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TEMPORAL METRIC SPACES IN RADIATIVE TRANSFER THEORY

V. Local Temporal Diameters

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Temporal Metric Spaces in Radiative Transfer Theory

V. Local Temporal Diameters

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INTRODUCTION

In this paper, the last of the present series on temporal metric spaces, we conclude our study of temporal diameters begun in paper IV with a detailed look at their local counterparts: the local temporal diameters. The temporal diameters describe the time required for a transient n-ary light field to come to steady state throughout a bounded medium. This time was found to depend, among other things, on the location of the source p^* within the medium. In this paper we consider, in addition to p^*, a receiving point $p^\#$, and inquire: for a given p^* and a given $p^\#$, at what time will the n-ary radiance field attain steady state at $p^\#$? Apparently this time depends on the location of not only p^*, but also that of $p^\#$, and to point up this fact, we will designate the n-ary local temporal diameter by $T_n(p^*, p^\#)$.

The main emphasis here as in all the preceding papers of this series, is on the study of the time-dependent radiant flux problem with the emphasis on basic knowledge rather than immediate applications.

As in paper IV, the present geometric setting is an arbitrary closed bounded optically convex subset Φ of $E_3 \times \Xi$ with respect to a temporal metric t. But as promised in IV, we will also briefly consider the problem of defining temporal diameters of unbounded carrier spaces. At the conclusion of this paper we will have fulfilled the program of investigation outlined in paper I.

LOCAL n-ARY TEMPORAL DIAMETER

We continue to use the notation established in paper IV. Specifically, we now consider the subset S'_{n-1} of S_{n-1} which consists of all points of S_{n-1} whose ith and (i+1)th coordinates are adjacent in $\Phi \times \Phi$. That is, S'_{n-1} is the collection of all of the following special kind of (n-1)-tuples: $\{(\rho_1,\rho_2), (\rho_2,\rho_3), \ldots, (\rho_{n-1}, \rho_n)\}$ of points of $\Phi \times \Phi$.

Definition 1. Let ρ^* be a local source in Φ, and $\rho^\#$ an arbitrary point of Φ (the general observation point). Then the local n-ary temporal diameter $T_n(\rho^*, \rho^\#)$ of Φ with respect to ρ^* and $\rho^\#$ is defined for each $n \geq 1$ as:

$$T_n(\rho^*, \rho^\#) = \sup_{S'_{n-1}} \left\{ t(\rho^*, \rho_1) + \sum_{j=1}^{n-1} t(\rho_j, \rho_{j+1}) + t(\rho_n, \rho^\#) \right\}. \tag{1}$$

It is evident that $T_n(\rho^*, \rho^\#)$ gives the least local epoch time (relative to ρ^*) at which $\rho^\#$ is receiving n-ary scattered radiant flux from all points of Φ.

If one were to stand at $\rho^{\#}$ and wait for n-ary scattered radiance from ρ^{*}, the first burst would be carried by the flux traveling along the natural path from ρ^{*} to $\rho^{\#}$. Then as time progressed, other points off the natural path would be contributing n-ary scattered flux to $\rho^{\#}$. More precisely, the subregion of Φ that is actively contributing n-ary scattered flux to $\rho^{\#}$ at local epoch time $T_{\rho^*}(\rho^{\#})$ is $\mathcal{E}(\rho^{*}, \rho^{\#}; T_{\rho^*}(\rho^{\#}))$ --the characteristic ellipsoid of the source-receiver pair $(\rho^{*}, \rho^{\#})$.

If, for example, ρ^{*} were the fundamental source, and emitted radiant flux steadily, then the primary scattered flux at $\rho^{\#}$ would build up until $T_{\rho^*}(\rho^{\#}) = -t(\rho^{*}, \rho^{\#}) + T_1(\rho^{*}, \rho^{\#})$. For $T_{\rho^*}(\rho^{\#})$ greater than this value, the primary radiation field would be in the steady state with respect to the source ρ^{*}. This would mean, geometrically, that the characteristic ellipsoid of the point pair $(\rho^{*}, \rho^{\#})$ would contain Φ as a proper subset for such times $T_{\rho^*}(\rho^{\#})$ that exceed $-t(\rho^{*}, \rho^{\#}) + T_1(\rho^{*}, \rho^{\#})$.

