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Stochastic optimization and its applications in time- varying wireless systems

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Stochastic Optimization and Its Applications in Time-Varying Wireless Systems.

A dissertation submitted in partial satisfaction of the requirements for the degree
Doctor of Philosophy

in

Electrical Engineering (Communication Theory and Systems)

by

Yih-Hao Lin

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Professor Rene L. Cruz, Chair
Professor Patrick Fitzsimmons
Professor Tara Javidi
Professor Laurence B. Milstein
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The dissertation of Yih-Hao Lin is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

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ABSTRACT OF THE DISSERTATION

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Professor Rene L. Cruz, Chair

With the advent of third generation wireless cellular systems, new functionalities are deployed to support dynamic adjustments of system parameters and operating points. How to effectively manage the resources using these functions with system state information, such like queue dynamics, packet delay and channel fluctuation, is very critical to the system performance.

For delay tolerant data, instead of continuously sending it in every slots, allowing some tolerable delay between consecutive transmissions could greatly increase the chance of encountering better channel states, and helps in achieving better average resource utilization. In this dissertation, we investigate the optimal

channel-aware scheduling policy for applications concerned with the long-term average performance (such as average power consumption and throughput, etc.), and the realization of the policy in various contexts. A broad class of scheduler design problems can be expressed as optimal stochastic control problems concerned with the time and ensemble average of controlled processes. In light of this, we devise a framework for stochastic dynamic control utilizing a mathematical tool, referred to as Stochastic Optimization. Leveraging the duality approach developed for convex optimization, we simplify the constrained Stochastic Optimization problem into an unconstraint one. Furthermore, we develop an online algorithm for solving the stochastic optimization problem. For a broad class of stationary stochastic processes which satisfy a set of mixing conditions, the behavior of the algorithm can be approximately characterized by a projected differential inclusion. Exploring the trajectory of the projected differential inclusion, it is proved that the long-term average of the control variables generated by the proposed algorithm along its recursive steps converges asymptotically to the optimal one.

To demonstrate the use cases of the established framework, we study the power-optimized routing problem in multi-hop wireless networks. A distributed implementation of the algorithm is devised. Applying the same framework, we also look into the problem of joint source distortion management and wireless downlink scheduling, in which we aim to minimize the maximum average distortion of the data requested by each user. The proposed algorithm comes with the properties of finite receiver buffer occupation as well as the negligible packet-drop ratio.

Chapter 1

Introduction

With the advance of modem communication theory, it was noticed that instead of attempting to compensate for the channel impairments, dynamically allocating resources according to the channel states provides better system performance in terms of the average throughput or power comsumption [30] [24] [50]. This new viewpoint of the varying channels founds the basis of this dissertation, in which we seek the optimal scheduling rule for resource management adapting to the continuous changes in the channel states.

1.1 Stochastic Optimization

Due to different degree of user mobility and the movement of surrounding obstructions, the variations of the wireless channel states occur in different time scales. In this dissertation, we focus on the slow time varying channel so that the channel states can be accurately measured or estimated. One widely acknowledged approach for describing the slow time scale channel varying behavior is the block fading channel model [19, p.p. 102], in which the time series of channel states is represented by a random process $\{\xi^n,\ n = 1, 2, 3, \ldots\}$. For designing an effective scheduling algorithm, the most important question to be answered is how to address the uncertainties and variations inherent in $\{\xi^n,\ n = 1, 2, 3, \ldots\}$. From the literature, we notice that many of the channel aware scheduling problems (e.g. [24] [30] [48] [11]) can be expressed as optimizations with respective to the time and/or ensemble average of controllable random sequences. For example, to maintain the expected average rate at 1Mbps over a wireless link using the minimum average transmit power, the optimal scheduler defines a solution to the following optimization:

$$\text{minimize} \qquad \limsup \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{P^n\} \tag{1.1}$$

$$\text{subject to} \qquad \liminf \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{R^n\} \geq 1 \tag{1.2}$$

$$R^n \leq \phi(h^n P^n / \eta) \quad \text{(Mbps)}, \tag{1.3}$$

where P^n is the scheduled transmit power at time n, h^n is the channel gain at time n, η is the noise power of additive white Gaussian noise at the receiver, R^n is the transmit rate in time n, and $\phi(\cdot)$ is the capacity function of the channel. The use of limsup and liminf in the above equation is to ensure the existence of the limits.

This type of problem settings are within the scope of Stochastic Optimization, which is an extension to the classical convex optimization with infinite dimensional control variables. Specifically, unlike traditional nonlinear optimization which deals with finite dimensional deterministic variables, stochastic optimization problems work with basic variables representing the time and ensemble average of randomized control sequences. The generic setting of stochastic optimization problem is displayed in the following form:

$$\text{minimize} \quad \limsup_{N} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{X^n\} \tag{1.4}$$

$$\text{subject to} \quad \limsup_{N} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{\vec{Y}^n\} \leq 0 \tag{1.5}$$

$$(X^n, \vec{Y}^n) \in \mathcal{D}(\xi^n), \tag{1.6}$$

where $X^n \in \mathbb{R}$ and $\vec{Y}^n \in \mathbb{R}^r$ are control variables, $\{\xi^n\}$ represents the underlying random process, and $\mathcal{D}(\xi^n)$ is a set with its elements determined by ξ^n. The solution to this problem is a policy controlling the value of X^n and $\vec{Y}^n$ along the time index $n = 0, 1, 2, \ldots, \infty$. The generic stochastic optimization problem is investigated in detail in Chapter 2, and a framework for solving the problem is established via rigorous mathematical analysis. An important class of online recursive algorithms for solving the optimization are presented in Chapter 2, in which we demonstrate that the algorithm converges asymptotically to the optimal solution if $\{\xi^n\}$ belongs to a particular set of ϕ-mixing processes.

1.2 Multi-hop Wireless Networks: Routing, Link Scheduling, and Power Control

To demonstrate the utilization of Stochastic Optimization in practical problems, we look into power efficient routing for wireless networks.

Along with the growing interest of ad-hoc wireless network, power efficient scheduling algorithms has become an important research topic. In [14], the authors studied the power efficient scheduling assuming fixed wireless channels; in [8], the authors studied the optimal tradeoff between the average delay and the average power consumption; and in [48], the authors studied the power efficient scheduling for a time-varying multi-access network. Each of these works touched a piece of the ultimate scheduling problem from the perspectives of network routing, power control, link capacity, and scheduling, respectively. In this dissertation, we seek the optimal scheduling policy which jointly considers routing, scheduling, and power control under the time-varying channels.

Usually, wireless devices are short of steady power supplies, hence saving transmit power is crucial to extending the device's life time. To address this issue, we focus on the power efficient routing problems in Chapter 3 and 4, where we assume the block fading channel model for characterizing the channel variations. In routing problems, the data originating from a set of nodes are sent to their destination in a stored-and-forward fashion with the assistance of intermediate nodes. The data traffic belonging to the same source and destination pairs constitute a end-to-end data stream called flow, and a minimum average rate requirement is assumed for each flow. Similar to the network flow optimization with deterministic capacity constraint [17], the optimal selection of routes in a network can be expressed as a set of flow rate assignments. A flow rate assignment is feasible if the average ingressive and egressive flow rates at each node are equal. This requirement constitutes the flow conservation constraint. In addition to the flow conservation constraint, we have link capacity constraints accounting for the limit of maximum aggregated flow rates borne on each link, and the channel capacity constraints accounting for the feasible transmit rate vectors under the instantaneous channel conditions. Subject to these constraints, the problem of optimal power efficient routing is formulated as the minimization of average power consumption. The details are discussed in Chapter 3.

Exploiting the stochastic approximation algorithm developed in Chapter 2, we derive the asymptotically optimal strategy that minimizes the long-term average transmit power under average rate constraints for each flow. From the numerical results in Chapter 3, we observe that to combat the interference, for low data rate demands, the system should work in a time division multiplexing (TDM) manner to avoid excessive co-channel interferences. However, as the rate demands increase, more frequently, a large number of links are activated concurrently to support the required QoS.

Although our theoretical results substantiate the optimality of the algorithm, it still leaves some room for further enhancement. This can be seen from the following observations: 1) the realization of the algorithm generally relies on centralized controllers, and 2) the complexity of the scheduler raises up along with the number of users.

To construct a scalable scheduler, in Chapter 4, we consider decomposing the system into multiple isolated subsystems so that the transmit signals are orthogonal at the receivers. A distributed algorithm is proposed in Chapter 4, which is proven to be asymptotically optimal under the given decomposition. Note that decomposition is the approach we take to address the complexity issue; it may result in inefficient use of system resources, and sacrificing system performance. However, it provides some good effects compensating the loss. One important gain contributed by the decomposition is the provision of more structured model for designing systems, which makes problem more manageable in the sense that each node relies only on the parameters exchanged with the nearby nodes to make decisions.

1.3 Rate Distortion Management and Downlink Transmission

As the second application of our stochastic optimization framework, in Chapter 5, we investigate the joint source distortion management and transmit rate control problem in a multi-user downlink wireless system. Similar research works include [37] [16] [21] [39]. All of them are concerned with systems consisting of single user or link and exclude the gains of employing multi-user communication [20] [49]. Our work starts with a generic setting considering the capacity achieving schemes for the broadcast channel. Although, at the first place, we do not consider delay and buffer size in the problem formulation, surprisingly, we can build on this to construct a system that performs well in terms of average delay and buffer size.

We assume a set of sources generate sequence of symbols stored at the access point. The rate distortion function $R(D_{avg})$ for each source is available so that, given the target average distortion D_{avg}, the scheduler knows the average number of bits needed to represent the quantized data. The requested data is transmitted to and buffered at the receiver, and the received data is used by the target application at a constant rate (data consumption rate). If user's buffered data reach zero and the arrival rate is less than the data consumption rate, a buffer underflow occurs. In this work, we focus on applications which are distortion tolerant. In each scheduling interval, the scheduler chooses the appropriate parameter for quantizing the transmit data sequence, and compresses the processed data before sending it onto the channel. The scheduler needs to balances the distortion caused by the quantization and the distortion caused by buffer underflows. Specifically, the goal we set for this source distortion management and transmit rate control problem is to minimize the maximum long-term average distortion among users. Built on top of the fundamental results established in Chapter 2 for Stochastic Optimization, we derive the scheduler reaching this end. Moreover, with the recognition of the equivalence between the evolution of queue size and that of the dual variables [22] using the subgradient methods, we obtain the property of boundedness on the user's queue size for our algorithm, which gives asymptotically negligible buffer underflow frequency.

1.4 Main Contributions

The main contributions of this work are summarized in the following list.

- We develop a systematic framework using Stochastic Optimization to design and analyze channel aware scheduling algorithms.
- Under the framework, we propose a generic online recursive algorithm to reach the optimal solution asymptotically.
- We prove, through Stochastic Approximation, that the asymptotic behavior of the proposed algorithm follows the solution of a projected differential inclusion in expectation, and the trajectory of the solution converges to the optimal point of the optimization.
- We develop an online algorithm to handle scheduling and power control for power efficient multi-hop wireless networks, and we propose a heuristic extension to implement the algorithm in a distributed manner.
- We develop an online algorithm for jointly managing the source distortion and scheduling the wireless downlink transmissions. The algorithms ensure the stability of the system.

To explain in detail, we begin with establishing the fundamental basis and framework for Stochastic Optimization, and move on later to the applications using this framework.

Chapter 2

Stochastic Approximation and Stochastic Optimization

In this chapter, we introduce an optimization framework for stochastic systems, namely stochastic optimization, which is motivated by various interesting stochastic dynamic control problems to be discussed in this dissertation. The formulation of stochastic optimization problems have very similar appearance as that of the classical convex optimization problems. In addition, many important concepts in stochastic optimization are adopted from the classical convex optimization. The main difference between the two is that unlike the classical convex optimization, which considers finite dimensional deterministic variables, stochastic optimization works with controllable stochastic decision processes. Specifically, stochastic optimization deals with the average behavior (time and the ensemble) of the stochastic processes which can be dynamically controlled via adjusting a set of parameters. An example for the controllable stochastic decision process is the sequence of transmit rates on a time varying channel.

Our contribution in this subject focuses on the convergent proof of an adaptive recursive algorithm which reaches the solution of the stochastic optimization problem asymptotically. Moreover, to update the value in each time instant, this recursion only requires the observation of the underlying exogenous stochastic process which is a process unaffected by the control actions (e.g. channel conditions). Therefore, this algorithm can be run concurrently as the system goes, and provides online scheduling information to the system. The proof for the algorithm's convergent behavior heavily relies on the fundamental basis developed for a particular class of recursive algorithms called stochastic approximation. The feasibility and optimality of the proposed recursive algorithm follows the convergent behavior we established in this chapter. To explain in detail, we start with the stochastic approximation (SA) in the first section of this chapter. Later on, we will built the framework of stochastic optimization on top of SA.

2.1 Stochastic Approximation Algorithms

Basic stochastic approximation methods are generally known as a family of recursive algorithms attempting to find the zeros or the extremes of functions when only the noisy observation of the function values are available. The outcome of the recursive algorithm is a sequence which approximates the zeros or the extreme point of the function. The well-known Robbins-Monro algorithm [41] and

Kiefer-Wolfowitz algorithm [23] belong to this category.

The generic stochastic approximation algorithms consist of three major components:

1. *Step Size*: $\epsilon > 0$
2. *State Variable*: $\theta_\epsilon^n = (\theta_{\epsilon,1}^n, \cdots, \theta_{\epsilon,r}^n) \in \mathbb{R}^r$, which is updated in each iteration
3. *Search Direction*: $Y_\epsilon^n = (Y_{\epsilon,1}^n, \cdots, Y_{\epsilon,r}^n) \in \mathbb{R}^r$; Y_ϵ^n is a stochastic process that depends on an exogenous process $\{\xi_\epsilon^n\}$.

More detailed explanations on the terminologies and presentation about these three quantities are described below.

- A stochastic process is called *exogenous* with respect to the system if its future evolution, given the past history of the system, depends only on the past history of ξ_ϵ^n. More precisely, let $\{\mathcal{F}_n^\epsilon\}$ be a sequence of nondecreasing σ-algebras, where $\mathcal{F}_n^\epsilon$ measures at least $\{\theta_\epsilon^j, Y_\epsilon^{j-1}, \xi_\epsilon^j; j \le n\}$, then a exogenous process satisfies

$$P(\xi_\epsilon^k \mid \mathcal{F}_n^\epsilon) = P(\xi_\epsilon^k \mid \xi_\epsilon^j, j = 0, 1, \ldots, n) \text{ for } k \ge n. \tag{2.1}$$

  For applications in wireless communications, channel state is a major source contributing the randomness. Suppose that the system operates in discrete time, one can regard ξ_ϵ^n as the vector representing the states of all channels in the system at time n.

- The variables are subscripted with step size ϵ to emphasize the fact that the selection of these variables may depend on the step size that we take.

Recursive Algorithm

The evolution of state variable θ_ϵ^n is determined by the projected stochastic difference equation given below,

(**SA**)

$$\theta_\epsilon^{n+1} = \Pi_H\left(\theta_\epsilon^n + \epsilon Y_\epsilon^n\right) \tag{2.2}$$

where $\Pi_H(\cdot)$ denotes the projection of input onto the constraint set $H = \{\theta:\ 0 \leq \theta_i \leq K_u\}$ with certain fixed constant K_u. Function $\Pi_H(\cdot)$ translates its argument to the closest point in constraint set H. The displacement from the vector v to its projection $\Pi_H(v)$ is called the *reflection term*. Define the reflection term $Z_\epsilon^n = (Z_{\epsilon,1}^n, \cdots, Z_{\epsilon,r}^n) \in \mathbb{R}^r$ by rewriting (2.2) as

$$\theta_\epsilon^{n+1} = \theta_\epsilon^n + \epsilon Y_\epsilon^n + \epsilon Z_\epsilon^n. \tag{2.3}$$

Since the constraint set is restrict to $\mathbb{R}^{+,r}$, θ_ϵ^n is componentwise nonnegative. The output of (2.2) is a sequence $\{(\theta_\epsilon^n, Z_\epsilon^n),\ n = 0, 1, 2, \ldots\}$, and stochastic approximation is concerned with the asymptotic behavior ($n \to \infty$) of θ_ϵ^n and Z_ϵ^n with small step size. Before we delve into the analytical sections, we first review some fundamental probability concepts and terminologies which are used frequently in this work.

Definition 2.1. *Let $\{A_n\}$ be a sequence of $\mathbb{R}^k$-valued random variables on a common probability space $(\Omega, P, \mathcal{F})$ with $(a_{n,i}, i = 1, \ldots, k)$ being the real-valued components of A_n. Let P_n denote the measures on the Borel set of $\mathbb{R}^k$ determined by A_n, and let $x = (x_1, \ldots, x_k)$ denote the canonical variable in $\mathbb{R}^k$. If there is a*

$\mathbb{R}^k$-valued random variable A with real-valued components $(a_1, \ldots, a_k)$ such that

$$P\{a_{n,1} < \alpha_1, \ldots, a_{n,k} < \alpha_k\} \to P\{a_n < \alpha_1, \ldots, a_n < \alpha_k\} \tag{2.4}$$

for each $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$ at which the right side of is continuous , then we say that A_n converges to A in distribution. Let P_A denote the measure on the Borel sets of $\mathbb{R}^k$ determined by A. An equivalent definition is that

$$EF(A_n) = \int F(x)dP_n(x) \to EF(A) = \int F(x)dP_A(x) \tag{2.5}$$

for each bounded and continuous real-valued function $F(\cdot)$ on $\mathbb{R}^k$. Convergence in distribution is also called weak convergence, we use the following notation to denote the weak convergence.

$$A_n \Rightarrow A \tag{2.6}$$

2.1.1 Interpolated Process

The recursive algorithm (**SA**) (2.2) generates a sequence $\{\theta_\epsilon^n\}$. We are particularly interested in the evolution of $\{\theta_\epsilon^n\}$ in two extreme cases. First, how does the sequence $\{\theta_\epsilon^n\}$ behave as the step size ϵ goes to zero? Second, does the sequence converge as n goes to infinity? These questions can be answered by looking into the interpolated process of sequence $\{\theta_\epsilon^n\}$, $\{Y_\epsilon^n\}$ and $\{Z_\epsilon^n\}$. The interpolated process of a sequence with respect to step size ϵ is the "continuous-time" interpolations[1] of the sequence with the interpolation interval equal toϵ.

[1]Please be aware that the term "continuous-time" is not the real "time", but a virtual continuous index used to describe the asymptotic behavior of the interpolated process.

Define interpolations $\theta_\epsilon(t)$, $Y_\epsilon(t)$ and $Z_\epsilon(t)$ as follows.

$$\theta_\epsilon(t) = \begin{cases} \theta_\epsilon^0 & \text{for } t < 0 \\ \\ \theta_\epsilon^n & \text{on the time interval } [\,n\epsilon, n\epsilon + \epsilon). \end{cases} \tag{2.7}$$

$$Y_\epsilon(t) = \begin{cases} \epsilon \sum_{i=0}^{t/\epsilon - 1} Y_\epsilon^i, & \text{where } t \geq 0 \\ \\ 0 & \text{for } t < 0 \end{cases} \tag{2.8}$$

$$Z_\epsilon(t) = \begin{cases} \epsilon \sum_{i=0}^{t/\epsilon - 1} Z_\epsilon^i, & \text{where } t \geq 0 \\ \\ 0 & \text{for } t < 0 \end{cases} \tag{2.9}$$

The integer part[2] of t/ϵ is always used in describing the scope of summations when they are defined. An example of $\theta_\epsilon(t)$ is illustrated in Figure 2.1

We are most interested in the asymptotic behavior of $(\theta_\epsilon(t), Z_\epsilon(t))$ with small step size. In the next subsection, we will present a theorem about the convergent results. Precisely, it states that under certain appropriate conditions, the interpolated process $(\theta_\epsilon(t), Z_\epsilon(t))$ converges weakly to a limiting processes $(\theta(t), Z(t))$ along a subsequence of step size $\{\epsilon_k\}$. Before the theorem is revealed, we first present a set of assumptions which are sufficient to ensure the existence of the limiting processes $(\theta(t), Z(t))$.

[2]The integer part of a real number is defined to be the largest integer less than or equal to it.

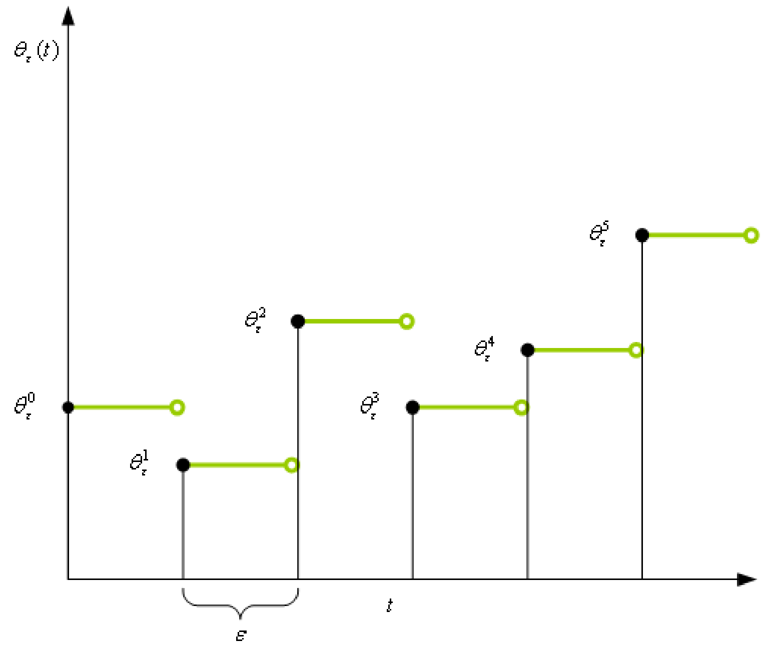


Figure 2.1: Interpolated Process $\theta_\epsilon(t)$

2.1.2 Limiting Process and Its Behavior

Assumptions

Let $\{\xi_\epsilon^n;\ n \geq 0\}$ be random variables over certain complete and separable metric space Ξ; $\{\mathcal{F}_n^\epsilon\}$ be a sequence of nondecreasing σ-algebras, where $\mathcal{F}_n^\epsilon$ measures at least $\{\theta_\epsilon^j, Y_\epsilon^{j-1}, \xi_\epsilon^j; j \leq n\}$; and $\mathbb{E}_n^\epsilon$ be the expectation conditioned on $\mathcal{F}_n^\epsilon$. In the following, we list a set of assumptions served as sufficient conditions for the convergence of our algorithm.

(A.1) $\{Y_\epsilon^n; \epsilon, n\}$ is uniformly integrable.

(A.2) There are measurable functions $g_\epsilon^n(\cdot)$ such that

$$\mathbb{E}_n^\epsilon Y_\epsilon^n = g_\epsilon^n(\theta_\epsilon^n, \xi_\epsilon^n) \tag{2.10}$$

(A.3) For each $\delta > 0$ there is a compact set $A_\delta \subset \Xi$ such that

$$\inf_{n,\epsilon} P\{\xi_\epsilon^n \in A_\delta\} \geq 1 - \delta. \tag{2.11}$$

(A.4) For each θ, the sequences

$$\{g_\epsilon^n(\theta_\epsilon^n, \xi_\epsilon^n);\, \epsilon, n\}\,, \quad \{g_\epsilon^n(\theta, \xi_\epsilon^n);\, \epsilon, n\} \tag{2.12}$$

are uniformly integrable.

(A.5) There is a set-valued function[3] $\mathcal{G}(\cdot)$ that is upper semi-continuous, and for each compact set A, and any sequence α, α_i^n, satisfying

$$\lim_{n,m\to\infty} \sup_{n\leq i\leq n+m} |\alpha_n^i - \alpha| = 0, \tag{2.13}$$

we have

$$\lim_{n,m} \text{distance}\left[\frac{1}{m}\sum_{i=n}^{n+m-1} \mathbb{E}_n^\epsilon g^i(\alpha_n^i, \xi^i), \mathcal{G}(\alpha)\right] I_{\{\xi^n\in A\}} = 0 \quad \text{w.p.1.} \tag{2.14}$$

Specifically, the upper semi-continuous property referred in (A.5) is defined below.

Definition 2.2. ***Upper semi-continuity*** *Let $N_\delta(x)$ be a δ-neighborhood of x. A set-valued function $\mathcal{B}(\cdot)$ is said to be* upper-semi-continuous *if it satisfies*

$$\bigcap_{\delta>0} \text{co}\left[\bigcup_{y\in N_\delta(x)} \mathcal{B}(y)\right] = \mathcal{B}(x), \tag{2.15}$$

where $\text{co}(A)$ *denotes the convex hull of the set A.*

We note that:

[3]In our work, a set-valued function is a mapping from $\mathbb{R}^r$ to the subsets of $\mathbb{R}^r$.

Remark 2.1. *Assumption (A.5) holds under simple mixing conditions, M-dependence, or ergodicity type conditions on* $\{\xi_\epsilon^n\}$.

Keeping assumptions (A.1-5) in mind, we now introduce the convergent theorem for stochastic approximation.

Theorem 2.1. *[27, Chapter 8, Theorem 2.5] Assume (A.1-5), for any nondecreasing sequence of integers* q_ϵ, *for each subsequence of* $\{\theta_\epsilon(\epsilon q_\epsilon + \cdot), Z_\epsilon(\epsilon q_\epsilon + \cdot), \epsilon > 0\}$, *there exists a further subsequence and a process* $(\theta(\cdot), Z(\cdot))$ *such that*

$$(\theta_\epsilon(\epsilon q_\epsilon + \cdot), Z_\epsilon(\epsilon q_\epsilon + \cdot)) \Rightarrow (\theta(\cdot), Z(\cdot)) \tag{2.16}$$

as $\epsilon \to 0$ *through the convergent subsequence, where* $\theta(t)$ *and* $Z(t)$ *have Lipschitz continuous paths with probability one. In addition, there is an integrable* $z(\cdot)$ *such that*

$$Z(t) = \int_0^t z(s)\, ds, \text{ where } z(t) \in -C(\theta(t)) \text{ for almost all } t, \omega \tag{2.17}$$

and

$$\dot{\theta} \in \mathcal{G}(\theta) + z, \text{ for almost all } t, \omega. \tag{2.18}$$

For completeness, we summarize some key properties of $C(x)$ in the following list.

Remarks on $C(x)$

- The set-valued function $C(x)$ is defined for $x \in H$ (The constraint set defined for the recursive algorithm (**SA**)). If $x \in H^o$, the interior of H, $C(x)$ contains only the zero element; if $x \in \partial H$, the boundary of H, $C(x)$ is the infinite convex cone generated by the outer normals at x of the faces on which x lies.

- Since an "infinitesimal" change in x does not increase the number of active constraints, $C(\cdot)$ is upper-semi-continuous.

In brief, Theorem 2.1 provides a rule to check if the updated sequence generated by the stochastic approximation algorithm follows the trajectory of a projected differential inclusion.

Essentially, the stochastic optimization problems are convex (This will be shown shortly.). Therefore, to solve the problem, we expect certain "gradient-type" recursive algorithms (originally designed for convex optimization) will work. This conjecture can be verified using those fundamental basis developed in Section 2.1.

In general, the functions that we deal with are not necessarily differentiable. In other words, the gradient of the function may not exist. To be cautious, in order to continue applying the "gradient-type" algorithm, we need to consider a generalization of gradient called "subgradient". The definition of subgradient for nondifferentiable convex functions is defined as follows.

Definition 2.3. *Given a convex function $f : \mathbb{R}^n \mapsto \mathbb{R}$, we say that a vector $d \in \mathbb{R}^n$ is a* subgradient *of f at a point $x \in \mathbb{R}^n$ if*

$$f(z) \geq f(x) + (z-x)'d, \quad \forall z \in \mathbb{R}^n. \tag{2.19}$$

If instead f is a concave function, we say that d is a subgradient of f at x if $-d$ is a subgradient of the convex function $-f$ at x. The set of all subgradients of a convex (or concave) function f at $x \in \mathbb{R}^n$ is called the subdifferential *of f at* *x, and is denoted by $\partial f(x)$. Note that subdifferentials are upper-semi-continuous set-valued functions [27, p. 25]. (For more details about the subgradient, please refer to [9, p.p. 711].)*

Some Comments on $\mathcal{G}(\theta)$

The set-valued function $\mathcal{G}(\theta)$ defined in assumption (A.5) is the key element determining the asymptotic behavior of state variable θ_ϵ^n for small step size. In the remainder of this section, we will focus on the case when $\mathcal{G}(\theta) = \partial V(\theta)$, the subdifferential of a concave function $V(\theta)$ at θ. The lessons we learned from this class of $\mathcal{G}(\theta)$ form a strong basis for solving the stochastic optimization problems when the exogenous process $\{\xi^n\}$ is ϕ-mixing [15] (uniformly mixing [10]).

Mixing condition is a stochastic property which limits the difference between the conditional and the unconditional probability distribution. A process is called ϕ-mixing (uniformly mixing) if there exists a sequence $\{\phi_k,\ k = 0, 1, 2, \ldots\}$ such that the difference between the following two quantities is bounded uniformly by ϕ_k.

(i) The marginal distribution of the process at time $i + k$

(ii) The distribution of the process at time $i + k$ conditioned on the history of the process up to time i.

Moreover, the bounding sequence $\phi_k \to 0$ as k goes to ∞. The formal definition is summarized below.

Definition 2.4. *Let $\mathcal{B}_m^n$ be the σ-algebra generated by the random variables*

$\{\xi^m, \xi^{m+1}, \ldots, \xi^n\}$. Define ϕ_k by

$$\phi_k \triangleq \sup_i \sup_{A\in\mathcal{F}_0^{i+k}, B\in\mathcal{F}_0^i} |\, P\{A|B\} - P\{A\}\,| \tag{2.20}$$

If $\lim_k \phi_k = 0$, then $\{\xi^k\}$ is called a ϕ-mixing process.

For a ϕ-mixing process $\{\xi_\epsilon^n\}$, the difference between the expectation of a $\mathcal{B}_n^\infty$ measurable function and its expectation conditioned on $\{\xi_\epsilon^m,\ 0 \le m \le n\}$ is negligible if the time separation k is large. In other words, it means that:

Lemma 2.1. *[26, Chapter 6, Lemma 4]*

Let ξ^n be ϕ-mixing, and let g^n be measurable on $\mathcal{F}_n^\infty$ with $|g^n| \le K$, then

$$\Big|\, \mathbb{E}[\, g^{n+k}\, |\mathcal{B}_0^n\,] - \mathbb{E}[\, g^{n+k}\,]\, \Big| \le 2K\phi_k \tag{2.21}$$

The following theorem presents an important class of functions satisfying assumption (A.5).

Theorem 2.2. *Consider a bivariate function $V(\cdot,\cdot)$ which is concave in the first argument. Let $\{\xi_\epsilon^n\}$ be a stationary ϕ-mixing stochastic process, and $g_\epsilon^n(\cdot, \xi_\epsilon^n)$ be a subgradient of $V(\cdot, \xi_\epsilon^n)$. Assume $|g_\epsilon^n(\cdot, \xi_\epsilon^n)| \le K$ for some $K > 0$. For any sequence α_i^n, α satisfying*

$$\lim_{n,m\to\infty} \sup_{n\le i\le n+m} |\alpha_n^i - \alpha| = 0, \tag{2.22}$$

we have

$$\lim_{n,m} \frac{1}{m} \sum_{i=n}^{n+m-1} \mathbb{E}_n^\epsilon g_\epsilon(\alpha_n^i, \xi_\epsilon^i) \in \mathcal{G}(\alpha) \triangleq \partial\, \mathbb{E}^\epsilon V(\alpha, \xi_\epsilon^0), \tag{2.23}$$

where $\mathbb{E}^\epsilon$ denotes the unconditional expectation, and $\partial\, \mathbb{E}^\epsilon V(\alpha, \xi_\epsilon^0)$ denotes the subdifferential of $\mathbb{E}^\epsilon\{V(\alpha, \xi_\epsilon^0)\}$ at α.

Proof.

$$\lim_{n,m} \frac{1}{m} \sum_{i=n}^{n+m-1} \mathbb{E}_n^{\epsilon} g_{\epsilon}(\alpha_n^i, \xi_{\epsilon}^i) \tag{2.24}$$
$$= \lim_{n,m} \frac{1}{m} \sum_{i=n}^{n+m-1} \mathbb{E}^{\epsilon} g_{\epsilon}(\alpha_n^i, \xi_{\epsilon}^i) + \lim_{n,m} \frac{1}{m} \sum_{i=n}^{n+m-1} (\mathbb{E}_n^{\epsilon} g_{\epsilon}^i(\alpha_n^i, \xi_{\epsilon}^i) - \mathbb{E}^{\epsilon} g^i(\alpha_n^i, \xi_{\epsilon}^i))$$

The second limit in the equality above goes to zero, since

$$\lim_{n,m} \frac{1}{m} \sum_{i=n}^{n+m-1} \left| \mathbb{E}_n^{\epsilon} g_{\epsilon}(\alpha_n^i, \xi_{\epsilon}^i) - \mathbb{E}^{\epsilon} g_{\epsilon}(\alpha_n^i, \xi_{\epsilon}^i) \right| \overset{(a)}{\leq} 2K \lim_{n,m} \frac{1}{m} \sum_{i=n}^{n+m-1} \phi_{i-n} \tag{2.25}$$
$$= 2K \lim_{m} \frac{1}{m} \sum_{i=0}^{m-1} \phi_i \tag{2.26}$$
$$\overset{(b)}{=} 0 \tag{2.27}$$

The inequality (a) above follows Lemma 2.1, and the equality (b) holds because the Cesàro mean of a convergent sequence leads to the same limit of that sequence. The proof is then completed with the use of the following fact.

$$\lim_{n,m} \frac{1}{m} \sum_{i=n}^{n+m-1} \mathbb{E}^{\epsilon} g(\alpha_n^i, \xi_{\epsilon}^i) \overset{(c)}{=} \lim_{n,m} \frac{1}{m} \sum_{i=n}^{n+m-1} \mathbb{E}^{\epsilon} g(\alpha_n^i, \xi_{\epsilon}^0) \overset{(d)}{\in} \partial\, \mathbb{E}^{\epsilon} V(\alpha, \xi_{\epsilon}^0), \tag{2.28}$$

where (c) follows the the stationarity of ξ_{ϵ}^n and (d) follows the the upper-semi-continuous property of the subdifferentials. □

2.1.3 Asymptotic Behavior of the Limiting Process

In Theorem 2.1, it has been demonstrated that limiting behavior of the stochastic approximation algorithm is governed by a projected differential inclusion. In this subsection, we look into the cases when the right-hand-side of the projected differential inclusion (2.18) is equal to the subdifferential of a certain concave function.

