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Axions in Cosmology

By

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ABSTRACT

This thesis is based on two main projects completed during my graduate studies. The first project consists of showing that hybrid inflationary theories are still viable meaning that there is a parameter space that give us cosmological parameters in agreement with the Planck data.
The second project explore the mixing between gravity wave, electromagnetic waves and axion waves in a curved space time as a way to potentially detect dark matter.

We revisit two-field hybrid inflation as an effective field theory for low-scale inflation with sub-Planckian scalar field ranges. We focus on a prototype model by Stewart because it allows for a red spectral tilt, which still fits the current data. We describe the constraints on this model imposed by current CMB measurements. We then explore the stability of this model to quantum corrections. We find that for relevant, marginal, and at least a finite set of irrelevant operators, some additional mechanism is required to render the model stable to corrections from both quantum field theory and quantum gravity. We outline a possible mechanism by realizing the scalars as compact axions dual to massive 4-form field strengths, and outline how natural hybrid inflation may be supported by strong dynamics in the dual theory.

We describe bosonic (scalar, electromagnetic and gravitational) wave mixing in curved spacetime. Curved spacetime adds a new length scale, the Schwarzschild radius, which significantly alters the oscillation probabilities in comparison to the standard flat spacetime computations. The alterations are analogous to the Mikheyev-Smirnov-Wolfenstein (MSW) effect for neutrinos and are "frozen- in" as the outgoing gravitational and/or electromagnetic wave propagates away from a compact object. Although we consider the axion and axion-like particles, our computations are largely model independent and applicable for generic spin-zero dark matter. We describe the probabilities for axions and generic bosonic dark matter oscillations. We describe some of the observational consequences of the mixing including the energy and polarization of the waves exiting the compact object.

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Glossary

CMB : Cosmic Microwave Background.

DE: Dark Energy.

DM: Dark Matter.

EFT: Effective Field Theory.

EOM : Equations Of Motion.

IR : Infrared.

MSW : Mikheyev-Smirnov-Wolfenstein.

NDA : Naïve dimensional analysis.

QCD : Quantum Chromodynamics.

RGE : Renormalization Group Equation.

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CHAPTER 1
INTRODUCTION

We live in the modern era of cosmology. Thanks to the technology available nowadays, we have been able to probe the sky from all the way to the early stages of our universe. The discovery of the Cosmic microwave background (CMB), which is an electromagnetic radiation remnant from an early time of the universe, discovered in 1965 by American astronomers Arno Penzias and Robert Wilson and granted them the Nobel prize in 1978, gives us a glimpse of the physics of the very early universe. The WMAP and Planck satellites, produced maps of the universe at the time of the CMB ie $t = 380000$ years after the Big Bang. Those maps revealed crucial information about the history of the universe as well as its composition. The current stage of the scientific knowledge agrees that the universe is made of 68% of Dark Energy (DE), 27% of Dark Matter(DM) and 5% of baryonic matter. Despite this consensus, the nature of DE and DM is still unknown to the scientific community.

The CMB maps themselves have puzzled the community for decades. Indeed, those maps indicates that the universe has always been very homogeneous: at the time of the CMB, it appears that part of the universe that seemingly had never been able to communicate, share some common information. For example, if one were to warm up a soup in the microwave, one would expect a non uniform temperature distribution. One can wonder why the CMB temperature appears to be so uniform, up to $\Delta T = \pm 0.00335K$. One way to answer this cosmological puzzle is to theorize a very short period of accelerated expansion right after the Big Bang. We call it Inflation.

Inflation last $10^{32}s$ during which the universe expands as much as it did between the end of inflation up to today. Although there are multiple arguments in favor of inflation, the mechanism allowing for a period of accelerated expansion remains unknown.

During my PhD, I worked on a possible explanation for inflation, where the main fuel for this expansion takes the form of a scalar field called an axion.

Axions are pseudo scalar particle, that could also be used to explain the nature of both DM and DE.

I have also investigated theories which involve axions as DM candidate, by looking at the mixing between axion waves, gravitational waves and electromagnetic wave near a Kerr black hole.

In addition to the previous projects, I worked on a model of Dark Energy Quintessence where DE is described by an axion that mimics the behavior of the dominant component of the universe during the different era of our universe and start behaving like DE at late time.

My journey as a theoretical cosmologist has taken me from the very beginning of the universe to looking at the nature of DE, the dominant component of the late universe.

CHAPTER 2
INFLATION

2.1 Motivations for Inflationary theories

We live in the golden age of observational cosmology. Telescopes such as COBE (1989-1993) , WMAP (2001-2010) and more recently, Planck (2009-2013) allowed the scientific community to probe the early universe. WMAP in particular gave us the first measurement of the Cosmic microwave background (CMB) and the Planck satellite provided physicists with very precise measurements of cosmological parameters.

The CMB, or Cosmic Microwave Background, is the ”oldest map” of the universe. Having access to the temperature fluctuations at $t = 380000$ years after the Big Bang is a window into the physics of the very early universe. We found the temperature to be $T = 2.73 \pm 0.00335K$ which indicates that the universe was very homogeneous at that time.

We know that the universe is very old, almost 14 billions years old, the universe is very homogeneous, $\frac{d\rho}{\rho} \sim 10^{-5}$, it is almost perfectly flat $\frac{k}{a^2 H_0} \leqslant 10^{-2}$, allows for galaxies to form and is void of problematic cosmological structures such as domain walls or magnetic monopoles.

However, these observations raise a few cosmological questions. We will only present a few of them: The Horizon problem and the flatness problem, and we will show how a theory of inflation solves those cosmological problems in the remaining part of this section:

2.1.1 The Horizon Problem

The CMB tells us that the universe is very homogeneous. Parts of the universe that are far away from one another exhibit a similar temperature up to $\Delta T = 0.00335K$ (see fig.2.1). This observation is puzzling as some parts of the universe are causally disconnected and yet still present the same temperature. It is possible to evaluate the number of causally disconnected regions in the universe by evaluating the ratio of the particle horizon, L_h, to the apparent horizon, l at the time of the CMB. We can assume a radiation dominated universe as a first order approximation. In such a universe, the scale factor scales as follows $a = a_0\sqrt{\frac{t}{t_0}}$ and we get :

$$\begin{aligned} L_h &= a(t)\int_0^t \frac{dt'}{a(t')} = 2t \\ l &= \frac{1}{H} = \frac{1}{H_0}\frac{t}{t_0} \end{aligned} \tag{2.1}$$

The ratio between the particle horizon and the apparent horizon is given by (taken into account one spatial dimension):

$$\frac{l}{L_h} = \frac{1}{2tH_0}\sqrt{\frac{t}{t_0}} = \frac{T}{T_0} \sim 10^3 \tag{2.2}$$

Hence there are $(10^3)^3 = 10^9$ causally disconnected regions with the same temperature (up to $\Delta T = 0.00335K$).

One way to solve the horizon problem is to consider a universe dominated with a fluid with a constant equation of state $w = P/\rho$ prior to the original $t = 0$s from the original Big Bang model.

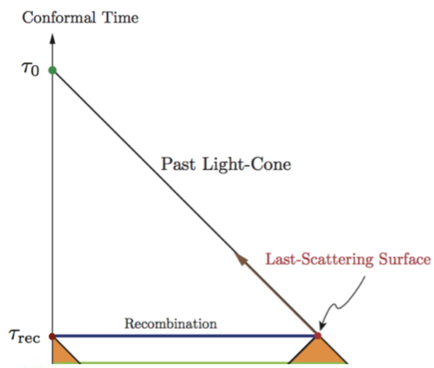

Figure 2.1: **Causally disconnected regions in old BIG-BANG models**
Here the two red points have always been causally disconnected since their past light cone, here in orange, never intersect.

The conformal time is defined as:

$$\tau \equiv \int \frac{dt}{a(t)} \tag{2.3}$$

For a universe dominated by a fluid with state parameter w, we know that the scale factor is given by :

$$a = a_0 t^{\frac{2}{3(w+1)}} \tag{2.4}$$

and we get:

$$\tau = \int \frac{dt}{a_0 t^{\frac{2}{3(w+1)}}} = \frac{1}{a_0} \frac{2}{3(w+1)} a^{\frac{1+3w}{2}} \tag{2.5}$$

Hence, if $w \leqslant -1/3$, $\tau \to \infty$ which allows for any two points to have been causally connected earlier in time (see fig. 2.2).

We can also rephrase this solution looking at the dynamics of the comoving Hubble sphere, defined as $R_H = (aH)^{-1}$ where $H = \dot{a}/a$, is the Hubble parameter. Indeed, in order to ensure that points were in causal contact earlier in time, we can theorize a period of increasing comoving

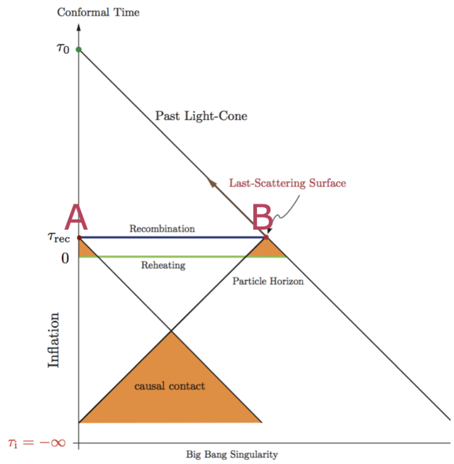

Figure 1.2: *Conformal diagram for inflationary cosmology.*

Figure 2.2: **Causally connected regions in inflation models**
Here the two red points have always been causally connected earlier in time

Hubble sphere between the beginning of time until the end of inflation:

$$\frac{d}{dt}(aH)^{-1} < 0 \tag{2.6}$$

A period of decreasing comoving Hubble sphere is equivalent to having a period of accelerated expansion:

$$\begin{aligned} &\frac{d}{dt}(aH)^{-1} = \frac{d}{dt}(\dot{a})^{-1} = -\frac{\ddot{a}}{(\dot{a})^2} \text{ hence} \\ &\frac{d}{dt}(aH)^{-1} < 0 \implies \ddot{a} > 0 \end{aligned} \tag{2.7}$$

2.1.2 The Flatness Problem

The flatness problem a fine tuning problem. It is the statement that at $t = 1s$ we observe $|1 - \Omega| < 10^{-60}$, where Ω is the density parameter, defined as follow:

$$\Omega = \frac{\text{actual mass density}}{\text{critical mass density}} \tag{2.8}$$

Using the Friedmann equation:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\mathcal{G}}{3}\rho - \frac{kc^2}{a^2} \tag{2.9}$$

we can express Ω as:

$$|\Omega - 1| = \frac{k}{a^2H^2} \tag{2.10}$$

We have shown in the previous section that inflation is a period of accelerated expansion that solves the horizon problem. In the case of the flatness problem, a decreasing comoving Hubble sphere, $R_H = (aH)^{-1}$, would ensure that Ω is close to 1 up to 10^{-60} at $t = 1$s. This statement is equivalent to a period of accelerated expansion, as shown previously.

2.2 Slow Roll Inflation

Now that we understand why inflation, a period of accelerated expansion, solves both the horizon and the flatness problem, we need to develop models that can explain the mechanism that allows for such an expansion. Because the universe is isotropic, it is natural to consider a scalar field coupled to gravity as the motor of inflation as follows:

$$\mathcal{S} = \int d^4x\sqrt{g}\left(\frac{M_{pl}^2}{2}R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right) \tag{2.11}$$

In order to control the amount of inflation that is needed to solve the previously mentioned cosmological problems, we need a potential that is flat for sometime. For that purpose, we can define the following slow roll parameters:

$$\begin{aligned} \epsilon &\equiv \frac{m_{pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \\ \eta &\equiv \frac{m_{pl}^2}{8\pi}\left(\frac{V"}{V}\right) \end{aligned} \tag{2.12}$$

Where m_{pl} is the Planck mass and $M_{pl} = m_{pl}^2/(8\pi)$ is the reduced Planck mass.

It should be clear that ϵ controls the flatness of the potential and η controls how long the potential stays flat.

More precisely, we have established in the previous section, that for a universe filled with a fluid with state parameter w, inflation is satisfied for $w < -1/3$ (or $w \to -1$ is even better).

The density parameter and pressure for a scalar field are defined as follows:

$$\rho = \frac{\dot{\phi}}{2} + V$$
$$p = \frac{\dot{\phi}^2}{2} - V \tag{2.13}$$

Hence the state parameters for a scalar field is given by:

$$w = \frac{p}{\rho} = \frac{\frac{\dot{\phi}}{2} - V}{\frac{\dot{\phi}^2}{2} + V} \approx -1 + \frac{2}{3}\epsilon \tag{2.14}$$

We can see that given a potential $V(\phi)$, we will have the condition for inflation if: $|\epsilon| < 1$ and $|\eta| < 1$. Those are the conditions to realize slow roll inflation.

2.3 A Model of Inflation: Chaotic Inflation

Let us look at a simple model of inflation. The potential is given by:

$$V_{chaotic}(\phi) = \frac{m^2}{2}\phi^2 \tag{2.15}$$

Where ϕ is the inflaton and m, its mass.

The slow-roll parameters are given by:

$$\epsilon_V(\phi) = \eta_V(\phi) = \left(\frac{m_{pl}}{4\pi}\right)^2 \frac{1}{\phi^2} \tag{2.16}$$

Hence in order to satisfy the slow-roll condition i.e $\epsilon_V << 1$ and $\eta_V << 1$, we get the following condition on the scalar field:

$$\phi^2 >> \frac{m_{pl}^2}{4\pi} = 2M_{pl}^2, \text{ hence}$$
$$\phi >> \sqrt{2}M_{pl} \tag{2.17}$$

We can see that in order to realize a slow-roll inflation in a chaotic inflation scenario, we need to ensure that the field ϕ starts with super Planckian values.

Such a theory will be sensitive to quantum gravity effect and would need some mechanism protecting it from such effects.

CHAPTER 3
AXION MONODROMY INFLATION

3.1 Introduction

Inflation is the leading candidate for explaining why the universe is so old, flat and smooth. Explicit models rely on a local quantum field theory (QFT) of the agent behaving like a transient but long lived vacuum energy, which forced the universe to rapidly expand early on. In a local Poincare-invariant QFT, this requires scalar fields with a very flat potential. An immediate question is whether such models are natural or fine-tuned. Leaving aside the conundrum of initial conditions, and merely focusing on the naturalness of the parameters of the local QFT, the inflationary epoch is obviously an arena where the ideology of naturalness may be tested observationally.

The simplest QFT models of inflation involve a single scalar field which needs both a very flat potential and super-Planckian field variations [1], [2]. These large-field models typically produce observable primordial gravitational waves. Thus they are increasingly in tension with observations although they are not yet ruled out. However the super-Planckian field ranges also present a theoretical challenge, a large number of Planck-suppressed operators must be fine-tuned absent a mechanism for suppressing them [3]–[19]. Since data is starting to take a bite out of these models, one may be tempted to avoid the issue of Planck-suppressed operators by studying small-field models, or by studying even more exotic alternatives where naturalness of the QFT of the inflaton is altogether abandoned. These latter alternatives are also not safe from the predations of quantum gravity. Therefore we will take a very conservative attitude, ignore exotic inflationary models altogether, and only consider inflationary small-field QFTs which might be natural. As we will find, such small-field models in fact require not so small fields after all, and are still threatened by Planck-suppressed irrelevant operators[1]. Some mechanism is still required to ensure that the quantum gravity corrections are under control.

In this paper we will revisit two-field hybrid inflation models [21]–[23], and explore their stability to both quantum field theoretic and quantum gravity corrections. The 'classic' model of hybrid inflation [21]–[23] predicts a blue spectrum of CMB perturbations that is now ruled out by observations [24]. However, this does not shut the door on hybrid inflation. For example, a variant due to Stewart [25] (see also [26]–[28]) does produce a red spectrum (as we will review), and for some range of parameters can be made compatible with observations. We will find that additional quartic terms are induced by radiative corrections, which nevertheless can be kept naturally small. However – unsurprisingly – the quadratic mass terms suffer from the usual hierarchy problem afflicting fundamental scalars, and a mechanism is needed to keep them sufficiently small. Even with such a mechanism, we will find that the combination of data-based constraints and technical naturalness put the theory into a corner for which the range of the scalar fields are close in magnitude to the Planck scale. The main constraints are:

- We need to ensure that the inflaton rolls sufficiently slowly over a sub-Planckian range to generate N-efoldings of inflation.
- The two fields in the hybrid model can be roughly classified as "inflaton" and "waterfall" fields. The former rolls during inflation leading to a spectral index. The latter is locked by the large value of the inflaton during inflation, and provides the bulk of the vacuum energy, and condenses at the end of inflation, ending vacuum dominance and initiating reheating. We ask that the field displacement of the "waterfall" field is sub-Planckian, while being high enough to give the right scale of density perturbations when the couplings remain weak

[1] This has been noted in similar contexts before, such as in [20].

enough for radiative stability.

The upshot is that two-field hybrid models that fit the data, are technically natural, and protected from quantum gravity corrections are very nontrivial to realize. In addition to requiring hierarchies in masses and renormalizable couplings, the nearly-Planckian field ranges mean that there are a large but finite number of Planck-suppressed irrelevant operators which may spoil slow-roll inflation unless there is a mechanism to make their dimensionless coefficients sufficiently small[2]. Furthermore, satisfying the demands placed on inflation models with sub-Planckian field ranges puts additional pressure on the EFT which one uses to describe the dynamics, because the inflaton – which must be light during inflation – quickly becomes very heavy after inflation ends. This is merely a matter of continuity: if the field rolls on a flat and short plateau to give 60 efoldings of inflation, that plateau must be very flat, and the post-plateau minimum very narrow, and hence with a large curvature. This pushes reheating out of the EFT used to describe inflation: the UV embedding must be understood.

After exploring these issues, we will argue that an embedding of the Stewart model exists which can protect the required hierarchies in masses and couplings from both QFT and quantum gravity corrections. To do this we will use a two-field version of the axion monodromy effective field theories developed in [7], [8], [12], [14], [16], [17], [19]. In these theories the two fields that drive inflation and reheating are considered as axion-like pseudoscalars, dual to longitudinal modes of two massive 4-form field strengths. The scalar mass maps to the gauge field mass [16]; as with masses for Abelian vector fields, the 4-form mass is stable to quantum corrections. Additional small hierarchies needed to support inflationary dynamics may then arise from dimensionless couplings in the strong coupling regime. For models which match CMB data one must either tune the value of the mass or find an additional mechanism, beyond EFT, to explain its smallness from

[2] In principle, it is possible that the individually dangerous operators may sum up to tamer contributions to the effective potential, if there are approximate shift symmetries in some regimes of phase space.

first principles. Either way, the small mass is at least technically natural[3]. We will also see that the requirements of naturalness and bounds from data push the dual 4-form theory into the strong coupling regime. However since the theory has only longitudinal propagating modes, in the inflationary regime the theory looks weakly coupled because each explicit power of the inflaton ϕ comes with a factor $\propto \mu/\mathcal{M}$, where $\mathcal{M}$ is the cutoff and μ the inflaton mass, thus providing additional suppression factors that make the theory appear weakly coupled in the axial gauge, and also tame quantum gravity corrections to the irrelevant operators of the theory.

The current data put additional pressure on this EFT, suggesting that a full theory of hybrid inflation really requires two different EFTs to consistently describe it, one for the slow roll regime and another for reheating, when the inflaton becomes very heavy (and presumably other degrees of freedom become very light). Providing the details of the complete first-principles construction of such a model is beyond the scope of this work. Here our approach is bottom-up, reverse-engineering a low energy theory in order to glimpse how it can be derived from a UV-complete model. Hence we cannot provide a detailed contents of the full spectrum of the theory, which is necessary to see how various fields transition to below and above the cutoff. Nevertheless, our analysis indicates that such constructions exist, and serves as a guide for how to search for more complete models which realize such dynamics.

We close with a summary of what is new in this chapter:

- First, a detailed consideration of matching the Stewart model to the most up-to-date data has not been done before; nor has the stability of this model to either QFT or quantum gravity corrections been explored. We do both in §3. While the model fits data well, we will find that as it stands, it in fact is unnatural: the model suffers from the mass hierarchy problem, since the dimensional scales in the theory are very sensitive to the UV completion. This requires at

[3]Lower bounds on the mass of 1-form gauge fields have been conjectured in [29]: we believe inflation is well away from such bounds.

least a small $\mathcal{O}(0.1)$ hierarchy between mass scales. A protection mechanism is necessary to stabilize this against quantum corrections and render the theory at least technically natural. We will also see that the theory has many irrelevant operators which are individually large, and a dynamical explanation of why they do not spoil inflation is needed.

- Furthermore, we will also see that the theory is additionally strained by the fact that the post-inflationary mass of the inflaton is near the cutoff of the infationary EFT, meaning that to reliably describe exit from inflation and reheating, we really should use a different EFT in that regime.

- Finally, our proposal for realizing the model as the theory of coupled axions dual to 4-form field strengths is new. This provides a UV completion of the hybrid inflation model which addresses the hierarchy problem and the subsequent fine tunings, rendering it both natural and protected from quantum gravity corrections. To do this we take an approach that is distinct from standard axionic model-building. Typically axions have not been considered for hybrid models because their nonderivative couplings are constrained by nonrenormalization theorems, and taken to be simple trigonometric functions. This assumes that the axion potentials are generated by instantons in some sort of dilute gas approximation. However, it has been known for some time, if underappreciated, that this approximation often fails and axion potentials can be more complex, looking more like perturbative polynomial interactions, if we assume that the axion potential is multivalued [30]–[32]. The central lesson of [7], [8], [12], [14], [16], [17], [19] is that the resulting effective field theories are still constrained by gauge symmetries, and 'dimensional transmutation' between dual pictures allows for small couplings that are protected from QFT loops and quantum gravity corrections. In §3 we develop and deepen this story by expanding it to multifield models and explore its strong coupling r

3.2 Hybrid Inflation with a Red Spectrum

3.2.1 The Model

The pioneering model of hybrid inflation [21] has the form:

$$V(\phi, \sigma) = \frac{\lambda}{4}\left(\sigma^2 - \tilde{M}^2\right)^2 + \mu^2\phi^2 + \frac{1}{4}g^2\sigma^2\phi^2\,. \tag{3.1}$$

Here ϕ is the inflaton. If $\phi^2 > \lambda\tilde{M}^2/g$, then $\sigma = 0$ is a minimum of the potential for fixed ϕ, leading to the effective potential

$$V_{eff} = \tilde{M}^4 + \mu^2\phi^2\,. \tag{3.2}$$

During inflation, if the second term dominates, the theory is effectively the standard chaotic inflation, which requires a super-Planckian range for ϕ and which is ruled out by existing bounds on gravitational waves. If the M^4 term dominates the potential, then

$$\left(\frac{\delta\rho}{\rho}\right)^2 = \frac{V^3}{24\pi^2 M_{pl}^6 (V')^2} \sim \frac{1}{\phi^2}\,, \tag{3.3}$$

grows larger as ϕ decreases, leading to a blue spectral index, which is ruled out by CMB measurements [24]. More precisely the spectral index is

$$n_s = 1 + \frac{2}{N} > 1\,, \tag{3.4}$$

where N is the number of efoldings.

An alternate model by Stewart [25], [26], defined by

$$V(\phi, \sigma) = \frac{m^2}{2}\left(\sigma - \sqrt{2}M\right)^2 + \frac{g^2}{4}\phi^2\sigma^2\,, \tag{3.5}$$

which does not have degenerate post-inflationary minima, and so also no dangers of any stable defect production after inflation, does have a red spectrum:

$$n_s \sim 1 - \frac{3}{2N}\,. \tag{3.6}$$

As we will show in §2.2, this model can be made compatible with current CMB data, while maintaining sub-Planckian field ranges that are one of the strong motivations for hybrid inflation.

As far as we know, a full discussion of the quantum stability of these models has never been carried out[4]. This is one of the main goals of this paper. We thus write a modification of Eq. (3.5) containing all of the relevant and marginal operators that could be generated by quantum corrections:

$$V(\phi, \sigma) = \frac{m^2}{2}\left(\sigma - \sqrt{2}M\right)^2 + \frac{\lambda}{4}\left(\sigma^2 - \tilde{M}^2\right)^2 + \frac{\mu^2}{2}\phi^2 + \frac{g^2}{4}\phi^2\sigma^2 + \frac{\lambda'}{4}\phi^4\,. \tag{3.7}$$

Since the first term breaks the $_2$ symmetry $\sigma \to -\sigma$ one may be tempted to include terms proportional to $\sigma\phi^2$ or σ^3. However, if we compute quantum corrections to this potential following [34], linear terms in σ can be absorbed into the source term used to explore the full theory space, and thus these terms are never generated by loop corrections. In the effective field theory (EFT) language, the reason is that the parity symmetry is softly broken, only by a relevant operator, and thus is invisible to the UV effects in the loops.

[4]We are aware of a somewhat incomplete attempt in [33].

In the EFT context, we would naturally expect that all of the dimensionful parameters in this model should be below the cutoff. However, we will find in the next section that for a realization of this model via massive 4-forms, this does not have to be the case for M, for which there is a see-saw formula involving the ratio of the cutoff to a fundamental mass scale. A hint that this might be the case comes from noting that if σ is a pseudoscalar axion, as it (along with ϕ) is a dual of a massive 4-form field strength in the embedding we described below, M is a spurion for the breaking of CP, and so it might naturally be related to the initial expectation value of one of the longitudinal modes of massive dual forms; we will see that this is so in §3. There is no a priori reason that the expectation value of an axion field should be below the cutoff [35]. In principle, since it is controlled by the initial flux of a 4-form field strength it may even be $\sim M_{pl}$.

The first and fifth terms are absent in [21]; while Ref. [25] ignores the second, third and fifth terms. To realize the latter model with its red spectrum of fluctuations, we must ensure that the terms controlled by μ, λ, λ' are subdominant, so that the Stewart model is a good approximation. We will argue in §2.3 that the suppression of λ, λ' can be made technically natural. Finding a phenomenologically acceptable value of m and suppressing μ enough that the ϕ^2 term is subdominant requires more work, as scalar masses are afflicted by the hierarchy problem. Rendering these masses even technically natural requires a mechanism, such as axion monodromy, whose application to hybrid inflation we will describe in §3.

For now and in §2.2, we will take (3.7) as given and explore the predictions of the model in the limit well described by the Stewart model. As in [21], [25], at sufficiently large ϕ, the $g^2\phi^2\sigma^2$ term renders σ massive and we can integrate it out. At the classical level we simply solve the equation

$$\frac{\partial V}{\partial \sigma} = m^2\left(\sigma - \sqrt{2}M\right) + \lambda\sigma^3 - \lambda\tilde{M}^2\sigma + \frac{g^2}{2}\phi^2\sigma = 0\,, \tag{3.8}$$

for σ at large ϕ. We demand that the term $\lambda\sigma^3$ be subdominant. Ignoring this term, we find that:

$$\sigma_{min} = \frac{\sqrt{2}m^2 M}{m^2 - \lambda\tilde{M}^2 + \frac{g^2}{2}\phi^2} \,. \tag{3.9}$$

It is instructive to neglect the term $\propto\lambda$, and in this limit plug equation (3.9) back into (3.7). So, using $\sigma_{min} = \sqrt{2}m^2 M/(m^2 + \frac{g^2}{2}\phi^2) + \mathcal{O}(\lambda)$, this yields

$$V_{eff}(\phi) \simeq m^2 M^2 \frac{\frac{g^2\phi^2}{2}}{m^2 + \frac{g^2}{2}\phi^2} + \frac{\mu^2}{2}\phi^2 + \frac{\lambda'}{4}\phi^4 + \mathcal{O}(\lambda) \,. \tag{3.10}$$

If $\phi^2 \gg 2m^2/g^2$, then $\sigma \simeq \frac{2\sqrt{2}m^2 M}{g^2\phi^2}$, and in this limit the potential (3.10) becomes

$$V_{inflation} \simeq m^2 M^2 \left(1 - \frac{2m^2}{g^2\phi^2}\right) + \frac{\mu^2}{2}\phi^2 + \frac{\lambda'}{4}\phi^4 + \mathcal{O}(\lambda) \,, \tag{3.11}$$

after expanding the fraction in the first term. Clearly, as ϕ grows the first term becomes flatter. This is the inflationary plateau. The terms $\propto\mu, \lambda'$ limit it, since they make the potential convex again as they take over. Essentially, the regime with positive spectral slope is therefore between the two inflection points of the potential (3.10), when ϕ is in the regime

$$\sqrt{\frac{2}{3}}\frac{m}{g} \leqslant \phi \leqslant (12)^{1/4}\sqrt{\frac{M}{g\mu}}m \,. \tag{3.12}$$

The hybrid inflation regime, with sub-Planckian ranges for ϕ and a red spectrum, will require that the terms controlled by μ, λ' also be sufficiently subdominant, during the first 10 efoldings of the visible epoch of inflation (epochs which leave imprints on the CMB). The precise details depend on how small μ is. If μ is larger than the critical value $\mu_* = \sqrt{12}\frac{Mm^2}{gM_{pl}^2}$, the upper bound in (3.12) is sub-Planckian, and so the period of inflation with a red spectrum is generically shorter with all other parameters being fixed. If μ is smaller than μ_*, the upper bound is super-Planckian. At any rate we will insist that the final 50 efoldings must occur over a sub-Planckian range of the inflaton, while the inflaton lies below the upper bound in (3.12). In other words, if we define the maximum value of ϕ at 50 efolds before the end of inflation as αm_{pl}, we require $\alpha < 1$. The upper bound in ϕ in (3.12) then implies $\sqrt{12} M m^2 \alpha^2 g M_{pl}^2 \mu$. In this way, whichever option, any trans-Planckian field excursions are observationally irrelevant, offering assurance that our model of nature need not be too sensitive to the UV. We will however see that this puts stress on naturalness.

We will further require that $\mu/m < 1$, so that we can consistently integrate out the σ field during inflation, and ignore its fluctuations. This also ensures that the σ fluctuations during inflation are suppressed; as a consequence we can ignore isocurvature perturbations, which are anyway strongly constrained by the data. We will however find that when other bounds are met, $\mu/m < 1$ is automatically satisfied.

The limitations of an EFT are set by the cutoff $\mathcal{M}$. While the effective potential (3.7) involves only renormalizable operators, and so it gives no indication of any intrinsic UV cutoff, we can place a lower bound on the cutoff of the two-field model by noting that $\mathcal{M}$ should be of the order of the effective mass of σ during inflation. This mass will provide a natural UV cutoff for the effective theory, Eqs. (3.10) and (3.11) for ϕ, once we integrate out σ. This potential is obtained for large ϕ by integrating the field σ when it is effectively near zero. Taking the second derivative of (3.8) gives the effective mass of σ in that regime,

$$m_\sigma^2 \simeq m^2 + \frac{g^2 \phi^2}{2} + \mathcal{O}(\lambda)\,. \tag{3.13}$$

To get an idea about the limit on the inflaton ϕ range in our EFT we conservatively use the range $\sim M$ of σ as a guideline[5]. Imposing $M < M_{pl}$, we thus take the cutoff $\mathcal{M} = \sqrt{4\pi}\mathcal{M}_*$ to be

[5]This assumes a lack of hierarchy between σ, ϕ; in practice we expect ϕ to have a somewhat larger range, as we

$\mathcal{M} \sim gM$. Here $\mathcal{M}_*$ is the strong coupling scale, used to normalize the EFT operator expansion in the Naïve Dimensional Analysis (NDA) framework [36], [37], a framework which was deployed to organize the EFT of large field inflation in [16], [17]. To get a sharper estimate of the cutoff, we can also produce a lower bound by recalling that in NDA, the overall dimensional scale of the potential terms in EFT is $\mathcal{M}_*^4$ at strong coupling. This sets an upper bound $V \mathcal{M}_*^4$ on the potential energy during inflation [16], [17]. From (3.11), we then find $\mathcal{M}_* \sqrt{mM}$. Thus the cutoff lies in the range

$$\sqrt{4\pi m M} \mathcal{M} \sim gM \,. \tag{3.14}$$

For values of these parameters which satisfy constraints we develop below, we find $gM > \sqrt{mM}$ so we take gM as a conservative estimate of $\mathcal{M}$.

Note, however, that as the inflaton ϕ rolls towards the end of inflation, its effective (tachyonic) mass on the plateau, $|m_\phi^2| \sim 24 \frac{m^2}{\mathcal{M}_*^2} m^2 \ll m^2$ changes to

$$m_\phi^2 \sim g^2 M^2 \sim \mathcal{M}^2 \,. \tag{3.15}$$

Thus the EFT of inflation and the stage right after inflation breaks down as the theory proceeds to reheating at the true vacuum $\sigma = \sqrt{2}M$. As serious as this is, by itself it is *not* a fundamental flaw of the theory. It indicates that as inflation nears the end, the theory is undergoing a phase transition where the energy density stored in the very flat potential tends to dissipate quickly, and nonperturbatively, at very short distances. Some new fields, including σ – which was very heavy during inflation – become light and need to be integrated in. Others, like the inflaton ϕ, become heavy. Further, since the ϕ mass is tachyonic, as its magnitude becomes larger the tachyonic instability becomes faster near the end of inflation and operates at sub-horizon scales. So the important

are working to flatten its potential.

lesson from this is a *warning*: we should not expect to have a single EFT of hybrid inflation, but instead count on having the late stages of the exit and reheating as separate descriptions from slow roll inflation.

3.2.2 Matching to Data

We next want to demonstrate that our model (3.10) has the capacity to match existing CMB data, focusing on:

- The number N of efoldings of inflation, constrained (up to details of reheating, which we do not model here) by bounds on spatial curvature to be at least on the order of $50-60$. For the sake of convenience we take $N = 50$.
- The power in scalar density (or CMB temperature) fluctuations at an appropriate pivot point, set by experiments to be of order $\frac{\delta\rho}{\rho} \sim 5 \times 10^{-5}$.
- The tensor-scalar ratio r, with a current upper bound at $r \leqslant 0.056$ [24]. We will in fact demand here that the inflaton ϕ has a sub-Planckian field range over the last N efoldings of inflation; this ensures that r is well below the above bound [33], [38].
- The scalar spectral index α_S, between 0.95 and 0.985, depending on r.