For the same reasons as those used in the discussion of $T(\Phi)$ and $T_n(\rho^{*})$ (namely continuity of t on $\Phi \times \Phi$) we may represent $T_n(\rho^{*}, \rho^{\#})$ by a finite sum of the kind:

$$T_n(\rho^{*}, \rho^{\#}) = t(\rho^{*}, \rho^{\#}) + \sum_{j=1}^{n-1} t(\rho_j, \rho_{j+1}) + t(\rho_n, \rho^{\#}) ,$$

This observation forms the basis for the following theorem:

<u>Theorem 1. For all $\rho^{\#} \varepsilon \Phi$ and any fixed local source $\rho^{*} \varepsilon \Phi$, $T_n(\rho^{*}, \rho^{\#}) \leq T_n(\rho^{*})$, $n = 1, 2, \ldots$</u> .

<u>Proof</u>: Using the finite sum representations of $T_n(\rho^{*})$ and $T_n(\rho^{*}, \rho^{\#})$, we have:

$$T_n(\rho^{*}) = t(\rho^{*}, \rho_1) + \cdots + t(\rho_n, \rho_{n+1}),$$

$$T_n(\rho^{*}, \rho^{\#}) = t(\rho^{*}, \rho_1') + \cdots + t(\rho_n', \rho^{\#}). \tag{2}$$

But $T_n(\rho^{*})$ is the maximum of a set of sums which includes (2) as a special case, hence the theorem follows.

<u>Corollary 1</u>: $T_n(\rho^{*}) = \sup \{ T_n(\rho^{*}, \rho^{\#}) : \rho \in \Phi \}$

<u>Corollary 2</u>: By Theorem 3, paper IV, and the preceding Theorem, we have $T_n(\rho^{*}, \rho^{\#}) \leq t_m(\rho^{*}) + (n + \frac{1}{2}) T(\Phi)$ so that $T_n(\rho^{*}, \rho^{\#})$ is bounded for all ρ^{*}, $\rho^{\#}$ and $n \geq 1$.

The preceding corollaries give explicit upper bounds on $T_n(\rho^{*}, \rho^{\#})$. In particular, Corollary 1 states that $T_n(\rho^{*})$ is never less than $T_n(\rho^{*}, \rho^{\#})$, and is generally appreciably greater than $T_n(\rho^{*}, \rho^{\#})$. However, there is an obvious upper limit to the difference between $T_n(\rho^{*})$ and $T_n(\rho^{*}, \rho^{\#})$ which is given by

<u>Theorem 2. $T_n(\rho^{*})$ cannot exceed $T_n(\rho^{*}, \rho^{\#})$ by more than $T(\Phi)$ for all n and all ρ^{*} in Φ, i.e., $T_n(\rho^{*}) \leq T_n(\rho^{*}, \rho^{\#}) + T(\Phi)$</u>.

<u>Proof</u>: Representing $T_n(\rho^*)$ as a finite sum:

$$T_n(\rho^*) = t(\rho^*, \rho_1) + \cdots + t(\rho_n, \rho_{n+1}),$$

we can use part of this representation, namely

$$t(\rho^*, \rho_1) + \cdots + t(\rho_{n-1}, \rho_n)$$

as a typical sum occurring in the definition of $T_n(\rho^*, \rho^\#)$. That is, let

$$T_n'(\rho^*, \rho^\#) = \left[t(\rho^*, \rho_1) + \cdots + t(\rho_{n-1}, \rho_n)\right] + t(\rho_n, \rho^\#).$$

Then

$$T_n(\rho^*) - T_n'(\rho^*, \rho^\#) = t(\rho_n, \rho^\#) \leq T(\Phi).$$

But $T_n(\rho^*, \rho^\#)$ is the supremum of all such values as $T_n'(\rho^*, \rho^\#)$. Hence the desired inequality follows by taking the supremum of each side of the preceding inequality over S_{n-1}'.

