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Connecting the Multiplicative Field with Student Mathematical Thinking

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Connecting the Multiplicative Field with Student Mathematical Thinking

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in Education

by

Brandon McMillan

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ABSTRACT OF THE DISSERTATION

Connecting the Multiplicative Field with Student Mathematical Thinking

by

Brandon McMillan

Doctor of Philosophy in Education

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Professor Megan Loef Franke, Chair

Even though algebraic ideas are addressed across a number of grades, algebra continues to serve as a gatekeeper to upper mathematics and degree attainment because of the high percentage of students that fail algebra classes and become halted in their educational progress. One reason for this is students not having the opportunity to build on their own thinking to connect mathematical ideas from elementary through middle school. The multiplicative field consists of major components of the mathematical ideas learned in these important years, but research within the multiplicative field has focused primarily on the mathematics and not enough on connections within student mathematical thinking. This study focuses on examining the connections in student strategies between whole number, fraction, and two-step rate problems, as well as, how students' ideas of grouping connect to graphing their strategies. Findings add to previous research of student strategies with multiplication and division by detailing some of the nuance in students' use of grouping. A focus on grouping strategies reveals students' progression in understanding the mathematical properties from implicit to explicit to purposeful planning of use of the distributive and associative properties of multiplication. Progression within strategies occurs not as a trajectory, but as part of a constellation of ideas. Additionally, student grouping strategies provided a context to begin to connect solutions to graphing.

Implications from this research indicate the need for researchers and teachers to uncover what students know and examine use of grouping to support connections across mathematical concepts within the multiplicative field.

The dissertation of Brandon McMillan is approved.

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2018

DEDICATION

For Lucy, Filipe, Samuel, Lesieli, Kolei and my parents Glenn and Kathy.

TABLE OF CONTENTS

LIST OF TABLES

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Chapter 1: Introduction

The middle grades (4-8) are critical years for children. They are a formative time in which students are developing notions of self and autonomy. At the same time, students are transitioning from elementary school to middle school, preparing for high school, and is the time when many students begin to be lost or left behind in education (Balfranz, 2007, 2009; Bruce et al., 2011). Meeting the needs of adolescent children has been the subject of research for over 100 years (Juvonen et al., 2004) and has demonstrated the difficulty of middle grades. Many students experience a decline in self-esteem from 6th grade to 8th grade (Reddy et al., 2003) and feel less support from their teachers and greater hassles in daily school life (Seidman et al., 1994). When compared to elementary school teachers, middle school teachers grade more stringently (Eccles and Midgley, 1989) and tougher grading and expectations relate to lower grades and academic self-perceptions (Murdock et al., 2000). As well, interpersonal relationships are demonstrably less positive with greater amount of negative behaviors such as cruelty and meanness (Merten, 1997). Each of these contribute to the difficulty in the transition from elementary school to middle school. Alspaugh (1998) found the loss in achievement to be significant among students transitioning to middle school.

Research indicates that in high poverty areas, middle school experiences highly impact students (Balfranz, 2007, 2009; Bruce et al., 2011). At 6th grade, if students have at least one of the following indicators; failing a mathematics or English class, less than 80% attendance, or receiving a poor behavior grade in a core class, they have a 10-20% chance of graduating high school on time (Balfranz, 2009). In mathematics, it is far too common for students to struggle, and among the above listed indicators, mathematics is the focus of this study.

According to the National Educational Assessment Program (NAEP) results for 2015, 40% of 4th graders across the United States were proficient in mathematics while 33% of students in 8th reached proficiency (NCES, 2015). Across each subset of race or geography (urban, city, rural, town), proficiency from 4th grade to 8th grade dropped. Asian/Pacific Islander students outperform all groups, while White students scored higher than African American and Hispanic students. The trends in the achievement difference between 8th grade African American and White students has not changed significantly from 1990 to 2015, although scores for both subsets have increased. That trend is consistent with Hispanic and White students from 1990 to 2015 as well (NCES, 2015).

Additionally, the difference in scores between those eligible for the National School Lunch Program and those not eligible has been the same from 2003 to 2015, the years tracked for these subsets. Although scores have increased in both groups, students from wealthier families consistently score higher (NCES, 2015). Although grades were a better predictor for high school graduation than standardized test scores, these differences by racial and economic subsets are alarming (Balfranz, 2009). These assessments are an example of one indicator for mathematical proficiency and although the assessments do not give a complete understanding of students' mathematical abilities, they highlight the need to examine student understanding of mathematics[1].

As evidenced by the NAEP data, mathematics achievement has not been equitable among racial groups and has become a civil rights issue (Moses & Cobb, 2001). Efforts to reform

[1]Using the term mathematical understanding can be interpreted in several ways. Understanding is a complex measure, but can be defined as, "we understand something if we see how it is related or connected to other things we know" (Hiebert et al, p 4, 1997)

mathematics education and improve student outcomes has been occurring for decades. The National Council of Teachers of Mathematics (NCTM) distributed the *Curriculum and Evaluation Standards for School Mathematics* in 1989 as goals for mathematical instruction inthe classroom. The *Standards* claimed to push for equity in the classroom with a renewed focus on mathematics teaching and learning. NCTM published the statement:

> The social injustices of past schooling practices can no longer be tolerated…Mathematics has to become a critical filter for employment and full participation in our society. We cannot afford to have the majority of our population mathematically illiterate. Equity has become an economic necessity. (Martin & Larnell, 2013, p. 383; NCTM, 1989, p. 4)

Although efforts have been and continue to be made to address this issue of mathematics achievement since this statement in 1989, there is still much to be done. One example of reform to create change has been the Common Core State Standards (CCSS) (National Governors Association, 2010). While the CCSS might offer some possibilities for working together across schools and districts, it does not solve the problem of making schools better for students, in and of itself. The standards can be a supportive tool if current school policies are changed and school environments are created where student thinking in mathematics is more valued in driving instruction than textbook choice.

Within the mathematics content standards of grades 4-8, building rational number understanding and algebraic thinking is central. However, algebra has been described as a gatekeeper to upper mathematics and degree attainment because of the high percentage of students that fail algebra classes and become halted in their educational progress (Moses, 2001). Failure in algebra though is not a result of one class or year, but a result of many years of mathematical misunderstanding compounded by increasingly difficult content. Often, we blame the algebra class or the students themselves, however the failure rate is related to a variety of issues not due to the student or even the teacher. One potential challenge is the way textbooks have structured content across the grades to build mathematical concepts rather than building on student thinking (Fosnot & Dolk, 2001). Students have not had as much opportunity to develop the connections in their thinking between important mathematical ideas and developing the link between formal mathematical notation and informal mathematics problem solving.

> A single concept usually develops not in isolation but in relationship with other concepts, through several kinds of problems and with the help of several wordings and symbolisms. (Vergnaud, p, 141, 1988)

Developing algebraic thinking is a process that develops over time, not just in algebra class and can begin in the elementary classroom (Carpenter et al, 2003; Blanton, 2008).

In order to improve student outcomes in mathematics, learning must build from what students already know (Carpenter et al, 1999, 2015). However, the way in which mathematics is usually taught is in a discrete manner that separates ideas of multiplication & division to fractions to proportional reasoning and linear functions. These are thought of as completely different domains and ideas, when in fact they are all connected around multiplicative thinking. When taught in a discrete manner, however, students are not able to build a coherent understanding of the mathematics necessary for algebraic success. This multiplicative field (Vergnaud, 1988; Lamon, 2007, 2012) needs to be examined more in the ways in which classroom instruction can build the coherence between mathematical concepts. Multiplicative thinking takes time to develop (Clark & Kamii, 1996) and thus can be further developed through coherent instruction that builds on students' own thinking of the concepts and can lead to an ability to understand algebra.

If we look to the example of research in elementary mathematics, we know students can solve a variety of problems posed to them. Students solve problems in a variety of ways and want to share their strategies. Research on student thinking in a number of domains of elementary mathematics has led to developing trajectories of student thinking. This work has been done with single and multi-digit addition and subtraction, single and multi-digit multiplication and division, place value, algebraic thinking, and fraction understanding (Carpenter et al., 1999, 2003, 2015; Empson & Levi, 2011). Building on student thinking as students move through elementary school and into middle school can then allow teachers to link mathematical ideas to more formalized mathematics procedures that are often the focus of middle school. Instruction based upon student thinking also has the potential to position students differently in the classroom. Students no longer are the receptacles of knowledge filled by the teacher, but are positioned as sense makers and mathematicians with which to draw important ideas (Battey & Chan, 2010; Battey & Franke, 2015).

There is a continued need within research to understand student thinking within different mathematical domains and build frameworks for teachers to comprehend and use in their classroom instruction. Work is currently being conducted in the field of student thinking for these middle years (grades 4-8) school mathematics within the contents of proportional reasoning, fractions, and integers (Steinthorsdottir, 2009; Riehl & Steinthorsdottir, 2014; Empson & Levi, 2011; Pierson Bishop et al, 2014). This research continues to be developed to construct frameworks of student thinking to inform and guide instruction. These frameworks of student thinking provide a lens into the ways in which we can build connections to multiplicative thinking from multiplication through linear functions.

A means in which to build coherence within the mathematics is to find a context that can extend student thinking from multiplication through linear functions. One context that needs to be examined is equal grouping problems (Carpenter, 1999; Empson & Levi, 2011). Mathematical problems within the context of number of groups x amount per group = total amount in all groups has been shown to be an important foundation for multiplication and division of whole numbers as well as a means to develop fraction understanding (Baek, 2005; Empson & Levi, 2011). Through problem solving, students can develop the understanding of the relationship between the three quantities. This same relationship of the three quantities, groups, amount in each group, and total, is the same relationship developed in the equation representing proportional reasoning $y = kx$ and $y = mx$ for linear functions. In these cases the y represents the total amount in all groups, the x represents the number of groups, and k or m represent the amount per group. Having a foundation of multiplicative thinking within the context of grouping problems can allow for the connection to higher mathematics. This can then draw upon student understanding to build mathematical content knowledge.

This study investigated: how do students' varying understandings of grouping, in particular, their use of grouping strategies, support their ability to solve fraction and then linear function problems and thus begin to articulate a trajectory of student thinking from grouping to linear functions?

This question was examined through classroom observation and one on one conversations with students in a 5^{th} grade classroom. This study took place in a 5^{th} grade classroom due to the focus on grouping and the work on fractions with little attention to linear equations. The one on one mathematical conversations were conducted four times across the 2016-2017 school year. The first conversation elicited students' understanding of grouping as they solved whole number multiplication, measurement division, and partitive division problems. The second conversation addressed how students used their understandings of grouping to solve multiple group unit fraction problems, the third conversation targeted multiple group non-unit fraction problems, and the fourth conversation examined two-step rate problems, common problems used for linear equations. An important aspect of this research was to take a somewhat different approach to the one on one conversations with students and to use data collection approaches that recognized that mathematical thinking is developed in connection with students lived social and historical experiences, is contextual and assessing it requires attention to who students are and building relationships that support interaction. As such a goal of my research was to become a member of the community within the classroom by regularly visiting the classroom, collect evidence of students' mathematical understanding in the classroom context, and include opportunities in the conversation process to build relationships with students that enable the researcher and student to get to know each other.

This study was additionally developed because of my own experience as a 6^{th} and 8^{th} grade middle school mathematics teacher, as well as my current work with elementary school teachers. Additionally, I draw from my own experiences working with my son, who at the time of the study was in 5^{th} grade. When working with him, I recognize his great mathematical ideas, which do not seem to be unfolded and developed through a Common Core aligned, but still quite traditional mathematics curriculum. These experiences drive my research in this field. Looking to student thinking as a means to develop mathematical understanding is key to developing this research. Listening to and learning from students' thinking, I believe, is key to improving mathematics education.

Chapter 2: Literature Review and Theoretical Framework

As this research study is one that focuses on the development of student mathematical thinking of grouping in relation to multiplication and division, fractions, and linear equations, the literature review begins with understanding the idea of the importance of developing relational thinking. Developing relational thinking with students is key to be able to connect students' understandings within mathematics. Research on equal grouping problems will demonstrate this problem structure as a means for developing relational thinking. This study also builds upon the ideas of connecting the mathematical concepts within this multiplicative field of mathematics as well as developing this field through understanding student thinking. Additionally, when striving to understand student thinking, part of the conversation process will be getting to know the students and more of the experiences they bring to the classroom.

Developing Relational Thinking

In school mathematics much attention has been paid to the topics students need to cover rather than the understandings students develop. Middle grades (4-8) mathematics content standards include the development of understanding the concepts of multiplication and division of multi-digit whole numbers, fraction operations, and algebra (ie. proportional reasoning and linear equations) (National Governors Association, 2010). Fractions and algebra are topics that are essential to later mathematical success, yet difficult to learn (National Council of Teachers of Mathematics 1998, 2000). The difficulty in these concepts and increased attention to them, coincides with the decline in student mathematical performance (NCES, 2015). Students' struggles in algebra are typically blamed on their misunderstanding of fractions because students' fluency with fraction operations has been noted as essential to algebra success (U.S. Department of Education 2008).

However, an increased focus on the operations of fractions would not be sufficient in improving student outcomes with fractions or algebra (Empson, Levi, & Carpenter, 2011). Empson, Levi & Carpenter (2011) argue that striving for better procedural understanding in fractions is not the key to shifting success, but the key is in helping students to see the connections and coherence between the mathematics with whole numbers, fractions, and algebra. They argue this is accomplished by building procedural understanding through developing students' relational thinking.

> *Relational thinking* involves children's use of fundamental properties of operations and equality to analyze a problem in the context of a goal structure and then to simplify progress towards this goal. (p 411, Empson, Levi, Carpenter, 2011)

Relational thinking includes the ability for students to begin decomposing and recomposing numbers in ways that make sense to them based upon the mathematical properties. Empson, Levi, & Carpenter (2011) argue that the development of relational thinking is so critical to understanding in mathematics, they've stated: "*to understand arithmetic is to think relationally about arithmetic*" (p. 411). By developing relational thinking and the understanding and flexibility with mathematical properties, students will become better equipped to understand algebraic operations, thus translating to greater success in the classroom.

What exactly does relational thinking look like and how do students develop it? Relational thinking is part of a framework of student thinking in which students are able to make sense of problem solving in their own way (Carpenter, 2003, 2015; Empson & Levi, 2011). Students need to engage in problem solving tasks that allow them to progress along the framework, rather than learn procedures or steps disconnected to their own thinking. Memorization of procedures to solve mathematics does not produce the thinking necessary to build the relational understanding needed. The context of equal grouping problems for both multiplication and division problems with whole numbers and fractions has been important in the development of students' relational thinking (Carpenter, 2015; Empson & Levi, 2011). Further examination of equal grouping problems and students' thinking of these problems provides more context for what relational thinking looks like and how it can be developed.

The basis for using equal grouping problems

Research on multiplication and division has distinguished four problem types: equal grouping problems, multiplicative comparison problems, Cartesian products, and array/area (Greer, 1992). The research of this study focuses on one type, equal grouping, as the means to understand and develop student thinking. Grouping problems are often thought of as just repeated addition, but the context of grouping focuses on the unitizing of the group and thinking of groups of items being added while also retaining the idea that the groups contain an individual number of items. This is an important conceptual hurdle for students when working with multiplicative ideas. (Van de Walle, 2007; Clark & Kamii, 1996; Kouba, 1989; Steffe, 1988). The focus of this study was based upon equal grouping problems because of simplicity of context and situations encountered by students (Baek, 2005) as well as the structure of the context that allows for development of various mathematical concepts and relational thinking.

The structure of grouping problems is based upon the relationship of three quantities. Depending upon which of the quantities is unknown determines whether it is a multiplication or division problem. The three quantities within each problem consist of: the number of groups (multiplier), the amount in each group (multiplicand), and the total amount in all groups (product) (Baek, 2005). Vergnaud (1988) describes this as part of "simple proportion problems" in which f(x) is the total, f(1) is the amount in 1 group, and x is the amount of groups. The type of problem depends on which of the three quantities is unknown. Within a multiplication problem, the unknown is the total number of items for all groups. If the number of groups is unknown, it is a measurement or quotative division problem. If the amount in each group is unknown, it is a partitive division problem (Greer, 1992; Carpenter et al, 1999, 2015). This study looks at these three problem types within equal grouping: multiplication, measurement division, and partitive division as important components of instruction. Each of these problem types helps to build important mathematical ideas around the relationships of the three quantities that can connect mathematical concepts.

Table 1

Grouping problem types and structures

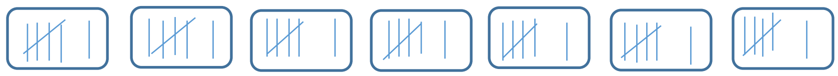
Number of groups x amount in each group = total for all groups
Multiplication: Lesieli has 7 packages of gum. There are 6 pieces of gum in each package. How many pieces of gum does Lesieli have? 7 packages of gum x 6 pieces per package = Unknown total pieces of gum for all packages
Measurement Division: Lesieli has 42 pieces of gum. There are 6 pieces of gum in each package. How many packages of gum does she have? Unknown numbers of groups x 6 pieces per package = 42 total pieces of gum for all packages
Partitive Division: Lesieli has 42 pieces of gum. The gum is in 7 packages, with the same amount in each package. How many pieces of gum are in each package? 7 packages of gum x Unknown number of pieces per package = 42 total pieces for all packages

The distinction of problems is important because of the way in which students think about and solve multiplication, measurement division, and partitive division problems. Student strategies will differ depending on the unknown (Carpenter et al, 1999, 2015; Van Putten et al, 2005). For example, in a single digit multiplication problem the initial strategy in student thinking is direct modeling. Direct modeling of the problem is correlated with the context and unknown of each, where students will physically represent each quantity. If the problem represented 7 x 6 a student would draw the 7 groups and represent the 6 in each group in some manner.

As students begin to think about the context of grouping and unitize the amount in each group, this allows for students to connect direct modeling strategies to counting strategies. Rather than count singular items in the group a student may count 7 x 6 as 7 groups of 6, which is 6, 12, 18, 24, 30, 36, 42. As students begin to develop their understanding of multiplication facts they are able to use what facts they know to help with others. This is known as using derived facts. An example of derived facts would be if a student is unsure of 7 x 6, but knows 6 x 6 = 36 so 7 x 6 = 6 x 6 + 6 = 36 + 6 = 42. Students typically begin to remember certain facts that can help in finding others (ie. times 2, times 10, times itself). Thus, in the trajectory of student thinking, when students are able to use derived facts to solve problems they are developing and employing their relational thinking by decomposing the multiplier into smaller groups to work with facts they already know (Carpenter et al, 1999, 2015). As in the previous case, the student decomposes the multiplier (number of groups), 7, into 6 + 1 in order to facilitate an easier calculation based on what the student knows.

Table 2

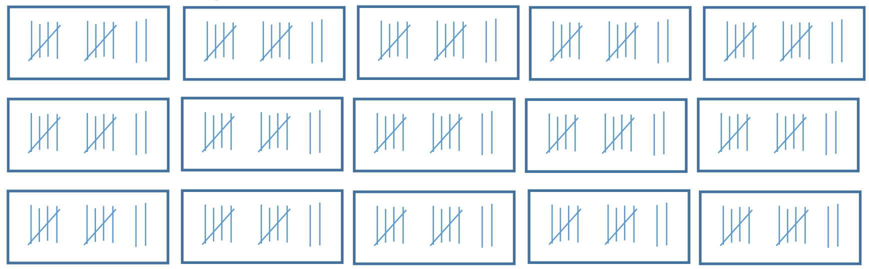
Single digit multiplication student solution strategies

Single Digit Multiplication Trajectory (7x6)	
Direct Modeling	
Counting/Adding Strategies	7 x 6 = 6 + 6 + 6 + 6 + 6 + 6 + 6 = 42 OR 7 x 6 => 6, 12, 18, 24, 30, 36, 42
Derived Facts (Relational Thinking)	Not sure of 7 x 6, but knows 6 x 6 = 36 so, 7 x 6 = (1 + 6) x 6 = 6 x 6 + 1 x 6 = 6 x 6 + 6 = 36 + 6 = 42

This same progression of student thinking in grouping problems extends into multi-digit multiplication and division (Baek, 2005, Van Putten et al, 2005). Students begin with direct modeling the quantities in the problems to represent the story. From direct modeling, students will begin to use addition strategies and subtraction strategies to solve. As students become more experienced with addition and subtraction and develop more relational thinking, sophisticated doubling and invented algorithms begin to be used as students make connections between groups, the number of items in each group, and total items in all groups. This may include doubling strategies in which students will begin to double the amount of groups to add or subtract in less steps. Another strategy that students use is grouping of groups by partitioning the factor related to number of groups into more manageable quantities. This is usually accomplished with breaking down groups into non-decade amounts or by decade amounts which uses the base-ten number system.

Table 3

Multi-digit multiplication student solution strategy

<table>
<tr>
<td rowspan="3">Samuel has 15 packs of Pokémon cards. There are 12 cards in each pack. How many cards does Samuel have?</td>
<td>Direct Modeling
</td>
</tr>
<tr>
<td>Counting/Adding Strategies
12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 = 180
OR
12 + 12 = 24
12 + 12 = 24   24 + 24 = 48
12 + 12 = 24
12 + 12 = 24   24 + 24 = 48   48 + 48 = 96
12 + 12 = 24
12 + 12 = 24   24 + 24 = 48   96 + 48 + 24 + 12 = 180
12 + 12 = 24
12
OR
12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144, 156, 168, 180
OR
10+10+10+10+10+10+10+10+10+10+10+10+10+10+10=150
2+ 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 30
150 + 30 = 180</td>
</tr>
<tr>
<td>Invented Algorithms/Mathematical properties/Relational Thinking -Beginning use:
5 x 12 + 5 x 12 + 5 x 12 = 60 + 60 + 60 = 180
Or
2 x 12 = 24</td>
</tr>
</table>

	4 x 12 = 48 8 x 12 = 96 16 x 12 = 192 15 x 12 = 16 x 12 - 1 x 12 = 192 - 12 = 180 Relational Thinking- Strategic Use: (Distributive property of multiplication over addition) 15 x 12 = 15 x 10 = 150 15 x 10 + 15 x 2 = 120 +60 =180 15 x 2 = 30 150 + 30 = 180 Or (Distributive property of multiplication over addition) 15 x 12 = 10 x 12 = 120 10 x 12 + 5 x 12= 120+60=180 5 x 12 = 60 150 + 30 = 180 Or (Associative property of multiplication) 15 x 12 = 3 x 5 x 12 = 3 x 60 180 *Beginning use and strategic use distinction in relational thinking developed by Linda Levi.

