.56	1.64
Means			
3.00	1.80	2.10	2.40

Table 2
Simulation Results Based on Chi-Square (Eq. 16) and F (Eq. 17) Tests

	$\hat{T}_{\mu\|\Sigma}$			$\hat{F}_{\mu\|\Sigma}$		
Sample Size	Rejections	Mean	Std Dev	Rejections	Mean	Std Dev
100	48	3.646	3.184	40	1.191	1.040
200	36	3.390	2.799	35	1.119	0.924
400	40	3.340	2.704	39	1.108	0.897
800	27	3.157	2.522	27	1.050	0.838
1600	24	2.983	2.363	22	0.993	0.787

Table 3
Simulation Results Based on Modified Chi-Square (Eq. 18) Test

	$\hat{T}^{*}_{\mu\|\Sigma}$		
Sample Size	Rejections	Mean	Std Dev
100	39	3.431	2.814
200	34	3.297	2.645
400	39	3.295	2.635
800	27	3.137	2.489
1600	22	2.974	2.348

Table 4

Simulation Results Based on $\hat{T}$ and $\hat{F}$ Tests

	$\hat{T}$			$\hat{F}$		
Sample Size	Rejections	Mean	Std Dev	Rejections	Mean	Std Dev
100	478	62.496	57.054	475	20.411	18.634
200	480	58.789	49.857	480	19.399	16.452
400	469	58.837	48.250	469	19.514	16.003
800	468	55.512	44.684	468	18.458	14.857
1600	471	52.962	43.617	470	17.632	14.521

Table 5
Simulation Results Based on $\tilde{\mu}_\xi$ Applied in $T_{\mu|\Sigma}$

	$\hat{T}_{\mu\|\Sigma}(\tilde{\mu}_\xi)$			$\hat{F}(\tilde{\mu}_\xi)$		
Sample Size	Rejections	Mean	Std Dev	Rejections	Mean	Std Dev
100	204	11.500	17.515	197	3.756	5.720
200	193	10.144	12.317	190	3.347	4.065
400	204	10.035	11.626	202	3.328	3.856
800	193	9.041	9.556	192	3.006	3.177
1600	190	8.858	9.935	189	2.949	3.308