In analogy to Theorem 4 of paper IV, we have

<u>Theorem 3</u>. <u>If $T_n(\rho^*, \rho^\#)$ is the local n-ary temporal diameter of Φ for the point pair $(\rho^*, \rho^\#)$, then</u>

$$\lim_{n \to \infty} \frac{T_n(\rho^*, \rho^\#)}{n} = T(\Phi),$$

for all ρ^*, $\rho^\# \in \Phi$.

Proof: The method of proof follows closely that used to establish Theorem 4 of paper IV, and therefore need not be given here.

Also, in analogy to Theorem 5, paper IV, we have the corresponding result for $T_n(\rho^*, \rho^\#)$:

Theorem 4. The sequence $\{t(\rho_{n-1}, \rho_n)\}$ of summands in the finite sum representation of $T_n(\rho^*, \rho^\#)$ converges to $T(\Phi)$ for all given ρ^*, and $\rho^\#$.

Proof: Follows the proof of Theorem 5, paper IV. Observe that the summand $t(\rho_n, \rho^\#)$ is excepted, since the path associated with this extremal time is constrained to have an endpoint at the fixed point $\rho^\#$.

The following simple but useful theorem locates those points of Φ used in the finite sum representation of $T_n(\rho^*, \rho^\#)$:

Theorem 5. For spaces Φ of constant index of refraction, the points $\rho_1, \ldots, \rho_n$ in the finite sum representation of $T_n(\rho^*, \rho^\#)$ (and hence of $T_n(\rho^*)$) all lie on the boundary of Φ.

Proof: Because of the hypothesized constancy of the index of refraction, the proof reduces to a simple exercise in analytic geometry, and will be omitted here.

Examples of Local n-ary Diameters

Example 1. Let Φ be a spheroid of diameter D (in the usual metric), and let the index of refraction be a constant on Φ, (Figure 1). Let $\rho^{\#}$ be at a distance r_0 from the fundamental source ρ_0 located at the center of Φ. To calculate $T_1(\rho_0,\rho^{\#})$ we use Theorem 5 to restrict attention to the boundary of Φ. To locate the position of ρ_1 on the boundary we imagine it in motion on the boundary of Φ and then observe that $d(\rho_1,\rho^{\#})$ can increase as ρ_1 approaches the diameter of Φ defined by ρ_0 and $\rho^{\#}$. $d(\rho_1,\rho^{\#})$ attains a maximum when ρ_1 is on the endpoint of the diameter farthest from $\rho^{\#}$ (i.e., such that ρ_0 is in the segment defined by ρ_1 and $\rho^{\#}$). Hence it follows that

$$T_1(\rho_0,\rho^{\#}) = (r_0 + D)/v,$$

where v is the speed of light in Φ.

To compute $T_2(\rho_0,\rho^{\#})$ we once again use Theorem 5 to initially locate ρ_1 and ρ_2 on the boundary of Φ (Figure 2). We see that, by holding ρ_1 fixed, $d(\rho_1,\rho_2)$ is a maximum if the natural path $P(\rho_1,\rho_2)$ is coincident with a diameter of Φ. The remaining path $P(\rho_2,\rho^{\#})$ has its length maximized by requiring $P(\rho_2,\rho^{\#})$ to be in the diameter of Φ determined by ρ_0 and $\rho^{\#}$ such that ρ_0 is in $P(\rho_2, \rho^{\#})$. Thus,

$$T_2(\rho_0, \rho^{\#}) = (r_0 + 2D)/v.$$

In a similar manner, it is easy to show that

$$T_n(\rho_0, \rho^{\#}) = (r_0 + nD)/v.$$

For this special space, we see that

$$T_i(\rho_0, \rho^{\#}) - T_j(\rho_0, \rho^{\#}) = (i-j)\,T(\Phi),$$

where $T(\Phi) = D/v$.