This ability to decompose the number of groups or the amount in each group exemplifies the development of the relational thinking students need to develop as preparation for more sophisticated mathematics. Baek (2005) gives the example of a non-decade partition of groups from a student trying to solve 35 groups of 23 (35 x 23) by breaking it down to 7 groups of 5 groups of 23. A decade partition is exemplified in the student strategy of 43 groups of 24 (43 x 24) in which the 43 is partitioned into 4 groups of 10 groups of 24 added to 3 groups of 24. Students' understanding of groups and the ability to decompose the groups in decade or non-decade partitions contributes to their development of relational thinking (Empson, Carpenter, & Levi, 2011). This is also evident in the invented algorithm strategy of compensation. Baek (2005) demonstrates this with a student example for 47 groups of 34 (47 x 34). The student finds 5 groups of 34 by taking half of 10 groups. Then the student finds 50 groups of 34 with 10 times as many as the 5 groups. Since 50 groups is too much, the student subtracts 3 groups of 34 to get 47 groups of 34.

An important note to these progressions of student thinking is that these strategies do not need to be taught explicitly to students, but that students are able to progress and develop more mathematically sophisticated ideas as they are allowed to experience problem solving in ways that make sense to them, as well as discuss and share strategies with classmates and the teacher. This grouping context allows students to develop a fluidity to working with groups of items that is not developed in rote or procedural learning of multiplication and division and is a basis of informal understanding of more formal mathematical properties. In these previous examples students are employing the use of the associative property of multiplication, the distributive property of multiplication over addition, and the distributive property of multiplication over subtraction, all important properties to understand for algebraic learning. Although used, the students may not recognize the use of the properties because they are thinking of the context of groups and the relationships of numbers they understand. Students engaging in problem solving in this manner are primed to give the formal mathematical names to these strategies only after developing their use first.

The use of a grouping relationship has been beneficial in developing fraction understanding as well (Empson & Levi, 2011). Research has shown that partitive grouping problems, in particular equal sharing problems, support students understanding of fractions as a quantity, based on the relationship between items and sharers (Empson & Levi, 2011; Lamon, 2007). Rather than develop fractional understanding from shading in parts of a whole or using the terminology of parts out of the whole, which inherently uses understanding of fractions as a part of a set, equal sharing partitive division problems develop an important foundation with fraction sense. This type of problem is structured such as “4 children are sharing 6 brownies so that each child gets the same amount, how much brownie would each child get?” In this case, students develop the understanding of the relationship between the 6 brownies shared with 4 children, which means each child gets 6/4 or 1 1/2. This amount per person solidifies the understanding of fraction as a quantity, in this case the quantity each group receives.

Once students are able to understand fractions as quantities through partitioning problems, students are ready to begin work with operations with fractions (Empson, Levi, & Carpenter, 2011). Multiplication and measurement division problems extend the understanding from partitioning problems into both multiplication and division as students’ progress through their thinking from informal strategies to more formal mathematical representations (Empson & Levi, 2011). An example of multiplication would be, “Filipe is baking bread. He uses 1/4 cup of butter for each loaf. How much butter is needed to bake 12 loaves?” In this case, the 12 loaves represent the groups, 1/4 cup of butter is the amount per group, and the unknown is the amount of butter for all 12 loaves. Empson, Levi, & Carpenter (2011) demonstrate that using the context of grouping aids in the development of relational thinking in fractions as Baek (2005) did for multi-digit multiplication and division. An example of relational thinking for this problem would be using the associative property of multiplication in knowing that if 4 loaves with 1/4 cup of butter each would be one cup of butter, then 12 loaves is three times that, so 3 cups of butter (12 x 1/4 = (3 x 4) x 1/4 = 3 x (4 x 1/4) = 3 x 1 = 3). The development of relational thinking with fractions allows students to flexibly use fractions operations, which are critical to success in algebra.

Baek (2005) demonstrated that through problem solving based on student thinking rather than following a textbook series of lessons, students were able to develop relational thinking within whole number multiplication and division problems. Empson & Levi (2011) were able to do likewise with fractions. What is not known from research is whether students' understanding of grouping is connected between the mathematical concepts of multiplication and division with whole numbers and fractions. Specifically, how do students along different aspects of the trajectory within the whole number multiplication and division use that understanding within operations of fractions? For example, how do students using relational thinking with whole numbers begin to use it with fraction operations after developing understanding of fractions as a quantity? Additionally, how does students' thinking along the trajectory within whole numbers and fractions project to other mathematics such as proportional reasoning and linear functions. In both the cases of Baek (2005) and Empson & Levi (2011) the basis of grouping was used as a distinct context of problems for the particular math concept. What is now needed is the comprehension of how students understanding of grouping in multiplication and division of whole numbers connects to their ability to solve fraction problems and then how understanding of grouping connects to problems of proportional reasoning and linear functions.

Multiplicative Field

Student thinking can then be an avenue to investigate the multiplicative field (Vergnaud, 1988) of mathematics.

> …the conceptual field of multiplicative structures consists of all situations that can be analyzed as simple and multiple proportion problems and for which one usually need to multiply or divide…Among these concepts are linear and n-linear functions, vector spaces, dimensional analysis, fraction, ratio, rate, rational number, and multiplication and division. (p. 141, Vergnaud, 1988)

Research continues to be done within this multiplicative field to determine ways in which to improve student outcomes in mathematics (Lamon, 2007; Lobato & Ellis, 2010). Lamon (2007) describes the importance of several interconnected ideas that build students' comprehension within the multiplicative field: Unitizing, Relative thinking, Sharing & Comparing, Quantities & Covariation, Reasoning up & down, Measurement, and the five sources of meaning of a/b. Though not exhaustive in this list, it does capture the distinct yet interrelated ideas that need to be developed to understand the rational number system and proportional reasoning, key ideas for algebra and higher mathematics (Lobato & Ellis, 2010). This study will examine the ways in which grouping problems facilitate learning between the different concepts within this multiplicative field.

Grouping problems connect mathematical concepts because of the relationships within the three quantities discussed previously; the number of groups (multiplier), the amount in each group (multiplicand), and the total for all groups (product). In measurement division the total is divided by the amount in each group. In partitive division, the total is divided by the number of groups to find the amount per group. The notion of consistently working on these problems can potentially be mapped to proportional reasoning and linear equations. These same three quantities can be algebraically represented as $y = kx$ for a proportional relationship where y is the total for all groups, k is the proportional rate or amount per group, and the x is the number of groups. This also connects to the representation of $y = mx$, the formal mathematical notation for a linear function. Again, the y represents the total, the m, known as the slope, is the amount per group, and x is the number of groups. Essentially when students are working with partitive division problems they are finding the slope of an equation. This is true with both whole number and fraction partitive problems. Thus, students are working on slope long before discussing in that manner. This idea is represented $m = y/x$, which is finding the slope and also the representation of a partitive division problem.

Based on this connection of mathematical ideas, my hypothesis was that if students were able to develop relational thinking within whole number multiplication/division as well as with equal sharing problems, this would contribute to students' strategies with multiplication and division with fractions and then linear equation problems. Increased student experience with grouping problems would build student understanding of slope and algebraic ideas informally, which could then be connected in a more formal manner. Student understanding of slope within grouping problems could then potentially be mapped out on the coordinate plane as a graph of the relationship between y and x. If possible to connect in this manner, algebraic topics of slope and graphing would no longer be as abstract and disconnected topics as they have traditionally been. Aspects of this hypothesis did occur during the study and will be explained further within the results.

The benefits of attention to student mathematical thinking

Understanding student thinking has been an important tool for teachers and for student success (Carpenter et al 1989, 1999, 2003, 2015; Empson & Levi, 2011; Fuson et al 1997; Kamii, 1989; Labinowicz, 1985). Carpenter et al. (1989) assessed teachers' classroom practice based on the use of a framework of knowledge of students' thinking with addition & subtraction problems and teachers' change in beliefs and knowledge (Franke & Kazemi, 2001). Articulating a framework for addition and subtraction problem types and detailing different student mathematical strategies was a means for teachers to use student thinking to guide instruction in the classroom. This work came to be known as Cognitively Guided Instruction (CGI) (Carpenter et al., 1999, 2003, 2015).

Carpenter et al (1989, 1999, 2003, 2015; Empson & Levi, 2011) has expanded teachers notions of what students are able to do mathematically. Teachers' expectations of their students' mathematical thinking has changed immensely. Teachers have come to recognize the varied strategie+s their students use in problem solving (Franke & Kazemi, 2001). Teachers have become learners from their students because of the unexpected range of mathematical strategies and need to develop practices and questions to elicit student thinking (Franke, Kazemi, & Battey, 2007). Listening to student mathematical ideas has become a central aspect of classrooms implementing these principles (Franke & Kazemi, 2001). In order to facilitate the elicitation of student mathematical ideas, classroom practice has changed for the teachers. The practice began to include posing word problems to solve, eliciting and sharing of multiple student strategies, and using what was heard from students to make instructional decisions (Franke, Kazemi, & Battey, 2007).

The significance of Cognitively Guided Instruction work is it has revealed the opportunity for teachers to become experts in their students' thinking and to experiment in their teaching practice (Carpenter et al, 1996). Teachers practice drew upon three ideas; detailed knowledge of the development of student thinking, organization of student thinking connected to the mathematics content, and the notion that they could continually learn from their practice and students (Franke, Kazemi & Battey, 2007). The aspects of engaging children differently, listening to students thinking, and having a research base of student knowledge and problem types, contributes to teachers' pedagogical content knowledge (Shulman, 1986). The Carpenter et al (1989) study indicated students who engaged in this manner had significantly better mathematical problem solving skills, while at the same time, keeping on par with mathematical facts skills, as the control group. The success of the study demonstrated students engaged in Cognitively Guided Instruction (CGI) allowed for greater insight into student thinking.

Since the Carpenter et al. (1989) study, research on teacher learning of student thinking has consistently demonstrated a change in teacher beliefs and practice. This has resulted in a change in student outcomes (Carpenter, Fennema, & Franke, 1996; Carpenter et al., 1999; Fennema et al., 1993, 1996; Franke, Kazemi, & Battey, 2007). As a result of this change in teaching practice, teachers continue to investigate in their classroom instruction with eliciting student thinking. This has led to new understandings of student thinking in other mathematical domains (Carpenter et al, 1993, 1999, 2003; Greer, 1992; Fuson et al., 1997; Empson & Levi, 2011).

As teachers began recognizing different ways in which to elicit children's thinking, some of the research shifted from individual interviews to examining teacher's beliefs and identities (Fennema et al., 1996, Franke et al. 1998, 2001). This shift in examining teacher identities and practice draws upon the theory of communities of practice (Lave & Wenger, 1991; Wenger, 1998). This led to examination of how student thinking is developed in the context of the classroom (Maldonado et al., 2009; Turner, Dominguez, Empson, & Maldonado, 2013; Webb et al., 2008, 2009).

The case for understanding and basing instruction on student mathematical thinking is substantial (Carpenter et al., 1989; Carpenter et al., 1996; Carpenter et al., 1999, 2003, 2015, Empson & Levi, 2011). It is imperative that this work continues, by studying and developing understanding of students thinking and their link from elementary school concepts through middle school. The study of student thinking in the context of middle grades (4-8) has potential for improving teacher understanding of student thinking at this important age and mathematical time of development to bridge greater success in high school mathematics and beyond.

Getting to know students

In addition to the achievement benefits of basing instruction on students thinking, teachers and researchers now recognize the valuable mathematical thinking students bring to school. Attending to the mathematical ideas students bring to school has the potential to shift the power dynamics of the classroom. Students can be positioned as sense-makers and as assets to the instruction. Understanding student thinking has potential to challenge deficit notions teachers may bring to the classroom and have historically held with students of color by focusing on what students can do rather than what they cannot and recognizing students as mathematically intelligent. The positioning of students in this manner allows students to view themselves as mathematicians, problem solvers, and capable of being successful in mathematics. Instruction in this manner has the potential to change the dominant view that only certain people are and will be successful in mathematics. (Battey & Franke, 2015; Battey & Chan, 2010)

Students' mathematical thinking is developed in connection with the lived social and historical experiences of the students. This mathematical thinking occurs in the home, community, and in the classroom. Although not specifically studying the lived experiences of the students, this research recognizes that understanding students' mathematical thinking requires getting to know the student and building relationships with them.

Here, I take the stance that mathematics teaching and learning is a social practice in which educators, parents, and students participate (Boaler & Greeno, 2000). Lave and Wenger (1991, 1998) argue that learning occurs through participation within communities of practice. Lave and Wenger (1991) define community of practice as

> ...a set of relations among persons, activity and world, over time and in relation with other tangential and overlapping communities of practice (p. 98).

Students participate in overlapping communities of practice around mathematical learning. The classroom is one particular community as is the conversation session. Within a community of practice the student brings multiple identities based on their social, cultural, and political histories (Gutierrez; 2009; Martin, 2007). These identities are continually constructed and shaped through interactions (Holland et al, 1998). One of these identities is mathematical identity. Martin (2007) defines mathematical identity as:

> …the dispositions and deeply held beliefs that individuals develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics to change the conditions of their lives. A mathematics identity encompasses a person's self-understandings and how they are seen by others in the context of doing mathematics (p.150).

This definition highlights students' perceptions of their ability to participate in the mathematics education community. Students' identities shape how they participate and in turn their participation shapes their identity. Drawing on students' cultural practice, recognizing their experiences and knowledge they bring to the classroom, supports their participation and the development of their identities as capable and knowledgeable mathematics students.

There is an imperative need for educators and researchers to recognize the importance of positioning students as intellectual resources within the community in which funds of knowledge of participants can be drawn upon. Funds of knowledge are the historical and cultural skills and knowledge that have been developed to enable a household to function within a given society (Moll et al, 1992). With student engagement built upon these mathematical funds of knowledge, participation in school mathematics education will change. Wenger (1998) depicts the need to create:

> …inventive ways of engaging students in meaningful practices, of providing access to resources that enhance their participation, of opening their horizons, so they can put themselves on learning trajectories they can identify with, and of involving them in actions, discussions, and reflections that make a difference to the communities they value. (p. 10)

Individuals are shaped by their interactions within the cultural and historical experiences of their lives (Lave & Wenger 1991; Nasir, 2002; Wenger 1998). Focusing solely on student mathematical thinking in one instance does not directly address the socio-historical and cultural identities of students and how they contribute to the understandings students bring to and demonstrate in the classroom. Spending time to get to know students and understand their participation in the classroom provides insight into who students are and what they know. Knowing this can allow researchers to create settings to support students to share their mathematical thinking and position students as competent. The goal here is to move closer to blending the fields of student thinking and student identities productively. If race, culture, and gender historical experiences are not taken into consideration, mathematics education will continue to play a part in the inequities that exist in society (Franke, Kazemi, & Battey, 2007; Gutstein, 2003; Moses & Cobb, 2001)

> Just as knowing how individual students' thinking is supported by knowing trajectories of students within particular content domains, knowing students' identities in relation to school and mathematics is supported by knowing the communities in which they have participated…Teachers need to recognize the forces inside and outside that shape the multiple identities for students as they engage in mathematics. Teachers need to know how to draw students' identities into the mathematical work… (Franke, Kazemi, & Battey, p 248, 2007)

The goal of this work was thus two-fold; to examine both the mathematical thinking students bring to the classroom within this multiplicative field, but to also to conduct one on one conversations and participate in the classroom in a manner that allowed for the researcher to come to know students as part of the process of understanding their mathematical thinking.

Chapter 3: Methods

This chapter outlines how students' understanding of grouping problems was examined to understand how they solved whole number, fraction, and linear equation problems. The study took place in a 5th grade classroom where the teacher worked to support her students to develop ideas of grouping and where I spent time building relationships with students and collecting student work. Over four conversations, students transitioned from solving whole number multiplication and division grouping problems, to unit fraction and non-unit fraction grouping problems, and finally worked with two-step rate problems with the structure of $y = mx + b$.

In this chapter, first, I describe the participants of the study with details around the students and classroom teacher. Next, I discuss the data collection sources, in particular, the four conversations with students over the school year. Finally, I describe the process of data analysis used.

Participants

Students in this study came from the 5th grade classroom of Ms. Beaumont (pseudonyms used for all participants in the study). The class consisted of 27 students. The public school is located in a primarily low income predominantly Latino community of southern California. Student demographics of this school include 80.4% of students that identify as Latino, 8.3% identify as White, 5.9% identify as African American, 2.6% identify as Asian, 0.9% identify as Pacific Islander, and 0.3% identify as Native American. The students had participated in mathematics from a traditional textbook format in their years prior to 5th grade at this school. Teachers at the school over the previous few years had been trying to implement more of the standards for mathematical practice (National Governors Association, 2010) and the school had recently adopted a CCSS aligned textbook.

The teacher in this study, Ms. Beaumont was in her 16th year of classroom experience. Ms. Beaumont has an extensive understanding of students' mathematical thinking based upon her years of training and practice with Cognitively Guided Instruction (Carpenter et al, 2015, 2003; Empson & Levi, 2011). Ms. Beaumont teaches in a way that uses problem solving to elicit student thinking, values students' understanding, and builds upon students' own mathematical sense making. Additionally, her classroom is a place in which students develop understanding by engaging with each other's ideas and are able to learn from each other because of their development in being able to justify and explain their reasoning. Ms. Beaumont also had experience teaching 4th and 5th grade in which she has developed students' ideas of grouping through problem solving.

Ms. Beaumont's 5th grade classroom was chosen because of her experience in developing students' understanding of grouping problems and because 5th grade students are most likely in a place to be developing the mathematical concepts of focus and will not have been taught many of the formalisms. Fifth grade students are preparing in mathematical content to transition to middle school and more algebraic content. 5th grade content standards (National Governors Association, 2010) contain operations of whole numbers, operations with fractions, and beginnings of graphing on a coordinate plane. This allows for an examination of students' understanding of the link between whole numbers, fractions, and linear functions with algebra.

Throughout the 2016-2017 school year, students in Ms. Beaumont's class were engaged in mathematics based on the Common Core Standards for 5th grade (National Governors Association, 2010), through the use of Cognitively Guided Instruction (Carpenter et al, 2003;

Carpenter et al, 2015; Empson & Levi, 2011). Ms. Beaumont did not use a particular textbook curriculum in her classroom, but designed problem-solving tasks based on her understanding of the Common Core State Standards and her understanding of her students thinking. Ms. Beaumont posed story problems to students with the goal of continually developing more sophisticated understanding of the mathematics. The class curriculum generally followed the outline of the equal grouping problems discussed for the four mathematical conversations. First, students engaged in multiplication and division whole number problems, then transitioned to equal sharing partitive division fraction problems. Next, students engaged in multiplication and division of fractions and then adding and subtraction of fractions. Additionally, students worked with decimals, place value, and geometry content. Ms. Beaumont structured her class in a way that allowed students to solve problems in a way that made sense to them, gave them time to solve for themselves or with a partner, and then connected students' ideas with a discussion or share-out of ideas (Smith & Stein, 2011; Kazemi & Hintz, 2014). Students consistently participated with partners, small groups, or whole class in discussions that elicited and built upon students' thinking

Data Sources and Collection

To understand student thinking and to get to know students better in the classroom, visits occurred on a weekly basis for the duration of the study. The classroom time allowed me to get to know the students and their participation and thinking with mathematical problems. Additionally, more time in the classroom provided time to develop relationships with the students. Student work was also collected on a weekly basis as a means of examining student thinking.

Consent forms were given to all 27 students in the class, with 24 of the student families giving consent. These 24 students consisted of 12 female students and 12 male students. Many of these students had been placed in Ms. Beaumont's class by the principal because they had struggled in mathematics and schooling previously.

A series of four one on one conversations were conducted with each of the 24 students throughout the 2016-2017 school year. These conversations lasted between 30-45 minutes depending on student strategies for the problems. The one on one conversations were recorded by video, while students also recorded their strategy on paper. Recognizing that the specific moment of the conversation was only one representation of students' ideas and strategies and doesn't necessarily always capture the strategies students may employ, student work was also collected and used as a tool to confirm ways that students were thinking about the problem types. This class work ended up only being used to check some of the findings and to look at where students began the school year prior to the conversations.

One on One Conversations

Each conversation consisted of two parts. First, in order to get to know the students, each student and I played a game and talked for about the first 10 minutes. I used the time to talk with the student to ask them questions to learn more about their likes/dislikes and experiences. This consisted of questions about their families, what they did over the weekend, over the winter break, what activities they liked to do, how were they doing with their video games, etc. The games we played consisted of card games called Make 10 or Make 24 and/or a game students showed me that they called "Sticks," which is a strategic numbers game played with fingers of the two players. Students enjoyed playing these games and later into the sessions some students even asked what we were going to play and shared that they were excited to try and beat me at the game. These discussions combined with the visits to the classroom were an important part to engage in before discussing the mathematics because of the opportunity to build relationships where students felt comfortable with sharing their experiences and ideas. This discussion time even led to a student sharing about how he learned that his father who had been gone a long time in Mexico had died and he had suspected, but his family finally told him. Examples such as this demonstrate the importance of getting to know the students and their lived experiences. After playing the games and talking with students, I posed the mathematical story problems.

