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A new invariant of 4-manifolds

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We define an integer invariant L_X of a smooth, compact, closed 4-manifold X by minimizing a certain complexity of a trisection of X over all trisections. The good feature of L_X is that when $L_X = 0$ and X is a homology 4-sphere, then X is diffeomorphic to the 4-sphere. Naturally, L is hard to compute.

trisections | 4-manifolds | Heegaard splittings | curve complex

We define an integer invariant L_X of a smooth, compact, closed 4-manifold X by minimizing a certain complexity of a trisection of X over all trisections.

Loops in the Cut Complex

Let X be a closed, orientable, smooth 4-manifold. In ref. 1, Gay and Kirby show that X has a trisection into three 4-dimensional handlebodies and prove that any two trisections of X are stably equivalent under a suitable notion of stabilization. We exploit these results to define a new 4-manifold invariant L_X and prove that $L_X = 0$ if and only if X is a connect sum of copies of $S^1 \times S^3$, $S^2 \times S^2$, CP^2, and S^4 (the case of the empty connect sum). If $L_X \leq 1$, we obtain the same 4-manifolds, so L_X is never one.

Definition 1: A$(g; k_1, k_2, k_3)$-trisection of a closed, oriented 4-manifold X (where $0 \leq k_i \leq g, i = 1, 2, 3$) is a decomposition $X = X_1 \cup X_2 \cup X_3$, where (*i*) each $X_i \cong \natural^{k_i} S^1 \times B^3$, (*ii*) each $X_i \cap X_j \cong \natural^g S^1 \times B^2$ (for $i \neq j$), and (*iii*) $X_1 \cap X_2 \cap X_3 \cong \#^g S^1 \times S^1$.

Definition 2: A $(g; k_1, k_2, k_3)$-trisection diagram is a 4-tuple $(\Sigma, \alpha, \beta, \gamma)$ such that each of (Σ, α, β), (Σ, β, γ), and (Σ, γ, α) are genus g Heegaard diagrams of $\#^{k}{}_{i} S^1 \times S^2, i = 1, 2, 3$, respectively. A trisection diagram for a given trisection $X = X_1 \cap X_2 \cap X_3$ is a trisection diagram $(\Sigma, \alpha, \beta, \gamma)$, where Σ is diffeomorphic to $X_1 \cap X_2 \cap X_3$, α is a cut system for $X_1 \cap X_2$, β for $X_2 \cap X_3$, and γ for $X_3 \cap X_1$.

The stabilization operation for a balanced trisection increases the genus of the central surface Σ by 3. It can be understood in terms of the trisection diagram by taking the connect sum of $(\Sigma, \alpha, \beta, \gamma)$ with the standard genus three trisection diagram of S^4.

An unbalanced trisection can be "balanced" by taking the connect sum with genus one trisections of the 4-sphere.

The topology of each of the three pieces of X is completely determined by a single integer k_i, and the topology of each of the overlaps between pieces is determined by another integer g. If $k = k_1 = k_2 = k_3$, the trisection is called balanced.

Given a trisection of X^4, we have a central surface $\Sigma = X_0 \cap X_1 \cap X_2$ in X bounding three 3-dimensional handlebodies $X_i \cap X_j$, which fit together in pairs to form Heegaard splittings of three 3-manifolds in X, and these 3-manifolds in turn uniquely bound three 4-dimensional 1-handlebodies. We can thus specify a trisection by considering systems of curves on Σ.

Definition 3: A cut system for a closed surface Σ of genus g is an unordered collection of g simple closed curves on Σ that cut Σ open into a $2g$-punctured sphere.

Definition 4: A genus g Heegaard diagram for a closed orientable 3-manifold is a triple (Σ, α, β), where Σ is a closed orientable genus g surface and each of α and β is a cut system for Σ.