From the particular representation of $T_n(\rho_0, \rho^{\#})$ above, we may illustrate the general contents of Theorems 3 and 4. Specifically,

$$\lim_{n\to\infty} \frac{T_n(\rho_0, \rho^{\#})}{n} = \lim_{n\to\infty}\left(\frac{r_0}{nv} + \frac{D}{v}\right) = T(\Phi)$$

and

$$t(\rho_{n-1}, \rho_n) = T(\Phi), \quad n \geq 2.$$

Example 2. Let Φ be a rectangular parallelepiped of diameter D (in the usual metric) and with constant index of refraction (Figure 3). Let the fundamental source ρ_0 be at the center of Φ . Let r_m be the maximum distance from $\rho^{\#}$ to the eight vertices of Φ . Then it is easy to see that

$$T_n(\rho_0, \rho^{\#}) = \left[r_m + \frac{(2n-1)}{2} \right] \Big/ v, \quad n \geq 1,$$

and that

$$T_i(\rho_0, \rho^{\#}) - T_j(\rho_0, \rho^{\#}) = (i-j)\, T(\Phi),$$

where $T(\Phi) = D/v$.

Again, we have particular examples of Theorems 3 and 4:

$$\lim_{n\to\infty} \frac{T_n(\rho_0, \rho^{\#})}{n} = \lim_{n\to\infty} \left[\frac{r_m}{nv} + \frac{(2n-1)D}{2nv} \right] = T(\Phi)$$

and

$$t(\rho_{n-1}, \rho_n) = T(\Phi), \quad n \geq 2 .$$

In addition, the preceding examples serve also to give simple illustrations of the corollary to Theorem 4, paper IV, which deals with the bounds on the difference $\frac{T_n(\rho^*)}{n} - T(\Phi)$. The same corollary has obvious extensions to $T_n(\rho^*, \rho^{\#})$.

TIME CONSTANTS FOR UNBOUNDED SPACES

General Remarks

The various temporal diameters $T(\Phi)$, $T_n(\rho^*)$, and $T_n(\rho^*, \rho^\#)$ studied so far all become infinite in unbounded spaces. From an operational point of view this means that experiments conducted with ideal instruments in such spaces would reveal that no true steady state is ever attained. Specifically, the instruments would record a continuous build-up or decay of radiance over all time: the light field would approach but never actually attain an extreme value such as a maximum or minimum.

In reality, it is impossible to have infinite spaces, ideal instruments, and continuous evolution of the light field to extreme values. Of these three idealizations, the most impractical--for the purposes of radiative transfer studies in real media--is the ideal instrument. While there are supportable objections to the infinite-space and continuous-light idealizations, these objections carry less weight than those directed at ideal instruments.

Each real flux-measuring instrument has a certain absolute threshold of sensitivity. If the amount of radiant flux impinging on its light-sensitive element is below this threshold, the instrument will record that no flux is incident on it. More generally, at each level P of incident irradiation, the instrument has a threshold $S_0(P)$ of sensitivity to change in the level of irradiation. Below this level it can no longer sense the relative change in the amount of incident radiant flux.

Thus, if an amount $P > 0$ of flux is incident on the instrument's receptor, and the amount is changed to $P' = P + \Delta P$, then if the absolute magnitude of the relative change $|\Delta P / P|$ is less than $S_o(P)$, the instrument will say that $P = P'$. The sensitivity threshold $S_o(P)$ generally depends on P . A <u>good instrument</u> is defined here as one for which S_o is independent of P , and such that $S_o \leq \frac{1}{20}$.

In this section we briefly consider the problems of defining the counterparts to the local temporal diameter $T_n(\rho^*, \rho^\#)$ and the temporal diameter $T(\Phi)$ in an unbounded space. The definition will be made with the aid of a good instrument whose absolute threshold radiance sensitivity is N_o and whose relative threshold (radiance) sensitivity is S_o for a given wavelength. To keep the discussion free of irrelevant complexities we will assume that the location space of Φ is E_3 --infinite euclidean three-space. Furthermore, Φ will have a constant index of refraction n, a constant volume attenuation function α , but a volume scattering function whose angular dependence is arbitrary. Generalizations of the following results to other types of infinite spaces (half-spaces, slabs, etc.), more general source conditions, and attenuation conditions offer no essential obstacles, but will not be considered here.