The mathematical goals and problems are detailed in Table 4. The first conversation was to capture students' strategies with multiplication and division with whole numbers. The first conversation occurred in December 2016. Students were already a few months into the school year and had been working on grouping ideas since the beginning of school. The conversation consisted of multiplication problems with 12 groups of 15 and then with higher numbers with 32 groups of 25. That was done to gauge strategies as the numbers increased. Next, a measurement division problem was posed. This was posed to identify how students figure out the number of groups of 48 in 912. With measurement division, students are given the amount in each group, so this problem will show how they work with groups of 48. The last problem type was a partitive division problem. This problem gave the total students of 132 and ask students to split that between 4 groups. This gave insight into how students constructed groups without initially knowing how much in each. These problems provided information about the ways in which students can manage their strategies across problem situation and as a base with which to compare strategies used later in the year.

The second conversations occurred in the beginning of January 2017. The mathematical purpose of the second conversation was to begin to examine student's understandings of fractions, especially towards the beginning of their fraction experiences in the class. The problems focused on student understanding of unit fractions. Students had already started working with fractions before the winter break, mostly with partitive division problems with equal sharing contexts. The first problem was an equal sharing partitive division fraction problem with 4 students sharing 10 items. This allowed students to demonstrate their understanding of fractions as a quantity that each person gets. Next, students were given a measurement division problem. This problem gives evidence of students' understanding of fraction as a quantity, but also shows students ideas about how to group fractional amounts together. To connect the ideas of grouping fractional amounts, a multiplication of fractions problem was also posed. This problem asked how much 12 groups of ¼ was. Table 4 also provides examples of extra number sets that were posed to some of the students after the initial number set to examine further understanding.

The third conversation occurred at the end of April 2017. The mathematical goal of the third conversation was to see how students were continuing to develop their grouping strategies with larger whole numbers and with non-unit fractions. Students had been working with both of these ideas for several months up to that point. Also, two of the problems in this conversation incorporated a rate context, to see how students' ideas connected between grouping contexts to rate contexts. The first problem was a multiplication problem of 132 x 24 to revisit students whole number grouping strategies to compare to Conversation 1. Second, was a multiplication problem of 16 x 2/3 to examine students' multiplication with a non-unit fraction to compare to their unit fraction work in Conversation 2. The third question examined non-unit fraction ideas with a measurement division problem of 24 ÷ ¾. After students solved this problem, students were asked if they could represent their solution strategies on a graph. Students had done this once in class with a measurement division problem, and this was the initial attempt to graph within our conversations together, to begin to see how students connect ideas between grouping and graphing.

The fourth conversation occurred at the end of May 2017 and beginning of June 2017. This was at the end of the school year and had a goal of being able to capture students use of grouping that they had developed over the year to work on problems that they hadn't experienced yet. The two problems posed were two-step rate problems that captured the problem structure of y=mx+b. One was with an increasing slope and the other with a decreasing slope, to see how students grouping strategies would connect to solving each of the problems. After each of the problems, students were asked if they could represent their solution on a graph. In each of these graphing problems, students were provided with a graph that had the axes labeled and numbered, since they had not discussed scaling previously.

Table 4

Problems used in four conversations

Conversation	Goal	Questions/Activities
1	A) Get to know students B) Understand students' strategies with whole number multiplication & division (measurement & partitive)	A) Get to know you game – while playing ask: - What games do you like to play at home? Who do you play them with? -What else do you like to do for fun? - What does your family like to do for fun? B) Multiplication - (enter student's name) has 12 packs of Pokémon cards. There are 15 cards in each pack. How many cards does (enter student's name) have? C) Multiplication- (enter student's name) has 32 packs of Pokémon cards. There are 25 cards in each pack. How many cards does (enter student's name) have? D) Partitive Division –There are 132 students in the 5th grade at Green elementary and four classrooms. How

		many students are in each class, if there are the same amount? Potential Prompt for more sophisticated strategies, need to know what they've done in class, "I remember seeing you use ____ in class, could you use that here?" E is a little harder problem, use whatever strategy makes sense to you E) Measurement Division- A school is going on a field trip to the California Science Center. There are 912 students in the school. 48 students can ride on each bus. How many buses will be needed? "Could you use that ____strategy in this problem?"
2	A) Further get to know students B) Understand students' strategies for solving with unit fractions. C) To see if students are employing similar strategies with multiplication and division with whole numbers as with fractions.	A) Get to know you game – questions to ask: - -What's your favorite subject in school? Do you like to do math? Why or why not? - Do you think you're good at math? Why or why not? - What kinds of math do you do outside of school? B) Partitive Division Fraction – ___ kids are sharing ___ pan dulce so that each get the same amount. How much does each child get? (4, 10) (8, 18) (3, 3/4) (4, 2/3) C) Measurement Division Fraction – Our family is making some tamales for Christmas. We have 12 cups of masa in a bowl. If we use ___ cup of masa for each tamal, how many tamales can we make? (1/3) (2/3) D) Multiplication with Fraction – Filipe feeds his dog 1/4 cup of dog food each meal. How much dog food is needed for 12 meals? (2/4, 12) (3/4, 12)
3	A) Further get to know students B) Examine student strategies to see how students are using strategies of equal sharing fraction problems and multiplication of	A) Get to know you activity B) Pose Problems 1) The central library has 132 shelves of books. If there are 24 books on each shelf, how many books does the library have? 2) Rate problem – If Samuel walked 2/3 of a mile to school each morning. How far will Samuel walk in 16 mornings?

	fractions with non-unit fractions. C) Examine how students connect strategy solutions to graphing.	3) If Filipe walks 3/4 of a mile per day this week and after, how many days will it take to reach 24 miles? Pose graph questions: How would you represent your strategy from this problem on the graph?
4	A) Get to know student more B) To examine student strategies in solving two-step rate problems with the y=mx+b structure. C) Examine how students connect strategy solutions to graphing.	A) Get to know you activity B) Linear Equation problems – Two step rate problems 1) Samuel is planting a garden. He plants a tomato plant that is 4 inches tall. If the tomato plant grows 4/5 inch per week. How tall will the tomato plant be after 15 weeks? How would you represent your strategy on a graph? 2) If you have 13 cans of dog food and you feed your dog 1/3 can of food each day. How much dog food will you have left after 27 days? How would you represent your strategy on a graph?

Data Analyses

The data for this analysis consists of the student written work and video from the four conversations. This analysis was to understand students use of grouping within their strategies. When solving grouping multiplication and division problems, students are asked to group and to develop ideas of grouping based on the structure of the problem. Even when using direct modeling strategies and count/add students are developing notions of grouping because of the structure of the problem, but this study looks at how students begin to marshal their understanding of grouping to help them solve the different problem types, but also more complex problems. When they marshal their understanding of grouping they're using relational thinking because they're stepping back from the problem and making a decision about how to group the groups.

Coding of the student work took several iterations to examine the different ways students were using grouping. The student strategies were first coded using the progression of strategies indicated from previous research (Baek, 2008; 2003; Empson & Levi, 2011; Carpenter et al., 2015). These included the categories of invalid strategies, direct modeling, count/add strategies, and invented algorithms. The strategies were then coded to capture some of the nuances of grouping that students were using in their strategies. For example, Table 5 depicts the nuances of the grouping that were coded to capture from student strategies.

Table 5

Student strategy coding for grouping

Question 2 –Multiplication - 32 x 25
0 – Invalid Strategy
1 – Direct Modeling
2 – Taught Strategy from previous grades/Area Model, Lattice Method
3 – Counting/Adding – Skip counting or adding without grouping
4 – Counting/Adding – Groups of groups with adding
5 – Beginning Use Invented Algorithm – Associative and/or distributive property use (groups of groups)
6 – Strategic Use Invented Algorithm – Associative and or distributive property use (groups of groups)

Note: See Appendix H for coding of all problems

After coding, I wanted to examine the strategies used by individual students across problems and conversations. To do this, I created a table of individual students' strategies across multiplication problems from each conversation (See Appendix E) and then across division problems from each conversation (See Appendix F). Additionally, I wanted to look at whole class strategies across problems. To do this, I created a table of strategies using the coding categories from previous research (See Table 7) and then with the more nuanced coding (See Table 10). These tables helped to capture what was occurring with student strategies across the problems and patterns within the strategies the class was demonstrated with their work.

As evidence of the grouping strategies that students demonstrated in their solutions, I collected examples of student work from the conversations. These examples depict the nuances in the grouping ideas from students and demonstrate students graphing strategies from Conversations 3 and 4.

With the tables of student work over the conversation, I wanted to look at student strategy use over time in a couple of ways. One way was to create line plots of student strategies over time to visually capture the movement in student strategies across problems. Another way was to compare student strategy use from Conversation 1 to the other conversations. This was done by performing chi-square tests of association to compare students use of grouping strategies in Conversation 1 in relation to later conversations. The goal in looking at the associations across conversations was not to make claims based upon individual significance, but to see patterns in strategy across students and across time. These tables of student strategies, table of associations, and student work provide evidence of the grouping students demonstrated in their solutions across the four conversations.

Chapter 4: Results

The goal of this research was to examine students' understanding of grouping and how grouping use predicted their ability to solve more algebraic and complex problems. The students solved a range of grouping multiplication and division whole number, fraction, and rate problems across the school year. The strategies themselves indicate a level of understanding of grouping, however it is also the case that looking at strategies across problems enables us to see how students leverage their understanding of grouping to think relationally and solve more complex problems.

Previous research examining children's thinking for multiplication and division problems focused on direct modeling, counting/adding, and invented algorithm strategies when detailing student solutions, and the student responses in this research mirrored those same categories (Carpenter et al, 2015; Empson & Levi, 2011). Thinking relationally within the use of an invented algorithm on the multiplication and division problems, as they worked with the number of groups or amount in each group, was then used as an indicator of deep understanding of grouping.

While students' use of grouping ideas was prevalent in the data, the pattern that emerged was somewhat different than originally hypothesized. Student strategies on multiplication and division problems did not necessarily lead to a clear developmental trajectory connecting whole number to fraction to linear function ideas, but rather the data suggest that student thinking from multiplication and division with whole numbers to fractions to linear functions involves a constellation of ideas that students develop in relation to each other. Students' constellation of ideas contributes to developing flexibility with problem solving in relation to grouping contexts in working with both whole number and fractions and begin to relate ideas to rates and graphing linear relationships.

These findings will be detailed as follows:

1) Solving a range of problems with particular number choices showed students use different strategies across the problems.

2) Examining students use of grouping within strategy provides additional information about their developing use of the mathematical properties.

3) Grouping of groups early is helpful in connecting ideas across problems, but is not necessary.

4) Developing a strong understanding of grouping is related to building understanding across mathematical concepts.

Solving a range of problems with particular number choices showed students use different strategies across the problems

Across the problems posed, students chose varying strategies across direct modeling, counting/adding, and invented algorithms. This section will examine the use of these strategies across the four conversations.

Overview of strategies along the trajectory of student thinking

Student strategies varied across the different multiplication and division problems as well as within problem type when the number choices varied. Table 6 shows that student strategies over the four conversations involved the full range of strategies from direct modeling to invented algorithms and included invalid strategies. Ninety-five percent of direct modeling strategies occurred with the problems involving fractional amounts in each group, while count/add and invented algorithm strategies were used regularly across the problems.[2]

Table 6

Number of student strategies across the four conversations

	Invalid strategy or (missing data)	Previously Shown Algorithm or tool	Direct Modeling	Count/Add	Invented Algorithm
Conversation 1					
Mult 12 x 15	1	1	1	13	8
Mult 32 x 25	2	2	1	9	10
MD 912/48	4	2	0	8	10
PD 132/4	5	2	0	1	16
Conversation 2					
Mult 12 x ¼	0	0	2	11	11
MD 12 ÷ 1/3	3	0	8	5	8
PD 10/4	0	0	22	0	2
Conversation 3					
Mult 132 x 24	3	6	0	3	12
Mult 16 x 2/3	(1)	0	0	10	13
MD 24 ÷ ¾	4	0	0	5	15
Conversation 4					
Y= 4/5x + 4	1	0	0	10	13
Y = 13 – 1/3x	0	0	9	6	9

As seen in Conversation 1, even within the first three months of school, 79% of the students were using valid strategies on all the problems. In addition, at least 70.8% of the students used either a count/add strategy or an invented algorithm and 66.6% of the students used an invented algorithm on at least one of the problems. What makes this striking is that on a pre-assessment given by Ms. Beaumont within the second week of school, 29% of the students used a valid strategy on a multidigit multiplication problem of 21 x 66. Of the 7 students that did use

[2] Within the whole number problems, some students used two previously taught strategies from 4th grade, both the open array for multiplication and a partial quotient algorithm for division. Students did not exhibit a taught strategy or algorithm with the fraction problems as they did not have experience with one.

a correct strategy, students either used a count/add strategy or a previously taught algorithm such as the standard algorithm for multiplication or the lattice method. All the students tried one of these three strategies, but 17 of the students were not able to use any of these methods with validity. In a whole number partitive division problem of 168 ÷ 5, three students had a valid strategy: one used a previously taught algorithm, one passed out items by fives and ones, and one used relationships with grouping.

Number choice mattered in Conversation 1. Within the same conversation, students chose different strategies for the same multiplication problem type with different number choices. With the lower number choice (12,15) in multiplication, more students chose to use a count/add strategy of adding up 15's, but when dealing with 32 groups of 25, students tried other strategies including invented algorithms. Their familiarity with the 25 and the large number of groups encouraged some students to try a different strategy. In looking at the division problems compared to the multiplication, there was less count/add use and more invented algorithm and invalid strategy use. On the partitive division (132/4) problem 66% of the students used an invented algorithm, more than any other Conversation 1 problem. The numbers provided an opportunity for the students to use number relationships they knew such as passing out 25 to each of the four groups and then pass out the remaining 32, or by passing out 20, then 10, then 3 to each of the 4 groups.

In looking at Conversation 2 where problems included a whole number of groups and fractional amount in each group, particularly using unit fractions, students had fewer invalid strategies and used more direct modeling strategies. Twenty-two students used a direct modeling strategy for the partitive division problem of 4 children sharing 10 pan dulce, so that they each get the same amount.

In Conversation 3, when whole number choices increased in quantity, and problems involved a non-unit fractional amount in each, there was an increase in invalid strategies, compared to Conversation 2, but there was also no direct modeling use. Students were primarily using count/add strategies and invented algorithms. Conversation 3 also introduced a measurement division problem of 24 ÷ 3, with a rate context. Sixty-three percent of students were able to use an invented algorithm on this problem even with a slight change in context.

Conversation 4 detailed student strategies with two problem types that students had not previously solved in class. These were rate problems, but added a constant or second step of adding. Only one student used an invalid strategy on either of the problems. In the problem involving a plant being planted at 4 inches and growing 4/5 inch per week, one student used an invalid strategy and 23 students used count/add or invented algorithms to solve for how tall the plant will be in 15 weeks (see Table 6). Whereas when the problem involved the total decreasing by the amount in each group each day (with the structure $y= -1/3x + 13$ or $y = 13- 1/3x$ when $x = 27$) 9 students direct modeled, 6 students used count/add, and 9 used an invented algorithm. Taking away the amount in each group for each day (using 1/3 can of dog food per day) led to more students using a direct modeling or a count/add strategy.

These data suggest that the posing of larger numbers around easily accessible quantities can support students to begin to develop these strategies. Additionally, the range of problems matter because these different problems afford different opportunities to work on these strategies.

Examining students use of grouping within strategy provides additional information about their developing use of the mathematical properties

Students use of groups of groups coincided with their development of more strategic use of the mathematical properties and thus indicates students use of relational thinking. This section examines student use of groups of groups within the count/adding and invented algorithm strategies and re-examines the students' strategies to address these more nuanced ideas of grouping.

Grouping of groups within counting/adding strategies

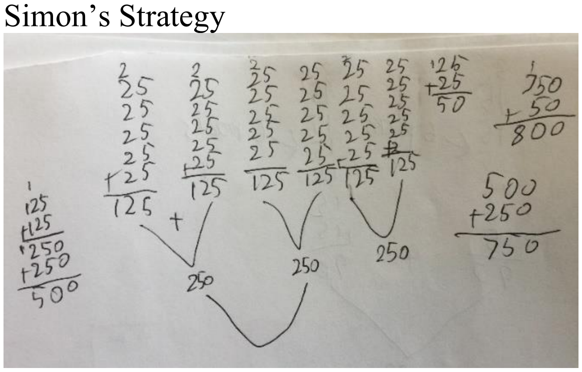
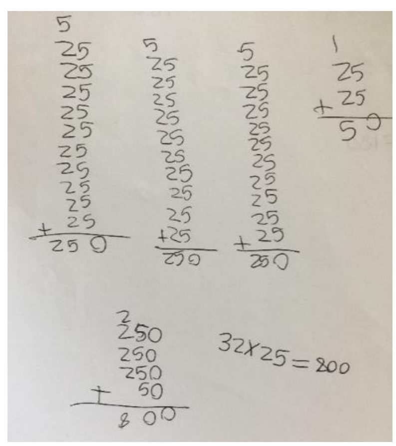
Count/add strategies begin when students begin the transition of abstracting from their direct modeling representations to using the quantities in the story to add or subtract. In multiplication and division problems, often students naturally do this by counting by the number repeatedly in a skip counting manner or by adding up with a repeated addition or repeated subtraction strategy, still representing the groups from the structure of the problem individually. Previous research has noted that some students begin to use a (Baek, 2008; 2005) simple doubling strategy which is a sign of students beginning to use some basic grouping of groups in pairs to help them add more quickly. In this doubling strategy, students still represent all the number of groups from the problem, but start to pair them up to add.

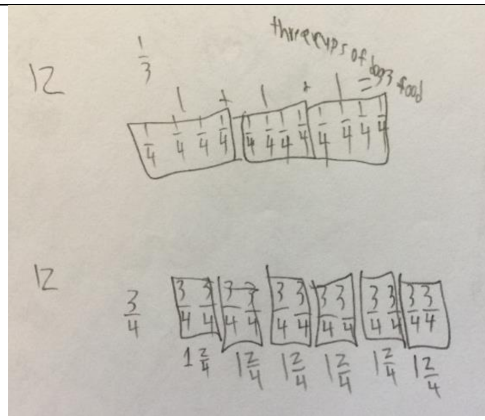
Within count/add strategies there begins to be two types of strategies, one in which students use the groups that are part of the structure of the problem and one where we start to see students use the groups of groups thinking to help them solve problems. For example, in 32 x 25, students wrote all the groups of 25, but then added 4 groups of 25, repeatedly to make 100, then added up the hundreds to find the result. In 912 divided by 48, some students added up 3 groups of 48 to get to 144 and then add up groups of 144 to get close to 912. Students would often add up the amount in each group until that got close to or more than 100, they would then repeatedly add that amount until getting close to 912. As students developed their understanding of grouping, they tried to become more sophisticated in ways to add the groups to make the adding more efficient. Students are essentially grouping the groups to help in their adding. They didn't always know how many to group or relationships to use initially, but sometimes just added groups together, figured out the total for that many groups, and then added groups of that group to continue. This is the groups of groups idea with adding.

Table 7 shows student examples of using groups of groups thinking with adding, with at least one example across each conversation to see an example of the range of ways that students used this idea.