We will do this here under the assumption that the first two terms in (3.11) dominate in our model. The above constraints can then be phrased in terms of bounds on m, M, g and the maximum excursion $\alpha M_{pl} \equiv \phi_{max}$ of the scalar field during the visible epoch. We will find that the data supports a range for these parameters. The further question of whether there is a range for these parameters and for $\lambda, \tilde{M}, \mu, \lambda'$ for which the remaining terms in (3.11) are subdominant and the whole theory is stable under quantum corrections is the central point of the rest of this paper.

We first impose the constraint that inflation last for a sufficient number of efoldings. Assuming that the magnitude of the potential energy V is dominated by the first term in (3.11), the number N of efoldings is given by:

$$N = -\frac{1}{M_{pl}^2}\int \frac{V}{V'}\mathrm{d}\phi \simeq \frac{1}{M_{pl}^2}\frac{g^2}{16m^2}\phi_{max}^4\,, \tag{3.16}$$

Using $\phi_{max} = \alpha M_{pl}$, we find:

$$\frac{m}{gM_{pl}} \simeq \frac{\alpha^2}{4\sqrt{N}}\,. \tag{3.17}$$

Note that we are interested in $\alpha \leqslant 1$; thus, $m \ll gM_{pl}$. Since we also wish to impose a sub-Planckian range of σ, and that range is set by M, this condition is at least compatible with $m < \mathcal{M} \sim gM$.

Our bound assumed that V is dominated by the constant term $\sim m^2M^2$ during the first ~ 10 efoldings of the visible epoch of inflation (the period which leads to observed CMB fluctuations). We need to check that this is self-consistent. 10 efoldings corresponds to to approximately $\frac{\phi_{max}}{N}\Delta N \sim \frac{2.5}{N}\phi_{max}$ which is a small variation, so we need simply check the dominance of the constant term at $\phi \sim \phi_{max}$. Under these conditions,

$$\frac{m}{g\phi_{max}} = \frac{m}{\alpha g M_{pl}} \simeq \frac{\alpha}{4\sqrt{N}} \ll 1\,, \tag{3.18}$$

so our assumption of the dominance of the constant term in V is self-consistent. Note that in light of the discussion in the previous section, this demonstrates that for parameters that yield the requisite number of efoldings, $m \ll \mathcal{M}$. Also note that it is straightforward to show that the slow roll parameters $\eta \sim M_{pl}^2 V''/V$, $\sim M_{pl}^2(V'/V)^2$ are both small given (3.17) together with $\alpha \leqslant 1$

We next check the scalar spectral index:

$$n_S = 1 - 3M_{pl}^2 \left(\frac{V'}{V}\right)^2 + 2M_{pl}^2 \frac{V''}{V} = 1 - 24M_{pl}^2 \frac{m^2}{g^2\phi^4}\left(\frac{2m^2}{g^2\phi^2} + 1\right) \simeq 1 - \frac{3}{2N} \,. \qquad (3.19)$$

In the last term in the RHS we are assuming that $\frac{2m^2}{g^2\phi^2} \ll 1$: this is the same as the demand that V in (3.11) is dominated by the leading constant term m^2M^2. If we take N between 50 and 60, n_s varies between 0.97 and 0.975. This is within current bounds, particularly for $N \sim 50$. However future observations might be able to constrain this more strongly, and perhaps even falsify the model. In any case, as we noted, in the rest of this paper we will take $N = 50$ as the pivot point to match the model to the data.

Thirdly, we impose a constraint on m, M, g, α from the power in scalar fluctuations:

$$\frac{\delta\rho}{\rho} = \frac{1}{2\pi}\frac{H^2}{\dot{\phi}} = \frac{1}{2\pi\sqrt{3}M_{pl}^3}\frac{V^{3/2}}{V'} = \frac{g^2M\phi^3}{8\pi\sqrt{3}mM_{pl}^3} \,, \qquad (3.20)$$

where we assume, following the above discussion, that the first term in (3.11) dominates the magnitude of V. Using Eqs. (3.17) and (3.20), and that $\delta\rho/\rho \approx 5 \times 10^{-5}$, we find two constraints on m, M, g, α. We can express any two in terms of the other two. For convenience, we will use these equations to express m and M in terms of the two remaining parameters g and α, which we will treat as independent parameters, at $N = 50$. We find

$$\frac{m}{M_{pl}} \simeq \frac{\alpha^2 g}{28} \,, \qquad \frac{M}{M_{pl}} \simeq \frac{8 \cdot 10^{-5}}{\alpha g} \,, \qquad (3.21)$$

where we substituted the first equation into (3.20) to obtain the second one. We note that the parameters g and α cannot be completely arbitrary. In addition to $g < 1$ and $\alpha \leqslant 1$, their choice must be made to maintain $M/M_{Pl} \leqslant 1$, and ensure that the approximation where we neglected μ

and λ'-dependent terms is self-consistent, even when we include quantum corrections. We will see that those requirements yield non-trivial restrictions on g and α.

Finally, the tensor-scalar ratio r is given by:

$$r = 6M_{pl}^2 \left(\frac{V'}{V}\right)^2 = 6M_{pl}^2 \frac{16m^4}{g^4\phi^6} = \frac{3m}{2gM_p}\frac{1}{N^{3/2}} \,. \tag{3.22}$$

Using (3.17), we find

$$r = \frac{3\alpha^2}{8N^2} \simeq 1.5 \times 10^{-4}\alpha^2 \,, \tag{3.23}$$

well below the current bounds, and likely unobservable, given $\alpha < 1$. Thus we see that if, e.g, r were observed, and if $n_S \sim 0.95$, our model would be challenged by data.

Using the above constraints, we wish to display more explicitly the region in the space of parameters m, M, g, α that satisfy them. First, the scale of the potential, and therefore the lower limit (3.14) on the cutoff $\mathcal{M}$ is

$$V^{1/4} \simeq \sqrt{mM} \simeq \sqrt{\alpha} \times 10^{-3} M_{pl}\mathcal{M}/\sqrt{4\pi} \,. \tag{3.24}$$

As we noted above, we will impose the requirement that all the fields remain sub-Planckian, in addition to the dimensional parameters that appear in the relevant operators of the theory. The field range for σ is $\sim \sqrt{2}M$ so we demand $M < M_{pl}$ as well as $\alpha < 1$. This is what helps keep the tensor power low. Furthermore, in principle this might have helped with keeping quantum gravity effects under control and allowing inflation and reheating to take place in the same effective field theory. As have seen and will see, these conditions are sufficient for neither. We have already argued that reheating takes place through a phase transition during which energies and momenta of order the cutoff become activated. We will show below that the Stewart model will be at best barely sub-Planckian, so that a large number of irrelevant operators must have small coefficients, even if they appear as Planck-suppressed. Note also that when it is helpful to gain a conceptual handle on the constraints of our theory, we will impose a field space "democracy" with $\sigma, \phi M$ up to $\mathcal{O}(1)$ factors.

With these in mind, we find that the second equation in (3.21) combined with $M/M_{pl} < 1$ yields a bound

$$g \gtrsim \frac{8 \cdot 10^{-5}}{\alpha} \, . \tag{3.25}$$

It is worth noting that this bound makes $\mu/m < 1$ consistent. This is needed for ϕ to be a good candidate for the inflaton with the fluctuations of σ suppressed during inflation. In particular if we impose $\mu < \mu_*$, so that the plateau lasts for at least a Planck scale in range before quantum gravity cuts it off, then combining $\mu_* = \sqrt{12}\,\frac{Mm^2}{gM_{pl}^2}$ with the first of Eqs. (3.21) and using inequality (3.25) gives $\mu_* \simeq 3.4 \times 10^{-7} \alpha M_{pl}$, and so, using (3.25),

$$\frac{\mu}{m} < 0.1 \, . \tag{3.26}$$

Consistency of our EFT also demands that $m \ll \mathcal{M} \sim gM$ (using Eq. (3.14)). Combining the two Eqs. (3.21) we find $m/M = 500\,\alpha^3 g^2$, and thus $m/gM = 500\,\alpha^3 g$. We already noted however that $m/gM \sim m/\mathcal{M} \ll 1$, and will recheck it in the next section that requiring radiative stability of the marginal operators in the theory yields an independent bound $g \leqslant 1.6 \times 10^{-3}$, which guarantees $m/gM < 1$ as well as $m \ll M$.

In the end there is a nontrivial subspace of the parameters m, M, g, α satisfying our bounds. In Fig. 1 we have plotted a region in the space of g, α, which as we noted we treat as independent parameters. For completeness we have also included the bound $g \leqslant 1.6 \times 10^{-3}$, which we will derive in the next section. Suppressing corrections to make the theory radiatively stable favors weaker couplings, $g < 1$. However, as seen from Eq. (3.25), g cannot be arbitrarily small as long as $\alpha < 1$. Decreasing g mandates increasing M and/or α to keep $\delta\rho/\rho$ fixed. Conversely, lowering α and/or M/M_{pl} puts pressure on the perturbativity of g. The theory therefore does *not* really work at arbitrarily low scales, as is clear from Fig. 1. Nonetheless, there is still a nontrivial window here; in particular, for e.g. $\alpha, M/M_{pl} \sim 0.1$, we can have $g \sim 0.001$, and so on.

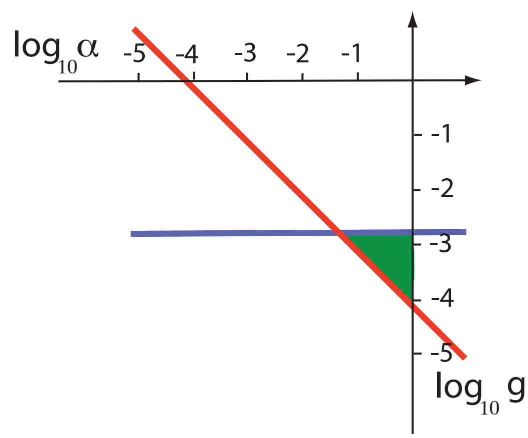

Figure 3.1: Constraints on the inflation parameters g, α due to data and the naturalness of marginal operators. The inclined line corresponds to the bound of Eq. (3.25), which follows from $M/M_{pl} \leqslant 1$. The vertical coordinate axis is the bound $\alpha \leqslant 1$. The horizontal bound comes from $g \leqslant 1.6 \times 10^{-3}$ which follows from imposing naturalness of quartic operators (see §2.3.2). The green shaded region is the regime for which the Stewart model is consistent with all of these bounds. Note that this figure ignores the bounds from naturalness of the mass terms, which are problematic for the Stewart model (3.5). The reason we are ignoring these bounds is because ultimately they may be addressed by embedding (3.5) in dual form monodromy, as we will outline in §3

We forewarn the reader that in the plots in Fig. 1 we have ignored the "elephant in the room" of the Stewart model (3.5): the power law divergent contributions to the relevant operators, specifically the mass terms. As we will argue, these require additional fine tunings, unless a mechanism is found which cancels them. Our goal in this paper is to indeed provide an example of such a mechanism, an embedding of (3.5) into a theory of dual forms, so that ϕ, σ are pseudoscalars exhibiting axion monodromy. However even after the cancellation, the bounds discussed here remain. In particular we stress that the EFT of hybrid inflation can be <u>at best</u> barely sub-Planckian in order to be both technically natural and consistent with data. The specter of quantum gravity still haunts hybrid inflation.

The reader may be disturbed that M may well be the UV cutoff of many string compactifications. As we have discussed above and will expand on in more detail in §3, this does not place the theory outside of the bounds of effective field theory below said cutoff.

3.2.3 Quantum Stability

In this section we discuss the degree to which the Stewart model (3.5) can be rendered technically natural. The model will pick up corrections that include all of the terms in (3.7). If we take the Stewart model as the tree-level action, the corrections will be generated through loops by the coupling g^2. We will thus assume that the additional relevant and marginal couplings in (3.7) that are absent from (3.5) will be of the order of these loop corrections. The field theory corrections to irrelevant terms are also dangerous. For these terms, we will note that quantum gravity is expected to generate Planck-suppressed operators with order $\mathcal{O}(1)$ coefficients in a theory of quantum gravity, and we will find that these dominate the corrections generated by QFT loops. We will then show that despite the sub-Planckian field ranges of the Stewart model, a large but finite number of irrelevant operators must have small coefficients in order for the Stewart model to be a good description of the dynamics, so that an additional mechanism or fine tuning is needed to suppress irrelevant operators. In §3, we will show that axion monodromy can provide it, as it does for large field inflation [12].

The relevant terms in the Stewart model comprise the linear term $\propto m^2 M\sigma$, which does not get corrected, as we have discussed, and the scalar masses m, μ. As with the Standard Model, the masses suffer from a potential hierarchy problem. We can realistically expect field-theoretic corrections to give $\delta m^2, \delta\mu^2 \sim \frac{g^2}{16\pi^2}\mathcal{M}^2$ where $\mathcal{M}$ is the cutoff, under the generous assumption that ϕ, σ couple with strength g to ultraviolet degrees of freedom. This encodes the well-known ultraviolet sensitivity of the masses. There is also a weaker version of the hierarchy problem, which is that the loops of the heavy field σ can correct the light mass of ϕ by large terms.

In general, we can tame these corrections to a certain degree if we assume that a softly broken shift symmetry holds up to the fundamental scale in the limit $g \to 0$. This of course is technical naturalness. Current lore is that *nonperturbative* quantum gravity effects spoil such symmetries.[6] An open question is whether these effects include relevant operators. As case studies of related Euclidean wormhole effects on the Peccei-Quinn symmetry of axion potentials show, the complete resolution of these questions remains open [39], [40] (see also [41], [42]). All we will do for this paper is merely parameterize the problem, leaving the issue of the existence of softly broken symmetries conceptually unresolved. We will instead argue in §3 that gauge symmetries of the dual formulation in terms of massive 4-form field strengths may accomplish the same goal.

3.2.3 Scalar Masses

Mass terms for ϕ, σ at the cutoff $\mathcal{M}$ would clearly invalidate our EFT, pushing all dynamics to the cutoff. Furthermore, if we estimate $\mathcal{M} \sim gM$, we have shown that constraints from data imply that $m/\mathcal{M} \sim 500\alpha^3 g$. If we impose the constraints on $g \lesssim 10^{-3}$ from technical naturalness, and demand that inflation occur over a sub-Planckian field range, then some mechanism must keep m well below the cutoff. Furthermore, we must also impose $\mu \ll m$. If we require that the second

[6]Perturbative quantum gravity effects preserve the shift symmetry of scalars: see [2], [12].

derivative of $\mu^2\phi^2$ be subdominant to the second derivative of the second term in (3.11), then, if $m, \mu \sim \mathcal{M}$, we find that this condition combined with (3.17) implies that

$$M \gg \left(\frac{\mu}{m}\right) \times \sqrt{\frac{4N}{3}} M_{pl} \sim \left(\frac{\mu}{m}\right) \times 8 M_{pl} \tag{3.27}$$

so that a sub-Planckian field range for σ requires $\mu < 0.12\, m$.

Now let us assume for a moment that the mass m is pushed up to $m \sim g\mathcal{M}/4\pi \sim g^2 M/4\pi$ by UV corrections. Using Eq. (3.17) we find $gM/M_{pl} = \alpha^2\pi/\sqrt{N}$. Combining this with the second equation in (3.21), we can solve for α to find $\alpha \sim 0.06$. Looking at Figure 1, there is still a small range of g for which this barely sub-Planckian theory is viable, in the far left corner of the shaded region.

Note that $\mathcal{M} \sim gM$ is at most a lower bound on the scale of new physics. In practice, new physics could appear up to the Planck scale, and thus push the technically natural scale for m farther up still. In this case, the above argument shows that α is pushed up towards 1. As with the Higgs mass, there is no clear barrier to new physics appearing up to the fundamental Planck scale, be this the 4d or 10d Planck scale. If M is close to the Planck scale, $gM \lesssim 10^{15}\,\mathrm{GeV}$ (using the constraints on g we will discuss below), so a higher fundamental scale indicates a more serious hierarchy problem beyond the existing need for a little hierarchy between $\mu\,, m$. This little hierarchy in itself is still a problem, albeit possibly a weaker one.

In the end, controlling $\mu\,, m$ in the face of QFT and quantum gravity corrections requires unnatural fine-tunings or an explicit mechanism. In §3 we discuss one such mechanism, axion monodromy. This promotes the scalar masses to gauge field masses which are much less sensitive to UV corrections, whether from QFT or quantum gravity [16] (this is not to say that the mass is completely unconstrained [16], [29].) If the cutoff scale contributions $\sim g\mathcal{M}/4\pi$ to the masses

m, μ are prohibited by a mechanism such as monodromy, the theory could be natural, UV safe and consistent with the data. We will show explicitly how this happens in hybrid monodromy in §3.

3.2.3 Marginal Couplings

If we begin with the Stewart model (3.5) at tree level, we will induce not just mass terms but marginal quartic couplings $\frac{1}{4}\lambda\sigma^4$ and $\frac{1}{4}\lambda'\phi^4$. The induced couplings λ, λ' will be of order g^4 (times some factors we will discuss momentarily). If we write down our EFT with couplings of this order from the outset, these couplings are technically natural. If we choose g sufficiently small, we will see that these couplings will remain subdominant and (3.5) is a good approximation to a model that is stable to quantum corrections in the regime that generates phenomenologically acceptable epochs of inflation and reheating. Thus, the marginal couplings by themselves are under control, in principle - if we were to ignore the relevant operators, the marginal couplings could be natural.

We estimate the size of the quantum corrections following the discussion of [34]. The quantum corrections to the quartic terms will include the term

$$\mathcal{L}_{radiative} \ni \frac{g^4}{64\pi^2} \ln\left(\frac{g^2\phi^2}{\tilde{\mathcal{M}}^2}\right)\phi^4\,. \tag{3.28}$$

This set of terms arises from summing together the 1-PI irreducible diagrams renormalizing ϕ^4 terms due to the virtual σ modes. Here $\tilde{\mathcal{M}}$ is the subtraction scale, which is in principle arbitrary. For example, it can be taken to be the mass scale of the σ field during inflation, which is integrated out to generate the effective ϕ theory; or it might be the cutoff of the 2-field hybrid model (since, practically, we do not expect that they are far apart).

The log will give the largest contribution if the argument is small. This means, we will get the strongest correction to the scalar field potential for the smallest values of ϕ, σ during inflation – i.e., near the exit. Since we already noted that the EFT during inflation will fail at the exit, needing a different EFT to describe reheating, we will here only require these corrections to remain under control during the early stages of inflation. To illustrate this quantitatively, let us take g of order 10^{-3} and $\alpha \sim 0.1$, with the cutoff $\mathcal{M} \sim 10^{-2} M_{pl}$. In this case the log is of order unity; the loop factor $64\pi^2$ in the numerator gives a suppression factor of order 10^{-3}, so this gives a correction to ϕ^4, σ^4 of order $10^{-3} g^4$. Note that these values of g, α are within the allowed region in Fig. 1.

The strongest constraint on the ϕ^4 term is that its second derivative be subleading compared to the second derivative of the first two terms in (3.11). As we discussed previously, this means that the inflection point induced by the quartic corrections to (3.11) is far enough to allow the plateau to yield ~ 50 efolds of inflation. In our numerical example, $\ln(g^2\phi^2/\tilde{\mathcal{M}}^2) \sim 1$, and we find that this condition combined with Eqs. (3.17) gives

$$g \lesssim 1.6 \times 10^{-3} \,. \tag{3.29}$$

This is the bound we incorporated preemptively in Fig. 1. We see that this leaves a small region of parameter space that is consistent with both naturalness of the quartic couplings, and with sub-Planckian expectation values for ϕ, σ. As noted, these scalars cannot be hugely sub-Planckian: in this range one or both are within an order of magnitude of M_{pl}. Allowing for both scalars to have smaller ranges would require larger values of g. In turn this would require either finely tuned small quartic couplings for ϕ or some additional mechanism to suppress the self-coupling at all orders in the loop expansion. We of course assume that the theory is natural here, without large cancellations between regularized and bare terms in the loop expansion.

Note that in [12], we have discussed a very similar-looking problem with single field inflation, namely radiative corrections to the inflaton potential V. The point there was that the QFT corrections to $V(\phi)$ are automatically natural since they are of the form $F(\eta)V$, where $F(\eta)$ is an analytic function of $\eta \sim \frac{M_{Pl}^2}{V}\partial_\phi^2 V$ at $\eta = 0$, with $\mathcal{O}(1)$ expansion parameters. Thus during inflation, if the potential V is chosen to be flat, $\eta \ll 1$ and loop corrections will remain small. In this model the new issue arises from the presence of the new field σ, and its new couplings to ϕ. The Stewart model requires the biquadratic potential $\phi^2\sigma^2$ to dominate the ϕ^2, σ^4 terms. The bound (3.29) ensures precisely that.

Yet as we just stressed in the previous subsection, without some additional mechanism(s) to suppress the masses, and preserve a small hierarchy between them, it is not even possible to keep the field ranges below the Planck scale without fine tuning. And even if we tune the field range for M, we still need $\alpha < 0.12$, which in turn implies $g \sim 10^{-3}$, as designated in Figure 1. This shows just how fine tuned the theory must be – baring hierarchy protection mechanisms. One might hope that selecting a special scale in the log in Eq. (3.28), which makes the log very small might help [33], however this would only make sense if the UV corrections are tamed. Thus our discussion showcases just how desperately the model needs a mechanism in the UV to protect it from large quantum corrections to the masses.

3.2.3 Irrelevant Couplings

Finally, let us consider the irrelevant operators. In addition to being a potential problem in QFT, these are also a portal for the effects of quantum gravity to come in. Let us first note that the latter are far more dangerous. Basic dimensional considerations indicate that integrating out σ during inflation will give corrections of the form[7]

$$\delta\mathcal{L}_p \sim \frac{g^{4+2p}\phi^{4+2p}}{\mathcal{M}^{2p}} \tag{3.30}$$

Taking the lower bound $\mathcal{M} \sim gM$ and applying the second equation in (3.21), we find

$$\delta\mathcal{L}_p \sim g^4 \left(\frac{g\alpha}{8\times 10^{-5}}\right)^{2p} \frac{\phi^{4+2p}}{M_{pl}^{2p}} \tag{3.31}$$

For $g \sim 10^{-3}$, $\alpha \sim 0.1$, and noting that we are ignoring phase space and symmetry factors, this is generally smaller by a factor of g^4 as compared to Planck-suppressed operators with $\mathcal{O}(1)$ coefficients. We will therefore focus on the latter but note that in general QFT contributions will also need to be suppressed.

Estimates such as the one above can be overly pessimistic. We know that in QFT, a series of operators which individually look dangerous, can sum up in the effective action such that the relevant effective potential remains flat. Essentially this can happen when the loop expansion is an alternating series, with operators of the form ϕ^{p+4} having signs $(-1)^p$. As a result the sum total of all the operators which should be included in the EFT is merely a log correction to the leading term, as discussed in the previous section (see [12]). Thus the irrelevant operators then need not be a show-stopper, and indeed flattened potentials such as those discussed in [11], [17] depend on them.

However in the presence of the extra field σ, and with quantum gravity corrections having no known pattern, we will be maximally conservative, and instead outline the conditions which *guarantee* that operators irrelevant in the RG sense are also irrelevant in the sense of not contributing

[7] Aside from the overall prefactor $\sim 1/16\pi^2$ this term displays correct normalizations as per NDA. We will however ignore these factors in this section, since our main purpose here is to outline the issue. Such additional numerical factors may in fact be helpful.

during inflation. To this end, we will focus on the potentially the worst-behaved terms which are higher powers of the lighter field ϕ, normalized for convenience by the Planck scale M_{pl}. This is merely a matter of choice; a different normalization would yield apparently different numerical statements, but the contents would be exactly the same. So consider a coupling of the form

$$\delta V = \frac{\delta_p}{(p+4)!} \frac{\phi^{p+4}}{M_{pl}^p} \, . \tag{3.32}$$

If such corrections are too large, they behave as the inflaton mass term, shortening the width of the hybrid plateau of (3.10). More specifically, the magnitude as well as derivatives of such operators during inflation need to be smaller than the value and derivatives respectively of the tree-level potential in Eq. (3.11). Focusing on the second derivative, we find that

$$\delta_p \ll (p+2)! \, \alpha^{-p} \times 1.2 \times 10^{-13} \, . \tag{3.33}$$

Thus for $\alpha \sim 0.1$ we still need a mechanism which suppresses a finite set of irrelevant operators. This mechanism must suppress both QFT and quantum gravity corrections. To this end, a sub-Planckian axion, with or without monodromy, should be effective. We stress – as we noted above – that if there are cancellations between adjacent irrelevant operators in the EFT expansion of the effective potential, as in the case of large field models or Coleman-Weinberg theories where irrelevant operators comprise an alternating series, that could help too. Our analysis of individual operators nevertheless shows that this may be easier to realize with sub-Planckian field ranges and parameters. As we have stressed all along, a mechanism which subverts the UV sensitivity of the masses is beneficial, since it will also help with the irrelevant operators.

The upshot is that even though our model has sub-Planckian field ranges, the irrelevant operators are not automatically guaranteed to be parametrically suppressed relative to the Planck scale.

The Planck-suppressed irrelevant operators of sufficiently low dimension might in fact interfere with slow-roll inflation unless their dimensionless couplings are kept sufficiently small.

3.3 A Pseudoscalar Realization and its 4-Form Dual

Small field hybrid models of inflation, as exemplified by the Stewart model, face two serious issues:

- The scalar masses are UV sensitive within the confines of QFT;
- There are Planck-suppressed irrelevant operators which require very small dimensionless coefficients.

As we will review and develop here, both of these problems may be addressed by considering ϕ, σ as pseudoscalar axions dual to massive 4-form field strengths [16]. The masses μ, m are dual to the gauge theory masses, which are not UV sensitive; while corrections to the scalar potential are suppressed by additional powers of the ratio of the masses to the cutoff, $m/\mathcal{M}$, $\mu/\mathcal{M}$. These effects follow from the gauge symmetries of the model. In the duality frame described by the 4-form field strengths, these are a pair of nonlinearly realized $U(1)$ gauge symmetries, with Stückelberg fields restoring the gauge symmetry of the mass term. In the dual scalar theory the gauge group is discrete $\mathbb{Z} \times \mathbb{Z}$, with each factor acting on a scalar and on a discrete variable. These discrete variables are dual to 4-form flux, labelling distinct branches of a multivalued potential [8], [12], [16], and act as a sort of discrete Stückelberg field.

The symmetries are obscured on the scalar side by their nonlinear realization combined with gauge fixing, which follows from picking a specific branch where the scalar longitudinal modes of the massive 4-form field strengths reside. Nevertheless the gauge redundancies remain operational in the full phase space of the theory, ensuring technical naturalness and protecting the scalar dynamics from the perils of quantum gravity. In the end, our goal here is to rewrite the Stewart model as an example of a simple gauge fixed EFT of massive 4-forms, and demonstrate how the ills of the scalar theory may be healed by gauge symmetries.

We find that for scalar theories satisfying the constraints outlined in §2.2, with sub-Planckian field ranges, the dual theory appears to be very strongly coupled: dimensionless coefficients of the leading irrelevant operators, written as powers of the field strength, are pushed to be large. The duality map yields an NDA-like presentation of the pseudoscalar action in terms of which the functions defining the potential appear to have large coefficients in a Taylor series expansion. These apparent large couplings are the price we pay for our mechanism for controlling m, μ, while maintaining sub-Planckian field ranges in the face of constraints given by the data.

However, these coefficients may not be the proper measure of couplings, governing the scattering amplitudes of asymptotic states. Absent mass terms, the 4-forms are non-propagating; with masses, the propagating modes are the longitudinal ones, which come multiplied by powers of the gauge field mass. Thus, the physical asymptotic states have couplings that are suppressed by ratios like $\mu/\mathcal{M}$, $m/\mathcal{M}$. Wavefunction renormalizations (aka "seizing" [43]) can push down the effective coupling further. The irrelevant operators induced by the leading "large" coupling are natural in the naïve sense – that is, they have order $\mathcal{O}(\lesssim 1)$ dimensionless coefficients.

A complete exploration of naturalness and NDA for massive p-form gauge fields has not been carried out and we will not do so here. A UV completion of our 4-form theory, would be an important way to explore the range of validity of our EFT. We will outline the issues that we encounter and point the way to possible resolutions.

3.3.1 A word on the Duality

The formalism of axion monodromy as a theory of inflation has been studied extensively. The possible UV completion as a string theory and apparent simplicity of the effective field theory makes this theory an attractive candidate to explain the dynamics of inflation. In this note, we highlight the mechanism that allow for a duality between a theory of a massive three form with a shift symmetry and a theory of a massive scalar field exhibiting a shift symmetry.

We start by considering the Lagrangian of a massless pseudo scalar field with a shift symmetry:

$$\mathcal{L} = f^2(\partial_\mu \phi)^2 \tag{3.34}$$

We want to create a mass gap while maintaining the shift symmetry and with- out introducing any additional degrees of freedom.

In this note we show that a theory of a massive three form with a shift symmetry is dual to a theory of a massive scalar field with a shift symmetry. The number of degrees of freedom is equal to one in both theories.

Let us start by considering a dynamical two-form $B_{\mu\nu}$. (In this dual picture, the pseudo scalar ϕ is replaced by a two-form $B_{\mu\nu}$). The Lagrangian for B is:

$$\mathcal{L} = h_{\mu\nu\lambda}h^{\mu\nu\lambda} \tag{3.35}$$

where $h_{\mu\nu\lambda} = \partial_{[\mu}B_{\nu\lambda]}$ This theory exhibits a global shift symmetry:

$$B_{\mu\nu} \rightarrow B_{\mu\nu} + \Omega_{\mu\nu} \tag{3.36}$$

Now we want to promote that global shift symmetry to a gauge symmetry and create a mass gap for B without introducing additional degrees of freedom:

First let us review the dynamics of a theory of a four-form. The Lagrangian for such a theory is:

$$\mathcal{L} = F_{\mu\nu\lambda\rho}F^{\mu\nu\lambda\rho} \tag{3.37}$$

where

$$F_{\mu\nu\rho} = \partial_{[\mu}A_{\nu\lambda\rho]} \tag{3.38}$$

F is a totally antisymmetric tensor in 4 dimensions. Hence it is totally determined :

$$F_{\mu\nu\lambda\rho} = q(x^a)\epsilon_{\mu\nu\lambda\rho} \tag{3.39}$$

The equation of motion for F is:

$$\partial^\mu F_{\mu\nu\lambda\rho} = 0 \implies q(x^a) = cst \tag{3.40}$$

This is a theory of constant with one propagating degree of freedom. Now, let us consider the following Lagrangian, where $A_{\mu\nu\lambda}$ couples to the external conserved current $J_{\mu\nu\lambda}$:

$$\mathcal{L} = F_{\mu\nu\lambda\rho}F^{\mu\nu\lambda\rho} + A_{\mu\nu\lambda}J^{\mu\nu\lambda} \tag{3.41}$$

This Lagrangian is invariant under the gauge transformation:

$$A_{\mu\nu\rho} \rightarrow A_{\mu\nu\rho} + d_{[\mu}\Omega_{\nu\rho]} \tag{3.42}$$

It is worth noting that A contains no propagating degrees of freedom because of the gauge freedom. By coupling the three-form A to the two-form B we effectively gauge the shift symmetry:

$$\mathcal{L} = -\frac{1}{48}F^2_{\mu\nu\lambda\rho} + \frac{m^2}{12}(A_{\mu\nu\lambda} - h_{\mu\nu\lambda})^2 \tag{3.43}$$

We have successfully created a mass gap for B while maintaining the shift symmetry and without introducing any additional degree of freedom.
We now wish to promote $h_{\mu\lambda\rho}$ as a fundamental three-form by imposing the Bianchi identity:

$$\epsilon^{\mu\nu\lambda\rho}\partial_{[\mu}h_{\nu\lambda\rho]} = 0 \tag{3.44}$$

The Lagrangian for our theory becomes:

$$\mathcal{L} = -\frac{1}{48}F^2_{\mu\nu\lambda\rho} + \frac{m^2}{12}(A_{\mu\nu\lambda} -_{\mu\nu\lambda})^2 + \frac{m}{6}\phi\epsilon^{\mu\nu\lambda\rho}\partial_\mu h_{\nu\lambda\rho} \tag{3.45}$$

where the pseudo scalar ϕ appears as a Lagrange multiplier.
In the next part of that note, we wish to show how this theory is dual to a theory of a massive scalar field with a shift symmetry.
First, let us integrate out h :

$$\mathcal{L} = -\frac{1}{48}F^2_{\mu\nu\lambda\rho} + \frac{1}{2}(\partial\phi)^2 + \frac{m}{24}\phi\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu\lambda\rho} \tag{3.46}$$

We then enforce $F_{\mu\nu\lambda\rho} = 4\partial_{[\mu}A_{\nu\lambda\rho]}$ by introducing the Lagrange multiplier Q:

$$\mathcal{L} = -\frac{1}{48}F^2_{\mu\nu\lambda\rho} + \frac{1}{2}(\partial\rho)^2 + \frac{m\phi + Q}{24}\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu\lambda\rho} - \frac{Q}{6}\epsilon_{\mu\nu\lambda\rho}\partial^\mu A^{\nu\lambda\rho} \tag{3.47}$$

The equation of motion for F give us:

$$F_{\mu\nu\lambda\rho} = (m\phi + Q)\epsilon_{\mu\nu\lambda\rho} \tag{3.48}$$

We can integrate out F and we get:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(m\phi + Q)^2 + \frac{1}{6}\epsilon_{\mu\nu\lambda\rho}(\partial^\mu Q)A^{\nu\lambda\rho} \tag{3.49}$$

This is a theory of a massive scalar field with a shift symmetry:

$$\begin{aligned} \phi &\to \phi + \phi_0 \\ Q &\to Q - \frac{\phi_0}{\mu} \end{aligned} \tag{3.50}$$

3.3.2 Single Field Monodromy Inflation

Before we dive into the details of two fields hybrid inflationary theory, let's first look into the dynamics of a single field monodromy theory. We start with the following Lagrangian:

$$\mathcal{S} = -\frac{1}{48}\int F_{\mu\nu\rho\lambda}F^{\mu\nu\rho\lambda} \tag{3.51}$$

where F is $F_{\mu\nu\rho\lambda} = \partial_{[\mu}A_{\nu\rho\lambda]}$.