Theorem 2.3. *Let $V(\cdot)$ be a scalar valued concave function. The solution to the projected differential inclusion(**PSGI**)*

$$\dot{\theta} \in \partial V(\theta) + z, \quad \theta(0) = \theta^0, \; z(t) \in -C(\theta(t)) \tag{2.29}$$

is unique. (The definition of $C(\cdot)$ was given in Theorem 2.1.)

Proof. For ease of explanation, we only prove the one dimensional case, in which $\theta \in \mathbb{R}^1$. The proof is valid for two or higher dimensional space after minor modifications. Let $x(t)$ and $y(t)$ be two solutions of (2.29) with initial conditions x^0 and y^0, respectively. By definition, we have

$$x'(t) = g_x(t) + z_x(t) \tag{2.30}$$

$$y'(t) = g_y(t) + z_y(t), \tag{2.31}$$

where $g_x(t) \in \partial V(x(t))$ and $g_y(t) \in \partial V(y(t))$. The derivative of $1/2\|x(t) - y(t)\|^2$ satisfies:

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|x(t) - y(t)\|^2 &= (x(t) - y(t)) \cdot (x'(t) - y'(t)) && (2.32)\\ &= (x(t) - y(t)) \cdot (g_x(t) - g_y(t) + z_x(t) - z_y(t)) && (2.33)\\ &\leq 0. && (2.34) \end{aligned}$$

The last inequality can be proved using the following two facts. To make it clear, without loss of generality, we assume that $x(t) \leq y(t)$.

1. By the definition of subgradient, we have: (i) $V(y(t)) - V(x(t) \leq g_x(t)(y(t) - x(t))$, and (ii) $V(x(t)) - V(y(t)) \leq g_y(t)(x(t) - y(t))$. Summing up (i) and (ii), we end up with the inequality $(x(t) - y(t)) \cdot (g_x(t) - g_y(t)) \leq 0$

2. If $x(t) = y(t)$, we have $(x(t) - y(t)) \cdot (z_x(t) - z_y(t)) = 0$. Otherwise, we have $(x(t) - y(t)) < 0$ and $(z_x(t) - z_y(t)) \geq 0$. Both cases give us the inequality $(x(t) - y(t)) \cdot (z_x(t) - z_y(t)) \leq 0$.

Since $\|x(t) - y(t)\|$ has non-positive derivative, $\|x(t) - y(t)\|$ must be a non-increasing function in t. Comparing with the initial conditions, we have:

$$\|x(t) - y(t)\| \leq \|x^0 - y^0\| \text{ for } t \geq 0. \tag{2.35}$$

Moreover, if $x^0 = y^0 = \theta^0$, the monotonicity of $\|x(t) - y(t)\|$ implies that $x(t) = y(t) \ \ \forall t \geq 0$. This proves the uniqueness of the projected differential inclusion. □

In Theorem 2.3, the uniqueness of the solution to the projected differential inclusion (2.18) has been established when $\mathcal{G}(\cdot)$ is the subdifferential of a concave function $V(\cdot)$. As we move on, the next step is to demonstrate that the trajectory of the projected differential inclusion (**PSGI**) defined in Theorem 2.3 converges to a maximizer (if there are many) of $V(\cdot)$ over $H = \{\theta \mid 0 \leq \theta_i \leq K_u\}$. To achieve this goal, we need the following lemma.

Lemma 2.2. *If $\theta(t)$ is a continuous solution to (**PSGI**) defined in Theorem 2.3, its derivative $\dot{\theta}(t)$ is right continuous for all but a countable set of t.*

Proof. Let's focus on the i^{th} component of $\theta(t) = (\theta_1(t), \theta_2(t), \ldots, \theta_r(t))$. Since $\theta_i(t)$ is continuous, $I_i = \{t \mid \theta_i(t) \in (0, K_u)\}$ is an open set. Moreover, I_i can be expressed as a countable union of open intervals [6, Theorem 10.1.9]. Specifically, there exist disjoint open intervals $I_{ij} = (a_{ij}, b_{ij})$ such that $I_i = \bigcup_j I_{ij}$. For convenience, we further denote the boundary points a_{ij} and b_{ij} ($i = 1, \ldots, r,\ j =$

$1, 2, \ldots)$ by an ordered sequence $\{0 < c_1 < c_2 < c_3 < \cdots\}$, which partitions $\mathbb{R}^+$. In each interval $[c_k \; c_{k+1}]$, one of the following three conditions must hold.

1. $\theta_i(t) = 0$ for all t in $(c_k \; c_{k+1})$, or
2. $\theta_i(t) \in (0, K_u)$ for all t in $(c_k \; c_{k+1})$, or
3. $\theta_i(t) = K_u$ for all t in $(c_k \; c_{k+1})$.

Our strategy is to show that if the solution of

$$\dot{\theta} \in \partial V(\theta) + z, \quad \theta(0) = \theta^0 \tag{2.36}$$

in time interval $[c_k \; c_{k+1}]$ has invariant components, by taking away those invariant parts from the solution, we arrive at a lower dimensional function, which is the solution to a differential inclusion without projection. Equation (2.36) can be rewritten in the following form:

$$\begin{pmatrix} \dot{\theta}_1(t) \\ \dot{\theta}_2(t) \\ \vdots \\ \dot{\theta}_r(t) \end{pmatrix} = \begin{pmatrix} g_1\left(\theta_1(t), \ldots, \theta_r(t)\right) \\ g_2(\theta_1(t), \ldots, \theta_r(t)) \\ \vdots \\ g_r(\theta_1(t), \ldots, \theta_r(t)) \end{pmatrix} + \begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_r(t) \end{pmatrix}, \tag{2.37}$$

where

$$\begin{pmatrix} g_1\left(\theta_1(t), \ldots, \theta_r(t)\right) \\ g_2(\theta_1(t), \ldots, \theta_r(t)) \\ \vdots \\ g_r(\theta_1(t), \ldots, \theta_r(t)) \end{pmatrix} \in \partial V(\theta). \tag{2.38}$$

For ease of explanation, let's assume that $\theta_2(t)$ is the only component which is invariant in the interval $\in [c_k \ c_{k+1}]$. Without loss of generality, we assume that $\theta_2(t) = 0$. Note that our proof can be generalized to the case when $\theta(t)$ has multiple invariant components in $[c_k \ c_{k+1}]$. In the current setting, for $t \in (c_k \ c_{k+1})$, we have $\dot{\theta}_2(t) \equiv 0$, and $z_i(t) = 0$ for $i \neq 2$. This results from the facts that $\theta_2(t) \equiv 0$ and $\theta_i(t) > 0$ for $i \neq 2$ over the interval $(c_k \ c_{k+1})$. One can check that

$$\begin{pmatrix} g_1(\theta_1(t), 0, \theta_3(t) \ldots, \theta_r(t)) \\ g_3(\theta_1(t), 0, \theta_3(t) \ldots, \theta_r(t)) \\ \vdots \\ g_r(\theta_1(t), 0, \theta_3(t) \ldots, \theta_r(t)) \end{pmatrix} \in \partial\hat{V}(\theta_1(t), \theta_3(t), \ldots, \theta_r(t)) \text{ over } (c_k \ c_{k+1}), \tag{2.39}$$

where $\hat{V}(\theta_1(t), \theta_3(t), \ldots, \theta_r(t)) = V(\theta_1(t), 0, \theta_3(t), \ldots, \theta_r(t))$ is a concave function of $(\theta_1(t), \theta_3(t), \ldots, \theta_r(t))$. Moreover, by Theorem 2.3, $(\theta_1(t), \theta_3(t), \ldots, \theta_r(t))$ is the unique solution to the new differential inclusion without projection:

$$\begin{pmatrix} \dot{\theta}_1(t) \\ \dot{\theta}_3(t) \\ \vdots \\ \dot{\theta}_r(t) \end{pmatrix} \in \partial\hat{V}(\theta_1(t), \theta_3(t), \ldots, \theta_r(t)) \tag{2.40}$$

given the same initial condition of (2.36) at $t = c_k$. The remainder proof follows the fact that the derivative of the solution to *non-projected* gradient inclusion is right continuous [4, p. 147, Theorem 1]. Using the equivalence between (2.37) and (2.40), we conclude that $\dot{\theta}(t)$ is right continuous at all but a countable number of points in $\{c_1, c_2, \cdots\}$. $\square$

Lemma 2.3. *Let $\theta(t)$ be the unique solution of the projected differential inclusion (**PSGI**) (2.29) defined in theorem 2.3. The trajectory of $\theta(t)$ converges to a point. That is*

$$\lim_{t\to\infty} \theta(t) = \theta^*, \tag{2.41}$$

where θ^* is a maximizer of $V(\cdot)$ over H.

Proof. Let $\theta(t)$ be the solution to the projected differential inclusion

$$\dot{\theta} \in \partial V(\theta) + z, \text{ for almost all } t, \omega. \tag{2.42}$$

with $\theta(0) = \theta^0$. Since $\dot{\theta}(t) - z(t)$ is a subgradient of $V(\theta(t))$, by definition, we have:

$$\begin{aligned} V(\theta(t+s)) - V(\theta(t)) &\leq \langle \dot{\theta}(t) - z(t), (\theta(t+s) - \theta(t)) \rangle & (2.43) \\ &\overset{(a)}{\leq} \langle \dot{\theta}(t), (\theta(t+s) - \theta(t)) \rangle & (2.44) \end{aligned}$$

The last inequality (a) holds because $z(t) \cdot (\theta(t+s) - \theta(t)) \leq 0$. This results from the following facts, for simplicity, we focus on the one dimensional case. The same argument can be generalized to higher dimensional space.

1. If $0 < \theta(t) < K_u, z(t) = 0$, hence $z(t) \cdot (\theta(t+s) - \theta(t)) = 0$.

2. If $\theta(t) = K_u$, $z(t) \leq 0$, $(\theta(t+s) - \theta(t)) \leq 0$, hence $z(t) \cdot (\theta(t+s) - \theta(t)) \geq 0$.

3. If $\theta(t) = 0$, $z(t) \geq 0$, $(\theta(t+s) - \theta(t)) \geq 0$, hence $z(t) \cdot (\theta(t+s) - \theta(t)) \geq 0$.

Similarly, we have

$$V(\theta(t)) - V(\theta(t+s)) \leq \langle \dot{\theta}(t+s)), (\theta(t) - \theta(t+s)) \rangle \tag{2.45}$$

Divide both sides of (2.44) and (2.45) by s, and let $s \downarrow 0$. Exploiting the fact that $\dot{\theta}(\cdot)$ is right continuous (Lemma 2.2) at all but a countable number of points, we end up with this equality

$$\frac{d}{dt}V(\theta(t)) = \|\dot{\theta}(t)\|^2, \tag{2.46}$$

which holds at all but a countable set of t. Integrating this equality from 0 to t, we end up with

$$V(\theta(t)) - V(\theta^0) = \int_0^t \|\dot{\theta}(s)\|^2 ds. \tag{2.47}$$

Following the same arguments given in [4, p. 160, Theorem 2], we can prove that

$$\lim_{t} \theta(t) = \theta^*, \tag{2.48}$$

where θ^* is a maximizer of $V(\cdot)$ over H. □

The following theorem states that the trajectory of interpolated process $\theta_\epsilon(t)$ follows its limiting process in expectation uniformly over finite time interval.

Theorem 2.4. *Under the same setting of Theorem 2.1, in which $\mathcal{G}(\theta)$ is equal to the subdifferential of a concave function $V(\theta)$ at θ, the interpolated process of the stochastic approximation $\theta_\epsilon(t)$ and its limiting process $\theta(t)$ have the following convergence property. For any $T > 0$, we have*

$$\lim_{\epsilon \to 0} \mathbb{E}\{\|\theta_\epsilon(t) - \theta(t)\|\} = 0, \quad \text{uniformly on } [0\ T] \text{ almost surely,} \tag{2.49}$$

*where $\theta(t)$ is the unique solution of (**PSGI**)* (2.29) *with initial condition $\theta(0) = \theta^0$.*

Proof. We start with the proof of convergence. Our strategy is to take the proof in contrapositive form. Assuming that

$$\limsup_{\epsilon} \mathbb{E}\{\|\theta_\epsilon(t) - \theta(t)\|\} = \alpha, \tag{2.50}$$

for some $t \in [0,T]$ $\alpha > 0$, one can find a subsequence $\epsilon_k \to 0$ such that

$$\lim_{k} \mathbb{E}\{\|\theta_{\epsilon_k}(t) - \theta(t)\|\} = \alpha \tag{2.51}$$

and $\theta_{\epsilon_k}(t) \Rightarrow \theta(t)$ (Theorem 2.1). Note that, by uniqueness, $\theta(t)$ is a constant if t is fixed. Since weak convergence to a constant implies the convergence in probability [40, Proposition 8.5.2], it follows that

$$\| \theta_{\epsilon_k}(t) - \theta(t) \| \to 0 \;\text{ in probability as } k \to \infty. \tag{2.52}$$

For simplicity, we assume that $\|Y_\epsilon^n\| \leq B_1$ uniformly in n and ϵ for some constant B_1. However, the proof can be generalized to the case when Y_ϵ^n is uniformly integrable. From the definition of interpolated process $\theta_\epsilon(t)$, we have the following bound.

$$\|\theta_\epsilon(t+s) - \theta_\epsilon(t)\| \leq B_1 s + B_1 \epsilon, \tag{2.53}$$

where the left-hand-side results from $s/\epsilon + 1$ times of interpolations multiplied by the bound ϵB_1. Since $\theta_{\epsilon_k}(t)$ is uniformly bounded by $tB_1 + \epsilon_k B_1$, through [40, Theorem 6.6.1 (c)], it follows that $\mathbb{E}\{\| \theta_{\epsilon_k}(t) - \theta(t) \|\}$ converges to 0 in L^1 at time t, which contradicts to (2.51). Hence we complete the proof of convergence.

The uniformity of the convergence follows the extended Arzela-Ascoli theorem [27, Theorem 4.2.2]. □

Theorem 2.5. *Under the same setting of Theorem 2.1 with the additional assumption that the set-valued function $\mathcal{G}(\theta)$ is taken from the subdifferential of a concave function $V(\theta)$ at θ, the sequence $\{\theta_\epsilon^n\}$ generated by the stochastic approximation algorithm* (2.2) *has the following property:*

Suppose that θ^* is the unique maximizer of $V(\cdot)$ over H, given $\delta > 0$, there exists $\hat{\epsilon} > 0$ such that, for any $\epsilon < \hat{\epsilon}$,

$$\mathbb{E}\{ \| \theta_\epsilon^n - \theta^* \| \} < \delta \text{ for all but a finite number of times in } n. \tag{2.54}$$

Proof. Let $\theta(t)$ be the weak limiting process of $\theta_\epsilon(t)$ with $\theta(0) = \theta_\epsilon^0$. Theorem 2.3 ensures that $\theta(t)$ is unique. Moreover, from lemma 2.3, we know that $\lim_{t\to\infty} \theta(t) = \theta^*$ with probability one. By the dominated convergence theorem, it follows that $\lim_{t\to\infty} \mathbb{E}\{ \|\theta(t) - \theta^*\| \} = 0$. As a result, there exists $T > 0$ such that

$$\mathbb{E}\{ \|\theta(t) - \theta^*\| \} \leq \delta/2 \;\; \text{for all } t \geq T. \tag{2.55}$$

We apply the method of proof in contrapositive form. If, for every $\sigma > 0$, there exists $\epsilon \in (0, \sigma)$ such that $\mathbb{E}\{\| \theta_\epsilon^n - \theta^* \|\} > 2\delta$ infinitely often in n, we will be able to find a sequence $\epsilon_k \to 0$ such that $\mathbb{E}\{\|\theta_{\epsilon_k}(T) - \theta^* \|\} \leq \delta$ (by Theorem 2.4 and (2.55)) and $\mathbb{E}\{ \|\theta_{\epsilon_k}^n - \theta^* \|\} > 2\delta$ infinitely often in n. Let η_{ϵ_k} be the first time after time T such that $\mathbb{E}\{ \|\theta_{\epsilon_k}(\eta_{\epsilon_k}) - \theta^* \|\} > 2\delta$ and τ_{ϵ_k} the last time before time η_{ϵ_k} that $\mathbb{E}\{ |\theta_{\epsilon_k}(\tau_{\epsilon_k}) - \theta^* |\} \leq \delta$. Defining $q_{\epsilon_k} = \tau_{\epsilon_k}/\epsilon_k$, since q_{ϵ_k} is a sequence of nonnegative integers, one can always find a subsequence which has q_{ϵ_k} nondecreasing[4]. Let $\chi_{\epsilon_k} = \eta_{\epsilon_k} - \tau_{\epsilon_k}$, by Theorem 2.1, there exists a further

[4]This can be done by picking the $\inf_{k\geq n} q_{\epsilon_k}$

subsequence which converges weakly

$$(\theta_{\epsilon_k}(\epsilon_k q_{\epsilon_k} + \cdot), Z_{\epsilon_k}(\epsilon_k q_{\epsilon_k} + \cdot)) \Rightarrow (\tilde{\theta}(\cdot), \tilde{Z}(\cdot)). \tag{2.56}$$

If there exists a further subsequence $\{\epsilon_{k_i};\ i \geq 0\}$ such that $\chi_{\epsilon_{k_i}} \to \tilde{T}$ for some $\tilde{T} > 0$, we get

$$\mathbb{E}\{\, \|\tilde{\theta}(0) - \theta^*\,\|\} \leq \delta \text{ and } \ \mathbb{E}\{\, \|\tilde{\theta}(\tilde{T}) - \theta^*\,\|\} > 2\delta. \tag{2.57}$$

Otherwise, we have $\chi_{\epsilon_{k_i}} \to \infty$ and $\delta < \mathbb{E}\{\, \|\tilde{\theta}(t) - \theta^*\,\|\} < 2\delta$ for $t > 0$. Both cases contradict to the fact that *the difference* $\|\tilde{\theta}(t) - \theta^*\,\|$ *is monotonically decreasing in time* t. This monotonicity can be reasoned as follows. In the proof of Theorem 2.3, we have shown that the difference between two solutions are monotone decreasing. Since θ^* is the unique maximizer, if we take θ^* as the initial value of the differential inclusion (2.29), the solution is $x(t) = \theta^*$ for $t \geq 0$. Therefore, $\|\tilde{\theta}(t) - \theta^*\,\| = \|\tilde{\theta}(t) - x(t)\|$ decreases in time t. □

Briefly speaking, Theorem 2.5 states that, if $\mathcal{G}(\cdot)$ is the subdifferential of concave function $V(\cdot)$, which has unique maximizer over H, the sequence generated by the stochastic approximation algorithm converges in expectation to a neighborhood of this unique maximizer.

Although, in general, $V(\cdot)$ may have multiple maximizers , if the set consisting of the maximizers of $V(\cdot)$ is compact, the sequence generated by the stochastic approximation still converges to the set of all maximizers in expectation . This result is summarized in the following theorem. Since its proof is very similar to that for Theorem 2.5, the details are omitted.

Theorem 2.6. *Let Θ be the set of maximizers of $V(\cdot)$ over H. Under the same setting in Theorem 2.5, except the uniqueness of the maximizer, we have the following result. Given $\delta > 0$, there exists $\hat{\epsilon} > 0$ such that, for any $\epsilon < \hat{\epsilon}$,*

$$\mathbb{E}\{\,\mathrm{dist}(\,\theta_\epsilon^n, \Theta\,)\,\} < \delta \text{ for all but a finite number of times in } n, \tag{2.58}$$

where $\mathrm{dist}(\theta, A)$ *denotes the distance from point θ to set A.*

Remark to Theorem 2.6

Note that $\mathrm{dist}(\theta, A)$ is a convex function of θ if A is a convex set [12, p.p. 88]. The concavity of $V(\cdot)$ implies the convexity of Θ. By Jensen's inequality, we therefore have $\mathrm{dist}(\mathbb{E}\{\,\theta_\epsilon^n\}, \Theta) < \delta$.

Lastly, we have two important results for stochastic approximation, referring to the reflection terms.

Lemma 2.4. *Under the same setting in Theorem 2.5, let $Z(t)$ be the limiting process of the interpolated process defined for the reflection term, and let Θ be the set of maximizers of $V(\cdot)$ over H, we have the following boundedness property. If $\Theta \subset H^o$ (interior of H) and $\|Y_\epsilon^n\| \leq B_1$ for some constant B_1, there exists B_2 such that $\|Z(t)\| \leq B_2$ for all $t \geq 0$.*

Proof. For the sake of simplicity and clarity, we only prove the case when the maximizer of $V(\cdot)$ is unique, which means that $\Theta = \{\theta^*\}$. The proof for set Θ consisting of more than one point can be done under the same rationale, and is omitted.

Consider one realization of the limiting process $(\theta(t), Z(t))$. Since H is compact, it implies there exist p points, $\{a_1, a_2, \cdots, a_p\}$, in H such that $\bigcup_{i=1}^{p} N_{\delta/2}(a_i) = H$, where $N_{\delta/2}(a_i)$ denotes the neighborhood of a_i of radius $\delta/2$. Moreover, under the initial condition $\theta^0 = a_i$, the limiting process converges asymptotically. In other words, we have $\lim_{t\to\infty} \theta(t) = \theta^*$. This ensures the existence of a positive time $T' > 0$ such that $\|\theta(t) - \theta^*\| \leq \delta/2$ for all $t > T'$ under the initial condition $\theta^0 \in \{a_i;\ i = 0, 1, 2, \ldots, p\}$. Recall that, in the proof of Theorem 2.3, we demonstrated that the difference between two solutions decreases with time. Since every point in H is covered by a neighborhood $N_{\delta/2}(a_i)$ (due to the compactness of H), via the triangular inequality and the decreasing distance between solutions along the time index, we can obtain the equality $\|\theta(t)-\theta^*\| \leq \delta$ for all $t > T'$ which holds for any initial point in H. This means that $\theta(t)$ locates in the interior of H after time T', and no reflection occurs after time T'. In other words, $Z(t) = Z(T')$ for $t \geq T'$.

To complete this proof, it remains to show that $Z(T')$ is bounded. By the definition of reflection terms, we have the inequality

$$\|Z_\epsilon(t+s) - Z_\epsilon(t)\| \leq \sum_{i=t/\epsilon}^{(t+s)/\epsilon} \epsilon \|Y_\epsilon^i\| \leq (s+\epsilon)B_1. \tag{2.59}$$

Let $\epsilon \to 0$. For each sample path of the limiting process $Z(t)$, we have

$$\|Z(t+s) - Z(t)\| \leq sB_1 \text{ with probability one.} \tag{2.60}$$

The boundedness of $Z(T')$ follows this inequality, and the proof is done by setting $B_2 = T'B_1$ □

Theorem 2.7. *Assume the same setting of Theorem 2.1 with two additional terms*

(i) The set valued function $\mathcal{G}(\theta)$ is the subdifferential of a concave function $V(\theta)$ at θ,

(ii) The maximizers of $V(\theta)$ constitutes a compact set Θ, and $\Theta \subset H^o$.

Given $\delta > 0$ there exists $\hat{\epsilon} > 0$ such that $\limsup_t \frac{\mathbb{E}\{Z_\epsilon(t)\}}{t} \leq \delta$ for all $\epsilon \leq \hat{\epsilon}$.

Proof. We use proof in contrapositive form. Suppose that the conclusion of the theorem is false, then for any given $T > 0$, there must exist a decreasing sequence $\epsilon_k \to 0$ and a nondecreasing sequence q_k such that

$$\frac{\mathbb{E}\{Z_{\epsilon_k}(\epsilon_k q_k + T)\}}{T} > \delta \tag{2.61}$$

By Theorem 2.1, there is a subsequence of step size $\{\epsilon_{k_i}\}$ such that

$$Z_{\epsilon_{k_i}}(\epsilon_{k_i} q_{k_i} + T) \Rightarrow Z(T) \tag{2.62}$$

along the subsequence. Via the definition of weak convergence, the following relation holds.

$$\mathbb{E}\{Z_{\epsilon_{k_i}}(\epsilon_{k_i} q_{k_i} + T)\} \to \mathbb{E}\{Z(T)\} \tag{2.63}$$

This implies that $\mathbb{E}\{Z(T)\} > T\delta$, which contradicts to Lemma 2.4 when we set $T = \frac{B_2+1}{\delta}$ using the same constant B_2 defined in Lemma 2.4. Therefore, the conclusion made by the theorem is true. □

Remarks on Theorem 2.7

1. The definition of interpolated process $Z_\epsilon(t)$, we have

$$\left\| \frac{Z_\epsilon(t)}{t} - \frac{\sum_{i=0}^{n-1} Z_\epsilon^i}{n} \right\| \leq B_1/n \quad \text{for } t/\epsilon - 1 < n \leq t/\epsilon, \tag{2.64}$$

where B_1 is the same constant defined in Lemma 2.4. As a result, given $\delta > 0$, there exists $\hat{\epsilon} > 0$ such that $\limsup_n \frac{\sum_{i=0}^{n-1} \mathbb{E}\{Z_\epsilon^i\}}{n} \leq \delta$ for all $\epsilon \leq \hat{\epsilon}$.

2. Recall that, for our applications, the constraint set is defined as $H = \{\theta \parallel 0 \leq \theta_i \leq K_u\}$. Before, we assumed the interpolated process $Z_\epsilon(t)$ counts all the reflections either at the boundary below at 0 or at the boundary above at K_u. Alternatively, we can separate these reflection terms according to whether they touch the lower or the upper boundaries of H. We define the reflection term from above $\check{Z}_\epsilon^{\,n} \geq 0$ and the reflection term from below $\hat{Z}_\epsilon^{\,n} \geq 0$ by rewriting the recursive algorithm (2.2) as

$$\theta_\epsilon^{n+1} = \theta_\epsilon^n + \epsilon(Y_\epsilon^n + \hat{Z}_\epsilon^{\,n} - \check{Z}_\epsilon^{\,n}). \tag{2.65}$$

Also, we define the interpolated processes $\hat{Z}_\epsilon(t)$ and $\check{Z}_\epsilon(t)$ analogously to $Z_\epsilon(t)$ using respectively $\hat{Z}_\epsilon^{\,n}$ and $\check{Z}_\epsilon^{\,n}$ in lieu of Z_ϵ^n.

On these definitions, all the results proved so far for Z_ϵ, $Z(t)$, and $Z_\epsilon(t)$ are still applicable for the terms defined for the upper or the lower reflections.

2.2 Stochastic Optimization

In this section, we focus on establishing the framework for stochastic optimization. Mathematically, stochastic optimization can be viewed as a discrete time stochastic dynamic control problem, in which the actions and the resulting performance is controlled by a policy. To explain in detail, we start with an overview of control policies.

2.2.1 Control Policy

A control policy consists of a sequence of control actions performed to the system based on the observations of system states and other available information. In each discrete time instant, the control action is restricted to a set described below.

Feasible Set

We enumerate the sequence using the time index n $(= 0, 1, 2, \ldots)$. Two sequences of control variables $X^n \in \mathbb{R}$ and $\vec{Y}^n \in \mathbb{R}^r$ are used in the problems. At each time n, the feasible choices of X^n and $\vec{Y}^n$ are coupled together by a set determined by the exogenous process $\{\xi^n\}$. As we mentioned earlier, the evolution of sequence $\{\xi^n\}$ is unaffected by the control actions. Assume that the statistics of $\{\xi^n\}$ is unknown, but the outcome $\xi^n(\omega)$ at each time instant n is perfectly observed by the controller. The scheduler can therefore choose X^n and $\vec{Y}^n$ utilizing the information of $\xi^n(\omega)$. The selected X^n and $\vec{Y}^n$ are called feasible if the vector $(X^n, \vec{Y}^n) \in \mathbb{R}^{1+r}$ locates in the set $\mathcal{D}(\xi^n)$. We note that $\mathcal{D}(\cdot)$ is a set-valued mapping from the range space of ξ^n to the subsets of $\mathbb{R}^{r+1}$. We also assume that the elements in $\mathcal{D}(\xi^n(\omega))$ are bounded in norm uniformly in n and ω. Specifically, there is a constant B_1 such that if $(x, \vec{y}) \in \mathcal{D}(\xi^n(\omega))$, we have $\|x\| \leq B_1$ and

$\|\vec{y}\| \leq B_1$, which is true for any n and ω,.

Feasible Control Policy

A control policy π decides the values of X^n and $\vec{Y}^n$ at time n $(= 0, 1, 2, \ldots)$. If the control policy chooses X^n and $\vec{Y}^n$ from the set $\mathcal{D}(\xi^n)$, it is called a *feasible policy*. In this work, we consider the randomized control policies which make decisions based on the perfect observations (possibly non-causal) of the sample path $\{\xi^n(\omega),\ n = 0, 1, 2, \ldots\}$. For a non-randomized control policy, given a realization $\{\xi^n(\omega)\}$, the output of the decision is a deterministic sequence. However, for a randomized control policy, the output corresponding to the same realization is a random sequence of vectors $(X^n, \vec{Y}^n)$. More precisely, a randomized control policy produces the distribution of random process $\{(X^n, \vec{Y}^n),\ n = 0, 1, \ldots\}$. Clearly, the scope of randomized control polices is a broader than that of the non-randomized control polices.

Different randomized control polices may result in different distributions for the process $\{\xi^0, X^0, \vec{Y}^0, \xi^1, X^1, \vec{Y}^1, \xi^2, X^2, \vec{Y}^2, \ldots\}$. To identify the control sequences generated by different policies, we denote X_π^n and $\vec{Y}_\pi^n$ as the value of control variables chosen by the randomized policy π at time n.

System Design Criterion

In this work, we seek the feasible randomized control policy which aims to minimize the long-term average of $\mathbb{E}\{X_\pi^n\}$ while keeping the long-term average of $\mathbb{E}\{\vec{Y}_\pi^n\}$ nonnegative. For this reason, we call X_π^n the cost control variable and

$\vec{Y}_\pi^n$ the demand control variable. Note that the expectation $\mathbb{E}\{\cdot\}$ is taken over the probability space established from $\{\xi^n\}$ and the chosen randomized policy π.

To get a clear idea about this framework, lets consider a wireless downlink system. Suppose the system design criterion is to minimize the long-term average of total transmit power and keep a long-term average rate vector $\vec{C} = (C_1, C_2, \ldots, C_L)$ for user 1 to L, one can apply the framework by setting $X^n = \sum_{l=1}^{L} P_l^n$ as the instantaneous transmit power, and $\vec{Y}^n = \vec{C} - \vec{R}^n$ as the difference between the instantaneous transmit rate and the target rate, where $\vec{R}^n = (R_1^n, R_1^n, \ldots, R_L^n)$ denotes the instantaneous rate vector to users $1, 2, \ldots, L$. The instantaneous channel gains for user 1 to L constitutes the exogenous process $\xi^n = (h_1^n, h_2^n, \ldots, h_L^n)$. If the peak transmit power is limited, for every time slot n, the feasible choices of X^n and $\vec{Y}^n$ must be restricted to a set $\mathcal{D}(\xi^n)$ determined by the instantaneous channel gains ξ^n.

2.2.2 Stochastic Optimization Problem

The generic form of stochastic optimization is displayed in the following set of equations.

Problem 2.1 (**Stochastic Optimization**).

(SOP)

$$\text{minimize} \quad \limsup_{N} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{X_\pi^n\} \tag{2.66}$$

$$\text{subject to} \quad \limsup_{N} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{\vec{Y_\pi^n}\} \leq 0 \tag{2.67}$$

$$\text{policy } \pi \text{ such that } (X_\pi^n, \vec{Y_\pi^n}) \in \mathcal{D}(\xi^n) \tag{2.68}$$

Before delving into the solution, we remarks two important information regarding the problem formulation.

1. We use "limsup" instead of "lim" because the limsup of a sequence always exists, but the limit does not. In this dissertation, "the limsup of a sequence of vector is defined componentwise". Given a sequence of vector $\{\vec{v}^n, n = 1, 2, 3, \ldots\}$, where $\vec{v}^n = (v_1^n, v_2^n, \ldots, v_r^n)$, then

$$\limsup_n \vec{v}^n \triangleq (\limsup_n v_1^n, \limsup_n v_2^n, \ldots, \limsup_n v_r^n).$$

2. The elements in $\mathcal{D}(\xi^n(\omega))$ are bounded in norm uniformly in n and ω. In other words, the norms of any feasible control variables X^n and $\vec{Y}^n$ are bounded by a certain fixed constant. This is a reasonable assumption for real applications. For example, let ξ^n denote the channel gains, and let X^n and $\vec{Y}^n$ denote respectively the transmit power and rates in a wireless downlink system. Although in theory unbounded channel gains are allowed (e.g. Rayleigh fading channel model), in practice, the feasible transmission rate over channels can never go beyond certain constant bound irrespective of the realization of the channel. This bound is determined by the applicable modulation schemes and the peak transmit power limited by the amplifier.