Table 7

Student count/add strategies with grouping

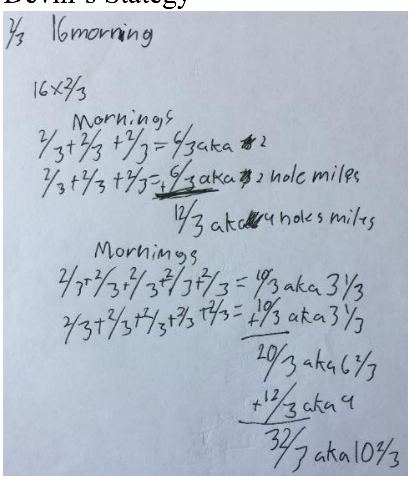
<table>
<tr><td colspan="2">Conversation 1 – 32 x 25 - Samuel has 32 packs of Pokémon cards. There are 25 cards in each pack. How many cards does Samuel have?</td></tr>
<tr><td>Simon's Strategy
</td><td>Lisa's Strategy
</td></tr>
<tr><td colspan="2">Conversation 2 – 12 x ¼ & 12 x ¾ - Filipe feeds his dog 1/4 cup of dog food each meal. How much dog food is needed for 12 meals?</td></tr>
</table>



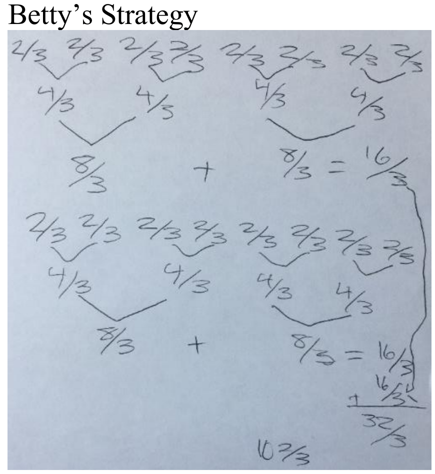
Israel's Strategy

Conversation 3 – 16 x 2/3 - *If Samuel walked 2/3 of a mile to school each morning. How far will Samuel walk in 16 mornings?*

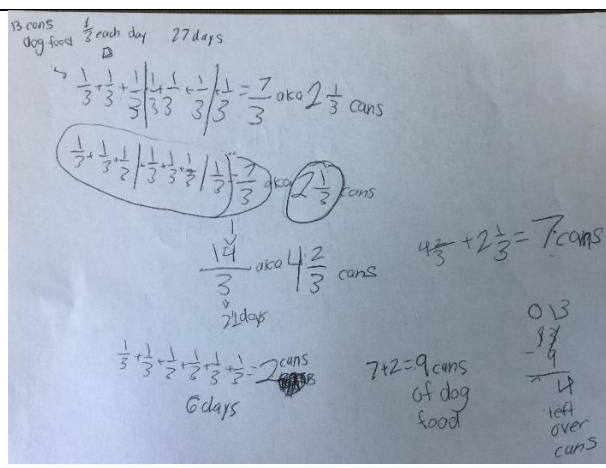
Devin's Stategy



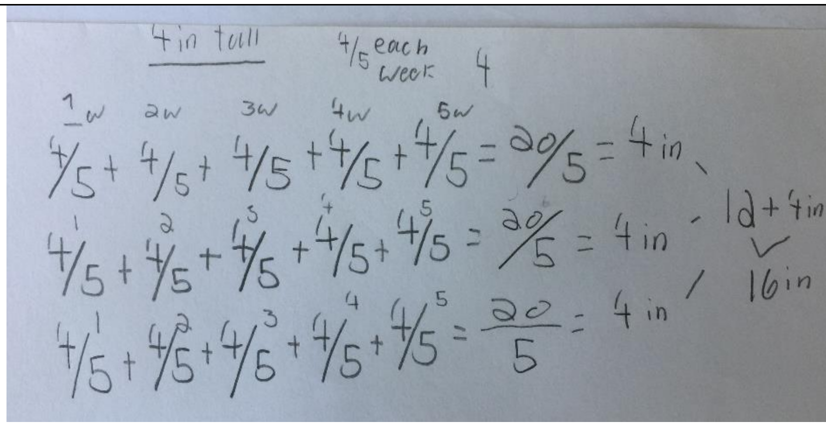
Betty's Strategy



Conversation 4 – y = -1/3x + 13 or y = 13 - 1/3x when x = 27 - *If you have 13 cans of dog food and you feed your dog 1/3 can of food each day. How much dog food will you have left after 27 days?*



Julie's Strategy

Conversation 4 – y = 4/5x + 4 when x = 15 - *Samuel is planting a garden. He plants a tomato plant that is 4 inches tall. If the tomato plant grows 4/5 inch per week. How tall will the tomato plant be after 15 weeks?*
 Seba’s Strategy

Note: See Appendix H for explanation of student strategies

Table 7 shows students’ strategies with adding where students begin to draw on their understanding of grouping groups to support their strategy use. This use of beginning grouping in count/add strategies was more evident in this study for the multiplication problems, but there was also evidence of this in the measurement division problems. It was more evident in the work with whole numbers and when working with unit fractions, than with non-unit fractions (See Table 9).

Students in these examples are implicitly beginning to use the mathematical properties, but are still extensively using adding strategies that operate on the groups individually. The groups of groups with count/add strategies is more implicit than compared to when students begin using groups of groups with invented algorithms. As students begin to use representations that are more multiplicative and keep track of the groups and total at the same time, they begin to use the mathematical properties more explicitly until they eventually are able to purposefully plan their use. This was more evident in the invented algorithm strategies.

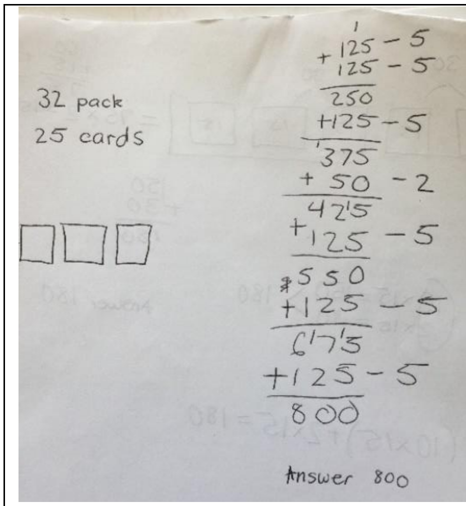
Beginning Use and Strategic Use of mathematical properties in invented algorithms

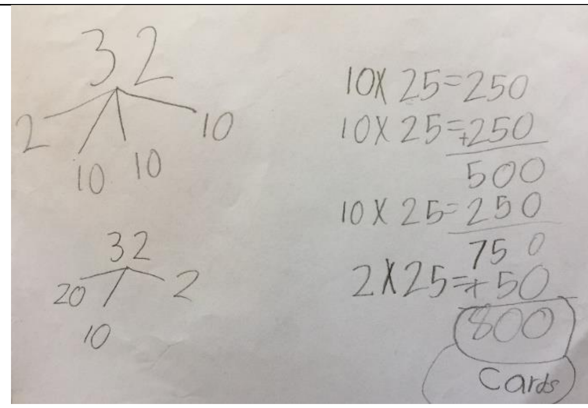
As students use invented algorithms they extend the grouping ideas used in count/add strategies in ways that show explicit and intended use of the ideas of grouping of groups and support relational thinking and the use of the mathematical properties. Invented algorithms can then be categorized into beginning use and strategic use of the associative and/or distributive properties of multiplication. Strategic use involves planned use of the associative or distributive properties. This occurs when a student decides prior to starting the strategy how to decompose the numbers in a manner that enables them to be efficient or when they recognize a relationship and use that relationship to help them draw on the mathematical properties. Beginning Use describes the early and less explicit planning to use the properties as students use relationships to group in ways that help them to solve, but often arrive at the decomposition of the desired numbers through solving not through purposeful or planned use of them. Often the distinction between Beginning Use and Strategic Use can be identified in the student written work however at times student explanation provides more insight into how they used the mathematical properties to support their solution. Table 8 will examine examples looking at the differences in Beginning Use and Strategic Use invented algorithms.

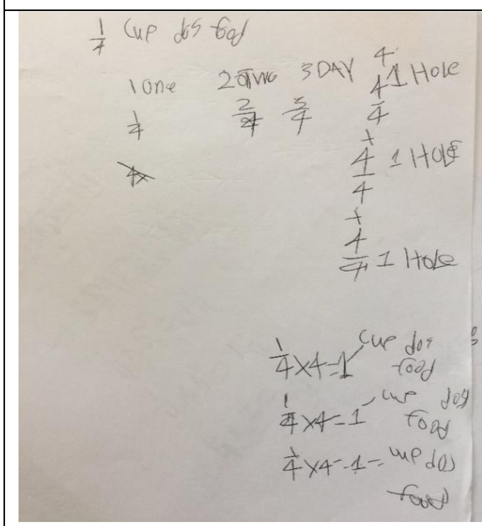
Table 8

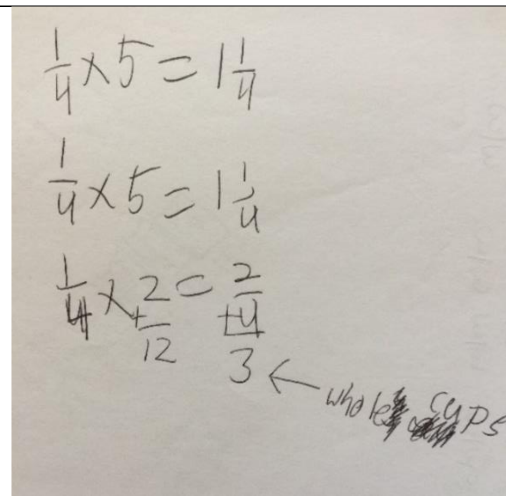
Illustrations of distinctions between Beginning Use and Strategic Use within problems

Beginning Use	Strategic Use
Conversation 1 – 32x25 - *Samuel has 32 packs of Pokémon cards. There are 25 cards in each pack. How many cards does Samuel have?*	

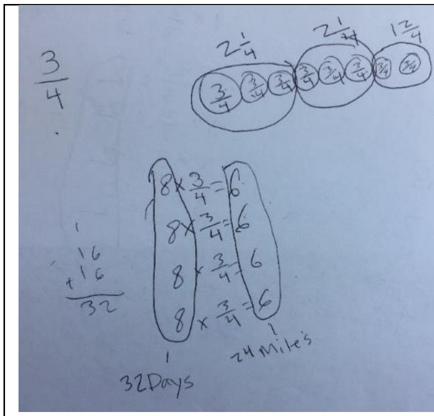
<table>
<tr>
<td>

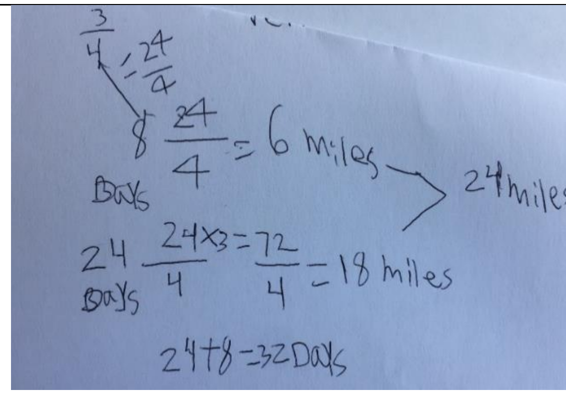
Esther’s Strategy</td>
<td>

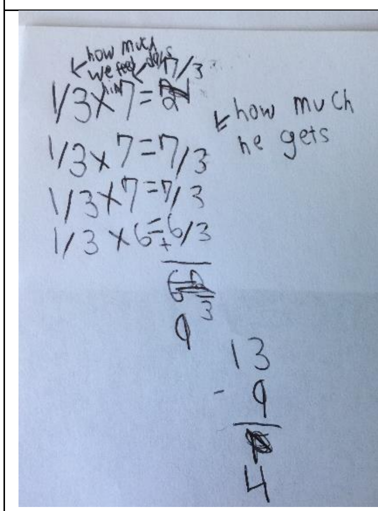
Adan’s Strategy</td>
</tr>
<tr>
<td colspan="2">Conversation 2 - 12 x ¼ - Filipe feeds his dog 1/4 cup of dog food each meal. How much dog food is needed for 12 meals?</td>
</tr>
<tr>
<td>

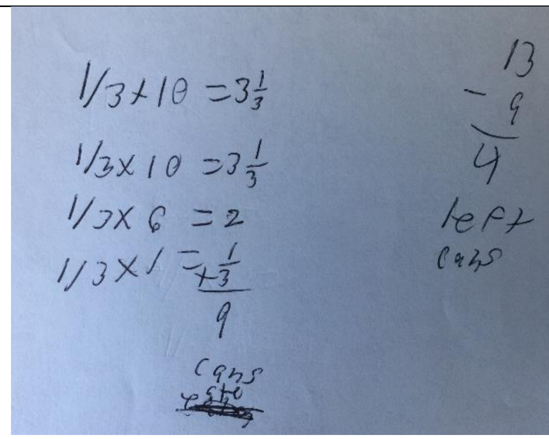
Teodoro’s Strategy</td>
<td>

Daniel’s Strategy</td>
</tr>
<tr>
<td colspan="2">Conversation 3 - 24 ÷ ¾ - If Filipe walks 3/4 of a mile per day this week and after, how many days will it take to reach 24 miles?</td>
</tr>
</table>

<table>
<tr>
<td>

Alissa’s Strategy</td>
<td>

Teodoro’s Strategy</td>
</tr>
<tr>
<td colspan="2">Conversation 4 Problem 2 – y = -1/3x + 13 or y = 13 – 1/3x when x = 27 - If you have 13 cans of dog food and you feed your dog 1/3 can of food each day. How much dog food will you have left after 27 days?</td>
</tr>
<tr>
<td>

Eric’s Strategy</td>
<td>

Daniel’s Strategy</td>
</tr>
</table>

Note: See Appendix I for explanation of student strategies

Across the strategies and problems in Table 8, the Strategic Use strategies demonstrate more explicit and planned use of the mathematical properties. For example, although the strategies from Eric and Daniel (Table 8) look similar in that the students decomposed the twenty-seven into different groups, Eric’s would still be considered beginning use based upon his explanation. He explained that he chose seven groups of 1/3 because he used the same number from a problem solved previously and then kept adding the seven groups until he realized he needed to make an adjustment at the end. The 27 was not decomposed strategically along the way, but was composed out of building up with any number of groups until the student arrived at the desired quantity. Eric is getting closer to Strategic Use, but would be categorized as Beginning for this study because of how he arrived at the 27. Daniel's strategy was categorized as Strategic because he decomposed the 27 into 10, 10, 6, and 1, planning the number of groups before and along the way. He started with the 10 groups and then 10 groups, but instead of the 7 groups together as originally thought he decided to break apart the 7 because he knew that 6 groups of 1/3 is 2 and there would be 1 group left. Understanding the difference between Beginning Use and Strategic Use in this example demonstrates the need to elicit and listen to student explanation of how they are composing or decomposing the groups or amount in each group to determine students developing use of the mathematical properties.

Coding for count/add with groups of groups, beginning use, and strategic use strategies

Table 9 parses out the use of groups of groups in the strategies students used and shows that the majority of students used groups of groups to help facilitate their solutions, whether with adding or with an invented algorithm. In each of the problems, except the partitive division of 4 students sharing 10 pan dulce in Conversation 2, as expected, 50% or more of the students used groups of groups ideas in their strategies. In Conversation 1 there are few students using Strategic Use strategies, but majority of students are using groups of groups within count/add and Beginning Use. In Conversations 3 & 4 there is still a significant amount of students using groups of groups with Count/Add and Beginning Use, but there is some shifting to more use of Strategic Use groups of grouping and to the other end of Count/Add without groups of groups.

Table 9

Categories of student strategies across the four conversations with grouping ideas

	Invalid Strategy or missing data	Previously Shown Algorithm or tool	Direct Modeling	Count/ Add	Count/ Add W/ Groups	Invented Algorithm -Beginning Use	Invented Algorithm –Strategic Use
Conversation 1							
Mult 12 x 15	1	1	1	4	9	5	3
Mult 32 x 25	2	2	1	3	6	6	4
MD 912/48	4	2	0	4	4	10	0
PD 132/4	5	2	0	1	0	11	5
Conversation 2							
Mult 124 x ¼	0	0	2	4	7	6	5
MD 12 ÷ 1/3	3	0	8	0	5	6	2
PD 10/4	0	0	22	0	0	0	2
Conversation 3							
Mult 132 x 24	3	6	0	2	1	4	8
Mult 16 x 2/3	1	0	0	7	3	6	7
MD 24 ÷ ¾	4	0	0	4	1	13	2
Conversation 4							
Y= 4/5x + 4	1	0	0	6	4	4	9
Y = 13 – 1/3x	0	0	9	3	3	2	7

Table 9 can also be examined for details in differences of students' groups of groups use within strategies and within problem types. Within count/add strategies in Conversations 1 and 2 more students are using groups of groups to help them add. However, in Conversations 3 and 4 more students use count/add without groups of groups to solve. Across the conversations in both whole number and with fractions, students used more beginning use strategies and count/add with groups of groups with the measurement division. However, in the multiplication problems is where students were more comfortable with using Strategic Use strategies. Across Conversations 2 through 4, the problems with the highest number of Strategic Use strategies are multiplication problems.

Grouping of groups early is helpful in connecting ideas across problems, but is not always necessary.

Students who had developed understandings of groups of groups early used it in other conversations and many students began to develop these understandings of grouping over the course of the study. Since students engaged with the problems differently based upon previous experiences of making sense of these problems, only 1 student in the class used groups of groups with invented algorithms in every problem across all four conversations, and this student still moved between beginning use and strategic use. Groups of groups across problems will be further examined through the individual student strategies across multiplication problems, associations of groups of group strategies from the first conversation to groups of groups strategies in others, and the student work of Teodoro.

Patterns of students' strategy use over time

Seventy-nine percent of the students used groups of groups thinking in two or more multiplication problems. Sixty-three percent of the students used groups of groups with count/add or invented algorithms with one of the whole number multiplication problems in Conversation 1 and with multiplication from Conversation 4. Sixty-seven percent of the students used groups of groups thinking in at least one of the whole number problems and in at least one of the fraction multiplication problems.

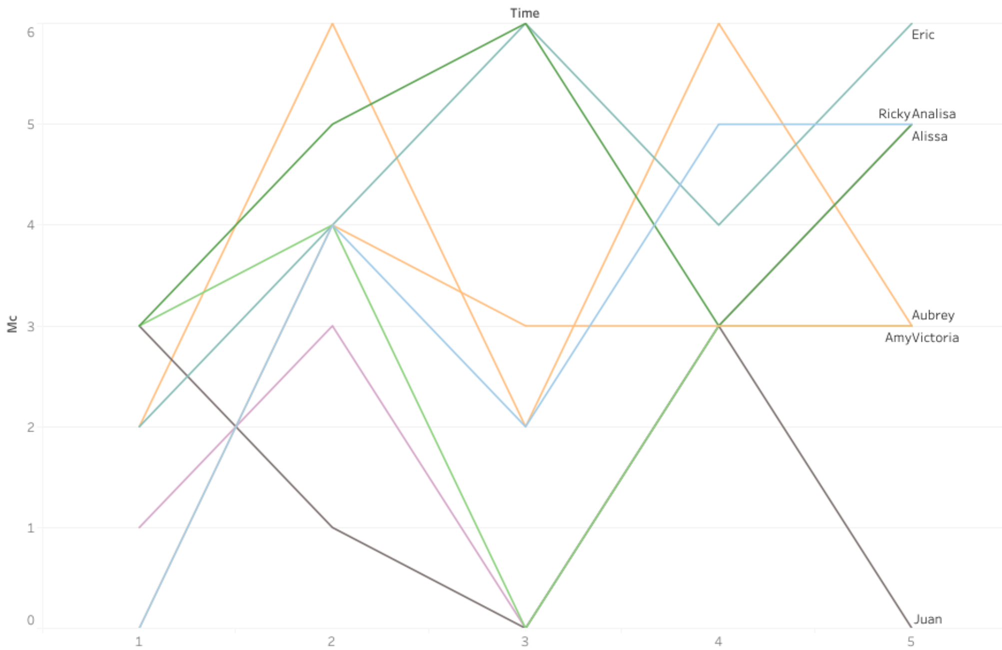
Across the division problems from the four conversations there were a few notable aspects of students' strategies. Fifty-four percent of the students used a groups of groups count/add or invented algorithm strategy on one of the division problems from Conversation 1 and on at least one other division problem in a later conversation. Additionally, one student used groups of groups across all the division problems and 29% of the students used a groups of groups strategy across all division problems except the partitive division problem of sharing pan dulce. An interesting aspect of the division problems is that with the problem of 4 kids sharing 10 pan dulces or doughnuts, twenty out of the twenty-four students used an emergent anticipatory direct modeling strategy, two used an anticipatory strategy (strategic use), and two used a non-anticipatory direct modeling strategy (Empson & Levi, 2011) rather than the more sophisticated strategies.

As seen in these percentages presented, there is repeated use within the groups of groups strategies across problems, however there are not necessarily consistent patterns for individual students. Students were able to use groups of groups strategies throughout, but that was dependent on the problem, the number choices, and the individual student. We will examine the grouping strategies across the multiplication problems by individual student to understand this more. Multiplication problems are being used to examine the consistency of groups of groups strategies because there were multiplication problems solved across every conversation.

Figures 1 and 2 depict students' individual strategies across multiplication problems from each conversation. Figure 1 depicts strategies for students that used an invalid strategy, direct modeling, a previously taught strategy, or a count/add without groups of groups strategy on the problem from Conversation 1 and details what strategies they used in the following conversations. Figure 2 focuses in on only the students who used count/add with groups of groups, beginning use and strategic use invented algorithm strategies in the first conversation and the strategies they used in the other conversations.