Because F is totally anti-symmetric, F can only be the levi-civita tensor:

$$F_{\mu\nu\rho\lambda} = q(x^a)\epsilon_{\mu\nu\rho\lambda} \tag{3.52}$$

The equation of motion gives us:

$$D^\mu F_{\mu\nu\rho\lambda} = 0 \implies q(x^a) = cst \tag{3.53}$$

We can see that a theory of a dynamical four form is nothing but a theory of a constant.

Now, let us couple this four form to a dynamical scalar field ϕ as follows:

$$\mathcal{S} = \int d^4x\sqrt{g}\left(\frac{M_p^2}{2}R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{48}F^2_{\mu\nu\lambda\sigma} + \frac{\mu\phi}{24}\frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{g}}F_{\mu\nu\lambda\sigma} + ...\right) \tag{3.54}$$

This Lagrangian exhibits a shift symmetry, which is desired in order to control the flatness of the potential in the case of inflationary theories:

$$\begin{aligned} &\phi \to \phi + c \\ &\mathcal{L} \to \mathcal{L} + c\mu\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu\rho\lambda}/24 \end{aligned} \tag{3.55}$$

Although the Lagrangian is shifted by $c\mu\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu\rho\lambda}/24$, this term is a total derivative and will be absorbed by the boundary conditions.

Next, we can look at the tree level corrections to the propagator and we can see explicitly that this theory is equivalent to a theory of a propagating massive scalar field ϕ:

$$\frac{1}{p^2} + \frac{1}{p^2}\mu^2\frac{1}{p^2} + \frac{1}{p^2}\mu^2\frac{1}{p^2}\mu^2\frac{1}{p^2} + ... = \frac{1}{p^2 - \mu^2} \tag{3.56}$$

More generally, we can show that a theory of a dynamical four-form coupled to a scalar field exhibiting a shift symmetry is dual to a theory of a dynamical massive scalar field also exhibiting a shift symmetry (see section 3.4).

$$\mathcal{L} = -\frac{1}{48}F^2_{\mu\nu\lambda\sigma} + \frac{m^2}{12}(A_{\mu\nu\lambda} - h_{\mu\nu\lambda})^2 + \frac{m}{6}\phi\epsilon^{\mu\nu\lambda\rho}\partial_\mu h_{\nu\lambda\rho} \quad (3.57)$$

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(m\phi + Q)^2 + \frac{1}{6}\epsilon_{\mu\nu\lambda\sigma}(\partial^\mu Q)A^{\nu\lambda\sigma} \quad (3.58)$$

In models of inflation, although we want to control the flatness of the potential by having a shift symmetry, we also want to break that shift symmetry, as inflation eventually has to come to an end. The shift symmetry is broken by choosing a value for the mass of our scalar field. In other words, we choose a path and we realize a monodromy (from the greek mono:single , dromy:path).

3.3.3 Dual of a Two Field Hybrid Model

We now turn to illustrating how to embed the family of two interacting scalar field theories that can support hybrid inflation, and contain the Stewart model as a limit into a dual theory of two interacting 4-form field strengths. In this case, in addition to the complications arising due to nonlinear terms, the model also includes a term linear in σ. In principle, we could try to just shift the field until the linear term is absorbed away. However, $\sigma \sim 0$ gives the "instantaneous" vacuum (in the Born-Oppenheimer sense) in the σ sector at the beginning of inflation. As we noted above, in our version of hybrid inflation, the inflaton (ϕ) mass goes up to the cutoff at the end of inflation; while the σ mass drops below the cutoff as $\phi \to 0$, as seen in Eq. (3.13). Since we regard the pseudoscalar-4-form duality as an IR duality, with field and operator dimensions computed from the IR fixed point, we must carefully specify the EFT and thus the scalar field value about which we perform the dualization.

In this section we will compute the dual form theory following these steps:

- We will establish the dual correspondence of the canonical variables on the two sides, using

NDA-normalized variables; in order to connect σ and ϕ with their dual forms, we invoke the sector of the theory just at the end of inflation, but before σ relaxes to the true vacuum; here both scalars are light.

- We will show that the scalar kinetic terms dualize to mass terms for the dual forms.
- We will then identify the NDA-normalized non-linear couplings.
- We will establish the mapping between the operators on the two dual sides in the weak coupling; we will identify the lower bound on the cutoff of the theory in terms of the dimensional parameters in the EFT. By weak coupling we mean that dimensionless couplings for operators normalized by the cutoff are small, and the theory is in some sense close to Gaussian.

We are limiting this section to weak coupling for illustrative purposes. In §3.3 we will find that the bounds from naturalness and from observations, which we explored on the scalar side, appear to imply that the dual form theory must be in strong coupling during inflation. We will discuss the possible implications of this observation there. The main result here will be a somewhat telegraphic walk through the duality transformation, skipping some of the explicit steps above; the reader can however readily fill in the missing steps of the complete analysis.

Let us begin by establishing the canonical transformation between the scalar and 4-form pictures, setting up the 'dictionary' for transitioning from one side to the other. To simplify our formulae, we will use the dimensionless zero-form duals of the 4-forms:

$$\mathtt{F} = -\frac{1}{4!\mathcal{M}_*^2}\epsilon_{\mu\nu\lambda\sigma}F^{\mu\nu\lambda\sigma}\,, \qquad \mathtt{G} = -\frac{1}{4!\mathcal{M}_*^2}\epsilon_{\mu\nu\lambda\sigma}G^{\mu\nu\lambda\sigma}\,, \tag{3.59}$$

Here $F = dA$, $G = dB$ locally; globally, the values A, B in different charts of the cover of spacetime may be related by 2-form gauge transformations $A \to A - d\Lambda_A$, $B \to B - d\Lambda_B$ in the overlap between the charts.

The scale $\mathcal{M}_*$ in (3.59), included for dimensional reasons here, is the strong coupling of the scalar EFT (3.7), as per NDA. We will link it to the theory's dimensional parameters below. Note that since we are interested in dualizing a model of hybrid inflation for which higher-derivative terms do not contribute to the dynamics, we restrict our attention to only the terms which are quadratic in derivatives. Note that we will ignore terms of higher than quadratic order in ϕ, σ. In standard slow roll inflation, these terms are kept small by the dynamics [17]. As in that work there could be other regimes of the theory in which higher-derivative terms could also assist a slow-roll phase of the theory. A hybrid model with these higher-derivative terms activated would be an interesting topic for future work.

We expect that the dual scalar theory in general takes the form

$$\mathcal{L} = \frac{1}{2}\mathcal{Z}(\frac{\mu\phi}{\mathcal{M}_*^2}, \frac{m\sigma}{\mathcal{M}_*^2})(\partial\phi)^2 + \frac{1}{2}\hat{\mathcal{Z}}(\frac{\mu\phi}{\mathcal{M}_*^2}, \frac{m\sigma}{\mathcal{M}_*^2})(\partial\sigma)^2 - \mathcal{M}_*^4\, \mathcal{V}(\frac{\mu\phi}{\mathcal{M}_*^2}, \frac{m\sigma}{\mathcal{M}_*^2})\,. \tag{3.60}$$

As in the single-field case, we introduce the axions φ and χ, which are related to the hybrid inflation scalars ϕ and σ via:

$$\mu\phi = \mu\varphi + Q\,, \qquad m\sigma = m\chi + P\,. \tag{3.61}$$

This is to say that the discrete gauge symmetry $\mathbb{Z} \times \mathbb{Z}$ shifts Q, φ and P, χ so that ϕ, σ remain unchanged. The compact scalars φ, χ are the duals to the longitudinal modes of the full massive form system: more precisely $d\varphi, d\chi$ are dual to the 3-form Stückelberg field strengths. They can be absorbed into the local fluctuations of the massive 3-form potential via gauge fixing.

The functional form (3.60), in which non-derivative couplings of ϕ, σ come multiplied by factors of μ, m, and normalized by $\mathcal{M}_*$, is based on our experience with the single-field case. In the weak coupling limit we study here, this form is justified in that it produces a natural theory of 4-forms, in the sense of NDA. It should be possible to justify this combination of mass parameters and scalar field values entirely within the scalar frame by utilizing discrete gauge invariances of the model, with the explicit discrete Stückelberg fields P, Q, but we leave this for future work.

To formally dualize the scalar theory, we have found it useful to employ the dimensionless variables

$$\Phi = \frac{\mu\varphi + Q}{\mathcal{M}_*^2} \,, \qquad \mathcal{X} = \frac{m\chi + P}{\mathcal{M}_*^2} \,, \tag{3.62}$$

and then rewrite the potential in (3.60) in terms of them. We call these "NDA-normalized variables" as the potential $\mathcal{V}$ and the kinetic functions $\mathcal{Z}$, $\hat{\mathcal{Z}}$, can be written in terms of them. Furthermore, imagine for the moment that we can approximate $\mathcal{Z}, \hat{\mathcal{Z}} \simeq 1$. Adding and subtracting the Lagrange multiplier terms $\Phi\mathtt{F} + \mathcal{X}\mathtt{G}$, the "chimera" Lagrangian that ensues is

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}(\partial\chi)^2 + \frac{\mu\varphi}{4!}\epsilon_{\mu\nu\lambda\sigma}F^{\mu\nu\lambda\sigma} + \frac{m\chi}{4!}\epsilon_{\mu\nu\lambda\sigma}G^{\mu\nu\lambda\sigma} - \mathcal{M}_*^4\Big(\mathcal{V}(\Phi,\mathcal{X}) - \Phi\,\mathtt{F} - \mathcal{X}\,\mathtt{G}\Big) \,, \tag{3.63}$$

which, after integrating the scalar-4-form bilinears by parts, becomes

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}(\partial\chi)^2 - \frac{\mu}{3!}\epsilon_{\mu\nu\lambda\sigma}\,\partial_\mu\varphi\, A^{\nu\lambda\sigma} + \frac{m}{3!}\epsilon_{\mu\nu\lambda\sigma}\,\partial_\mu\chi\, B^{\nu\lambda\sigma} - \mathcal{M}_*^4\Big(\mathcal{V}(\Phi,\mathcal{X}) - \Phi\,\mathtt{F} - \mathcal{X}\,\mathtt{G}\Big) \,. \tag{3.64}$$

The first four terms dualize to mass terms for A, B: we can complete the squares for the scalar derivatives and integrate them out, with the remaining terms being precisely $\mu^2 A^2$ and $m^2 B^2$. This part is straightforward because this contribution to the Lagrangian is bilinear at weak coupling.

What remains is to dualize the effective potential, and replace the variables Φ and $\mathcal{X}$ with $\mathtt{F}$ and $\mathtt{G}$ defined in (3.59). We can treat the last four terms in (3.64) independently from the rest because Φ, are combinations of the discrete Stückelberg fields Q, P, and so can be varied independently of φ, σ. The 4-form dual of the final term in brackets in (3.64) is the Legendre transform of the effective potential. In practice, we integrate out the fields $\Phi, \mathcal{X}$ to replace them with $\mathtt{F}, \mathtt{G}$, which means inverting [44], [45]

$$\mathtt{F} = \partial_\Phi \mathcal{V}\,, \qquad \mathtt{G} = \partial_\mathcal{X} \mathcal{V}\,, \tag{3.65}$$

and substituting $\Phi = \Phi(\mathtt{F}, \mathtt{G})$, $\mathcal{X} = \mathcal{X}(\mathtt{F}, \mathtt{G})$ into $K = \Phi\, \mathtt{F} + \mathcal{X}\, \mathtt{G} - \mathcal{V}(\Phi, \mathcal{X})$. In doing this, we bear in mind that during inflation the scalar σ is much heavier than ϕ – in fact it may be heavier than the cutoff $\mathcal{M} \sim gM$. Since it is changing very slowly, with initial value $\sigma \simeq 0$, the field σ remains displaced from its true minimum at $\sqrt{2}M$ for a period after inflation ends, when σ is much lighter than during inflation. Thus we can pick the transient value of $\sigma \simeq 0$ as a "pivot" to dualize the scalar theory at it. This means that we should dualize σ around zero during inflation and right after inflation, and around $\sqrt{2}M$ at the very late stages after inflation when much of reheating takes place, when σ oscillates around the true minimum. The region near $\sigma = \sqrt{2}M$, about which the theory reheats, and the "plateau" at $\sigma \sim 0$ describing inflation and its end prior to reheating, are different phases; they are best treated as distinct EFTs as we already explained in §2. In the latter, σ remains heavier than ϕ. This is true even for small ϕ when we choose $m \gg \mu$, as we do if we wish ϕ to be the inflaton. Near $\sigma = \sqrt{2}M$, ϕ becomes the heavy field with a mass at or near the cutoff. These phases must in general be connected inside a UV completion.

We compute the duality transformation for the case that $\mathcal{V}$ is has the functional form (3.7): in terms of Φ, $\mathcal{X}$, this is:

$$\mathcal{V} = \frac{1}{2}\Phi^2 + \frac{1}{2}\big(\mathcal{X} - \gamma\big)^2 + \frac{\bar{g}^2}{4}\Phi^2\mathcal{X}^2 + \frac{\bar{\lambda}}{4}\Big(\mathcal{X}^2 - \delta^2\Big)^2 + \frac{\bar{\lambda}'}{4}\Phi^4 + \dots\,, \tag{3.66}$$

where the rescaled couplings are:

$$\gamma = \sqrt{2}\frac{mM}{\mathcal{M}_*^2}\,, \;\; \delta = \frac{m\tilde{M}}{\mathcal{M}_*^2}\,, \;\; \bar{\lambda} = \lambda\left(\frac{\mathcal{M}_*}{m}\right)^4, \;\; \bar{g}^2 = g^2\left(\frac{\mathcal{M}_*^2}{m\mu}\right)^2, \;\; \bar{\lambda}' = \lambda'\left(\frac{\mathcal{M}_*}{\mu}\right)^4. \tag{3.67}$$

These are the couplings that appear in the 4-form dual; we have not written out the operators that would be irrelevant in the scalar frame. We expect that a proper formulation of NDA for this theory will involve potentials and coefficients of $(\phi)^2$, $(\sigma)^2$ that are functions of Φ, $\mathcal{X}$, $\bar{g}^2/16\pi^2$, $\lambda/16\pi^2$, $\lambda'/16\pi^2$. The appearance of couplings with factors of $1/16\pi^2$ generally arises in effective actions due to phase space factors in loop integrals [37].

In this section, to provide an explicit example, we will work in the approximation $\bar{\lambda}, \delta, \bar{\lambda}' \ll^2 1$, and drop terms proportional to $,', \delta$. This limit is consistent with the functional form of the Stewart model. Note that this "weak coupling" assumption is stronger than the assumption that $\lambda, \delta, \lambda' \ll g^2 1$; as we will see when we impose consistency with the constraints of §2, data pushes $\bar{g}^2$ to be large. Yet it can remain the domain of strong coupling below the cutoff of the NDA-normalized action. We will retain only the lowest order irrelevant operators on the dual form side, as they are duals of the marginal operators on the scalar side. We will neglect writing out explicitly the higher dimension irrelevant operators here; we will however need to discuss them in §3.3, when we go to comparatively large ϕ in order to describe the epoch of inflation that imprints on the CMB.

To solve Eqs. (3.65), we solve the nonlinear equations one at a time, and perform the expansion of the algebraic inversions of (3.65) in Taylor series in the couplings (3.67). This means, formally, that we take the "instantaneous vacuum" to be controlled by the root of the nonlinear equations (which, to any finite order in couplings, are polynomials), dominated by the linear terms. The solutions remain perturbative in the coupling constants, and thus consistent with the EFT description.

Using (3.66) in (3.65), inverting, and expanding in the couplings, we find:

$$\begin{aligned} \Phi &= \mathtt{F}\Big(1-\frac{\bar g^2}{2}\mathcal{X}^2\Big)+\mathcal{O}(\bar\lambda,\bar\lambda',\bar g^4) \\ \mathcal{X} &= \Big(\mathtt{G}+\gamma\Big)\Big(1-\frac{\bar g^2}{2}\Phi^2\Big)+\mathcal{O}(\bar\lambda,\bar\lambda',\bar g^4)\,. \end{aligned} \tag{3.68}$$

Since we want solutions which are perturbative in the coupling, we can use each of these equations to $\mathcal{O}(1)$ to replace the terms to $\mathcal{O}(\bar g^2)$ in the other, and find the correct answer to the order $\bar g^2$. Thus, the inversion formulas are

$$\begin{aligned} \Phi &= \mathtt{F}\Big(1-\frac{\bar g^2}{2}\big(\mathtt{G}+\gamma\big)^2\Big)+\mathcal{O}(\bar\lambda,\bar\lambda',\bar g^4)\,, \\ \mathcal{X} &= \Big(\mathtt{G}+\gamma\Big)\Big(1-\frac{\bar g^2}{2}\mathtt{F}^2\Big)+\mathcal{O}(\bar\lambda,\bar\lambda',\bar g^4)\,. \end{aligned}$$

Substituting these into $K=\Phi\,\mathtt{F}+\mathcal{X}\,\mathtt{G}-\mathcal{V}(\Phi,\mathcal{X})$ and using (3.66) finally yields

$$K=\frac{1}{2}\mathtt{F}^2+\frac{1}{2}\Big(\mathtt{G}+\gamma\Big)^2-\frac{1}{2}\gamma^2-\frac{\bar g^2}{4}\mathtt{F}^2\Big(\mathtt{G}+\gamma\Big)^2+\mathcal{O}(\bar\lambda,\bar\lambda',\bar g^4)\,. \tag{3.69}$$

Using (3.59), the total Lagrangian in the dual variables and at weak coupling is therefore

$$\mathcal{L}=\mathcal{M}_*^4\Big\{\frac{1}{2}\mathtt{F}^2+\frac{1}{2}\Big(\mathtt{G}+\gamma\Big)^2-\frac{1}{2}\gamma^2-\frac{\bar g^2}{4}\mathtt{F}^2\Big(\mathtt{G}+\gamma\Big)^2\Big\}+\frac{\mu^2}{12}A^2_{\nu\lambda\sigma}+\frac{m^2}{12}B^2_{\nu\lambda\sigma}+\mathcal{O}(\bar\lambda,\bar\lambda',\bar g^4)\,. \tag{3.70}$$

where again $F=dA$, and $G=dB$, and $\mathtt{F},\mathtt{G}$ are defined in terms of F,G in Eqs. (3.59).

Note that from the first of Eqs. (3.69), the vacuum of the theory $\mathcal{X}=0$ maps to $\mathtt{G}+\gamma=0+\mathcal{O}(\bar g^2)$. This means that $\mathtt{G}$ does not correctly describe the fluctuations of the dual form sector B,G around this vacuum. Instead the correct canonical variable to use is

$$\mathcal{G}=\mathtt{G}+\gamma\,. \tag{3.71}$$

This constant shift of the magnetic dual of the 4-form field strength implies a shift of the electric 4-form $\mathcal{G}_{\mu\nu\lambda\sigma} = \mathcal{M}_*^2 \, \epsilon_{\mu\nu\lambda\sigma} \, \mathcal{G}$ by

$$\mathcal{G}_{\mu\nu\lambda\sigma} = G_{\mu\nu\lambda\sigma} + \gamma \mathcal{M}_*^2 \, \epsilon_{\mu\nu\lambda\sigma} \, , \tag{3.72}$$

or in the form notation, $\mathcal{G} = \mathtt{G} + \gamma\mathcal{M}_*^2 \, \Omega_4$, where Ω_4 is the space-time volume 4-form. If we write $\mathcal{G} = d\mathcal{B}$, then the 3-form potentials are are related by

$$\mathcal{B}_{\mu\nu\lambda} = B_{\mu\nu\lambda} + h_{\mu\nu\lambda} \tag{3.73}$$

Here h is defined locally, by integrating

$$dh = \gamma \mathcal{M}_*^2 \, \Omega_4 \, . \tag{3.74}$$

to yield

$$h = \gamma \mathcal{M}_*^2 \, t \, \Omega_3 \, , \tag{3.75}$$

where Ω_3 is the volume form of the constant comoving time hypersurfaces.

With all of this, the renormalizable and leading irrelevant terms in the dual theory at weak coupling around the transient vacuum $\sigma \simeq 0$ are:

$$\mathcal{L} = -\frac{1}{2\cdot 4!} F_{\mu\nu\lambda\sigma}^2 - \frac{1}{2\cdot 4!} \mathcal{G}_{\mu\nu\lambda\sigma}^2 - \frac{1}{2}\gamma^2 \mathcal{M}_*^4 - \frac{\bar{g}^2}{4\cdot (4!)^2 \mathcal{M}_*^4} F_{\mu\nu\lambda\sigma}^2 \mathcal{G}_{\mu\nu\lambda\sigma}^2 \\ + \frac{\mu^2}{12} A_{\nu\lambda\sigma}^2 + \frac{m^2}{12}\Big(\mathcal{B}_{\mu\nu\lambda} - h_{\mu\nu\lambda}\Big)^2 + \mathcal{O}(\bar{\lambda}, \bar{\lambda}', \bar{g}^4) \, .$$

This is the dual form EFT of the Stewart model limit (3.5) of hybrid inflation in the regime $\sigma \simeq 0$ and $g\phi < m$ (in the case that $\overline{\lambda}, \overline{\lambda}' \ll \bar{g}^2 \ll 1$). This region of field space, which is reached right after inflation ends, is readily accessible to simple analytical tools to construct the dual. Note that this regime can be readily extrapolated down to $\phi = 0$ as long as $\sigma \simeq 0$. However the extrapolation down to $\sigma = \sqrt{2}M$ requires a different EFT, because at $\sigma = \sqrt{2}M$ the ϕ field gets Higgsed by the σ *vev*, having the mass $\mu_{eff}^2 \simeq g^2 M^2 \sim \mathcal{M}^2$ (3.15). Next we wish to discuss the limit of the theory that extends beyond this regime and supports inflation – which requires $g\phi \gg m$. As we have stated, and will see in detail shortly, that forces us to go beyond weak coupling and the action (3.76).

Note, that the situation here is analogous to what we encountered in the single field case. The scalar and massive 4-form duals are related simply in the weak coupling limit, with the scalar being the light and weakly coupled longitudinal mode of the massive 4-form field strength. Yet to produce inflation that fits the data the theory must be pushed into the strong coupling, beyond the simple perturbative picture, where the connection between the scalar and the form fields becomes very nonlinear. Nevertheless, we take the attitude that the existence of the dual form picture suffices for our purpose since it is a 'home' for the gauge symmetries that protect the scalar mass. Once that is in force, to study inflation one can work on the scalar side alone, and evolve the theory to strong coupling.

Before continuing with the foray into the inflationary regime, let us clarify the role of the γ-dependent terms in Eq. (3.76). These are the positive vacuum energy $\frac{1}{2}\gamma^2\mathcal{M}_*^4$, and the 3-form h inside the mass term for $\mathcal{B}$. The former is the initial value of the vacuum energy driving inflation. In the dual scalar picture, this comes from the σ mass term near $\sigma = 0$, and assumes that any additional contribution to the vacuum energy is negligible during inflation. The additional cosmological contributions which we neglected here would be the vacuum energy in the true vacuum, and neglecting them corresponds to choosing the vacuum energy in the true vacuum to be very small. This is just the cosmological constant problem, which is notoriously difficult to address in any EFT. This appearance of the vacuum energy as a constant in our EFT points to the fact that this appears to be fine tuned. This fine tuning could be addressed via saltatory variation of the cosmological constant by nucleation of membranes charged under the 4-forms, following [46]–[48]; regions with a different initial vacuum energy but the same couplings otherwise will either inflate forever, or collapse too soon.

Absent the mass term here, the only way to relax the initial vacuum energy towards zero is via membrane nucleation. In our theory, the mass term allows this vacuum energy to *also* be relaxed by the slow rolling of the longitudinal mode of the massive form. Here the 3-form h in the mass term ensures that the pivot point $\mathcal{G}, \mathcal{B} = 0$ correctly describes the onset of inflation in terms of variables with canonical kinetic terms. It appears as a source term in the equation of motion $d^*\mathcal{G} + m^2\mathcal{B} = m^2h$, driving the longitudinal mode to evolve so that the vacuum energy is lowered.

3.3.4 Scales and Couplings for Hybrid Monodromy EFT

We now turn to considering pseudoscalar axions in the parameter regime controlled by the dual requirements that the theory have sub-Planckian field ranges and be consistent with current data. Our aim is to write an effective action which respects NDA normalizations, in which we are assured that the mass parameters μ, m are UV-insensitive, and the irrelevant operators are suppressed.

The first thing we might try is to perform the duality map outlined in the previous section. The masses μ, m map to 4-form masses which we know are UV-insensitive; while past experience in the more extreme case of large-field inflation is that the irrelevant operators do not spoil slow-roll inflation [8], [12], [14], [16], [17]. From the discussion above, one would expect that a natural effective action of 4-forms following [37] would be a function of $F/\mathcal{M}_*^2$, $G/\mathcal{M}_*^2$, m, μ, $\bar{g}^2/16\pi^2$, $\overline{\lambda}/16\pi^2$, $\overline{\lambda}'/16\pi^2$, with the factors of $1/16\pi^2$ arising from loop factors.

As we noted, however, and in full analogy with the single large field theories, in the parameter regime of the Stewart model pointed to by the data, the duality map is no longer controlled by the leading linear map $F \sim \mu\phi$, $\mathcal{G} \sim m\sigma$. We can see from Eq. (3.68) that this would require $\Phi < \sqrt{2}$; we will see that this does not hold during the epoch of inflation that imprints on the observed CMB.

Inspired by the map at weak coupling, however, we could hope that there is an effective action in the scalar frame which satisfies some version of NDA. At weak coupling, the duality map indicates that μ, m are not renormalized (though we currently lack an argument directly in the scalar frame). It is then natural to suppose that the effective action would be a function of $\mathcal{M}_*$, Φ, $\mathcal{X}$, $\bar{g}^2/16\pi^2$, $\overline{\lambda}/16\pi^2$, $\overline{\lambda}'/16\pi^2$ with $\mathcal{O}(1)$ coefficients, as the weak-coupling duality points in this direction.

However, we will see that in the regime allowed by the data, $\overline{g}^2/16\pi^2$ is large, and when the field ranges are lowered by an order of magnitude below the Planck scale and more, perturbativity of the theory appears to be in jeopardy. A concern is that the effective action is strongly coupled and out of control. On the other hand, we will see that with $\alpha\mathcal{O}(\lesssim 1)$ we can still meet the data and suppress irrelevant operators enough while keeping $\overline{g}^2/16\pi^2$ barely within the range of EFT. One possible take from this is that identifying $\overline{g}^2$ defined in Eq. (3.67) as the coupling may be overly naïve. On the scalar side, Φ, $\mathcal{X}$ are of course not canonically normalized, and moving to canonical variables demonstrates that the natural couplings are g^2, λ, λ'. On the 4-form side, at weak coupling, the massive 4-forms F only propagate in the presence of a mass term for the gauge fields: the two-point functions $\langle {}^*F^*F\rangle \propto \mu$, $\langle {}^*G^*G\rangle \propto m$ up to contact terms, so that even $\frac{\overline{g}^2}{16\pi^2} \gtrsim 1$ can induce $\mathcal{O}(1)$ values of $\overline{\lambda}$, $\overline{\lambda}'$ via quantum corrections. The large effective $\overline{g}^2$, which is required by combining naturalness and data, can come about as a result of nontrivial nonlinear mixings of various irrelevant operators on backgrounds with large form fluxes, that simulate the inflationary vacuum energy. In other words, the scale of the biquadratic operator $\sim g^2\phi^2\sigma^2$ on the scalar side is not set by a single large irrelevant operator in the dual theory but is a combination of many irrelevant operators of dimension higher than eight, which add together enhancing the effective coupling. In the end, it is an open question how to implement naturalness and NDA for massive 4-forms or their duals[8]. We leave this for future work.

In the remainder of this section we develop the above points in detail, and highlight possible paths towards ensuring hybrid inflation makes sense both phenomenologically and as a natural QFT safe from quantum gravity.

3.3.4 Identification of NDA Parameters

We open with identifying the range of parameters $\mathcal{M}_*$, $\overline{g}^2$, $\overline{\lambda}$, $\overline{\lambda}'$ in the potential (3.66). We dub this the NDA potential for the scalar fields. We will conjecture that the discrete gauge symmetries together with dimensional analysis demand that ϕ, σ appear in the form Φ, $\mathcal{X}$. We would further demand that the couplings of the theory are consistent with the values induced by quantum corrections; that is, that the model is self-consistently technically natural, or even simply natural. At present we do not know how this would work in practice, just as we do not completely understand NDA for massive 4-forms. Should the duality hold, there are nonrenormalization theorems in the scalar theory that we have not derived that will constrain the effective action. A simple guess, by rough analogy with [37], would be that if all couplings were $\mathcal{O}(16\pi^2)$, our theory would be technically natural if the effective potential could be writen as $V(\Phi, \mathcal{X})$ with $\mathcal{O}(1)$ couplings; if $\overline{g}^2/16\pi^2 \ll 1$, our theory could be natural for $\overline{\lambda}/16\pi^2$, $\overline{\lambda}'/16\pi^2$ to be small; and for $\overline{g}^2/16\pi^2 \gg 1$, the theory is strongly coupled and completely out of control.

Before turning to the couplings, our first question is the choice of the cutoff scale $\mathcal{M}$ and the

[8]And for that matter, NDA for massive vector fields.

strong coupling scale $\mathcal{M}_* = \mathcal{M}/\sqrt{4\pi}$. In a top-down theory these would of course be fundamental quantities. Here we are taking a bottom-up approach, asking what values of the cutoff give an action such that the dual form theory takes an NDA-like form. We thus identify $\mathcal{M}_*^4$ as the scale of the energy density over which the effective potential varies, and in particular the scale that drives inflation:

$$\mathcal{M}_*^2 \gtrsim \sqrt{2} m M \,. \tag{3.76}$$

The scale of the energy density driving inflation is thus $V \sim m^2 M^2 \mathcal{M}_*^4/2 = \mathcal{M}^4/32\pi^2$. Given the subtleties we are about to discuss in identifying expansion coefficients in NDA for this theory, there may be some wiggle room here. We will simply adopt our straightforward definition of $\mathcal{M}_*$ and see where it gets us. Note that this definition of $\mathcal{M}_*$ guarantees that the field range of $\mathcal{X}$ between the end of inflation and reheating is $\mathcal{O}(1)$.

Next, we wish to bound $\bar{g}^2$. Equation (3.16), giving the number of efolds of inflation in terms of the field displacement ϕ, gives (after a few lines of algebra):

$$N = \alpha^2 \pi^2 \frac{\bar{g}^2}{16\pi^2} \Phi^2 \,. \tag{3.77}$$

Demanding now that $\alpha < 1$ and $N \gtrsim 50$ immediately shows that to have inflation we must start with $\bar{g}^2\Phi^2$ which is initially at least as big as

$$\bar{g}^2 \Phi^2 \gtrsim \frac{16N}{\alpha^2} \gtrsim 800 \,. \tag{3.78}$$

If this were the only constraint, we could support sufficient inflation in a regime where the coupling $\bar{g}^2/16\pi^2$ is small. At weak coupling, $\Phi \sim 4\pi$ is a unitarity bound. This maps to the statement $F/\mathcal{M}_*^2 < 4\pi$, $F/\mathcal{M}^2 < 1$. If this bound on Φ extends past the regime that the duality map can be constructed perturbatively in powers of the fields, we can ask when the unitarity limit saturates the inequality in Eq. (3.78). This occurs when $\overline{g}^2 \geqslant 5$, or $\overline{g}^2/16\pi^2 \gtrsim 0.032$. Taking $\overline{g}^2/16\pi^2$ to be the right parameter for an NDA analysis of the scalar action, the effective action should still be under perturbative control even if the tree-level coupling looks strong. A further constraint on $\overline{g}$ comes from imposing the observed scalar power which combine with the number of efolds leads to Eq. (3.21). We begin with

$$\frac{g^2}{16\pi^2} = \frac{\overline{g}^2}{16\pi^2}\left(\frac{m\mu}{\mathcal{M}_*^2}\right)^2 , \tag{3.79}$$

and employ our bound (3.76). Furthermore, we will assume a hierarchy $\mu = m$; $= 1/8$ corresponds to the bound from M being sub-Planckian. Finally, using Eq. (3.76), and then Eq. (3.21) to write m/M in terms of α, g we find: The resulting lower bound on $\overline{g}^2$ is:

$$\frac{\overline{g}^2}{16\pi^2} \gtrsim \frac{6\times 10^{-8}}{{}^2\alpha^6 g^2} . \tag{3.80}$$

If we were to set $\epsilon = 1/8$, as required for $M < M_{pl}$, and choose $g \sim 1.6\times 10^{-3}$, the maximum value allowed for the Stewart model to be technically natural, then $\overline{g}^2/16\pi^2 \lesssim 1$ would require $\alpha > 1$. This can be satisfied with a just barely super-Planckian field displacement. The coupling becomes strong rather quickly if we lower α while fixing g^2, ϵ. There is thus a tension between keeping the scalar theory technically natural and sub-Planckian, and our criterion that the NDA couplings be $\mathcal{O}(\lesssim 1)$.