The first constraint (2.67) displayed in the stochastic optimization (**SOP**) can be regard as the linear constraint of the long-term average of sequence $\vec{Y_\pi^n}$. The second constraint (2.68) can be regard as the restrictions on the sequence $\{(X_\pi^n, \vec{Y_\pi^n})\}$ at every time n, which forms a constraint set for the long-term average of $\{(X_\pi^n, \vec{Y_\pi^n})\}$. For ease of explanation, we call (2.68) the instantaneous constraints. Constraints (2.67) and (2.68) limit the values taken by the sequence $\{(X_\pi^n, \vec{Y_\pi^n})\}$ in different time scale. To unify the expression of the constraints using the same time scale, we replace the instantaneous constraints with their relaxed version in the format of expected long-term average of $\{(X_\pi^n, \vec{Y_\pi^n})\}$. For convenience, we call this the *average relaxation* of the instantaneous constraint.

Mathematically, the average relaxation of the instantaneous constraint gives the following constraint set for the expected long-term average of the control sequence $\{(X_\pi^n, \vec{Y_\pi^n})\}$.

$$\bar{\mathcal{D}} \triangleq \left\{ (x, \vec{y}) \,\middle|\, \text{where } x = \limsup_N \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{X_\pi^n\}, \text{ and } \vec{y} = \limsup_N \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{\vec{Y_\pi^n}\} \right.$$
$$\left. \text{for some randomized policy } \pi \text{ with } (X_\pi^n, \vec{Y_\pi^n}) \in \mathcal{D}(\xi^n). \right\}$$

To solve (SOP), we first convert the problem to the domain consisting of expected long-term average of feasible control sequences. More precisely, we treat the time and ensemble average of control sequences as the new basic variables, and use the term "*Sequence Average Space*" to name the set of these new variables. By substituting X with $\limsup_N \frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}\{X_\pi^n\}$ and $\vec{Y}$ with $\limsup_N \frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}\{\vec{Y_\pi^n}\}$, we arrive at the following optimization problem in the sequence average domain.

Problem 2.2 (Optimization in the Sequence Average Space)**.**

(OSAS)

$$\text{minimize} \quad X \tag{2.69}$$

$$\text{subject to} \quad \vec{Y} \leq 0 \tag{2.70}$$

$$(X, \vec{Y}) \in \bar{\mathcal{D}} \tag{2.71}$$

Note that Problem 2.1 (SOP) and Problem 2.2 (OSAS) share the same optimal value. This is because Problem 2.2 is a relaxed version of Problem 2.1, and every solution of Problem 2.2 corresponds to a feasible randomized policy in Problem 2.1.

We can further extend the set $\bar{\mathcal{D}}$ to its convex hull $co(\bar{\mathcal{D}})$ without altering the solution to Problem 2.2. This is justified via the following lemma.

Lemma 2.5. *Given two point $(X_1, \vec{Y}_1)$ and $(X_2, \vec{Y}_2)$ in $\bar{\mathcal{D}}$ with $Y_1 \leq 0$ and $Y_2 \leq 0$. Considering the convex combination $(\ \lambda X_1 + (1-\lambda)X_2,\ \lambda\vec{Y}_1 + (1-\lambda)\vec{Y}_1\ )$ with $0 \leq \lambda \leq 1$, one can always find a point (X_3, Y_3)in $\bar{\mathcal{D}}$ which satisfies $\vec{Y}_3 \leq \lambda Y_1 + (1-\lambda)Y_2 \leq 0$ and also gives a smaller value of X_3 than $X_1 + (1-\lambda)X_2$.*

Proof. By the definition of $\bar{\mathcal{D}}$, there exist feasible policies π_1 and π_2 such that:

$$(X_1, \vec{Y}_1) = (\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X^n_{\pi_1}\}, \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{Y}^n_{\pi_1}\}) \tag{2.72}$$

$$(X_2, \vec{Y}_2) = (\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X^n_{\pi_2}\}, \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{Y}^n_{\pi_2}\}) \tag{2.73}$$

For $0 \leq \lambda \leq 1$, we have:

$$\begin{aligned}
&\lambda X_1 + (1-\lambda)X_2 && (2.74)\\
&= \lambda \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X_{\pi_1}^n\} + (1-\lambda)\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X_{\pi_2}^n\} && (2.75)\\
&= \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\lambda X_{\pi_1}^n\} + \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{(1-\lambda)X_{\pi_2}^n\} && (2.76)\\
&\overset{(a)}{\geq} \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\lambda X_{\pi_1}^n + (1-\lambda)X_{\pi_2}^n\} && (2.77)\\
&= \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X_{\pi_3}^n\}, && (2.78)
\end{aligned}$$

where π_3 is the randomized policy which chooses π_1 with probability λ, and π_2 with probability $1-\lambda$. The inequality (a) above holds because $\limsup a_n + \limsup b_n \geq \limsup a_n + b_n$ [42, p. 78]. Similarly,

$$\begin{aligned}
0 &\geq \lambda \vec{Y}_1 + (1-\lambda)\vec{Y}_2 && (2.79)\\
&= \lambda \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{Y}_{\pi_1}^n\} + (1-\lambda)\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{Y}_{\pi_2}^n\} && (2.80)\\
&= \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\lambda \vec{Y}_{\pi_1}^n\} + \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{(1-\lambda)\vec{Y}_{\pi_2}^n\} && (2.81)\\
&\geq \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\lambda \vec{Y}_{\pi_1}^n + (1-\lambda)\vec{Y}_{\pi_2}^n\} && (2.82)\\
&= \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{Y}_{\pi_3}^n\}. && (2.83)
\end{aligned}$$

Defining $X_3 \triangleq \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X_{\pi_3}^n\}$ and $\vec{Y}_3 \triangleq \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{Y}_{\pi_3}^n\}$, we therefore have $(X_3, \vec{Y}_3) \in \bar{\mathcal{D}}$ with $\vec{Y}_3 \leq \lambda Y_1 + (1-\lambda)Y_2 \leq 0$ and $X_3 \leq \lambda X_1 + (1-\lambda)X_2$. □

The relaxation from $\bar{\mathcal{D}}$ to $co(\bar{\mathcal{D}})$ gives Problem 2.3.

Problem 2.3.

$$\begin{aligned} \text{minimize} \quad & X && (2.84) \\ \text{subject to} \quad & \vec{Y} \leq 0 && (2.85) \\ & (X, \vec{Y}) \in co(\bar{\mathcal{D}}) && (2.86) \end{aligned}$$

Note that Problem 2.3 is a convex optimization problem. Its Lagrangian is defined as:

$$L(X, \vec{Y}, \vec{\theta}) = X + \vec{\theta} \cdot \vec{Y}, \tag{2.87}$$

where $\vec{\theta}$ is the vector of dual variables for the constraint $\vec{Y} \leq 0$. The dual function $V(\vec{\theta})$ is defined as the following optimization parameterized by $\vec{\theta}$

$$V(\vec{\theta}) = \min_{(X,\vec{Y}) \in co(\bar{\mathcal{D}})} L(X, \vec{Y}, \vec{\theta}), \tag{2.88}$$

which is a concave function of $\vec{\theta}$ [9, Proposition B.25(a)].

Problem 2.4. *The maximization of the dual function over* $\vec{\theta} \geq 0$ *is called the dual problem.*

$$\begin{aligned} \text{maximize} \quad & V(\vec{\theta}) \\ \text{subject to} \quad & \vec{\theta} \geq 0 \end{aligned}$$

Lemma 2.6. *For convex optimization problems, if the* Slater constraint qualification *[9, Chapter 5, Assumption 5.3.1] is satisfied, the strong duality theorem [9, Chapter 5, Prop. 5.3.1] assures that the optimal values of the primal problem and the dual problem are equal. That is* $V^* = X^*$.

This lemma shows that the minimum of the convex primal problem meets the maximum of the dual problem[5]. Moreover, if $(X^*, \vec{Y}^*, \vec{\theta}^*)$ satisfies the following properties, $(X^*, \vec{Y}^*)$ is a solution to the Problem 2.2.

Lemma 2.7 (Proposition 5.1.5, [9])**.** $(X^*, \vec{Y}^*, \vec{\theta}^*)$ *is an optimal solution-Lagrange multiplier pair if and only if*

$$\begin{array}{ll} \theta^* \in X, (X, \vec{Y}) \in co(\bar{\mathcal{D}}) & \text{(Primal Feasibility)} \\ \theta^* \geq 0 & \text{(Dual Feasibility),} \\ (X^*, \vec{Y}^*) = \arg\min_{(X,\vec{Y})\in co(\bar{\mathcal{D}})} L(X, \vec{Y}, \vec{\theta}^*) & \text{(Lagrangian Optimality)} \\ \theta_i Y_i = 0,\ \ i = 1, \ldots, r & \text{(Complementary Slackness)} \end{array} \tag{2.89}$$

Recall that any solution of the problem expressed in the sequence average domain can be achieved by certain feasible randomized policy π.

Since the dual problem has simpler constraints then the primal problem does, we will tackle the problem from the dual. The value of the dual function $V(\cdot)$ can be evaluated using the optimization in the original space. Specifically, it works this way:

Lemma 2.8.

$$V(\vec{\theta}) \triangleq \min_{(X,\vec{Y})\in co(\bar{\mathcal{D}})} L(X, \vec{Y}, \vec{\theta}) \tag{2.90}$$

$$= \min_{\pi\in\Gamma} \left\{ \limsup_N \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{X_\pi^n\} + \vec{\theta} \cdot \limsup_N \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{\vec{Y_\pi^n}\} \right\} \tag{2.91}$$

where Γ *is the set of feasible randomized policy for the optimization problem.*

[5]Actually, the minimum of the primal problem is always greater than or equal to the maximum of the dual problem. This is called the Weak Duality Theorem. This result is good enough for us to prove the optimality and feasibility .

Proof. Since $\bar{\mathcal{D}} \subset co(\bar{\mathcal{D}})$, we have

$$\min_{(X,\vec{Y})\in co(\bar{\mathcal{D}})} L(X,\vec{Y},\vec{\theta}) \leq \min_{(X,\vec{Y})\in \bar{\mathcal{D}}} L(X,\vec{Y},\vec{\theta}) \tag{2.92}$$

However, for $(X^*, \vec{Y}^*)$ that achieves the minimization of the left-hand side, it can be expressed as a convex combination of finite points from $\bar{\mathcal{D}}$. Lemma 2.5 ensures the existence of vector $(X^{\#}, \vec{Y}^{\#}) \in \bar{\mathcal{D}}$ such that $X^{\#} \leq X^*$ and $\vec{Y}^{\#} \leq \vec{Y}^* \leq 0$. Since $\vec{\theta}$ is componentwise nonnegative, this implies that

$$\min_{(X,\vec{Y})\in co(\bar{\mathcal{D}})} L(X,\vec{Y},\vec{\theta}) \geq \min_{(X,\vec{Y})\in \bar{\mathcal{D}}} L(X,\vec{Y},\vec{\theta}) \tag{2.93}$$

Combining (2.92) and (2.93), we have

$$\min_{(X,\vec{Y})\in co(\bar{\mathcal{D}})} L(X,\vec{Y},\vec{\theta}) = \min_{(X,\vec{Y})\in \bar{\mathcal{D}}} L(X,\vec{Y},\vec{\theta}) \tag{2.94}$$

To finish the proof, lastly, we need to show the equivalence between the right-hand side of (2.91) and the right hand side of (2.94).

By the definition of $\bar{\mathcal{D}}$, for each feasible randomized policy π, the corresponding expected long-term average of control sequences satisfy

$$\left(\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X_\pi^n\}, \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{Y}_\pi^n\}\right) \in \bar{\mathcal{D}}. \tag{2.95}$$

This inclusion gives the inequality:

$$\min_{(X,\vec{Y})\in \bar{\mathcal{D}}} L(X,\vec{Y},\vec{\theta}) \leq \tag{2.96}$$

$$\min_{\pi\in\Gamma}\left\{\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X_\pi^n\} + \vec{\theta}\cdot\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{Y}_\pi^n\}\right\}. \tag{2.97}$$

Conversely, for each $(X,\vec{Y}) \in \bar{\mathcal{D}}$, there exists a randomized feasible policy π such that

$$\left(\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X_\pi^n\}, \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{Y}_\pi^n\}\right) = (X,\vec{Y}). \tag{2.98}$$

This establishes the inequality in the opposite direction of (2.97) and completes the proof.

$$\min_{(X,\vec{Y})\in\bar{\mathcal{D}}} L(X,\vec{Y},\vec{\theta}) \geq \tag{2.99}$$

$$\min_{\pi\in\Gamma}\left\{\limsup_{N}\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X_{\pi}^{n}\}+\vec{\theta}\cdot\limsup_{N}\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{Y_{\pi}^{n}}\}\right\} \tag{2.100}$$

□

Note that the dual problem is a concave maximization problem, which can be solved using the *projected subgradient methods* [9, p. 610]. Based on the Danskin's Theorem [9, Proposition B.25(b)], the control variables chosen by the policy $\pi^{\#}$ that minimizes the left-hand side of (2.91) with respect to $\vec{\theta}$ constitutes a subgradient of $V(\vec{\theta})$ at $\vec{\theta}$, that is $\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{Y}_{\pi^{\#}}^{n}\}$.

Ideally, using the subgradient, we can apply projected subgradient methods numerically to obtain one optimal dual variable $\vec{\theta}^{*}$. The randomized feasible policy that satisfies the "Lagrangian optimality" and the "complementary slackness" of Lemma 2.7 with respect to $\vec{\theta}^{*}$ is the optimal policy.

Unfortunately, in general, the evaluation of (2.91) may require the knowledge of the probability distribution and the complete realization of $\{\xi^n,\ n = 0,1,2,\ldots\}$. Therefore, causally computing the subgradient may not be applicable. This suggests that the traditional subgradient methods may not be useful when instantaneous decision making is required. To address this issue, we first consider a function that bounds the dual function $V(\cdot)$ from below, and look into the policies based on this bounding function.

Subsidiary Bounding Function $\tilde{V}(\cdot)$

Following Lemma 2.8, we bound the dual function as follows.

$$V(\vec{\theta}) = \min_{\pi\in\Gamma}\left\{\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X_\pi^n\} + \vec{\theta}\cdot\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{Y_\pi^n}\}\right\} \quad (2.101)$$

$$\overset{(*)}{\geq} \min_{\pi\in\Gamma}\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X_\pi^n + \vec{\theta}\cdot\vec{Y_\pi^n}\} \quad (2.102)$$

The last inequality $(*)$ results from the fact "$\limsup_n a_n + \limsup_n b_n \geq \limsup_n (a_n + b_n)$", and the equality holds if the limits of the sequences exist. For notational simplicity, we define

$$\tilde{V}(\vec{\theta}) \triangleq \min_{\pi\in\Gamma}\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X_\pi^n + \vec{\theta}\cdot\vec{Y_\pi^n}\}, \quad (2.103)$$

$$V(\vec{\theta},\xi^n) \triangleq \min_{(X^n,\vec{Y}^n)\in\mathcal{D}(\xi^n)} X^n + \vec{\theta}\cdot\vec{Y}^n. \quad (2.104)$$

Please be aware that we slightly abuse the notation V to emphasize that $V(\theta,\xi^n)$ is a analog of $V(\theta)$ in each discrete time instant.

Following the definition, we derive the following results:

$$V(\vec{\theta}) \geq \tilde{V}(\vec{\theta}) = \min_{\pi\in\Gamma}\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X_\pi^n + \vec{\theta}\cdot\vec{Y_\pi^n}\} \quad (2.105)$$

$$= \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\min_{(X^n,\vec{Y}^n)\in\mathcal{D}(\xi^n(\omega))} X^n + \vec{\theta}\cdot\vec{Y}^n\} \quad (2.106)$$

$$= \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{V(\vec{\theta},\xi^n).\} \quad (2.107)$$

Remark 2.2 (Some Results about the Concavity and the Subgradient)**.**

By definition, $V(\vec{\theta},\xi^n)$ is concave in θ. Indeed, $\mathbb{E}\{V(\vec{\theta},\xi^n)\}$ and $\tilde{V}(\vec{\theta})$ are concave functions, too. This fact can be proved using the linearity and the equality (2.107) *for $\tilde{V}(\vec{\theta})$. According to [9, Proposition B. 25], the minimizer $Y_{\pi^*,\theta}^n$ of*

(2.109) *is a subgradient of* $V(\vec{\theta}, \xi^n)$*. This result implies that* $\mathbb{E}\{Y^n_{\pi^*,\theta}\}$ *is a subgradient of* $\mathbb{E}\{V(\vec{\theta}, \xi^n)\}$*. Hence, its long-term average,* $\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{Y^n_{\pi^*,\theta}\}$*, is a subgradient of* $\tilde{V}(\vec{\theta})$*.*

Note that it is legitimate to move the min operator inside the expectation in (2.106) because the feasible sets $\{\mathcal{D}(\xi^n),\ n = 0, 1, 2, \ldots\}$ are separable, which means that any decision made at time n has no impact on the feasible set $\mathcal{D}(\xi^m)$ of the control actions at time $m \neq n$. Therefore, the minimum of the right-hand side of (2.105) is achievable via the policy which minimizes $X^n + \vec{\theta} \cdot \vec{Y}^n$ over $\mathcal{D}(\xi^n(\omega))$ for each realization of ξ^n at each time n. Such policy is determined by the dual variable $\vec{\theta}$.

In our case, instead of solving the dual problem, we opt to find the maximizer of $\tilde{V}(\cdot)$. Since $\tilde{V}(\cdot)$ is also concave, ideally, we can apply numerical algorithm, referred to subgradient algorithm, to find the solution. For finding the maximizer of $\tilde{V}(\cdot)$, the outcome of the numerical algorithm is a sequence of dual variables $\{\vec{\theta}^n,\ n = 0, 1, 2, \ldots\}$. In the following, we define a class of policies which sequentially ($n = 0, 1, 2, \ldots$) assign the control values utilizing the solution to a specific minimization with respect to a sequence of dual variables and the realization of $\xi^n(\omega)$. This decision making only requires the instantaneous information about the dual variables and the $\xi^n(\omega)$. Therefore, it can be implemented as an online algorithm.

Definition 2.5. *For a sequence* $\{\theta^n,\ n = 0, 1, 2, \ldots\}$*, we define* π^* *as the policy* *which decides the control values at time n according to the following rule:*

$$(X^n_{\pi^*,\theta^n}, \vec{Y}^n_{\pi^*,\theta^n}) \in \arg \min_{(x,\vec{y})\in\mathcal{D}(\xi^n)} x + \vec{\theta}^n \cdot \vec{y}. \qquad (2.108)$$

In the degenerate case, when the sequence $\vec{\theta}^n$ is only the repetition of a fixed vector $\vec{\theta}$, π^* is defined as the policy which chooses the control values at time n with the following law:

$$(X^n_{\pi^*,\theta}, Y^n_{\pi^*,\theta}) \in \arg \min_{(x,\vec{y})\in\mathcal{D}(\xi^n)} x + \theta \cdot \vec{y}. \qquad (2.109)$$

Since $V(\vec{\theta}) \geq \tilde{V}(\vec{\theta})$, the following problem gives a lower bound on the optimal values of the dual problem and the primal problem:

Problem 2.5.

$$\max_{\vec{\theta}\geq 0} \tilde{V}(\vec{\theta}). \qquad (2.110)$$

Note that $\tilde{V}(\cdot)$ is concave , and $V(\vec{\theta}) = \tilde{V}(\vec{\theta})$ if the limit (instead of limsup) of the expected long-term average in (2.103) exists. Unlike the dual function $V(\cdot)$, for which the optimal policy of (2.91) is unknown, we have the close form of the policy π^* that minimizes (2.103) for $\tilde{V}(\cdot)$. But still, we have the problem of evaluating the subgradient for $\tilde{V}(\cdot)$ when $\{\xi^k(\omega)\}$ can only be observed causally, i.e. only the values of $\{\xi^k(\omega),\ k = 0, \ldots, n\}$ are known up to time n. For this reason, in general, the direct approach using subgradient method is not applicable. We need new approaches to find the solution. Hopefully, we can leverage some concepts inherent in the subgradient method to design the new algorithm. Most importantly, we need to find the best policy from those using only casual information. With these prerequisites in mind, we try the recursive algorithm below which updates the dual variables under the control policy π^*,

$$\vec{\theta}^{(n+1)} = \Pi_H[\vec{\theta}^n + \epsilon \vec{Y}^n_{\pi^*,\theta^n}], \tag{2.111}$$

where Π_H denotes the projection onto the set $H = \{\theta \mid 0 \leq \theta_i \leq K_u\}$. Note that this algorithm takes the place of the subgradient method, and it degenerates to the subgradient method if we replace $Y^n_{\pi^*,\theta^n}$ by a subgradient of $\tilde{V}(\vec{\theta})$. The behavior of this recursive algorithm are covered by the analytical work on stochastic approximation established in Section 2.1. In reality, the dynamics of $\vec{\theta}^n$, $\vec{Y}^n_{\pi^*,\theta^n}$ and $X^n_{\pi^*,\theta^n}$ depends on the step size ϵ. However, for notational simplicity, we omit the subscript ϵ which appeared in the stochastic approximation (2.2). Shortly, we will prove that if the assumptions (A.1-5) given in Theorem 2.1 are satisfied, the expected long term average of the controlled sequence $\{(X^n_{\pi^*,\theta^n}, Y^n_{\pi^*,\theta^n}),\ n = 0, 1, 2 \ldots\}$ actually reaches the optimal solution asymptotically.

2.2.3 Feasibility and Optimality

In this subsection, we will prove two important properties for our algorithm: the feasibility and optimality . We restrict our analysis to the case where $\{\xi^n\}$ is a stationary ϕ-mixing process. Applying Theorem 2.2 with $Y^n_{\pi^*,\theta}$ taking the place of g^n_ϵ, the assumptions (A.5) of Theorem 2.1 are satisfied if $\mathcal{G}(\theta) \triangleq \partial\tilde{V}(\theta)$. Therefore, under the same setting described in Theorem 2.1, all the results given in the last section for stochastic approximation can be applied to the recursion (2.111). To explain in detail, we begin with the proof of feasibility.

Asymptotic Feasibility

We assume that, for each maximizer of $\tilde{V}(\cdot)$, the values of its components are within the interval $[0, K_u)$. If the set of all maximizers of $\tilde{V}(\cdot)$ is compact, this assumption is valid when K_u is sufficiently large. In practice, one can set K_u to the maximal supported value used by the computing processor. We define the compensation terms from below $\hat{Z}^n \in \mathbb{R}^{r,+}$ and above $\check{Z}^n \in \mathbb{R}^{r,+}$ by rewriting the recursion (2.111) as

$$\vec{\theta}^{(n+1)} = \vec{\theta}^n + \epsilon\,(\vec{Y}^n_{\pi^*,\vec{\theta}^n} + \hat{Z}^n - \check{Z}^n). \tag{2.112}$$

The equation (2.112) can be rewritten as

$$\vec{\theta}^{(n+1)} - \vec{\theta}^n = \epsilon\,(\vec{Y}^n_{\pi^*,\vec{\theta}^n} + \hat{Z}^n - \check{Z}^n). \tag{2.113}$$

We add both sides of (2.113) from $n = 0$ to $n = N-1$. Summing the left-hand side of (2.113)from 0 to N forms a telescoping sum, this results the equation below.

$$\vec{\theta}^{(N)} - \vec{\theta}^0 = \epsilon \sum_{n=0}^{N-1} (\vec{Y}^n_{\pi^*,\vec{\theta}^n} + \hat{Z}^n - \check{Z}^n) \tag{2.114}$$

After taking the expectation both sides of (2.114) and dividing them by ϵN, we arrive at the equality

$$\frac{\mathbb{E}\{\vec{\theta}^{(N)} - \vec{\theta}^0\}}{\epsilon N} = \frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}\{(\vec{Y}^n_{\pi^*,\vec{\theta}^n} + \hat{Z}^n - \check{Z}^n)\} \overset{(a)}{\geq} \frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}\{(\vec{Y}^n_{\pi^*,\vec{\theta}^n} - \check{Z}^n)\}. \tag{2.115}$$

The inequality (a) above results from the fact $\hat{Z}^n \geq 0$. To get the end result, we take limsup on both sides of (2.115).

$$\limsup_{\epsilon} \limsup_{N} \frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}\{(\vec{Y}^n_{\pi^*,\vec{\theta}^n} - \check{Z}^n)\} \leq \limsup_{\epsilon} \limsup_{N} \frac{\mathbb{E}\{\vec{\theta}^{(N)} - \vec{\theta}^0\}}{\epsilon N} \overset{(b)}{=} 0 \tag{2.116}$$

The equality (b) holds because $\mathbb{E}\{\|\vec{\theta}^{(N)} - \theta^*\|\}$ is bounded (Theorem 2.5). Lastly, the proof of the feasibility is complete by showing

$$\limsup_{\epsilon} \limsup_{N} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{\check{Z}^n\} = 0. \tag{2.117}$$

This is what given in Remarks for Theorem 2.7.

<u>**Asymptotic Optimality**</u>

In this subsection, we move on to prove the asymptotic optimality of Algorithm (2.111), with the assumption that $\|\vec{Y}_{\pi}^n\| \leq B_1$ uniformly in n for any feasible policy π. For clarity, we first assume that the maximizer of $\tilde{V}(\theta)$ is unique. We note that the core of the proof relies on some bound on the cross product term $\vec{\theta}^n \cdot \vec{Y}^n_{\pi^*,\vec{\theta}^n}$. Taking square on both sides of (2.111), we have

$$\|\vec{\theta}^{n+1}\|^2 \leq \|\vec{\theta}^n\|^2 + 2\epsilon\vec{\theta}^n \cdot \vec{Y}^n_{\pi^*,\vec{\theta}^n} + \epsilon^2\|\vec{Y}^n_{\pi^*,\vec{\theta}^n}\|^2, \tag{2.118}$$

or equivalently we have

$$\|\vec{\theta}^{n+1}\|^2 - \|\vec{\theta}^n\|^2 \leq 2\epsilon\vec{\theta}^n \cdot \vec{Y}^n_{\pi^*,\vec{\theta}^n} + \epsilon^2\|\vec{Y}^n_{\pi^*,\vec{\theta}^n}\|^2. \tag{2.119}$$

For simplicity, we assume that $\vec{\theta}^0 = 0$. In this way, we do not sacrifice the generality of the proof since the initial value is asymptotically negligible after averaged over n. We add up both sides of (2.119) from $n = 0$ to $N - 1$, and divide them by ϵN. It follows that

$$\frac{1}{2\epsilon N}\mathbb{E}\{\| \vec{\theta}^{N+1} \|^2\} \leq \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{\theta}^n \cdot \vec{Y}^n_{\pi^*,\vec{\theta}^n}\} + \epsilon\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\| \vec{Y}^n_{\pi^*,\vec{\theta}^n} \|^2\}. \tag{2.120}$$

Note that the second term on the right-hand side of (2.120) is bounded by ϵB_1.

As a result, we have

$$\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{\theta}^n\cdot\vec{Y}^n_{\pi^*,\vec{\theta}^n}\}\geq-\epsilon B_1. \tag{2.121}$$

We denote $\vec{\theta}^*$ as the unique solution[6] of $\tilde{V}(\vec{\theta})$ over $\vec{\theta}\geq 0$. Recall that:

$$V(\vec{\theta}^n,\xi^n)=X^n_{\pi^*,\vec{\theta}^n}+\vec{\theta}^n\cdot\vec{Y}^n_{\pi^*,\vec{\theta}^n}. \tag{2.122}$$

Summing up both sides from $n=1$ to N, we get

$$\begin{aligned}
\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{V(\vec{\theta}^n_\epsilon,\xi^n)\} &= \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X^n_{\pi^*,\vec{\theta}^n}\}+\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{\theta}^n_\epsilon\cdot\vec{Y}^n_{\pi^*,\vec{\theta}^n}\}\quad(*)\\
&\leq \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X^n_{\pi^*_{\vec{\theta}^*}}\}+\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{\theta}^n\cdot\vec{Y}^n_{\pi^*_{\vec{\theta}^*_\epsilon}}\}\\
&= \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X^n_{\pi^*_{\vec{\theta}^*}}\}+\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\vec{\theta}^*\cdot\vec{Y}^n_{\pi^*_{\vec{\theta}^*}}\}\\
&\quad+\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{(\vec{\theta}^n-\vec{\theta}^*)\cdot\vec{Y}^n_{\pi^*_{\vec{\theta}^*}}\}\\
&= \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{V(\vec{\theta}^*,\xi^n)\}+\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{(\vec{\theta}^n-\vec{\theta}^*)\cdot\vec{Y}^n_{\pi^*_{\vec{\theta}^*}}\},\quad(**)
\end{aligned}$$

By comparing equations $(*)$ with $(**)$, and exploiting inequality (2.121), we have

$$\begin{aligned}
\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X^n_{\pi^*,\vec{\theta}^n}\}-\epsilon B_1 &\leq \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{V(\vec{\theta}^*,\xi^n)\}+\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{(\vec{\theta}^n-\vec{\theta}^*)\cdot\vec{Y}^n_{\pi^*_{\vec{\theta}^*}}\}\\
&\leq \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{V(\vec{\theta}^*,\xi^n)\}+\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{\|\vec{\theta}^n-\vec{\theta}^*\|\}B_1.
\end{aligned}$$

The proof is complete after taking $\lim_\epsilon\limsup_N$ on both sides of the equality.

$$\limsup_\epsilon\limsup_N\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X^n_{\pi^*,\vec{\theta}^n}\}\leq\limsup_N\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{V(\vec{\theta}^*,\xi^n)\}=\tilde{V}(\vec{\theta}^*)\leq V^*=X^* \tag{2.123}$$

[6]The proof can be accomplished in more general setting, in which we allow the solutions to the problem is not unique. The outline of the proof for this general case will be given at the end of this section.

Remark 2.3 (The Uniqueness of $\vec{\theta}^*$)**.** *In the previous proof of asymptotic feasibility and optimality, we assumed unique maximizer $\vec{\theta}^*$ of $\tilde{V}(\cdot)$. In general, the uniqueness of solution $\vec{\theta}^*$ may not hold. Nevertheless, as long as the set of $\vec{\theta}^*$ is included in a certain compact set A where for each elements of A, the values of its components are in $[0, K_u)$, the same results for optimality and feasibility are still valid. The outline of the proof is given in the Appendix of this chapter.*

The core results of stochastic optimization are summarized in the following theorem.

Theorem 2.8. *Consider a stochastic system created by the stationary ϕ-mixng exogenous process $\{\xi^n\}$. The stochastic optimization below are concerned with the optimal randomized control policy π over the control variables $X_\pi^n \in \mathbb{R}$ and $\vec{Y}_\pi^n \in \mathbb{R}^r$.*

$$\text{minimize} \quad \limsup_N \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{X_\pi^n\} \tag{2.124}$$

$$\text{subject to} \quad \limsup_N \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{\vec{Y}_\pi^n\} \leq 0 \tag{2.125}$$

$$\text{policy } \pi \text{ such that } (X_\pi^n, \vec{Y}_\pi^n) \in \mathcal{D}(\xi^n). \tag{2.126}$$

Assume that $\mathcal{D}(\xi^n) \subset \mathbb{R}^{r+1}$ is a set function determined by ξ^n, and there is a compact set $A \subset \mathbb{R}^{r+1}$ such that $\mathcal{D}(\xi^n) \subset A$ for all n. If, in addition, at time $n = 0, 1, 2, \ldots$, the following three conditions hold:

1. *the scheduler selects the control variables $(X^n_{\pi^*,\vec{\theta}^n}, \vec{Y}^n_{\pi^*,\vec{\theta}^n})$ under the policy π^* that follows the rule*

$$(X^n_{\pi^*,\vec{\theta}^n}, \vec{Y}^n_{\pi^*,\vec{\theta}^n}) \in \arg \min_{(x,\vec{y}) \in \mathcal{D}(\xi^n)} x + \vec{\theta}^n \cdot \vec{y}, \tag{2.127}$$

2. $\vec{\theta}^n$ *is updated according to the recursive algorithm*

$$\vec{\theta}^{(n+1)} = \Pi_H[\vec{\theta}^n + \epsilon \vec{Y}^n_{\pi^*,\vec{\theta}^n}], \quad H = \{\vec{\theta} \mid 0 \le \theta_i \le K_u\}, \tag{2.128}$$

3. the set of the maximizers to $\tilde{V}(\vec{\theta})$ *(defined in* (2.103)*) over* $\vec{\theta} \ge 0$ *does not touch the upper boundary*[7] *(*K_u*) of the constraint set* H*,*

then the expected long-term average of the control sequence $(X^n_{\pi^*,\vec{\theta}^n}, \vec{Y}^n_{\pi^*,\vec{\theta}^n})$ *converges to the optimal solution asymptotically as the step size* ϵ *goes to zero, that is:*

$$\limsup_{\epsilon} \limsup_{N} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{X^n_{\pi^*,\vec{\theta}^n}\} = X^* \tag{2.129}$$

$$\limsup_{\epsilon} \limsup_{N} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{\vec{Y}^n_{\pi^*,\vec{\theta}^n}\} \le 0 \tag{2.130}$$

2.3 Appendix

In this appendix, we outline the proofs for the asymptotic feasibility and optimality of the algorithm (2.111) when Problem 2.5 has more than one solutions. Let's denote Θ as the set consisting of all the solutions to Problem 2.5.