Figure 1

Student strategies across multiplication problems starting with invalid, DM, taught algorithm, or count/add

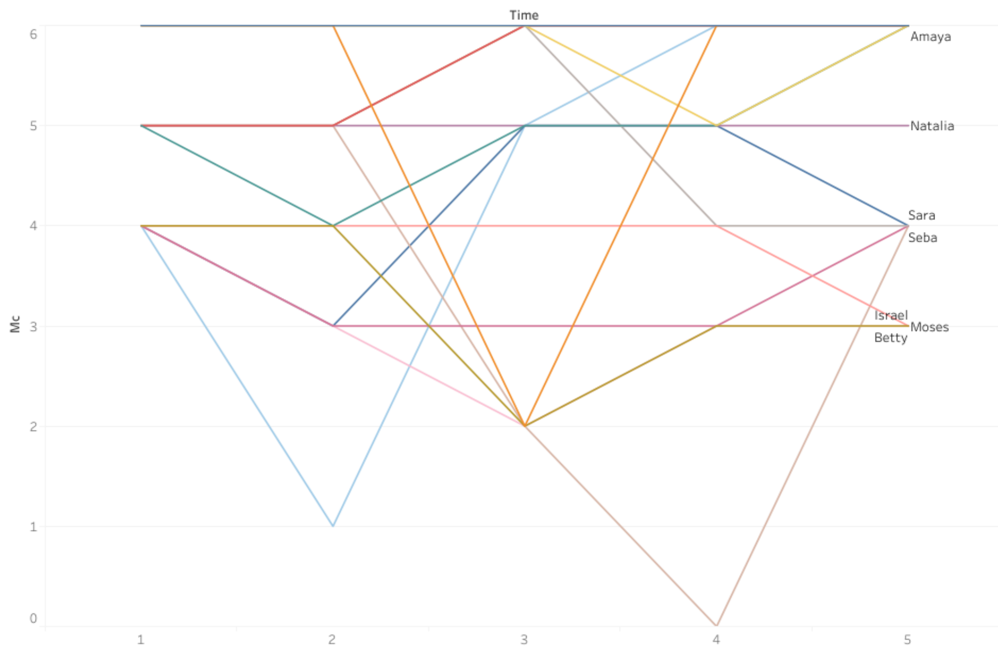


	Time				
Name	1	2	3	4	5
Alissa	0	4	2	5	5
Amy	0	4	3	3	3
Analisa	3	5	6	3	5
Aubrey	3	4	0	3	3
Eric	2	4	6	4	6
Juan	3	1	0	3	0
Ricky	1	3	0	3	5
Victoria	2	6	2	6	3

*Note: **Strategies(Mc):**0 – Invalid Strategy, 1- Direct Modeling, 2 – Previously taught strategy, 3- Count/Add, 4 – Count/Add w/groups, 5 – Beginning Use, 6 – Strategic Use; **Time**: 1 – Conversation 1- 32x25, 2- Conversation 2 – 12 x ¼, 3 – Conversation 3 – 132x24, 4- Conversation 3 – 16 x 2/3, 5- Conversation 4 – 15 x 4/5 +4*

Figure 2

Student strategies across multiplication problems starting with count/add with groups of groups, beginning use, and strategic use invented algorithms



	Time				
Name	1	2	3	4	5
Adan	6	6	6	6	6
Amaya	6	6	2	6	6
Betty	4	4	2	3	3
Daniel	6	6	6	5	6
Devin	5	4	5	5	6
Esther	5	5	6	6	6
Israel	4	4	4	4	3
Julie	5	5	6	4	4
Lisa	4	3	3	3	4
Moses	4	3	2	3	3
Natalia	5	5	5	5	5
Sam	6	6	6	6	6
Sara	5	5	2	0	4
Seba	4	3	5	5	4
Simon	4	1	5	6	6
Teodoro	5	5	6	6	6

*Note: **Strategies(Mc):**0 – Invalid Strategy, 1- Direct Modeling, 2 – Previously taught strategy, 3- Count/Add, 4 – Count/Add w/groups, 5 – Beginning Use, 6 – Strategic Use; **Time**: 1 – Conversation 1- 32x25, 2- Conversation 2 – 12 x ¼, 3 – Conversation 3 – 132x24, 4- Conversation 3 – 16 x 2/3, 5- Conversation 4 – 15 x 4/5 +4*

Each of the figures show the movement in individual strategies and use of groups of groups with varying problems. Across conversations, different strategies were used by the students in relation to the different multiplication problems. The variation includes movement in and out of groups of groups strategies and within groups of groups strategies from count/add to beginning use and strategic use invented algorithms. As seen in the figures, several students were able to use groups of groups strategies in multiple problems. As the complexity

(complexity meaning moving from whole number to unit fraction to non-unit fraction to y=mx+b structure and from grouping context to rate context) of problems increased, students continued to employ a range of strategies. Several used groups of groups strategies across problems and others continued to develop groups of groups over the study.

Association of groups of groups strategies across conversations

Table 10 details the results of chi-square tests examining the associations of students using groups of groups strategies from Conversation 1 compared to the later conversations. A few patterns arise in this table. What stands out is the significant associations between the strategies in the division problems from Conversation 1 and the strategies used in Conversations 3 and 4; this same pattern does not occur when looking at the associations with the multiplication problem from Conversation 1. When students used groups of groups strategies on the division problems in Conversation 1 they were likely to use a grouping of groups strategy for the problems in Conversations 3 and 4. Note that the problems in Conversations 3 and 4 involved non-unit fractions and that this pattern did not emerge for the unit fraction problems in Conversation 2.

Table 10

Chi-squared associations of groups of groups strategies from Conversation 1 to others

	32 x 25 - *Samuel has 32 packs of Pokémon cards. There are 25 cards in each pack. How many cards does Samuel have?* **Conversation 1**	**132 ÷ 4 -** *There are 132 students in the 5th grade and four classrooms. How many students are in each class, if they have the same amount?* **Conversation 1**	**912 ÷ 48 -** *A school is going on a field trip to the Science Center. There are 912 students in the school. 48 students can ride on each bus. How many buses will be needed?* **Conversation 1**
12 x ¼ - *Filipe feeds his dog 1/4 cup of dog food each meal. How much dog food is needed for 12 meals?* **Conversation 2**	P = 1.00 Cramer's V = .000	P = .317 Cramer's V = .204	P = .633 Cramer's V = .098
10 ÷ 4 - *10 kids are sharing 4 pan dulce so that each get the same amount. How much does each child get?* **Conversation 2**	P = .296 Cramer's V = .213	P = .296 Cramer's V = .213	P = .212 Cramer's V = .255
12 ÷ 1/3 - *Our family is making some tamales for Christmas. We have 12 cups of masa in a bowl. If we use 1/3 cup of masa for each tamal, how many tamales can we make?* **Conversation 2**	P = .386 Cramer's V = .177	P = 1.00 Cramer's V = .000	P = .408 Cramer's V = .169
132 x 24 - *The central library has 132 shelves of books. If there are 24 books on each shelf, how many books does the library have?* **Conversation 3**	P = .043* Cramer's V = .414	P = .004* Cramer's V = .591	P = .045* Cramer' V = .410
16 x 2/3 - *If Samuel walked 2/3 of a mile to school each morning. How far will Samuel walk in 16 mornings?* **Conversation 3**	P = .408 Cramer's V = .347	P = .007* Cramer's V = .548	P = .005* Cramer's V = .567
24 ÷ ¾ - *If Filipe walks 3/4 of a mile per day this week and after, how many days will it take to*	P = .032* Cramer's V = .438	P = .032* Cramer's V = .438	P = .019* Cramer's V = .478

reach 24 miles? **Conversation 3**			
Y = 15(4/5) + 4 - *Samuel is planting a garden. He plants a tomato plant that is 4 inches tall. If the tomato plant grows 4/5 inch per week. How tall will the tomato plant be after 15 weeks?* **Conversation 4**	P = .112 Cramer's V = .324	P = .000* Cramer's V = .907	P = .000* Cramer's V = .759
Y = 13 – 27(1/3) - *If you have 13 cans of dog food and you feed your dog 1/3 can of food each day. How much dog food will you have left after 27 days?* **Conversation 4**	P = .386 Cramer's V = .177	P = .009* Cramer's V = .530	P = .013* Cramer's V = .507

*Note: * signifies statistically significant p value of <.05*

So why would using a groups of groups strategy on a whole number partitive division problem be related to using a groups of groups strategy on a two-step rate problem later? The number choices encourage students to think about the groups of groups. In both cases, the grouping ideas are apparent and connected because students are able to use numbers they know to create their groups of groups strategy. The 132 they can decompose into 25s for instance, and with the 15 groups they can make 3 groups of 5 groups of 4/5. The types of problems are not so closely related, but the grouping strategies students use are related. The students can use what they know about grouping to help them think about how to solve the problems and manipulate the groups in ways that are necessary to solve.

Examining groups of groups ideas through Teodoro's strategies across problems

If we look at the following strategy from conversation 4, where the problem states, *Samuel is planting a garden. He plants a tomato plant that is 4 inches tall. If the tomato plant grows 4/5 inch per week. How tall will the tomato plant be after 15 weeks?* Teodoro used a

Strategic Use invented algorithm strategy of knowing that 15 groups of 4/5 is the same as 5 groups of 4/5, added three times as seen in Figure 4. This example is evidence of the development of Teodoro's groups of groups thinking with fractions that helped with this problem.

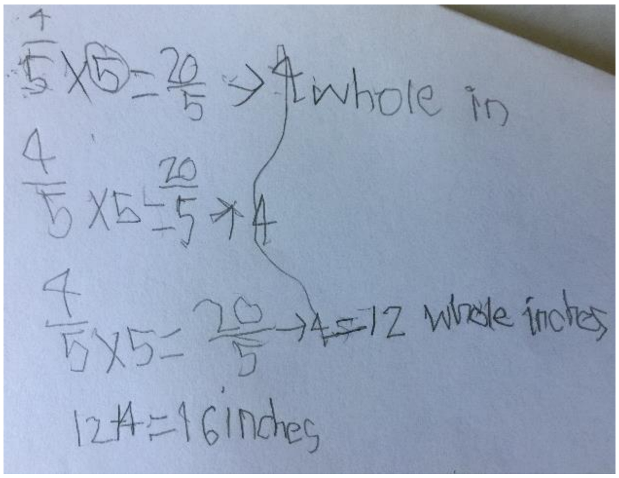


Figure 3 - Strategic Use – Teodoro

To get to this point of being able to solve in this manner Teodoro developed several key understandings evident in problem solving over the course of the school year. He developed the understanding that the number of groups or in this case the number of weeks can be decomposed into smaller quantities that make the fifteen (Figure 3). This number choice allows him to draw upon the understanding of three fives make 15. Teodoro developed the understanding of being able to break up the groups with the experience of working with whole number grouping problems. He has become particularly strategic with multiplication problems. This was evident across the multiplication problems starting with whole numbers. In conversation 1 he began with the 32 x 25, where there were 32 boxes of Pokémon cards and 25 in each box, by adding 25 x 12 + 25 x 12 + 25 x 4 + 25 x 4. He knew that he could break up the boxes in ways until he got to 32 boxes of Pokémon cards because of the work in class and his understanding of the relationship that 4 boxes of 25 is 100. He also knew that if 4 boxes of 25 is 100, then 12 boxes is 300 because it is three times as many boxes so three time as many cards. He was then able to add up the quantities of 12 boxes of 25 and 4 boxes of 25 until he reached 32 boxes of 25.

Teodoro again demonstrates his understanding of grouping that he develops with his work with fractions. This is noticeable in his development of knowing how many groups of a fraction to make a whole number and the relationship of grouping fractions together. In conversation 2 when working with multiplying 12 x 1/4, Teodoro first adds ¼ four times until it is one, recognizes that ¼ four times is one and then repeats that three times, getting twelve groups of ¼, which is 3 (Table 8). He recognizes the relationship of ¼ x 4 = 4/4 = 1 or 4 groups of ¼ is one. This develops into connecting that 5 groups of 4/5 is 20/5. He could add them together, but knows that it would be 20/5 by multiplying. Teodoro is connecting ways to use his number of groups of fractions to make whole numbers. Knowing both connections that 15 is 3 groups of 5 and 5 groups of 4/5 is 20/5, which is 4, allows Teodoro to engage in this problem from Conversation 4 with a Strategic Use strategy even though he hadn't dealt with a two-step problem like this previously.

Teodoro's development in grouping whole numbers, then grouping with fractions, helped to connect his ideas across problems and to be able to tackle an algebraic problem usually reserved for middle school. This same type of connections across the problems was evident with other students and the earlier that they developed groups of groups thinking, the more they were able to use it later and continue to develop it.

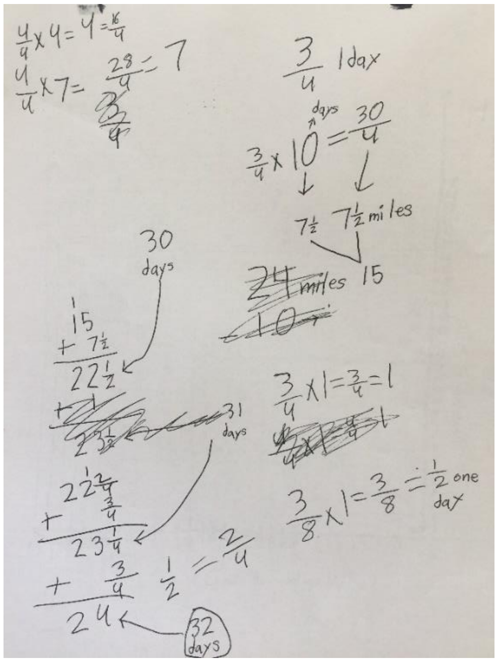
Developing a strong understanding of grouping is crucial to building student understanding across multiplicative mathematical concepts

A goal with this dissertation was to try and understand how ideas of grouping can support students as they reason about ideas central in algebra. These algebraic ideas include using mathematical properties in solutions, building understanding of linear equations, and relating their solutions to a graph. I've previously discussed students use of mathematical properties in their solutions and connecting to rate and solving problems with linear equations of y=mx+b structure. Students' sense making of grouping and in particular grouping of groups also supported them to more naturally take up representing their ideas on a graph.

This section then examines the power of grouping to build mathematical understanding across concepts by examining the strategies of two students: Adán and Natalia, and then see how other students connected their grouping strategies to graphing.

Grouping strategies of Adán and Natalia

Adán is an example of a student who used groups of groups strategies across multiple problems.



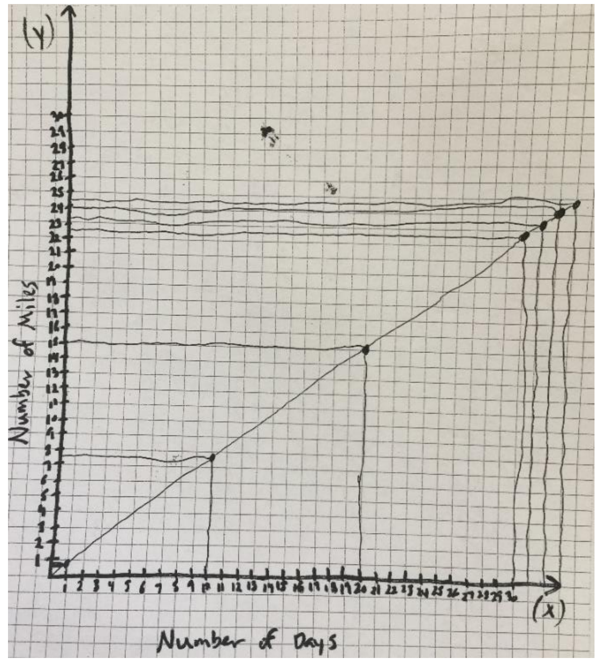
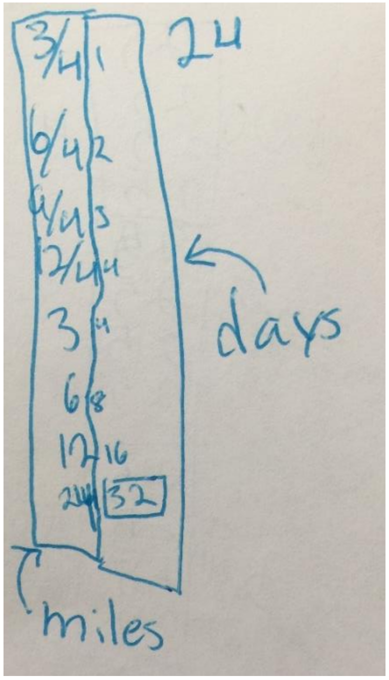
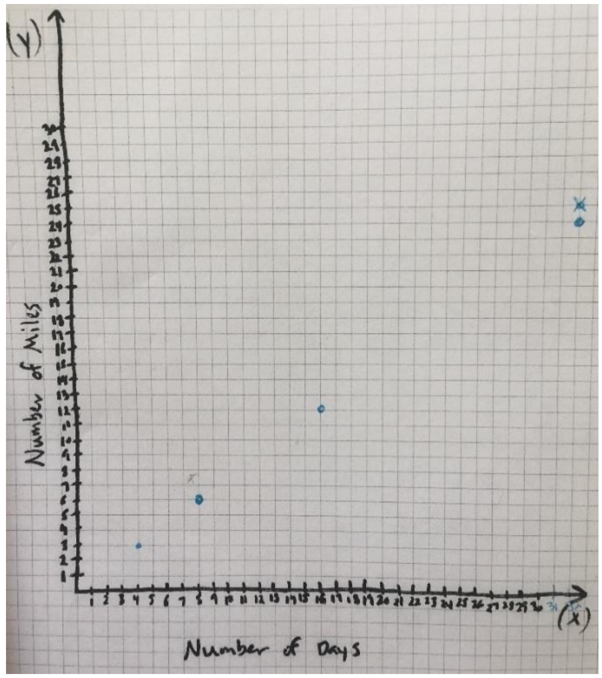


Figure 4 -Adán’s Strategy Figure 5 – Graph of Adán’s strategy

Adán’s use of grouping within the problems can be seen in how he kept track of the quantities for graphing. For solving the problem of 24 ÷ ¾ (See Figure 4), Adán used the relationship of 10 days for 7 ½ miles to find 20 days for 15 miles, 30 days for 22 ½ miles, and then added 1 more day and ¾ miles to get 31 days for 23 ¼ miles and 32 days for 24 miles. Adán’s graphing reflects this solution and maps coordinates of 10 days to 7 ½ miles and continues up to the solution (Figure 5). Adán also plotted the 1 day for ¾ miles and when asked if there were more points that would follow the relationship, he responded by saying if Filipe walked more days, then you could add more ¾ miles because it is increasing by ¾ each day. Adán demonstrates an emerging understanding of the constant of proportionality or slope in recognizing that a point can be placed for each increase in ¾ mile per each additional day.

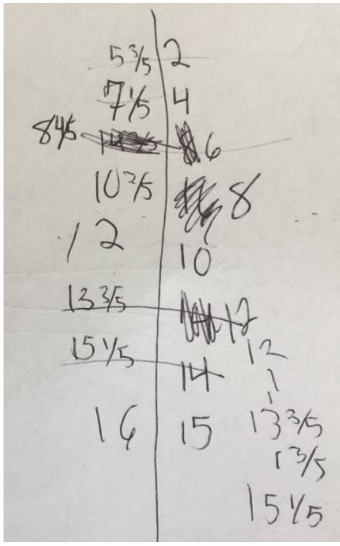




Figures 6 - Natalia's Strategy Figure 7 – Graphing of Natalia's strategy

Natalia also uses groups of groups thinking to help in her solving of problems. Natalia often used doubling in her strategies, but has developed her understanding of doubling the number of groups which also means doubling the total across the groups. This was evident particularly in the measurement division problems with both whole number and fractions. Natalia also started creating tables for her work to keep track of the quantities and her groupings. In the problem of 24 ÷ ¾, Natalia solved the problem by creating a table of the miles and days of adding the ¾ miles per day until finding 3 miles in 4 days and then continues to double that until finding 24 miles and 32 days (See Figure 6). When working with measurement division or multiplication, Natalia worked to group in ways to find whole numbers and then group those groups with doubles. To graph the coordinates, Natalia uses the whole number relationships found in the solution and graphed those points on the coordinate plane (See Figure 7). Without a class focus on creating tables then graphs, Natalia essentially started using a middle school standard (National Governor's Association, 2010) of connecting the relationship from a table to graph. Additionally, her proportional and ratio reasoning is evident with what is basically a ratio or x-y table that she created. She is very aware of the two contexts and as she increases one quantity, she needs to increase the other according to the appropriate ratio.

Natalia's strategy with doubling worked well for proportional problems. When students move to problems with structures of y=mx+b, which are no longer proportional, but still grow at a constant rate, they sometimes have trouble solving or connecting their strategies. In the problem with a structure of 15 x 4/5 + 4 = y, Natalia started at 4 and then began grouping two weeks of growth, which is 1 3/5 inches and adding that to the 4 inches. So, after 2 weeks the plant would be 5 3/5 inches tall (Figure 8). At four weeks it would be 7 1/5 inches tall. She began trying to double her quantities on the table from there, but realized it would not work. She instead kept adding her ratio of 2 more weeks would be an additional 1 3/5 inches, until the 16 weeks are up. When graphing, Natalia began by placing a point on the (0,4) and then placed a point at (1, 4 4/5), but then proceeded placing the points based on the table of points she created with every 2 weeks being 1 3/5 more inches (Figure 9). Natalia's ratio understanding from these two problems demonstrate the ability to double with groups of groups in proportional problems or use ratio understanding of adding ratios if the doubling didn't work. Her realization that she couldn't double anymore and her strategies for these problems lend to the understanding of the difference between proportional and non-proportional problems. It is also an opportunity to discuss constant rate with proportionality or not and what it means when starting at a number other than zero, key components of algebra.



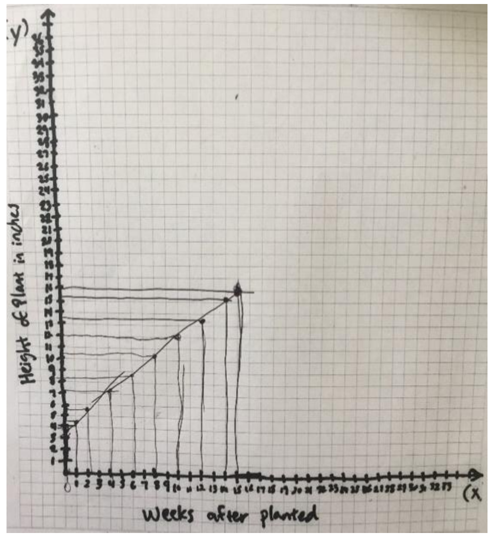


Figure 8 - Natalia’s Strategy Figure 9 – Graph for Natalia’s strategy

Both of these students exemplify the important algebraic mathematical understanding and flexibility in problem solving that is developed when given the opportunity to engage in grouping problems and solving problems in ways that make sense to them with groups of groups thinking. Both students are using mathematical properties to help them solve problems without direct instruction in how to use them and understand the relationships between the quantities of groups, amount in each group, and total across groups to support their problem solving and later to connecting to ideas of graphing, proportionality, and slope. The work of Adán and Natalia exemplifies some of the understandings that were being developed in the class by a number of the students. For example, 67% of students were able to graph multiple points on the 24 ÷ ¾ problem.

Chapter 5: Discussion

While researchers including Baek (2008; 2003), Empson & Levi (2011), and Carpenter et al. (2015) have articulated student strategies for whole number and fractional grouping multiplication and division problems, this study has been able to add to the detail and nuance. One of the things this study has done is to contribute to the field's documentation and elaboration of the kinds of strategies that young people use to solve multi-digit multiplication and division problems as well as multiplication and division with fractions. What's important about this nuance is that it gives more detail to what students are doing with grouping which turns out to support connections across number choice and problem types.