If we further input the technically natural scalings $\lambda, \lambda' \sim g^4$, we find

$$
\begin{aligned}
\bar{\lambda} &\sim g^4 \left(\frac{\mathcal{M}_*}{m}\right)^4 \sim \overline{g}^4 \left(\frac{\mu}{\mathcal{M}_*^4}\right) , \\
\bar{\lambda}' &\sim g^4 \left(\frac{\mathcal{M}_*}{\mu}\right)^4 \sim \overline{g}^4 \left(\frac{m}{\mathcal{M}_*}\right)^4 .
\end{aligned}
\tag{3.81}
$$

Thus $\overline{g}^2 \gtrsim 1$ is still consistent with $\mathcal{O}(\lesssim 1)$ couplings $\overline{\lambda}, \overline{\lambda}'$. If $\mu \sim 0.3 g \mathcal{M}_*$, then $\overline{\lambda}'/16\pi^2 \sim \mathcal{O}(1)$. For $m = 8\mu \sim 2 g \mathcal{M}_*$, $\overline{\lambda} \sim 2 \times 10^{-4}$.

We have computed these values of $\overline{\lambda}, \overline{\lambda}'$ from technically natural values of the couplings in the scalar theory, as discussed in §2. In terms of our NDA variables they indicate that a consistent application of the principles behind NDA – that couplings are of the same order as their quantum corrections – allows for some complicated structure of the action in the variables $\Phi, \mathcal{X}$. This should not be surprising as these variables are not canonically normalized, and their correlation functions will scale as positive powers of m, μ relative to those for ϕ, σ. But it is these variables which naturally map to the dual 4-forms.

Finally, we can ask what happens if operators such as Φ^{4+k} appear in the action with $\mathcal{O}(1)$ coefficients. From the discussion above, this could be overly pessimistic from the point of view of technical naturalness, but it is expected that a UV completion that includes quantum gravity will enhance irrelevant operators from their technically natural values. We are assuming that said completion will still give couplings that are functions of $\mu\phi$, $m\sigma$ weighted by powers of $\mathcal{M}_*$ or M_{pl}. Let us consider the former scale, to be maximally conservative within our set of conjectures. Then we find

$$
\delta\mathcal{L}_p \sim \frac{c_p}{(p+4)!} \frac{\mu^{4+p}\phi^{4+p}}{{}^{4+2p}_{*}} .
\tag{3.82}
$$

Comparing this to Eq. (3.32), we find

$$\delta_p \sim c_p \frac{M_{pl}^p\, \mu^{p+4}}{{}^{2p+4}_{*}} \,. \tag{3.83}$$

If we saturate our bound Eq. (3.76), let $\mu = m$, and impose the constraints Eq. (3.21), we find:

$$\delta_p \sim \left(\frac{\alpha g}{1.1 \times 10^{-4}}\right)^p \left(\frac{\alpha^3 g^{22}}{3.1 \times 10^{-3}}\right)^2 c_p \,. \tag{3.84}$$

This is consistent with Eq. (3.33) if

$$\left(\frac{\alpha^2 g}{1.1 \times 10^{-4}}\right)^p \left(\frac{\alpha^3 g^{22}}{3.1 \times 10^{-3}}\right)^2 c_p \ll (p+2)! \, \times 1.2 \times 10^{-13} \,. \tag{3.85}$$

If we adopt $g \sim 1.6 \times 10^{-3}$ and take $\epsilon \simeq 0.1$, this bound translates into

$$\left(1.45 \times \alpha^2\right)^p \alpha^6 \, c_p \ll (p+2)! \, \times 2 \times 10^{-3} \,, \tag{3.86}$$

and is readily achieved for all $p \geqslant 1$ even when $c_p \sim 1$ as long as $\alpha \leqslant 0.7$. In this regime, using Eq. (3.80), $\overline{g}^2/16\pi^2 \simeq 1.5/\alpha^6$, we find that for $\alpha \simeq 0.7$ the coupling becomes $\overline{g}^2/16\pi^2 \simeq \mathcal{O}(10)$. This demonstrates that there is some wiggle room with the numbers, where we can either adjust the cutoff $\mathcal{M}$ down by a factor of a few, or take a slightly larger dimensionless coefficient of the biquadratic operator to reduce the effective coupling $\overline{g}$ while maintaining control over irrelevant operators. So for our theory to hang together, all we need is to keep $c_p \sim 1$ even though $\overline{g}^2/16\pi^2$ is large, consistent with some notion of naturalness. From our discussion above, meeting this requirement does not seem out of reach. However we see rather dramatically how naturalness and data press the theory against Planck scale.

3.3.4 Comments on the Dual Massive 4-Form Theory

Our conjectures for writing an action for ϕ, σ consistent with NDA and the discrete gauge symmetries was inspired by the 4-form dual, in a regime for which the duality transformation can be computed and simply understood. However, the regime of the Stewart model supporting inflation consistent with data and sub-Planckian scalar fields is well out of this regime. We can see from the transformation (3.68) that our iterative procedure for constructing the duality map begins to break down when $\Phi > \sqrt{2}$. Indeed, (3.78) shows that 50 efolds before the end of inflation – the epoch during which inflaton fluctuations imprint on the CMB – we are well out of this range for $\alpha \leqslant 20$. This value of α would defeat the original purpose of hybrid inflation, and we will set it aside.

Thus, we do not have complete control of the 4-form dual when the parameters of the Stewart model are consistent with the data and when the field values are in the range which is relevant for the CMB. Nevertheless the scalar theory exists in this regime, and maps to the dual 4-form theory cleanly in the small field limit where we understand the duality. Thus, we could follow the theory in either frame as we increase the couplings and field values. The small field regime attained as inflation ends then serves as the anchor for this 'theory flow': as long as we are sufficiently close to $\phi, \sigma \sim 0$, then Φ will become arbitrarily small and the duality transformation is under control. In this regime we can derive insights into the pseudoscalar dual. Let us then discuss aspects of the duality map in this regime.

When the duality pertains, the effective action

$$\mathcal{L} = -\frac{1}{2\cdot 4!}F^2_{\mu\nu\lambda\sigma} - \frac{1}{2\cdot 4!}\mathcal{G}^2_{\mu\nu\lambda\sigma} - \frac{1}{2}\gamma^2\mathcal{M}_*^4 - \frac{c_1}{4\cdot(4!)^2\mathcal{M}_*^4}F^2_{\mu\nu\lambda\sigma}\mathcal{G}^2_{\mu\nu\lambda\sigma}$$
$$+ \frac{c_2}{(4!)^2\mathcal{M}_*^4}\Big(F^2_{\mu\nu\lambda\sigma}\Big)^2 + \frac{c_3}{(4!)^2\mathcal{M}_*^4}\Big(\mathcal{G}^2_{\mu\nu\lambda\sigma}\Big)^2 + \frac{\mu^2}{12}A^2_{\nu\lambda\sigma} + \frac{m^2}{12}\Big(\mathcal{B}_{\mu\nu\lambda} - h_{\mu\nu\lambda}\Big)^2 + \dots .$$

where we have included all dimension-8 operators consistent with the symmetries, should capture the pseudoscalar dynamics well. In this action, $c_1 \sim \overline{g}^2$; $c_2 \sim \overline{\lambda}$, $c_3 \sim \overline{\lambda}'$. The results which we find in this regime will be corrected at larger coupling and larger field values, but as long as the theory remains below the unitarity bound and the fluctuating light longitudinal modes have couplings suppressed by $\mu/\mathcal{M}, m/\mathcal{M}$, the qualitative insights gained by this analysis may continue to the phenomenologically relevant case of large $\overline{g}\Phi$.

First, the parameters μ, m are UV-insensitive and are at most logarithmically divergent. This follows from the arguments given in [16] for the single field case. We suspect that we can run this argument directly in the pseudoscalar dual, using the nonlinearly realized $\times$ discrete gauge symmetry. Realizing this would give further support for maintaining this feature of the theory in the large $g\Phi$ regime.

Secondly, we can treat M as a derived quantity. Recall this appears as a source term for σ in the dual theory, and sets the field range in σ between inflation and the end of reheating. Once we fix the cutoff $\mathcal{M}$, then if the bound (3.76), is saturated, $M = \mathcal{M}_*^2/\sqrt{2}m$ is given by a see-saw formula. In particular it is a derived quantity. In weak coupling, (3.76) can be translated into a bound on the maximal 4-form flux above which our EFT (3.76) breaks down. Since $\mathcal{M}^2 \sim 4\pi\, G_{max} \sim 4\pi\, \mathcal{N}_{max}\, e$, then Eq. (3.76) implies:

$$M \frac{\mathcal{M}^2}{\sqrt{2}m} \sim \mathcal{N}_{max}\, \frac{e}{\sqrt{2}m} \sim \mathcal{N}_{max}\, \frac{f}{\sqrt{2}}\,, \tag{3.87}$$

where e is the fundamental charge of membranes charged under $\mathcal{G}$ and $f = e/m$ is the period of the dual axion σ. The upshot is that as a derived quantity, the scale M can exceed the cutoff $\mathcal{M}$ without violating naturalness. Note here that $\mathcal{N}_{max} f$ represents the field range of σ during inflation; we are simply saying that field *ranges* need not be bounded by the cutoff, a fact we already understand for axions without monodromy (see for example [35]).

Finally, while it is tempting to identify $c_1 \sim \overline{g}^2$ as a coupling in terms of which we would write an effective action following the rules of NDA, the actual story is more complex. The essential point is that $F, \mathcal{G}$ only propagate because of the gauge field mass. Their two-point functions scale as μ^2, m^2 respectively. This can be seen directly by consistency with the duality transformations $F \sim \mu\phi$, $\mathcal{G} \sim m\sigma$. Alternatively, we can simply compute the propagator for $F, \mathcal{G}$. Given the propagator presented in [16] for a massive 3-form potential

$$\langle A_{\mu\nu\lambda}(p)A_{\mu'\nu'\lambda'}(-p)\rangle = \epsilon_{\mu\nu\lambda\rho}\epsilon_{\mu'\nu'\lambda'\rho'}\left(\frac{\frac{\xi}{2}\eta^{\rho\rho'}}{p^2-\frac{\xi\mu^2}{2}} + \frac{\left(1-\frac{\xi}{2}\right)p^\rho p^{\rho'}}{(p^2-\mu^2)(p^2-\frac{\xi\mu^2}{2})}\right) , \tag{3.88}$$

where ξ is a gauge-fixing parameter, the propagator for F is

$$\langle {}^*F(p)^*F(-p)\rangle = C\left(1+\frac{\mu^2}{p^2-\mu^2}\right) , \tag{3.89}$$

where C is a dimensionless constant comprised of symmetry factors. The first term in parentheses is a contact term; removing this by taking appropriate account of operator mixing, we find ${}^*F$ behaves μ times a scalar, consistent with the duality. Because propagators scale with the 4-form masses, calculations that are perturbative in c_k will come with additional powers of $m/\mathcal{M}_*$, $\mu/\mathcal{M}_*$. For example, let us ask whether $c_1 \gg c_3$ is consistent with quantum corrections. The F propagators yield

$$c_3 \sim^4 \left(\frac{\mu}{\mathcal{M}_*}\right)^4 \sim \left(\frac{g\mathcal{M}_*}{m}\right)^4 . \tag{3.90}$$

In this regime the scaling should come as no surprise – when $\mathcal{G} \sim m\sigma$, this is compatible with the technically natural value $\lambda \sim g^4$. Our conclusion is that in a proper treatment of the effective action for 4-forms, extending the principles of Naïve Dimensional Analysis, $c_1 \sim \overline{g}^2$ will appear in combination with powers of μ, m so that the natural coupling is some $\hat{g}^2 \ll \bar{g}^2$; and that a large value of is still compatible with a sensible effective action for which $c_{k>1}$ can be $\mathcal{O}(\lesssim 1)$.

Can we extend this observation to higher-dimension irrelevant operators in this regime? Consider the operators

$$\delta\mathcal{L}_k = c_k \frac{F^{2k}}{M_*^{4k-4}}\,, \tag{3.91}$$

which are the most important for the dynamics in a phase $\mathcal{G} \sim m\sigma \sim 0$, $F \neq 0$. At weak coupling, these terms are dual to a series of operators with leading terms (3.32), with $p = 2k - 4$. These operators will be generated after integrating out the heavy σ field, or on the dual side, from considering diagrams with $\mathcal{G}$ internal lines[9]. The simplest diagram generating (3.91) involves k vertex insertions $\propto \bar{g}^2 F^2 \mathcal{G}^2/\mathcal{M}_*^4$ in a $\mathcal{G}$ loop. This gives $\delta\mathcal{L}_k \simeq \bar{g}^{2k}\frac{F^{2k}}{\mathcal{M}_*^{4k}}\langle(\mathcal{G}^2)^k\rangle$, so that after the loop integral we find, using $\langle(\mathcal{G}^2)^k\rangle \simeq \mathcal{M}_*^4(m/\mathcal{M})^{2k}$ (note the cutoff in the numerator as opposed to the strong coupling scale),

$$\delta\mathcal{L}_k \simeq \mathcal{M}_*^4 \bar{g}^{2k}\frac{F^{2k}}{\mathcal{M}_*^{4k}}\Big(\frac{m^2}{4\pi\mathcal{M}_*{}^2}\Big)^k \simeq \mathcal{M}_*^4\Big(\frac{\bar{g}^2 m^2}{4\pi\mathcal{M}_*^2}\Big)^k\frac{F^{2k}}{\mathcal{M}_*^{4k}} \simeq \mathcal{M}_*^4\Big(\frac{g^2\mathcal{M}_*^2}{4\pi\mu^2}\Big)^k\frac{F^{2k}}{\mathcal{M}_*^{4k}}\,. \tag{3.92}$$

The last equality looks dangerous due to the to appearance of the small mass μ is the denominator. First, we note that that the limit $\mu \to 0$ is not a problem since $F \sim \mu\phi/\mathcal{M}_*^2$ so μ precisely cancels if we move back to the pseudoscalar frame. Furthermore, if $4\pi\mu^2 \gtrsim g^2\mathcal{M}_*^2$, it appears that $c_k \sim \mathcal{O}(1)$ is consistent even as $\bar{g}$ is increased.

More precisely, if we substitute $\mu \sim m/8$ the overall dimensionless factor multiplying the NDA-normalized part of the operator $\delta\mathcal{L}_k$ becomes, using Eq. (3.76), and substituting (3.21) for M/m,

$$c_k \sim \Big(\frac{1.6\times g\mathcal{M}_*}{m}\Big)^{2k}\Big(2.54\times\frac{g^2\sqrt{2}M}{m}\Big)^k \simeq \Big(\frac{0.2}{\alpha}\Big)^{3k}\,. \tag{3.93}$$

[9]Which contribute to the virtual momentum transfer due to the propagating longitudinal mode.

Remarkably this shows that the irrelevant operator contributions generated by integrating out σ remain safely small for even sub-Planckian field displacements $\alpha \gtrsim 0.2$, which as we noted we need to enforce to suppress their corrections to the inflationary plateau of (3.10). Further note that for $k = 2$, the loop-induced term is $c_2 F^4/\mathcal{M}_*^4$, i.e. just the radiative correction to $\bar{\lambda}'$. We therefore find that $\delta\bar{\lambda}' \sim (0.2/\alpha)^6$, which, again, is under control unless α is too small.[10] If this holds, then the resulting dual operators are precisely of the form (3.82), with coefficients $c_p \sim 1$, and these are subleading during inflation for $\alpha \lesssim 0.5$. Data pushes the theory towards the dangerous Planckian region, but there is still a consistent sub-Planckian regime in which the irrelevant operators are under control.

A rigorous understanding of the 4-form theory in the inflating regime is a matter of future work. Here we simply note that it is plausible to have a well-defined theory with irrelevant operators built from powers of F, $\mathcal{G}$, μA, and $m(\mathcal{B} - h)$ having $\mathcal{O}(\lesssim 1)$ dimensionless parameters when normalized via $\mathcal{M}_*$, even when the leading coefficient $c_1 \sim \overline{g}^2 \gg 1$. In this case we would need to show in this theory that the phenomenologically relevant phase of inflation would occur for $F < M^2 = 4\pi\mathcal{M}_*^2$, where we expect our effective theory to be well-defined.

To conclude this section, we have seen that the theory (3.87) and its special limit (3.76) whose strong coupling limit can realize hybrid inflation[11] are theories of two massive $U(1)$ gauge theories of 4-form field strengths in the unitary gauge, with 4-form kinetic mixings mediated by irrelevant operators. The mixing coefficients are controlled by natural parameters of $\mathcal{O}(1)$. In this form the theory is strongly coupled but natural: the unbroken gauge symmetries of (3.87) will protect the selection of the masses and couplings in (3.87) from UV effects in QFT. However, since the

[10] Here of course we are looking near $\phi = 0$ where the duality makes sense; α has meaning in the dual pseudoscalar theory as the range of ϕ in field space covered by the last 50 efolds of inflation, a range over which the duality map becomes complicated.

[11] Again, such a theory can generate inflation if for a given cutoff $\mathcal{M}_*$ the masses obey $\mu \ll m \ll \mathcal{M}_*$ and one of the form field strengths develops an initial CP-breaking flux on the background, controlled by $M_{cr} \sim \mathcal{M}_*^2/m$.

scales are all sub-Planckian, and the symmetries are gauged, the theory will be safe from quantum gravity corrections as well. The mass terms cannot receive large UV corrections since they are also couplings of the longitudinal modes, and are protected by gauge redundancies. We cannot predict what the values of these masses are from within the EFT itself. But once chosen, the gauge symmetries of the theory protect them from QFT and quantum gravity corrections, as in [16], [17]. Furthermore, while the masses are small relative to the cutoff, they will not be *too* small; they can be a few (< 10) orders of magnitude below the Planck scale. Thus the theory should be able to pass the bounds which one might find using arguments based on the weak gravity conjecture, and from other technical lamppost-based bounds coming from recent developments in string phenomenology.

3.3.5 Discussion

We have outlined how hybrid inflation might be made UV complete via dualizing it to a theory of two massive 4-form field strengths/3-form potentials. This UV completion contains gauge symmetries which explain the suppression of potentially dangerous operators which can adversely affect inflation [16], [17]. In a top-down approach to deriving hybrid inflation, this procedure would be reversed.

Indeed, imagine that in some UV-complete theory such as string theory, some of the higher rank forms yield massive 4-forms after compactification, with masses which are much smaller than the UV cutoff of the EFT of the 4-form systems. We believe that such constructions would be conceptually similar to the previous approaches [6], [9], [11], [49]–[54] where the main focus was on realizing the conditions for single large field inflation. In the present case they should involve multiple massive 4-forms below the cutoff. The EFTs of interest arise after integrating out the stabilized heavier fields, the KK states and the heavy string modes. The cutoff $\mathcal{M}$ demarcates this EFT from the full theory with those additional degrees of freedom, and could be viewed as scale where ignoring the lightest of the modes, which were integrated out to define the low energy EFT, would start yielding problems with unitarity. The irrelevant operators in (3.87) arise as the corrections generated by the virtual heavy modes as well as loops of the virtual light modes kept in the EFT. In the minimal case, these are the longitudinal modes of the massive 4-forms and the matter degrees of freedom, that the longitudinal modes decay into at the end of inflation. Perhaps the simplest manner in which these models can be realized is to imagine a theory with coupled p-forms, which includes higher-derivative corrections suppressed by a cutoff, and where after dimensional reduction the emergent 4-forms mix with pseudoscalar axions. These can be set to become longitudinal modes by a gauge fixing; after dualization, the higher-derivative operators are the potential for the longitudinal mode.

The longitudinal modes remain light because the 4-form/3-form potential gauge symmetries: continuous compact $U(1)$ and the discrete shift, $A \to A + da$, $\mathcal{B} \to \mathcal{B} + db$. These ensure that the dangerous corrections to the mass terms are absent. This extends to the quantum gravity corrections as well, which cannot break gauge symmetries. So as long as the operators in (3.87) are below the cutoff $\mathcal{M}$, the theory has a weak coupling expansion where a minor tuning of parameters realizes the regime which supports hybrid inflation.

This is manifest once one reverses the steps which led from (3.5) to (3.87). Indeed, inverting the steps in §3.2 will map the mass terms and irrelevant operators in (3.87) precisely on the potential (3.5). The mass terms are naturally small by gauge symmetry, whereas the marginal operator couplings in (3.5) are rendered small because they are controlled by the ratio of masses and the cutoff, and $m_{A,B} \ll \mathcal{M}$. Finally, the scale M in (3.5) appears to be larger than the cutoff because it is see-sawed by the mass: $M \sim \mathcal{M}^2/m$ as we discussed above. This can explain the origin of the Stewart limit of hybrid inflation naturally.

In this paper we have focused on realizing a model of hybrid inflation in the Stewart limit, controlled by only relevant and marginal operators. Our observation that the 4-form theory needs to be strongly coupled suggests that we consider models in which other higher-dimension operators play a significant role. In particular, it is possible that phenomenologically interesting and natural low-scale hybrid models could be realized with flattened potentials, following [6], [9], [11], [17]. It would be interesting to explore such more general models of hybrid inflation.

3.4 Summary

Many inflationary models have been severely constrained by the observations in the past decade or so. Specifically the improving bounds on the spectral index and on the tensor-scalar ratio have put pressure on the large field inflation models, which are arguably the simplest candidates for natural EFTs of inflation. These models are quite predictive as well, because they must occur at high scales to yield viable inflationary evolution, where their structure becomes sensitive to quantum gravity corrections. Thus the tightening constraints on large field models might be taken to imply that the prospects of learning something about quantum gravity from inflation have diminished. Moreover, by placing increasingly tighter constraints on large field models, observations might appear to favor more exotic, unnatural, proposals for inflation, or even more radical approaches to early universe cosmology.

Our results here suggest that these allusions are not a foregone conclusion yet. The pressures from observational bounds are significantly relieved by reducing the scale of inflation. This can be done in multifield inflation models, and it occurs in hybrid inflation with two non-degenerate fields, when the post-inflationary vacuum manifold is not degenerate. In this case, the tensor-scalar ratio is almost unobservably small, and the spectrum of perturbations is safely red, with the spectral index n_S between 0.97 and 0.975, which is still in agreement with the bounds. Further, the EFT of this variant of hybrid inflation is technically natural, and if it is realized as a dual for a theory with two massive 4-forms, which might be realized as an IR limit of string compactifications, it may also be protected from quantum gravity corrections although it involves almost Planckian field displacements. The UV safety of the theory is not a generic feature of all hybrid inflation proposals, as we have seen in detail. Yet it may arise in some constructions such as those which we outline here. However, remarkably, even in these cases the natural EFTs are still close to Planck scale. This, in our view, is quite interesting, since it keeps the possibility open that inflation, while being a viable EFT, might still be sensitive to some subleading corrections from quantum gravity, which while small might compete with the UV field theory effects.

CHAPTER 4
BOSONIC MIXING IN CURVED SPACE

In Collaboration with Professor Devin Walker (Dartmouth College), Nizar Ezroura (Michigan State University) and Bradley Shapiro (Dartmouth College) , we looked at the mixing between axions, as Dark Matter (DM) candidate, photons and gravitons in curved spacetime. We studied the mixing between those waves in the vicinity of a Kerr black hole. Our goal is to evaluate the conversion probability between these three types of waves. This would allow for a possible multi-messenger indirect detection of axions or generic Bosonic DM.

The mixing between gravitational waves and electromagnetic waves in the presence of a cosmological magnetic field in flat space has been studied by Dolgov and Ejlli [55]. In 1988, Raffelt and Stodolsky [56], and later in 2017, Masaki, Aoki and Soda [57] discussed the mixing and probability conversion between axion waves and electromagnetic waves in flat spacetime using a Chern-Simons term to model the coupling between axions and photons: $aF_{\mu\nu}F^{\mu\nu}$.
However the mixing between an axion waves and a gravitational waves has yet never been studied. The main reason being that the effects are negligible in flat spacetime.
In this work, we describe bosonic (scalar, electromagnetic and gravitational) wave mixing in curved spacetime. Curved spacetime adds a new length scale, the Schwarzschild radius, which significantly alters the oscillation probabilities in comparison to the standard flat spacetime computations. The alterations are analogous to the Mikheyev-Smirnov-Wolfenstein (MSW) effect for neutrinos and are “frozen-in” as the outgoing gravitational and/or electromagnetic wave propagates away from a compact object. Although we consider the axion and axion-like particles, our computations

are largely model independent and applicable for generic spin-zero dark matter. We describe the probabilities for axions and generic bosonic dark matter oscillations.

In some future work, we wish to describe some of the observational consequences of the mixing including the energy and polarization of the waves exiting the compact object.

In the next section, we present a preliminary version of the work that has been done so far.

4.1 Introduction

In flat space and in the presence of a static external electromagnetic field, axion waves can mix with electromagnetic and gravitational waves [56]. The gravitational wave mixing is not often considered because of Planck mass suppression in the mixing terms. However in the immediate environs of compact objects gravity is strong and gravitational wave mixing is necessary. In this work, we describe the mixing of axion waves with electromagnetic and gravitational waves in curved spacetime. The presence of non-trivial gravitational fields has observational consequences that are distinct from the well-known flat spacetime signatures.

4.1.1 Bosonic Mixing in Curved Spacetime

In order to compute the mixing, we use effective field theory techniques. These techniques are implicit for the standard flat space computation. The flat space computation requires a static external electromagnetic field that varies on very large length scales. This is in contrast to the important short wavelength, oscillating degrees of freedom of the electromagnetic and axion waves. Thus the calculation separates the important short degrees of freedom from the longer wavelength physics.

In this work, we consider a similar separation of length scales while also including gravitational waves.

For a patch of local spacetime near compact objects, we require the frequency of the cohered electromagnetic, gravitational and axion waves ($\bar{\lambda}$) to be large compared to the characteristic length scale of the background curvature (λ). It is therefore essential to explicitly average of the background curved spacetime at large distance scales in order to get a proper understanding of their effects on high-frequency waves. Any averaging scheme will lead to corrections to the non-linear Einstein equations at larger length scales. The averaged quantities are analogous to static external electromagnetic field in the flat spacetime computation. To be able to consider longer wavelength mixing, we can theoretically employ techniques analogous to renormalization group averaging (analogous IR RGE averaging). Some of the averaging schemes include Isaacson [58] which is based on Brill and Hartle [59]. Noonan [60] generalizes Isaacson in order to define a consistent gravitational energy-momentum pseudotensor in the presence of matter. Other efforts include Futamase [61], who performed spatial averaging in $3 + 1$ splitting of spacetime, Boersma [62], who constructed gauge-invariant averaging in perturbation theory as well as Kasai [63] and Zalaletdinov [64].

To summarize in general throughout this work, we use concepts from effective field theory in order to separate out the important from the unimportant, non-linear physics. At the length scales of interest and up to an error in the coupling, we mix the axion, electromagnetic and gravitational waves in order to place the equations of motions into their mass eigenstate form. The new, diagonalized equations of motion now do not have explicit interaction terms. Thus, the mass eigenstates carry information about the probability of conversion of an axion, electromagnetic and gravitational wave into an axion, electromagnetic and gravitational wave. We can now use tetrad vectors to take the equations of motion and propagate them through all of spacetime. The external electromagnetic field slowly and numerically varies as the tetrad vectors propagate the patch through all of spacetime. In order to in compute superradiance effects, we can construct and separate Newman-Penrose scalars into the master equations.

4.1.2 Review of Gravitational Wave Effective Theory

In this section, we review much of what is in [65], [66]. To understand gravity waves in curved spacetime, we employ standard effective field theory techniques to separate the wavelength of the metric perturbation from the background [65]–[67]. We review these techniques to establish notation and the mixing equations in the next section. We consider a locally flat patch of spacetime over which an external electromagnetic field is homogeneous and static. Importantly this locally flat patch of spacetime can be relatively close to compact objects. The need for a locally flat patch important for a variety of reasons: (1) To understand the axion, electromagnetic and gravitational wave oscillations, we must provide an orientation of the static, background electromagnetic field in relation to the freely falling body (the cohered mixed state). (2) The locally flat patch allows for an inertial frame to be constructed that allows for the mixing as well as providing the formalism to understand curvature corrections for the freely falling particle. To compute the mixing equations in the patch, we first consider the metric perturbation h_{ab},

$$g_{ab} = \overline{g}_{ab} + \epsilon\, h_{ab} + \frac{1}{2}\,\epsilon^2\, h_{ac}\, h^{c}{}_{b} + \mathcal{O}(\epsilon^3) \tag{4.1}$$

$$g^{ab} = \overline{g}^{ab} - \epsilon\, h^{ab} + \frac{1}{2}\,\epsilon^2\, h^{ac}\, h_{c}{}^{b} + \mathcal{O}(\epsilon^3), \tag{4.2}$$

where $g^{ab}g_{bc} = \delta^a_c$, ϵ is parametrized as $\epsilon \sim \mathcal{O}(/\mathcal{L})$, $= \lambda/2\pi$ is the reduced gravitational perturbation wavelength and $\mathcal{L}$ is the characteristic scale of variation of $\overline{g}_{ab}$, the background metric [67].

We justify the parametrization of ϵ below. Often the ϵ^2 term is notated by j_{ab} [65]. It is clear, in order to give a proper perturbative expansion, equation (4.1) requires $\ll \mathcal{L}$ which separates the length scales of the gravitational wave perturbation and the background metric. This separation of scales ultimately allows for a proper definition of a gravitational wave [1]. In addition to this expansion, we also note the expansions

$$\sqrt{-g} = \sqrt{-\overline{g}} + \frac{\epsilon}{2}\sqrt{-\overline{g}}\, h + \frac{\epsilon^2}{8}\sqrt{-\overline{g}}\, h^2 + \dots \tag{4.3}$$

$$\frac{1}{\sqrt{-g}} = \frac{1}{\sqrt{-\overline{g}}} - \frac{\epsilon}{2\sqrt{-\overline{g}}}\, h + \frac{\epsilon^2}{8\sqrt{-g}}\, h^2 + \dots \tag{4.4}$$

[1]To be sure our analysis applies to as many physical situations as possible, we also consider the parameterization $\epsilon \sim f_{\mathrm{b}}/f$ in parallel, where f is frequency of the gravitational wave perturbations and f_{b} is the maximal frequency of the background metric [66]. A priori f_{b} is not correlated with $\mathcal{L}_{\mathrm{b}}$. Moreover, often gravitational waves background are static. Thus, it may be possible to search for a wider variety of graviton-axion-electromagnetic waves. Ultimately, we will also consider graviton-axion-electromagnetic waves from a variety of astrophysical sources, including supermassive black holes, which will provide a variety of probes of the parameter space despite the $\ll \mathcal{L}_{\mathrm{b}}$ requirement.

which will be useful later. The expanded connection and Riemann tensor are now

$$\Gamma^{a}_{\ bc} = \overline{\Gamma}^{a}_{\ bc} + \frac{\epsilon}{2}\,\overline{g}^{am}\Big(\overline{\nabla}_c\, h_{mb} + \overline{\nabla}_b\, h_{mc} - \overline{\nabla}_m\, h_{bc}\Big) \tag{4.5}$$

$$+\ \frac{\epsilon^2}{4}\Big(\overline{g}^{am}\Big(\overline{\nabla}_c\,(h_{mq}\,h^{q}_{\ b}) + \overline{\nabla}_b\,(h_{mq}\,h^{q}_{\ c}) \tag{4.6}$$

$$-\ \overline{\nabla}_m\,(h_{bq}\,h^{q}_{\ c})\Big) - 2\,h^{am}\Big(\overline{\nabla}_c\, h_{mb} + \overline{\nabla}_b\, h_{mc} - \overline{\nabla}_m\, h_{bc}\Big)\Big) + \mathcal{O}(\epsilon^3)$$

$$R^{a}_{\ bcd} = \overline{R}^{a}_{\ bcd} + \frac{\epsilon}{2}\,\Big(\overline{g}^{am}\big(\overline{\nabla}_c\overline{\nabla}_d\, h_{mb} + \overline{\nabla}_c\overline{\nabla}_b\, h_{md} - \overline{\nabla}_c\overline{\nabla}_m\, h_{bd}\big) \tag{4.7}$$

$$-\ \overline{g}^{am}\big(\overline{\nabla}_d\overline{\nabla}_c\, h_{mb} + \overline{\nabla}_d\overline{\nabla}_b\, h_{mc} - \overline{\nabla}_d\overline{\nabla}_m\, h_{bc}\big)\Big) \tag{4.8}$$

$$+\ \frac{\epsilon^2}{4}\,\Big(\overline{g}^{am}\big(\overline{\nabla}_c\overline{\nabla}_d\,(h_{mr}\,h^{r}_{\ b}) + \overline{\nabla}_c\overline{\nabla}_b\,(h_{mr}\,h^{r}_{\ d}) - \overline{\nabla}_c\overline{\nabla}_m\,(h_{br}\,h^{r}_{\ d})\big) - \tag{4.9}$$

$$2\,\overline{\nabla}_c\big(h^{am}\big(\overline{\nabla}_d\, h_{mb} + \overline{\nabla}_b\, h_{md} - \overline{\nabla}_m\, h_{bd}\big)\big)$$

$$-\ \overline{g}^{am}\big(\overline{\nabla}_d\overline{\nabla}_c\,(h_{mg}\,h^{g}_{\ b}) + \overline{\nabla}_d\overline{\nabla}_b\,(h_{mg}\,h^{g}_{\ c}) - \overline{\nabla}_d\overline{\nabla}_m\,(h_{bg}\,h^{g}_{\ c})\big) - \tag{4.10}$$

$$2\,\overline{\nabla}_d\big(h^{am}\big(\overline{\nabla}_c\, h_{mb} + \overline{\nabla}_b\, h_{mc} - \overline{\nabla}_m\, h_{bc}\big)\big)$$

$$+\ \overline{g}^{eq}\,\overline{g}^{am}\big(\overline{\nabla}_e\, h_{mc} + \overline{\nabla}_c\, h_{me} - \overline{\nabla}_m\, h_{ce}\big)\big(\overline{\nabla}_d\, h_{qb} + \overline{\nabla}_b\, h_{qd} - \overline{\nabla}_q\, h_{bd}\big)$$

$$-\ \overline{g}^{em}\,\overline{g}^{aq}\big(\overline{\nabla}_e\, h_{qd} + \overline{\nabla}_d\, h_{qe} - \overline{\nabla}_q\, h_{de}\big)\big(\overline{\nabla}_c\, h_{mb} + \overline{\nabla}_b\, h_{mc} - \overline{\nabla}_m\, h_{bc}\big)\Big)$$

where $R_{\ bd} = R^{a}_{\ bcd}$ and $\overline{g}^{bd} R_{\ bd}$ is the Ricci tensor and Ricci scalar, respectively.