Formulation of the n-ary Time Constant Problem

Suppose a fundamental source at ρ_o in $\mathfrak{X}$ is turned on at epoch time $T = 0$, and radiates energy at a uniform rate for all subsequent time. Suppose further that a radiance meter is located at $\rho^{\#}$ in $\mathfrak{X}$ a distance $d = d(\rho_o, \rho^{\#})$ from ρ_o and that the meter can detect and record the n-ary radiance $N^n(\theta,\phi,T)$ in each direction (θ, ϕ) (with respect to some reference frame) at each epoch time T over a certain interval of time. If the steady state n-ary radiance distribution at $\rho^{\#}$ is designated by $N^n(\theta,\phi,\infty)$, let $S^n(\theta,\phi,T)$ be defined as

$$S^n(\theta,\phi,T) = \frac{N^n(\theta,\phi,\infty) - N^n(\theta,\phi,T)}{N^n(\theta,\phi,\infty)} \ .$$

Then in general $S^n(\theta,\phi,T) \to 0$ monotonically as $T \to \infty$ for all (θ,ϕ) and $n \geq 1$. Therefore, for each (θ,ϕ) and each $n \geq 1$, there is a unique time T_n such that

$$S^n(\theta,\phi,T_n) = S_o$$

Our problem is to determine T_n. This time T_n clearly depends on (θ,ϕ) in addition to d.

We can simplify the present problem of determining T_n and make its results more amenable to practical applications by redefining T_n so that it does not depend on (θ,ϕ). In order to free T_n of its (θ,ϕ) --dependence we agree to consider only the maximum value of

$N^n(\theta, \phi, T)$ over the unit sphere Ξ at time T. Let us designate this maximum by $\bar{N}^n(T)$. $\bar{N}^n(T)$ exists because $N^n(\cdot, \cdot, T)$ is generally a continuous function on the closed subset Ξ of E_3. T_n now depends only on the separation d between ρ_o and $\rho^{\#}$. The function $S^n(\theta, \phi, T)$ now takes the following form:

$$\bar{S}^n(T) = \frac{\bar{N}^n(\infty) - \bar{N}^n(T)}{\bar{N}^n(\infty)} ,$$

and $T_n(d)$ is defined as the solution of the equation

$$\bar{S}^n(T) = S_o . \tag{3}$$

We shall refer to $T_n(d)$ as the <u>time constant for n-ary radiance</u> in Φ with respect to ρ_o, $\rho^{\#}$, and the instrument whose relative threshold is S_o. $T_n(d)$ is the unbounded-space counterpart to the local n-ary temporal diameter $T_n(\rho_o, \rho^{\#})$ considered above. Because of the way in which $T_n(d)$ is defined and the type of space with which it is associated we expect its properties to have only a superficial resemblance to those of $T_n(\rho_o, \rho^{\#})$.

Derivation of the Formula for the n-ary Time Constant

In order to obtain an explicit formula for $T_n(d)$ we must solve the functional equation (3). which may be rewritten as:

$$\frac{\bar{N}^n(T)}{\bar{N}^n(\infty)} = 1 - S_0 . \tag{4}$$

Now, the results of an earlier work provide the basis for an explicit approximation of $\bar{N}^n(T)/\bar{N}^n(\infty)$ for all $n \geq 1$ and all T under the same conditions as those of the present problem.[2] This expression is:

$$\bar{N}^n(T)/\bar{N}^n(\infty) = \left[1 - exp\{-\alpha r(T)\}\right]^n , \quad (n \geq 1) \tag{5}$$

where

$$r(T) = (v T + d)/2 , \tag{6}$$

is simply the distance from $\rho^{\#}$ to the far end of the major diameter of the characteristic ellipsoid $\mathcal{E}(\rho_0, \rho^{\#}; T)$ (see paper III.). Replacing $\bar{N}^n(T)/\bar{N}^n(\infty)$ by its equivalent, (5), and then solving (4) for T , we have

$$T_n(d) = -2T_\alpha \ln\left[1 - \sqrt[n]{1 - S_0}\right] - \frac{d}{v} . \tag{7}$$

The expression $T_\alpha = 1/v\alpha$ is the time constant for reduced flux (scattering order $n = 0$) in Φ , and, of course,

$$t(p_0, p^{\#}) = d(p_0, p^{\#})/v = d/v,$$

where v is the speed of light in Φ .