Following Wajnryb (2) and Johnson (3), we define the following:

Definition 5: The cut complex $\mathcal{C}$ of Σ_g is a 1-complex with vertices corresponding to (isotopy classes) of cut systems. Two vertices α and α' in $\mathcal{C}$ are connected by an edge of type 0 if their corresponding cut systems $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_g\}$ and $\alpha' = \{\alpha'_1, \alpha'_2, \ldots, \alpha'_g\}$ agree on $g - 1$ curves and their final curves are disjoint. Two vertices α and α' are connected by an edge of type 1 if their corresponding cut systems α and α' agree on $g - 1$ curves and their final curves intersect in a single point. The distance between two vertices α and β, $d(\alpha, \beta)$, is the length of the shortest path (using the edge-metric) connecting them in the cut complex.

Notice that if α and α' are connected by a type 0 edge, then α can be obtained from α' by sliding α_g over some of $\alpha_1, \alpha_2, \ldots, \alpha_{g-1}$. $\mathcal{C}$ is connected (4).

Suppose we are given a $(g; k_1, k_2, k_3)$-trisection diagram $(\Sigma, \alpha, \beta, \gamma)$ for a trisection $\mathcal{T}$ of X.

Definition 6: Let Γ_α be the set of all vertices in $\mathcal{C}$ that are path connected to α by type 0 edges (generalized handle slides). Define Γ_β and Γ_γ similarly (see Fig. 1).

Definition 7: We say two cut systems α and β are in good position with respect to each other if we can order each, $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_g$, $\beta = \beta_1, \beta_2, \ldots, \beta_g$, so that for each i, either α_i is parallel to β_i (and we write $\alpha_i \mathcal{P} \beta_i$) or α_i intersects β_i in exactly one point (and we write $\alpha_i \mathcal{D} \beta_i$), and α_i is disjoint from β_j for all $i \neq j$. We say α_i and β_j are a good pair if they are either parallel or intersect in a single point and are disjoint from all other αs and βs.

Note that it is possible for α, β, γ to pairwise all be in a good position but not with respect to the same ordering. For example, in Fig. 2 all pairs are in a good position, but α_1 is paired with γ_2 and α_2 with γ_1.

Every vertex in Γ_α represents a different cut system describing the same handlebody $X_1 \cap X_2$.

We can calculate the length of the shortest path between Γ_α and Γ_β. We use a mild generalization of Waldhausen's theorem for Heegaard splittings of the 3-sphere (5):

Theorem 8: Let (Σ, α, β) be a genus g Heegaard diagram for $\#^k S^1 \times S^2$. Then there exist cut systems α' and β' that are connected to α and β, respectively, through type 0 edges such that α' and β' are in good position with respect to each other.

Significance

All known 4-manifolds invariants cannot distinguish a possible counterexample to the smooth 4-dimensional Poincare Conjecture from the standard 4-sphere. The L invariant, defined in this paper, can do so, for if it vanishes on a homotopy 4-sphere X, then X must be diffeomorphic to the 4-sphere. Unfortunately, it is very hard to calculate.

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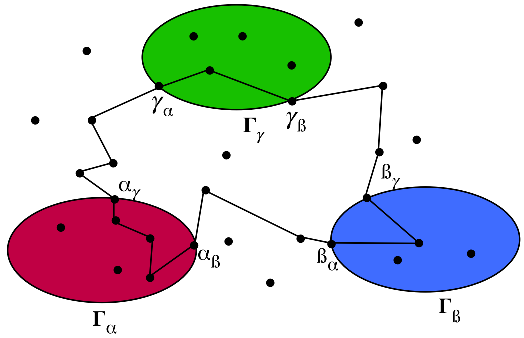

Fig. 1. $S^2 \times S^2$.

The details of this theorem appear in ref. 6, p. 313, together with a discussion of its relation to the isotopy question. We include an outline of the argument, which proceeds by induction, for the convenience of the reader. Proof: Use Haken's (7) lemma to find an essential separating 2-sphere S that intersects Σ in a single essential simple closed curve λ bounding imbedded disks E_α and E_β on both sides of Σ. Use an outermost arc on, say, E_α of intersections with the disks bounded by the αs to dictate handle slides on the αs to reduce the number of points of intersection between λ and the αs. This implies that any Heegaard splitting of $\#^k S^1 \times S^2$ is diffeomorphic to a good Heegaard splitting (but not necessarily isotopic to one).