4.1.3 Covariant Equations of Motion

The lagrangian in curved spacetime is

$$\begin{aligned}\mathcal{L} &= \sqrt{-g}\left(\frac{1}{2\,\kappa}\,R - \frac{1}{4}\,g^{ab}\,g^{cd}\,F_{ac}\,F_{bd} - \frac{1}{2}\left(g^{ab}\,\partial_a\phi\,\partial_b\phi + m^2\phi^2\right) + \frac{\tilde{\lambda}}{8\sqrt{-g}}\,\epsilon^{abcd}\,\phi\,F_{ab}\,F_{cd}\right) \qquad (4.11)\\ &+ \frac{\sqrt{-g}}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\left(g^{ab}\,g^{cd}\,F_{ac}\,F_{bd}\right)^2 + \frac{7}{4}\left(\frac{1}{2\sqrt{-g}}\,\epsilon^{abcd}\,F_{ab}\,F_{cd}\right)^2\right) + \ldots\end{aligned}$$

where $\kappa = 1/\overline{m}_{\rm pl}^2$, $\overline{m}_{\rm pl} = m_{\rm pl}/\sqrt{8\pi}$, $F_{\mu\nu}$ is the electromagnetic field strength tensor, ϕ is the axion, $\tilde{\lambda}$ is the axion-photon coupling and m_a is the axion mass. We also used $\tilde{\epsilon}_{abcd} = \sqrt{|g|}\,\epsilon_{abcd}$ and $\tilde{\epsilon}^{abcd} = \mathrm{sgn}(g)\,\epsilon^{abcd}/\sqrt{|g|}$ where ϵ_{abcd} is $+1$ for an even perturbation of $0123\ldots$ and -1 for an odd perturbation. We also define $\tilde{F}^{ab} = \epsilon^{abcd}\,F_{cd}/(2\,\sqrt{|g|})$. Throughout we use natural units [2]. There is a minus sign in front of the $\partial_\mu\phi\,\partial^\mu\phi$ term, because we are working in a $(-+++)$ signature. The terms on the second line is the lowest order, one-loop, correction from the Euler-Heisenberg Langrangian.

4.1.4 Effective Equations of Motion

The action is

$$S = \int d^4x\,\mathcal{L} = \int d^4x\,\sqrt{-g}\left(\frac{1}{2\,\kappa}\,R - \frac{1}{4}\,g^{ab}\,g^{cd}\,F_{ac}\,F_{bd} - \frac{1}{2}\left(g^{ab}\,\partial_a\phi\,\partial_b\phi + m^2\phi^2\right)\right. \qquad (4.12)$$

$$+ \frac{\tilde{\lambda}}{8\sqrt{-g}}\,\epsilon^{abcd}\,\phi\,F_{ab}\,F_{cd} \qquad (4.13)$$

$$\left. + \frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\left(g^{ab}\,g^{cd}\,F_{ac}\,F_{bd}\right)^2 + \frac{7}{4}\left(\frac{1}{2\sqrt{-g}}\,\epsilon^{abcd}\,F_{ab}\,F_{cd}\right)^2\right) + \ldots\right)$$

[2] See the definitions in, e.g., `https://www.seas.upenn.edu/~amyers/NaturalUnits.pdf`.

Here we define

$$S_1 = \frac{1}{2\,\kappa} \int d^4x\, \sqrt{-g}\, R = \frac{1}{2\,\kappa} \int d^4x\, \sqrt{-g}\, g^{bd}\, R^a_{\ bad} \tag{4.14}$$

$$S_2 = -\frac{1}{4} \int d^4x\, \sqrt{-g}\, g^{ab}\, g^{cd}\, F_{ac}\, F_{bd} \tag{4.15}$$

$$S_3 = -\frac{1}{2} \int d^4x\, \sqrt{-g}\, \left(g^{ab}\, \partial_a\phi\, \partial_b\phi + m^2\phi^2 \right) \tag{4.16}$$

$$S_4 = \frac{\tilde{\lambda}}{8} \int d^4x\, \epsilon^{abcd}\, \phi\, F_{ab}\, F_{cd} \tag{4.17}$$

$$S_5 = \frac{1}{90\, m_e^4} \left(\frac{e^2}{4\pi}\right)^2 \int d^4x\, \sqrt{-g} \left(\left(g^{ab}\, g^{cd}\, F_{ac}\, F_{bd}\right)^2 + \frac{7}{4} \left(\frac{1}{2\sqrt{-g}}\, \epsilon^{abcd}\, F_{ab}\, F_{cd} \right)^2 \right) \tag{4.18}$$

To evaluate these actions, we use the expansions from equations (4.1), (4.2), (4.3), (4.4) and (4.8).

4.1.4 Gravitational Perturbation Expansion

We can do the gravitational perturbative expansion for each term in the action. We define

$$\overline{h}_{ab} = h_{ab} - \overline{g}_{ab}\, h/2 \tag{4.19}$$

and therefore $\overline{h} = -h$ and substitute both into the action. Applying the gauge fixing terms,

$$\overline{\nabla}_a h^{ab} = 0 \qquad\qquad h = 0 \tag{4.20}$$

and using the following equations from the appendix,

$$\overline{\nabla}_a\overline{\nabla}_d\, h_{mb} = \overline{\nabla}_d\overline{\nabla}_a\, h_{mb} - \overline{R}^r{}_{mad}\, h_{rb} - \overline{R}^s{}_{bad}\, h_{sm}\,,$$

$$R = g^{bd}R_{bd} = \overline{R} + \tfrac{\epsilon}{2}\left(\overline{g}^{bd}\left(\overline{\nabla}_a\overline{\nabla}_b\, h^a_{\ d} + \overline{\nabla}_a\overline{\nabla}_d\, h^a_{\ b}\right) - \overline{g}^{bd}\,\overline{\nabla}_d\nabla_b\, h - \overline{g}^{bd}\,\overline{\Box}\, h_{bd}\right)$$
$$+\tfrac{\epsilon^2}{4}\Big(\overline{g}^{bd}\,\overline{\nabla}_b\, h_{ce}\overline{\nabla}_d\, h^{ce} + 2\, h^{ce}\left(\overline{g}^{bd}\,\overline{\nabla}_b\overline{\nabla}_d\, h_{ce} + \overline{g}^{bd}\,\overline{\nabla}_c\overline{\nabla}_e\, h_{bd} - \overline{g}^{bd}\,\overline{\nabla}_d\,\overline{\nabla}_e\, h_{cb} - \overline{g}^{bd}\,\overline{\nabla}_b\,\overline{\nabla}_e\, h_{cd}\right)$$
$$+2\,\overline{g}^{bd}\,\overline{g}^{eg}\,\overline{g}^{ch}\,\overline{\nabla}_g\, h_{dh}\left(\overline{\nabla}_e\, h_{cb} - \overline{\nabla}_c\, h_{eb}\right) - \overline{g}^{bd}\left(\overline{\nabla}_e\, h^{ce} - \tfrac{1}{2}\,\overline{g}^{ce}\,\overline{\nabla}_e\, h\right)\left(\overline{\nabla}_d\, h_{cb} + \overline{\nabla}_b\, h_{cd} - \overline{\nabla}_c\, h_{bd}\right)\Big)$$
$$+\,\mathcal{O}(\epsilon^3)\,.$$

We can now write out the gauge-fixed action. Dropping the bars off the h's, we have now

$$S_1 = \frac{1}{2\,\kappa}\int d^4x\,\sqrt{-\overline{g}}\left(\overline{R} + \frac{\epsilon^2}{2}\left(h^{bm}\, h_m^{\ d}\,\overline{R}_{bd} + h^{bd}\, h_s^{\ a}\, R^s{}_{bad}\right) - \frac{\epsilon^2}{4}\, h^{bd}\,\overline{\Box}\, h_{bd}\right) \tag{4.21}$$
$$S_2 = -\frac{1}{4}\int d^4x\,\sqrt{-\overline{g}}\left(\overline{g}^{ab}\,\overline{g}^{cd}F_{ac}\, F_{bd} - 2\,\epsilon\, h^{cd}\, T^{\text{em}}_{cd} + \epsilon^2\, h^{ab}\, h^{cd}F_{ac}\, F_{bd}\right) \tag{4.22}$$
$$S_3 = -\frac{1}{2}\int d^4x\,\sqrt{-\overline{g}}\left(\overline{g}^{ab}\,\partial_a\phi\,\partial_b\phi + m^2\phi^2 + \epsilon\, h^{cd}\, T^{\text{scalar}}_{cd} + \frac{\epsilon^2}{2}\, h^{af}\, h_f^{\ b}\,\partial_a\phi\,\partial_b\phi\right) \tag{4.23}$$
$$S_5 = \frac{1}{90\, m_e^4}\left(\frac{e^2}{4\pi}\right)^2\int d^4x\left(\sqrt{-\overline{g}}\left(\left(\overline{g}^{ab}\,\overline{g}^{cd}\, F_{ac}\, F_{bd}\right)^2 + \frac{7}{4}\left(\frac{1}{2\,\sqrt{-\overline{g}}}\,\epsilon^{abcd}\, F_{ab}\, F_{cd}\right)^2\right)\right. \tag{4.24}$$
$$\left. -\,4\,\epsilon\,\sqrt{-\overline{g}}\; h^{mn}\,\overline{g}^{ab}\,\overline{g}^{ef}\,\overline{g}^{gh}\, F_{eg}\, F_{fh}\, F_{am}\, F_{bn}\right).$$

where the energy-momentum tensors are

$$T^{\text{em}}_{cd} = \overline{g}^{ab}\, F_{ac}\, F_{bd} - \frac{1}{4}\overline{g}_{cd}\,\overline{g}^{ab}\,\overline{g}^{ef}\, F_{ae}\, F_{bf} \tag{4.25}$$
$$T^{\text{scalar}}_{cd} = \overline{g}_{cd}\left(\frac{1}{2}\,\overline{g}^{ab}\,\partial_a\phi\,\partial_b\phi + \frac{1}{2}\, m^2\phi^2\right) - \partial_c\phi\,\partial_d\phi \tag{4.26}$$

Here we have used the results of the expanded the Ricci scalar in the appendix. Equations (4.21), (4.22) and (4.24) match Dolgov and Ejilli equation 5 and 6 when the Minkowski limit is taken.

We express the actions into multiple forms that facilitate integration by parts. Specifically, the gravitational action is given by

$$S_1 = \frac{\epsilon^2}{4\,\kappa} \int d^4x \sqrt{-\overline{g}}\;\; h_{de} \Bigg(-\frac{1}{2} \overline{\Box}\, h^{de} - \Big(\overline{g}^{qw}\, \overline{R}^{d}_{\;\;qwa}\, h^{ae} - \,\overline{g}^{cd}\, \overline{R}^{e}_{\;\;rca}\, h^{ar} \Big) \Bigg) + \frac{1}{2\,\kappa} \int d^4x \sqrt{-\overline{g}}\, \overline{R}\,. \tag{4.27}$$

The parts of the electromagnetic action are

$$\mathrm{S}_2 = \int d^4x \sqrt{-\overline{g}}\;\, A_f \bigg(\tfrac{1}{2}\, \overline{g}^{ab}\, \overline{g}^{cf} (\nabla_b\, F_{ac}) - \tfrac{1}{2}\, \epsilon \left(\overline{g}^{cd}\, \overline{g}^{fe}\, \overline{g}^{ab} - \overline{g}^{cd}\, \overline{g}^{be}\, \overline{g}^{af} \right) \nabla_b (h_{de}\, F_{ac})$$
$$+ \tfrac{1}{4}\, \epsilon^2\, \overline{g}^{cd}\, \nabla_b \left(\overline{g}^{fe}\, h^{ab}\, h_{de}\, F_{ac} - \overline{g}^{be}\, h^{af}\, h_{de}\, F_{ac} \right) \bigg)$$

$$\mathrm{S}_4 = \int d^4x \sqrt{-\overline{g}}\;\, \tfrac{\tilde{\lambda}}{8\sqrt{-\overline{g}}}\, \epsilon^{abcd}\, \phi\, F_{ab}\, F_{cd}$$
$$= \int d^4x \sqrt{-\overline{g}}\;\, \tfrac{\tilde{\lambda}}{4\sqrt{-\overline{g}}}\, \epsilon^{abed}\, A_e\, \nabla_d \left(\phi\, F_{ab} \right)$$

$$S_5 = \int d^4x \sqrt{-\overline{g}}\;\, \tfrac{1}{90\, m_e^4} \left(\tfrac{e^2}{4\pi} \right)^2 \Bigg(\bigg(\left(\overline{g}^{ab}\, \overline{g}^{cd}\, F_{ac}\, F_{bd} \right)^2 + \tfrac{7}{4} \left(\tfrac{1}{2\sqrt{-\overline{g}}}\, \epsilon^{abcd}\, F_{ab}\, F_{cd} \right)^2 \bigg)$$
$$- \epsilon\, h_{de} \bigg(4\; \overline{g}^{md}\, \overline{g}^{ne}\, \overline{g}^{ab}\, \overline{g}^{kf}\, \overline{g}^{gh}\, F_{kg}\, F_{fh}\, F_{am}\, F_{bn} \bigg) \Bigg)$$
$$= \int d^4x \sqrt{-\overline{g}}\;\, \tfrac{1}{90\, m_e^4} \left(\tfrac{e^2}{4\pi} \right)^2 A_e \bigg(-2\, \nabla_g \left(\overline{g}^{ab}\, \overline{g}^{cd}\, \overline{g}^{gf}\, \overline{g}^{eh}\, F_{ac}\, F_{bd}\, F_{fh} \right)$$
$$- \tfrac{7}{16\,\overline{g}}\, \epsilon^{ebcd}\, \epsilon^{afgh}\, \nabla_b \left(F_{cd}\, F_{af}\, F_{gh} \right) - 8\, \epsilon\, \nabla_g \Big(\overline{g}^{cd}\, \overline{g}^{ab}\, \overline{g}^{ef}\, \overline{g}^{gh}\, h^{mn}\, F_{fh}\, F_{am}\, F_{bn} \Big) \bigg)$$
$$= \int d^4x \sqrt{-\overline{g}}\;\, \tfrac{1}{45\, m_e^4} \left(\tfrac{e^2}{4\pi} \right)^2 A_e\, \nabla_g \bigg(-F^2\, F^{ge} - \tfrac{7}{4}\, \widetilde{F}^{eg} \left(\widetilde{F}_{af}\, F^{af} \right)$$

$$- 4\,\epsilon\,\overline{g}^{cd}\,\overline{g}^{ab}\,\overline{g}^{ef}\,\overline{g}^{gh}\,h^{mn}\,F_{fh}\,F_{am}\,F_{bn}\Bigg)$$

which includes the Euler-Heisenberg corrections.

A reminder: The energy-momentum tensors are in equations (4.25) and (4.26). The scalar dark matter action is

$$S_3 = -\frac{1}{2}\int d^4x\sqrt{-\overline{g}}\,\left(\overline{g}^{ab}\,\partial_a\phi\,\partial_b\phi + m^2\phi^2 + h_{de}\left(\epsilon\,\overline{g}^{cd}\,\overline{g}^{je}\,T^{\text{scalar}}_{cj} + \frac{\epsilon^2}{2}\,\overline{g}^{ad}\,\overline{g}^{fe}\,h_f^{\;b}\,\left(\partial_a\phi\,\partial_b\phi\right)\right)\right) \tag{4.28}$$

$$S_3 = \int d^4x\sqrt{-\overline{g}}\;\phi\left(\frac{1}{2}\,\overline{g}^{ab}\,\nabla_a\nabla_b\,\phi - \frac{1}{2}m^2\phi - \frac{1}{2}\epsilon\,\left(-\frac{1}{2}\,\overline{g}^{ab}\,\overline{g}_{cd}\,\nabla_a\left(h^{cd}\,\nabla_b\,\phi\right) + \frac{1}{2}\,m^2 h\,\phi - \nabla_c\nabla_d\,\phi\right)\right.$$
$$\left. - \frac{\epsilon^2}{4}\,\nabla_a\left(h^{af}\,h_f^{\;b}\,\partial_b\,\phi\right)\right), \tag{4.29}$$

We can double check our results in the the Minkowski limit. Consider the gravitational wave equation of motion from the action

$$S = \int d^4x\sqrt{-\overline{g}}\;\;h_{de}\Bigg(-\frac{1}{2}\,\frac{\epsilon^2}{4\,\kappa}\overline{\Box}\,h^{de} - \frac{\epsilon^2}{4\,\kappa}\left(\overline{g}^{qw}\,\overline{R}^d_{\;qwa}\,h^{ae} - \overline{g}^{cd}\,\overline{R}^e_{\;rca}\,h^{ar}\right)$$
$$- \frac{1}{4}(-2\,\epsilon\,\overline{g}^{cd}\,\overline{g}^{fe}\,T^{\text{em}}_{cf}) - \frac{1}{4}\epsilon^2\,h^{ab}\,\overline{g}^{cd}\,\overline{g}^{fe}\,F_{ac}\,F_{bf}\Bigg) \tag{4.30}$$

which yields

$$\frac{1}{2}\overline{\Box}\,h^{de} + \left(\overline{g}^{qw}\,\overline{R}^d_{\;qwa}\,h^{ae} - \overline{g}^{cd}\,\overline{R}^e_{\;rca}\,h^{ar}\right) - \frac{2\,\kappa}{\epsilon}\,\overline{g}^{cd}\,\overline{g}^{fe}\,T^{\text{em}}_{cf} + \kappa\,h^{ab}\,\overline{g}^{cd}\,\overline{g}^{fe}\,F_{ac}\,F_{bf} = 0 \tag{4.31}$$

To match Dolgov and Ejlli, note we substitute $\kappa \to \kappa^2/2$ and $\epsilon \to \kappa$. Note Dolgov and Ejlli equation 7 leaves off the proper normalization of the kinetic term. The last term of the above equation is left off in Dolgov and Ejlli because it is higher order. The background Riemann tensors are zero in the Minkowski limit. As for the electromagnetic equation of motion from the action we have

$$S = \frac{1}{2}\int d^4x\sqrt{-\overline{g}}\; A_f\Bigg(\overline{g}^{ag}\,\overline{g}^{cf}(\nabla_g F_{ac}) - \epsilon\left(\overline{g}^{cd}\,\overline{g}^{fe}\,\overline{g}^{ab} - \overline{g}^{cd}\,\overline{g}^{be}\,\overline{g}^{af}\right)\nabla_b(h_{de}\,F_{ac}) \tag{4.32}$$
$$+ \frac{1}{2}\epsilon^2\,\overline{g}^{cd}\,\nabla_b\left(\overline{g}^{fe}\,h^{ab}\,h_{de}\,F_{ac} - \overline{g}^{be}\,h^{af}\,h_{de}\,F_{ac}\right) + \frac{\tilde{\lambda}}{2\sqrt{-\overline{g}}}\,\epsilon^{abfd}\,\nabla_d\left(\phi\,F_{ab}\right)$$
$$- \frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\nabla_g\left(F^2\,F^{gf}\right) - \frac{7}{2}\frac{1}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\nabla_g\left(\widetilde{F}^{fg}\,\widetilde{F}_{ah}\,F^{ah}\right)$$
$$- \frac{8}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\epsilon\,\overline{g}^{cd}\,\overline{g}^{ab}\,\overline{g}^{fk}\,\overline{g}^{gh}\nabla_g\left(h^{mn}\,F_{kh}\,F_{am}\,F_{bn}\right)\Bigg).$$

The equation of motion is

$$\nabla_g\left(F^{gf} - \frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(4\,F^2\,F^{gf} - 7\,\widetilde{F}^{fg}\,\widetilde{F}_{ah}\,F^{ah}\right)\right) \tag{4.33}$$
$$= \epsilon\,\nabla_b\left(h^{cf}\,F^b_{\ c} - h^{cb}\,F^f_{\ c}\right) - \frac{\tilde{\lambda}}{2\sqrt{-\overline{g}}}\,\epsilon^{abfd}\,\nabla_d\left(\phi\,F_{ab}\right) \tag{4.34}$$
$$+ \frac{8\,\epsilon}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\nabla_g\left(\overline{g}^{cd}\,h^{mn}\,F^{fg}\,F_{am}\,F^a_{\ n}\right)$$
$$- \frac{1}{2}\epsilon^2\,\overline{g}^{cd}\,\nabla_b\left(\overline{g}^{fe}\,h^{ab}\,h_{de}\,F_{ac} - \overline{g}^{be}\,h^{af}h_{de}\,F_{ac}\right)$$

This pretty much matches Dolgov and Ejlli.

4.1.4 Electromagnetic and Scalar Perturbation Expansions from the Action

Equations (4.1) and (4.2) separates the long and short wavelength gravitational waves. We can do the same for the electromagnetic and dark matter waves. We can make the following expansions

$$F_{ab} = \overline{F}_{ab} + \alpha\, F^{(1)}_{ab} + \dots \qquad\qquad \phi = \overline{\phi} + \beta\, \phi^{(1)} + \dots \tag{4.35}$$

For notational convenience, we may define $F^{(1)}_{ab} = f_{ab}$ and $\phi^{(1)} = \theta$. The expanded energy-momentum tensors are

$$\begin{aligned}
T^{\text{em}}_{cd} &= \overline{T}^{\text{em}}_{cd} + \alpha\, T^{\text{em}\,(1)}_{cd} + \alpha^2\, T^{\text{em}\,(2)}_{cd} + \dots && (4.36)\\
&= \left(\overline{g}^{ab}\overline{F}_{ac}\overline{F}_{bd} - \frac{1}{4}\,\overline{g}_{cd}\,\overline{g}^{ab}\,\overline{g}^{ef}\,\overline{F}_{ae}\overline{F}_{bf}\right) + \alpha\left(\overline{g}^{ab}\left(\overline{F}_{ac}F^{(1)}_{bd} + F^{(1)}_{ac}\overline{F}_{bd}\right)\right. && (4.37)\\
&\left. - \frac{1}{4}\,\overline{g}_{cd}\,\overline{g}^{ab}\,\overline{g}^{ef}\,\left(\overline{F}_{ae}F^{(1)}_{bf} + F^{(1)}_{ae}\overline{F}_{bf}\right)\right)\\
&+ \alpha^2\left(\overline{g}^{ab}F^{(1)}_{ac}\,F^{(1)}_{bd} - \frac{1}{4}\,\overline{g}_{cd}\,\overline{g}^{ab}\,\overline{g}^{ef}\,F^{(1)}_{ae}\,F^{(1)}_{bf}\right)
\end{aligned}$$

$$\begin{aligned}
T^{\text{scalar}}_{cd} &= \overline{T}^{\text{scalar}}_{cd} + \beta\, T^{\text{scalar}\,(1)}_{cd} + \beta^2\, T^{\text{scalar}\,(2)}_{cd} + \dots && (4.38)\\
&= \overline{g}_{cd}\left(\frac{1}{2}\,\overline{g}^{ab}\,\partial_a\overline{\phi}\,\partial_b\overline{\phi} + \frac{1}{2}\,m^2\overline{\phi}^2\right) - \partial_c\overline{\phi}\,\partial_d\overline{\phi}\\
&+ \beta\left(\overline{g}_{cd}\,\left(\overline{g}^{ab}\,\partial_a\overline{\phi}\,\partial_b\phi^{(1)} + m^2\,\overline{\phi}\,\phi^{(1)}\right) - \partial_c\,\overline{\phi}\,\partial_d\phi^{(1)} - \partial_d\,\overline{\phi}\,\partial_c\phi^{(1)}\right)\\
&+ \beta^2\left(\overline{g}_{cd}\left(\frac{1}{2}\,\overline{g}^{ab}\,\partial_a\phi^{(1)}\partial_b\phi^{(1)} + \frac{1}{2}\,m^2\,\phi^{(1)\,2}\right) - \partial_d\phi^{(1)}\,\partial_c\phi^{(1)}\right)
\end{aligned}$$

The relevant expanded actions, up to second order in the perturbative parameters, are now

$$S_2 = \frac{1}{4}\int d^4x\sqrt{-\overline{g}}\left(\overline{g}^{ab}\overline{g}^{cd}\left(\overline{F}_{ac}+\alpha F^{(1)}_{ac}\right)\left(\overline{F}_{bd}+\alpha F^{(1)}_{bd}\right)\right. \tag{4.39}$$

$$+ h_{de}\left(-2\epsilon\overline{g}^{cd}\overline{g}^{fe}\left(\overline{T}^{em}_{cf} +^{em(1)}_{cf} +\alpha^2 T^{em(2)}_{cf}\right) + \epsilon^2 h^{ab}\overline{g}^{cd}\overline{g}^{fe}\left(\overline{F}_{ac}+\alpha F^{(1)}_{ac}\left(\overline{F}_{bf}+\alpha F^{(1)}_{bf}\right)\right)\right) \tag{4.40}$$

$$= -\frac{1}{4}\int d^4x\sqrt{-\overline{g}}\left(A^{(1)}_d\left(-4\alpha\overline{g}^{ab}\overline{g}^{cd}\left(\overline{\nabla}_b\overline{F}_{ac}\right)-2\alpha^2\overline{g}^{ab}\overline{g}^{cd}\left(\overline{\nabla}_b F^{(1)}_{ac}\right)\right.\right. \tag{4.41}$$

$$+ 4\alpha\left(\overline{g}^{cg}\overline{g}^{de}\overline{g}^{md}-\overline{g}^{cg}\overline{g}^{ne}\overline{g}^{md}\right)\overline{\nabla}_n\left(h_{ge}\overline{F}_{mc}\right)+\overline{g}^{ab}\overline{g}^{cd}\overline{F}_{ac}\overline{F}_{bd} \tag{4.42}$$

$$+ h_{de}\left(-2\epsilon\overline{g}^{cd}\overline{g}^{fe}\overline{T}^{em}_{cf}+\epsilon^2\overline{g}^{cd}\overline{g}^{fe}h^{ab}\overline{F}_{ac}\overline{F}_{bf}\right) \tag{4.43}$$

We assume the background is constant on the length scales of interest, i.e. the covariant derivatives of background fields is zero. We can apply the gauge fixing condition

$$\overline{\nabla}_a A^{(1)\,a} = 0 \tag{4.44}$$

The action now becomes

$$S_2 = -\frac{1}{4}\int d^4x\sqrt{-\overline{g}}\,\Bigg(A_d^{(1)}\left(-2\,\alpha^2\,\overline{g}^{ab}\,\overline{g}^{cd}\left(\overline{\nabla}_b\,F^{(1)}_{ac}\right)\right. \tag{4.45}$$

$$+\,4\,\alpha\,\epsilon\left(\overline{g}^{cg}\,\overline{g}^{de}\,\overline{g}^{mn}-\overline{g}^{cg}\,\overline{g}^{ne}\,\overline{g}^{md}\right)\overline{\nabla}_n\left(h_{ge}\overline{F}_{mc}\right) \tag{4.46}$$

$$+\,\overline{g}^{ab}\,\overline{g}^{cd}\overline{F}_{ac}\overline{F}_{bd}-2\,\epsilon\,\overline{g}^{cd}\,\overline{g}^{fe}\,\overline{T}^{\mathrm{em}}_{cf}\,h_{de}+\epsilon^2\,\overline{g}^{cd}\,\overline{g}^{fe}\,\overline{F}_{ac}\,\overline{F}_{bf}\,h^{ab}\,h_{de}\Bigg)$$

$$= -\frac{1}{4}\int d^4x\sqrt{-\overline{g}}\,\Bigg(A_d^{(1)}\Big(-2\,\alpha^2\,\overline{g}^{ab}\,\overline{g}^{cd}\left(\overline{\nabla}_b\overline{\nabla}_a A^{(1)}_c-\overline{\nabla}_b\overline{\nabla}_c A^{(1)}_a\right) \tag{4.47}$$

$$+\,4\,\alpha\,\epsilon\left(\overline{g}^{cg}\,\overline{g}^{de}\,\overline{g}^{mn}-\overline{g}^{cg}\,\overline{g}^{ne}\,\overline{g}^{md}\right)\overline{\nabla}_n\left(h_{ge}\overline{F}_{mc}\right)\Big)$$

$$+\,\overline{g}^{ab}\,\overline{g}^{cd}\overline{F}_{ac}\overline{F}_{bd}-2\,\epsilon\,\overline{g}^{cd}\,\overline{g}^{fe}\,\overline{T}^{\mathrm{em}}_{cf}\,h_{de}+\epsilon^2\,\overline{g}^{cd}\,\overline{g}^{fe}\,\overline{F}_{ac}\,\overline{F}_{bf}\,h^{ab}\,h_{de}\Bigg)$$

where we applied the gravitational gauge fixing conditions. The equation (4.46) is consistent with equation (4.1.4.1). We can also use

$$\overline{\nabla}_b\overline{\nabla}_c\,A_a = \overline{\nabla}_c\overline{\nabla}_b\,A_a - \overline{R}^{r}{}_{abc}\,A_r \tag{4.48}$$

and write

$$S_2 = -\frac{1}{4}\int d^4x\sqrt{-\overline{g}}\,\Bigg(-2\,\alpha^2 A_d^{(1)}\left(\overline{g}^{cd}\left(\overline{\Box}A^{(1)}_c+\overline{g}^{ab}\,\overline{R}^{r}{}_{abc}\,A^{(1)}_r\right)\right. \tag{4.49}$$

$$-\,\frac{2\,\epsilon}{\alpha}\left(\overline{g}^{cg}\,\overline{g}^{de}\,\overline{g}^{mn}-\overline{g}^{cg}\,\overline{g}^{ne}\,\overline{g}^{md}\right)\overline{\nabla}_n\left(h_{ge}\overline{F}_{mc}\right)$$

$$+\,\overline{g}^{ab}\,\overline{g}^{cd}\overline{F}_{ac}\overline{F}_{bd}+h_{de}\left(-2\,\epsilon\,\overline{g}^{cd}\,\overline{g}^{fe}\,\overline{T}^{\mathrm{em}}_{cf}+\epsilon^2\,\overline{g}^{cd}\,\overline{g}^{fe}\,\overline{F}_{ac}\,\overline{F}_{bf}\,h^{ab}\right)\Bigg)$$

We also have the following Euler-Heisenberg action to second order in the perturbative parameters

$$\begin{aligned}
S_5 &= \int d^4x\sqrt{-\overline{g}}\ \frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \Bigg(\Big(\overline{F}^2 + 2\,\alpha\,\overline{F}F^{(1)} + \alpha^2\,F^{(1)}F^{(1)}\Big) \\
&\times \Big(\overline{F}^2 + 2\,\alpha\,\overline{F}\,F^{(1)} + \alpha^2\,F^{(1)}F^{(1)}\Big) \qquad (4.50)\\
&+ \frac{7}{4}\Bigg(\Big(\overline{F}\,\tilde{\overline{F}} + \alpha\,\Big(\overline{F}\tilde{F}^{(1)} + F^{(1)}\tilde{\overline{F}}\Big) + \alpha^2 F^{(1)}\tilde{F}^{(1)}\Big) \\
&\times \Big(\overline{F}\,\tilde{\overline{F}} + \alpha\,\Big(\overline{F}\tilde{F}^{(1)} + F^{(1)}\tilde{\overline{F}}\Big) + \alpha^2\,F^{(1)}\tilde{F}^{(1)}\,\Big)\Bigg) \\
&- 4\,\epsilon\,h_{de}\Big(\,\overline{F}^2\overline{F}_a{}^d\,\overline{F}^{ae} + \alpha\Big(\,\overline{F}^2\,\overline{F}_a{}^d F^{(1)\,ae} + \overline{F}^2 F^{(1)}_a{}^d\overline{F}^{ae} + 2\,\overline{F}\,F^{(1)}\overline{F}_a{}^d\overline{F}^{ae}\,\Big)\Big)\Bigg) \\
&= \int d^4x\sqrt{-\overline{g}}\ \frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \Bigg(\overline{F}^4 + \frac{7}{4}\,(\overline{F}\,\tilde{\overline{F}})(\overline{F}\,\tilde{\overline{F}}) - 4\,\epsilon\,h_{de}\,\overline{F}^2\,\overline{F}_a{}^d\,\overline{F}^{ae} \qquad (4.51)\\
&+ \alpha\left(8\,\overline{F}^2\,\overline{F}^{cd}\,\overline{\nabla}_c + \frac{14}{\sqrt{-\overline{g}}}\,(\overline{F}\,\tilde{\overline{F}})\,\epsilon^{abcd}\,\overline{F}_{ab}\overline{\nabla}_c\right) A_d^{(1)} \\
&+ \alpha^2\Bigg(4\,\overline{F}^2 F^{(1)\,cd}\,\overline{\nabla}_c\,A_d^{(1)} + 8\,\overline{F}^{gh}\overline{F}^{cd}\,F^{(1)}_{gh}\,\overline{\nabla}_c\,A_d^{(1)} + \frac{7}{\sqrt{-\overline{g}}}\,(\overline{F}\,\tilde{\overline{F}})\,\epsilon^{abcd}\,F^{(1)}_{ab}\overline{\nabla}_c A_d^{(1)} \qquad (4.52)\\
&+ \frac{14}{\sqrt{-\overline{g}}}\,\overline{F}\tilde{F}^{(1)}\,\epsilon^{abcd}\,\overline{F}_{ab}\overline{\nabla}_c A_d^{(1)}\Bigg) \\
&- 4\,\alpha\,\epsilon\,\Big(\,\overline{F}^2\,h_g{}^d\,\overline{F}^{ag}\,\overline{\nabla}_a - \overline{F}^2\,h_g{}^a\,\overline{F}^{dg}\,\overline{\nabla}_a + \overline{F}^2\,h^d{}_e\,\overline{F}^{ae}\,\overline{\nabla}_a - \overline{F}^2\,h^a{}_e\,\overline{F}^{de}\,\overline{\nabla}_a \qquad (4.53)\\
&+ 4\,h_{ge}\overline{F}_a{}^g\overline{F}^{ae}\overline{F}^{db}\,\overline{\nabla}_b\Big)A_d^{(1)}\Bigg)
\end{aligned}$$