Development of implicit, explicit, and planful use of grouping

My study provides details about several related ideas about students' thinking within the domain of multiplication and division grouping problems. One of the ideas is that a trajectory or pattern of development is complicated, students' strategies are driven by ways groups are used in the problem and the numbers chosen. Within the strategies of the class there was not a clean movement through all of the problems, but more of a constellation of student ideas working together to connect understanding. For students, it was more about using the underlying ideas of grouping, not using particular strategies for one problem, then a different strategy for other problems. The nuance showed students going from implicitly using grouping ideas, to explicitly using them, to purposeful planning their use. Seeing what students understand about grouping then is not about the strategy they use but about whether they implicitly group, explicitly use groups, or plan to use groups to maximize the fundamental properties. Their development toward planful use doesn't happen all at once: sometimes students plan ahead, other times they implicitly use grouping because the numbers do not lend themselves to it (which can be student dependent) so the students just get started.

The constellation of underlying ideas that students are working on cannot be articulated in a single pattern around strategy. The pattern is much more related to students' implicit use of the fundamental properties of arithmetic and grouping helps them to develop those ideas. Their use of grouping is what helps them to implicitly use these fundamental properties. The more students group the more they are explicit about their use of these fundamental properties to the point that they begin to plan to use them because they begin to see the power in them. So, being able to group matters for being able to use these fundamental properties for future mathematics and while we need to study that more, the preliminary data here suggests that students are developing these ideas. One could say that their productive use of planned grouping was due to their entry level knowledge and thus they continue using it as the mathematics becomes more sophisticated, but it's also the case that these students came in struggling with grouping and the fact they learned how to use it is consequential.

When examining a Count/Add strategy, one could think that it is not a very sophisticated strategy, but when students start to group within adding, such as 12 + 12 which is two groups, then double the groups and total, it isn't just powerful for the idea of doubling, but it's powerful for the idea of manipulating groups and being able to think about how the student might use the associative property later. The understanding that is being built when students develop their use of grouping in Count/Add highlights the need to not rush students through the strategies – moving quickly from Count/Add to Invented Algorithms – and see the tremendous power in the earlier strategies and spending time with them is important for developing understanding that leads to the use of the fundamental properties later. Using this lens also allows us to position all students competently because even if a student is using Strategic Use invented algorithms in one problem they are not finished learning and someone who is using Count/Add can still be seen as having important knowledge of grouping. Seeing the variety of mathematical ideas emerging across problems in somewhat different ways enables teachers to position each student as competent – and maybe competent in quite different ways. These ideas move us away from a hierarchy that leads to only some students to be positioned as competent.

Uncovering what students know

Students struggling in algebra class (NCES, 2015; Moses, 2001) are often described from a deficit view as lacking understanding and unable to learn particular mathematics. This study provides evidence that students who have previously struggled in mathematics can develop understandings that matter and may already possess understandings that are productive in the moment and the future. By examining students grouping ideas in direct modeling and count/add strategies and their use of groups of groups, in count/add and invented algorithm strategies we see how much students know and can do when given the opportunity to show it. The more we understand of students' ideas of grouping, the more we uncover details of what students do know, which can be leveraged in the classroom to build on mathematical understanding and change views of who can be successful in mathematics (Battey & Franke, 2015; Battey & Chan, 2010).

When students are positioned as mathematical sense-makers, it changes the view of students and what they are capable of (Franke, Kazemi, & Battey, 2007). This change can bring about further opportunities in mathematics as we allow students to continue to show us what they know about increasingly more complex mathematical ideas. As we do so, student's mathematical dispositions and identities change as they see themselves as mathematicians. The findings presented here suggest the need for additional research and suggest implications for teachers in the classroom.

It is possible that the way the study was constructed supported me to see all of these details of students' ideas. I spent time in the classroom getting to know the context, how instruction occurred and evolved, and the students and their ideas. I also began the conversations by trying to get to know the students better and show them that I was interested in them and their ideas. This may have helped students feel like they could share all of their ideas – partial, full, incorrect or incomplete to me; that they could try a problem that seems hard or even persist when unsure. This would need further study, but it is important to note that this is the context in which students shared what they knew.

Implications for research

This study provides a preliminary examination of one group of students who had the opportunity to consistently think about grouping but it is one group in one classroom with a very knowledgeable teacher – so we see what might be possible. However, we need more studies about the development of students' ideas around multiplication and division grouping problems and to think about how to better articulate what is important about these constellations of ideas that young people are developing as they move toward algebra.

This study focused on students' mathematical thinking and does not investigate everything that Ms. Lowry did to support the students' thinking about grouping. Research that addresses how teachers' facilitation of grouping influences what students learn and how they engage in the mathematics and with each other can help us understand the context in which learning can occur and potentially tell us much more about students' thinking. While not the focus of this study, it was clear from data collection across the year that Ms. Beaumont purposefully used language of "groups of" when reading a multiplication sentence. Rather than saying 5 x 6 as "Five times 6," she would say "Five groups of six" or "Five groups with six in each group." Ms. Beaumont also centered the majority of the problem-solving work on solving word problems and she incorporated the instructional activity of Counting Collections (Chan & Kern Schwerdtfeger, 2007; Franke, Kazemi, & Chan Turrou, 2018). When conferring with students about the problem, she referred to the context and to groups and was intentional to help students create equations based on the context of grouping. Additional studies can examine the affordances and challenges wtih practices like those seen in Ms. Beaumont's class in supporting students' participation and learning about grouping.

The connection mathematically between multiplication, division, fractions, and linear equations is recognized and evidenced in the Common Core State Standards (National Governors Association, 2010). However, there is limited evidence of the connection between these topics based on how students think about these domains. Further research is also needed in how the work with grouping multiplication and division problems connects to other mathematical concepts. For example, within this study, students began with extensive work with grouping and were later posed rate problems in the conversations and students used their grouping ideas on these problems as well. Research is needed to examine the connection from grouping to building understanding within other multiplication and division problem types, such as comparison problems, arrays/area, and combinations.

Further research to connect other mathematical domains around student thinking

There is also a need for increased understanding of student thinking across mathematical domains. This research demonstrates the need to understand students' connections across multiplicative thinking through linear functions. We can see from the students in this study that understanding of grouping can play a significant role in this understanding in ways that can be mapped to the strategies they use. Students development of grouping contributed to students understanding of mathematical properties crucial to later algebraic success. Although not all students in this classroom were able to use planned purposeful grouping consistently, grouping and its connections between multiplication and division to adding and subtracting, facilitated students' development of the mathematical properties. As part of the multiplicative field (Vergnaud, 1988), this research on grouping can be connected to a series of mathematical concepts which I will explain further.

This research has implications for research into connections to other mathematical concepts within the multiplicative field, such as ratios and proportions, linear functions, and the ways in which graphing has been traditionally taught, which is separately from students grouping strategies of problems with linear relationships. While students were able to think about graphing as a representation of their thinking, further research is needed to look at how positioning graphing as a tool to represent grouping strategies already developed may facilitate students understanding of graphing linear equations. Creating a table or graph from what students already have solved rather than from a prescribed manner may contribute to student connections to the formalisms of tables and graphs (Carpenter & Leher, 1999).

Another area of research to examine is students' opportunity to develop functional thinking as part of student strategies in grouping problems. As students solved the grouping problems, students needed to keep track of the number of groups, amount in each group, or total across groups, which contributed to them using informal tables or proportional reasoning strategies. This was particularly evident in the strategies for measurement division problems with both whole numbers and fractions as students kept track of the number of groups and the total across groups as they built up to the desired total. Students' iterations of the amount in each group and grouping of the amount in each group contributed to students developing ideas of slope, which is critical to functional thinking (Blanton & Kaput, 2011; Ellis, 2011). The importance of understanding constant rate or slope applies not only to helping students in algebra, but also extends to higher mathematics.

The mathematics concept of slope is something that is quite new to middle school students and connects all the way to calculus. It is a concept that is often taught directly using the idea of rise over run, with students counting the amount increased in y divided by the amount increased along the x axis. This can be taught with tools like the slope triangle or the slope formula or through other tools. Slope is often discussed as a rate of change, but is not necessarily connected to a context that develops this idea well. By examining connections of students' strategies with grouping multiplication and division problems to rate problems, we are able to see the development of building understanding of the relationship between the quantities in $y = kx$ and $y = mx + b$. When we examine students' ideas through grouping, we see that we are able to enable and support them to use these ideas that connect to slope and that the findings suggest that yes, young people's notions of grouping do actually provide a basis and foundation for them to understand slope in a way that's not about memorizing a set of procedures, but thinking more about rate of change. For example, Adán's grouping ideas with 24 ÷ ¾ help him to think about increasing by ¾ for each additional day, which is also represented with his graph. Further research is needed to press on the understandings that students are developing and see how to facilitate classrooms that help students to connect their informal understandings to the formal representations expected in algebra and higher mathematics.

Implications for teaching

Implications for the use of grouping problems in instruction

An implication for teaching based upon this research is the need for students to engage in grouping problems and build their understanding of groups with both whole numbers and fractions. Grouping problems can be engaged with early in the school curriculum to begin to develop students grouping strategies alongside other mathematical understandings in ways that will later support their use of the mathematical properties. The Common Core standards (National Governors Association, 2010) have students in second grade begin working with grouping problems, but if these problems were posed as early as Kindergarten and first grade it would provide students more opportunities to develop their grouping ideas in relation to their other understandings of the operations and place value. We know that young children can solve multiplication and division story problems, by modeling the actions in the problems and these grouping problems are beneficial to young children for developing place value understanding, which is critical in Kindergarten through 2^{nd} grade (Carpenter et al., 2015). This study adds to the affordances of providing these experiences by looking at how these ideas extend into multidigit whole numbers, fractions, rates, and graphing.

By engaging in grouping problems across the grades and paying attention to the language of "groups of…" students' strategies and thinking can become more connected across the number choices and problem types. An example of this was evident at the end of the school year in class, when Ms. Beaumont posed the question:

A zookeeper has ______pounds (lbs) of food prepared to feed animals. If the zookeeper must share the food equally for ______animals, how many pounds (lbs) of food will each animal get?

(1 20/100, 4) (13,616, 4) (6.68, 4)

When posing this problem, Ms. Beaumont had the discussion with students about what would it mean about the types of animals eating if you have 1 and 20/100 pounds of food to share with four versus 13,616 pounds to share? Students knew the relationship that sharing the initial amount must be smaller animals and shared out examples of small animals like small monkeys, whereas the 13,616 pounds of food must be for larger animals like elephants or rhinoceros.

Experience with grouping problems throughout the year allowed students opportunities to develop their strategies across the concepts of whole numbers, fractions, and decimals and to develop reasoning about the different quantities presented in this problem. Students' groups of groups strategies with this problem were similar across the different number sets as students developed connections across quantities. Students need opportunities to engage in grouping problems across the year and in different ways that can facilitate opportunities to build the understanding and flexibility within these different mathematical concepts. The examples from Teodoro, Adán, and Natalia in the results section (See Figures 4-8) demonstrate their flexibility in both the students' ability to decompose the number of groups as well as flexibility of connecting multiplication and addition.

This zoo problem also demonstrates the importance of number choices. Students need to experience the range of grouping problems, but also a range of number choices, to develop their grouping ideas around different number choices. Students ideas differed depending on their experience with the numbers in the problems and need to build their understandings of the grouping and mathematical properties as they relate to different number sets.

Connections to graphical representations

The connection between whole number, fraction, and decimal groups also applies to connections to tables and graphs. Grouping problems with a whole number of groups and either whole number, fraction, or decimal amounts per group are common problems that could be used to facilitate connections to tables and graphs. Tables and graphs are a way to represent the ways that students are composing the groups to get to their solutions. If students are able to connect tables and graphs to their own ideas, then tables and graphs have more meaning and connect to student thinking. Students would then be able to make more sense of the graphs and be able to build more mathematical reasoning based upon these tools with understanding. Middle school classrooms discuss in depth the ideas of graphing and tables to linear functions, but the question arises, how do teachers connect that to students' thinking for sense-making?

Starting with student thinking in this case is also equitable because it acknowledges the powerful mathematics that students know and allows them to build on that thinking by taking up tools and representations, such as tables or graphs, when they are ready to use them because their thinking with the mathematics affords them to do so. This suggests that we aren't just having kids use graphs and tables because the middle school standards (National Governors Association, 2010) say so, but because it naturally connects to kids thinking and they are ready to use them. Also, this aligns with the idea that the mathematics is connected across concepts and isn't just connected to one specific grade level or "content" (pre-alegbra vs. algebra).

This was evident initially with the measurement division problems posed in this study. Students grouped the whole numbers or fractional amounts in ways that built up to the desired amount of total, while keeping track of the number of groups. By keeping track of the two quantities of the total and the number of groups as they worked, these problems supported students to be focused and mindful of the importance of keeping track of which quantity they were working with. Ms. Beaumont noticed the need to focus on having students label the different quantities, because although they were developing nice uses of grouping fractions, some students were losing track with which quantity they were working on. These labeling encouraged students to be more mindful of the quantities in the problems and by noting the two varying quantities of groups and running total in solving the measurement division problems, students related that to the idea of increasing points on the graph. That relationship between the quantities of groups and running total could also be related to representing students thinking on a table. Some students started creating informal tables in their strategies for solving measurement division problems.

Often, multiplication problems are used initially in middle school curriculum to relate to a graph or table, but when students solved quickly by knowing 12 groups of ¼ is 12/4 or 3 then, it didn't produce the opportunity for multiple data points to connect to a graph or table. Division problems are usually moved through quickly and only in relation to multiplication, but more time spent with student strategies in these areas, as seen in our measurement division problems, may support students to connect to formal representations on tables and graphs initially based on their strategies used. Additionally, more work with partitive division problems may help with building ideas of unit rates and slope, while measurement division may help with groupings of that unit rate to build ratio and proportional reasoning.

Another mathematical extension based upon students connecting of grouping to a representation of a table and/or graph could also align with connections to functions. These same problems could be used to extend ideas of functions. For example, in the problem: *If Filipe walks 3/4 of a mile per day this week and after, how many days will it take to reach 24 miles?*, it is a measurement division problem, but the teacher could extend to functional thinking after students solve. This could be done by posing questions such as if that is how many days for 24 miles, how many days would it take for 48 miles? for 100 miles? for *n* number of miles? Students strategies could be used to connect to find the algebraic rule for any number of miles, which can be represented graphically.

Developing students' grouping thinking

This research adds to the work that student development of grouping is not a linear trajectory, but students developed over time or became more sophisticated grouping over time. It was not done with units that focused on multiplication, then division in separate ideas, but was based more on principles of engaging in whole number then fractions, while working with varying problems. The constellation of understandings around grouping was developed through varying instructional experiences. Students need to experience the relationships in working with the different grouping problems with different number choices to capture the varying opportunities to extend their thinking. Strategic number choices are a key piece to supporting students developing use of grouping.

The nuances in the grouping may help teachers to understand the progressions with student thinking with more detail and help teachers see areas where they can pose questions or problems that open opportunities to make further connections. Teachers often ask how to help students move from direct modeling to count/add to invented algorithms. There are a multitude of responses to this type of question, but one response is that helping teachers understand more of the nuances in the ways that students are thinking about grouping rather than strategy type can then help teachers facilitate learning, so students become more sophisticated with their grouping. Building students ways in which they group may lead to students developing use of the mathematical properties, which in turn is getting into the use of invented algorithms.

Creating opportunities for student participation

It is clear from this research that students were not solving problems in the same way, but with a variety of strategies. Students' strategies vary depending on how students work with groups of groups, size of groups, and number choices. Since students have many experiences and ideas of solving, this points to the importance of students needing to be able to share their ideas with their teacher and with the class. They need opportunities to be able to explain the nuances in the ways that they are solving problems. This is crucial for all students learning, but particularly students that haven't been as successful in mathematics or have been labeled as "not a math person" or with "low math ability". This often includes low income and/or students of color, that are consistently viewed through a deficit lens and are often left behind in opportunities to share and develop their mathematical thinking in this way. If equity is thought of as opportunities to participate in society and a community of practice (Lave & Wenger,1991), then teaching in a way that shares and builds upon children's natural ways of thinking about mathematics is leveraged because it is analogous to their participation in the classroom and greater society. Students being able to share their mathematical thinking is an important part of equitable practice in the mathematics classroom (Aguirre et al, 2013).

Conclusion

As a former middle school math teacher that was searching for ways to help students understand mathematical concepts, and as a father of four children that sees my children's thinking in different contexts, it is now more evident to me that children have so much understanding and potential that is often untapped in our educational system, particularly with historically marginalized students. Learning about student mathematical thinking drew me to this research work as the field works to continue to explore student mathematical thinking as it connects from elementary to middle school. As discussed in this paper, student thinking can be uncovered to unveil important mathematical ideas, such as those around grouping, and instruction and research needs to open opportunities to do so. Research in student thinking that highlights what students know, can be productive in supporting teachers and students. This project further cemented the importance of these ideas around student thinking into my own reasoning. As research and instruction change to build more on student thinking, it can support opportunities for more students to tap into their potential, which was evident in Ms. Beaumont's classroom.

Appendix A

Recording Sheet for Conversation 1

Student Name________________________ Conversation 1			
Problem Types	Classwork Examples	Work from conversation	Notes
Multiplication - (enter student's name) has 12 packs of Pokémon cards. There are 15 cards in each pack. How many cards does (enter student's name) have?			
Multiplication- (enter student's name) has 32 packs of Pokémon cards. There are 25 cards in each pack. How many cards does (enter student's name) have?			
Partitive Division – There are 132 students in the 5th grade at Green elementary and four classrooms. How many students are in each class, if there are the same amount			
Measurement Division- A school is going on a field trip to the California Science Center. There are 912 students in the school. 48 students can ride on each bus. How many buses will be needed?			
Conversation 1 Summary:			

Appendix B

Recording Sheet for Conversation 2

<table>
<tr><td colspan="4">Student Name________________________ Conversation 2</td></tr>
<tr><td>Problem Types</td><td>Classwork Examples</td><td>Work from conversation</td><td>Notes</td></tr>
<tr><td>Partitive Division Fraction –

___ kids are sharing ___ pan dulce so that each get the same amount. How much does each child get?

(4, 10) (8, 18) (3, 3/4) (4, 2/3)</td><td></td><td></td><td></td></tr>
<tr><td>Measurement Division Fraction –

Our family is making some tamales for Christmas. We have 12 cups of masa in a bowl. If we use ___ cup of masa for each tamal, how many tamales can we make?

(1/3) (2/3)</td><td></td><td></td><td></td></tr>
<tr><td>D) Multiplication with Fraction –

Filipe feeds his dog 1/4 cup of dog food each meal. How much dog food is needed for 12 meals?