Hence there exists a cut system in Γ_α that has distance precisely $g - k_1$, through $g - k_1$ type 1 edges, to the nearest cut system in Γ_β. Note that stabilizing the trisection increases the length of such a path in a straightforward way; if the initial trisection is balanced and the stabilization is balanced, the length of the path goes up by 2. For an unbalanced stabilization, the length goes up by either 0 or 1.

Definition 9: Let $l_{X,\mathcal{T}}$ be the length of the shortest closed path in $\mathcal{C}$ that intersects each of Γ_α, Γ_β, and Γ_γ, which also satisfies the following:

i) There are three pairs—$(\alpha_\beta, \beta_\alpha)$, $(\beta_\gamma, \gamma_\beta)$, and $(\gamma_\alpha, \alpha_\gamma)$—in

$$(\Gamma_\alpha, \Gamma_\beta), (\Gamma_\beta, \Gamma_\gamma), (\Gamma_\gamma, \Gamma_\alpha),$$

respectively, which are all good, so it takes $g - k_i$ type 1 moves to travel from the vertex corresponding to one element in the pair to the other.

ii) The subpath of $l_{X,\mathcal{T}}$ connecting α_β to α_γ (respectively, β_α to β_γ, γ_β to γ_α) remains within Γ_α (respectively, Γ_β, Γ_γ).

Normalize l by defining:

Definition 10: $L_{X,\mathcal{T}} = l_{X,\mathcal{T}} - 3g + k_1 + k_2 + k_3$. Note that this number can only decrease when we stabilize. Note also that this number is equal to the total number of type 0 moves in each of Γ_α, Γ_β, and Γ_γ.

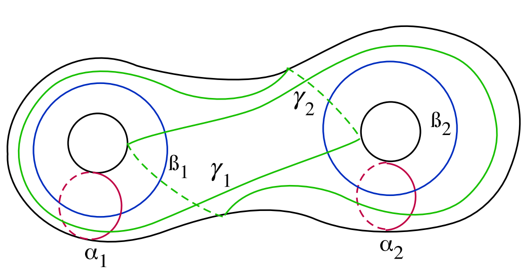

Fig. 2. $\mathcal{C}$.

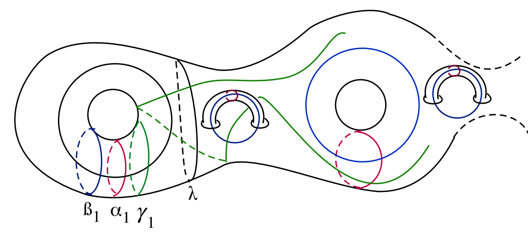

Fig. 3. $\alpha_1 \mathcal{P} \beta_1 \mathcal{P} \gamma_1$.

Definition 11: The length of X, denoted L_X, is the minimum value of $L_{X,\mathcal{T}}$ over all trisections $\mathcal{T}$ of X.

It follows immediately from the stable equivalence of balanced trisections (1) that L_X is well-defined.

We analyze the manifolds for which $L_X = 0$:

Theorem 12: $L_X = 0$ if and only if X is diffeomorphic to a connect-sum of copies of $S^1 \times S^3$, $S^2 \times S^2$, CP^2, and S^4 (in the case of an empty connect sum).

As an immediate corollary, we have the following:

Corollary 13: If X is a homology 4-sphere, then $L_X = 0$ if X is diffeomorphic S^4.

Proof of theorem: Let $(\Sigma, \alpha, \beta, \gamma)$ be a (g, k_1, k_2, k_3) trisection of X that realizes $L_{X,\mathcal{T}} = 0$.

Then, $g - k_1 = d(\alpha, \beta)$, $g - k_2 = d(\gamma, \beta)$, and $g - k_3 = d(\alpha, \gamma)$. Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, $(\beta_1, \beta_2, \ldots, \beta_n)$, $(\gamma_1, \gamma_2, \ldots, \gamma_n)$, and $n = 1, \ldots, g$ be the curves corresponding to the cut systems α, β, γ.