After integration by parts and assuming the background quantities are static,

$$+\,\alpha^2\Bigg(-4\,\overline{F}^2\,\overline{\nabla}_c\,F^{(1)\,cd} - 8\,\overline{F}^{gh}\overline{F}^{cd}\,\overline{\nabla}_c\,F^{(1)}_{gh} - \tfrac{7}{\sqrt{-\overline{g}}}\,(\,\overline{F}\,\tilde{\overline{F}}\,)\,\epsilon^{abcd}\,\overline{\nabla}_c F^{(1)}_{ab}$$

$$-\,\tfrac{14}{\sqrt{-\overline{g}}}\,(\overline{F}^{ef}\,\overline{\nabla}_c\,\tilde{F}^{(1)}_{ef})\,\epsilon^{abcd}\,\overline{F}_{ab}\Bigg)\,A_d^{(1)}$$

$$- 4\,\alpha\,\epsilon \left(-\overline{F}^2\,\overline{F}^{ag}\,\overline{\nabla}_a\,h_g{}^d + \overline{F}^2\,\overline{F}^{dg}\,\overline{\nabla}_a\,h_g{}^a - \overline{F}^2\,\overline{F}^{ae}\,\overline{\nabla}_a\,h^d{}_e + \overline{F}^2\,\overline{F}^{de}\,\overline{\nabla}_a\,h^a{}_e \right.$$

$$\left. \left. - 4\,\overline{F}_a{}^g\overline{F}^{ae}\overline{F}^{db}\,\overline{\nabla}_b\,h_{ge} \right) A_d^{(1)} \right)$$

$$= \int d^4x\sqrt{-\overline{g}}\;\; \tfrac{1}{90\,m_e^4} \left(\tfrac{e^2}{4\pi}\right)^2 \left(\overline{F}^4 + \tfrac{7}{4}\,(\overline{F}\,\widetilde{\overline{F}})(\overline{F}\,\widetilde{\overline{F}}) - 4\,\epsilon\,h_{de}\,\overline{F}^2\,\overline{F}_a{}^d\,\overline{F}^{ae} \right.$$

$$+ \alpha^2 \left(-8\,\overline{F}^{gh}\overline{F}^{cd}\,\overline{\nabla}_c\,F^{(1)}_{gh} - \tfrac{7}{\sqrt{-\overline{g}}}\,(\,\overline{F}\,\widetilde{\overline{F}}\,)\,\epsilon^{abcd}\,\overline{\nabla}_c F^{(1)}_{ab} - \tfrac{14}{\sqrt{-\overline{g}}}\,(\overline{F}^{ef}\,\overline{\nabla}_c\,\widetilde{F}^{(1)}_{ef})\,\epsilon^{abcd}\,\overline{F}_{ab} \right) A_d^{(1)}$$

$$\left. + 4\,\alpha\,\epsilon \left(2\,\overline{F}^2\,\overline{F}^{ag}\,\overline{\nabla}_a\,h_g{}^d + 4\,\overline{F}_a{}^g\overline{F}^{ae}\overline{F}^{db}\,\overline{\nabla}_b\,h_{ge} \right) A_d^{(1)} \right)$$

As for the scalar action, we can expand to find

$$S_3 = -\frac{1}{2}\int d^4x\sqrt{-\overline{g}}\ \Bigg(\overline{g}^{ab}\,\partial_a\overline{\phi}\partial_b\overline{\phi} + 2\beta\,\overline{g}^{ab}\,\partial_a\overline{\phi}\partial_b\phi^{(1)} + \beta^2\,\overline{g}^{ab}\,\partial_a\phi^{(1)}\partial_b\phi^{(1)} \tag{4.54}$$

$$+ m^2\,\overline{\phi}^2 + 2\,m^2\,\beta\,\overline{\phi}\,\phi^{(1)} \tag{4.55}$$

$$+ \beta^2\,m^2\,{\phi^{(1)}}^2 + \epsilon\,\overline{g}^{cd}\,\overline{g}^{je}\,h_{de}\,\left(\overline{T}^{\text{scalar}}_{cj} + \beta\,T^{(1)\text{ scalar}}_{cj} + \beta^2\,T^{(2)\text{ scalar}}_{cj}\right)$$

$$+ \frac{\epsilon^2}{2}\,\overline{g}^{ad}\,\overline{g}^{fe}\,h_{de}\,h_f^{\ b}\,\left(\partial_a\overline{\phi}\,\partial_b\overline{\phi}\right)\Bigg)$$

$$= -\frac{1}{2}\int d^4x\sqrt{-\overline{g}}\ \Bigg(\overline{g}^{ab}\,\partial_a\overline{\phi}\partial_b\overline{\phi} + m^2\,\overline{\phi}^2 + \epsilon\,h_{de}\,\overline{T}^{de}_{\text{scalar}} + 2\beta\,\overline{g}^{ab}\,\partial_a\overline{\phi}\partial_b\phi^{(1)}$$

$$+ \beta^2\,\overline{g}^{ab}\,\partial_a\phi^{(1)}\partial_b\phi^{(1)} \tag{4.56}$$

$$+ 2\,m^2\,\beta\,\overline{\phi}\,\phi^{(1)} + \beta^2\,m^2\,{\phi^{(1)}}^2 + \beta\,\epsilon\,h_{de}\,\overline{g}^{de}\left(\overline{g}^{ab}\,\partial_a\overline{\phi}\,\partial_b\phi^{(1)} + m^2\,\overline{\phi}\,\phi^{(1)}\right) \tag{4.57}$$

$$-\,2\,\beta\,\epsilon\,h^{cj}\,\partial_c\,\overline{\phi}\,\partial_j\phi^{(1)} + \frac{\epsilon^2}{2}\,h^a_{\ e}\,h^{eb}\,\partial_a\overline{\phi}\,\partial_b\overline{\phi}\Bigg)$$

$$= -\frac{1}{2}\int d^4x\sqrt{-\overline{g}}\ \Bigg(\overline{g}^{ab}\,\partial_a\overline{\phi}\,\partial_b\overline{\phi} + m^2\,\overline{\phi}^2 + \phi^{(1)}\Big(-2\,\beta\,\Box\,\overline{\phi} + 2\,m^2\,\beta\,\overline{\phi}$$

$$-\,\beta^2\,\Box\,\phi^{(1)} + \beta^2\,m^2\,\phi^{(1)} \tag{4.58}$$

$$+\,2\,\beta\,\epsilon\,h^{cj}\,(\partial_j\partial_c\,\overline{\phi}) + \epsilon\,h_{de}\,\overline{T}^{de}_{\text{scalar}} + \frac{\epsilon^2}{2}\,h^a_{\ e}\,h^{eb}\,\partial_a\overline{\phi}\,\partial_b\overline{\phi}\Big)\Bigg)$$

which is truncated at the perturbative parameter squared. As for the axion-photon coupling

$$S_4 = \int d^4x \sqrt{-\overline{g}}\ \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}}\epsilon^{abcd}\ \left(\overline{\phi} + \beta\,\phi^{(1)}\right)\ \left(\overline{F}_{ab} + \alpha\,F^{(1)}_{ab}\right)\left(\overline{F}_{cd} + \alpha\,F^{(1)}_{cd}\right) \tag{4.59}$$

$$= \int d^4x \sqrt{-\overline{g}}\ \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}}\epsilon^{abcd}\ \left(\overline{\phi} + \beta\,\phi^{(1)}\right)\ \left(\overline{F}_{cd}\,\overline{F}_{ab} + 2\,\alpha\,F^{(1)}_{ab}\overline{F}_{cd} + \alpha^2\,F^{(1)}_{ab}F^{(1)}_{cd}\right) \tag{4.60}$$

$$S_4 = \int d^4x \sqrt{-\overline{g}}\ \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}}\epsilon^{abcd}\left(\overline{F}_{cd}\,\overline{F}_{ab}\,\overline{\phi} + 2\,\alpha\,F^{(1)}_{ab}\,\overline{F}_{cd}\,\overline{\phi} + \alpha^2\,F^{(1)}_{ab}\,F^{(1)}_{cd}\,\overline{\phi}\right. \tag{4.61}$$

$$\left. + \phi^{(1)}\left(\beta\,\overline{F}_{cd}\,\overline{F}_{ab} + 2\,\alpha\,\beta\,F^{(1)}_{ab}\overline{F}_{cd}\right)\right) \tag{4.62}$$

$$S_4 = \int d^4x \sqrt{-\overline{g}}\ \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}}\epsilon^{abcd}\left(\overline{F}_{cd}\,\overline{F}_{ab}\,\overline{\phi} - A^{(1)}_b\,\overline{\nabla}_a\left(4\,\alpha\,\overline{F}_{cd}\,\overline{\phi} + 2\,\alpha^2\,F^{(1)}_{cd}\,\overline{\phi}\right.\right.$$
$$\left.\left. + 4\,\alpha\,\beta\,\phi^{(1)}\overline{F}_{cd} + \beta\,\phi^{(1)}\,\overline{F}_{cd}\,\overline{F}_{ab}\right)\right.$$

4.1.5 High-Frequency Expanded Equations of Motion

The action with the gravitational perturbation factored out is

$$S = \int d^4x \sqrt{-\overline{g}}\ \ h_{de}\Bigg(-\frac{1}{2}\left(\frac{\epsilon^2}{4\,\kappa}\right)\overline{\Box}\,h^{de} - \left(\frac{\epsilon^2}{4\,\kappa}\right)\left(\overline{g}^{qw}\,\overline{R}^d{}_{qwa}\,h^{ae} - \overline{g}^{cd}\,\overline{R}^e{}_{rca}\,h^{ar}\right) \tag{4.63}$$
$$+ \frac{1}{2}\,\epsilon\,\overline{g}^{cd}\,\overline{g}^{fe}\,T^{\text{em}}_{cf} - \frac{1}{4}\epsilon^2\,h^{ab}\,\overline{g}^{cd}\,\overline{g}^{fe}\,\overline{F}_{ac}\,\overline{F}_{bf} - \frac{\epsilon}{2}\,\overline{g}^{cd}\,\overline{g}^{je}\,T^{\text{scalar}}_{cj}$$
$$- \frac{\epsilon^2}{4}\,\overline{g}^{ad}\,\overline{g}^{fe}\,h_f^{\ b}\ (\partial_a\phi\,\partial_b\phi) - \frac{2\,\epsilon}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\ F^2\,F_a^{\ d}\,F^{ae}\Bigg)$$

The resulting gravitational equations of motion are

$$\frac{1}{2}\overline{\Box}\,h^{de} + \left(\overline{g}^{qw}\,\overline{R}^d{}_{qwa}\,h^{ae} - \overline{g}^{cd}\,\overline{R}^e{}_{rca}\,h^{ar}\right) - \frac{2\,\kappa}{\epsilon}\,\overline{g}^{cd}\,\overline{g}^{fe}\,T^{\text{em}}_{cf} + \kappa\,h^{ab}\,\overline{g}^{cd}\,\overline{g}^{fe}\,F_{ac}\,F_{bf}$$
$$+ \frac{2\,\kappa}{\epsilon}\,\overline{g}^{cd}\,\overline{g}^{je}\,T^{\text{scalar}}_{cj} + \kappa\,\overline{g}^{ad}\,\overline{g}^{fe}\,h_f^{\ b}\ (\partial_a\phi\,\partial_b\phi) + \frac{8\,\kappa}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\frac{1}{\epsilon}\right)\ F^2\,F_a^{\ d}\,F^{ae} = 0$$

where

$$T^{\text{em}}_{cd} = \overline{g}^{ab}\, F_{ac}\, F_{bd} - \frac{1}{4}\overline{g}_{cd}\, \overline{g}^{ab}\, \overline{g}^{ef}\, F_{ae}\, F_{bf} \tag{4.64}$$

$$T^{\text{scalar}}_{cd} = \overline{g}_{cd} \left(\frac{1}{2}\, \overline{g}^{ab}\, \partial_a\phi\, \partial_b\phi + \frac{1}{2}\, m^2 \phi^2 \right) - \partial_c\phi\, \partial_d\phi \tag{4.65}$$

The action with the electromagnetic perturbation factored out is

$$\begin{aligned} S_2 = -\frac{1}{4} \int d^4x \sqrt{-\overline{g}}\, \Big(& -2\,\alpha^2 A^{(1)}_d \left(\overline{g}^{cd} \left(\overline{\Box} A^{(1)}_c + \overline{g}^{ab}\, \overline{R}^{r}{}_{abc}\, A^{(1)}_r \right) \right. \\ & - \frac{2\,\epsilon}{\alpha} \left(\overline{g}^{cg}\, \overline{g}^{de}\, \overline{g}^{mn} - \overline{g}^{cg}\, \overline{g}^{ne}\, \overline{g}^{md} \right) \overline{\nabla}_n \left(h_{ge} \overline{F}_{mc} \right) \\ & + \overline{g}^{ab}\, \overline{g}^{cd} \overline{F}_{ac} \overline{F}_{bd} + h_{de} \left(-2\,\epsilon\, \overline{g}^{cd}\, \overline{g}^{fe}\, \overline{T}^{\text{em}}_{cf} + \epsilon^2\, \overline{g}^{cd}\, \overline{g}^{fe}\, \overline{F}_{ac}\, \overline{F}_{bf}\, h^{ab} \right) \Big) \end{aligned} \tag{4.66}$$

$$S = \int d^4x \sqrt{-\overline{g}}\, \left(A^{(1)}_d \left(\frac{\alpha^2}{2} \overline{g}^{cd} \left(\overline{\Box} A^{(1)}_c + \overline{g}^{ab}\, \overline{R}^{r}{}_{abc}\, A^{(1)}_r \right) \right. \right. \tag{4.67}$$

$$- \frac{\alpha^2}{2} \frac{2\,\epsilon}{\alpha} \left(\overline{g}^{cg}\, \overline{g}^{de}\, \overline{g}^{mn} - \overline{g}^{cg}\, \overline{g}^{ne}\, \overline{g}^{md} \right) \overline{\nabla}_n \left(h_{ge} \overline{F}_{mc} \right) \tag{4.68}$$

$$\frac{1}{2}\, \overline{g}^{ab}\, \overline{g}^{cd}\, \nabla_a F_{bd} + \epsilon \left(\overline{g}^{ab}\, \nabla_a \left(h^{cd}\, F_{bd} \right) - \frac{1}{4}\, \overline{g}_{mn}\, \overline{g}^{ab}\, \overline{g}^{cf}\, \nabla_a \left(h^{mn}\, F_{bf} \right) \right) + \frac{1}{2}\, \epsilon^2\, \nabla_a \left(h^{ab}\, h^{cd}\, F_{bd} \right) \tag{4.69}$$

$$\begin{aligned} & + \frac{\tilde{\lambda}}{4\sqrt{-\overline{g}}} \epsilon^{abcd}\, \nabla_d \left(\phi\, F_{ab} \right) + \frac{1}{90\, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \left(-2\, \nabla_g \left(\overline{g}^{ab}\, \overline{g}^{cd}\, \overline{g}^{gf}\, \overline{g}^{eh}\, F_{ac}\, F_{bd}\, F_{fh} \right) \right. \\ & \left. - \frac{7}{2\,\overline{g}}\, \epsilon^{ebcd}\, \epsilon^{afgh}\, \nabla_b \left(F_{cd}\, F_{af}\, F_{gh} \right) - 8\, \epsilon\, \nabla_g \left(\overline{g}^{cd}\, \overline{g}^{ab}\, \overline{g}^{ef}\, \overline{g}^{gh}\, h^{mn}\, F_{fh}\, F_{am}\, F_{bn} \right) \right) = 0 \end{aligned} \tag{4.70}$$

The equation of motion is

$$\overline{g}^{cd}\left(\overline{\Box}A^{(1)}_{c} + \overline{g}^{ab}\,\overline{R}^{r}{}_{abc}\,A^{(1)}_{r}\right) - \frac{2\,\epsilon}{\alpha}\left(\overline{g}^{cg}\,\overline{g}^{de}\,\overline{g}^{mn} - \overline{g}^{cg}\,\overline{g}^{ne}\,\overline{g}^{md}\right)\overline{\nabla}_{n}\left(h_{ge}\overline{F}_{mc}\right) = 0 \tag{4.71}$$

The scalar equations of motion are

$$\frac{1}{2}\,\overline{g}^{ab}\,\nabla_a\nabla_b\,\phi - \frac{1}{2}m^2\phi - \frac{1}{2}\epsilon\left(-\frac{1}{2}\,\overline{g}^{ab}\,\overline{g}_{cd}\,\nabla_a\left(h^{cd}\,\nabla_b\,\phi\right) + \frac{1}{2}\,m^2 h\,\phi - \nabla_c\nabla_d\,\phi\right)$$
$$- \frac{\epsilon^2}{4}\,\nabla_a\left(h^{af}\,h_f^{\;b}\,\partial_b\,\phi\right) + \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}}\,\epsilon^{abcd}\,F_{ab}\,F_{cd} = 0$$

4.1.5 Electromagnetic and Scalar Perturbation Expansions

Equations (4.1) and (4.2) separates the long and short wavelength gravitational waves. We can do the same for the electromagnetic and dark matter waves. We can make the following expansions

$$F_{ab} = \overline{F}_{ab} + \alpha\, F^{(1)}_{ab} + \dots \qquad\qquad \phi = \overline{\phi} + \beta\, \phi^{(1)} + \dots \tag{4.72}$$

For notational convenience, we may define $F^{(1)}_{ab} = f_{ab}$ and $\phi^{(1)} = \theta$. The expanded energy-momentum tensors are

$$T^{\rm em}_{cd} = \overline{T}^{\rm em}_{cd} + \alpha\, T^{{\rm em}\,(1)}_{cd} + \alpha^2\, T^{{\rm em}\,(2)}_{cd} + \dots \tag{4.73}$$

$$= \left(\overline{g}^{ab}\overline{F}_{ac}\overline{F}_{bd} - \frac{1}{4}\,\overline{g}_{cd}\,\overline{g}^{ab}\,\overline{g}^{ef}\,\overline{F}_{ae}\overline{F}_{bf}\right) + \alpha\left(\overline{g}^{ab}\left(\overline{F}_{ac}F^{(1)}_{bd} + F^{(1)}_{ac}\overline{F}_{bd}\right)\right. \tag{4.74}$$

$$\left. - \frac{1}{4}\,\overline{g}_{cd}\,\overline{g}^{ab}\,\overline{g}^{ef}\,\left(\overline{F}_{ae}F^{(1)}_{bf} + F^{(1)}_{ae}\overline{F}_{bf}\right)\right)$$

$$+ \alpha^2\left(\overline{g}^{ab}F^{(1)}_{ac}\,F^{(1)}_{bd} - \frac{1}{4}\,\overline{g}_{cd}\,\overline{g}^{ab}\,\overline{g}^{ef}\,F^{(1)}_{ae}\,F^{(1)}_{bf}\right)$$

$$T^{\rm scalar}_{cd} = \overline{T}^{\rm scalar}_{cd} + \beta\, T^{{\rm scalar}\,(1)}_{cd} + \beta^2\, T^{{\rm scalar}\,(2)}_{cd} + \dots \tag{4.75}$$

$$= \overline{g}_{cd}\left(\frac{1}{2}\,\overline{g}^{ab}\,\partial_a\phi\,\partial_b\phi + \frac{1}{2}\,m^2\phi^2\right) - \partial_c\phi\,\partial_d\phi \tag{4.76}$$

$$S_2 = -\frac{1}{4} \int d^4x \, \sqrt{-\overline{g}} \left(\overline{g}^{ab} \, \overline{g}^{cd} \, \overline{F}_{ac} \, \overline{F}_{bd} + 2 \, \alpha \, \overline{g}^{ab} \, \overline{g}^{cd} \, \overline{F}_{ac} \, F^{(1)}_{bd} + \alpha^2 \, \overline{g}^{ab} \, \overline{g}^{cd} \, F^{(1)}_{ac} \, F^{(1)}_{bd} \right. \tag{4.77}$$

$$\left. - \, 2 \, \epsilon \, h^{cd} \, T^{\text{em}}_{cd} + \epsilon^2 \, h^{ab} \, h^{cd} \left(\overline{F}_{ac} \, \overline{F}_{bd} + 2 \, \alpha \, \overline{F}_{ac} \, F^{(1)}_{bd} + \alpha^2 \, F^{(1)}_{ac} \, F^{(1)}_{bd} \right) \right)$$

$$S_3 = -\frac{1}{2} \int d^4x \, \sqrt{-\overline{g}} \left(\overline{g}^{ab} \, \partial_a \phi \, \partial_b \phi + m^2 \phi^2 + \epsilon \, h^{cd} \, T^{\text{scalar}}_{cd} + \frac{\epsilon^2}{2} \, h^{af} \, h_f^{\ b} \, \left(\partial_a \phi \, \partial_b \phi \right) \right) \tag{4.78}$$

$$S_4 = \frac{\tilde{\lambda}}{8} \int d^4x \; \epsilon^{abcd} \, \phi \, F_{ab} \, F_{cd} \tag{4.79}$$

$$S_5 = \frac{1}{90 \, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \int d^4x \, \sqrt{-\overline{g}} \left(\left(\left(\overline{g}^{ab} \, \overline{g}^{cd} \, F_{ac} \, F_{bd} \right)^2 + \frac{7}{4} \left(\frac{1}{\sqrt{-\overline{g}}} \, \epsilon^{abcd} \, F_{ab} \, F_{cd} \right)^2 \right) \right. \tag{4.80}$$

$$\left. + \, \epsilon \, \left(-4 \, h^{mn} \, \overline{g}^{ab} \, \overline{g}^{ef} \, \overline{g}^{gh} \, F_{eg} \, F_{fh} \, F_{am} \, F_{bn} \right) \right)$$

4.1.6 Expanded Lagrangian

The first term in the lagrangian, equation (4.11), can be expanded as

$$\frac{1}{2\,\kappa}\left(\sqrt{-g}\,R\right) = \frac{1}{2\,\kappa}\left(\sqrt{-\overline{g}} + \frac{\epsilon}{2}\sqrt{-\overline{g}}\;\overline{g}^{cd}\,h_{cd} + \ldots\right)$$

$$\times\left(\overline{R} + \frac{\epsilon}{2}\left(\overline{g}^{bd}\,\overline{\nabla}_a\overline{\nabla}_{\{b}\,h^a{}_{d\}} - \overline{g}^{bd}\,\overline{\nabla}_d\overline{\nabla}_b\,h - \overline{g}^{bd}\,\overline{\Box}\,h_{bd}\right)\right. \tag{4.81}$$

$$+\frac{\epsilon^2}{4}\left(\overline{g}^{bd}\,\overline{\nabla}_b\,h_{ce}\overline{\nabla}_d\,h^{ce} + 2\,\overline{g}^{bd}\,h^{ce}\left(\overline{\nabla}_b\overline{\nabla}_d\,h_{ce} + \overline{\nabla}_c\overline{\nabla}_e\,h_{bd} - \overline{\nabla}_d\,\overline{\nabla}_e\,h_{cb} - \overline{\nabla}_b\,\overline{\nabla}_e\,h_{cd}\right)\right. \tag{4.82}$$

$$+\,2\,\overline{g}^{bd}\,\overline{g}^{eg}\overline{g}^{ch}\,\overline{\nabla}_g\,h_{dh}\left(\overline{\nabla}_e\,h_{cb} - \overline{\nabla}_c\,h_{eb}\right) - \overline{g}^{bd}\left(\overline{\nabla}_e\,h^{ce}\right. \tag{4.83}$$

$$\left.\left.-\frac{1}{2}\,\overline{g}^{ce}\,\overline{\nabla}_e\,h\right)\left(\overline{\nabla}_d\,h_{cb} + \overline{\nabla}_b\,h_{cd} - \overline{\nabla}_c\,h_{bd}\right)\right)\right)$$

$$= \frac{1}{2\,\kappa}\sqrt{-\overline{g}}\left(\overline{R} + \frac{\epsilon}{2}\,\overline{g}^{bd}\left(\overline{\nabla}_a\overline{\nabla}_{\{b}\,h^a{}_{d\}} - \overline{\nabla}_d\overline{\nabla}_b\,h - \overline{\Box}\,h_{bd}\right)\right. \tag{4.84}$$

$$+\frac{\epsilon^2}{4}\left(2\,h^{ce}\left(\overline{\Box}\,h_{ce} + \overline{\nabla}_c\overline{\nabla}_e\,h - \overline{\nabla}_d\,\overline{\nabla}_e\,h_c{}^d - \overline{\nabla}_b\,\overline{\nabla}_e\,h_c{}^b\right) + \overline{g}^{bd}\,\overline{\nabla}_b\,h_{ce}\overline{\nabla}_d\,h^{ce}\right.$$

$$+\,2\,\overline{g}^{bd}\,\overline{g}^{eg}\overline{g}^{ch}\,\overline{\nabla}_g\,h_{dh}\left(\overline{\nabla}_e\,h_{cb} - \overline{\nabla}_c\,h_{eb}\right) - \overline{g}^{bd}\left(\overline{\nabla}_e\,h^{ce}\right. \tag{4.85}$$

$$\left.\left.-\frac{1}{2}\,\overline{g}^{ce}\,\overline{\nabla}_e\,h\right)\left(\overline{\nabla}_d\,h_{cb} + \overline{\nabla}_b\,h_{cd} - \overline{\nabla}_c\,h_{bd}\right)\right)\right)$$

$$+\frac{\epsilon}{4\,\kappa}\sqrt{-\overline{g}}\;\overline{g}^{lm}\,h_{lm}\left(\overline{R} + \frac{\epsilon}{2}\,\overline{g}^{bd}\left(\overline{\nabla}_a\overline{\nabla}_{\{b}\,h^a{}_{d\}} - \overline{\nabla}_d\overline{\nabla}_b\,h - \overline{\Box}\,h_{bd}\right)\right)$$

$$
\begin{aligned}
\frac{1}{2\,\kappa}\left(\sqrt{-g}\,R\right) = \frac{1}{2\,\kappa}\sqrt{-\overline{g}}\,\Bigg(&\overline{R} + \frac{\epsilon}{2}\,\overline{g}^{bd}\left(\,\overline{\nabla}_a\overline{\nabla}_{\{b}\,h^a{}_{d\}} - \overline{\nabla}_d\overline{\nabla}_b\,h - \overline{\Box}\,h_{bd}\right) \\
&+ \frac{\epsilon^2}{4}\,\Big(\,2\,h^{ce}\Big(\overline{\Box}\,h_{ce} + \overline{\nabla}_c\overline{\nabla}_e\,h - \overline{\nabla}_d\,\overline{\nabla}_e\,h_c{}^d - \overline{\nabla}_b\,\overline{\nabla}_e\,h_c{}^b\Big) \\
&+ \overline{g}^{bd}\,\overline{\nabla}_b\,h_{ce}\overline{\nabla}_d\,h^{ce} + 2\,\overline{g}^{eg}\,\overline{\nabla}_g\,h^{bc}\,\overline{\nabla}_e\,h_{cb} \\
&+ \frac{1}{2}\,\overline{\nabla}_e\,h\,\overline{\nabla}_d\,h^{ed} + \frac{1}{2}\,\overline{\nabla}_e\,h\,\overline{\nabla}_b\,h^{eb} - \overline{\nabla}_e\,h^{ce}\overline{\nabla}_c\,h \\
&- 2\,\overline{\nabla}_g\,h^{bc}\overline{\nabla}_c\,h_b{}^g - \overline{\nabla}_e\,h^{ce}\,\overline{\nabla}_d\,h_c{}^d - \overline{\nabla}_e\,h^{ce}\,\overline{\nabla}_b\,h_c{}^b + \frac{1}{2}\,\overline{g}^{ce}\,\overline{\nabla}_e\,h\,\overline{\nabla}_c\,h\Big)\Bigg) \\
&+ \frac{\epsilon}{4\,\kappa}\sqrt{-\overline{g}}\,\overline{g}^{lm}\,h_{lm}\Big(\,\overline{R} + \frac{\epsilon}{2}\,\overline{g}^{bd}\left(\,\overline{\nabla}_a\overline{\nabla}_{\{b}\,h^a{}_{d\}} - \overline{\nabla}_d\overline{\nabla}_b\,h - \overline{\Box}\,h_{bd}\right)\Big)
\end{aligned} \tag{4.86}
$$

after the gravitational wave expansion is

$$
\begin{aligned}
\mathcal{L} \;=\;& \left(\sqrt{-\overline{g}} + \frac{\epsilon}{2}\sqrt{-\overline{g}}\,\overline{g}^{cd}\,h_{cd} + \ldots\right)\left(\frac{1}{2\,\kappa}\Big(\overline{R} + \frac{\epsilon}{2}\left(\overline{g}^{bd}\,\overline{\nabla}_a\overline{\nabla}_b\,h^a_{\ d} + \overline{g}^{bd}\,\overline{\nabla}_a\overline{\nabla}_d\,h^a_{\ b}\right.\right.\\
-\;& \left.\overline{g}^{bd}\,\overline{\nabla}_d\overline{\nabla}_b\,h - \overline{g}^{bd}\,\overline{\Box}\,h_{bd}\right)\\
+\;& \frac{\epsilon^2}{4}\Big(\overline{g}^{bd}\,\overline{\nabla}_b\,h_{ce}\overline{\nabla}_d\,h^{ce} + 2\,\overline{g}^{bd}\,h^{ce}\left(\overline{\nabla}_b\overline{\nabla}_d\,h_{ce} + \overline{\nabla}_c\overline{\nabla}_e\,h_{bd} - \overline{\nabla}_d\,\overline{\nabla}_e\,h_{cb} - \overline{\nabla}_b\,\overline{\nabla}_e\,h_{cd}\right)\\
+\;& 2\,\overline{g}^{bd}\,\overline{g}^{eg}\,\overline{g}^{ch}\,\overline{\nabla}_g\,h_{dh}\,\overline{\nabla}_e\,h_{cb}\\
-\;& 2\,\overline{g}^{bd}\,\overline{g}^{eg}\,\overline{g}^{ch}\,\overline{\nabla}_g\,h_{dh}\,\overline{\nabla}_c\,h_{eb} - \overline{g}^{bd}\,\overline{\nabla}_e\,h^{ce}\overline{\nabla}_d\,h_{cb} - \overline{g}^{bd}\,\overline{\nabla}_e\,h^{ce}\overline{\nabla}_b\,h_{cd} + \overline{g}^{bd}\,\overline{\nabla}_e\,h^{ce}\overline{\nabla}_c\,h_{bd}\\
+\;& \frac{1}{2}\,\overline{g}^{bd}\,\overline{g}^{ce}\,\overline{\nabla}_e\,h\,\overline{\nabla}_d\,h_{cb}\\
+\;& \frac{1}{2}\,\overline{g}^{bd}\,\overline{g}^{ce}\,\overline{\nabla}_e\,h\overline{\nabla}_b\,h_{cd} - \frac{1}{2}\,\overline{g}^{bd}\,\overline{g}^{ce}\,\overline{\nabla}_e\,h\overline{\nabla}_c\,h_{bd}\Big) + \ldots\Big)\\
-\;& \frac{1}{4}\left(\overline{g}^{ab} - \epsilon\,h^{ab} + \frac{1}{2}\,\epsilon^2\,h^{ac}\,h_c^{\ b} + \ldots\right)\left(\overline{g}^{cd} - \epsilon\,h^{cd} + \frac{1}{2}\,\epsilon^2\,h^{ce}\,h_e^{\ d} + \ldots\right)F_{ac}\,F_{bd}\\
-\;& \frac{1}{2}\left(\left(\overline{g}^{ab} - \epsilon\,h^{ab} + \frac{1}{2}\,\epsilon^2\,h^{ac}\,h_c^{\ b} + \ldots\right)\partial_a\phi\,\partial_b\phi + m^2\phi^2\right)\\
+\;& \frac{\tilde{\lambda}}{8}\left(\frac{1}{\sqrt{-\overline{g}}} - \frac{\epsilon}{2\sqrt{-\overline{g}}}\,\overline{g}^{cd}\,h_{cd} + \ldots\right)\epsilon^{abcd}\,\phi\,F_{ab}\,F_{cd}\Big)\\
+\;& \frac{1}{90\,m_e^4}\left(\sqrt{-\overline{g}} + \frac{\epsilon}{2}\sqrt{-\overline{g}}\,\overline{g}^{cd}\,h_{cd}\right)\\
\times\;& \left(\frac{e^2}{4\pi}\right)^2\Bigg(\left(\left(\overline{g}^{ab} - \epsilon\,h^{ab}\right)\left(\overline{g}^{cd} - \epsilon\,h^{cd}\right)F_{ac}\,F_{bd}\right)^2\\
+\;& \frac{7}{4}\left(\left(\frac{1}{\sqrt{-\overline{g}}} - \frac{\epsilon}{2\sqrt{-\overline{g}}}\,\overline{g}^{cd}\,h_{cd}\right)\epsilon^{abcd}\,F_{ab}\,F_{cd}\right)^2\Bigg) + \ldots
\end{aligned}
\tag{4.87}
$$