Asymptotic Feasibility

From the case of unique maximizer, we learned that the proof of feasibility requires: (i) the boundedness of $\mathbb{E}\{\vec{\theta}^n\}$ and (ii) the negligible contribution of the reflection term from above. Since A is compact and does not touch the upper boundary of H, meaning for each element in A its component values $\in [0, K_u)$,

[7] *The upper boundary is defined componentwise for* $\vec{\theta}$ *at* K_u*.* $\vec{\theta}$ *does not touch the upper boundary of* H *if* $\theta_i < K_u$ *for* $i = 1, \ldots, r$*.*

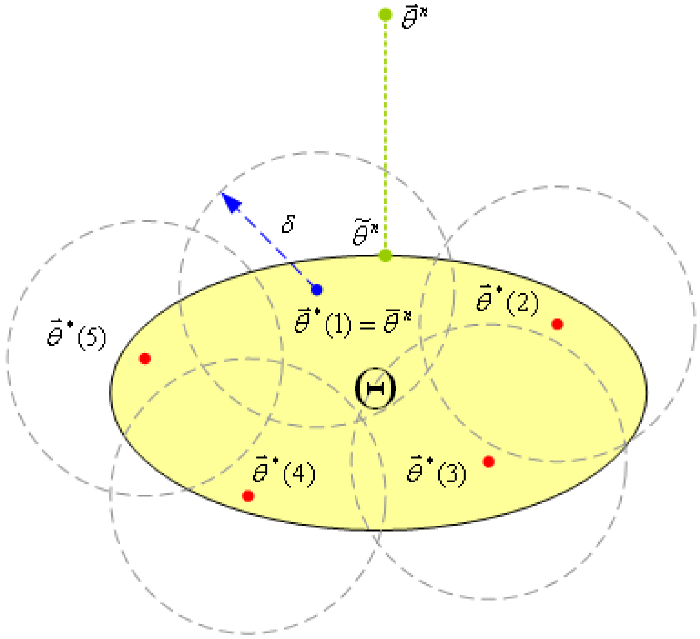


Figure 2.2: Projected Point and Partition Points.

(i) and (ii) hold. The proofs for these two facts are vary similar to what we did in Theorem 2.5 before for the stochastic approximation when $\mathcal{G}(\cdot)$ is a subdifferential, and hence is omitted.

<u>**Asymptotic Optimality**</u>

Since Θ is bounded by a compact set A, given $\delta > 0$, one can find a finite set of points $\{\vec{\theta}^*(1), \vec{\theta}^*(2), \ldots, \vec{\theta}^*(r_\delta)\}$ in Θ such that $\Theta \subset \bigcup_{i=1}^{r_\delta} N_\delta(\vec{\theta}^*(i))$.

For notational simplicity, we define $\tilde{\theta}^n \triangleq \Pi_\Theta(\vec{\theta}^n)$, which is the projection of $\vec{\theta}^n$ onto the solution set Θ of Problem 2.5. In the proof, we need to identify which neighborhood $N_\delta(\vec{\theta}^*(i))$ does $\vec{\theta}^n$ lie in. To this end, we denote $\bar{\theta}^n$ as the point in $\{\vec{\theta}^*(1), \ldots, \vec{\theta}^*(r_\delta)\}$ that is closest to $\tilde{\theta}^n$. Figure 2.2 illustrates an example in which the point $\vec{\theta}^n$ is projected onto the set Θ at $\tilde{\theta}^n$, the set Θ is covered by $\bigcup_{i=1}^{5} N_\delta(\vec{\theta}^*(i))$, and $\bar{\theta}^n = \vec{\theta}(1)$ is the point in $\{\vec{\theta}^*(1), \ldots, \vec{\theta}^*(5)\}$ which is closet to

$\tilde{\theta}^n$.

Before the proof of asymptotic optimality, we establish the following facts.

⟨**Fact 1**⟩

$$\frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{X^n_{\pi^*,\bar{\theta}^{mL}}\}+\frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{\bar{\theta}^{mL}\cdot\vec{Y}^n_{\pi^*,\bar{\theta}^{mL}}\}\leq X^*+\frac{1}{L}\sum_{n=0}^{L-1}2B_1(B_2+1)\phi_n,$$

where $B_2=\max\{\ \|\vec{\theta}^*(1)\|,\ldots,\|\vec{\theta}^*(r_\delta)\|\}$.

Proof. Since the right-hand side of the inequality above is a constant, it is sufficient to show that the inequality still holds if the expectation is replaced by the conditional expectation. If this conjecture is true for the conditional expectation, by taking the expectation over the conditional expectation, we then end up with the same result. The proof is as follows.

$$\begin{aligned}
&\frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{X^n_{\pi^*,\bar{\theta}^{mL}}\mid\bar{\theta}^{mL}=\vec{\theta}^*(i)\ \}+\frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{\bar{\theta}^{mL}\cdot\vec{Y}^n_{\pi^*,\bar{\theta}^{mL}}\mid\bar{\theta}^{mL}=\vec{\theta}^*(i)\ \}\\
&=\frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{X^n_{\pi^*,\vec{\theta}^*(i)}\mid\bar{\theta}^{mL}=\vec{\theta}^*(i)\ \}\\
&\quad+\frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{\vec{\theta}^*(i)\cdot\vec{Y}^n_{\pi^*,\vec{\theta}^*(i)}\mid\bar{\theta}^{mL}=\vec{\theta}^*(i)\ \}\\
&\overset{(1)}{\leq}\frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{X^n_{\pi^*,\vec{\theta}^*(i)}\}+\frac{1}{L}\sum_{n=mL}^{mL+L-1}2B_1\phi_{(n-mL)}\\
&\quad+\frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{\vec{\theta}^*(i)\cdot\vec{Y}^n_{\pi^*,\vec{\theta}^*(i)}\}+\frac{1}{L}\sum_{n=mL}^{mL+L-1}\|\vec{\theta}^*(i)\|2B_1\phi_{(n-mL)}\\
&\overset{(2)}{\leq}X^*+\frac{1}{L}\sum_{n=0}^{L-1}2B_1(B_2+1)\phi_n.
\end{aligned}$$

The inequality (1) above follows Lemma 2.1. More precisely, it follows the facts

$$\|\mathbb{E}\{X^n_{\pi^*,\vec{\theta}^*(i)}\ |\bar{\theta}^{mL}=\vec{\theta}^*(i)\}-\mathbb{E}\{X^n_{\pi^*,\vec{\theta}^*(i)}\}\|\leq 2B_1\phi_{(n-mL)}\qquad(2.131)$$

and

$$\|\mathbb{E}\{Y^n_{\pi^*,\vec{\theta}^*(i)}\ |\bar{\theta}^{mL} = \vec{\theta}^*(i)\} - \mathbb{E}\{Y^n_{\pi^*,\vec{\theta}^*(i)}\}\| \leq 2B_1\phi_{(n-mL)}. \tag{2.132}$$

The inequality (2) above is a consequence of the following result due to the stationarity of ξ^n and the definition of $\tilde{V}(\cdot)$:

$$\frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{X^n_{\pi^*,\vec{\theta}^*(i)}\} + \frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{\vec{\theta}^*(i)\cdot\vec{Y}^n_{\pi^*,\vec{\theta}^*(i)}\} = \tilde{V}(\vec{\theta}^*(i)) \leq X^*. \tag{2.133}$$

□

⟨**Fact 2**⟩

There exists $L' > 0$ such that, for all $L > L'$,

$$\frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{V(\vec{\theta}^n,\xi^n)\} \leq \quad X^* + 2B_1(B_2+1)\delta + \epsilon\frac{L+1}{2}B_1^2 + \mathbb{E}\{\mathrm{dist}(\vec{\theta}^{mL},\Theta)\}B_1 + \delta B_1$$

Proof. Assume that L is sufficiently large so that $\frac{1}{L}\sum_{n=0}^{L-1}\phi_n \leq \delta$. We have

$$\begin{aligned}
&\frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{V(\vec{\theta}^n,\xi^n)\} \\
&= \frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{X^n_{\pi^*,\vec{\theta}^n}\} + \frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{\vec{\theta}^n\cdot\vec{Y}^n_{\pi^*,\vec{\theta}^n}\} \\
&\leq \frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{X^n_{\pi^*,\bar{\theta}^{mL}}\} + \frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{\vec{\theta}^n\cdot\vec{Y}^n_{\pi^*,\bar{\theta}^{mL}_\epsilon}\} \\
&= \frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{X^n_{\pi^*,\bar{\theta}^{mL}}\} + \frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{\bar{\theta}^{mL}\cdot\vec{Y}^n_{\pi^*,\bar{\theta}^{mL}}\} \\
&\quad + \frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{(\vec{\theta}^n - \bar{\theta}^{mL})\cdot\vec{Y}^n_{\pi^*,\bar{\theta}^{mL}}\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{X^n_{\pi^*,\bar{\theta}^{mL}}\} + \frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{\bar{\theta}^{mL}\cdot \vec{Y}^n_{\pi^*,\bar{\theta}^{mL}}\} \\
&\quad + \frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{(\vec{\theta}^n - \vec{\theta}^{mL})\cdot \vec{Y}^n_{\pi^*,\bar{\theta}^{mL}}\} + \frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{(\vec{\theta}^{mL} - \bar{\theta}^{mL})\cdot \vec{Y}^n_{\pi^*,\bar{\theta}^{mL}}\} \\
&\overset{(3)}{\leq} X^* + \frac{1}{L}\sum_{n=0}^{L-1} 2B_1(B_2+1)\phi_n + \epsilon\frac{L+1}{2}B_1^2 + \mathbb{E}\{\|\vec{\theta}^{mL} - \bar{\theta}^{mL}\|\}B_1, \\
&\overset{(4)}{\leq} X^* + 2B_1(B_2+1)\delta + \epsilon\frac{L+1}{2}B_1^2 + \mathbb{E}\{\|\vec{\theta}^{mL} - \tilde{\theta}^{mL}\|\}B_1 + \mathbb{E}\{\|\tilde{\theta}^{mL} - \bar{\theta}^{mL}\|\}B_1 \\
&\overset{(5)}{\leq} X^* + 2B_1(B_2+1)\delta + \epsilon\frac{L+1}{2}B_1^2 + \mathbb{E}\{\mathrm{dist}(\vec{\theta}^{mL},\Theta)\}B_1 + \delta B_1,
\end{aligned}
$$

The inequality (3) above results from Fact 1 and the facts $\|\vec{Y}^n_{\pi^*,\bar{\theta}^{mL}}\| \leq B_1$, $\|\vec{Y}^n_{\pi^*,\vec{\theta}^n}\| \leq B_1$ and $\|\vec{\theta}^n - \vec{\theta}^{mL}\| \leq \epsilon(n-mL)B_1$. The triangular inequality and the assumption $\frac{1}{L}\sum_{n=0}^{L-1}\phi_n \leq \delta$ give the inequality (4) above. By the definition of $\bar{\theta}^{mL}$, we know that $\bar{\theta}^{mL} \in N_\delta(\tilde{\theta}^{mL})$ and $\|\tilde{\theta}^{mL} - \bar{\theta}^{mL}\| \leq \delta$. This fact in turn gives the inequality (5) above. □

⟨**Fact 3**⟩

$$
\begin{aligned}
&\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{V(\vec{\theta}^n,\xi^n)\} \\
&= \limsup_M \frac{1}{M}\sum_{m=0}^{M}\frac{1}{L}\sum_{n=mL}^{mL+L-1}\mathbb{E}\{V(\vec{\theta}^n,\xi^n)\} \\
&\leq X^* + 2B_1(B_2+1)\delta + \epsilon\frac{L+1}{2}B_1^2 + \limsup_M \frac{1}{M}\sum_{m=0}^{M}\mathbb{E}\{\mathrm{dist}(\vec{\theta}^{mL},\Theta)\}B_1 \\
&\quad + \delta B_1,
\end{aligned}
$$

Using the bound on the cross product term (2.121) and the definition of $V(\vec{\theta}^n,\xi^n)$, we have

$$
\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{X^n_{\pi^*,\vec{\theta}^n}\} - \epsilon B_1 \leq \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\{V(\vec{\theta}^n,\xi^n)\}. \tag{2.134}
$$

Combining (2.134) with Fact 3 and the fact

$$\limsup_{\epsilon} \limsup_{M} \mathbb{E}\{\text{dist}(\vec{\theta}^{mL}, \Theta)\} = 0 \tag{2.135}$$

provided by Theorem 2.6, it leads to the following inequality:

$$\limsup_{\epsilon} \limsup_{N} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{X^n_{\pi^*, \vec{\theta}^n}\} \leq X^* + 2B_1(B_2 + 1)\delta + \delta B_1. \tag{2.136}$$

The end result is hence achieved if we let δ goes to 0.

$$\limsup_{\epsilon} \limsup_{N} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{X^n_{\pi^*, \vec{\theta}^n}\} \leq X^*. \tag{2.137}$$

Chapter 3

Routing, Link Scheduling and Power Control in Multi-hop Wireless Networks over Time Varying Channels

In this chapter, we investigate the power efficient resource allocation policy in multi-hop wireless networks with multi-commodity flows. In each time slot, a scheduler has three decisions to make: 1) How much power should be spent by each node to transmit signals, 2) How much data should be transmitted on each channel, and 3) What fraction of the data sent over each channel belong to each flow. This result in three variables to be controlled by the scheduler. The challenge of assigning these variables comes from the coupling of feasibility. For example, the transmit power and the channel condition determine the achievable transmit rates on the channels; the achievable transmit rate on a channel limits the aggregate rate of all flows sent over it; and each flow has its own QoS target rate to meet. Therefore, any changes made to decision on one of them will affect the value of the others.

In this work, we seek a control rule that manages these resources (power, link rate, and flow rate) efficiently so that the predefined end-to-end flow rate requirements are satisfied using minimal average transmit power. This is essentially an optimal control problem, and we apply the stochastic optimization to solve it. Leveraging the techniques developed in Chapter 2, an online algorithm is proposed, which allocates the resource adapting to the observation of instantaneous time-varying channel conditions.

Chapter 3 is organized as follows. We begin with the introduction to channel aware schedulers. After the overview of related research, in Section 3.2, we introduce the concepts of link, flow, and the time-varying property of channels. In Section 3.3, we define the system model, notation, and the constraints which form the basis of analysis framework of this chapter. In Section 3.4, we formulate our primary problem, power efficient routing, as a stochastic optimization and solve the problem by a recursive algorithm. This is the most important part of this chapter. In Section 3.5, we examine our algorithm using an example of two flows. Lastly, we conclude this chapter with some remarks.

3.1 Introduction

The growing interest in multi-hop wireless network raises the demand for efficient resource allocation mechanisms for wireless systems. To maintain quality of service (QoS) and manage the cost of average resource consumptions, the scheduler must coordinate the communications taking place within the system. To be clear, in each time slot, a scheduler specifies the transmit power and rate on each data link to fulfill the QoS requirements considering the effect of mutual interference. Wireless channels are subject to dynamic changes caused by user mobility and the movements of surrounding obstructions. Maintaining a constant transmission rate continuously over a channel may result in excessively high power consumptions for severe channel conditions. An efficient scheduler should exploit the *channel state information* (CSI) and makes decisions accordingly.

Many research works studied the problem of utility maximization with respect to link rates under time-varying channels. In [47], Tse proposed the proportional fair scheduler (PFS), which exploits multi-user diversity in a time-varying environment. Under the assumption of stationarity on the channels, it was demonstrated that the throughput vector $[R_1^{avg}, \ldots, R_L^{avg}]$ of PFS achieves the maximum of logarithmic utility function $\sum_{l=1}^{L} \log(X_l^{avg})$ [28]. Agrawal [2] and Stolyer [45] further generalized this idea by considering the utility functions in the format $\sum_{i=1}^{L} f_l(X_l^{avg})$, where the function $f_l(\cdot)$ is concave and differentiable.

The key disadvantage of using this utility based scheduler is that the optimal scheduler of this kind is likely to favor the users experiencing better channel conditions. This behavior implicitly lowers the priority of users with poor channel quality. However, in practice, certain QoS performance guarantee is required for each link, which is hard to achieve using utility-optimized scheduler. To address this issue, Lee et al. [29] reconsidered the throughput maximization problem with additional minimum throughput constraint on each user. An algorithm is proposed by them to tackle this refined problem. However, their proposed solution is only sub-optimal if the number of users is only finite. In their algorithm, the scheduling decisions are ruled by a parameter, which is updated continuously via a recursive equation using a “subgradient-type” vector. If the channel state is deterministic and invariant, this subgradient-type vector is equal to a certain subgradient of the dual function to their optimization problem. However, for random time-varying channel, this is not necessary true. In their proof, there is a problematic statement which claims that, given the history of the system, the conditional expectation of this “subgradient-type” vector forms a subgradient if the channel process is stationary. Indeed, their proof works when the underlying process is *independent and identically distributed* (IID); however, we can find counter examples to their proof, utilizing the correlation between different samples of the stationary channel state process at different time.

In this work, the structure of asymptotically optimal scheduler for power efficient policies under time-varying channels is investigated. In particular, we aim to minimize the average power consumptions while maintaining minimum average end-to-end data rate for multi-commodity flows. The problem is formulated as an optimization concerned with time and ensemble average of controllable random sequences. A recursive scheduling algorithm is proposed for this problem, and its performance (in terms of power efficiency) can be tuned arbitrarily close to the optimal value. By means of stochastic optimization, we justify that our algorithm converges to the optimum irrespective of the system size.

3.2 Channels, Links and Flows

Time-Varying Channel

Due to the user mobility and the movements of surrounding obstructions, wireless channels are random and time-varying in nature. For a wireless network with user devices equipped with single omni-directional antenna, the magnitude of power loss over all channels fully determine the network channel capacity. The power loss gain of a channel can be expressed using a product of factors $G_T G_P G_c G_L G_M G_R$, where G_T and G_R denote the transmit and receive antenna gain, G_P denotes the path loss gain, G_c denotes the processing gain, G_L denotes the shadowing and G_M denotes the multi-path fading. Among these factors, G_M changes most frequently. The frequency of this variation can be roughly quantified as follows. The mobility of the mobile nodes causes Doppler shift in the carrier frequency, and the reciprocal of the Doppler shift gives the coherence time T_c of the channel. Note that the value of T_c is channel dependent. Within any time interval of length T_c, the power loss gain of the channel is approximately constant. Motivated by this property, in this work, we assume that the system divides the time into equally spaced intervals called slots. For analytic simplicity, we also assume that the power loss gains over all channels are fixed within a slot and can change at the boundaries of a slot. In other words, the block fading model for time-varying channels is assumed in this work. In reality, channel states can change within a slot. However, if the minimum coherence time among all channels is much larger than the slot size, the fraction of time slots in which channel variation violets the block fading model is negligible. This assumption is valid for a system with low mobility.

Links and Link Capacity

In wireless networks, a link is defined by an ordered pair of nodes (a, b) where node a is called transmit node and node b is called receive node. As a transmitter, node a encodes the information bit prepared for node b, modulates the codeword into signal, and sends the signal over the wireless channel connecting node a to node b. On receipt of the signal, node b performs demodulation and detection, and extracts the information from the coded symbol. A link denotes a point-to-point connection bearing the information directly from the transmitter to the receiver without the help of intermediate nodes.

In wireless setting, the transmit signal power from one node reaches all other nodes at different levels of attenuation. A signal in the air causes exogenous interference to a node if the signal does not bear information intended for that node. From Shannon's analysis in channel capacity [44], we know that the capacity for memoryless additive white gaussian noise (AWGN) channel is a function of the received signal to noise power ratio (SNR). However, the exact capacity formula of interfering channels is only available in some simplified cases. Nevertheless, from the receiver's perspective, intuitively, interferences play a similar role to gaussian noise, which deteriorates the reception quality. It has been examined in many communication systems, such as CDMA [51], that the quality of reception is determined by the ratio between the received signal power and the interference plus noise power. This ratio is called the signal-to-interference and noise power ratio (SINR). In this chapter, we assume that the maximum information bit rate to be transmitted over a link is a function $\phi(\cdot)$ of the SINR value at the receiving end of that link. From high-level view, this assumption is valid in the following two aspects: first, if the transmitted signals are independent, the aggregate interferences act like gaussian noise. This argument follows the central limit theorem. Second, if gaussian signaling (entropy achieving code) is assumed for channel coding, the aggregate interference is still gaussian. In both cases, if we include the interferences into the noise term, by Shannon's capacity formula for AWGN channel, the maximum achievable rate can be expressed as a function of SINR.

To achieve link capacity with zero error, in theory, it requires a set of codewords with infinite length. In a bandlimited wireless system, however, this amounts to infinite times of transmissions. For real applications, delay requirements are restrictive; hence, it is not possible to transfer error free information over the noisy link at its capacity rate in a finite time duration. Nevertheless, given any SINR value and physical layer design, there is an achievable bound on the link rate satisfying the predefined negligible probability of error (e.g. less than 10^{-5}). In the remainder of this chapter, we regard this achievable bound as the operational link capacity. Unless we state otherwise, the terms "link capacity" and "operational link capacity" are used interchangeably. By this definition of link capacity, the occurrences of errors are negligible. Therefore, when designing a scheduler, one can neglect these nuisance rare error events and assume perfect information transfer at a rate below the operational link capacity.

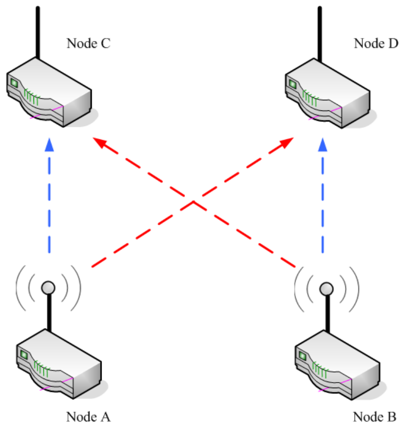
Link rate function $\phi(\cdot)$ is a strictly increasing function. To maintain transmission rate at R_t on each link, it amounts to keep the receiving SINR above the corresponding threshold γ_t. Essentially, there are two approaches to raise the SINR on a link. The first approach is to increase the transmit power on that link, the second approach is to reduced the interferences from other links. From the aspect of system level impact, these two approaches conflict to each other since by increasing the transmit power it increases the interferences to other nodes as well.

If the requested transmit rates need to be satisfied within each single time slot, the solution to the optimal power allocation reduces to the classical power control, which adjusts the transmit power to maintain the SINR for the requested transmit rate.

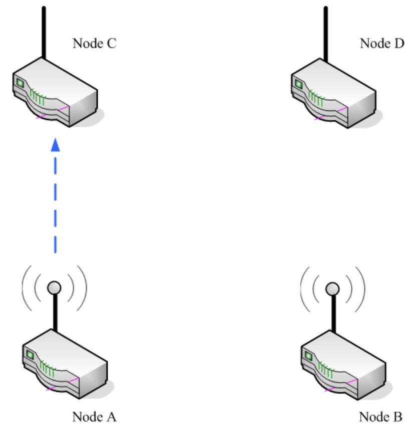
However, if the application is delay tolerant, the performance can be significantly improved if the scheduler exploits the time domain freedom and appropriately assign the link rates, flow rates, and transmit power over consecutive time slots. Consider the example in Figure 3.2, in which appropriate link scheduling improves the capacity. The channel gains in this example are 1 and the peak transmit power limit normalized by the noise power is 10. To maximize the data rate that is concurrently achievable on the links $A \rightarrow C$ and $B \rightarrow D$ in each time slot, both links must transmit at the peak power to overcome the interferences from A to D and B to C. Assuming linear rate function $\phi(SINR) = SINR$, the maximum rate achievable on both link is only $1/1.1$. However if we relax the delay requirement to two slots, the maximum rate concurrently achievable on both links is at least 5. This can be done by alternatively sending data on each link using maximum power. This example demonstrates that time sharing is more efficient then concurrent transmissions in high interference circumstances.

In this chapter, we look into the extreme case when there is no delay restrictions on the data, meaning the requested rates are allowed to met after averaging over infinite time slots. Due to this nature, our solution may not be suitable for the delay sensitive applications.

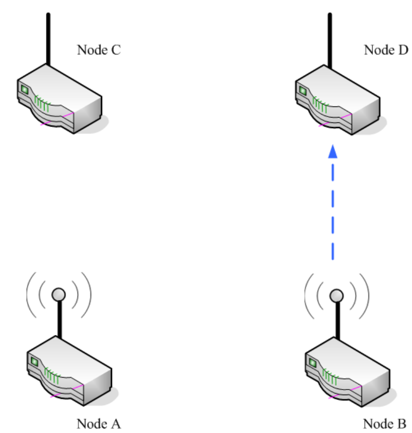
Flows and Routing Conceptually, the links and the nodes in the system can be described by a directed graph $\mathcal{G}(\mathcal{V}, E)$, where $\mathcal{V}$ is the vertex set consisting of all nodes and E is the directed edge set consisting of all links. The capacity of each directed edge is the link rate. A flow is a end-to-end connection representing a stream of bits originating from the same port of one node and destined for the same port of another node. Specifically, each flow is described by a 5-tuple (*source node, source port, destination node, destination port, demand data rate*). For analytical simplicity, we assume that for each ordered node pair there is at



(a) Concurrent Transmissions



(b) Transmission occurs in odd slots.



(c) Transmission occurs in even slots.

Figure 3.1: Transmission Scheduling of Two Parallel Links

most one flow taking place. Nevertheless, the analysis and schemes used in this chapter can be applied to the cases with multiple flows over the same source and destination nodes. Under this assumption, a flow is identified by an ordered pair (s, d) together with the rate demand v, where node s generates the data sent to node d at a rate v. A flow relies on routing over links to accomplish the information transfer. In networks with multiple flows, routings are regarded as multi-commodity flow assignment problem [3, p. 649]. In a static environment, the optimal flow assignment is fixed [17]. However, under time-varying channels, the existence of a static path is not assured. How to assign the flow rate adapting to channel variations is the key question to be answered in this chapter.

As we mentioned above, nodes and links form a directed graph, the information bits from a source node reaches its destination node through the help of other nodes in a store and forward fashion. Each node can be a source node of some flows and the helper node of other flows. As a helper node, it maintains individual queues for each flow that arrives at it. The helper nodes are scheduled to relay the flow data to the next helper nodes. Data of flows are transmitted from one helper node to another until they reach their destinations. We do not exclude the possibility of multiple routes, the data belonging to the same flow can be forwarded to the destination node through different paths. This behavior provides the multi-route diversity.

3.3 Notations and System Model

We consider a wireless network consisting of M nodes, L links and J flows. Each node is equipped with one omni-directional antenna. We label each node with unique index taken on the set $\{1, 2, \ldots, M\}$. The i^{th} link is denoted by $l_i = (a_i, b_i)$, where a_i and b_i are the indices of the transmitter and receiver of the i^{th} link, respectively. The j^{th} flow is denoted by $f_j = (s_j, d_j)$, where s_j and d_j are the indices of the source and the destination nodes of the j^{th} flow, respectively. The rate demand of flow f_j is $v(f_j)$. Please be aware that we slightly abuse the notations of l_i and f_j so that a link or flow can be identified or described either using the index or the ordered node pair.

System time is divided into equally spaced intervals called time slots, which is the basic scheduling time unit. The allocated transmission power and rates over all links in slot n are given in the vectors $\vec{P}^n = [P^n(l_1), P^n(l_2), \ldots, P^n(l_L)]$ and $\vec{R}^n = [R^n(l_1), R^n(l_2), \ldots, R^n(l_L)]$, respectively. From the data sent over the i^{th} link at time n, the amount of those belonging to the j^{th} flow is denoted by $C^n(f_j, l_i)$. The set $\{C^n(f_j, l_i), j = 1, \ldots, J,\ i = 1, \ldots, L\}$ is described by a $j \times i$ matrix $\mathbf{C}^{(n)}$, in which the (j, i)-th element is equal to $C^n(f_j, l_i)$. At the beginning of time slot n $(= 0, 1, 2, \ldots)$, the scheduler specifies the resource allocation in array $\vec{V}^n = [\vec{P}^n, \vec{R}^n, \mathbf{C}^{(n)}]$. The underlying time-varying phenomena, such as mobility and channel variations, are modeled by a stochastic process $\{\xi^n, n \geq 0\}$.

The power loss gain from the transmitter of link l_j to the receiver of link l_i is denoted by $G^n_{l_j, l_i}$, which is a random variable measured by the σ-algebra generated by $\{\xi^k,\ 0 \le k \le n\}$. On link l_i, the receiving end experiences noise power η_{l_i}. The signal to interference and noise ratio (*SINR*) can be evaluated using the formula $\gamma_{l_i}^n = \frac{G_{l_i,l_i}^n P^n(l_i)}{\sum_{j\neq i} G_{l_j,l_i}^n P^n(l_i) + \eta_{l_i}^n}$. For notational simplicity, we denote the set of all outgoing links at node m by $\mathcal{E}(m)$ and the set of all incoming links at node m by $\mathcal{F}(m)$.

Assuming that the maximum instantaneous data rate sent over link l_i is a function $\phi(\gamma)$ of *SINR* γ, under the peak transmission power limit $P_m^{\max}$ to node m, a resource allocation policy $\vec{V}^n$ is feasible if and only if $\vec{P}^n$, $\vec{R}^n$ and $\mathbf{C}^{(n)}$ satisfy the following sets of constraints.

(**Power Constraints**)

$$\begin{aligned} &P^n(l_i) \ge 0 && \text{for } i = 1, \dots, L \\ &\textstyle\sum_{l_i \in \mathcal{E}(m)} P^n(l_i) \le P_m^{\max} && \text{for } m = 1, \dots, M \end{aligned} \tag{3.1}$$

(**Link Rate Constraints**)

$$\begin{aligned} &R^n(l_i) \ge 0 && \text{for } i = 1, \dots, L \\ &R^n(l_i) \le \phi(\gamma_{l_i}) && \text{for } i = 1, \dots, L \end{aligned} \tag{3.2}$$

(**Capacity Constraints**)

In time slot n $(= 0, 1, 2, \dots)$

$$\begin{aligned} &\textstyle\sum_{j=1}^{J} C^n(f_j, l_i) \le R^n(l_i) && , \text{ for } i = 1, \dots, L \\ &C^n(f_j, l_i) \ge 0 && , \text{ for } j = 1, \dots, J, \text{ and } i = 1, \dots, L. \end{aligned} \tag{3.3}$$

For ease of explanation, we define the feasible set $\mathcal{D}(\xi^n)$ as the collection of all arrays $\vec{V}^n = [\vec{P}^n, \vec{R}^n, \mathbf{C}^{(n)}]$ satisfying the power constraints, link rate constraints, and the capacity constrains at time n under the channel condition ξ^n. Note that since the channel process $\{\xi^n\}$ is exogenous[1], control actions have no effect on the set $\mathcal{D}(\xi^n)$, and hence $\mathcal{D}(\xi^m)$ and $\mathcal{D}(\xi^n)$ are decoupled for $m \neq n$. In the following subsection, we examine the framework developed in Chapter 2 by applying it on the power efficient routing problem. As we mentioned earlier, a routing problem can be regard as a multi-commodity flow assignment problem. Its degenerate case, single commodity flow with static links, were studied in [17], in which it points out an important fact regarding the feasibility of flow allocation. That is the average ingressive and egressive rate of a flow at each node must be equal. Note that one may further relax this constraint by allowing the net flow rate entering a node to be smaller than or equal to zero. This relaxation does not alter the answer to the optimization problems since the objective cost function (total average transmit power)increases as excess transmit rates than necessary are scheduled. This argument works for a wireless network scheduling problem if the objective function is the aggregate average transmit power of the entire system. If we assume that the limit of long-term average flow rate exists, the discussion above can be translated into the following mathematical equations.

(**Flow Conservation Constraints**)

[1]A stochastic process is called exogenous if the control actions have no effect on the evolution of the process.

For node m $(= 1, \ldots, M)$ and flow f_j $(j = 1, \ldots, J)$

$$\lim_N \sum_{n=0}^{N-1} \left\{ \sum_{l \in \mathcal{E}(m)} \mathbb{E}(C^n(f_j, l_i)) - \sum_{l \in \mathcal{F}(m)} \mathbb{E}(C^n(f_j, l_i)) \right\} = \begin{cases} v(f_j), & \text{if } m = d_j \\ -v(f_j), & \text{if } m = s_j \\ 0, & \text{otherwise} \end{cases} \quad (3.4)$$

In general, the existence of the limiting long-term average is not assured. Therefore, we resort to the following relaxed constraint. Note this relaxed constraint does not alter the optimal value, since feasible solutions which over qualify the constraints result in excess power consumption. The relaxed version of the flow conservation constraints is of the following form.