(2/4, 12) (3/4, 12)</td><td></td><td></td><td></td></tr>
<tr><td colspan="4">Conversation 2 Summary:</td></tr>
</table>

Appendix C

Recording Sheet for Conversation 3

Student Name________________________ Conversation 3			
Problem Types	Classwork Examples	Work from conversation	Notes
1) The central library has 132 shelves of books. If there are 24 books on each shelf, how many books does the library have?			
2) Rate problem – If (enter student name) walked 2/3 of a mile to school each morning. How far will (enter student name) walk in 16 mornings?			
3) If Filipe walks 3/4 of a mile per day this week and after, how many days will it take to reach 24 miles? Pose graph questions: How would you represent your strategy from this problem on the graph?			
Conversation 3 Summary:			

Appendix D

Recording Sheet for Conversation 4

Student Name_________________________ Conversation 4			
Problem Types	Classwork Examples	Work from conversation	Notes
Proportion problems 1) Samuel is planting a garden. He plants a tomato plant that is 4 inches tall. If the tomato plant grows 4/5 inch per week. How tall will the tomato plant be after 15 weeks? How would you represent your strategy on a graph?			
2) If you have 13 cans of dog food and you feed your dog 1/3 can of food each day. How much dog food will you have left after 27 days? How would you represent your strategy on a graph?			
Conversation 4 Summary:			

Appendix E

Individual student strategies across multiplication problems

Students	Multiplication-Conv 1 32x25	Multiplication Conv 3 132 x 24	Multiplication-Conv 2 12x1/4	Multiplication Conv 3 16x2/3	Multiplication-Conv 4 Y= 4/5x + 4, X = 15
Daniel	Strategic	Strategic	Strategic	Beginning	Strategic
Sam	Strategic	Strategic	Strategic	Strategic	Strategic
Adan	Strategic	Strategic	Strategic	Strategic	Strategic
Amaya	Strategic	Taught	Strategic	Strategic	Strategic
Eric	Taught	Strategic	C/A w/groups	C/A w/groups	Strategic
Teodoro	Beginning	Strategic	Beginning	Strategic	Strategic
Devin	Beginning	Beginning	C/A w/groups	Beginning	Strategic
Victoria	Taught	Taught	Strategic	Strategic	Count/Add
Natalia	Beginning	Beginning	Beginning	Beginning	Beginning
Sara	Beginning	Taught	Beginning	Missing Data	C/A w/groups
Julie	Beginning	Strategic	Beginning	C/A w/groups	C/A w/groups
Esther	Beginning	Strategic	Beginning	Strategic	Strategic
Seba	C/A w/groups	Beginning	Count/Add	Beginning	C/A w/groups
Simon	C/A w/groups	Beginning	Direct Model	Strategic	Strategic
Alissa	Invalid	Taught	C/A w/groups	Beginning	Beginning
Betty	C/A w/groups	Taught	C/A w/groups	Count/Add	Count/Add
Analisa	Count/Add	Strategic	Beginning	Count/Add	Beginning
Lisa	C/A w/groups	Count/Add	Count/Add	Count/Add	C/A w/groups
Ricky	Direct Model	Invalid	Count/Add	Count/Add	Beginning
Israel	C/A w/groups	C/A w/groups	C/A w/groups	C/A w/groups	Count/Add
Aubrey	Count/Add	Invalid	C/A w/groups	Count/Add	Count/Add
Moses	C/A w/groups	Taught	Count/Add	Count/Add	Count/Add
Amy	Invalid	Count/Add	C/A w/groups	Count/Add	Count/Add
Juan	Count/Add	Invalid	Direct Model	Count/Add	Invalid

Note: Yellow highlights indicate incorrect solutions

Appendix F

Individual student strategies across division problems

Students	Convo 1: Measurement Division 912/48	Convo 1: Partitive Division 132/4	Convo 2: Measurement Division 12/1/3	Convo 2: Partitive Division 10/4	Convo 3: Measurement Division 24/3/4
Daniel	Beginning	Beginning	Beginning	DM- Emerg	Beginning
Sam	C/A w/groups	Strategic	Beginning	DM- Emerg	Beginning
Adan	Beginning	Beginning	Strategic	Strategic	Beginning
Amaya	Beginning	Strategic	Strategic	DM-Emerg	Beginning
Eric	Beginning	Strategic	Beginning	DM-Emerg	Strategic
Teodoro	Beginning	Strategic	Beginning	DM-Emerg	Strategic
Devin	Beginning	Beginning	Beginning	DM-Emerg	Beginning
Victoria	Taught	Taught	Beginning	DM-Emerg	Beginning
Natalia	Beginning	Beginning	Count/Add	DM-Emerg	Beginning
Sara	Beginning	Beginning	Direct Model	DM-Emerg	Beginning
Julie	Count/Add	Beginning	Direct Model	DM-Emerg	Beginning
Esther	Beginning	Strategic	Invalid	DM-Emerg	Beginning
Seba	Beginning	Beginning	Direct Model	DM-Emerg	Invalid
Jose	C/A w/groups	Beginning	C/A w/groups	DM-Emerg	Beginning
Alissa	C/A w/groups	Beginning	Direct Model	Strategic	Beginning
Betty	Taught	Taught	C/A w/groups	DM-Emerg	Beginning
Analisa	Count/Add	Beginning	Direct Model	DM-Emerg	Invalid
Lisa	C/A w/groups	Beginning	Invalid	DM-Non Ant	Invalid
Ricky	Invalid	Invalid	C/A w/groups	DM- Non Ant	Count/Add
Israel	Count/Add	Invalid	Invalid	DM-Emerg	Invalid
Aubrey	Invalid	Count/add	Direct Model	DM-Emerg	Count/Add
Moses	Count/Add	Invalid	C/A w/groups	DM-Emerg	C/A w/groups
Amy	Invalid	Invalid	Direct Model	DM-Emerg	Count/Add
Juan	Invalid	Invalid	Direct Model	DM-Emerg	Count/Add

Note: Yellow highlights indicate incorrect solution

Appendix G

Student strategies and graphing for conversation 4

Student	Mult – y = 4/5x + 4, x = 15	Graphed 1 point, multiple points, or no points from own solution	Mult y= -1/3x +13 x=27	Graphed 1 point, multiple points, or no points from own solution
Daniel	Strategic	Multiple points (3-4)	Strategic	Multiple points (2)
Sam	Strategic	Multiple points (5+)	Strategic	Multiple Points (5+)
Adan	Strategic	Multiple points (5+)	Strategic	Multiple Points (5+)
Amaya	Strategic	One point	Direct Model	One point
Eric	Strategic	Multiple points (3-4)	Beginning	Multiple Points (5+)
Teodoro	Strategic	Multiple points (3-4)	Strategic	Multiple Points (3-4)
Devin	Strategic	Multiple points (3-4)	Strategic	Multiple Points (2)
Victoria	Count/Add	Multiple points (3-4)	Strategic	One point
Natalia	Beginning	Multiple points (5+)	Beginning	Multiple Points (3-4)
Sara	C/A w/groups	One point	C/A w/groups	Multiple Points (2)
Julie	C/A w/groups	Multiple points (2)	C/A w/groups	Multiple Points (3-4)
Esther	Strategic	Multiple points (3-4)	Count/Add	Multiple Points (5+)
Seba	C/A w/groups	Multiple points (5 +)	C/A w/groups	Multiple Points (5+)
Simon	Strategic	Multiple points (3-4)	Direct Model	One point
Alissa	Beginning	One point	Strategic	One point
Betty	Count/Add	Multiple points (5+)	Direct Model	Multiple Points (5+)
Analisa	Beginning	One point	Direct Model	One point
Lisa	C/A w/groups	One point	Count/Add	One point
Ricky	Beginning	No points	Direct Model	One point
Israel	Count/Add	Multiple points (5+)	Count/Add	One point
Aubrey	Count/Add	No points	Direct Model	No points
Moses	Count/Add	No points	Direct Model	No points
Amy	Count/Add	No points	Direct Model	One point
Juan	Invalid	No points	Direct Model	One point

Note: Yellow highlights indicate incorrect solution and/or incorrect plotting of points

Appendix H

Strategy Coding for Each Problem

Conversation 1

Question 1 –Multiplication - 12 x 15

0 – Invalid Strategy

1 – Direct Modeling

2 – Taught Algorithm Strategy from previous grades/Area Model, Lattice Method

3 – Counting/Adding – Skip counting or adding without grouping

4 – Counting/Adding – Groups of groups with adding

5 – Beginning Use Invented Algorithm – Associative and/or distributive property use (groups of groups)

6 – Strategic Use Invented Algorithm – Associative and or distributive property use (groups of groups)

Question 2 –Multiplication - 32 x 25

0 – Invalid Strategy

1 – Direct Modeling

2 – Taught Algorithm Strategy from previous grades/Area Model, Lattice Method

3 – Counting/Adding – Skip counting or adding without grouping

4 – Counting/Adding – Groups of groups with adding

5 – Beginning Use Invented Algorithm – Associative and/or distributive property use (groups of groups)

6 – Strategic Use Invented Algorithm – Associative and or distributive property use (groups of groups)

Question 3 – Partitive Division 132/4

0 – Incorrect Strategy – tried multiplication or other operation

1 – Direct Modeling – Passing out by 10's & 1's

2 – Taught Algorithm Strategy from previous grades/Area Model

3 – Counting/Adding – Trial & Error Skip Counting or Adding

4 – Counting/Adding – Groups of groups adding

5 – Beginning Use Invented Algorithm – Associative and/or distributive property use (groups of groups)

6 – Strategic Use Invented Algorithm – Associative and or distributive property use (groups of groups)

Question 4 – Measurement Division – 912/48

0 – Invalid Strategy – tried multiplication or other operation

1 – Direct Modeling

2 – Taught Algorithm Strategy from previous grades/Area Model

3 – Counting/Adding – Skip counting or just adding up

4 – Counting/Adding – Adding or subtracting, but beginning to group in ways they add or subtract

5 – Beginning Use – Distributive and/or Associative Properties - Complex Doubling, Ratio build up

6 – Strategic Use – Distributive and/or Associative Properties or knew multiplicative relationship

Interview 2

Question 1 – Multiplication – 12 x 1/4, 12 x 2/4, 12 x 3/4

0 – Invalid Strategy

1 – Direct Modeling

2 – Taught Algorithm Strategy from previous grades/Area Model

3 – Counting/Adding – Skip counting or adding without grouping

4 – Counting/Adding – Groups of groups with adding

5 – Beginning Use Invented Algorithm – Associative and/or distributive property use (groups of groups)

6 – Strategic Use Invented Algorithm – Associative and or distributive property use (groups of groups)

Question 2 – Measurement division – 12/ 1/3, 12 / 2/3

0 – Invalid Strategy – tried multiplication or other operation

1 – Direct Modeling

2 – Taught Algorithm Strategy from previous grades/Area Model

3 – Counting/Adding – Skip counting or just adding up

4 – Counting/Adding – Adding or subtracting, but beginning to group in ways they add or subtract

5 – Beginning Use – Distributive and/or Associative Properties - Complex Doubling, Ratio build up

6 – Strategic Use – Distributive and/or Associative Properties or knew multiplicative relationship

Question 3 – Equal Sharing Fraction problem

0 – Incorrect Strategy

1 – Direct Modeling – Non – anticipatory

2 – Direct Modeling – Emergent Anticipatory – Sharing by 1's

3 – Direct Modeling – Emergent Anticipatory – Sharing by groups

4 – Count/Add – Trial & Error

6 – Strategic Use - Anticipatory/Multiplicative Relationship

Conversation 3

Question 1 – Multiplication – 132 x 24

0 – Invalid Strategy

1 – Direct Modeling

2 – Taught Algorithm Strategy from previous grades/Area Model

3 – Counting/Adding – Skip counting or adding without grouping

4 – Counting/Adding – Groups of groups with adding

5 – Beginning Use Invented Algorithm – Associative and/or distributive property use (groups of groups)

6 – Strategic Use Invented Algorithm – Associative and or distributive property use (groups of groups)

Question 2 – Multiplication – 16 x 2/3

0 – Invalid Strategy

1 – Direct Modeling

2 – Taught Algorithm Strategy from previous grades/Area Model

3 – Counting/Adding – Skip counting or adding without grouping

4 – Counting/Adding – Groups of groups with adding

5 – Beginning Use Invented Algorithm – Associative and/or distributive property use (groups of groups)

6 – Strategic Use Invented Algorithm – Associative and or distributive property use (groups of groups)

Question 3a – Measurement division – 24/ 3/4

0 – Invalid Strategy – tried multiplication or other operation

1 – Direct Modeling

2 – Taught Algorithm Strategy from previous grades/Area Model

3 – Counting/Adding – Skip counting or just adding up

4 – Counting/Adding – Adding or subtracting, but beginning to group in ways they add or subtract

5 – Beginning Use – Distributive and/or Associative Properties - Complex Doubling, Ratio build up

6 – Strategic Use – Distributive and/or Associative Properties or knew multiplicative relationship

Question 3b – Graphing of Measurement Division

0 – Graphed incorrectly or did not graph

1 - Graphed 1 point – either answer or beginning ratio

2 - Graphed 2 points – starting ratio and answer

3 – Graphed 3 -4 points

4 – Graphed 5 or more points

5 – Graphed 2-4 points and explains relationship between points

6 – Graphed 5+ points and explains relationship

Interview 4

Question 1 – Y= 4/5x + 4

0 – Invalid Strategy

1 – Direct Modeling

2 – Taught Algorithm Strategy from previous grades/Area Model

3 – Counting/Adding – Skip counting or adding without grouping

4 – Counting/Adding – Groups of groups with adding

5 – Beginning Use Invented Algorithm – Associative and/or distributive property use (groups of groups)

6 – Strategic Use Invented Algorithm – Associative and or distributive property use (groups of groups)

Question 1b – Graphing – Y= 4/5x + 4

0 – Graphed incorrectly or did not graph

1 - Graphed 1 point – either answer or beginning ratio

2 - Graphed 2 points – starting ratio and answer

3 – Graphed 3 -4 points

4 – Graphed 5 or more points

5 – Graphed 2-4 points and explains relationship between points

6 – Graphed 5+ points and explains relationship

Question 2 – Y = -1/3x + 13

0 – Invalid Strategy

1 – Direct Modeling

2 – Taught Algorithm Strategy from previous grades/Area Model

3 – Counting/Adding – Skip counting or adding without grouping

4 – Counting/Adding – Groups of groups with adding

5 – Beginning Use Invented Algorithm – Associative and/or distributive property use (groups of groups)

6 – Strategic Use Invented Algorithm – Associative and or distributive property use (groups of groups)

Question 2b – Graphing – Y=-1/3x + 13

0 – Graphed incorrectly or did not graph

1 - Graphed 1 point – either answer or beginning ratio

2 - Graphed 2 points – starting ratio and answer

3 – Graphed 3 -4 points

4 – Graphed 5 or more points

5 – Graphed 2-4 points and explains relationship between points

6 – Graphed 5+ points and explains relationship

Appendix I

Explanation of Student Strategies for Table

Count/add with grouping strategies
Conversation 1 – 32 x 25 - *Samuel has 32 packs of Pokémon cards. There are 25 cards in each pack. How many cards does Samuel have?*
Figure 1 - Simon's Strategy Figure 2 -Lisa's Strategy In these examples the students are trying to find ways to groups the 25's to add them more easily then repeatedly adding 25, 32 times. Simon decides to add them in groups of five 25's which makes 125, then adds two of the 125's together to make 250 with each 125 that can be paired and finally adds all the 250's together and adds the two extra groups of 25 for another 50. Lisa decides to group the 25's into 10 rather than in fives and finds out that it is 250. Lisa then adds the 250's together along with the 50 from the remaining two groups of 25.
Conversation 2 – 12 x ¼ & 12 x ¾ - *Filipe feeds his dog 1/4 cup of dog food each meal. How much dog food is needed for 12 meals?*
Figure 3 - Israel's strategy In this work Israel lays out all of the 12 groups of ¼ and 12 groups of ¾ and then tries to find a way to add them up. In the first problem Israel groups by counting the ¼'s to get 1 and then boxes those four ¼'s. Israel repeats that for the other ¼'s and finds out that there are three cups. In the second problem Israel decides to group two of the ¾'s together to make 1 2/4 and then proceeds to add up all the 1 2/4.
Conversation 3 – 16 x 2/3 - *If Samuel walked 2/3 of a mile to school each morning. How far will Samuel walk in 16 mornings?*
Figure 4 -Devin's Strategy Figure 5 - Betty's Strategy In both of these examples the students begin to group the 2/3's. Devin does three groups of 2/3 twice, and then 5 groups of 2/3 twice. Devin is connecting the groups to the 16 days in the problem, but not sure what those would be and uses the adding strategy to figure out by adding the groupings, then adding all the groupings together. Betty on the right is like what Baek (2008; 2005) calls the simple doubling where the students begin with all the 2/3's, but then combines 2 groups of 2/3, then 2 groups of 4/3, then 2 groups of 8/3, and finally 2 groups of 16/3. Less attention is paid to the number of groups each time it is doubled, but still thinking of grouping to support the addition.
Conversation 4 – $y = -1/3x + 13$ or $y = 13 - 1/3x$ when $x = 27$ - *If you have 13 cans of dog food and you feed your dog 1/3 can of food each day. How much dog food will you have left after 27 days?*
Julie's Strategy In Julie's work, the 1/3's were added together in groups of 7. Julie did this twice and combined those, which is 4 2/3 cans. Julie also realized that they did not need to add another 7 1/3's but could add another 2 1/3 to the 4 2/3. That aspect of not needing to represent the last group of 7 1/3's demonstrates the student is transitioning to some of the beginning use ideas of relational thinking within invented algorithms. Julie then adds six more thirds and counts it to two. The two is combined with the 7 other cans to make 9.

Conversation 4 – y = 4/5x + 4 when x = 15 - *Samuel is planting a garden. He plants a tomato plant that is 4 inches tall. If the tomato plant grows 4/5 inch per week. How tall will the tomato plant be after 15 weeks?*
Figure 7-Seba's Strategy In Seba's example he decides to add the 4/5's in groups of 5. Adding the 5 groups of 4/5 gets them to 20/5, that is easier to add together. Seba knows the 20/5's equal 4 and then adds the 4's together to get 12. He then adds that to the 4 inches the plant started with to get 16 inches.

Appendix J

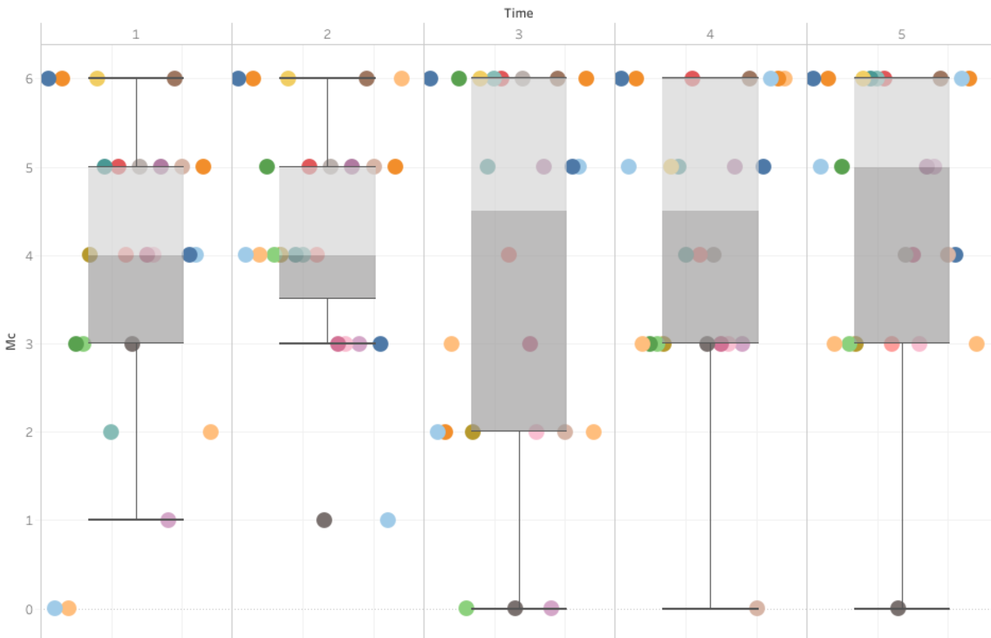
Explanation of student strategies from Table 8

Beginning use vs Strategic use strategies
Beginning Use – Esther   Strategic Use - Adán Esther knows the relationship of 5 groups of 25 is 125 and continues adding that relationship and the relationship of 2 groups of 25 is 50 until arriving at the composition of the 32 groups. Adán's strategic use work demonstrates he decomposed the 32 groups into 10 groups, 10 groups, 10 groups, and 2 groups based upon the place value of 32 and then multiplies each of those groups by the 25 in each group. Esther's strategy is a beginning use of the distributive property of multiplication over addition, with her adding 5 groups of 25 and 2 groups of 25, until she has 32 groups, whereas Adán's purposeful decomposition of the 32 groups is a strategic use of the distributive property of multiplication over addition based upon place value understanding of breaking the 32 into 3 tens and 2, multiplying those each be 25 and then adding them together.
Beginning Use - Teodoro   Strategic Use - Daniel In looking at both of these examples, Teodoro added the ¼ four times to make the 1 whole and then combined the four groups of ¼ for 1 repeatedly until making the 12 groups. In the strategic use example, Daniel decomposed the 12 groups into 5, 5, & 2 and knew the quantities of 5 groups of ¼ and 2 groups of ¼ and then combined them. Again the purposeful decomposition of the groups in Daniel's, is an example of a strategic use of the distributive property of multiplication over addition. Teodoro's is interesting in that his repeated grouping of 4 groups of ¼ is added like in the distributive property, but lends itself to developing the associative property of multiplication.
Beginning Use - Alissa   Strategic Use - Teodoro Alissa's strategy adds up the ¾ repeatedly until Alissa notices a connection that 8 groups of ¾ is 6. Alissa then uses that to add the groups until arriving at the 24 miles and 32 days. Alissa added up the 6 miles and 8 days as they went along until arriving at the correct composition of 24 miles. Teodoro, with the strategic use, also knew 8 groups of ¾ was 6, so that it took 8 days for 6 miles. Teodoro then knew that if he needed 24 miles that he needed 18 more miles, which is three times as much as the 6 miles, so he would need three times as many days, which is 24. He then added the 6 miles and 18 miles to make 24 and then added the 24 days and 8 days to get the 32 days. Both students use the distributive property, but Teodoro also uses proportional reasoning to know that 18 miles is three times as much as 6 miles, so he would need three times as many days.
Beginning Use - Eric   Strategic Use - Daniel Although these strategies look very similar in that the students decomposed the twenty-seven into different groups Eric was less strategic in how he decomposed it. Although not apparent in his written work, it is more evident in his explanation. He explained that he chose seven groups of 1/3 because he used the same number from a problem solved previously and then kept adding the seven groups until he realized he needed to make an adjustment at the

end. The twenty-seven was not decomposed strategically along the way, but was composed out of building up with any number of groups until the student arrived at the desired quantity. Eric is getting closer to strategic use, but would be categorized as beginning for this study because of how he arrived at the twenty-seven. Daniel's strategy was categorized as strategic because he decomposed the twenty-seven into 10, 10, 6, and 1, planning the number of groups before and along the way. He started with the 10 groups and then 10 groups, but instead of the 7 groups together as originally thought, he decided to break apart the 7 because he knew that 6 groups of 1/3 is 2 and there would be 1 group left. Understanding the difference between beginning use and strategic use in this example demonstrates the need to elicit and listen to student explanation of how they are composing or decomposing the groups or amount in each group to determine students use of relational thinking.