$$
\begin{aligned}
\mathcal{L} &= \sqrt{-\overline{g}}\,\Bigg(\frac{1}{2\,\kappa}\Big(\overline{R}+\frac{\epsilon}{2}\left(\overline{g}^{bd}\,\overline{\nabla}_a\overline{\nabla}_b\,h^a_{\ d}+\overline{g}^{bd}\,\overline{\nabla}_a\overline{\nabla}_d\,h^a_{\ b}-\overline{g}^{bd}\,\overline{\nabla}_d\overline{\nabla}_b\,h-\overline{g}^{bd}\,\overline{\Box}\,h_{bd}\right) \\
&+ \frac{\epsilon^2}{4}\Big(\overline{g}^{bd}\,\overline{\nabla}_b\,h_{ce}\overline{\nabla}_d\,h^{ce}+2\,\overline{g}^{bd}\,h^{ce}\left(\overline{\nabla}_b\overline{\nabla}_d\,h_{ce}+\overline{\nabla}_c\overline{\nabla}_e\,h_{bd}-\overline{\nabla}_d\,\overline{\nabla}_e\,h_{cb}-\overline{\nabla}_b\,\overline{\nabla}_e\,h_{cd}\right) \\
&+ 2\,\overline{g}^{bd}\,\overline{g}^{eg}\,\overline{g}^{ch}\,\overline{\nabla}_g\,h_{dh}\,\overline{\nabla}_e\,h_{cb} \\
&- 2\,\overline{g}^{bd}\,\overline{g}^{eg}\,\overline{g}^{ch}\,\overline{\nabla}_g\,h_{dh}\,\overline{\nabla}_c\,h_{eb}-\overline{g}^{bd}\,\overline{\nabla}_e\,h^{ce}\overline{\nabla}_d\,h_{cb}-\overline{g}^{bd}\,\overline{\nabla}_e\,h^{ce}\overline{\nabla}_b\,h_{cd}+\overline{g}^{bd}\,\overline{\nabla}_e\,h^{ce}\overline{\nabla}_c\,h_{bd} \\
&+ \frac{1}{2}\,\overline{g}^{bd}\,\overline{g}^{ce}\,\overline{\nabla}_e\,h\,\overline{\nabla}_d\,h_{cb} \\
&+ \frac{1}{2}\,\overline{g}^{bd}\,\overline{g}^{ce}\,\overline{\nabla}_e\,h\overline{\nabla}_b\,h_{cd}-\frac{1}{2}\,\overline{g}^{bd}\,\overline{g}^{ce}\,\overline{\nabla}_e\,h\overline{\nabla}_c\,h_{bd}+\ldots\Big) \\
&- \frac{1}{4}\left(\overline{g}^{ab}-\epsilon\,h^{ab}+\frac{1}{2}\,\epsilon^2\,h^{ac}\,h_c^{\ b}+\ldots\right)\left(\overline{g}^{cd}-\epsilon\,h^{cd}+\frac{1}{2}\,\epsilon^2\,h^{ce}\,h_e^{\ d}+\ldots\right)F_{ac}\,F_{bd} \\
&- \frac{1}{2}\left(\left(\overline{g}^{ab}-\epsilon\,h^{ab}+\frac{1}{2}\,\epsilon^2\,h^{ac}\,h_c^{\ b}+\ldots\right)\partial_a\phi\,\partial_b\phi+m^2\phi^2\right) \\
&+ \frac{\tilde{\lambda}}{8}\left(\frac{1}{\sqrt{-\overline{g}}}-\frac{\epsilon}{2\sqrt{-\overline{g}}}\,\overline{g}^{cd}\,h_{cd}+\ldots\right)\epsilon^{abcd}\,\phi\,F_{ab}\,F_{cd}\Bigg) \\
&+ \left(\frac{\epsilon}{2}\sqrt{-\overline{g}}\;\overline{g}^{cd}\,h_{cd}\right)\Bigg(\frac{1}{2\,\kappa}\Big(\overline{R}+\frac{\epsilon}{2}\left(\overline{g}^{bd}\,\overline{\nabla}_a\overline{\nabla}_b\,h^a_{\ d}+\overline{g}^{bd}\,\overline{\nabla}_a\overline{\nabla}_d\,h^a_{\ b}-\overline{g}^{bd}\,\overline{\nabla}_d\overline{\nabla}_b\,h-\overline{g}^{bd}\,\overline{\Box}\,h_{bd}\right) \\
&- \frac{1}{4}\left(\overline{g}^{ab}-\epsilon\,h^{ab}+\frac{1}{2}\,\epsilon^2\,h^{ac}\,h_c^{\ b}+\ldots\right)\left(\overline{g}^{cd}-\epsilon\,h^{cd}+\frac{1}{2}\,\epsilon^2\,h^{ce}\,h_e^{\ d}+\ldots\right)F_{ac}\,F_{bd} \\
&- \frac{1}{2}\left(\left(\overline{g}^{ab}-\epsilon\,h^{ab}+\frac{1}{2}\,\epsilon^2\,h^{ac}\,h_c^{\ b}+\ldots\right)\partial_a\phi\,\partial_b\phi+m^2\phi^2\right) \\
&+ \frac{\tilde{\lambda}}{8}\left(\frac{1}{\sqrt{-\overline{g}}}-\frac{\epsilon}{2\sqrt{-\overline{g}}}\,\overline{g}^{cd}\,h_{cd}+\ldots\right)\epsilon^{abcd}\,\phi\,F_{ab}\,F_{cd}\Bigg) \\
&+ \frac{1}{90\,m_e^4}\left(\sqrt{-\overline{g}}+\frac{\epsilon}{2}\sqrt{-\overline{g}}\;\overline{g}^{cd}\,h_{cd}\right) \\
&\times \left(\frac{e^2}{4\pi}\right)^2\Bigg(\left(\left(\overline{g}^{ab}-\epsilon\,h^{ab}\right)\left(\overline{g}^{cd}-\epsilon\,h^{cd}\right)F_{ac}\,F_{bd}\right)^2 \\
&+ \frac{7}{4}\left(\left(\frac{1}{\sqrt{-\overline{g}}}-\frac{\epsilon}{2\sqrt{-\overline{g}}}\,\overline{g}^{cd}\,h_{cd}\right)\epsilon^{abcd}\,F_{ab}\,F_{cd}\right)^2\Bigg)+\ldots
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L} \;=\;& \sqrt{-\overline{g}}\left(\frac{1}{2\,\kappa}\overline{R} + \frac{\epsilon}{4\,\kappa}\left(\overline{g}^{bd}\,\overline{\nabla}_a\overline{\nabla}_{\{b}\,h^a{}_{d\}} - \overline{g}^{bd}\,\overline{\nabla}_d\overline{\nabla}_b\,h - \overline{g}^{bd}\,\overline{\Box}\,h_{bd}\right)\right.\\
&- \frac{1}{4}\,\overline{g}^{ab}\,\overline{g}^{cd}F_{ac}\,F_{bd} + \frac{\epsilon}{2}\,\overline{g}^{ab}\,h^{cd}F_{ac}\,F_{bd} - \frac{1}{2}\left(\overline{g}^{ab}\partial_a\phi\,\partial_b\phi + m^2\phi^2\right) + \left.\frac{\epsilon}{2}\,h^{ab}\partial_a\phi\,\partial_b\phi\right)\\
&+ \frac{\epsilon}{2}\sqrt{-\overline{g}}\,\overline{g}^{cd}\,h_{cd}\left(\frac{1}{2\,\kappa}\overline{R} - \frac{1}{4}\,\overline{g}^{ab}\,\overline{g}^{cd}\,F_{ac}\,F_{bd} - \frac{1}{2}\left(\overline{g}^{ab}\partial_a\phi\,\partial_b\phi + m^2\phi^2\right)\right)\\
&+ \frac{\tilde{\lambda}}{8}\,\epsilon^{abcd}\,\phi\,F_{ab}\,F_{cd}\\
&+ \frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\sqrt{-\overline{g}}\left(\left(\overline{g}^{ab}\,\overline{g}^{cd}\,F_{ac}\,F_{bd} - \epsilon\,\overline{g}^{ab}\,h^{cd}\,F_{ac}\,F_{bd} - \epsilon\,h^{ab}\overline{g}^{cd}\,F_{ac}\,F_{bd}\right)^2\right.\right.\\
&+ \left.\frac{7}{4}\left(\frac{1}{\sqrt{-\overline{g}}}\epsilon^{abcd}\,F_{ab}\,F_{cd} - \frac{\epsilon}{2\sqrt{-\overline{g}}}\,\left(\overline{g}^{ef}\,h_{ef}\right)\epsilon^{abcd}\,F_{ab}\,F_{cd}\right)^2\right)\\
&+ \left.\frac{\epsilon}{2}\sqrt{-\overline{g}}\,\overline{g}^{cd}\,h_{cd}\left(\left(\overline{g}^{ab}\overline{g}^{cd}F_{ac}\,F_{bd}\right)^2 + \frac{7}{4}\left(\frac{1}{\sqrt{-\overline{g}}}\epsilon^{abcd}\,F_{ab}\,F_{cd}\right)^2\right)\right)
\end{aligned}
\tag{4.88}
$$

$$
\begin{aligned}
\mathcal{L} &= \sqrt{-\overline{g}}\left(\frac{1}{2\,\kappa}\overline{R}-\frac{1}{4}\,\overline{g}^{ab}\,\overline{g}^{cd}F_{ac}\,F_{bd}-\frac{1}{2}\left(\overline{g}^{ab}\partial_a\phi\,\partial_b\phi+m^2\phi^2\right)\right)\\
&+ \sqrt{-\overline{g}}\;\frac{\epsilon}{4\,\kappa}\left(\overline{g}^{bd}\,\overline{\nabla}_a\overline{\nabla}_{\{b}\,h^a{}_{d\}}-\overline{g}^{bd}\,\overline{\nabla}_d\overline{\nabla}_b\,h-\overline{g}^{bd}\,\overline{\Box}\,h_{bd}\right) \qquad (4.89)\\
&+ \sqrt{-\overline{g}}\;\frac{\epsilon}{2}\,h^{cd}\left(\overline{g}^{ab}\,F_{ac}\,F_{bd}+\partial_c\phi\,\partial_d\phi-\frac{1}{4}\,\overline{g}_{cd}\,\overline{g}^{ab}\,\overline{g}^{ef}\,F_{ae}\,F_{bf}-\frac{1}{2}\,\overline{g}_{cd}\left(\overline{g}^{ab}\partial_a\phi\,\partial_b\phi+m^2\phi^2\right)\right)\\
&+ \sqrt{-\overline{g}}\;\frac{1}{2\,\kappa}\frac{\epsilon}{2}\,\overline{g}^{cd}\,h_{cd}\,\overline{R}\\
&+ \frac{\tilde{\lambda}}{8}\,\epsilon^{abcd}\,\phi\,F_{ab}\,F_{cd}\\
&+ \frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\sqrt{-\overline{g}}\left(\left(\overline{g}^{ab}\,\overline{g}^{cd}\,F_{ac}\,F_{bd}-\epsilon\,\overline{g}^{ab}\,h^{cd}\,F_{ac}\,F_{bd}-\epsilon\,h^{ab}\overline{g}^{cd}\,F_{ac}\,F_{bd}\right)^2\right.\right.\\
&+ \left.\left.\frac{7}{4}\left(\frac{1}{\sqrt{-\overline{g}}}\epsilon^{abcd}\,F_{ab}\,F_{cd}-\frac{\epsilon}{2\sqrt{-\overline{g}}}\,\left(\overline{g}^{ef}\,h_{ef}\right)\epsilon^{abcd}\,F_{ab}\,F_{cd}\right)^2\right)\right.
\end{aligned}
$$

$$
\left. + \,\epsilon/2\times\sqrt{-\overline{g}}\;\overline{g}^{cd}\,h_{cd}\left(\left(\overline{g}^{ab}\overline{g}^{cd}F_{ac}\,F_{bd}\right)^2+\tfrac{7}{4}\left(\tfrac{1}{\sqrt{-\overline{g}}}\epsilon^{abcd}\,F_{ab}\,F_{cd}\right)^2\right)\right)
$$

(4.90)

We can substitute equation (4.1) into Einstein's equations, $G_{ab} = \kappa\,T_{ab}$, which we write in the trace-reversed form

$$
R_{ab} = \kappa\,\left(T_{ab}-\frac{1}{2}\,g_{ab}\,T\right)\,, \qquad (4.91)
$$

where T is the trace of the energy-momentum tensor. We derive the energy-momentum tensor in the next section. As a reminder, the Ricci tensor can be written schematically as

$$R_{ab} = \overline{R}_{ab} + \epsilon\, R^{(1)}_{ab} + \epsilon^2\, R^{(2)}_{ab} + \dots \tag{4.92}$$

where $R^{(1)}_{ab}$ and $R^{(2)}_{ab}$ are linear and quadratic in h_{ab}. $\overline{R}_{ab}$ is composed solely of the background metric and therefore only has the low frequency modes. $R^{(1)}_{ab}$ is linear in h_{ab} and therefore contains only the high-frequency modes. Because $R^{(2)}_{ab}$ is quadratic in the metric perturbation, it can have both high- and low-frequency modes. Note, two high-frequency modes can combine to generate a low-frequency mode. In terms of the low- and high-frequency modes, we can rewrite the Einstein equations as

$$\begin{aligned} \overline{R}_{ab} + \epsilon^2\, R^{(2)}_{ab}\Big|_{\text{low}} &= \kappa\, \left(T_{ab} - \frac{1}{2}\, g_{ab}\, T\right)\Bigg|_{\text{low}} \\ &= \kappa\, \left(\overline{T}_{ab} - \frac{1}{2}\, g_{ab}\, \overline{T}\right) + \epsilon^2\, \kappa\, \left(T^{(2)}_{ab} - \frac{1}{2}\, g_{ab}\, T^{(2)}\right)\Bigg|_{\text{low}} \quad \textbf{(low-frequency)} \end{aligned} \tag{4.93}$$

$$\begin{aligned} \epsilon\, R^{(1)}_{ab}\Big|_{\text{high}} + \epsilon^2\, R^{(2)}_{ab}\Big|_{\text{high}} &= \kappa\, \left(T_{ab} - \frac{1}{2}\, g_{ab}\, T\right)\Bigg|_{\text{high}} \\ R^{(1)}_{ab}\Big|_{\text{high}} &= \frac{\alpha}{\epsilon}\, \kappa\, \left(T^{(1)}_{ab} - \frac{1}{2}\, g_{ab}\, T^{(1)}\right)\Bigg|_{\text{high}} + \kappa\, \left(T'_{ab} - \frac{1}{2}\, g_{ab}\, T'\right)\Bigg|_{\text{high}} \quad \textbf{(high frequency)} \\ &\quad + \epsilon\, \kappa\, \left(T^{(2)}_{ab} - \frac{1}{2}\, g_{ab}\, T^{(2)}\right)\Bigg|_{\text{high}} - \epsilon\, R^{(2)}_{ab}\Big|_{\text{high}} \end{aligned} \tag{4.94}$$

where $R^{(1)}$ and $R^{(2)}$ is given in equation (4.107). The first equation impacts the energy-momentum tensor. The second equation will become the gravity wave equation in curved spacetime.

Given the separation of scales $< \mathcal{L}$, we now chose a scale $\bar{l}$ so that $< \bar{l} < \mathcal{L}$. For the length scale $\bar{l}$, the low-frequency modes are effectively static. We can therefore spatially average over many reduced wavelengths, , of the metric perturbation [3]. Equation (4.93) now becomes

$$\begin{aligned} \overline{R}_{ab} &= -\epsilon^2 \left\langle R^{(2)}_{ab} \right\rangle + \kappa \left\langle T_{ab} - \frac{1}{2}\, \overline{g}_{ab}\, T \right\rangle \\ &= -\epsilon^2 \left\langle R^{(2)}_{ab} \right\rangle + \kappa \left(\overline{T}_{ab} - \frac{1}{2}\, \overline{g}_{ab}\, \overline{T} \right) \end{aligned} \tag{4.95}$$

where we have defined

$$\langle T_{ab} \rangle = \overline{T}_{ab} \qquad\qquad \langle g_{ab}\, T \rangle = \overline{g}_{ab}\, \overline{T}\,, \tag{4.96}$$

which are the low frequency components of the energy-momentum tensor and metric. If we want to consider a different , where e.g. $\lesssim \mathcal{L}$, we can use an renormalization group analysis to relate the length scales [66]. We can define

$$R^{(2)} = \overline{g}^{ab} R^{(2)}_{ab} \tag{4.97}$$

as well as

$$\bar{t}_{ab} = -\frac{1}{\kappa} \langle R^{(2)}_{ab} - \frac{1}{2} \overline{g}_{ab}\, R^{(2)} \rangle \qquad\qquad \bar{t} = \overline{g}^{ab}\, \bar{t}_{ab} = \frac{1}{\kappa} \langle R^{(2)} \rangle\,. \tag{4.98}$$

[3] If we take $\epsilon \sim f_{\mathrm{b}}/f$, a temporal average over several $1/f$ periods of the metric perturbation is needed.

Combing the two equations in equation (4.98) generates

$$-\langle R_{ab}^{(2)} \rangle = \kappa \left(\bar{t}_{ab} - \frac{1}{2} \overline{g}_{ab} \, \bar{t} \right) \tag{4.99}$$

and substituting in to equation (4.95) generates

$$\begin{aligned} \overline{R}_{ab} &= \epsilon^2 \kappa \left(\bar{t}_{ab} - \frac{1}{2} \overline{g}_{ab} \, \bar{t} \right) + \kappa \left(\overline{T}_{ab} - \frac{1}{2} \overline{g}_{ab} \, \overline{T} \right) \\ &= \kappa \left(\overline{T}_{ab} + \epsilon^2 \, \bar{t}_{ab} \right) - \frac{\kappa}{2} \overline{g}_{ab} \left(\overline{T} + \epsilon^2 \, \bar{t} \right) \end{aligned} \tag{4.100}$$

Contracting the above yields $\overline{R} = -\kappa \, (\epsilon^2 \, \bar{t} + \overline{T})$ and our final form

$$\overline{R}_{ab} - \frac{1}{2} \, g_{ab} \, \overline{R} = \kappa \left(\overline{T}_{ab} + \epsilon^2 \, \bar{t}_{ab} \right) . \qquad \textbf{(low frequency)} \tag{4.101}$$

Here it is clear $\bar{t}_{ab}$ is the energy-momentum contribution from the background curvature. It is also clear the $\overline{T}_{ab}$ sources the long-wavelength modes. This is how the low-frequency modes are sourced by the low-frequency gravitational waves and the energy-momentum tensor.

4.1.7 Gravitational Perturbative Expansion

Using equation (4.1) and (4.2), we can now compute the connection, Ricci scalar and Ricci tensors in terms of background quantities and the curvature perturbation h_{ab}.

The expanded connection is

$$
\begin{aligned}
\Gamma^{a}_{\ bc} = \overline{\Gamma}^{a}_{\ bc} + \frac{\epsilon}{2}\,\overline{g}^{am}\Big(\overline{\nabla}_c\, h_{mb} + \overline{\nabla}_b\, h_{mc} - \overline{\nabla}_m\, h_{bc}\Big) \\
+ \frac{\epsilon^2}{4}\Big(\overline{g}^{am}\Big(\overline{\nabla}_c\left(h_{md}\, h^{d}_{\ b}\right) + \overline{\nabla}_b\left(h_{md}\, h^{d}_{\ c}\right) - \overline{\nabla}_m\left(h_{bd}\, h^{d}_{\ c}\right)\Big) \\
- 2\, h^{am}\Big(\overline{\nabla}_c\, h_{mb} + \overline{\nabla}_b\, h_{mc} - \overline{\nabla}_m\, h_{bc}\Big)\Big) + \mathcal{O}(\epsilon^3)\,.
\end{aligned}
\tag{4.102}
$$

where

$$
\overline{\Gamma}^{m}_{\ ij} = \frac{1}{2}\overline{g}^{mk}\left(\partial_j \overline{g}_{ki} + \partial_j \overline{g}_{ki} - \partial_j \overline{g}_{ki}\right) \tag{4.103}
$$

Recall $\epsilon \sim \mathcal{O}(/\mathcal{L})$. Also ϵ, tracks the number of h_{ab} that are in each expression. The corresponding Riemann tensor is [65]

$$
\begin{aligned}
R^{a}_{\ bcd} &= \overline{R}^{a}_{\ bcd} + \epsilon\, R^{a\,(1)}_{bcd} + \epsilon^2\, R^{a\,(2)}_{bcd} + \ldots \\
&= \overline{R}^{a}_{\ bcd} + \frac{\epsilon}{2}\Big(\overline{\nabla}_c\overline{\nabla}_d\, h^{a}_{\ b} + \overline{\nabla}_c\overline{\nabla}_b\, h^{a}_{\ d} - \overline{\nabla}_d\overline{\nabla}_c\, h^{a}_{\ b} \qquad (4.104) \\
&- \overline{\nabla}_d\overline{\nabla}_b\, h^{a}_{\ c} + \overline{g}^{am}\overline{\nabla}_d\overline{\nabla}_m\, h_{bc} - \overline{g}^{am}\overline{\nabla}_c\overline{\nabla}_m\, h_{bd}\Big) \qquad (4.105) \\
&+ \frac{\epsilon^2}{4}\Big(\overline{\nabla}_c\overline{\nabla}_b\left(h^{a}_{\ r}\, h^{r}_{\ d}\right) - \overline{\nabla}_d\,\overline{\nabla}_b\left(h^{a}_{\ r}\, h^{r}_{\ c}\right) + \overline{g}^{am}\overline{\nabla}_d\,\overline{\nabla}_m\left(h_{br}\, h^{r}_{\ c}\right) - \overline{g}^{am}\overline{\nabla}_c\overline{\nabla}_m\left(h_{br}\, h^{r}_{\ d}\right) \\
&\quad - 2\left(\overline{\nabla}_c\, h^{am}\right)\left(\overline{\nabla}_d\, h_{mb} + \overline{\nabla}_b\, h_{md} - \overline{\nabla}_m\, h_{bd}\right) - 2\, h^{am}\left(\overline{\nabla}_c\overline{\nabla}_b\, h_{md} - \overline{\nabla}_c\overline{\nabla}_m\, h_{bd}\right) \\
&\quad + 2\left(\overline{\nabla}_d\, h^{am}\right)\left(\overline{\nabla}_c\, h_{mb} + \overline{\nabla}_b\, h_{mc} - \overline{\nabla}_m\, h_{bc}\right) + 2\, h^{am}\left(\overline{\nabla}_d\,\overline{\nabla}_b\, h_{mc} - \overline{\nabla}_d\,\overline{\nabla}_m\, h_{bc}\right) \\
&\quad + \left(\overline{\nabla}_e\, h^{a}_{\ c} + \overline{\nabla}_c\, h^{a}_{\ e} - \overline{g}^{am}\overline{\nabla}_m\, h_{ce}\right)\left(\overline{\nabla}_d\, h^{e}_{\ b} + \overline{\nabla}_b\, h^{e}_{\ d} - \overline{g}^{eq}\,\overline{\nabla}_q\, h_{bd}\right) \\
&\quad - \left(\overline{\nabla}_e\, h^{a}_{\ d} + \overline{\nabla}_d\, h^{a}_{\ e} - \overline{g}^{aq}\overline{\nabla}_q\, h_{de}\right)\left(\overline{\nabla}_c\, h^{e}_{\ b} + \overline{\nabla}_b\, h^{e}_{\ c} - \overline{g}^{em}\,\overline{\nabla}_m\, h_{bc}\right)\Big) + \mathcal{O}(\epsilon^3)\,.
\end{aligned}
$$

where

$$
\overline{R}^{a}_{\ bcd} = \partial_c \overline{\Gamma}^{a}_{\ bd} - \partial_b \overline{\Gamma}^{a}_{\ cd} + \overline{\Gamma}^{a}_{\ ce}\,\overline{\Gamma}^{e}_{\ bd} - \overline{\Gamma}^{a}_{\ be}\,\overline{\Gamma}^{e}_{\ cd}\,. \tag{4.106}
$$

The first line in the above equation again matches the $\mathcal{O}(\epsilon)$ result given in [68]. The expanded Ricci tensor is now

$$
\begin{aligned}
R_{bd} &= \overline{R}_{bd} + \epsilon\, R^{(1)}_{bd} + \epsilon^2\, R^{(2)}_{bd} + \dots \\
&= \overline{R}_{bd} + \frac{\epsilon}{2}\left(\overline{\nabla}_a \overline{\nabla}_{\{b}\, h^a{}_{d\}} - \overline{\nabla}_d \overline{\nabla}_b\, h - \overline{\Box}\, h_{bd}\right) \\
&+ \frac{\epsilon^2}{4}\Bigg(\overline{\nabla}_b\, h_{ce} \overline{\nabla}_d\, h^{ce} + 2\, h^{ce}\left(\overline{\nabla}_b \overline{\nabla}_d\, h_{ce} + \overline{\nabla}_c \overline{\nabla}_e\, h_{bd} - \overline{\nabla}_d\, \overline{\nabla}_e\, h_{cb} - \overline{\nabla}_b\, \overline{\nabla}_e\, h_{cd}\right) \\
&+ 2\, \overline{g}^{eg} \overline{g}^{ch}\, \overline{\nabla}_g\, h_{dh}\left(\overline{\nabla}_e\, h_{cb} - \overline{\nabla}_c\, h_{eb}\right) - \left(\overline{\nabla}_e\, h^{ce} - \frac{1}{2}\, \overline{g}^{ce}\, \overline{\nabla}_e\, h\right)\left(\overline{\nabla}_d\, h_{cb} + \overline{\nabla}_b\, h_{cd} - \overline{\nabla}_c\, h_{bd}\right)\Bigg) \\
&+ \mathcal{O}(\epsilon^3)
\end{aligned}
\tag{4.107}
$$

where $\overline{R}_{bd}$ is the standard Ricci tensor

$$
\overline{R}_{bd} = \partial_a \overline{\Gamma}^a{}_{bd} - \partial_b \overline{\Gamma}^a{}_{ad} + \overline{\Gamma}^a{}_{ac} \overline{\Gamma}^c{}_{bd} - \overline{\Gamma}^a{}_{bc} \overline{\Gamma}^c{}_{ad}
\tag{4.108}
$$

which matches the equation in [65]. We will use this equation in trace-reversed Einstein equations. The trace-reversed Einstein equation also requires the energy-momentum tensor. For future convenience, we can modify the ϵ term by using the identity

$$
\frac{1}{2}\, \overline{\nabla}_a \overline{\nabla}_{\{b}\, h^a{}_{d\}} = \frac{1}{2}\, \left(\overline{\nabla}_b \overline{\nabla}_a\, h^a{}_d + \overline{R}_{sb}\, h^s{}_d - \overline{R}^s{}_{dab}\, h^a{}_s + \overline{\nabla}_d \overline{\nabla}_a\, h^a{}_b + \overline{R}_{rd}\, h^r{}_b - \overline{R}^s{}_{bad}\, h^a{}_s\right)
$$

The Ricci tensor is now

$$R_{bd} = \overline{R}_{bd} + \frac{\epsilon}{2}\left(\overline{\nabla}_b\overline{\nabla}_a\, h^a{}_d + \overline{\nabla}_d\overline{\nabla}_a\, h^a{}_b - \overline{\nabla}_d\overline{\nabla}_b\, h - \overline{\Box}\, h_{bd} + \overline{R}_{sb}\, h^s{}_d - \overline{R}^s{}_{dab}\, h^a{}_s + \overline{R}_{rd}\, h^r{}_b - \overline{R}^s{}_{bad}\, h^a{}_s\right) \tag{4.109}$$

$$+ \frac{\epsilon^2}{4}\left(\overline{\nabla}_b\, h_{ce}\overline{\nabla}_d\, h^{ce} + 2\, h^{ce}\left(\overline{\nabla}_b\overline{\nabla}_d\, h_{ce} + \overline{\nabla}_c\overline{\nabla}_e\, h_{bd} - \overline{\nabla}_d\,\overline{\nabla}_e\, h_{cb} - \overline{\nabla}_b\,\overline{\nabla}_e\, h_{cd}\right)\right.$$
$$+ 2\,\overline{g}^{eg}\overline{g}^{ch}\,\overline{\nabla}_g\, h_{dh}\left(\overline{\nabla}_e\, h_{cb} - \overline{\nabla}_c\, h_{eb}\right)$$
$$\left. - \left(\overline{\nabla}_e\, h^{ce} - \frac{1}{2}\,\overline{g}^{ce}\,\overline{\nabla}_e\, h\right)\left(\overline{\nabla}_d\, h_{cb} + \overline{\nabla}_b\, h_{cd} - \overline{\nabla}_c\, h_{bd}\right)\right) + \mathcal{O}(\epsilon^3)$$

Using the parametrization described in the previous section, we can determine the most relevant terms. The derivatives in each expression yield factors of and $\mathcal{L}$ when they act on low-frequency and high-frequency objects. For example,

$$\overline{\nabla}_b\overline{\nabla}_a\, h^a{}_d + \overline{\nabla}_d\overline{\nabla}_a\, h^a{}_b - \overline{\nabla}_d\overline{\nabla}_b\, h - \overline{\Box}\, h_{bd} \sim \mathcal{A}/^2 \tag{4.110}$$

$$\overline{R}_{sb}\, h^s{}_d - \overline{R}^s{}_{dab}\, h^a{}_s + \overline{R}_{rd}\, h^r{}_b - \overline{R}^s{}_{bad}\, h^a{}_s \sim \mathcal{A}/\mathcal{L}^2 \tag{4.111}$$

where we follow the convention of MTW [65] and parametrize the dimensionless amplitude of the metric perturbation with $\mathcal{A}$. Because $\ll \mathcal{L}$, then $1/\mathcal{L} \ll 1/$ and the second line in the above two equations is far more suppressed. The other terms have a approximate parametrization of

$$
\begin{aligned}
R_{bd} = \; & \underbrace{\overline{R}_{bd}}_{1/\mathcal{L}^2} \\
& + \frac{\epsilon}{2}\left(\underbrace{\overline{\nabla}_b\overline{\nabla}_a\, h^a{}_d + \overline{\nabla}_d\overline{\nabla}_a\, h^a{}_b - \overline{\nabla}_d\overline{\nabla}_b\, h - \overline{\Box}\, h_{bd}}_{\mathcal{A}/{}^2} + \underbrace{\overline{R}_{sb}\, h^s{}_d + \overline{R}_{rd}\, h^r{}_b}_{\mathcal{A}^3/{}^2} - \underbrace{\overline{R}^s{}_{dab}\, h^a{}_s - \overline{R}^s{}_{bad}\, h^a{}_s}_{\mathcal{A}/\mathcal{L}^2}\right) \\
& + \frac{\epsilon^2}{4}\left(\underbrace{\overline{\nabla}_b\, h_{ce}\overline{\nabla}_d\, h^{ce} + 2\, h^{ce}\left(\overline{\nabla}_b\overline{\nabla}_d\, h_{ce} + \overline{\nabla}_c\overline{\nabla}_e\, h_{bd} - \overline{\nabla}_d\,\overline{\nabla}_e\, h_{cb} - \overline{\nabla}_b\,\overline{\nabla}_e\, h_{cd}\right)}_{\mathcal{A}^2/{}^2}\right. \\
& \left. + \underbrace{2\,\overline{g}^{eg}\,\overline{g}^{ch}\,\overline{\nabla}_g\, h_{dh}\left(\overline{\nabla}_e\, h_{cb} - \overline{\nabla}_c\, h_{eb}\right) - \left(\overline{\nabla}_e\, h^{ce} - \frac{1}{2}\,\overline{g}^{ce}\,\overline{\nabla}_e\, h\right)\left(\overline{\nabla}_d\, h_{cb} + \overline{\nabla}_b\, h_{cd} - \overline{\nabla}_c\, h_{bd}\right)}_{\mathcal{A}^2/{}^2}\right) \\
& + \mathcal{O}(\epsilon^3)
\end{aligned}
\tag{4.112}
$$

which, combined with the ϵ factors, have roughly $1/\mathcal{L}^2$, $1/(\ \mathcal{L})$, $/\mathcal{L}^3$, $1/\mathcal{L}^2$ and $1/\mathcal{L}^2$ suppression factors, respectively. Because $\ll \mathcal{L}$, that means $/\mathcal{L}^3 \ll 1/\mathcal{L}^2$. Also from this equation, $R^{(2)}_{ab} \sim \mathcal{A}^2/{}^2$. Thus, from equation (4.99) and (4.100) as well as $\overline{T}_{ab} = 0$, it is clear $\overline{R}_{ab} \sim \mathcal{A}^2/{}^2$, which is not naively obvious.