(**Relaxed Version of Flow Conservation Constraints**)

For node m $(= 1, \ldots, M)$ and flow f_j $(j = 1, \ldots, J)$

$$\limsup_N \frac{1}{N} \sum_{n=0}^{N-1} \left\{ \sum_{l \in \mathcal{E}(m)} \mathbb{E}(C^n(f_j, l_i)) - \sum_{l \in \mathcal{F}(m)} \mathbb{E}(C^n(f_j, l_i)) \right\} \leq \begin{cases} v(f_j), & \text{if } m = d_j \\ -v(f_j), & \text{if } m = s_j \\ 0, & \text{otherwise} \end{cases} \quad (3.5)$$

3.4 Power Efficient Routing

Based on the discussions in the last section, the power efficient routing problems can be formulated as an optimization problem of the following appearance.

$$\begin{aligned} \text{minimize} \quad & \limsup_{N} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{i=1}^{L} \mathbb{E}(P^n(l_i)) \\ \text{subject to} \quad & \text{Relaxed Flow Conservation Constraints satisfies (3.5)} \\ & \vec{V}^n \in \mathcal{D}(\xi^n) \ \text{ for } n = 0, 1, 2, \ldots \end{aligned} \tag{3.6}$$

The proposed optimization problem fits into the framework of stochastic optimization, so that we can use the recursive algorithm (2.111) in Chapter 2 to design the scheduling policy. It is asymptotically optimal. The connection between (3.6) and stochastic optimization becomes more obvious after we replace the control variable X^n in the stochastic optimization with the aggregate power consumption $\sum_{i=1}^{L} \mathbb{E}(P^n(l_i))$ and the demand control variable Y^n with the terms $\left(\sum_{l_i \in \mathcal{E}(m)} C^n(f_j, l_i) - \sum_{l_i \in \mathcal{F}(m)} C^n(f_j, l_i) + \nu_{mj}\right)$, where

$$\nu_{mj} = \begin{cases} -v(f_j), & \text{if } m = d_j \\ v(f_j), & \text{if } m = s_j \\ 0, & \text{otherwise.} \end{cases} \tag{3.7}$$

Following the recursive algorithm for solving stochastic optimization, in each time slot, the scheduler π^* (given in Definition 2.5 in Chapter 2) solves the following subproblem and makes decisions in accordance with the corresponding minimizers.

$$\min_{\vec{V}^n \in \mathcal{D}(\xi^n)} \sum_{i=1}^{L} P^n(l_i) + \sum_{m=1}^{M} \sum_{j=1}^{J} \beta_{mj} \left(\sum_{l_i \in \mathcal{E}(m)} C^n(f_j, l_i) - \sum_{l_i \in \mathcal{F}(m)} C^n(f_j, l_i) + \nu_{mj} \right) \tag{3.8}$$

To evaluate (3.8), first, we rearrange the terms immediately behind the double summation on the right of (3.8) with respect to $C^n(f_j, l_i)$. The new appearance of the problem is displayed below,

$$\min_{\vec{V}^n \in \mathcal{D}(\xi^n)} \sum_{i=1}^{L} P^n(l_i) + \sum_{j=1}^{J}\sum_{i=1}^{L} \sigma_{ij} C^n(f_j, l_i) \ + \ \sigma_0, \text{ where } \sigma_0 = \sum_{j=1}^{J}\sum_{m=1}^{M} \beta_{mj}\nu_{mj}, \tag{3.9}$$

and σ_{ij} is the new coefficient of variable $C^n(f_j, l_i)$ after the rearrangement. To be clear, σ_{ij} has the following form

$$\sigma_{ij} = \sum_{m=1}^{M} \beta_{mj}(I_{\{l_i \in \mathcal{E}(m)\}} - I_{\{l_i \in \mathcal{F}(m)\}}), \tag{3.10}$$

where $I_{\{A\}}$ is the indicator function of event A. If we fix variables $\vec{P}$ and $\vec{R}$, equation (3.9) can be regarded as a linear programming on $C^n(f_j, l_i)$ with a polytope constraint set formed by the capacity constraints (3.3). Therefore, one can find solutions to this problem at the extreme points of (3.3). Using this observation, the optimal assignments of flow rates on each link in time slot n has the following presentation.

$$C^{*n}(f_j, l_i) = \begin{cases} R^n(l_i) & \text{if } j = \arg\min_{j=1,\ldots,J} \sigma_{ij}, \text{ and } \sigma_{ij} < 0 \\ & \text{(If there is a tie, choose one randomly)} \\ 0 & \text{else.} \end{cases} \tag{3.11}$$

Substituting (3.11) into (3.9) and rearranging the terms according to $R^n(l_i)$, we arrive at the minimization below,

$$\min_{\vec{V}^n \in \mathcal{D}(\xi^n)} \sum_{i=1}^{L} P^n(l_i) + \sum_{i=1}^{L} \alpha_i R^n(l_i) + \sigma_0, \tag{3.12}$$

where α_i is the corresponding coefficient after the rearrangement.

$$\alpha_i = \min(\min_{j=1,\ldots,J} \sigma_{ij}, 0) \tag{3.13}$$

Similarly, if we fix the variable $\vec{P}$, the optimal transmission rate on link l_i has the following form.

$$R^{*n}(l_i) = \begin{cases} \phi(\gamma^n_{l_i}) & \text{if } \alpha_i < 0 \\ 0 & \text{if } \alpha_i \geq 0. \end{cases} \tag{3.14}$$

Lastly, substituting (3.14) into (3.12), and grouping the terms according to $\phi(\gamma^n_{l_i})$, we arrive at the following optimization involving only the transmission power vector,

$$\min_{\vec{V}^n \in \mathcal{D}(\xi^n)} \sum_{i=1}^{L} P^n(l_i) + \sum_{i=1}^{L} \lambda_i \phi(\gamma^n_{l_i}) + \sigma_0,$$

where $\lambda_i = \min(\alpha_i, 0)$.

According to Theorem 2.8, the solution of (3.6) can be obtained through the following steps.

Algorithm 3.1.

At time slot n, the scheduler assigns the transmission power $P^{*n}(l_i)$ for link l_i following the rule

$$\arg \min_{\vec{P}^n \in (3.1)} \left\{ \sum_{i=1}^{L} P^n(l_i) + \sum_{i=1}^{L} \lambda_i \phi \left(\frac{G^n_{l_i,l_i} P^n(l_i)}{\sum_{j \neq i} G^n_{l_j,l_i} P^n(l_i) + \eta^n_{l_i}} \right) \right\},$$

where the minimum is taken over the power constraints (3.1)*. The optimal data rate $R^{*n}(l_i)$ and flow rate $C^{*n}(f_j, l_i)$ can be deduced from (3.14) and (3.11). The dual variables are updated recursively as follows*

$$\beta^{n+1}_{mj} = \beta^n_{mj} + \epsilon \, \Pi_H \left[\sum_{l_i \in \mathcal{E}(m)} C^{*n}(f_j, l_i) - \sum_{l_i \in \mathcal{F}(m)} C^{*n}(f_j, l_i) + \nu_{mj} \right]$$

, where $H = \{\xi \mid 0 \leq \xi_i \leq K_u\}$ contains the optimal dual variables $\vec{\beta}^*$ in its interior.

Note that the selection of appropriate value of K_u is a technical issue. For practical applications, it is sufficient to set K_u to the limit supported by the computing device.

Let $\vec{V}^{*n} = [\vec{P}^{*n}, \vec{R}^{*n}, \mathbf{C}^{(*n)}]$, according to Theorem 2.8, if the process $\{\xi^n,\ n = 0, 1, 2, \ldots\}$ is ϕ-mixing , the long-term average of $\vec{P}^*(k), \vec{R}^*(k)$ and $\vec{C}^*(k)]^\top$ converge asymptotically to the optimal solution of (3.6) as the step size ϵ diminishes to zero. The following theorem summarizes the key results of this chapter, which can be deduced directly from Theorem 2.8.

Theorem 3.1 (Asymptotic Feasibility and Optimality)**.**

1. (Optimality)
$$\limsup_{\epsilon} \limsup_{N} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{i=1}^{L} P^{*n}(l_i) \le P^* \tag{3.15}$$

2. (Feasibility)
$$\limsup_{\epsilon} \limsup_{N} \frac{1}{N} \sum_{n=0}^{N-1} \left\{ \sum_{l \in \mathcal{E}(m)} \mathbb{E}(C^{*n}(f_j, l_i)) - \sum_{l \in \mathcal{F}(m)} \mathbb{E}(C^{*n}(f_j, l_i)) + \nu_{mj} \right\} \le 0$$

In the next section, we examine a numerical example for power efficient routing, assuming the linear rate function $R(\gamma_{l_i}^n) = W' \gamma_{l_i}^n$. The cost function in the evaluation of (3.15) becomes

$$\min_{\vec{P}^n \in (3.1)} \left\{ \sum_{i=1}^{L} P^n(l_i) + \sum_{i=1}^{L} \lambda_i W' \left(\frac{G_{l_i,l_i}^n P^n(l_i)}{\sum_{j \ne i} G_{l_j,l_i}^n P^n(l_i) + \eta_{l_i}^n} \right) \right\},$$

where the minimum is taken over the power constraints (3.1). According to the observation in [14], this objective function is componentwise concave in $P^n(l_i)$, and hence there must exist solutions to the minimization at the extreme points of power constraints (3.1). In other words, the solution of $P^{*n}(l_i)$ is either 0 or $P^{\max}$.

3.5 Numerical example

This example consists of 7 nodes and 8 links. The topology is illustrated in Figure 3.2. There are two flows, each requires a minimum throughput of $\tilde{C}$. Flow 1 originates at node 1 and ends at node 5, it exploits links 1, 3, 4, 5, and 7 to route the traffic. Flow 2 originates at node 3 and is destined to node 7, it use link 2, 3, 4, 6, and 8 to route the traffic. The peak transmit power limit $P_m^{\max}$ is set to 50 watts and the linear rate function factor W' is set to 50 MHz. For each link, we model the channel states as a IID stochastic process. The channel states of different channels are independent. The background noise power ξ_n is the square of a standard normal random variable. The channel gain $G_{l_1 l_2}$ is given by $e/d^2(l_1, l_2)$, where e is an exponential random variable with unit mean, which models the Rayleigh fading. The notation $d(l_1, l_2)$ denotes the distance between the transmitter of link l_1 and the receiver of link l_2. The step size is set to $\epsilon = 0.0005$. To investigate how system performs along with the value of $\tilde{C}$, we gradually increase the value of $\tilde{C}$ from 0.5 to 30 Mbps. The traces of the average flow rates carried on each link are plotted in Figure 3.4. Defining C_{ij} as the long-term average data rate which belongs to flow j and passes link i. We plot the curves of the average flow rate C_{ij} versus the demand flow rate in Figure 3.4. The upper curve in Figure 3.4 actually contains eight overlapped traces of the time averages of $\{C_{11}, C_{22}, C_{31}, C_{32}, C_{41}, C_{42}, C_{51}, C_{62}\}$ and the lower curve contains two overlapped traces of the time averages of $\{C_{71}, C_{82}\}$. Since the topology of the network is symmetric, we only focus on the behavior of flow 1. It is noticed that when the requested throughput is below 5 Mbps, the optimal route of flow 1 is $\{1 \rightarrow 2 \rightarrow 4 \rightarrow 6 \rightarrow 5\}$; as the requested throughput grows, the help from the direct path $\{1 \rightarrow 5\}$ starts taking effect. Although both $\{1 \rightarrow 2 \rightarrow 4 \rightarrow 6 \rightarrow 5\}$ and $\{1 \rightarrow 5\}$ are energy efficient paths, on route $\{1 \rightarrow 2 \rightarrow 4 \rightarrow 6 \rightarrow 5\}$, each link contributes less interference to the system, and the possibility that all of the links are in deep fade at the same time is small. Therefore, the optimal scheduler shall prefer route $\{1 \rightarrow 2 \rightarrow 4 \rightarrow 6 \rightarrow 5\}$ over $\{1 \rightarrow 5\}$. In Figure 3.3, we gather the data from slot 5001 to slot 25000 and plot the average power consumption with respect to $\tilde{C}$. It is evident from Figure 3.4 that when $\tilde{C}$ is small, the scheduler works like a TDMA system. In other words, in every slot, at most one of the links is activated, and the scheduler refrains from transmitting on a link unless that channel is in good condition. Therefore, the total power consumptions increase linearly in the low throughput region. However, as the throughput request $\tilde{C}$ goes up to 20MBps, more links are needed to participate the routing and forwarding in a slot. The total power consumption then go up nonlinearly due to the interferences.

3.6 Conclusions

In this chapter, we have shown how a joint link scheduling, power control, and routing optimization problem over time-varying channels can be solved using stochastic optimization. In addition, we proposed an recursive algorithm which exploits the *CSI* to schedules the transmission. We have proved that the optimality and feasibility of our proposed algorithm are reached asymptotically. From the

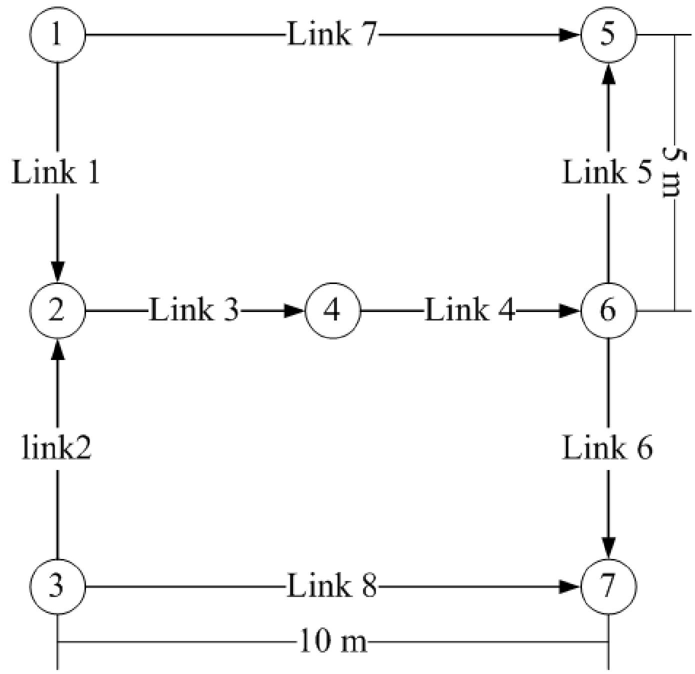


Figure 3.2: Network Topology

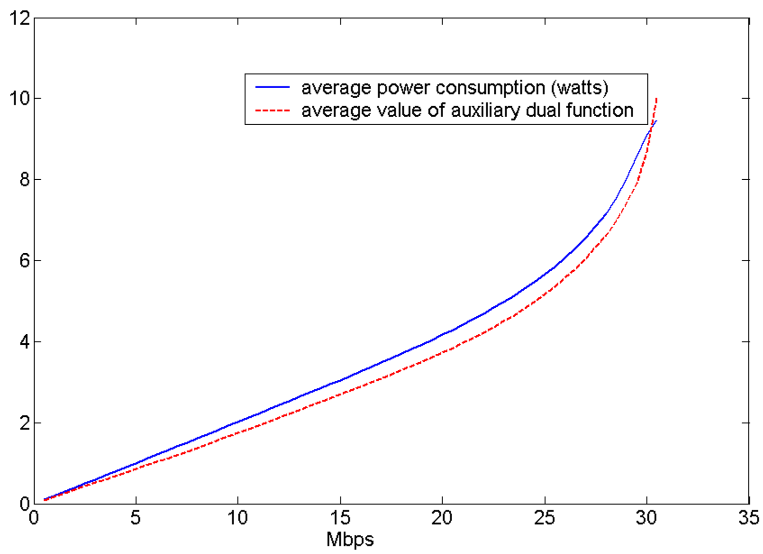


Figure 3.3: Average Transmit Power versus Requested Average Flow Rate

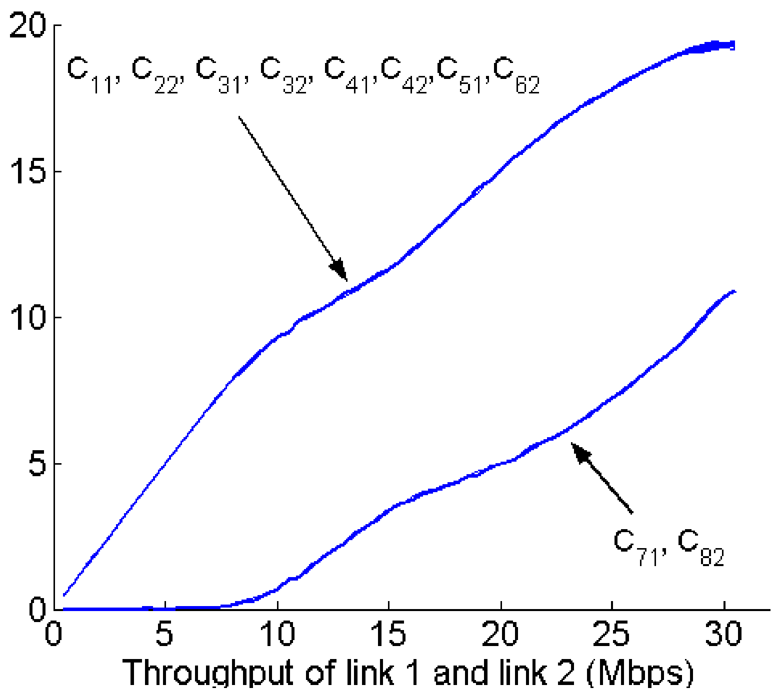


Figure 3.4: Average Link Rates of Data Belonging to Each Flow.

numerical results, we observed that for flows with minimum throughput requirements, to save power consumption the scheduler will opportunistically routes the traffic through links with better channel conditions. This gives us a guideline for designing the system with low rate requirements.

3.7 Acknowledgement

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Chapter 4

Distributed Scheduling, Power Control and Routing for Multi-hop Wireless MIMO Networks

In this chapter, we develop a cross-layer, yet distributed, resource allocation mechanism for multi-hop wireless MIMO networks, which works efficiently over time-varying channels. The design criterion for this work is to use minimum total power to transfer the data for all end-to-end connections at their requested rates. To reduce the complexity of scheduling algorithm growing along with the system size, we decompose the global system into multiple MIMO broadcast subsystems, where the communications taking place within each subsystems are orthogonalized to those occurring outside the subsystem. Under this setting, in every time slot, each subsystem can independently decide the allocation of transmit power, the antenna weights, the transmission rates, as well as the forwarding rules for each end-to-end traffic. The decisions may depend on the local channel state information and certain parameters communicated by its neighboring subsystems. Based on this configuration, we propose an distributed adaptive scheduling algorithm, and its performance is proven to be asymptotically optimal under the decomposition rule. Apart from the scalability, the proposed decomposition allows us to quantify the gains associated with multi-user techniques in a MIMO ad-hoc network against traditional link scheduling solutions. Numerical results are provided to quantify the advantage of MIMO multiuser techniques under various network conditions.

4.1 Introduction

Modern wireless communication systems, such as WCDMA and 1xEV-DO, are capable of dynamically adjusting the operating parameters and the resource allocations. This feature motivated the extensive research of opportunistic scheduling in cellular networks [24] [2] [1] [7] [11] [29] [35] [36], which studied efficient rate allocation adapting to the changes in channel conditions. Meanwhile, given the success of the IEEE 802.11 wireless local area networks, there is growing interest in ad-hoc wireless networks. Since more coordinations are required for ad-hoc wireless networks, the channel aware algorithms developed for cellular networks is not adequate for this use.

Traditional network design philosophy separates the system into decoupled layers in accordance with functionality, where the system optimization is performed in a layer-by-layer fashion. This usually leads to over-design and inefficient use of resources. To operate at system's best performance, an efficient scheduler adjusts the resource across all layers in accordance with the QoS requirements driven by the applications. Since the operations in higher layers are realized through the functionalities provided by lower layers, this optimization may require the cross-layer coordination. The cross-layer dynamic resource allocation problems for wireless networks with *time invariant* channels have been studied in [14] [32] [31]. However, the scope of these research works are limited to stationary nodes. To quantify the impacts of channel variation, we consider a dynamic scheduling approach where channel state information is assumed to be known at the transmitter as well as the receiver.

To compensate for those insufficiencies described above, we consider a cross-layer approach to the question of optimal scheduling in MIMO ad-hoc networks over random time-varying channels. In particular, we seek the optimal routing in terms of total transmit power consumption which sends data from a set of source nodes to their destination nodes at a required average rate. The methodology used for this work is motivated by the achievements in the following three research areas.

1. *Interfering Link Scheduling Problem*

Research on this subject focuses on how to coordinate the transmissions within a wireless network to fulfill the target *point-to-point* transmission rates, using a minimal allocation of power. It originated from the power control problem in cellular networks [5], which later was extended to the context of ad-hoc networks [46] [34].

2. *Minimum Cost Multi-commodity Flow Optimization Problem*

   This research topic is concerned with the efficiency in terms of the resource utilization for moving the information bits from a set of source nodes to their destination nodes at the requested rate. The simpler version of this problem (single commodity flow with constant link capacity) was first investigated by Ford and Fulkerson [17], and later on was extended to a cross-layer context. The minimum cost flow assignment problem in wireless communications, accounting for the interference using omni-antennas, was investigated in [14], and was later extended to include time-varying channels in [33].

3. *Multi-user Communication*

   The third motivating factor for our work is the recognition of the significant improvements offered by multi-user communication techniques [47]. In particular, we are interested in techniques such as dirty paper coding or successive interference cancelation that allow for simultaneous transmissions of data to multiple users while keeping the impact of interference at minimal. Our extension of link constraints to the capacity region ones, in effect, would generalize the work on interfering links to allow for some level of interference cancelation in form of dirty paper coding.

This paper can be viewed as a work combining the above three research topics, and includes the joint consideration of link scheduling, routing and power control. Due to co-channel interference, the feasible resource allocations at nodes are coupled to each other. For instance, all interfering links must cooperate to determine the transmit power so that the target instantaneous rate on each link can be satisfied. Consequently, the complexity of the optimal cross-layer scheduling algorithm grows exponentially with the system size [46] [14]. Therefore, scalability may become an issue in implementing cross-layer scheduling algorithms. In light of this, we take a divide-and-conquer approach to address the growing complexity. Specifically, we divide the system into multiple MIMO broadcast subsystems so that the communications taking place within a MIMO broadcast subsystem do not interfere with those occurring outside the subsystem. This can be achieved by exploiting the orthogonality in the time domain, the frequency domain and/or the code domain. With a MIMO broadcast system, though, we allow for interference cancelation via dirty paper coding. Note that the proposed decomposition may sacrifice system performance although it provides scalability for scheduling in return. In summary, the contribution of our work is two fold: 1) we extend previous work in cross-layer optimization using omni-antennas to a MIMO system where antenna resources are used to enable some level of interference cancelation; 2) we devise a locally centralized scheduling algorithm to coordinate the working of MIMO broadcast subsystems such that the best performance under the decomposition is achieved.

In addition to the main contributions mentioned above, we would like to emphasize our contribution from the analytical aspect. In this chapter, we utilize stochastic optimization, an extension of classical convex optimization, to devise our optimal and localized algorithm. The proposed stochastic optimization is concerned with the time and ensemble average of the stochastic decision processes subject to 1) the instantaneous resource allocation constraints limited by the instantaneous channel condition, and 2) the long-term average QoS requirements. Stochastic optimization was used in previous works to deal with scheduling in wireless networks with dynamic channel changes. However, the proofs provided in these papers are either oversimplified or inaccurate . For example, in [29], the authors studied the opportunistic downlink scheduling problem under time-varying channels, in which they applied the duality technique, and used the stochastic subgradient method for solving the problem. The stochastic subgradient algorithm recursively generates a sequence of dual variables. In every time slot, the scheduler observes the channel state and updates accordingly the dual variable by adding to it a scaled "subgradient-like" vector. The proof of optimality for the algorithm given in [29] depends on an critical assumption: the conditional expectation of this "subgradient-like" vector given the past history forms a subgradient of the dual function. In general, this may or may not be true, because the conditional expectation of the "subgradient-like" vector is, in general, not a subgradient of the dual function. Some degree of dependency between the instantaneous channel condition and the scheduler's previous decisions may remain. The authors of [29] overlooked this subtle effect.

We propose an online recursive algorithm for solving the stochastic optimization problem. Moreover, rigorous mathematical proofs of the optimality and feasibility are provided for our algorithm.

Lastly, we note that in [38] the authors studied a very similar problem to the one we consider here.

To clarify and distinguish between our contributions, we summarize the key aspects below.

- The first major distinction between the two comes from methodology. We tackle the problem directly using classical convex optimization framework and the duality technique. All the proofs in our work follow the convergent proof for the recursive algorithm on the dual variables on the basis of stochastic approximation and the projected differential inclusions. In [38], the author approaches the problem from the aspect of stability, where the Lyapunov type penalty function is used as the cost of violating the QoS requirements. The Lyapunov drift analysis is applied to prove the stability; and the feasibility and optimality follow the result of stability.

  One major advantage of using stochastic approximation over a Lyapunov approach is the generality of the channel and arrival processes. Lyapunov techniques usually rely on the independent renewals over time while the convergence of stochastic approximation algorithm only requires mild mixing conditions.

- Second, unlike the framework developed in [38], which largely relies on the existence of relative interior feasible point, and hence is not applicable for equality constraints, our approach can be generalized to accommodate equality constraints. Consider a simple example of maximizing the aggregate transmit rates on two links with the requirement of equal rates on both links. The equal rate constraint contains no relative interior point. Hence, it is not covered by [38]. However, since our proof does not require the interior point assumption, it is more general.

- Last, in our work, we utilize MIMO and interference cancelation, while, in [38], the author assumes a single antenna system. In particular, we identify the advantages of interference cancelation as well as spatial multiplexing, under various networks topologies and traffic conditions.

This chapter is organized as follows. We begin with the notation and system models in Section 4.2, in which we introduce 1) the constraints on the transmission power and rate for a given set of instantaneous channel conditions, and 2) the constraints over the long-term average. Following the overview of the system operations, in Section 4.3 we formulate the problem as a stochastic optimization. We point out the difficulties in solving the problem in the absence of knowledge of the probability distribution of the channel conditions, and propose a subgradient-type online recursive algorithm. Under the mixing assumption on the channel dynamics, we show that the proposed algorithm asymptotically leads to an optimal solution. In Section 4.4, we present numerical examples demonstrating the performance of the proposed algorithm using a small network of 15 nodes. Finally, we conclude this work with a discussion of promising future research topics in Section 4.5.

4.2 Notation and System Models

4.2.1 Network Description

The system consists of M nodes, each equipped with n_t transmit antennas and n_r receive antennas. We identify each node with a unique integer index taken from the set $\{1, 2, \ldots, M\}$. A MIMO link denotes a logical connection from one node to another over the matrix channel established by the multiple antennas at the transmitter and the receiver, i.e., each MIMO link corresponds to an $n_t \times n_r$ channel matrix. The i^{th} $(i = 1, 2, ..., L)$ link is denoted by $l_i \triangleq (a_i, b_i)$, where a_i and b_i denote the indices of the transmitter and receiver of the i^{th} link, respectively. Using this notation, we are able to enumerate the link using the subscript i, or specify the group of links that originate from the same transmit node (a_i), or end at the same receive node (b_i). For simplicity, the terms "MIMO link" and "link", are used interchangeably.

System time is divided into equally spaced unit intervals called time slots. We use the notation n $(= 0, 1, 2, \ldots)$ to index the slots. At time slot n, the channel matrix from node a to node b is described by a random matrix $\mathbf{H}^{(n)}((a,b)) \in \mathbb{C}^{n_t \times n_r}$, where the (i,j)-th element in $\mathbf{H}^{(n)}((a,b))$ denotes the channel gain from the i^{th} transmit antenna of node a to the j^{th} receive antenna of node b. The channel matrix of link l_i is $\mathbf{H}^{(n)}(l_i)$. Note that we slightly abuse the notation, since $\mathbf{H}^{(n)}(l_i)$ and $\mathbf{H}^{(n)}((a_i,b_i))$ represent the same entity. We assume that $\mathbf{H}^{(n)}(l_i)$ and $\mathbf{H}^{(n)}(l_j)$ are mutually independent if $i \neq j$. Furthermore, we assume for any link l_i, the random sequence of channel matrices $\{\mathbf{H}^{(n)}(l_i),\ n \geq 0\}$ is stationary and uniform mixing [15, p.p. 345] (ϕ-mixing [27, p.p. 356]). The definition of uniform mixing (ϕ-mixing) is given below.

Definition 4.1. *Let $\mathcal{B}_m^n$ be the σ-*algebra *generated by the random variables $\{\xi^m, \xi^{m+1}, \ldots, \xi^n\}$. Define ϕ_k by*

$$\phi_k \triangleq \sup_i \sup_{A \in \mathcal{F}_0^{i+k}, B \in \mathcal{F}_0^i} |\, P\{A|B\} - P\{A\}\,| \tag{4.1}$$

If $\lim_k \phi_k = 0$, then $\{\xi^k\}$ is called a uniform (ϕ-mixing) process.

For notational simplicity, we denote the set of all outgoing links at node m by $\mathcal{E}(m)$ and the set of all incoming links at node m by $\mathcal{F}(m)$. Further, we define the set $\Pi^{(n)}(a) = \{\mathbf{H}^{(n)}(l_i),\ i \in \{1,2,\ldots,L\},\ l_i \in \mathcal{E}(a)\}$ to denote the set of channel matrices of the links that originate at node a, and the global *channel state information* (CSI) is represented by the matrix array $(\Pi^{(n)}(1), \Pi^{(n)}(2), \ldots, \Pi^{(n)}(M))$.

In multi-hop wireless systems, data are transferred from their source node toward their destination nodes through single or multiple routes supported by the links. The abstraction of this end-to-end connection is called a flow. A number of J

flows in the system are assumed. The j^{th} flow is denoted by $f_j \triangleq (s_j, d_j)$, where s_j and d_j indicate the source and destination nodes of the j^{th} flow, respectively. The rate required by flow f_j is denoted by $v(f_j)$. Also, to specify the rate allocation, we denote $C^n(f_j, l_i)$ as the amount of data belonging to flow f_j directly transmitted from node a_i to node b_i at time n. Nodes are equipped with J queues storing information bits of each flow.

At any given time n, and with the full knowledge of the global channel state $(\Pi^{(n)}(1), \ldots, \Pi^{(n)}(M))$, a controller (in general, centralized) is responsible to assign or allocate the following:

System Resource

1. $\Phi^{(n)} = (\mathbf{S}^{(n)}(l_1), \mathbf{S}^{(n)}(l_2), \ldots, \mathbf{S}^{(n)}(l_L))$: The array of positive semi-definite covariance matrices[1] $\mathbf{S}^{(n)}(l_i)$ of the signal vector transmitted over link l_i ($i = 1, \ldots, L$) at time n.

   For ease of explanation, we denote the array of covariance matrices of all links that originate at node a by $\Phi^{(n)}(a) = \{\mathbf{S}^{(n)}(l_i),\ i = 1, \ldots, L,\ l_i \in \mathcal{E}(a)\},\ \ a = 1, \ldots, M$

2. $\vec{P}^n = [P^n(1), P^n(2), \ldots, P^n(M)]$: The array of the transmit powers P^n_m of node m $(= 1, 2, \ldots, M)$ at time n.

   Note that $P^n_m = \sum_{l_i \in \mathcal{E}(m)} \mathrm{Tr}(\mathbf{S}^{(n)}(l_i))$.

3. $\vec{R}^n = [R^n(l_1), R^n(l_2), \ldots, R^n(l_L)]$: The array of the transmit link rates $R^n(l_i) \geq$

[1]The covariance matrix of a signal vector is positive semi-definite.

0 on link l_i at time n.

4. $\mathbf{C}^n = \{C^n(f_j, l_i);\ j = 1, \ldots, J,\ i = 1, \ldots, L\}$: The array of the transmit rate over link l_i belonging to flow f_j at time n.

   Note that $C^n(f_j, l_i) \geq 0$ and $\sum_{j=1}^{J} C^n(f_j, l_i) \leq R^n(l_i)$.

Moreover, the resource allocations described above are subject to the following constraints:

<u>**Resource and QoS Constraints**</u>

C1− (**Physical Layer Constraint I**)

Given channel states $\{\Pi^{(n)}(1), \ldots, \Pi^{(n)}(M)\}$ and covariance matrix allocation $\Phi^{(n)}$, the scheduled transmission rate vector $\vec{R}^n$ is selected from the capacity region

$\mathcal{C}_{\text{MIMO}}(\Pi^{(n)}(1), \ldots, \Pi^{(n)}(M), \Phi^{(n)})$, which is a function of channel states and covariance matrices. We assume that any rate vector in

$\mathcal{C}_{\text{MIMO}}(\Pi^{(n)}(1), \ldots, \Pi^{(n)}(M), \Phi^{(n)})$ can be achieved with negligible bit error rate ($e.g., < 10^{-6}$).

C2− (**Physical Layer Constraint II**)

The peak transmit power of node m is limited to $P_m^{\max}$, that is,

$0 \leq P_m^n \leq P_m^{\max}$.

C3− (**Queue Stability**)

The queue build-up at each relaying node m is stable, i.e., the long-term average of the information rate of flow f_j entering node m equals the long-term average of information bit rate of flow f_j leaving node m. This requirement is satisfied by fulfilling the equation:

$$\lim_N \frac{1}{N} \sum_{n=0}^{N-1} \left\{ \sum_{l \in \mathcal{E}(m)} \mathbb{E}(C^n(f_j, l_i)) - \sum_{l \in \mathcal{F}(m)} \mathbb{E}(C^n(f_j, l_i)) \right\} = 0. \tag{4.2}$$

C4– (**Minimum Flow Rate Constraint**)

Each flow f_j ($j = 1, \ldots, J$) is guaranteed a long-term average rate equal to the target rate $v(f_j)$. That is,

$$\lim_N \frac{1}{N} \sum_{n=0}^{N-1} \left\{ \sum_{l \in \mathcal{E}(m)} \mathbb{E}(C^n(f_j, l_i)) - \sum_{l \in \mathcal{F}(m)} \mathbb{E}(C^n(f_j, l_i)) \right\} = \begin{cases} v(f_j), & \text{if } m = d_j \\ -v(f_j), & \text{if } m = s_j \end{cases} \tag{4.3}$$

Objective: Given a set of control policies that satisfy (**C1**)-(**C4**), we are interested in the one which also minimizes the long-term average transmit power, $\limsup_N \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=1}^{M} \mathbb{E}(P_m^n)$

Relaxation of (**C3**) and (**C4**)

We first note that to minimize the transmit power consumption, a smart controller should not schedule excessive transmit rate than necessary to support the demand. Hence, (**C3**) and (**C4**) can be replaced by their relaxed versions given below:

(**C3′**)

$$\limsup_N \frac{1}{N} \sum_{n=0}^{N-1} \left\{ \sum_{l \in \mathcal{E}(m)} \mathbb{E}(C^n(f_j, l_i)) - \sum_{l \in \mathcal{F}(m)} \mathbb{E}(C^n(f_j, l_i)) \right\} \leq 0, \tag{4.4}$$

and

(**C4′**)

$$\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\left\{\sum_{l\in\mathcal{E}(m)}\mathbb{E}(C^n(f_j,l_i))-\sum_{l\in\mathcal{F}(m)}\mathbb{E}(C^n(f_j,l_i))\right\} \tag{4.5}$$

$$\leq \begin{cases} v(f_j), & \text{if } m=d_j \\ -v(f_j), & \text{if } m=s_j \end{cases} \tag{4.6}$$

Note that we use the *limsup* operator instead of the *lim* operator because the *limsup* of a sequence always exists, but the *limit* does not. Given a sequence of vectors $\{\vec{A}^n=(A_1^n,A_2^n,\ldots,A_r^n)\mid n=0,1,2,\ldots\}$, its *limsup* is defined as:

$$\limsup_n \vec{A}^n \triangleq \left(\limsup_n A_1^n, \limsup_n A_2^n, \ldots, \limsup_n A_r^n\right).$$

4.2.2 Reduction of Complexity: System Decomposition using MIMO techniques

Identifying the necessary and sufficient conditions for (**C1**), in general, is a non-trivial task, as it is closely related to the information theoretic capacity of an ad-hoc networks, which is an open problem. Moreover, as the number of nodes, M, increases, the complexity of any controller will grow and will depend on an ever-growing overhead cost of collecting information about system states, such as channel gain and queue length. Instead, we introduce a more structured model where the capacity region is known, and the scope of information exchange of system states, such as the channel conditions and the queue size, is limited to some locality.