Since $(\Sigma, \alpha, \beta, \gamma)$ realizes $L_{X,\mathcal{T}} = 0$, we may assume that $\alpha_1, \alpha_2, \ldots, \alpha_g$, $\beta_1, \beta_2, \ldots, \beta_g$ are in good position with respect to each other and that $\alpha_1, \alpha_2, \ldots, \alpha_g$, $\gamma_1, \gamma_2, \ldots, \gamma_g$ are in good position with respect to each other (any ordering of α determines one for β and for γ). Note that the βs and γs would also be "good" if we allowed reordering of subindices. Consider the example of $S^2 \times S^2$ where β_1 and γ_2 are good, as are β_2 and γ_1.

We may also assume that $\alpha_i \mathcal{P} \beta_i$ for $i = 1, \ldots, k_1$.

After possible relabeling, we have the following cases:

Case 1 : $\alpha_1, \beta_1, \gamma_1$ are all parallel (see Fig. 3).

No other curve from $\alpha \cup \beta \cup \gamma$ intersects $\alpha_1, \beta_1, \gamma_1$. Let δ be a simple closed curve intersecting α_1 (also β_1, γ_1) transversely in a single point, chosen to be disjoint from all other αs and βs. α_1 and δ together have a neighborhood that is a punctured torus T. We say that T is defined by α_1 and δ. Let $\lambda = \partial T$. λ is disjoint from all αs and βs but may intersect $\gamma_2 \cup \ldots \cup \gamma_g$. However, we can slide these γs over γ_1 to remove these intersections, obtaining $\gamma_2', \ldots, \gamma_g'$, which are disjoint from ∂T. Let $\gamma' = \gamma_1, \gamma_2', \ldots, \gamma_g'$. Since this operation has no effect on the intersections of curves with subindices $2, \ldots, g$, $(\Sigma, \alpha, \beta, \gamma')$ also realizes $L = 0$. In $(\Sigma, \alpha, \beta, \gamma')$, ∂T is a splitting curve—that is, a separating simple closed curve—disjoint from all curves in all three cut systems, which splits the diagram into two subdiagrams, each with $L = 0$. The subdiagram containing α_1 yields an $S^1 \times S^3$ summand that we can

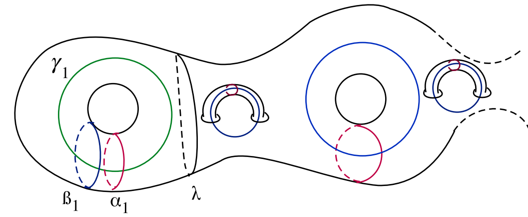

Fig. 4. $\alpha_1 \mathcal{P} \beta_1 \mathcal{D} \gamma_1$.

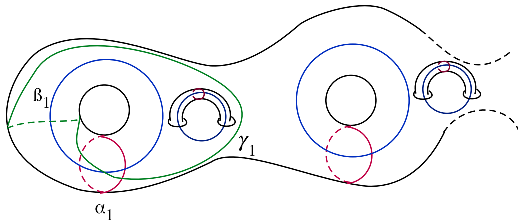

Fig. 5. $\alpha_1 \mathcal{D} \beta_1 \mathcal{D} \gamma_1 \mathcal{D} \alpha_1$.

split off and proceed to consider the smaller genus remaining subdiagram.

Case 2 : $\alpha_1 \mathcal{P} \beta_1$ and γ_1 intersects each in exactly one point (see Fig. 4).

As before, we can find a punctured torus T containing $\alpha_1, \beta_1, \gamma_1$, which in this case is automatically disjoint from all curves in $\alpha \cup \beta \cup \gamma$. Hence, there is an obvious S^4 summand in the trisection diagram, which we split off to reduce the genus and again proceed on the remainder.

Case 3 : No pair of curves from α, β, γ is parallel. In particular, $\alpha_1 \mathcal{D} \beta_1$ and $\alpha_1 \mathcal{D} \gamma_1$. Let λ be the boundary of the torus T defined by α_1 and β_1.