Appendix K

Box Plot of student multiplication strategies over the four conversations



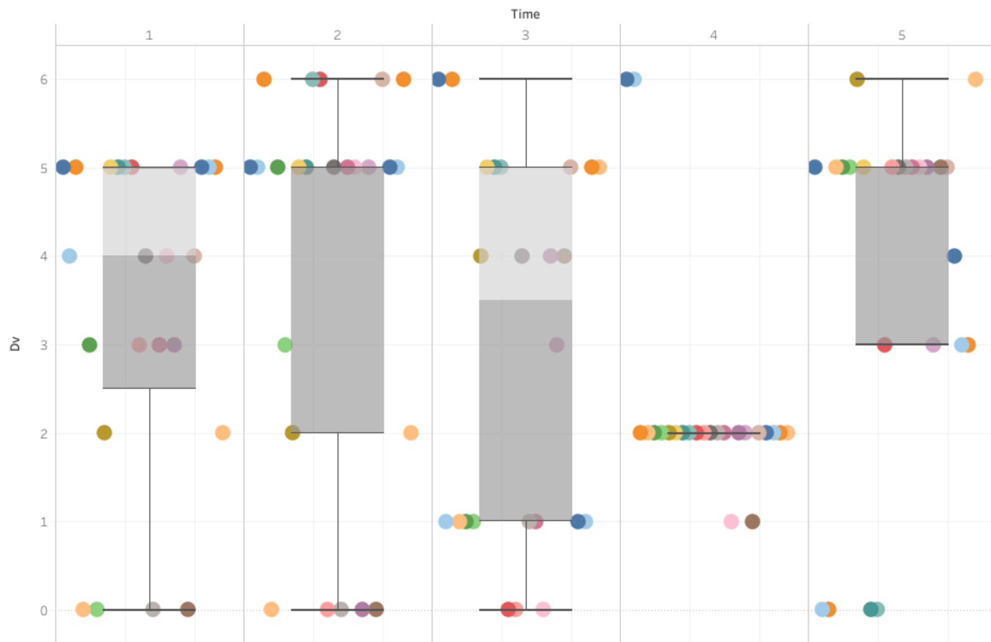
	Time				
Name	1	2	3	4	5
Adan	6	6	6	6	6
Alissa	0	4	2	5	5
Amaya	6	6	2	6	6
Amy	0	4	3	3	3
Analisa	3	5	6	3	5
Aubrey	3	4	0	3	3
Betty	4	4	2	3	3
Daniel	6	6	6	5	6
Devin	5	4	5	5	6
Eric	2	4	6	4	6
Esther	5	5	6	6	6
Israel	4	4	4	4	3
Juan	3	1	0	3	0
Julie	5	5	6	4	4
Lisa	4	3	3	3	4
Moses	4	3	2	3	3
Natalia	5	5	5	5	5
Ricky	1	3	0	3	5
Sam	6	6	6	6	6
Sara	5	5	2	0	4
Seba	4	3	5	5	4
Simon	4	1	5	6	6
Teodoro	5	5	6	6	6
Victoria	2	6	2	6	3

*Note: **Strategies(Mc):**0 – Invalid Strategy, 1- Direct Modeling, 2 – Previously taught strategy, 3- Count/Add, 4 – Count/Add w/groups, 5 – Beginning Use, 6 – Strategic Use; **Time**: 1 – Conversation 1- 32x25, 2- Conversation 2 – 12 x ¼, 3 – Conversation 3 – 132x24, 4- Conversation 3 – 16 x 2/3, 5- Conversation 4 – 15 x 4/5 +4*

Figure 3 gives a detailed look at all of the individual students' strategies across multiplication problems from Conversation 1 through Conversation 4. With each color dot representing the strategy for each student, there is not a clear pattern to how students solved the problems, but there is movement to more strategic use seen in the increase of the median and less variability within strategies over the conversations. The strategies are not following a trajectory, evident in the variation of strategies across the problems, but less variability in strategy over time with multiplication, shows evidence of students' using more sophisticated strategies. Greater sophistication indicates more use of groups of groups within multiplication. This variation, but more groups of groups over time, says that students strategies will vary depending on the problem, but there is still progression to more sophisticated grouping, signaling building understanding of use of the mathematical properties.

Appendix L

Box Plot of student division problems across conversations



	Time				
Name	1	2	3	4	5
Adan	5	5	6	6	5
Alissa	4	5	1	6	0
Amaya	5	6	6	2	0
Amy	0	0	1	2	5
Analisa	3	5	1	2	5
Aubrey	0	3	1	2	5
Betty	2	2	4	2	6
Daniel	5	5	5	2	5
Devin	5	5	5	2	0
Eric	5	6	5	2	0
Esther	5	6	0	2	3
Israel	3	0	0	2	5
Jose	4	5	4	2	5
Juan	0	0	1	2	5
Julie	3	5	1	2	5
Lisa	4	5	0	1	5
Moses	3	0	4	2	5
Natalia	5	5	3	2	3
Ricky	0	0	4	1	5
Sam	4	6	5	2	5
Sara	5	5	1	2	4
Seba	5	5	1	2	3
Teodoro	5	6	5	2	3
Victoria	2	2	5	2	6

Note: ***Division Strategies (DV):*** *0 – Invalid Strategy, 1 – Direct Modeling, 2 – Previously Taught Algorithm, 3 – Count/Add, 4 – Count/Add w/ groups, 5 – Beginning Use, 6 – Strategic Use;* ***On time 4:*** *1 – Non- Anticipatory Direct Modeling, 2 – Emergent Anticipatory Direct Modeling, 6 – Anticipatory thinking;* ***Time:*** *1- Conversation 1 -912/48, 2 – Conversation 1- 132/4, 3 – Conversation 2 – 12/ 1/3, 4 – Conversation 2 – 10/4, 5 – Conversation 3 – 24/ ¾*

Appendix M

Comparing strategies across problems from Conversation 4

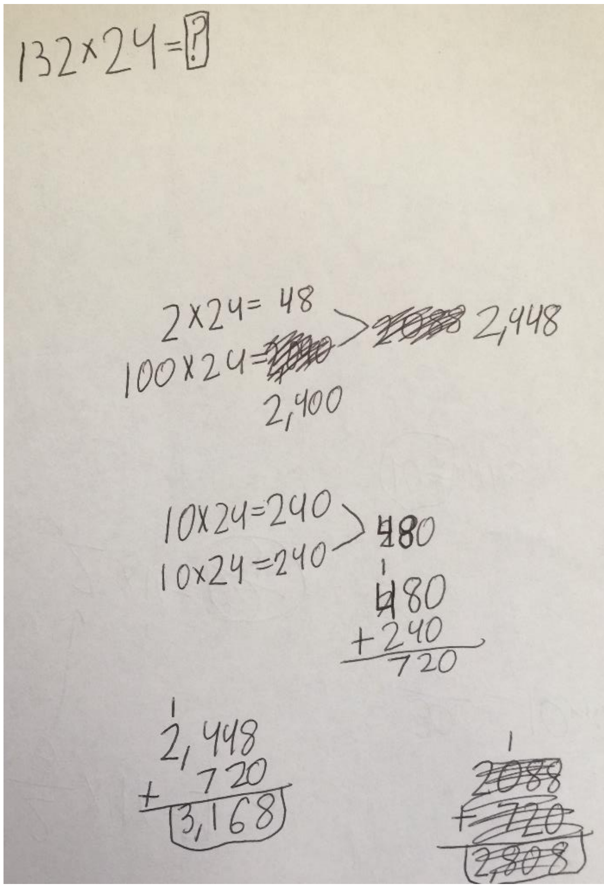
		Y= 13 – 1/3x strategies			
		1.00	2.00	4.00	Total
Y=4/5x+4 strategies	0.00	1	0	0	1
	1.00	0	0	0	0
	2.00	4	5	1	10
	4.00	4	1	8	13
Total		9	6	9	24

Note:0-Invalid Strategy, 1-Direct Modeling, 2-Count/Add, 3-Previously Taught strategy/algorithm, 4-Invented Algorithms

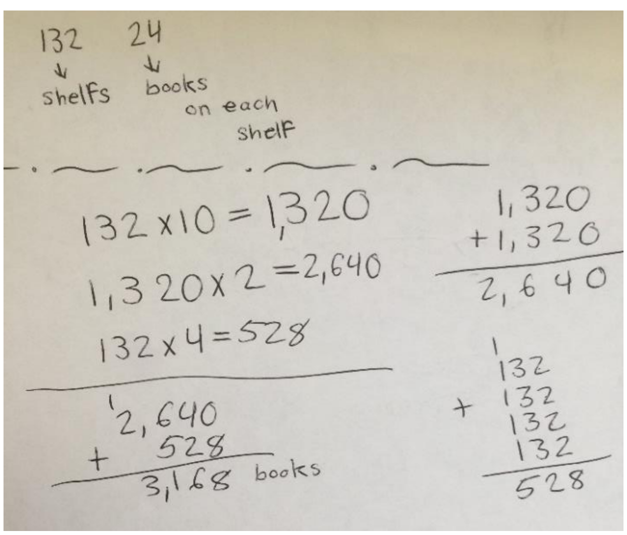
The table demonstrates students' different strategies used in the problems with the structure of y=mx +b. As shown in the table, students varied in their strategies that they used and moved between strategies with the two problems. In thinking about students' connection to these types of problems, only four students answered the 4/5 x 15 + 4 problem incorrectly and one student answered the -1/3 x 27 + 13 problem incorrectly, even though this was the first time that they had engaged in these types of problems. As noted in this table, nine students used a direct modeling strategy in the y=-1/3x + 13 problem, while six used a counting strategy, and eight used an invented algorithm. Of those nine that used a direct modeling strategy, four had used an invented algorithm for the problem of y = 4/5x + 4, four had used a counting strategy and one had an invalid strategy. Eight of the students using an invented algorithm in the first problem used it in the second and five that used a counting strategy used it in the second. The interesting note is the movement within the direct modeling, in which no one used it the first problem, but nine used it the second.

Appendix N

Example Whole Number Multiplication groups of groups strategies



Adán's Strategy

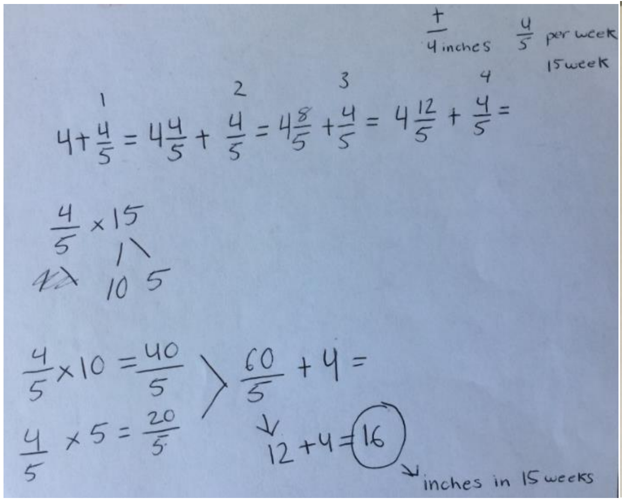


Esther's Strategy

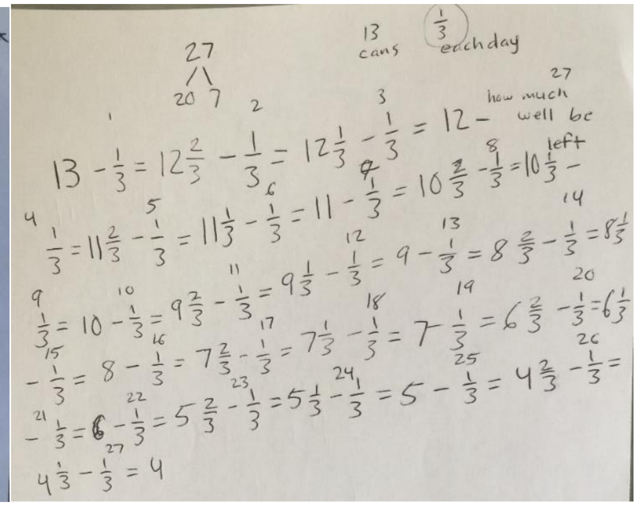
Appendix O

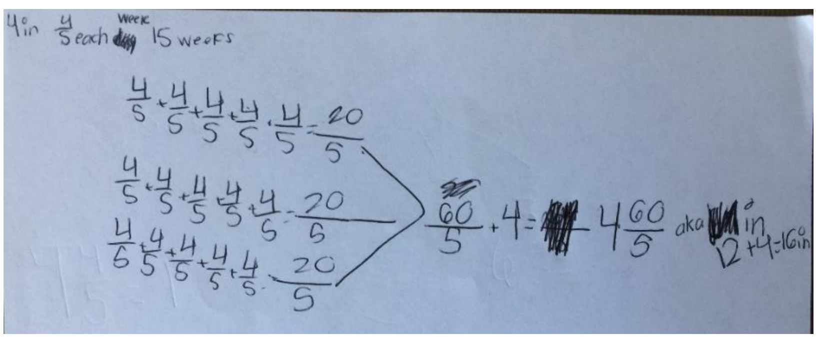
Samples of student strategies for y=mx+b problems

Esther's work for 15 x 4/5 + 4

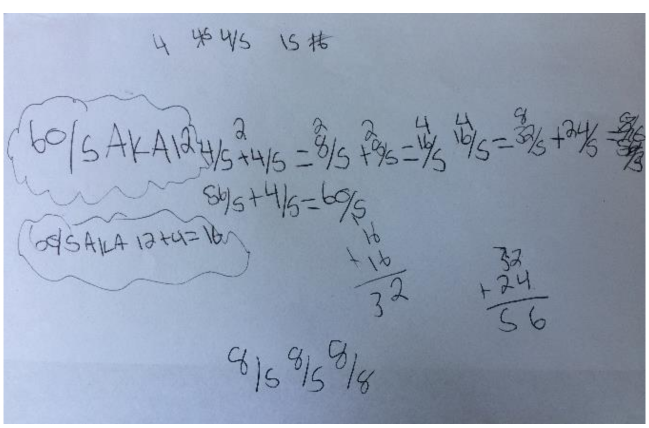


Esther's work for 13 – 27 x 1/3





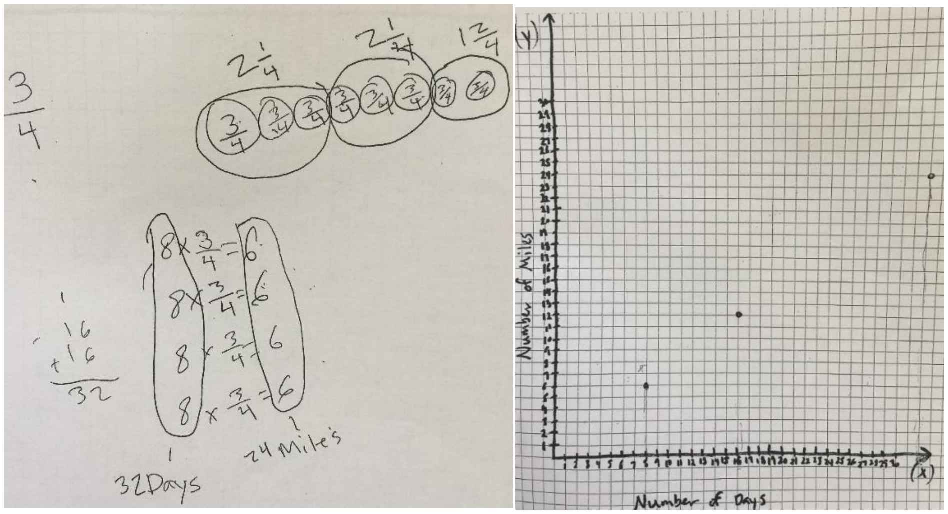
Julie's strategy



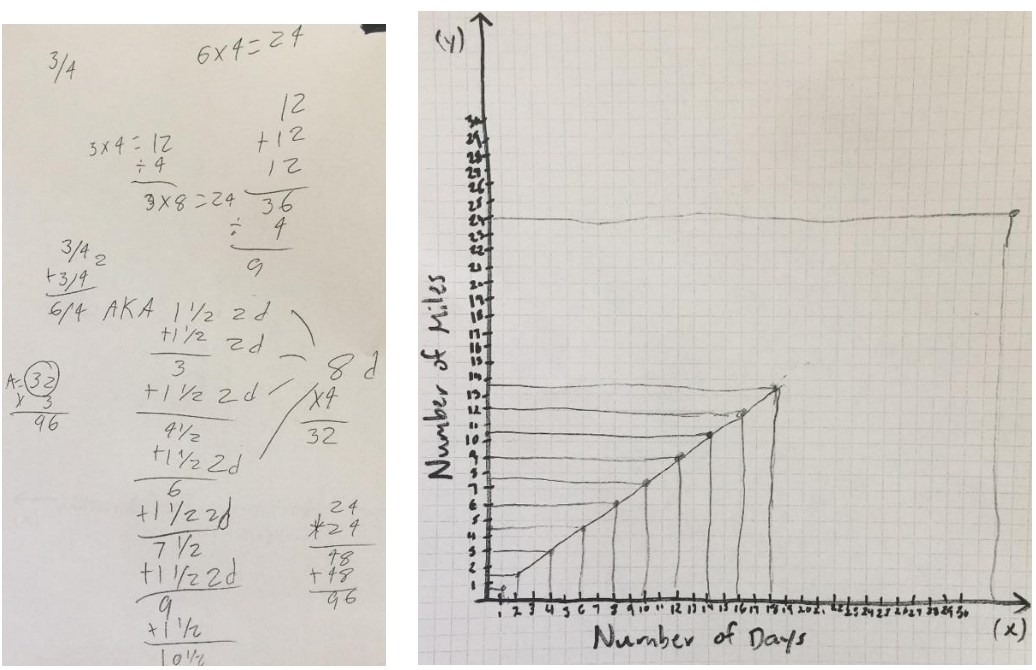
Analisa's Strategy

Appendix P

Student samples for strategies for graphing 24 ÷ ¾



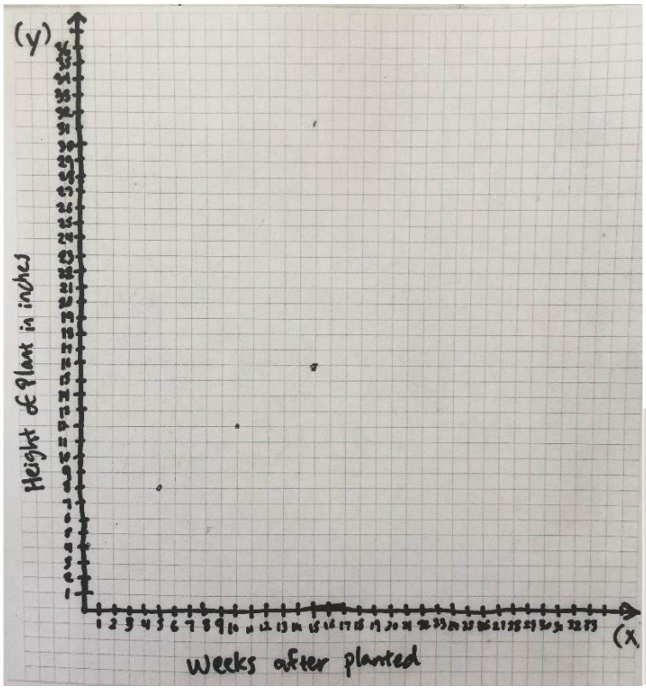
Alyssa's strategy & graph

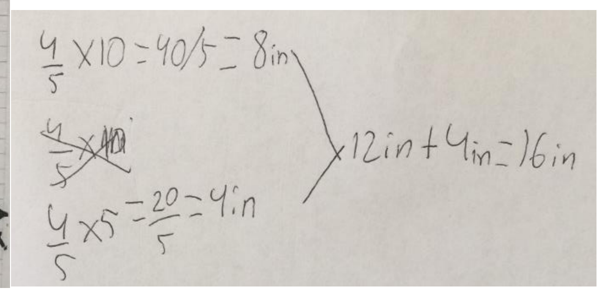


Sam's Strategy & graph

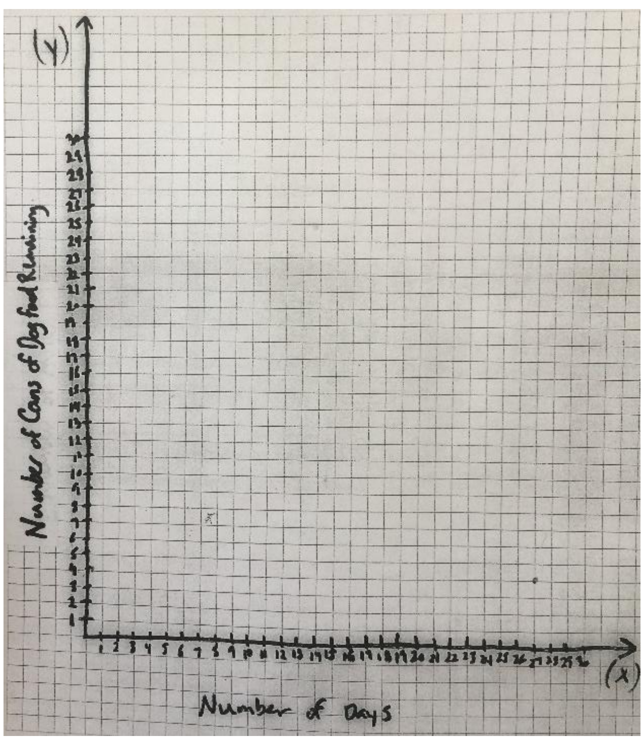
Appendix Q

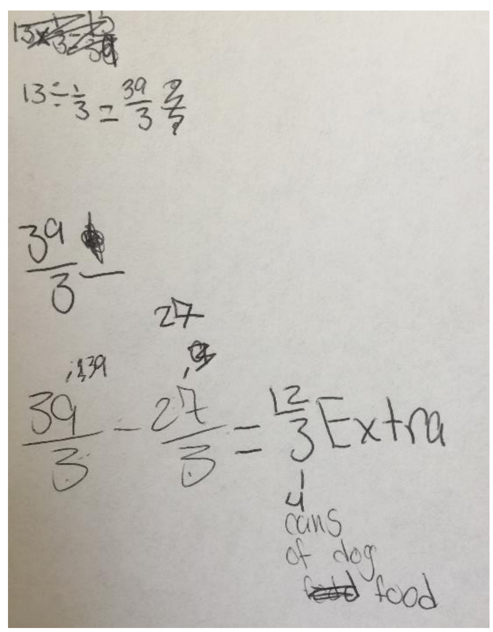
Samples of student strategies and graphs of y=mx+b problems





Jose's Strategy & graph





Victoria's Strategy & graph

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