In other words, instead of solving the open problem of the capacity of wireless ad-hoc networks, we take a divide-and-conquer approach. We intentionally partition the system into a number of manageable subsystems, and enforce some rules to operate each subsystem so that the mutual influence between subsystems is negligible. Admittedly, in doing this, we may sacrifice optimality, depending on the choice of system subblocks and the manner in which subblocks are decoupled.

The challenge in identifying and achieving capacity in a given ad-hoc network is in handling interference. One recent simple and elegant capacity result relates to a MIMO broadcast (MIMO-BC) when dirty paper coding is used. In this paper, leveraging the BC, we decompose the system into MIMO-BC subsystems, and assume the use of ad hoc rules to avoid interference between each BC unit. In this way, we partially remove the effect of co-channel interference from different BC subsystems, but keep some degrees of freedom to manage the co-channel interference within each BC subsystem. It will become evident shortly that such a decomposition benefits the scheduler design in reducing the amount of information exchange of the optimal scheduling policy to a level which requires only the cooperation of neighboring nodes.

However, even under such conditions, optimizing the systems at their best performance is still a challenging and interesting topic. Note that the choice of ad-hoc rules via which the MIMO-BC subsystems are decoupled is beyond the scope of this paper.The investigation of tradeoff between different decomposition schemes is an interesting area for future studies.

Ideally, the above MIMO-BC decomposition will satisfy the following:

Assumptions on the Decomposition

(A.1) Given each node a, in each time slot, there exists a set of nodes $\mathcal{N}(a)$ with whom node a forms a MIMO broadcast subsystem in which node a is the transmitter.

(A.2) In time slot n, for node $b \notin \mathcal{N}(a)$, the interference caused by node a at b is negligible. This is achieved by orthagonalizing the transmissions across MIMO-BC subsystems. For mathematical convenience, we can assume that $\mathbf{H}^{(n)}((a, b)) \approx 0$ for $b \notin \mathcal{N}(a)$

(A.3) If node $c \in \mathcal{N}(a) \bigcap \mathcal{N}(b)$, then the transmission of signals from node a and node b are orthogonal at node c.

The MIMO-BC decomposition along with assumptions (A.1)-(A.3) create a grouping of nodes in the network $\{\mathcal{N}(1), \mathcal{N}(2), \ldots, \mathcal{N}(N)\}$. We refer to this decomposition as the network topology in time slot n. Note that we do not exclude the possibility that the decomposition can change dynamically based on the channel state. In our work, we assume that the protocol for accomplishing the decomposition is known by all users, and that it depends only on knowledge of the instantaneous channel states and node positions. Note that this knowledge is only needed locally. In the remainder of this work, we resort to such model and schedule the resources for the given decomposition.

Now given the above decomposition, the nodes and the links in the system form a directed graph $G(V, E)$, where V is the vertex set consisting of all nodes, and E is the set of directed edges consisting of all links representing the ability of a node to participate in a MIMO-BC subsystem. In other word, the set $\mathcal{N}(a)$ appearing in assumptions (A.1)-(A.3) denotes the neighboring nodes of node a on the graph $G(V, E)$.

Now under assumptions (A.1)-(A.3), we can simplify constraint C1 as follows:

C1′− (**Physical Layer Constraint I**)

$(\vec{P}^n, \mathbf{C}^n)$ is said to be feasible under the channel realization $(\Pi^{(n)}(1), \ldots, \Pi^{(n)}(M))$ if and only if for each node m there exists a link rate schedule $\{R^n(l_i) \mid i = 1, \ldots, L;\ l_i \in \mathcal{E}(m)\}$ such that for any node m,

$$\sum_{j=1}^{J} C^n(f_j, l_i) \leq R^n(l_i) \ \ \forall\ l_i \in \mathcal{E}(a),\ i = 1, \ldots, L \tag{4.7}$$

and

$$\{R^n(l_i) \mid i = 1, \ldots, L;\ l_i \in \mathcal{E}(m)\} \in \ \mathcal{C}_{\text{BC}}(P_a^n, \Pi^{(n)}(m)), \tag{4.8}$$

where $\mathcal{C}_{\text{BC}}(P_a^n, \Pi^{(n)}(m))$ is the feasible link rate region of the MIMO gaussian broadcast channels formed by node m and all of the links originating at node m. In theory, the capacity region of the MIMO gaussian broadcast channels [49] can be characterized by the total power P_m^n transmitted on the broadcast channels and the channel states $\Pi^{(n)}(m)$.

Philosophically, our decomposition, along with assumptions (A.1)-(A.3), provides a generalization of link-based abstraction of a network [43] [18] with non-interfering links, in which we allow for multi-user techniques to be integrated in the joint link scheduling, routing and power control scheduler. In other words, the contribution of our work remains at the networking layer. We are attempting to modify network mechanisms of link scheduling, power and routing to take advantage of broadcast capacity achieving techniques such as dirty paper coding, etc.. Even though we do not consider the realization of assumptions (A.1)-(A.3) as the main focus or contribution of our work, we do discuss some possible candidate schemes in Section 4.4.2.

4.3 Problem Formulation

The objective of this work is to develop a locally centralized cross-layer scheduling algorithm which minimizes the average transmit power consumption while maintaining minimum end-to-end throughput request for time-varying channels. Mathematically, the optimization problem described above can be formulated as a constrained dynamic program with an expected average cost criterion [25]. However, by relaxing condition (C1′) to an average constraint, we consider an alternative optimization problem (**P**) whose solution is a lower bound for the original problem. We then provide a distributed recursive algorithm to solve the optimization (**P**). The important property of this algorithm is that at each instant it satisfies C1′, hence is a solution to the original problem. To describe this in detail, we start with the formulation of the optimization (**P**) under conditions (C1′),(C2),(C3′) and (C4′).

Primal problem (P)

$$\begin{aligned}
\text{minimize} \quad & \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=1}^{M}\mathbb{E}(P_m^n) \\
\text{subject to} \quad & \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\left\{\sum_{l\in\mathcal{F}(m)}\mathbb{E}(C^n(f_j,l_i)) - \sum_{l\in\mathcal{E}(m)}\mathbb{E}(C^n(f_j,l_i))\right\} \le \nu_j(m) \\
& \text{for } n = 0,1,2,\ldots \\
& \begin{cases} C^n(f_j,l_i) \ge 0 & \text{, for } j=1,\ldots,J, \text{ and } i=1,\ldots,L \\ \sum_{j=1}^{J} C^n(f_j,l_i) \le R^n(l_i) & \text{, for } i=1,\ldots,L \\ 0 \le P_m^n \le P_m^{\max} & \text{, for } m=1,\ldots,M \\ \{R^n(l_i) \mid i=1,\ldots,L;\ l_i\in\mathcal{E}(m)\} \in\ \mathcal{C}_{\text{BC}}(P_m^n,\Pi^{(n)}(m)), & \end{cases}
\end{aligned} \tag{4.9}$$

where

$$\nu_j(m) = \begin{cases} v(f_j), & \text{if } m = d_j \text{ (Destination node of the } j^{th} \text{ flow)} \\ -v(f_j), & \text{if } m = s_j \text{ (Source node of the } j^{th} \text{ flow)} \\ 0, & \text{otherwise} \end{cases} \tag{4.10}$$

For notational simplicity, we denote the sequence $\{v^n \mid n = 0,1,2,\ldots\}$ by $\langle v^n\rangle$. Note that in general , $\langle P_m^n\rangle$, $\langle C^n(f_j,l_i)\rangle$ and $\langle R^n(l_i)\rangle$ are randomized control sequences. One possible solution to (P) gives the optimal distributions of these random sequences in order to constitute the optimal randomized control rule. Other solutions, such as the one we define here, will use an online scheme to provide non-stationary solutions whose time variations maps to the optimal marginal distributions.

4.3.1 Dual problem

For convenience, we define $\mathcal{D}(\Pi^{(n)}) = \mathcal{D}(\Pi^{(n)}(1),\ldots,\Pi^{(n)}(M))$ as the feasible set jointly determined by the four constraints of (4.9). Note that, for set $\mathcal{D}(\Pi^{(n)})$, the dependency on $(\Pi^{(n)}(1),\ldots,\Pi^{(n)}(M))$ is established through the broadcast capacity region $\mathcal{C}_{\text{BC}}(P_a^n, \Pi^{(n)}(m))$ in ($\mathbf{C1}'$).

Following the standard optimization approach, the Lagrangian of the primal problem ($\mathbf{P}$) is defined below:

$$\begin{aligned} L(\langle\mathbf{C}^n\rangle, \langle\vec{P}^n\rangle, \vec{\theta}) \;\triangleq\; & \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}[\sum_{m=1}^{M} P_m^n] \\ & + \sum_{m=1}^{M}\sum_{j=1}^{J}\theta_j(m)\left\{\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\,[\sum_{l_i\in\mathcal{F}(m)} C^n(f_j,l_i)\right. \\ & \left. - \sum_{l_i\in\mathcal{E}(m)} C^n(f_j,l_i) - \nu_j(m)]\right\}, \end{aligned} \tag{4.11}$$

where $\theta_j(m) \geq 0$ is the dual variable associated with the corresponding flow conservation constraint for flow f_j at node m. The dual function is defined as

$$V(\vec{\theta}) = \min_{(\mathbf{C}^n,\vec{P}^n)\in\mathcal{D}(\Pi^{(n)});\ \forall n} L(\langle\mathbf{C}^n\rangle, \langle\vec{P}^n\rangle, \vec{\theta}), \tag{4.12}$$

which is concave in $\vec{\theta}$ [9, p.p. 592]. In optimization terminology, the maximization of the dual function over the space of the nonnegative dual variables is called the dual problem.

Dual Problem

$$\begin{aligned} &\text{maximize} && V(\vec{\theta}) \\ &\text{subject to} && \vec{\theta} \geq 0. \end{aligned} \tag{4.13}$$

The weak duality theorem ensures that the maximum V^* of (4.13) is less than or equal to the minimum of the primal problem. Although it is not required in the subsequent proofs, we can demonstrate that the primal problem (**P**) is essentially a convex problem. Hence the strong duality theorem [9, p.p. 504, Proposition 5.2.1] ensures that V^*, the maximum of the dual problem, is equal to the minimum of the primal problem. Motivated by this equivalence, we now move on to find the optimal scheduling policy from the dual problem.

Since the dual function is concave, in theory, we can use the projected subgradient method [9, p.p. 610] to solve the dual problem (4.13). The projected subgradient method is an iterative procedure between the two steps of

(a) the evaluation of a subgradient g for the dual function $V(\cdot)$ at $\vec{\theta}^n$, and

(b) the dual variable update

$$\vec{\theta}^{n+1} = \Pi_H[\vec{\theta}^n + \epsilon g], \quad H = \{\vec{\theta}\ | 0 \leq \theta_i \leq K_u\}, \tag{4.14}$$

where $\epsilon > 0$ is called the step size, and K_u is a positive value which is large enough to include the maximizers of $V(\cdot)$ in H. Note that this assumption on K_u is for technical reason in the proof. In reality, K_u can be set to the largest positive number supported by the calculating device, which, in general, should be large enough to qualify the requirement. A subgradient of the dual function $V(\cdot)$ at $\vec{\theta}^k$ can be obtained by solving the minimization in (4.12) [9, Proposition B.25]. Specifically, let the control sequences $\langle \mathbf{C}^{*n}_{\theta^k} \rangle, \langle \vec{P}^{*n}_{\theta^k} \rangle$ be the solution to (4.12) and

$C^{*n}_{\theta^k}(f_j, l_i)$ be the (i, j)-th component of $\mathbf{C}^{*n}_{\theta^k}$. One subgradient of $V(\cdot)$ at θ^k is

$$\vec{g}(\vec{\theta^k}) = (g_{mj}; m = 1, \ldots, M;\ j = 1, \ldots, J) \tag{4.15}$$
$$g_{mj}(\vec{\theta^k}) \triangleq \limsup_N \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\left[\sum_{l_i \in \mathcal{F}(m)} C^{*n}_{\theta^k}(f_j, l_i) - \sum_{l_i \in \mathcal{E}(m)} C^{*n}_{\theta^k}(f_j, l_i) - \nu_j(m)\right].$$

However, we are not able to evaluate g unless we know the distributions of the channel process. To address this issue, instead of dealing with $V(\cdot)$ directly, we consider a subsidiary function $\tilde{V}(\cdot)$ defined as follows:

$$\tilde{V}(\vec{\theta}) \triangleq \limsup_N \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\left\{ \min_{(\mathbf{C}^n, \vec{P}^n) \in \mathcal{D}(\Pi^{(n)})} [\sum_{m=1}^{M} P^n_m] + \sum_{m=1}^{M} \sum_{j=1}^{J} \theta_j(m) [\sum_{l_i \in \mathcal{F}(m)} C^n(f_j, l_i) - \sum_{l_i \in \mathcal{E}(m)} C^n(f_j, l_i) - \nu_j(m)] \right\}$$

We first note that $\tilde{V}(\vec{\theta}) \leq V(\vec{\theta})$. This is because, for any two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$, $\limsup_n a_n + b_n \leq \limsup_n a_n + \limsup_n b_n$. Moreover, if the limit exists (instead of limsup), $\tilde{V}(\vec{\theta}) = V(\vec{\theta})$.

$\tilde{V}(\vec{\theta})$ is an acceptable approximation of $V(\vec{\theta})$. We exploit this fact and solve the following problem:

$$\text{maximize} \quad \tilde{V}(\vec{\theta}) \tag{4.16}$$

$$\text{subject to} \quad \vec{\theta} \geq 0 \tag{4.17}$$

By definition, $\tilde{V}(\theta)$ is a concave function. Danskin's theorem [9, Proposition B.25] provides us a subgradient of $\tilde{V}(\theta)$ at $\vec{\theta}$, which is obtained using the policy π^* which selects the power and flow rate matrix with respect to the value of $\vec{\theta}$ by the rule

$$
\begin{aligned}
(\mathbf{C}^n_{\pi^*\vec{\theta}}, \vec{P}^n_{\pi^*\vec{\theta}}) &= \arg \min_{(\mathbf{C}^n, \vec{P}^n) \in \mathcal{D}(\Pi^{(n)})} [\sum_{m=1}^{M} P^n_m] + \sum_{m=1}^{M} \sum_{j=1}^{J} \theta_j(m) [\sum_{l_i \in \mathcal{F}(m)} C^n(f_j, l_i) \\
&\quad - \sum_{l_i \in \mathcal{E}(m)} C^n(f_j, l_i) - \nu_j(m)].
\end{aligned}
$$

It follows that the subgradient of $\tilde{V}(\cdot)$ at $\vec{\theta}$ is $\tilde{g} = (\tilde{g}_{mj};\ m = 1, \ldots, M;\ \jmath = 1, \ldots, J)$, where

$$
\tilde{g}_{mj}(\theta) \triangleq \limsup_N \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E} \left[\sum_{l_i \in \mathcal{F}(m)} C^n_{\pi^*\vec{\theta}}(f_j, l_i) \; - \sum_{l_i \in \mathcal{E}(m)} C^n_{\pi^*\vec{\theta}}(f_j, l_i) - \nu_j(m) \right]. \tag{4.18}
$$

Note that π^* is a function of the channel states. If the channel state process is stationary and ergodic, we can take away the expectation from (4.18) and write

$$
\begin{aligned}
\tilde{g}_{mj}(\theta) &\overset{(i)}{=} \mathbb{E} \left[\sum_{l_i \in \mathcal{F}(m)} C^n_{\pi^*\vec{\theta}}(f_j, l_i) \; - \sum_{l_i \in \mathcal{E}(m)} C^n_{\pi^*\vec{\theta}}(f_j, l_i) - \nu_j(m) \right] && (4.19) \\
&\overset{(ii)}{=} \lim_N \frac{1}{N} \sum_{n=0}^{N-1} \left[\sum_{l_i \in \mathcal{F}(m)} C^n_{\pi^*\vec{\theta}}(f_j, l_i) \; - \sum_{l_i \in \mathcal{E}(m)} C^n_{\pi^*\vec{\theta}}(f_j, l_i) - \nu_j(m) \right], && (4.20)
\end{aligned}
$$

where equality (i) above follows the stationarity, and equality (ii) above follows the ergodicity. However, even with the ergodicity assumption, we still cannot causally compute the exact subgradient $\tilde{g}$ since the channel realizations are not known in advance. Thereby, subgradient method is not applicable for solving the minimization (4.16) online using the subgradient method. To address this issue, we consider the following iterative algorithm which consists of two steps:

(a′) the dual variable update

$$
\vec{\theta}^{n+1} = \Pi_H[\vec{\theta}^n + \epsilon\ \vec{g}_{\pi^*\vec{\theta}^n}], \quad H = \{\vec{\theta}\ |0 \le \theta_i \le K_u\} \tag{4.21}
$$

and

(b′) the evaluation of the search direction $\vec{g}_{\pi^*\vec{\theta}^n}$

$$g_{\pi^*\vec{\theta}^n}(m,j) \triangleq \left[\sum_{l_i \in \mathcal{F}(m)} C^n_{\pi^*\vec{\theta}^k}(f_j, l_i) \; - \sum_{l_i \in \mathcal{E}(m)} C^n_{\pi^*\vec{\theta}^k}(f_j, l_i) - \nu_j(m) \right]. \quad (4.22)$$

If the channel state process is ϕ-mixing, the local average of $\vec{g}_{\pi^*\vec{\theta}^n}$ approximates the subgradient of $\tilde{V}(\cdot)$ when the step size ϵ is small. Consequently, as the step size goes to zero, the expectation of the sequence $\{\vec{\theta}^n\}$ generated iteratively via (4.21) and (4.22) converges asymptotically in mean to the set of the maximizers of $\tilde{V}(\cdot)$ over the constraint set H . This result is summarized in the following theorem. To emphasize the fact that the dynamics of θ^n depends on the step size ϵ, we subscript the process θ^n with ϵ in the remainder section.

Theorem 4.1. *Let Θ be the set of the maximizers of of $\tilde{V}(\cdot)$ over H. Suppose that the channel process $\{(\Pi^{(n)}(1), \ldots, \Pi^{(n)}(M)) \mid n = 0, 1, 2, \ldots\}$ is stationary and ϕ-mixing. Then we have the following result.*

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \mathbb{E}\{ \operatorname{dist}(\theta^n_\epsilon, \Theta) \} = 0 \quad (4.23)$$

where $\operatorname{dist}(\theta, A)$ *denote the distance from point θ to the set A. In other words, $\forall \delta > 0$, there exists $\hat{\epsilon} > 0$ such that, $\forall \epsilon < \hat{\epsilon}$,*

$$\mathbb{E}\{ \operatorname{dist}(\theta^n_\epsilon, \Theta) \} < \delta \text{ for all but a finite number of times in } n, \quad (4.24)$$

Proof. The proof follows Theorem 2.6. □

Based on this fundamental convergence result, we can further prove that the long-term expected average of the sequence $(\mathbf{C}^n_{\pi^*\vec{\theta}^k}, \vec{P}^n_{\pi^*\vec{\theta}^k})$ generated by the policy π^* converges asymptotically to the optimal solution of the primal problem.

Theorem 4.2. *Assume that the channel process* $\{(\Pi^{(n)}(1), \ldots, \Pi^{(n)}(M)) \mid n = 0, 1, 2, \ldots\}$ *is* ϕ*-mixing and let* Θ *be the set of maximizers of* $\tilde{V}(\vec{\theta})$ *over* $\vec{\theta} \geq 0$. *In addition, the following three conditions are assumed,*

(i) In every time slot k*, the scheduler assigns the control variables under the policy* π^* *with respect to* $\vec{\theta}^k$*, that is,*

$$\begin{aligned}(\mathbf{C}^n_{\pi^*\vec{\theta}^n}, \vec{P}^n_{\pi^*\vec{\theta}^n}) \;&=\; \arg \min_{(\mathbf{C}^n, \vec{P}^n) \in \mathcal{D}(\Pi^{(n)})} [\sum_{m=1}^{M} P^n_m] + \sum_{m=1}^{M}\sum_{j=1}^{J} \theta_j(m)^n [\sum_{l_i \in \mathcal{F}(m)} C^n(f_j, l_i) \\ &\quad - \sum_{l_i \in \mathcal{E}(m)} C^n(f_j, l_i) - \nu_j(m)] \end{aligned} \tag{4.25}$$

(ii) In every time slot n*,* θ^n *is updated according to the recursive algorithm*

$$\theta_j^{(n+1)}(m) = \Pi_H \left[\theta^n_j(m) + \epsilon \; (\sum_{l_i \in \mathcal{F}(m)} C^n_{\pi^*\vec{\theta}^n}(f_j, l_i) - \sum_{l_i \in \mathcal{E}(m)} C^n_{\pi^*\vec{\theta}^n}(f_j, l_i) - \nu_j(m)) \right], \tag{4.26}$$

$$H = \{\theta \mid 0 \leq \theta \leq K_u\},$$

(iii) The set Θ *is contained in the set* $\{\theta \mid 0 \leq \theta_j(m) < K_u\}$.

Under these assumptions, the long-term average of the expectation of sequences $(\mathbf{C}^n_{\pi^*\vec{\theta}^n}, \vec{P}^n_{\pi^*\vec{\theta}^n})$ *converges to the optimal solution asymptotically if the step size* ϵ *is infinitesimally small. That is,*

$$\limsup_{\epsilon} \limsup_{N} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=1}^{M} \mathbb{E}(P^n_{\pi^*\vec{\theta}^n, m}) = P^*$$

$$\limsup_{\epsilon} \limsup_{N} \frac{1}{N} \sum_{n=0}^{N-1} \left\{ \sum_{l \in \mathcal{F}(m)} \mathbb{E}(C^n_{\pi^*\vec{\theta}^n}(f_j, l_i)) - \sum_{l \in \mathcal{E}(m)} \mathbb{E}(C^n_{\pi^*\vec{\theta}^n}(f_j, l_i)) \right\} \leq \nu_j(m),$$

where P^* *is the minimum of the primal problem (**P**).*

Proof. The proof follows Theorem 2.8 in Chapter 2. □

The recursive algorithm (4.26) solves the maximization of $\tilde{V}(\theta)$ asymptotically utilizing the solution to the minimization of (4.25) in every single time slot. To identify these two optimization problems, we call the maximization of $\tilde{V}(\theta)$ (4.16) the *outer optimization* and (4.25) the *inner optimization*. The recursive algorithm (4.26) is also called the *outer optimization loop*.

As it can be seen, the complexity of the algorithm mainly comes from solving the inner optimization. In the following sections, we investigate the solutions to the inner optimization problem dedicated for a MIMO ad-hoc network under the proposed decomposition and assumptions (A.1)-(A.3).

4.3.2 Inner Optimization Problem

Given our proposed decomposition, the inner and outer optimizations can be solved in a distributed manner. To explain in detail, we start with the inner optimization problem, assuming the knowledge of the dual variable $\vec{\theta}$.

After rearranging some terms, the inner optimization can be rewritten as:

$$\text{minimize} \quad \sum_{m=1}^{M} P_m^n + \sum_{m=1}^{M}\sum_{i=1}^{L}\sum_{j=1}^{J}(\theta_j(b_i) - \theta_j(m)) I_{\{l_i \in \mathcal{E}(m)\}} C^n(f_j, l_i) + \sigma_0 \tag{4.27}$$

$$\text{subject to} \quad \mathcal{D}(\Pi^{(n)}(1), \ldots, \Pi^{(n)}(M)) \tag{4.28}$$

where $\sigma_0 = -\sum_{j=1}^{J}\sum_{m=1}^{M}\theta_j(m)\nu_j(m)$. Further, from (A.1)-(A.3), we have that the $\mathcal{D}(\Pi^{(n)}(1), \ldots, \Pi^{(n)}(M))$ can be written as a product form. Therefore, the solution to the inner optimization above can be obtained by solving the M separated optimization subproblems below at node m for each subsystem $\mathcal{N}(m)$.

(P.1) Optimization Subproblem I [Flow, Power and Link Rate]

$$\begin{aligned}
\text{minimize} \quad & P_m^n + \sum_{i=1}^{L}\sum_{j=1}^{J}(\theta_j(b_i) - \theta_j(m)) I_{\{l_i \in \mathcal{E}(m)\}} C^n(f_j, l_i) \\
\text{subject to} \quad & \sum_{j=1}^{J} C^n(f_j, l_i) \leq R^n(l_i), \text{ for } l_i \in \mathcal{E}(m),\ i = 1, \ldots, L && (4.29) \\
& C^n(f_j, l_i) \geq 0 && (4.30) \\
& 0 \leq P_m^n \leq P_m^{\max}, && (4.31) \\
& \{R^n(l_i) \mid i = 1, \ldots, L;\ l_i \in \mathcal{E}(m)\} \in \mathcal{C}_{\text{BC}}(P_m^n, \Pi^{(n)}(m)), && (4.32)
\end{aligned}$$

Note that (P.1) can be solved independently by node m if $\theta_j(b_i)$ and $\theta_j(a_i)$ are available at node m for all links $l_i(= (a_i, b_i))$ belong to $\mathcal{E}(m) \bigcup \mathcal{F}(m)$. In other words, we assume an out-of-band signaling mechanism for exchanging the dual variable $\theta_j(b_i)$ between the transmit node a_i and receive node b_i for each link l_i.

If we fix all the variables in (P.1) except $C^n(f_j, l_i)$ ($i = 1, \ldots, M,\ j = 1, \ldots, L$), problem (P.1) can be viewed as a linear programming on $C^n(f_j, l_i)$ with the linear constraints given by (4.29) and (4.30). These linear constraints form a polyhedral set of $(C^n(f_1, l_i), \ldots, C^n(f_J, l_i))$ with the extreme points

$$\{(R^n(l_i), 0, \ldots, 0),\ (0, R^n(l_i), 0, \ldots, 0),\ \ldots, (0, \ldots, 0, R^n(l_i))\}. \qquad (4.33)$$

In other words, one can apply the extreme point solutions [9, Proposition B.21(c)] to arrive at

(**R1**)

$$C^{*n}(f_j, l_i) = \begin{cases} R^n(l_i) & \text{if } j = \arg\min_{j'} (\theta_{j'}(b_i) - \theta_{j'}(m)) I_{\{l_i \in \mathcal{E}(m)\}} \\ & \text{(If there is a tie, choose the one with the smallest index.)} \\ 0 & \text{otherwise,} \end{cases}$$

for $l_i \in \mathcal{E}(m)$. If $(\theta_{j'}(m) - \theta_{j'}(b_i)) I_{\{l_i \in \mathcal{E}(m)\}} \geq 0$, the trivial solution of $C^{*n}(f_j, l_i)$ $C^{*n}(f_j, l_i) = 0$ is used. Therefore, (4.29) and (4.30) in (P.1) can be replaced by the appropriate versions as functions of $C^{*n}(f_j, l_i)$.

Defining

$$\gamma_{l_i} = \min\{ \min_{j=1,\dots,J} (\theta_j(b_i) - \theta_j(m)) I_{\{l_i \in \mathcal{E}(m)\}}, 0\}, \tag{4.34}$$

and substituting (**R1**) with the flow variables $C^n(f_j, l_i)$, problem (**P.1**) is further reduced to the following optimization concerning only the transmit power and link rate.

(P.2) Optimization Subproblem II [Power and Link Rate]

$$\begin{aligned} \text{minimize} \quad & P_m^n + \sum_{l_i \in \mathcal{E}(m)} \gamma_{l_i} R^n(l_i) \\ \text{subject to} \quad & 0 \leq P_m^n \leq P_m^{\max} \\ & \{R^n(l_i) \mid i = 1, \dots, L;\ l_i \in \mathcal{E}(m)\} \in \mathcal{C}_{\text{BC}}(P_m^n, \Pi^{(n)}(m)). \end{aligned} \tag{4.35}$$

Notice that now we have a power-rate optimization problem in the context of a MIMO Gaussian broadcast channel. The information theoretic result shows that the capacity region of $\mathcal{C}_{\text{BC}}(P_m^n, \Pi^{(n)}(m))$ is convex. Thereby, we can exploit existing numerical convex optimization techniques to solve (**P.2**) efficiently. One common method to solve this problem is to apply the MAC-BC duality of [49].

Please note that the framework developed so far can be extended to more general cases. For example, we can group the nodes so that only one node in each group is allowed to transmit. This is a *time division multiplexing* (TDM) scheme within each group. In this case, the solution to the inner optimization can be obtained by solving the subproblem independently at each node, and selecting from each group the node with the minimal optimal value. We will see the impact of such restrictions in Section 4.4.

Next, we briefly explain the procedure to implement the outer loop in a distributed manner.

4.3.3 Distributed Implementation of the Outer Optimization Loop

Recall that the outer optimization loop updates the dual variables $\vec{\theta}$ as follows:

$$\theta_j^{n+1}(m) = \theta_j^n(m) + \epsilon\, \Pi_H \left[\sum_{l_i \in \mathcal{F}(m)} C^{*n}(f_j, l_i) - \sum_{l_i \in \mathcal{E}(m)} C^{*n}(f_j, l_i) + \nu_j(m) \right] \quad (4.36)$$

where the superscript '$*$' denotes the optimal solution obtained from the inner optimization.

Note that the updates can be accomplished locally at each node a by exchanging the values of $C^{*n}(f_j, l_i)$ with its predecessor and successor nodes in the directed graph $G(E, V)$ which describes the system topology. Previously, we have shown that under the decomposition assumptions (A.1)-(A.3), the inner optimization can be solved in a distributed manner. Therefore, the overall algorithm can be implemented in a distributed way by exchanging with neighboring nodes certain parameters, namely the updated dual variables and the optimal flow rates to the inner optimization .

For completeness, we summarize the proposed online scheduling algorithm as follows.

Algorithm 4.1.

1. At the beginning of the time slot n, each node exchanges the values of its dual variables $\theta_j^n(m),\ j = 1, \ldots, J$ with the nodes in the set $\mathcal{N}(m)$.

2. Based on this information exchange, each node, m independently, solves the optimization subproblem (P.1) via (P.2).

3. The scheduler then takes the solution of (P.1) as its decision of the resource allocation.

4. At the end of the time slot n, node m exchanges the solution of (P.1) with the nodes $\mathcal{N}(m)$, then updates the dual variables according to (4.36).

5. Repeat steps 1 to 5 in the next time slot.

4.4 Numerical Example

In this section, we use the developed algorithms to identify the benefits of multi-user detection in MIMO Ad-hoc networks.

4.4.1 Virtual Geographical Cells

In this subsection, similar to [38], we look into an example in which the geographic plane is virtually partitioned into isolated cells. Exploiting the information of its instantaneous position, each node is aware of which cell it belongs to. This can be done with the aid of the *global positioning system* (GPS). Each cell is assigned a frequency band. Two cells that are adjacent to each other are assigned non-overlapping frequency bands. The nodes in the same cell can transmit using the frequency band assigned to that cell. However, in each cell, the associated frequency band is used exclusively by only one node to send data. In other words, within each cell, the transmissions are scheduled in a time division multiplexing (TDM) fashion. In Section 4.4.2, we briefly discussed the impact of such constraints. We define the neighboring cells of node a as the set consisting of the cell where node a is present and the cells adjacent to it. For each transmit node, its potential one-hop receivers lie in this set of neighboring cells. In other words, for each node a, the set $\mathcal{N}(a)$ consists of all other nodes in the cell to which a belongs and all nodes in its adjacent cells. In practice, all of the transmissions taking place outside a given node's neighboring cells cause interference. Here, though, we assume that with appropriate frequency planning, co-channel interference is negligible compared to the transmitted signal. We assume that the geographic cell partition requires K non-overlapping frequency bands to avoid neighboring cells using the same frequency band[2]. To implement this system, each node is equipped with K transceivers tuned to the K frequency bands. If a node is selected to transmit, it uses one frequency band to send data, and the remaining $(K-1)$ frequency bands to receive the data sent from the $(K-1)$ transmitters in its neighboring cells (except its own cell). If a node is not selected to transmit, it listens to all K frequency bands to receive signals from the transmitters in its neighboring cells.

We remark that, in this approach, we lose frequency efficiency by a factor of K, however, assumptions (A.1)-(A.3) are satisfied so that we can implement our scheduling algorithm in a decentralized manner.

Based on Algorithm 4.1, in each time slot, each node solves the optimization problem (**P.1**) for the MIMO gaussian broadcasts channels consisting of the receivers in $\mathcal{N}(a)$. Since in each cell the scheduling is TDM, an arbitration rule is needed to select the transmitter. The criterion is to minimize the inner optimization. After quickly commutating (**P.1**), each node broadcast its cost value of (**P.1**) to all other nodes in the same cell. In each cell, the node with the minimal solution to (**P.1**) is selected to transmit (this will give the minimum to the inner optimization). If there is a tie, it is broken arbitrarily. We note that this process can be done in a distributed manner with local information exchange in a cell.

[2]For example, for the hexagon cell, this requires 7 frequency bands.