Observations on $T_n(d)$

(a). Equation (7) is applicable as long as $\bar{N}^n(T)$ exceeds N_o the absolute threshold radiance of the instrument. According to (5), $\bar{N}^n(\omega) > \bar{N}^n(T)$, so that if $\bar{N}^n(\omega) > N_o$, Equation (7) can generally yield a positive, finite value for $T_n(d)$. But, according to the results of reference 2, the order of magnitude of $\bar{N}^n(\omega)$ is $\bar{N}^o \omega_o^n$, where $\bar{N}^o$ is the maximum inherent radiance of the source, and $\omega_o = s/\alpha \leq 1$, where s is the volume total scattering coefficient. It follows that there is generally an integer M such that for all $n > M$, $N^n(\omega) < N_o$. For these values of n , the instrument will not detect $N^n(\theta, \phi, T)$ for any T and (θ, ϕ) , and Equation (7) will not apply.

(b). For a given d , $T_n(d) \to \omega$ monotonically as $n \to \infty$. This property agrees qualitatively with the conclusion of Theorem 3. However, $T_n(d)$ goes to ∞ at a much slower rate than $T_n(p_0, p^{\#})$. The latter eventually increases linearly with n , while $T_n(d)$ eventually increases logarithmically with n .

(c). For very large $t(\rho_0, \rho^{\#})$ (or, equivalently, large $d = d(\rho_0, \rho^{\#})$) and relatively small T_α (optically dense media) it is possible for $T_n(d)$ to be negative for many of the smaller scattering orders. This indicates that, for sufficiently small S_0, the instrument at $\rho^{\#}$ can sense only highly diffuse light from the source ρ_0 --subject to the conditions of Observation (a).

(d). According to (a), (b) and (c) above, there will generally be, for a given medium, source, and d , only a finite number of $T_n(d)$, $n = n_1, \ldots, n_p$, for which $T_n(d)$ is finite and non-negative, indicating that the scattering orders $n = n_1, \ldots, n_p$ are the only ones "detectable" for the given conditions. For good instruments (7) may be approximated by

$$T_n(d) \cong -2 T_\alpha \ln \left[\frac{S_0}{n} \right] - \frac{d}{v}$$

Derivation of the Formula for the Basic Time Constant

In formulating a definition of the time constant associated with n-ary radiance we managed to find an unbounded-space counterpart to $T_n(\rho^*, \rho^{\#})$. We now introduce a concept which has no counterpart in the bounded-space theory of temporal diameters as developed so far, though we could, if required, easily find a suitable candidate for the part in that theory. This concept is the basic <u>time constant</u> $T(d)$

for Φ , and is defined as follows. Let

$$\bar{S}(T) = \frac{\bar{N}(\infty) - \bar{N}(T)}{\bar{N}(\infty)} ,$$

where

$$\bar{N}(T) = \sum_{j=0}^{\infty} \bar{N}^{j}(T) .$$

Then $T(d)$ is defined as the solution of the equation

$$\bar{S}(T) = S_0 . \tag{8}$$

Since $\bar{S}(T)$ monotonically decreases toward zero as $T \to \infty$, there will generally exist for every $S_0 > 0$ a unique time $T(d)$ satisfying (8). For epoch time $T > T(d)$, the instrument will sense no further temporal change in the radiance distribution about $\rho^{\#}$. With respect to such an instrument, the light field will be in steady state at $\rho^{\#}$ for all $T \geq T(d)$.

Now, according to the results of reference 2, the ratio $\bar{N}(T)/\bar{N}(\infty)$ may be approximated by the expression:

$$\bar{N}(T)/\bar{N}(\infty) = \frac{1 - \omega_0}{1 - \omega_0\left[1 - \exp\{-\alpha R(T)\}\right]} , \tag{9}$$

where $r(T)$ is defined in (6) above, and $\omega_0 = s/\alpha \leq 1$ is the albedo for single scattering already encountered in Observation (a) above.