Subcase a: $\gamma_1 \mathcal{D} \beta_1$ (see Fig. 5).

Claim: Then we can split off a $\pm CP^2$ summand.

Proof: If γ_1 does not lie in T, then we can move it there by a type 0 move. Then, ∂T will be a splitting curve.

Subcase b: γ_1 is disjoint from β_1 (see Fig. 6). Then, we can assume (by relabeling as needed) that $\gamma_1 \mathcal{D} \beta_2$ and γ_1 are disjoint from all other curves in α and β.

Claim: Then, we can split off a $S^2 \times S^2$ summand.

Proof: We analyze the remainder of the γ_is and show there must exist a γ_2 such that

- $\gamma_2 \mathcal{D} \beta_1$,
- $\gamma_2 \cap \alpha_1$ is empty,
- $\gamma_2 \mathcal{D} \alpha_2$, and
- $\gamma_2 \cap \beta_2$ is empty.

This follows because exactly one γ, which we label γ_2, is dual to β_1, and it links γ_1 in λ when both intersect λ. That forces γ_2 to intersect α_2 in one point. By type 0 moves on γ_1 and γ_2, we can arrange that all curves with indices 1 or 2 are outside the punctured $S^2 \times S^2$, whose boundary is a splitting curve. This concludes the proof.

We now prove the stronger theorem:

Theorem 14: If there exists a trisection $\mathcal{T}$ such that $L_{X,\mathcal{T}} = 1$, then $L_X = 0$, and X is again diffeomorphic to a connect sum of copies of $S^1 \times S^3$, $S^2 \times S^2$, and CP^2.

Assume $(\Sigma, \alpha, \beta, \gamma)$ realizes $L_{X,\mathcal{T}} = 1$. We may also assume that $\alpha_1, \alpha_2, \ldots, \alpha_g$, $\beta_1, \beta_2, \ldots, \beta_g$ are in good position with respect to each other and that $\alpha_1, \alpha_2, \ldots, \alpha_g$, $\gamma_1, \gamma_2, \ldots, \gamma_g$ are in good position with respect to each other. Note that the βs and γs would also be good if we allowed reordering of

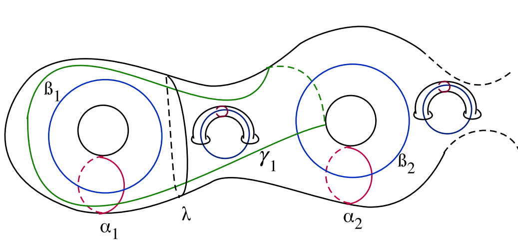

Fig. 6. $\alpha_1 \mathcal{D} \beta_1$, $\gamma_1 \mathcal{D} \alpha_1$.

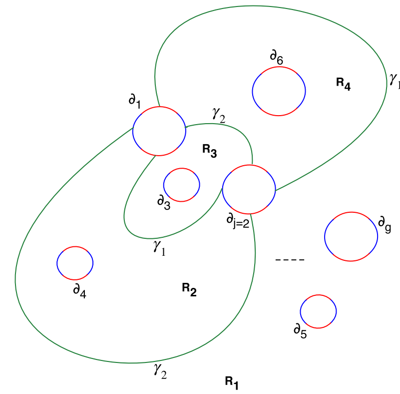

Fig. 7. *P*.

subindices, with the exception of a single γ_j. Hence there exists a cut system $\Gamma' = \gamma_1, \gamma_2, .., \gamma_j', .., \gamma_g$ that is distance one from $\gamma_1, \gamma_2, \ldots, \gamma_g$ and that is good with respect to $\beta_1, \beta_2, \ldots, \beta_g$ after reordering.

The arguments in cases 1 and 2 of theorem 1 work as before if $\alpha_i \mathcal{P} \beta_i$ or $\alpha_i \mathcal{P} \gamma_i$ for any i, or $\beta_i \mathcal{P} \gamma_k$ for any $k \neq j$, or $\beta_i \mathcal{P} \gamma_j'$; that is, we can assume that the trisection is balanced and that $g = k$ so that each X_i is a 4-ball.