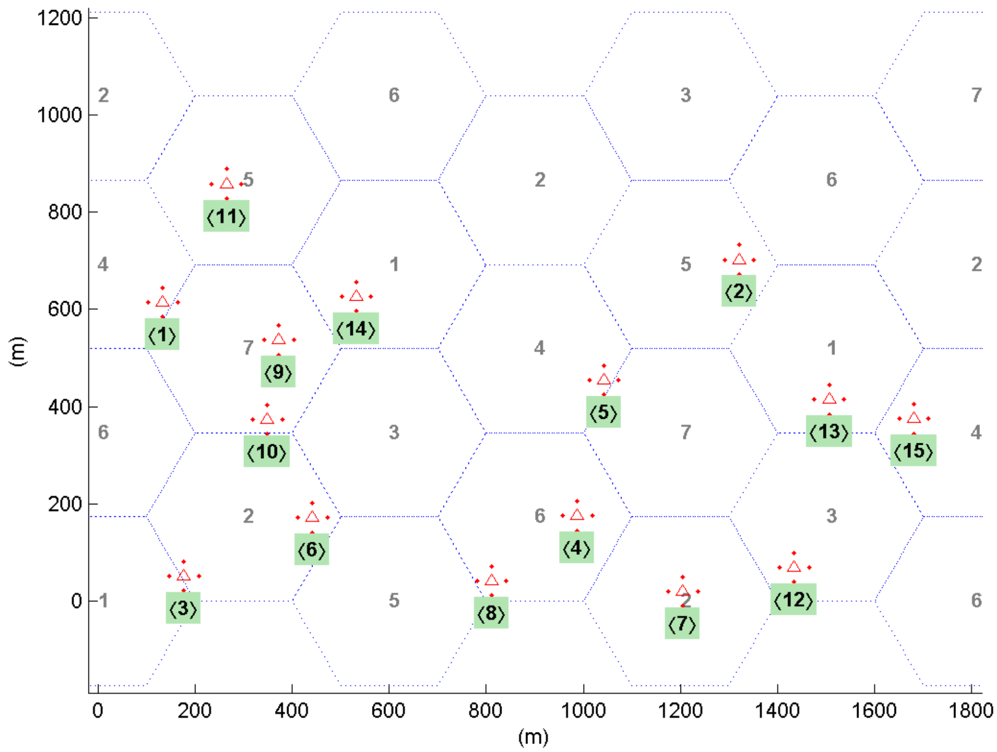

Figure 4.1: System Topology with Partition of Hexagonal Cells.

4.4.2 Numerical Results

We consider a MIMO ad-hoc network with 15 nodes, and the network topology is depicted in Figure 4.1. In the center of each cell we label the frequency band from 1 to 7. The cells labeled with the same number use the same frequency band. The small triangles in Figure 4.1 represent the starting locations of nodes. Nodes are identified by their index. Each node is equipped with 4 antennas represented by 4 dots around the triangle in Figure 4.1. The peak transmit power of each node is limited to 1Watt. We set the pathloss exponent to 4, the available bandwidth to 1MHz, and the carrier frequency to 2.4GHz. We assume additive white gaussian noise with the noise power equal to 5.7e-16Watts. To model the time-varying behavior, we let each node randomly move around the neighboring cells of the cell where it is present when the simulation starts. This model captures variations in the channel gains while keeping the system topology not altering in small fine scales. To model the effect of multipath, we assume each channel is subjected to five multipaths reflected from the randomly positioned scatterers. We consider two network traffic scenarios, each with 4 end-to-end flows. The simulations are run for equal flow demands at 3Mbps, 4Mbps and 5Mbps. Three MIMO schemes are considered in order to capture the impact of multi-user detection:*MVDR beamforming* (BF), point-to-point *spatial multiplexing* (SM), and the *dirty paper coding* (DP) for MIMO broadcast channels.

In the first example, there are four end-to-end flow commodities: node 1 to node 15, node 3 to node 2, node 11 to node 7, and node 9 to node 12. In this setting, the overall traffic has a left to right flow.

The average power consumption under our algorithms are given in Figure 4.2. As the flow rate demand is low, the three MIMO techniques (BF, SM, DP) perform approximately the same. This suggests that TDM is near optimal at low flow rate demands. However, as the flow rate demand increases, spatial multiplexing and dirty paper coding outperform the beamforming, but the spatial multiplexing remains comparable to the dirty paper coding.

In the second example, we reverse the directions of two flows in the first example, i.e. The flows are between nodes 1 and 15, nodes 3 and 2, nodes 7 and

11, and nodes 12 and 9. In this case, two flows send data from the right toward the left of the network while the remaining two send data from the left toward the right. The total power consumption versus the requested throughput under different MIMO techniques is plotted in Figure 4.3. Here we see that at high rates, the SM outperforms the BF. This is because while in this example SM can use up to 4 eign-modes of the MIMO channel to transmit data, BF uses only the best one. Further, Dirty Paper Coding significantly outperforms BF and SM. This is due to the fact that flows traverse across the network in opposite directions. More specifically, while beamforming and spatial multiplexing are limited to only point-to-point link transmission, the dirty paper coding scheme allows transmission to take place in both directions concurrently, resulting in significant improvements over BF and SM.

4.5 Discussion

In this paper, we have developed distributed algorithms which minimize the average transmit power of the system, while maintaining end-to-end flow rates, if feasible. The numerical results shows that the benefit of multi-user transmission schemes is significant when the average flow rate demand is high.

We emphasis that in this work we treat the MIMO broadcast decomposition and neighbor associations as given in order to derive cross-layer optimal solutions. The cost-benefit analysis of such decomposition, on the other hand, remains largely untouched. In other words, optimizing the decomposition scheme

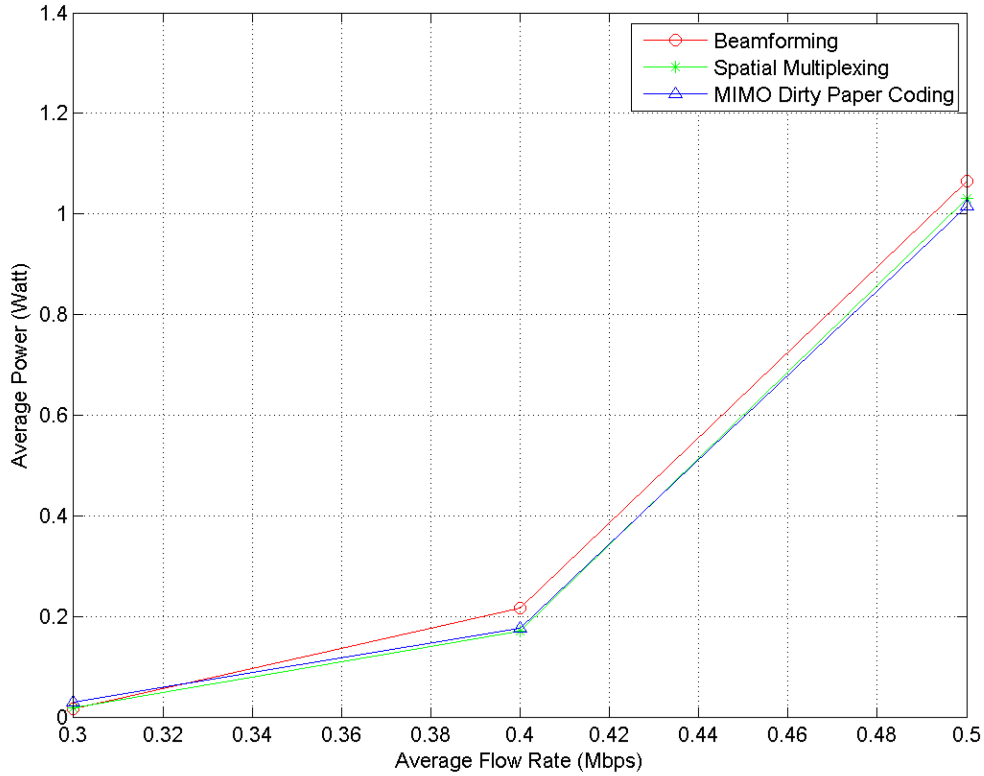

Figure 4.2: Transmit Power Consumptions of Four Flows (Case 1).

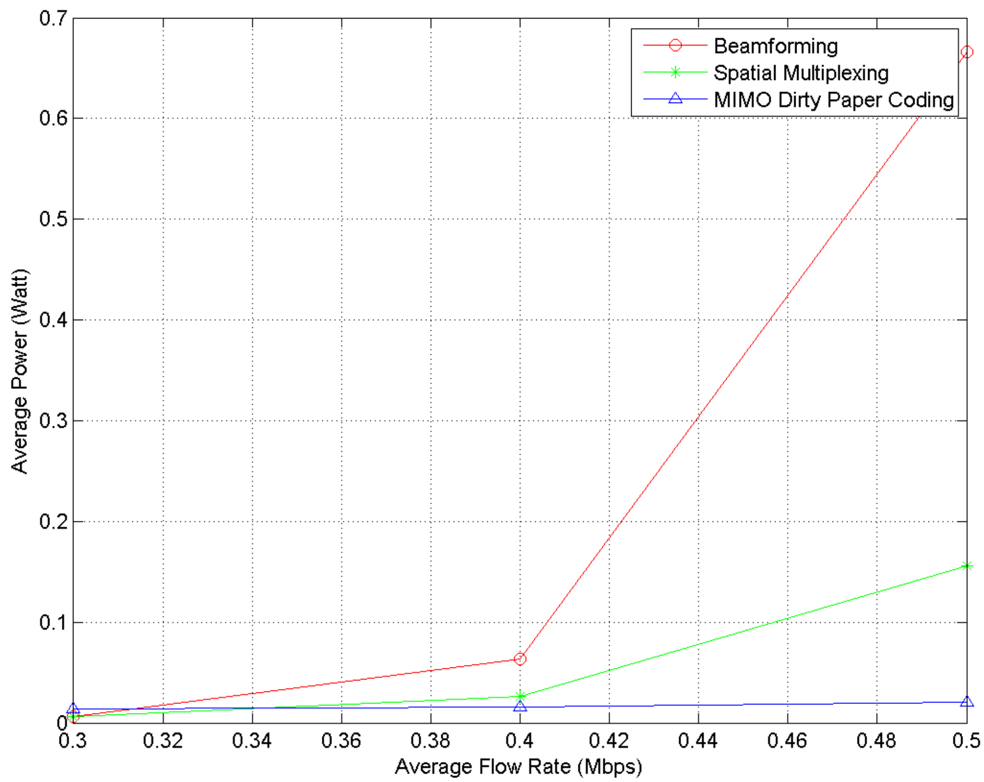

Figure 4.3: Transmit Power Consumptions of Four Flows (Case 2)

itself is an interesting area of future research. It will be interesting to see how to adaptively change the decomposition with respect to the channel states and the queue backlogs.

In our setting, the information flows are passed over links in a decode-store-forward fashion. Incorporating ideas such as cooperative relaying as well as network coding is another interesting and challenging topic for future research.

4.6 Acknowledgement

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Chapter 5

Opportunistic Source Distortion Management in a Multi-user Downlink System

5.1 Introduction

In this chapter, we investigate the joint source distortion management and data transmission scheduling policies in multi-user wireless downlink systems. We assume there are L sources of data prepared to serve the users, and each of them generates sequence of data symbols to be used sequentially by the applications running on the users' equipments. For simplicity, we assume that each data sequence is of infinite length, and each user requests one of the L data sequences to use. After appropriate processing, the requested data are transmitted from the *access point* (AP) to the users over time-varying wireless channels. At the user sides, the received data are stored in the buffer and consumed by the applications at a very high rate in unit of data blocks. Each block consists of a fixed number of source symbols. We assume that data are generated in advance by the sources and stored at AP so that users can pre-fetch excessive amount of data than needed to maintain constant supplies to the application. This provision mitigates the performance degradation resulting from the deficiencies in the user's buffer.

In this work, we target the applications which are delay sensitive but data distortion tolerant. The video and audio applications are the typical examples of this kind. For delay sensitive data, each output of the source is associated with a specific deadline. If the server fails to deliver a symbol to its client by the deadline, the symbol is considered lost. Instead of delay, packet losses can be characterized in terms of the dynamics of reserve buffer at each user. A symbol lost occurs if the number of pre-fetched data in the reserve buffer plus the instantaneous arrival of the data is less than the amount of data consumed by the application in each scheduling interval. Given any channel state, we assume that the achievable transmit rate on each downlink is solely determined by the transmit power. Under appropriate transmit rate and power assignments, the perfect reception of transmitted signals is assumed. Thereby, whenever the channel conditions are so bad that it can not support all of the users supplying the data to the applications at the required rate, the scheduler inevitably has to make trade-off and temporarily stops serving some users who actually need the data immediately. This results in buffer underflow to these users. The buffer underflow event is analogous to packet drops, which results the maximum average distortion to the data. A effective scheduler should keep the frequency of such events as small as possible.

Quantization can be regarded as a lossy process which removes the details of the information inherent in the data to a controllable granularity. To lower down the number of buffer underflows, scheduler can do quantization on data and use fewer bits to represent the quantized data. In this way, the average distortion of data is slightly increased, but, at the same time, the packing efficiency (in terms of data symbols) is improved by stuffing more data symbols into each transmit packet. This in turn reduces the possibility of buffer underflow.

The objective of this work is to seek the optimal joint source distortion and data transmission scheduling rule which balances the tradeoffs between buffer underflows and the controlled data distortion from quantization error. In particular, we opt to design the scheduler which minimizes the maximum weighted average distortion of data among all users. Beside the average distortion, we also want to maintain the stability of users' queues. In this work, stability is defined in terms of the expected size of the queue. If the expected queue size is finite, we call it stable. To cope with these challenging requirements, we proposed an online recursive algorithm that dynamically adjusts its decisions with respect to channel variations. Based on the theoretical results established in Chapter 2, we prove that the algorithm is asymptotically feasible and its performance is asymptotically optimal.

Our work is motivated by the research in the following two fields.

The first research subject concerns joint source and channel coding of "single user" systems, which has been comprehensively studied in video processing society [21] [37]. The main objective of the works in this group is to decide the quantization level applied on the source using the information of channel condition and the reserve buffer at the user so that a set of source data are delivered to the user by a "common deadline". The most common approach used to tackle this type of problems is *stochastic dynamic programming* (SDP). The solutions to SDP rely on the probability distribution of the wireless channels. The are a few weakness in these works, one of them can be seen in the example of video transmission. For video transmission, deadlines are essentially designated on a per video frame basis. The single deadline setting is meaningful only for the set of data segments which make up one video frame. In other words, the works done by [21] [37] are not optimized for scheduling a long sequence of video frames. For applications in which video sources are pre-recorded and stored, the system can benefit from pre-fetching larger amount of data under good channel condition. This feature requires scheduler make decisions across consecutive video frames, and hence is not covered in [21] [37].

The second research subject is the pure video transmission scheduling problem, in which the encoding rate of video is determined in advance and not changeable. The scheduler is only responsible for deciding the transmit rate for each user. The research works under this setting include [39] and [16]. More

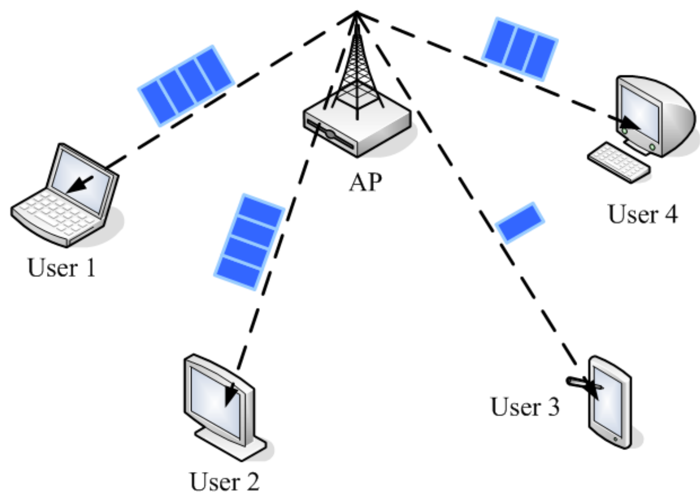

Figure 5.1: Broadcast Downlink System

precisely, following their algorithms, the scheduler sends packets to the user with the fewest pre-fetched data reserves. Later in this paper, we will demonstrate that the dynamics of the pre-fetched data reserves at each user has the notation of dual variables for certain constraints on the arrival rates. The scheduling policy that sends data to the user with the shortest queue is equivalent to the policy that minimizes buffer underflow using TDMA transmission techniques.

Our problem setting is very similar to that of [39] [16] with the enhancement of the ability to control the coding rate of the data. The results given in [39] [16] are heuristic and lack rigorous proof. To accurately estimate the performance, we adopt a new methodology, "stochastic optimization", which provides a solid analytic basis than these given in [39] [16].

To explain in detail, we start with the system model and notation used in this work.

5.2 System Model and Notation

We consider a wireless downlink system with one AP and L users. The system operations are supervised by a control entity called scheduler, which is responsible for allocating resources to maximize the data reconstruction quality and the fairness among all users. The scheduling is performed periodically. Specifically, the scheduler divides time into equally spaced intervals, called slots, and makes control decisions at the beginning of every time slot. For ease of explanation, we index the slots by n $(= 0, 1, 2, \ldots, \infty)$. During the bootstrap steps, each user sends request to the AP for a sequence of source data, the AP then redirect the requests to the server.

Since the AP and the server is connected by the wired infrastructure, the communication taking place between server and AP is much faster and reliable than those over the wireless downlink channels. Hypothetically, one can regard the server and the AP as co-located, and limit the scope of this work to optimizing the transmissions between the AP and the users over the wireless channels.

5.2.1 Wireless Channel and Rate

The first and the most important factor affecting the performance of the system comes from the communication over the wireless medium. The system has a broadcast wireless channel from the AP to the users. The capacity and the reliability of wireless channels depend on the channel impairments such like pathloss, shadowing and noise. Moreover, due to mobility and the movements of surrounding obstructions, the channel gains between the AP and the users can vary dynamically. In our work, we characterize these impairments and their time-varying behaviors by a stochastic process $\{\xi^n,\ n = 0, 1, 2, \ldots\}$. Note that, under this abstraction, the block fading channel model is implicitly assumed, which means that the channel state is fixed within a time slot and could change at the boundaries of each slot. For realistic situations, channel condition actually can vary in the middle of a slot. Yet, for a low mobility system, which is what assumed in the work, the coherence time of a channel is much longer than the size of a slot. Consequently, the relative frequency of the occurrence of channel variation within a slot is negligible.

From the aspect of control mechanism, the channel process ξ^n is an "exogenous" process. More precisely, the statistics of ξ^n conditioned on the past of the system evolution depends only on $\{\xi^m,\ m < n\}$. No control actions can change the behavior of the stochastic process $\{\xi^n\}$.

In time slot n, the scheduler decides transmit rate vector $(X_1^n, \ldots, X_L^n)$, which represents the amount of data sent from AP to the users. Each AP is subject to the peak transmit power limit denoted by $P^{\max}$. The set of feasible transmit rate vectors supported by the system form the capacity region of broadcast channels. The Capacity region under the peak power limit can be characterized by the instantaneous channel states ξ^n [49]. We denote this set by $\mathcal{D}(\xi^n)$. Note that $\mathcal{D}(\xi^n)$ is a set-valued mapping from the space of the channel process ξ^n into a bounded set in $\mathbb{R}_+^L$.

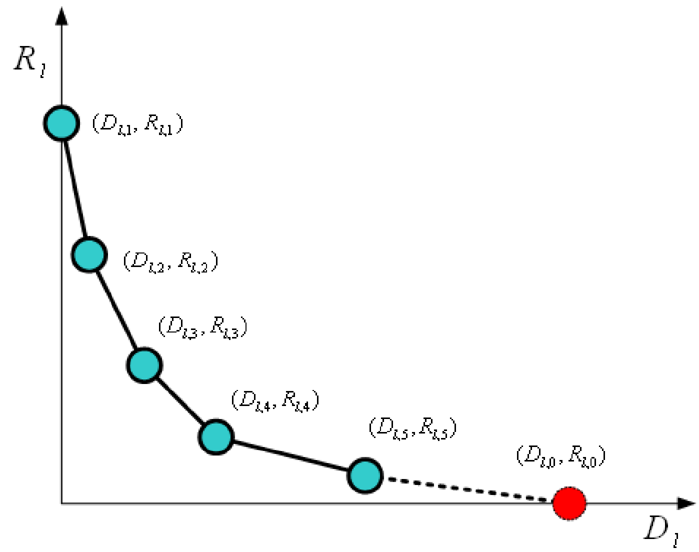

Figure 5.2: Operation Rate-Distortion Curves with $J_l = 5$ Operational Points, the Red Circle Denotes the null block.

5.2.2 Operational Rate-Distortion Relation

The second factor that affects system performance is the way we follow to describe the source data. In our work, we assume a fairly generic stochastic model for the sources. It only requires that the source of data sequence $l(= 1, \ldots, L)$ is a stationary ergodic stochastic process $\{\chi_l^k\}$ with the sequence index[1] k. When the system is running, at user sides, data are consumed in unit of blocks per slot, where each block consists of L_B consecutive symbols generated by the source. Adapting to the channel variations, AP processes the data before sending it over the channel. In the process, a scheduler selects appropriate number of data blocks from the source data sequence and passes the chosen blocks through a quantizer followed by a entropy coder (data compressor). The quantizer parameters, such

[1]Note that the source sequence index is different from the time index. The source sequence index denotes the order that source sequence to be consumed by the application at the user.

as the granularity of quantization levels, are chosen from a finite selections offered by the pre-determined quantizers. In this work, we assume the quantizer is given. We do not opt to find out the optimal source encoder. We only focus on how to dynamically adjust the parameters to optimize the system performance. The basic scheduler structure is given in Figure 5.2.

For a given source, each quantizer parameter corresponds to a particular average *rate distortion* (R-D) pair, called operational R-D point. Operational R-D curve is defined as the convex envelop of a finite set of achievable operation points. With this definition, it follows that operational R-D curve is piecewise linear and convex. Since the operational R-D curve is convex, to design an efficient scheduler, it suffices to consider only the points lying on the operational R-D curve. For user l, we assume its operational R-D curve is defined with J_l operational R-D points denoted by the set

$$\Gamma_l = \{(R_{l,1}, D_{l,1}), \ldots, (R_{l,J_l}, D_{l,J_l})\}\,, \tag{5.1}$$

where

$$R_{l,1} > R_{l,2} > \cdots > R_{l,J_l} > 0, \tag{5.2}$$

$$0 = D_{l,1} < D_{l,2} < \cdots < D_{l,J_l}. \tag{5.3}$$

An example of operational rate-distortion curve is plotted in Figure 5.2.

We define a null block as a processed transmit data block containing no information with respect to the source. Consequently, a null block corresponds to an operational point with maximum average distortion $D_l^{\max}$ and zero information.

We do not include the rate-distortion point of null block in the set Γ_l. Note that null block is a virtual concept, we do not really send null blocks over the channel. In implementation, sending null blocks is equivalent to dropping the same amount of blocks at the AP on purpose. Therefore, sending a null block effectively does not consume any transmit rate. Due to this unique property, we handle null blocks separately from Γ_l. However, to unify the notations, we denote

$$D_{l,0}^n = D_l^{\max}, \; R_{l,0} = 0 \tag{5.4}$$

as the average distortion and rate for the null blocks of user l.

Following the stationary ergodic assumption about the sources, rate distortion curves are invariant [13] with respect to source sequence index. Consequently, no matter how many symbols has been sent in the current slot, it will not alter the rate-distortion property of the sources in later time slots. Note that the stationary ergodicity of the sources is a strong assumption, which may not hold in general. Here we point out one promising systems where approximations can be applied to rationalize this assumption to some extent. For example, for video coding that uses wavelet transform, the distribution of wavelet coefficients in each subband can be approximated by Laplacian distribution [52]. If the power in each subband is approximately constant for a long run of video frames, the distribution of wavelet coefficients can be regarded as identical across these video frames. With high consumption rates, the scheduler tends to send long sequence of data in each transmission, which makes the source coding effective. Under these conditions, the sources of subband wavelet coefficients fit our model. It will become clear shortly that our proposed scheduling algorithm adapts to the variations dynamically. Even R-D curves may vary along the sequence index k, if we update the instantaneous R-D curves to the scheduler frequently to catch the variation of the statistics of source data, we conjecture that our algorithm will achieves some neighborhood of the optimal solution, even though our proof assumes the fixed R-D characteristics.

5.2.3 System Operation

This subsection gives an overview of system operations. We start with the AP (server) side. At the beginning of time slot n, scheduler draws a number of data blocks from the storage of each source elements, and classifies these blocks into multiple groups such that the blocks of the same group is encoded with the same R-D parameter. For user $l(= 1, \ldots, L)$, we denote $A^n_{l,i}$ $(i = 0, \ldots, J_l,)$ as the number of blocks to be encoded using the target average distortion parameter $D^n_{l,i}$. At the output of the source encoders, the scheduler collects $\sum_{i=0}^{J_l} A^n_{l,i}$ encoded blocks for each user $l(= 1, \ldots, L)$, packs the coded data blocks into transmit packets, and sends the packets over the wireless link to user l.

At client sides, user $l = (1, \ldots, L)$ pulls out C_l blocks from its receiving buffer at the end of a time slot, decodes the blocks, and deliver the reconstructed blocks up to the applications. If, at the instant, the number of buffered blocks Q^n_l plus the arrival minus C_l reaches 0, a buffer underflow event occurs. One could regard buffer underflow events as the server's sending null blocks to fill up the deficit in user's pre-fetch buffer. As we explained earlier, a null block corresponds to the largest average distortion. To reduce the frequency of buffer underflow, one can dynamically assign the amount of bits used to expressing each block. The assignment is accomplished by selecting different operational R-D parameters for the encoder at the server.

Following the definition, at time n, the aggregate distortion incurred in the transmission to user l is $\sum_{i=0}^{J_l} A_{l,i}^n D_{l,i}^n$, and the aggregate information bits sent to user l are $\sum_{i=1}^{J_l} A_{l,i}^n R_{l,i}^n$. To account for the priorities among users, we weighted the distortion of each source by the factor ω_l. The weighted aggregate distortion is $\omega_l \sum_{i=0}^{J_l} A_{l,i}^n D_{l,i}^n$.

For notational simplicity, we define

$$A_l^n \triangleq \sum_{i=0}^{J_l} A_{l,i}^n, \tag{5.5}$$

as the aggregated number of blocks scheduled to transmit for user l at time n. The average distortion in a received block of user l at time n is defined by

$$D_l^n \triangleq \frac{\sum_{i=0}^{J_l} A_{l,i}^n D_{l,i}}{A_l^n} \tag{5.6}$$

The average bits to represent a distorted data block of user l at time n is defined as

$$R_l^n = \frac{\sum_{i=1}^{J_l} A_{l,i}^n R_{l,i}.}{A_l^n}. \tag{5.7}$$

5.2.4 The Feasible Set

A scheduler is responsible for efficiently managing system resource, such as the operational R-D parameters of source encoders, the number of blocks sent to each user, and the downlink transmit rate to each user. In this subsection, we introduce the intrinsic constraints that determines the allowable system operation. We first note that the selections of these resources are coupled together. The channel state determines the capacity of downlink transmissions to each user in each time slot; the downlink capacity posts a upper bound on the product of the number of data blocks sent to each user and the average bits to describe these data blocks; the average data blocks sent to user l must be higher than the data consumption rate C_l.

In the following, we express the constraints present above in mathematical equations. Note that, in this work, we focus on the applications working in the regime of high data consumption rate C_l (blocks/slot). Therefore, after normalization, we can regard the arrival process $A_{l,i}^n$ as nonnegative continuous variables. In other words, the fluid model for arrival processes is assumed.

The constraints present below are effective for $l = 1, \ldots, L$ and $n = 1, 2, 3, \ldots$.

Feasibility Constraints

(C.1) The achievable transmit rate vector $(X_1^n, \ldots, X_L^n)$ at time n belongs to the capacity region $\mathcal{D}(\xi^n)$ determined by the channel gain process $\{\xi^n,\ n = 0, 1, 2, \ldots\}$. It is represented by the inclusion

$$(X_1^n, \ldots, X_L^n) \in \mathcal{D}(\xi^n). \tag{5.8}$$

(C.2) The aggregate information bits sent to user l in unit time slot can not exceed

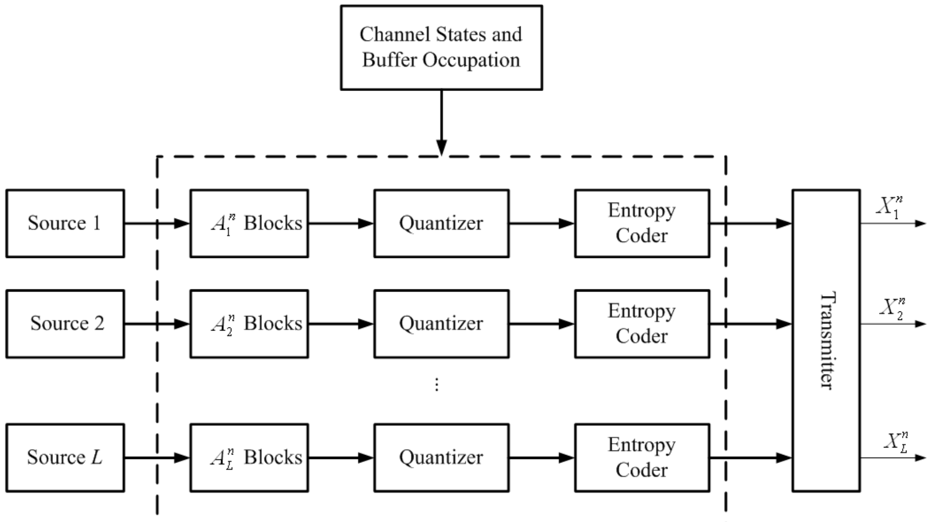

Figure 5.3: Scheduler Structure

the scheduled downlink transmit rate.

$$\sum_{i=1}^{J_l} A_{l,i}^n R_{l,i} \leq X_l^n \tag{5.9}$$

$$0 \leq A_{l,i}^n \text{ for } i = 1, \ldots, J_l. \tag{5.10}$$

(**C.3**) Since block consumption rate is C_l, a smart scheduler shall not send over C_l null blocks in a single time slot. Therefore, we can include the following constraint without affecting the overall performance.

$$0 \leq A_{l,0}^n \leq C_l. \tag{5.11}$$

(**C.4**) The expected long term average of the data blocks sent to user l must be greater than or equal to the consumption rate C_l.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{E}\left[\sum_{i=0}^{J_l} A_{l,i}^n\right] \geq C_l, \tag{5.12}$$

5.3 Problem Formulation

The primary objective of this work is to determine the scheduling policy that minimizes the maximum weighted average distortion among users. In addition, we also expect that the policy can be implemented without consuming infinite buffer spaces.

Intuitively, to minimize the average distortion, the scheduler should not sends excessive blocks than necessary to users. If user's backlog length is long, the user's buffered data can supply the regular consumption for a longer duration, hence the scheduler can wait longer for a better channel condition to transmit the subsequent data blocks. Our strategy is to solve the primary problem first disregarding the queue dynamics, and demonstrate later that, our algorithm ensures that the expectation of buffer dynamics is bounded.

To explain in detail, we begin with the mathematical formulation.

MinMax Distortion Optimization

Recall that our objective is to minimize the maximum average distortion among all users. To reach this goal, we first consider the following stochastic optimization problem constrained to (**C.1**)-(**C.4**),.

(**P.1**)

$$
\begin{aligned}
\text{minimize} \quad & \max_{l=1,\ldots,L} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\omega_l \mathbf{E}\left[\sum_{i=0}^{J_l} A_{l,i}^n D_{l,i}^n\right]}{C_l} \\
\text{subject to} \quad & \text{for } l = 1, \ldots, L \\
& \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{E}\left[\sum_{i=0}^{J_l} A_{l,i}^n\right] \geq C_l, \\
& \text{for } \ n = 0, 1, 2, \ldots \\
& \begin{cases}
\sum_{i=1}^{J_l} A_{l,i}^n R_{l,i} \leq X_l^n \\
0 \leq A_{l,i}^n \text{ for } i = 1, \ldots, J_l \\
0 \leq A_{l,0}^n \leq C_l \\
(X_1^n, \ldots, X_L^n) \in \mathcal{D}(\xi^n)
\end{cases},
\end{aligned}
$$

where the expectation $\mathbb{E}\{\cdot\}$ is taken over the probability space generated by $\{\xi^n,\ n = 0, 1, 2, \ldots\}$ under any selected decision rule. Note that the constraints behind the large brace in (**P.1**) need to be satisfied in every single time slot.

To reach the solution of (**P.1**), we introduce nonnegative subsidiary variables Y, which bounds the long-term average distortions for all users from above. After substituting the objective function with Y, we arrive at a new stochastic optimization problem (**P.2**), in which the solution reaches the same minimum as that of (**P.1**).

<u>**Equivalent Optimization**</u> I

(**P.2**)

$$
\begin{aligned}
&\text{minimize} \quad Y \\
&\text{subject to} \quad \text{for } l = 1, \ldots, L \\
&\qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{E}\left[C_l - \sum_{i=0}^{J_l} A_{l,i}^n \right] \le 0 \\
&\qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{E}\left[\frac{\omega_l \sum_{i=0}^{J_l} A_{l,i}^n D_{l,i}^n}{C_l} - Y \right] \le 0 \\
&\qquad \text{for} \quad n = 0, 1, 2, \ldots \\
&\qquad \begin{cases}
0 \le Y \le Y^{\max} = \max\{\omega_1 D_1^{\max}, \ldots, \omega_L D_L^{\max}\} \\
\sum_{i=1}^{J_l} A_{l,i}^n R_{l,i} \le X_l^n \\
0 \le A_{l,i}^n \text{ for } i = 1, \ldots, J_l \\
0 \le A_{l,0}^n \le C_l \\
(X_1^n, \ldots, X_L^n) \in \mathcal{D}(\xi^n)
\end{cases}
\end{aligned}
$$

We can further replace Y by the long-term average of auxiliary sequence $\{Y^n,\ n = 0, 1, 2, \ldots\}$, this replacement gives another equivalent problem (**P.3**) with the same minimum value. The equivalence comes from the fact that, for every Y, there exists an auxiliary sequence $\{Y^n,\ n = 0, 1, 2, \ldots\}$ with long-term average equal to Y and vice versa.