With this expression for $\bar{N}(T)/\bar{N}(\infty)$, Equation (8) is readily solved. The solution is:

$$T(d) = -2T_\alpha \ln\left[\frac{S_o(1-\omega_o)}{\omega_o(1-S_o)}\right] - \frac{d}{v} \quad (10)$$

Observe how $T(d)$ depends jointly on ω_o and S_o. If, for example, S_o is small, $T(d)$ is large. If on the other hand ω_o were small, then $T(d)$ would be small. Each of these features are quite what one would expect of $T(d)$. Further observations can be made on $T(d)$. They would for the most part parallel those of $T_n(d)$ above.

It is of interest to observe that, while the concepts of the n-ary diameters for bounded spaces could not be carried over into the unbounded context, the methods developed here for the various <u>time constants</u> $T_n(d)$ and $T(d)$ can be applied, mutatis mutandis, to the bounded context. We shall not, however, pursue this interesting problem of time constants in general any further for the present, for we have already accumulated sufficient evidence of the power and scope of the n-ary scattering approach.

Thus we conclude a brief study of the theoretical and practical possibilities inherent in the n-ary approach to the time-dependent multiple scattering problem.

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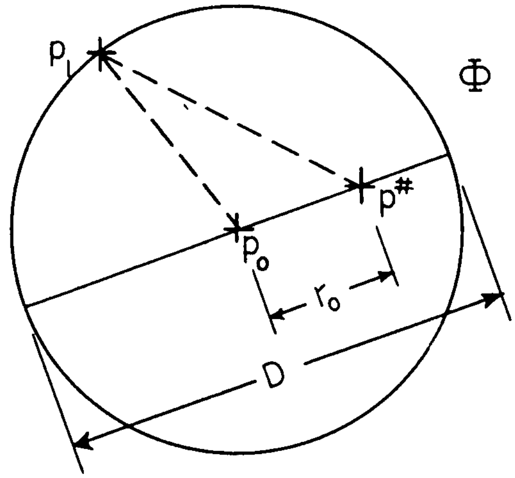

Figure 1

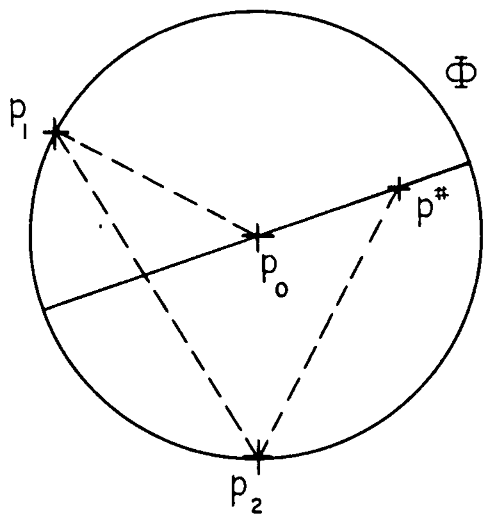

Figure 2

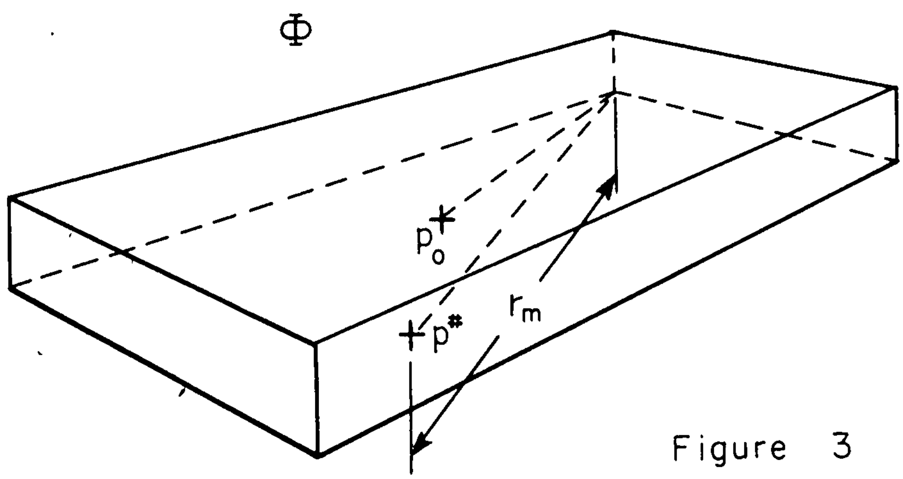

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Figure 3