If $g = 2$, the theorem follows from ref. 8. Assume $g > 2$. We relabel so $\Gamma = \gamma_1, \gamma_2, .., \gamma_g$ and $\Gamma' = \gamma_1, \gamma_2, .., \gamma_g'$.

Hence, we have the following string of relations:

$$\beta_1 \mathcal{D} \alpha_1 \mathcal{D} \gamma_1 \mathcal{D} \beta_j \mathcal{D} \alpha_j.$$

If $j = 1$, we are back in case 3, subcase a of the previous argument. Assume $j \neq 1$. Then, we can continue our string to the left,

$$\gamma_a \mathcal{D} \beta_1,$$

and to the right,

$$\alpha_j \mathcal{D} \gamma_b.$$

If $a = b$, then $a \neq g$. If $a \neq b$, then either a or b (or both) is not equal to g. In any case, we obtain a slightly longer string by adding on to the left or to the right, say to the left,

$$\gamma_a \mathcal{D} \beta_1 \mathcal{D} \alpha_1 \mathcal{D} \gamma_1 \mathcal{D} \beta_j \mathcal{D} \alpha_j,$$

where $a \neq g$ and γ_g is not in the string.

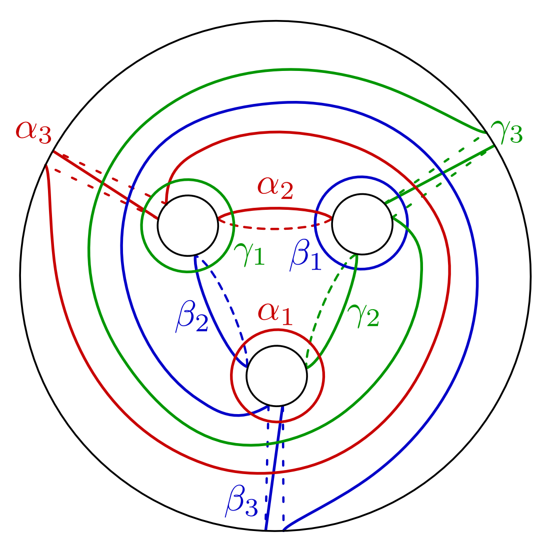

Fig. 8. *Q*.

Claim: $a = j$.

Proof of claim: Cut Σ open along all αs and βs to obtain a planar surface P with g boundary components, $\partial_1, \ldots, \partial_g$, with the labelling inherited from the αs. The remnants of γ_1 in P are two properly imbedded arcs connecting ∂_1 to ∂_j. The remnants of γ_a in P are also two properly imbedded arcs, whose endpoints are linked on ∂_1 with the endpoints of γ_1. We know γ_a is dual to exactly one α and disjoint from all others; the only available α that yields a single connected curve is α_j. Hence, $a = j$ (see Fig. 7).

We now relabel so $j = 2$ and summarize our findings thus far:

$$\gamma_2 \mathcal{D} \beta_1 \mathcal{D} \alpha_1 \mathcal{D} \gamma_1 \mathcal{D} \beta_2 \mathcal{D} \alpha_2 \mathcal{D} \gamma_2.$$

Definition 15: Call such a set $\gamma_2, \beta_1, \alpha_1, \gamma_1, \beta_2, \alpha_2, \gamma_2$ a good sextet.

The remnants of γ_1 and γ_2 in P cut P into four regions, one of which, R_1, contains ∂_g.

Suppose the other three regions, R_2, R_3, R_4, are disks—that is, contain no other boundary components of P.

Claim: Either γ_g or γ_g' is disjoint from $R_2 \cup R_3 \cup R_4$.