Equivalent Optimization II

(**P.3**)

$$\text{minimize} \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{E}\left[Y^n\right]$$

$$\text{subject to} \quad \text{for } l = 1, \ldots, L$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{E}\left[C_l - \sum_{i=0}^{J_l} A_{l,i}^n\right] \le 0 \tag{5.13}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{E}\left[\frac{\omega_l \sum_{i=0}^{J_l} A_{l,i}^n D_{l,i}^n}{C_l} - Y^n\right] \le 0 \tag{5.14}$$

$$\text{for} \quad n = 0, 1, 2, \ldots$$

$$\begin{cases} 0 \le Y^n \le Y^{\max} \\ \sum_{i=1}^{J_l} A_{l,i}^n R_{l,i} \le X_l^n \\ 0 \le A_{l,i}^n \text{ for } i = 1, \ldots, J_l \cdot \\ 0 \le A_{l,0}^n \le C_l \\ (X_1^n, \ldots, X_L^n) \in \mathcal{D}(\xi^n) \end{cases} \tag{5.15}$$

The formulation of (**P.3**) is covered by the framework developed for stochastic optimization in Chapter 2.

To solve (**P.3**), we use the duality approach presented in Chapter 2. Let $\alpha = (\alpha_1, \ldots, \alpha_L)$ and $\beta = (\beta_1, \ldots, \beta_L)$ be the dual variables corresponding to the constraints (5.13) and (5.14). Based on the framework developed for stochastic optimization in Chapter 2, the solution of the dual problem can be achieved asymptotically through the recursive algorithm defined below:

Recursive Algorithm for Solving the Dual Problem.

(**RA**)

For $n = 0, 1, 2, \ldots,$

$$\alpha^{n+1} = \Pi_H \left[\alpha^n + \epsilon \mathbf{v}_\alpha^n\right], \quad \beta^{n+1} = \Pi_H \left[\beta^n + \epsilon \mathbf{v}_\beta^n\right], \; H = \{(x_1, \ldots, x_L) | 0 \le x_l \le B_l\},$$

where $|\alpha_l^*| < B_l$ and $|\beta_l^*| < B_l$;

$$\mathbf{v}_\alpha^n = \begin{pmatrix} \frac{\omega_1 \sum_{i=1}^{J_1} A_{1,i}^{n,*} D_{1,i}}{C_1} - Y^{n,*} \\ \vdots \\ \frac{\omega_L \sum_{i=1}^{J_L} A_{L,i}^{n,*} D_{L,i}}{C_L} - Y^{n,*} \end{pmatrix}, \; \mathbf{v}_\beta^n = \begin{pmatrix} C_l - \sum_{i=1}^{J_1} A_{1,i}^{n,*} \\ \vdots \\ C_l - \sum_{i=1}^{J_L} A_{L,i}^{n,*} \end{pmatrix};$$

and $Y^{n,*}$, $A_{l,i}^{n,*}$'s and $X_l^{n,*}$ are one of the minimizers to the following minimization called Inner Optimization.

Inner Optimization

(**P.4**)

$$\begin{aligned} \text{minimize} \quad & Y^n + \sum_{l=1}^{L} \left\{ \alpha_l \left(\frac{\omega_l \sum_{i=0}^{J_l} A_{l,i}^n D_{l,i}^n}{C_l} - Y^n \right) \right. \\ & \left. + \beta_l \left(C_l - \sum_{i=0}^{J_l} A_{l,i}^n\right) \right\} \\ \text{subject to} \quad & 0 \le Y^n \le Y^{\max} \\ & \sum_{i=1}^{J_l} A_{l,i}^n R_{l,i} \le X_l^n \\ & 0 \le A_{l,i}^n \text{ for } i = 1, \ldots, J_l \\ & 0 \le A_{l,0}^n \le C_l \\ & (X_1^n, \ldots, X_L^n) \in \mathcal{D}(\xi^n) \end{aligned}$$

Inner Optimization is an intermediate step required to update the recursion (**RA**). Optimization (**P.4**) can be further decomposed into three decoupled subproblems.

Subproblem I

$$\begin{aligned} \text{minimize} \quad & (1-\sum_{l=1}^{L}\alpha_l)Y^n && (5.16)\\ \text{subject to} \quad & 0 \le Y^n \le Y^{\max} && (5.17)\end{aligned}$$

The solution of Y^n is fairly simple. If $(1-\sum_{l=1}^{L}\alpha_l) > 0$, $Y^{n,*}=0$, otherwise $Y^{n,*}=Y^{\max}$.

Subproblem II

$$\begin{aligned} \text{minimize} \quad & \sum_{l=1}^{L}\left(\frac{\alpha_l\omega_l D_{l,0}^n - C_l\beta_l}{C_l}\right)A_{l,0}^n && (5.18)\\ \text{subject to} \quad & 0 \le A_{l,0}^n \le C_l && (5.19)\end{aligned}$$

The solution of $A_{l,0}^n$ is trivial. If $\frac{\alpha_l\omega_l D_{l,0}^n - C_l\beta_l}{C_l} > 0$, $A_{l,0}^{n,*}=0$, otherwise $A_{l,0}^{n,*}=C_l$.

Subproblem III

(**P.5**)

$$\begin{aligned} \text{minimize} \quad & \sum_{l=1}^{L}\left\{\sum_{i=1}^{J_l}(\frac{\alpha_l\omega_l D_{l,i}^n - C_l\beta_l}{C_l})A_{l,i}^n\right\}\\ \text{subject to} \quad & \sum_{i=1}^{J_l}A_{l,i}^n R_{l,i} \le X_l^n\\ & 0 \le A_{l,i}^n \text{ for } i=1,\dots,J_l\\ & (X_1^n,\dots,X_L^n) \in \mathcal{D}(\xi^n)\end{aligned}$$

The solution to Subproblem (**P.5**) can obtained in two steps. First, we fix transmit rate vector $(X_1^n,\dots,X_L^n)$. Under this condition, (**P.5**) reduces to the linear programming on variables $A_{l,i}^n$. The solution to this degenerated problem is very clear, it is to fill up the capacity of the scheduled rate X_l^n by the blocks encoded with the R-D parameter $(R^n_{l,i^*_l}, D^n_{l,i^*_l})$ based on i^*_l chosen according the rule

$$i^*_l = \arg \min_{i=1,\ldots,J_l} \frac{\alpha_l \omega_l D^n_{l,i} - C_l \beta_l}{C_l R_{l,i}}. \tag{5.20}$$

Specifically, we summarize the solutions to (**P.5**) for the fixed transmit rate vector.

$$(\mathbf{S.1}) \qquad \begin{cases} A^{n,*}_{l,i^*_l} = \frac{R_{l,i^*_l}}{X^n_l} \\ A^{n,*}_{l,i} = 0 \text{ if } i \neq i^*_l. \end{cases} \tag{5.21}$$

After clarifying the appearance of optimal $A^n_{l,i}$, we substitute (**S.1**) into (**P.5**) and solve the optimal transmit rate vector $(X^{n,*}_1, \ldots, X^{n,*}_L)$ from the optimization below.

(**P.6**)

$$\begin{aligned} \text{minimize} \quad & \sum_{l=1}^{L} \left\{ \left(\frac{\alpha_l \omega_l D^n_{l,i^*_l} - C_l \beta_l}{R_{l,i^*_l} C_l}\right) X^n_l \right\} \\ \text{subject to} \quad & (X^n_1, \ldots, X^n_L) \in \mathcal{D}(\xi^n) \end{aligned}$$

Note that (**P.6**) is equivalent to a downlink weighted sum capacity maximization problem. If $\mathcal{D}(\xi^n)$ forms a convex set, this problem can be solved efficiently using numerical techniques. In particular, if the MIMO broadcast Gaussian channel and the capacity achieving channel coding are assumed, one can exploit two very effective methods, *broadcast to multi-access* (BC-MAC) duality [49] and the polymatroid structure [48], to approach and solve problem (**P.6**).

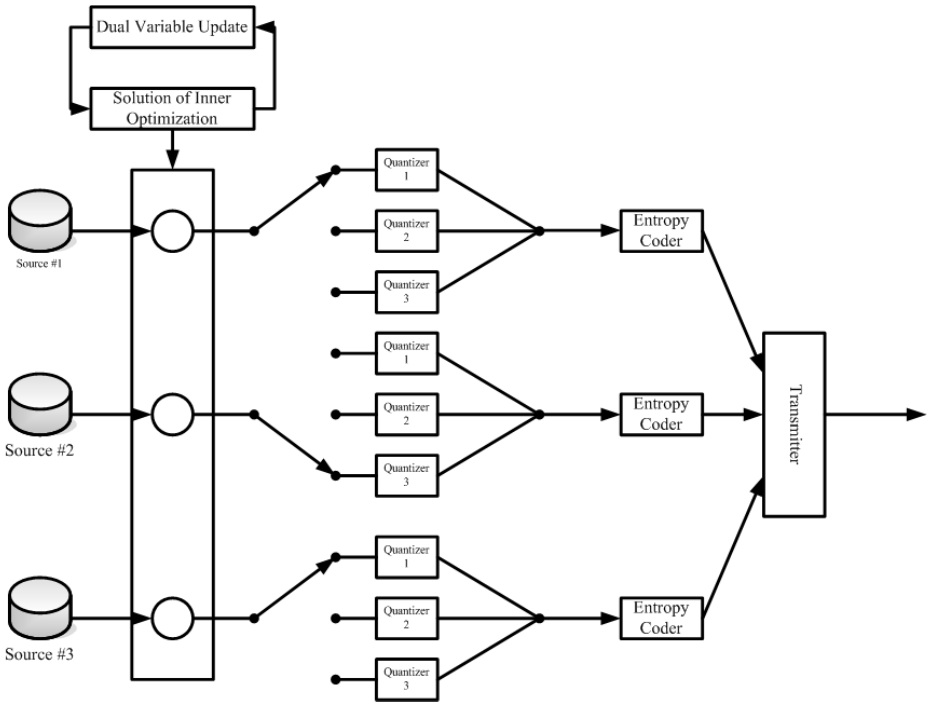

Figure 5.4: Scheduler

5.4 Online Scheduling Algorithm

At our current position, we understand the necessary steps to solve the dual problem of (**P.1**). Through Theorem 2.8 (see Chapter 2), it is justified that the sequences $\{A_{l,i}^{n,*},\ n = 0, 1, 2, \ldots\}$ and $\{X_l^{n,*},\ n = 0, 1, 2, \ldots\}$ obtained from solving subproblem I-III converge asymptotically to solution of (**P.1**) as step size ϵ diminishes to zero.

We note that, however, the algorithm based on these minimizers ($\{A_{l,i}^{n,*},\ n = 0, 1, 2, \ldots\}$ and $\{X_l^{n,*},\ n = 0, 1, 2, \ldots\}$) does not take the buffer underflow event into account. In other words, (**P.1**) does not include the distortion caused by buffer underflow in the objective function. Therefore, additional processing and modification on $\{A_{l,i}^{n,*},\ n = 0, 1, 2, \ldots\}$ and $\{X_l^{n,*},\ n = 0, 1, 2, \ldots\}$ is necessary before completing our final algorithm. To explain in detail, we emphasis that for the recursive algorithm (5.16) its reflection term from above $\check{Z}_l^n \geq 0$ and its reflection term from below $\hat{Z}_l^n \geq 0$ can be defined by rewriting

$$\beta_l^{n+1} = \beta_l^n + \epsilon(C_l - A_l^{n,*} - \check{Z}_l^n + \hat{Z}_l^n). \tag{5.22}$$

Note that to maintain the same dynamics of β_l^n, among the $A_l^{n,*}$ blocks derived from (**P.5**), we only need to send those blocks encoded using the $(A_l^{n,*} - \hat{Z}_l^n)$ lowest average distortion. For notational simplicity, we define

$$\tilde{A}_l^{n,*} \triangleq A_l^{n,*} - \hat{Z}_l^n, \tag{5.23}$$

which serves as our new scheduling decision for the number of scheduled blocks. Since the evolution of β_l^n is unchanged with the new transmit block $\tilde{A}_l^{n,*}$, the optimality to (**P.1**) is preserved. Moreover, this new assignment $\tilde{A}_l^{n,*}$ does not affect the feasibility of (**C.1**)-(**C.3**), which can be easily examined through the definition. In particular, following the same proof of feasibility for stochastic optimization in Chapter 2, the new scheduled data blocks $A_l^{\hat{n},*}$ satisfies the constraint (**C.4**). With this in mind, we rewrite (5.22) into the the equation

$$\beta_l^{n+1} = \beta_l^n + \epsilon(C_l - \tilde{A}_l^{n,*} - \check{Z}_l^n). \tag{5.24}$$

and its projected form

$$\beta_l^{n+1} = \Pi_{[0\ B_l]}\left[\beta_l^n + \epsilon(C_l - \tilde{A}_l^{n,*} - \check{Z}_l^n)\right]. \tag{5.25}$$

Note that the reflection term $\check{Z}_l^n$ is an important variable used to setup a bound on the block loss caused by the buffer underflow.

5.5 Queue Dynamics

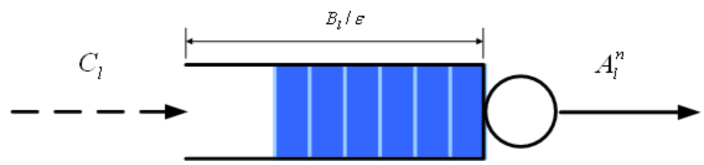

Figure 5.5: Inverse Queue: buffer size limited to B_l/ϵ

To investigate the net effect of buffer underflow to our algorithm in depth, we need to understand the evolution of the queue dynamics. We define $\{\hat{Q}_l^n,\ n = 0, 1, 2, \ldots\}$ as the queueing process with constant arrivals C_l and departure process $\{\tilde{A}_l^{n,*},\ n = 0, 1, 2, \ldots\}$. Since the role of the arrivals $\tilde{A}_l^{n,*}$ and departures C_l in this fictitious queuing system are switched relative to the real system, we name $\hat{Q}_l^n$ "inverse queue". The evolution of inverse queue is governed by the recursion:

$$\hat{Q}_l^{n+1} = \Pi_{[0,B_l/\epsilon]}\left[\hat{Q}_l^n + C_l - \tilde{A}_l^{n,*}\right], \tag{5.26}$$

where B_l/ϵ can be served as one upper bound on the inverse queue size.

Comparing the recursion of $\{\hat{Q}_l^n\}$ with that of $\{\beta_l^n\}$ (5.25), we induce the following important relation between β_l^n and $\hat{Q}_l^n$:

$$\beta_l^n = \epsilon\hat{Q}_l^n, \tag{5.27}$$

which implies that β_l^n has a notion of scaled inverse queue length. One important question to ask is how do we relate $\{\beta_l^n\}$ or $\{\hat{Q}_l^n\}$ with the queue dynamics $\{Q_l^n\}$ at each user? The answer to this question is based on the observation that the evolution of the queue size Q_l^n of user l obeys the following rule.

$$Q_l^{n+1} = [Q_l^n + \tilde{A}_l^{n,*} - C_l]^+. \tag{5.28}$$

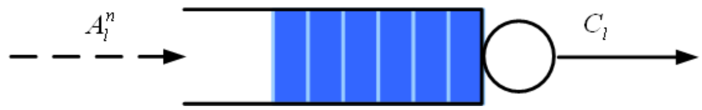

Figure 5.6: Queuing Structure

There are two important facts referring to Q_l^n.

- First, the boundedness of the queue, $Q_l^n \leq B_l/\epsilon$ (when $Q_l^0 = 0$).
- Second, the average block loss due to the buffer underflow is negligible as the step size ϵ goes to 0.

We start with an outline of the proof for the boundedness. Based on the definition of $\tilde{A}_l^{n,*}$, we observed that Q_l^n increases strictly as $\hat{Q}_l^n$ decreases strictly. If there exists n_1 such that $Q_l^{n_1} \geq B_l/\epsilon+\delta$ for some $\delta > 0$, then we can find out $n_2 < n_1$ such that n_2 is the last time before n_1 that $Q_l^{n_2} \leq \delta/2$. This implies that $Q_l^n < B_l/\epsilon$ for $n_2 \leq n \leq n_1$, as a result, the following result should hold: $Q_l^{n_1} - Q_l^{n_2} = \hat{Q}_l^{n_2} - \hat{Q}_l^{n_1} \geq B_l/\epsilon + \delta/2$. However, this cause a contradiction, hence $Q_l^n \leq B_l/\epsilon$.

Inspecting the recursions dominating the evolutions of Q_l^n and $\hat{Q}_l^n$, we notice that, for the dynamics of Q_l^n, the buffer underflow occurs *only when either the inverse queue* $\hat{Q}_l^n$ *hits a new peak or it reaches its upper bound* B_l. Further, based on this fact, we claim that the average block loss due to the buffer underflow is negligible. An outline of the proof is described below.

A reasonable bound on the block loss from the buffer underflow is determined by the following two facts.

- The aggregate block loss due to $\hat{Q}_l^n$'s trace hitting a new peak is bounded by

B_l.

- When $\hat{Q}_l^n$ hits the upper bounded B_l, the block loss is less than the reflection term $\check{Z}_l^n$.

Therefore, for user l, the total loss of blocks by time n due to the buffer underflows is smaller than or equal to $B_l/\epsilon + \sum_{k=0}^{n} \check{Z}_l^k$. By Theorem 2.7, the expected average blocks loss results from the buffer underflow events converges to zero asymptotically. This is verified through the equation below.

$$\limsup_{\epsilon} \lim_{N\to 0} \frac{B_l}{N\epsilon} + \limsup_{\epsilon} \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\{\check{Z}_l^n\} = 0 \qquad (5.29)$$

5.6 Numerical Example

In this section, we use a simple example to examine the performance and behavior of our algorithm. We consider a 10 user system with the rate distortion curves given in Figure 5.7. In the setup of the example, the maximum average distortion $D_l^{\max}$ is normalized to 1 by presetting the weights to $\omega_l = 1/D_l^{\max}$. To model the time-varying behavior of the channel, we use a 10 state Markov to describe the channel condition. The transition matrix of the channel state is given in (5.30), where the (i, j) entry of the matrix is the transition probability from channel gain $10^{0.1i}$ to $10^{0.1j}$. The channel gain is normalized by the noise power, and the peak transmit power is limited to 1 Watt. The target data consumption rate at each user is normalized to $C_l = 1$.

We assume that each user is allocated a dedicated subcarrier. Therefore,

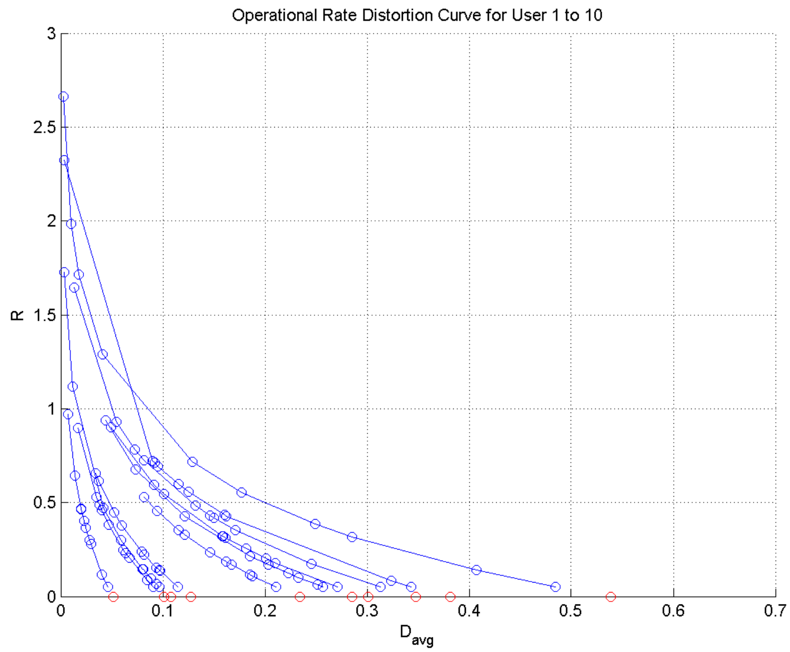

Figure 5.7: Operational Rate Distortion Curves

the downlinks are equivalent to a set of parallel gaussian channels. We perform the simulation for $30,000$ slots. Based on the empirical result, the algorithm converges in about $1,000$ slots. The queue dynamics of users 1 to 10 are plotted in Figure 5.9 to 5.13. In this setting, the weighted long-term average distortion is 0.2066 for all users.

From the result, we are particularly interested in the dynamics of the buffer underflow events. From Figure 5.9 to Figure 5.13, we can see that, except certain initial steps, the queue dynamics of each user rarely touches the zero at all, and its path oscillates around some constant level asymptotically. These phenomena suggest that our algorithm actually keeps very low frequency of buffer underflow occupancies. This agrees with our analytical results.

$$
\begin{pmatrix}
0.1822 & 0.1960 & 0.0020 & 0.0158 & 0.0297 & 0.1327 & 0.1465 & 0.1010 & 0.1149 & 0.0792 \\
0.1941 & 0.1584 & 0.0139 & 0.0277 & 0.0317 & 0.1446 & 0.1089 & 0.1129 & 0.1267 & 0.0812 \\
0.0079 & 0.1604 & 0.1743 & 0.0396 & 0.0436 & 0.1069 & 0.1109 & 0.1248 & 0.1386 & 0.0931 \\
0.1683 & 0.1723 & 0.0376 & 0.0416 & 0.0059 & 0.1188 & 0.1228 & 0.1366 & 0.1406 & 0.0554 \\
0.1703 & 0.1842 & 0.0495 & 0.0040 & 0.0178 & 0.1208 & 0.1347 & 0.1485 & 0.1030 & 0.0673 \\
0.0337 & 0.0475 & 0.1505 & 0.1644 & 0.1782 & 0.0832 & 0.0970 & 0.0515 & 0.0653 & 0.1287 \\
0.0455 & 0.0099 & 0.1624 & 0.1762 & 0.1802 & 0.0950 & 0.0594 & 0.0634 & 0.0772 & 0.1307 \\
0.1564 & 0.0119 & 0.0257 & 0.1881 & 0.1921 & 0.0574 & 0.0614 & 0.0752 & 0.0891 & 0.1426 \\
0.0198 & 0.0238 & 0.1861 & 0.1901 & 0.1545 & 0.0693 & 0.0733 & 0.0871 & 0.0911 & 0.1050 \\
0.0218 & 0.0356 & 0.1980 & 0.1525 & 0.1663 & 0.0713 & 0.0851 & 0.0990 & 0.0535 & 0.1168
\end{pmatrix}
\tag{5.30}
$$

Figure 5.8: Transition Matrix of the Channel State Process

5.7 Conclusions

In this chapter, we developed an online scheduling algorithm which minimizes the maximum long-term average distortion among users in a wireless downlink system. We demonstrated that the proposed algorithm converges asymptotically to the optimal solution. Moreover, we provided analytical and numerical results to point out the fact that, under our algorithm, the distortion (or data loss) due to the buffer underflow is negligible in the average. Note that the current version of our analysis applies to the source with fix rate-distortion behavior. In the following works referring to this paper, we will work towards the extensions

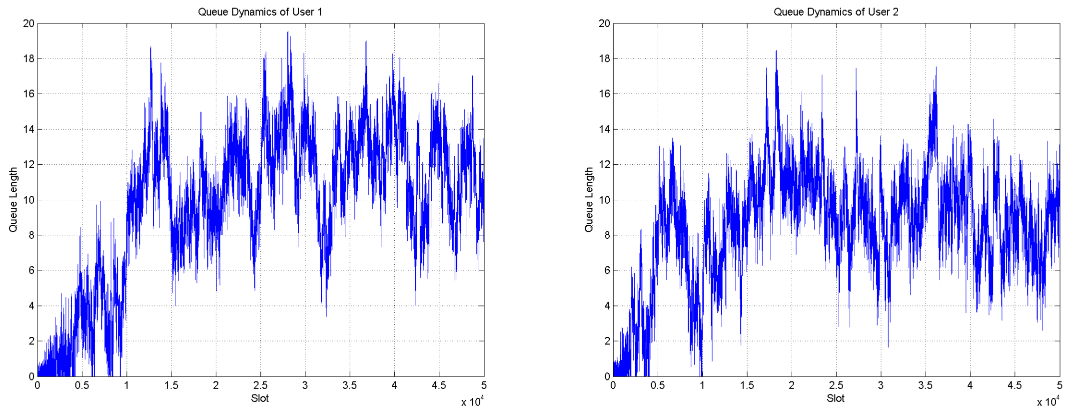

Figure 5.9: Queue Dynamics of User 1 and 2

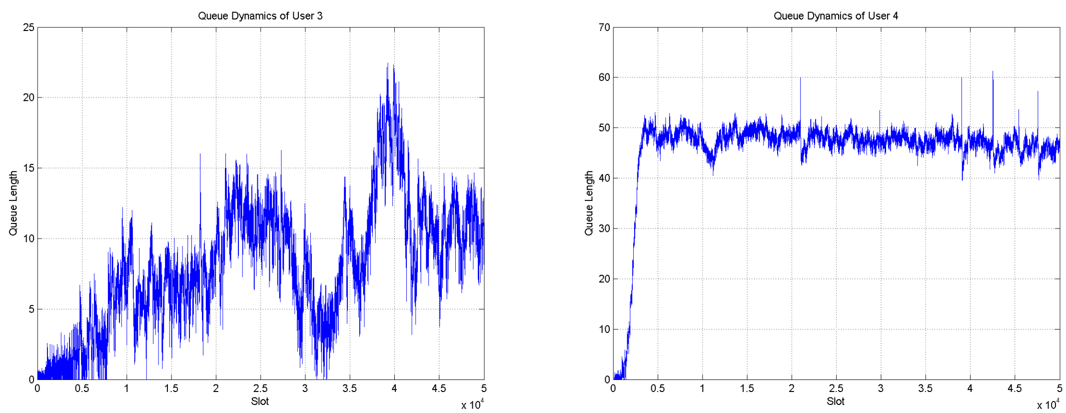

Figure 5.10: Queue Dynamics of User 3 and 4

of our current framework to include time-varying R-D characteristics and more general source signal models into discussion. It is our conjecture that even the source process are not stationary ergodic, the performance of our algorithm would be bounded within some neighborhood of the optimal solution.

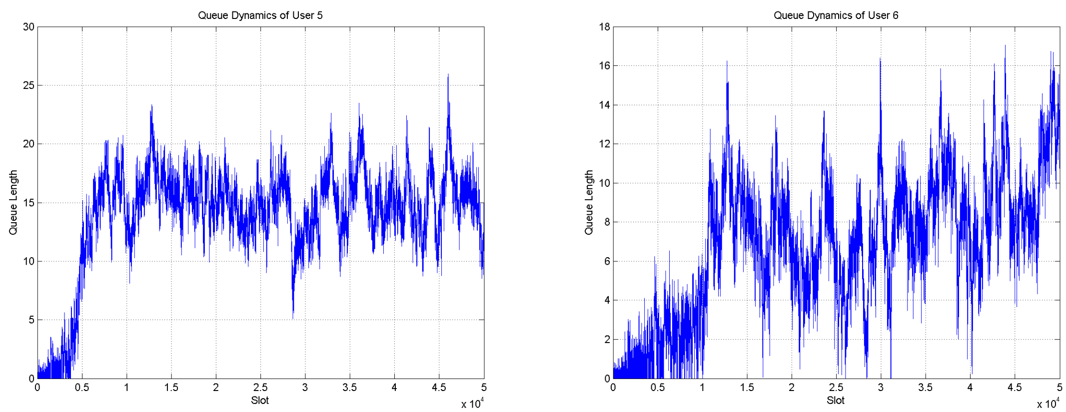

Figure 5.11: Queue Dynamics of User 5 and 6

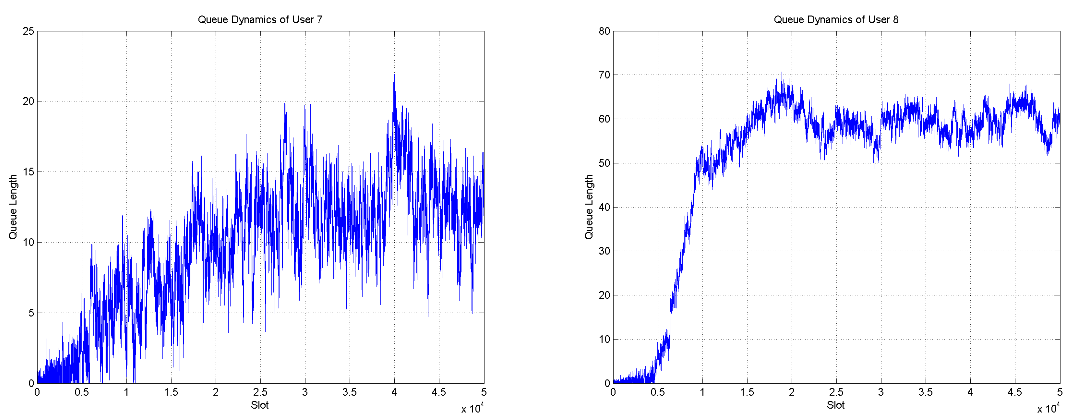

Figure 5.12: Queue Dynamics of User 7 and 8

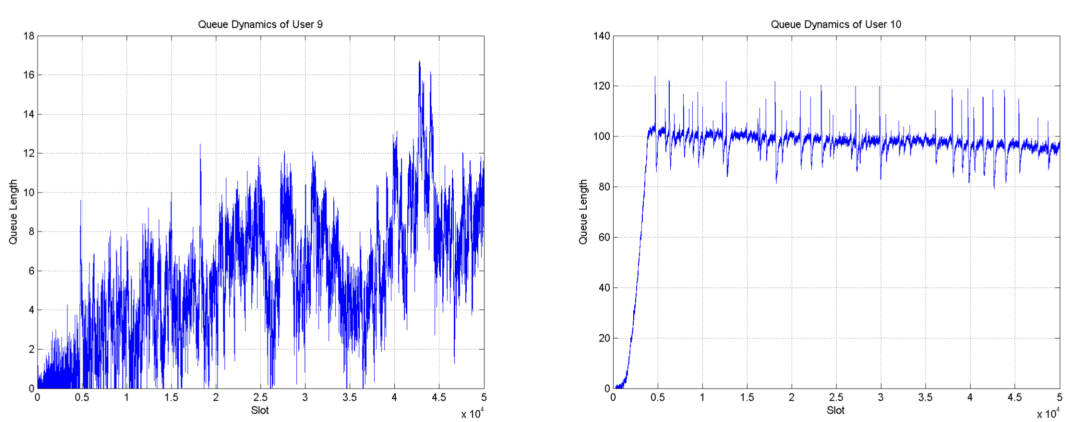

Figure 5.13: Queue Dynamics of User 9 and 10

Chapter 6

Conclusions

In this dissertation, we have developed a framework for solving stochastic optimization problems and analyzing the asymptotic behavior of the solutions. Moreover, we have demonstrated how to apply the framework to solve channel aware scheduling problems such as power efficient routing in wireless networks and the joint source distortion and downlink transmission management. To conclude this work, we remark some promising improvements for our framework, which lead to our future research plan.

Recall that in Chapter 2 we solved stochastic optimization problems using stochastic approximation algorithms. From the simulation results for the numerical examples in Chapter 3-5, we observed that the convergent time of those stochastic approximation algorithms increases as the step size decreases. However, the system performance improves when smaller step size is used. This results in a tradeoff between the convergent time and the system performance, which we can be roughly quantify by inspecting the limit process of the projected differential inclusion introduced in Chapter 2. We note that our proposed algorithm is not optimized for delay sensitive applications, which require shorter convergent time. How to incorporate delay sensitive data traffic into our framework is an major task to be accomplished in our future research plan.

The second improvable part in our current design comes from the assumption of perfect observation of channel states and perfect reception of transmit data. In reality, channel states are obtained at the receiver using the estimator for pilot signals sent periodically/continuously from the transmitter. Inevitably, the estimation incurs errors. Even it does not, the receivers could still make errors in detecting and decoding the received signals. How to include these realistic conditions and uncertainties into consideration is another important task to research on.

Finally, for all the applications presented in this dissertation, we ignored the cost and the overhead accompanied with the control signalling. Since the proposed online stochastic approximation algorithm makes decision relying on instantaneous channel state information at the transmitter and other parameters (e.g. dual variables) exchanged among the nodes, as the system becomes large, massive information transactions may overwhelm the system. To address this issue, we have done some preliminary research which show that it is not necessary to update the dual variables for the algorithm on a per slot basis. The time intervals between successive updates of the algorithm can be relaxed to a sequence of random variables. As long as the mean and the variance of this sequence of time intervals are finite, the proposed algorithm still converges. Nevertheless, in this way, the scheduler still needs the instantaneous channel states to make decisions. We note that to reduce the frequency of estimating the channel state, one promising approach is to incorporate more structured stochastic channel models into the framework. The model should rely on only certain parameters like mean and variance to determine the distribution of channel state. Instead of observing and reporting the channel state to the transmitters on a per slot basis, we can measure the channel state periodically with longer intervals and estimate the channel conditions in between the measurements. Accompanied with the imperfect channel estimation mentioned previously, it is anticipated that the issue of frequent channel state updates can be alleviated to some extent with the cost of minor capacity loss.

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