Proof of claim: Suppose γ_g intersects R_2. γ_g is disjoint from all αs except α_g and disjoint from all other γs, so γ_g can only intersect the pieces of ∂R_2 corresponding to remnants of the βs. Hence, (possibly after removing trivial intersections) γ_g intersects $\mathbb{R}_2$ in a collection of parallel arcs connecting the two β remnants on ∂R_2. This means that γ_g must also intersect R_3 and R_4 in a similar fashion. Recall that γ_g' is disjoint from γ_g, and γ_g' is disjoint from all βs except β_g. Then, by the same argument, if γ_g' intersects $\mathbb{R}_2$ at all, it must do so in a collection of parallel arcs connecting the two α remnants on ∂R_2. But any such arc would intersect an arc of γ_g, and γ_g' is disjoint from γ_g. So if γ_g intersects R_i, $i = 2, 3, 4$, then γ_g' cannot and vice versa.

Assume γ_g is disjoint from $R_2 \cup R_3 \cup R_4$.

Then, ∂R_1 is a splitting curve for α, β, γ, and we proceed by examining the smaller diagram inside R_1.

Suppose one of R_2, R_3, R_4 is not a disk, say R_2.

Using previous arguments, we can find another good sextet inside R_2.

This sextet also divides P into four components, one of which contains ∂_g.

If all other components are disks, we are done by the previous argument. Otherwise, select one that is not a disk, and repeat.

Eventually, we find a sextet such that one component of P defined by the sextet contains ∂_g, and all others are disks.

An Example with $L \leq 6$

Currently, the smallest nonzero $L_{X,\mathcal{T}}$ we know, namely 6, is achieved by a smooth orientable 4-manifold Q that is the quotient $(S^2 \times S^2)/Z/2$, where the group $Z/2$ acts by sending (x, y) to $(-x, -y)$. This allows the possibility that our theorem holds for $L \leq 5$, but we only conjecture the theorem can be strengthened to show that $L \leq 2$ implies $L = 0$. There is a notable lack of low-genus simply-connected, closed, smooth 4-manifolds (other than those with $L = 0$). In the nonspin case, there are connected sums of $\pm CP^2$, and in the spin case, there is the $K3$ complex surface. Many of these manifolds have exotic smooth structures (e.g., CP^2 with at least two points blown up; this means connected summing with $-CP^2$s), but these have complicated handlebody structures suggesting L is large. For π_1 nonzero, our Q is a fairly simple example and is a natural candidate for the smallest nonzero L.

There is a handlebody description of Q obtained by taking a simple description of the nonorientable disk bundle over RP^2 and doubling it to get Q (see ref. 9, p. 27). There are algorithms to turn this handlebody description into a trisection with genus three, but the diagram in Fig. 8 will most easily show that $L \leq 6$. This diagram was discovered independently by David Gay and by Jeff Meier, in the latter case as part of studying trisections of twist spun 3-manifolds.

By symmetry, it suffices to calculate how many type 0 moves are required to make the αs and βs standard. Ignore the γ curves, and observe that α_2 and β_1 are a pair that intersect each other once and are disjoint from all other αs and βs. Notice next that α_1 and β_2 would be a good pair if not for the fact that β_3 intersects α_1 twice. These intersections can be removed by two handle slides of β_3 over β_2. First, push the closer point of intersection clockwise along α_1 and then slide over β_2 to remove the point of intersection. Then, do the same with the further point of intersection, again moving clockwise and sliding over β_2.

We now have two pairs intersecting once each, and then one can check that α_3 and β_3 are in fact parallel on Σ, and thus, the $\alpha - \beta$ curves form a standard Heegaard spitting of $S^1 \times S^2$, as desired.

A sharp reader might observe that if the second handle slide had been done counterclockwise, then the two handle slides would combine into one type 0 move, suggesting that $l = 3$, but a sharper reader will realize that in this case α_3 and β_3 are no longer parallel, for the other pairs are stuck between the otherwise parallel curves.

Remarks

It seems likely that a complicated handlebody diagram for X would lead to a large value of L. But it is sobering to realize that the complex hypersurfaces such as the $K3$ surface are not connected sums of smaller 4-manifolds, yet if one connect sums with one copy of CP^2, the resulting complicated handlebody slides away to a connected sum of $\pm CP^2$s (refs. 10 or 11), showing that there must also be a way to do handle slides on the αs, βs, and γs to get $L = 0$.

ACKNOWLEDGMENTS. Both authors were supported in part by the National Science Foundation.

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