The $\overline{R}_{sb}\, h^s{}_d$ terms in the above equation matches 35.57 in MTW [65]. We can verify that $\epsilon \sim \mathcal{O}(/\mathcal{L})$ by taking $\kappa = 0$ and $\mathcal{A} = 1$ for equation (4.93). The proof is at [4]. We drop the $\mathcal{A}/\mathcal{L}^2$

[4] ϵ Parametrization: If $\kappa = 0$, then the low-frequency equation is parametrically,

$$\overline{R}_{ab} + \epsilon^2\, R^{(2)}_{ab}\Big|_{\text{low}} = 0 \tag{4.113}$$

$$\frac{1}{\mathcal{L}^2} + \frac{\epsilon^2}{{}^2} = 0 \tag{4.114}$$

which justifies $\epsilon \sim /\mathcal{L}$.

and $\mathcal{A}^3/^2$ terms (and those terms that are more suppressed). Assuming $\mathcal{A}$ is perturbative, we have

$$R_{bd} = \overline{R}_{bd} + \epsilon\, R^{(1)}_{bd} \tag{4.115}$$

where $\overline{R}_{bd}$ is given by equation (4.108) and

$$R^{(1)}_{bd} = \frac{1}{2}\left(\overline{\nabla}_b\overline{\nabla}_a\, h^a_{\ d} + \overline{\nabla}_d\overline{\nabla}_a\, h^a_{\ b} - \overline{\nabla}_d\overline{\nabla}_b\, h - \overline{\Box}\, h_{bd}\right). \tag{4.116}$$

4.2 Curved Spacetime Equations of Motion in the Presence of Non-Trivial Backgrounds

4.2.1 Perturbative Expansions

4.2.1 Bosonic and Electromagnetic Wave Expansion

We can split the physical fields into the long- and short-wavelength parts like the gravitational waves as well. In addition, there is another length scale of interest, $\mathcal{L}_c$, which is the characteristic scale over which the amplitude, polarization and/or wavelength of the boson(axion)/electromagnetic field changes substantially. We especially take this scale to be the length scale of the background static fields. We assume the boson (axion), electromagnetic and gravity waves are cohered. Thus, has the same value as with the gravitational wave expansion above. We also require $\ll \mathcal{L}_c$, where as a reminder is the reduced wavelength of the cohered axion, electromagnetic and gravity waves. We define $\alpha = /\mathcal{L}_c$ and assume $\mathcal{L}_c < \mathcal{L}$. Therefore, $\alpha > \epsilon$.

$$F_{ab} = \overline{F}_{ab} + \alpha\, F^{(1)}_{ab} \tag{4.117}$$

$$\phi = \overline{\phi} + \alpha\, \phi^{(1)} \tag{4.118}$$

where $F^{(1)}$ and $\phi^{(1)}$ are the high-frequency fields. $\overline{F}_{ab}$ and $\overline{\phi}$ are the background fields. This is consistent with Section 1.5.1 in [66]. Here we make the assumption that the expansion for the boson and electromagnetic waves are equivalent. Because the energy-momentum tensor electromagnetic and axion fields, we also need to expand it. The energy-momentum expansion is therefore

$$T_{ab} = \overline{T}_{ab} + \alpha\, T_{ab}^{(1,0)} + \epsilon\, T_{ab}^{(0,1)} + \alpha\, \epsilon\, T_{ab}^{(1,1)} + \alpha^2\, T_{ab}^{(2,0)} + \epsilon^2\, T_{ab}^{(0,2)} + \dots \tag{4.119}$$

where the a and b indices in $T^{(a,b)}$ represent the perturbative expansion in α and ϵ, respectively. Also,

$$\begin{aligned} \overline{T}_{ab} = {} & \underbrace{\overline{g}^{ef}\overline{F}_{ae}\overline{F}_{bf}}_{1/\mathcal{L}^2} + \underbrace{\partial_a\overline{\phi}\,\partial_b\overline{\phi}}_{1/\mathcal{L}^2} - \frac{2}{45\, m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \left(\underbrace{\overline{g}^{ef}\overline{F}^2\,\overline{F}_{ae}\overline{F}_{bf}}_{1/\mathcal{L}^4}\right) \\ & - \overline{g}_{ab}\Bigg(\frac{1}{4}\underbrace{\overline{F}^2}_{1/\mathcal{L}^2} + \frac{1}{2}\left(\underbrace{\overline{g}^{mn}\,\partial_m\overline{\phi}\,\partial_n\overline{\phi}}_{1/\mathcal{L}^2} + \underbrace{m^2\,\overline{\phi}^2}_{1}\right) \\ & - \frac{1}{90\, m_e^4}\left(\frac{e^2}{4\pi}\right)^2\Big(\underbrace{\overline{F}^4}_{1/\mathcal{L}^4} - \frac{7}{4\,\overline{g}}\underbrace{\left(\epsilon^{mnop}\overline{F}_{mn}\overline{F}_{op}\right)^2}_{1/\mathcal{L}^4}\Big)\Bigg) \end{aligned} \tag{4.120}$$

The expansion terms are

$$
\begin{aligned}
T_{ab}^{(1,0)} &= \underbrace{\overline{g}^{ef}\left(F_{ae}^{(1)}\,\overline{F}_{bf} + \overline{F}_{ae}\,F_{bf}^{(1)}\right)}_{\mathcal{A}/\mathcal{L}} + \underbrace{\partial_a\overline{\phi}\,\partial_b\phi^{(1)} + \partial_a\phi^{(1)}\partial_b\overline{\phi}}_{\mathcal{A}/\mathcal{L}} \\
&- \frac{4}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \overline{g}^{ef}\,\overline{g}^{mn}\,\overline{g}^{op}\left(\underbrace{F_{np}^{(1)}\,\overline{F}_{mo}\,\overline{F}_{ae}\,\overline{F}_{bf}}_{\mathcal{A}/\mathcal{L}^3}\right. \qquad (4.121) \\
&+ \left.\underbrace{\overline{F}_{np}\,\overline{F}_{mo}\,F_{ae}^{(1)}\,\overline{F}_{bf}}_{\mathcal{A}/\mathcal{L}^3}\right) - \overline{g}_{ab}\left(\frac{1}{2}\;\underbrace{\overline{g}^{mn}\overline{g}^{op}\,\overline{F}_{np}\,F_{mo}^{(1)}}_{\mathcal{A}/\mathcal{L}} + \underbrace{\overline{g}^{mn}\,\partial_m\overline{\phi}\,\partial_n\phi^{(1)}}_{\mathcal{A}/\mathcal{L}} + \underbrace{m^2\,\overline{\phi}\,\phi^{(1)}}_{\mathcal{A}}\right. \\
&- \frac{4}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\overline{g}^{mn}\,\overline{g}^{op}\,\overline{g}^{qr}\,\overline{g}^{st}\;\underbrace{F_{rt}^{(1)}\,\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}_{qs}}_{\mathcal{A}/\mathcal{L}^3} - \frac{7}{4\,\overline{g}}\,\epsilon^{hijk}\epsilon^{defg}\;\underbrace{F_{jk}^{(1)}\,\overline{F}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi}}_{\mathcal{A}/\mathcal{L}^3}\right) \\
&= \overline{g}^{ef}\left(F_{ae}^{(1)}\,\overline{F}_{bf} + \overline{F}_{ae}\,F_{bf}^{(1)}\right) + \partial_a\overline{\phi}\,\partial_b\phi^{(1)} + \partial_a\phi^{(1)}\partial_b\overline{\phi} - \overline{g}_{ab}\left(\frac{1}{2}\,\overline{g}^{mn}\overline{g}^{op}\,\overline{F}_{np}\,F_{mo}^{(1)}\right. \qquad (4.122) \\
&+ \left.\overline{g}^{mn}\,\partial_m\overline{\phi}\,\partial_n\phi^{(1)} + m^2\,\overline{\phi}\,\phi^{(1)}\right) + \ldots
\end{aligned}
$$

where on the second line we have kept the terms more suppress than $\mathcal{A}/\mathcal{L}^2$ and $\mathcal{A}^3/^2$.

$$T^{(0,1)}_{ab} = -\underbrace{h^{cd}\,\overline{F}_{ac}\overline{F}_{bd}}_{\mathcal{A}/\mathcal{L}^2} + \frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\underbrace{h^{cd}\,\overline{F}_{ac}\,\overline{F}_{bd}\,\overline{F}^2}_{\mathcal{A}/\mathcal{L}^4} + \underbrace{2\,\overline{g}^{cd}\,h^{mn}\overline{g}^{op}\,\overline{F}_{ac}\,\overline{F}_{bd}\,\overline{F}_{mo}\,\overline{F}_{np}}_{\mathcal{A}/\mathcal{L}^4}\right) \tag{4.123}$$

$$\begin{aligned}
&-\left(\frac{1}{4}\underbrace{h_{ab}\,\overline{F}^2}_{\mathcal{A}/\mathcal{L}^2} + \frac{1}{2}\left(\underbrace{\overline{g}^{mn}\,h_{ab}\,\partial_m\overline{\phi}\,\partial_n\overline{\phi}}_{\mathcal{A}/\mathcal{L}^2} + \underbrace{m^2\,h_{ab}\,\overline{\phi}^2}_{\mathcal{A}}\right)\right.\\
&-\frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\underbrace{h_{ab}\,\overline{F}^4}_{\mathcal{A}/\mathcal{L}^4} - \frac{7}{16\,\overline{g}}\underbrace{h_{ab}\,\left(\epsilon^{mnop}\overline{F}_{mn}\overline{F}_{op}\right)^2}_{\mathcal{A}/\mathcal{L}^4}\right)\Bigg)\\
&-\overline{g}_{ab}\left(-\frac{1}{2}\underbrace{h^{mn}\overline{g}^{op}\,\overline{F}_{np}\,\overline{F}_{mo}}_{\mathcal{A}/\mathcal{L}^2} - \frac{1}{2}\underbrace{h^{mn}\,\partial_m\overline{\phi}\,\partial_n\overline{\phi}}_{\mathcal{A}/\mathcal{L}^2} + \frac{4}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\underbrace{h^{mn}\,\overline{g}^{op}\,\overline{g}^{qr}\,\overline{g}^{st}\,\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}^2}_{\mathcal{A}/\mathcal{L}^4}\right)\\
&+\frac{\overline{g}_{ab}}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\frac{7}{16\,\overline{g}^2}\,\epsilon^{hijk}\epsilon^{defg}\underbrace{\mathrm{tr}\left(\overline{g}^{mn}\,h_{mn}\right)\overline{F}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi}\,\overline{F}_{jk}}_{\mathcal{A}/\mathcal{L}^4}\right)\\
&= -m^2\,h_{ab}\,\overline{\phi}^2 + \ldots
\end{aligned} \tag{4.124}$$

$$T^{(0,2)}_{ab} = -\frac{1}{4}\,m^2\,h_{ac}\,h^c_{\ b}\,\phi^2 + \ldots \tag{4.125}$$

$$\begin{aligned}
&= -\mathrm{h}^{ef}\left(F^{(1)}_{ae}\,\overline{F}_{bf} + \overline{F}_{ae}\,F^{(1)}_{bf}\right) - \left(\tfrac{1}{4}\,h_{ab}\,\overline{g}^{mn}\overline{g}^{op}\left(\overline{F}_{np}\,F^{(1)}_{mo} + \overline{F}_{mo}\,F^{(1)}_{np}\right)\right.\\
&\left. + \overline{g}^{mn}\,h_{ab}\,\partial_m\overline{\phi}\,\partial_n\phi^{(1)} + m^2\,h_{ab}\,\overline{\phi}\,\phi^{(1)}\right)\\
&-\overline{g}_{ab}\left(-\tfrac{1}{2}\,h^{mn}\overline{g}^{op}\left(\overline{F}_{np}\,F^{(1)}_{mo} + \overline{F}_{mo}\,F^{(1)}_{np}\right) - h^{mn}\,\partial_m\overline{\phi}\,\partial_n\phi^{(1)}\right) + \ldots
\end{aligned}$$

It is clear there are mixing terms between the gravity-electromagnetic wave and gravity-bosonic wave.

$$T_{ab}^{(2,0)} = \underbrace{\overline{g}^{cd}\,F_{ac}^{(1)}\,F_{bd}^{(1)}}_{\mathcal{A}^2/2} + \underbrace{\partial_a\phi^{(1)}\partial_b\phi^{(1)}}_{\mathcal{A}^2/2} - \frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \overline{g}^{cd}\overline{g}^{ef}\overline{g}^{gh}\Bigg(\underbrace{\overline{F}_{ac}\overline{F}_{bd}\overline{F}_{eg}^{(1)}\overline{F}_{fh}^{(1)}}_{\mathcal{A}^2/2\mathcal{L}^2} + \underbrace{F_{ac}^{(1)}\,F_{bd}^{(1)}\overline{F}_{eg}\overline{F}_{fh}}_{\mathcal{A}^2/2\mathcal{L}^2} \tag{4.126}$$

$$\begin{aligned}
&+ \underbrace{\left(\overline{F}_{ac}F_{bd}^{(1)} + F_{ac}^{(1)}\,\overline{F}_{bd}\right)\left(\overline{F}_{eg}F_{fh}^{(1)} + F_{eg}^{(1)}\overline{F}_{fh}\right)}_{\mathcal{A}^2/2\mathcal{L}^2}\Bigg) \\
&- \overline{g}_{ab}\Bigg(\underbrace{\frac{1}{4}\,\overline{g}^{ef}\overline{g}^{gh}\,F_{eg}^{(1)}F_{fh}^{(1)}}_{\mathcal{A}^2/2} + \frac{1}{2}\Big(\underbrace{\overline{g}^{cd}\partial_c\phi^{(1)}\partial_d\phi^{(1)}}_{\mathcal{A}^2/2} + \underbrace{m^2\,\phi^{(1)\,2}}_{\mathcal{A}^2}\Big) \\
&- \frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\Bigg(\underbrace{\left(\overline{g}^{ab}\,\overline{g}^{cd}\,\left(\overline{F}_{ac} + \alpha\,F_{ac}^{(1)}\right)\left(\overline{F}_{bd} + \alpha\,F_{bd}^{(1)}\right)\right)^2\Big|_{\alpha^2}}_{\mathcal{A}^2/2\mathcal{L}^2} \\
&+ \frac{7}{4}\underbrace{\left(\frac{1}{\sqrt{-g}}\,\epsilon^{abcd}\,\left(\overline{F}_{ac} + \alpha\,\overline{F}_{ac}^{(1)}\right)\left(\overline{F}_{bd} + \alpha\,F_{bd}^{(1)}\right)\right)^2\Big|_{\alpha^2}}_{\mathcal{A}^2/2\mathcal{L}^2}\Bigg)\Bigg) \\
&= \overline{g}^{cd}\,F_{ac}^{(1)}\,F_{bd}^{(1)} + \partial_a\phi^{(1)}\partial_b\phi^{(1)}
\end{aligned} \tag{4.127}$$

The energy-momentum tensor is now

$$T_{ab} = \overline{T}_{ab} + \alpha\,T_{ab}^{(1,0)} + \epsilon\,T_{ab}^{(0,1)} + \alpha\,\epsilon\,T_{ab}^{(1,1)} + \alpha^2\,T_{ab}^{(2,0)} + \epsilon^2\,T_{ab}^{(0,2)} + \dots \tag{4.128}$$

4.2.1 Low-Frequency Gravitational Wave Equation

From equation (4.101), the low frequency gravitational wave equation is

$$\overline{R}_{ab} - \frac{1}{2}\, g_{ab}\, \overline{R} = \kappa \left(\overline{T}_{ab} + \epsilon^2\, \bar{t}_{ab}\right). \qquad \textbf{(low frequency)} \qquad (4.129)$$

where $\overline{T}_{ab}$ is

$$\overline{T}_{ab} = \overline{g}^{ef}\overline{F}_{ae}\overline{F}_{bf} + \partial_a\overline{\phi}\,\partial_b\overline{\phi} - \frac{2}{45\, m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \overline{g}^{ef}\overline{F}^2\,\overline{F}_{ae}\overline{F}_{bf} \qquad (4.130)$$
$$- \overline{g}_{ab}\Bigg(\frac{1}{4}\,\overline{F}^2 + \frac{1}{2}\left(\overline{g}^{mn}\,\partial_m\overline{\phi}\,\partial_n\overline{\phi} + m^2\,\overline{\phi}^2\right) - \frac{1}{90\, m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\overline{F}^4 - \frac{7}{4\,\overline{g}}\left(\epsilon^{mnop}\overline{F}_{mn}\overline{F}_{op}\right)^2\right)\Bigg)$$

As a reminder $\bar{t}_{ab} = -\langle R^{(2)}_{ab} - \frac{1}{2}\overline{g}_{ab}\, R^{(2)}\rangle/\kappa$, equation (4.98), and is the contribution to the energy-momentum tensor from the curvature.

4.2.1 Expanded High-Frequency Gravitational Wave Equations

From equation (4.94), the high-frequency gravitational wave equation is

$$R^{(1)}_{ab}\Big|_{\text{high}} = \frac{\alpha}{\epsilon}\,\kappa\,\left(T^{(1,0)}_{ab} - \frac{1}{2}\, g_{ab}\, T^{(1,0)}\right)\Big|_{\text{high}} + \kappa\,\left(T^{(0,1)}_{ab} - \frac{1}{2}\, g_{ab}\, T^{(0,1)}\right)\Big|_{\text{high}} \qquad (4.131)$$
$$+ \alpha\,\kappa\,\left(T^{(1,1)}_{ab} - \frac{1}{2}\, g_{ab}\, T^{(1,1)}\right)\Big|_{\text{high}} \qquad (4.132)$$
$$+ \epsilon\,\kappa\,\left(T^{(0,2)}_{ab} - \frac{1}{2}\, g_{ab}\, T^{(0,2)}\right)\Big|_{\text{high}} + \frac{\alpha^2}{\epsilon}\,\kappa\,\left(T^{(2,0)}_{ab} - \frac{1}{2}\, g_{ab}\, T^{(2,0)}\right)\Big|_{\text{high}} - \epsilon\, R^{(2)}_{ab}\Big|_{\text{high}}.$$

where

$$R^{(1)}_{bd}\Big|_{\text{high}} = \frac{1}{2}\left(\overline{\nabla}_b\overline{\nabla}_a\, h^a{}_d + \overline{\nabla}_d\overline{\nabla}_a\, h^a{}_b - \overline{\nabla}_d\overline{\nabla}_b\, h - \overline{\Box}\, h_{bd} - \overline{R}^s{}_{dab}\, h^a{}_s - \overline{R}^s{}_{bad}\, h^a{}_s\right)\Big|_{\text{high}} \tag{4.133}$$

$$\begin{aligned} R^{(2)}_{bd}\Big|_{\text{high}} = \frac{1}{4}\Bigg(& \overline{\nabla}_b\, h_{ce}\overline{\nabla}_d\, h^{ce} + 2\, h^{ce}\left(\overline{\nabla}_b\overline{\nabla}_d\, h_{ce} + \overline{\nabla}_c\overline{\nabla}_e\, h_{bd} - \overline{\nabla}_d\,\overline{\nabla}_e\, h_{cb} - \overline{\nabla}_b\,\overline{\nabla}_e\, h_{cd}\right) \\ & + 2\,\overline{g}^{eg}\,\overline{g}^{ch}\,\overline{\nabla}_g\, h_{dh}\left(\overline{\nabla}_e\, h_{cb} - \overline{\nabla}_c\, h_{eb}\right) \\ & - \left(\overline{\nabla}_e\, h^{ce} - \frac{1}{2}\,\overline{g}^{ce}\,\overline{\nabla}_e\, h\right)\left(\overline{\nabla}_d\, h_{cb} + \overline{\nabla}_b\, h_{cd} - \overline{\nabla}_c\, h_{bd}\right)\Bigg)\Big|_{\text{high}} \end{aligned} \tag{4.134}$$

The high-frequency wave equation, before gauge fixing and up to $\mathcal{O}(\epsilon^3)$, is

$$\frac{1}{2}\left(-\overline{\Box}\, h_{bd} + \overline{\nabla}_b\overline{\nabla}_a\, h^a{}_d + \overline{\nabla}_d\overline{\nabla}_a\, h^a{}_b - \overline{\nabla}_d\overline{\nabla}_b\, h\right) \tag{4.135}$$

$$\begin{aligned} = & -\frac{1}{2}\left(-\overline{R}^s{}_{dab}\, h^a{}_s - \overline{R}^s{}_{bad}\, h^a{}_s\right) \\ & - \frac{\epsilon}{4}\Bigg(\overline{\nabla}_b\, h_{ce}\overline{\nabla}_d\, h^{ce} + 2\,\overline{g}^{eg}\,\overline{g}^{ch}\,\overline{\nabla}_g\, h_{dh}\left(\overline{\nabla}_e\, h_{cb} - \overline{\nabla}_c\, h_{eb}\right) \\ & + 2\, h^{ce}\left(\overline{\nabla}_b\overline{\nabla}_d\, h_{ce} + \overline{\nabla}_c\overline{\nabla}_e\, h_{bd} - \overline{\nabla}_d\,\overline{\nabla}_e\, h_{cb} - \overline{\nabla}_b\,\overline{\nabla}_e\, h_{cd}\right) \\ & - \left(\overline{\nabla}_e\, h^{ce} - \frac{1}{2}\,\overline{g}^{ce}\,\overline{\nabla}_e\, h\right)\left(\overline{\nabla}_d\, h_{cb} + \overline{\nabla}_b\, h_{cd} - \overline{\nabla}_c\, h_{bd}\right)\Bigg) \\ & + \kappa\Bigg(\frac{\alpha}{\epsilon}\, T^{(1,0)}_{ab} + T^{(0,1)}_{ab} + \alpha\, T^{(1,1)}_{ab} \\ & - \frac{1}{2}\, g_{ab}\left(\frac{\alpha}{\epsilon}\, T^{(1,0)} + T^{(0,1)} + \alpha\, T^{(1,1)}\right)\Bigg) \end{aligned} \tag{4.136}$$

where $T^{(1,0)}_{ab}$, $T^{(0,1)}_{ab}$ and $T^{(1,1)}_{ab}$ are defined in equations (4.121)- (4.255). The expanded trace of the energy-momentum tensor is

$$
\begin{aligned}
T^{(1,0)} &= 2\,\overline{g}^{ef}\,\overline{g}^{ab}\,F^{(1)}_{ae}\,\overline{F}_{bf} + 2\,\overline{g}^{ab}\,\partial_a\overline{\phi}\,\partial_b\phi^{(1)} - \frac{8}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \overline{g}^{mn}\overline{g}^{op}\,F^{(1)}_{np}\,\overline{F}_{mo}\,\overline{F}^2 \\
&\quad - 4\left(\frac{1}{2}\,\overline{g}^{mn}\overline{g}^{op}\,\overline{F}_{np}\,F^{(1)}_{mo} + \overline{g}^{mn}\,\partial_m\overline{\phi}\,\partial_n\phi^{(1)} + m^2\,\overline{\phi}\,\phi^{(1)}\right. \\
&\quad \left. - \frac{4}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\overline{g}^{qr}\,\overline{g}^{st}\,F^{(1)}_{rt}\,\overline{F}_{qs}\,\overline{F}^2 - \frac{7}{4\,\overline{g}}\,\epsilon^{hijk}\epsilon^{defg}\,F^{(1)}_{jk}\,\overline{F}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi}\right)\right) \\
&= -2\,\overline{g}^{ab}\,\partial_a\overline{\phi}\,\partial_b\phi^{(1)} - 4\,m^2\,\overline{\phi}\,\phi^{(1)} - \frac{14}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\frac{1}{\overline{g}}\,\epsilon^{hijk}\epsilon^{defg}\,F^{(1)}_{jk}\,\overline{F}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi}\right)
\end{aligned}
\tag{4.137}
$$

$$
\begin{aligned}
T^{(0,1)} &= -\overline{g}^{ab}\,h^{cd}\,\overline{F}_{ac}\overline{F}_{bd} + \frac{6}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 h^{cd}\,\overline{g}^{ab}\,\overline{F}_{ac}\,\overline{F}_{bd}\,\overline{F}^2 \\
&- h\left(\frac{1}{4}\,\overline{F}^2 + \frac{1}{2}\left(\overline{g}^{mn}\,\partial_m\overline{\phi}\,\partial_n\overline{\phi} + m^2\,\overline{\phi}^2\right)\right. \\
&- \left.\frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\overline{F}^4 - \frac{7}{16\,\overline{g}}\left(\epsilon^{mnop}\overline{F}_{mn}\overline{F}_{op}\right)^2\right)\right) \\
&- 4\left(-\frac{1}{4}\,h^{mn}\overline{g}^{op}\,\overline{F}_{np}\,\overline{F}_{mo} - \frac{1}{4}\,\overline{g}^{mn}h^{op}\,\overline{F}_{np}\,\overline{F}_{mo}\right. \\
&- \frac{1}{2}\,h^{mn}\,\partial_m\overline{\phi}\,\partial_n\overline{\phi} + \frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(h^{mn}\,\overline{g}^{op}\,\overline{g}^{qr}\,\overline{g}^{st}\,\overline{F}_{rt}\,\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}_{qs}\right. \\
&+ h^{mn}\,\overline{g}^{op}\,\overline{g}^{qr}\,\overline{g}^{st}\,\overline{F}_{rt}\,\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}_{qs} \\
&+ \left.\left. h^{mn}\,\overline{g}^{op}\,\overline{g}^{qr}\,\overline{g}^{st}\,\overline{F}_{rt}\,\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}_{qs} + h^{mn}\,\overline{g}^{op}\,\overline{g}^{qr}\,\overline{g}^{st}\,\overline{F}_{rt}\,\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}_{qs}\right)\right) \\
&+ \frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\frac{7}{16\,\overline{g}^2}\,\epsilon^{hijk}\epsilon^{defg}\,\mathrm{tr}\left(\overline{g}^{mn}\,h_{mn}\right)\overline{F}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi}\,\overline{F}_{jk}\right)
\end{aligned}
\tag{4.138}
$$

$$
\begin{aligned}
&= -\overline{g}^{ab}\,h^{cd}\,\overline{F}_{ac}\overline{F}_{bd} + \frac{6}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 h^{cd}\,\overline{g}^{ab}\,\overline{F}_{ac}\,\overline{F}_{bd}\,\overline{F}^2 \\
&- h\left(\frac{1}{4}\,\overline{F}^2 + \frac{1}{2}\left(\overline{g}^{mn}\,\partial_m\overline{\phi}\,\partial_n\overline{\phi} + m^2\,\overline{\phi}^2\right)\right. \\
&- \left.\frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\overline{F}^4 - \frac{7}{16\,\overline{g}}\left(\epsilon^{mnop}\overline{F}_{mn}\overline{F}_{op}\right)^2\right)\right) + h^{mn}\overline{g}^{op}\,\overline{F}_{np}\,\overline{F}_{mo} + \overline{g}^{mn}h^{op}\,\overline{F}_{np}\,\overline{F}_{mo} \\
&+ 2\,h^{mn}\,\partial_m\overline{\phi}\,\partial_n\overline{\phi} - \frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(h^{mn}\,\overline{g}^{op}\,\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}^2\right) \\
&+ \frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\frac{7}{16\,\overline{g}^2}\,\epsilon^{hijk}\epsilon^{defg}\,\mathrm{tr}\left(\overline{g}^{mn}\,h_{mn}\right)\overline{F}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi}\,\overline{F}_{jk}\right)
\end{aligned}
\tag{4.139}
$$

$$
\begin{aligned}
T^{(1,1)} &= -2\, h^{ef}\, \overline{g}^{ab} F^{(1)}_{ae}\, \overline{F}_{bf} + \frac{12}{45\, m_e^4}\left(\frac{e^2}{4\pi}\right)^2 h^{ef}\, \overline{g}^{ab}\, \overline{g}^{mn} \overline{g}^{op}\, \big(F^{(1)}_{np}\, \overline{F}_{mo}\, \overline{F}_{ea}\, \overline{F}_{fb} \\
&+ F^{(1)}_{em}\, \overline{F}_{fn}\, \overline{F}_{oa}\, \overline{F}_{pb} \\
&- h\bigg(\frac{1}{4}\, \overline{g}^{mn} \overline{g}^{op}\, \big(\overline{F}_{np}\, F^{(1)}_{mo} + \overline{F}_{mo}\, F^{(1)}_{np}\big) + \overline{g}^{mn}\, \partial_m \overline{\phi}\, \partial_n \phi^{(1)} + m^2\, \overline{\phi}\, \phi^{(1)} \\
&- \frac{1}{90\, m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \Big(\overline{g}^{mn}\, \overline{g}^{op}\, \overline{g}^{qr}\, \overline{g}^{st}\, \Big(F^{(1)}_{rt}\, \overline{F}_{mo}\, \overline{F}_{np}\, \overline{F}_{qs} + \overline{F}_{rt}\, F^{(1)}_{mo}\, \overline{F}_{np}\, \overline{F}_{qs} \\
&+ \overline{F}_{rt}\, \overline{F}_{mo}\, F^{(1)}_{np}\, \overline{F}_{qs} + \overline{F}_{rt}\, \overline{F}_{mo}\, \overline{F}_{np}\, F^{(1)}_{qs}\Big) \\
&- \frac{7}{16\, \overline{g}}\, \epsilon^{hijk} \epsilon^{defg}\, \Big(F^{(1)}_{jk}\, \overline{F}_{de}\, \overline{F}_{fg}\, \overline{F}_{hi} + \overline{F}_{jk}\, F^{(1)}_{de}\, \overline{F}_{fg}\, \overline{F}_{hi} \\
&+ \overline{F}_{jk}\, \overline{F}_{de}\, F^{(1)}_{fg}\, \overline{F}_{hi} + \overline{F}_{jk}\, \overline{F}_{de}\, \overline{F}_{fg}\, F^{(1)}_{hi}\Big)\Big)\bigg) - 4\bigg(-\frac{1}{2}\, h^{mn} g^{op}\, \big(\overline{F}_{np}\, F^{(1)}_{mo} + \overline{F}_{mo}\, F^{(1)}_{np}\big) \\
&- h^{mn}\, \partial_m \overline{\phi}\, \partial_n \phi^{(1)} \\
&+ \frac{1}{45\, m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \Big(h^{mn}\, g^{op}\, g^{qr}\, g^{st}\, \Big(F^{(1)}_{rt}\, \overline{F}_{mo}\, \overline{F}_{np}\, \overline{F}_{qs} + \overline{F}_{rt}\, F^{(1)}_{mo}\, \overline{F}_{np}\, \overline{F}_{qs} \\
&+ \overline{F}_{rt}\, \overline{F}_{mo}\, F^{(1)}_{np}\, \overline{F}_{qs} + \overline{F}_{rt}\, \overline{F}_{mo}\, \overline{F}_{np}\, F^{(1)}_{qs}\Big)\Big) \\
&+ \frac{1}{45\, m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \Big(g^{mn}\, g^{op}\, h^{qr}\, g^{st}\, \Big(F^{(1)}_{rt}\, \overline{F}_{mo}\, \overline{F}_{np}\, \overline{F}_{qs} \\
&+ \overline{F}_{rt}\, F^{(1)}_{mo}\, \overline{F}_{np}\, \overline{F}_{qs} + \overline{F}_{rt}\, \overline{F}_{mo}\, F^{(1)}_{np}\, \overline{F}_{qs} + \overline{F}_{rt}\, \overline{F}_{mo}\, \overline{F}_{np}\, F^{(1)}_{qs}\Big)\Big)
\end{aligned}
\tag{4.140}
$$

Minkowski Limit:. Taking the Minkowski limit and using the gauge fixing conditions,

$$
\partial^a h_{ab} = 0 \qquad\qquad h = 0 \tag{4.141}
$$

the high-frequency wave equation is therefore

$$
-\frac{1}{2}\, \overline{\Box}\, h_{bd} = \kappa\bigg(\frac{\alpha}{\epsilon}\, T^{(1,0)}_{ab} + T^{(0,1)}_{ab} + \alpha\, T^{(1,1)}_{ab} - \frac{1}{2}\, g_{ab}\left(\frac{\alpha}{\epsilon}\, T^{(1,0)} + T^{(0,1)} + \alpha\, T^{(1,1)}\right)\bigg) \tag{4.142}
$$

The axion terms are omitted for comparison. The remaining terms are

$$T^{(1,0)} = -\frac{14}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \epsilon^{hijk}\epsilon^{defg}\,F^{(1)}_{jk}\,\overline{F}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi} \tag{4.143}$$

$$T^{(0,1)} = \frac{4}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 h^{cd}\,\eta^{ab}\,\overline{F}_{ac}\,\overline{F}_{bd}\,\overline{F}^2 + \eta^{mn}h^{op}\,\overline{F}_{np}\,\overline{F}_{mo} \tag{4.144}$$

$$\begin{aligned} T^{(1,1)} &= 2\,h^{ef}\,\eta^{ab}F^{(1)}_{ae}\,\overline{F}_{bf} \\ &- \frac{4}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \eta^{ab}\,\eta^{mn}\eta^{op}\,h^{ef}\left(F^{(1)}_{np}\,\overline{F}_{mo}\,\overline{F}_{ea}\,\overline{F}_{fb} + F^{(1)}_{em}\,\overline{F}_{fn}\,\overline{F}_{oa}\,\overline{F}_{pb}\right) \end{aligned} \tag{4.145}$$

To compare with [55] equation 7, the RHS equation must have $T^{(1,1)}_{ab}$ and no axion. Deriving $T^{(1,1)}_{ab}$ from the energy-momentum tensor above reproduces equation 7 in [55]. We consider background fields which is a key difference with [55].

4.2.1 Expanded Axion Waves Equations

The expanded axion equation of motion is

$$\left(g^{ab}\,\partial_a\partial_b - g^{ab}\Gamma^c{}_{ab}\,\partial_c - m^2\right)\phi = \frac{\tilde{\lambda}}{8\sqrt{-g}}\,\epsilon^{abcd}\,F_{ab}F_{cd} \tag{4.146}$$

where the left-handed side is given by

$$
\begin{aligned}
\Big(g^{ab}\,\partial_a\partial_b - g^{ab}\Gamma^c{}_{ab}\,\partial_c - m^2\Big)\phi = &\left(\overline{g}^{ab}\,\partial_a\partial_b - \overline{g}^{ab}\,\overline{\Gamma}^c{}_{ab}\,\partial_c - m^2\right)\overline{\phi} \\
&+ \alpha\,\left(\overline{g}^{ab}\,\partial_a\partial_b - \overline{g}^{ab}\,\overline{\Gamma}^c{}_{ab}\,\partial_c - m^2\right)\phi^{(1)} \\
&+ \epsilon\Big(-h^{ab}\,\partial_a\partial_b + h^{ab}\overline{\Gamma}^c{}_{ab}\,\partial_c \\
&- \frac{1}{2}\,\overline{g}^{ab}\overline{g}^{cm}\left(\overline{\nabla}_b\,h_{ma} + \overline{\nabla}_a\,h_{mb} - \overline{\nabla}_m\,h_{ab}\right)\partial_c\Big)\overline{\phi} \\
&+ \alpha\,\epsilon\Big(-h^{ab}\,\partial_a\partial_b + h^{ab}\overline{\Gamma}^c{}_{ab}\,\partial_c \\
&- \frac{1}{2}\,\overline{g}^{ab}\,\overline{g}^{cm}\left(\overline{\nabla}_b\,h_{ma} + \overline{\nabla}_a\,h_{mb} - \overline{\nabla}_m\,h_{ab}\right)\,\partial_c\Big)\phi^{(1)}
\end{aligned}
\tag{4.147}
$$

The right hand side of the axion equation is

$$
\begin{aligned}
\frac{\tilde{\lambda}}{8\sqrt{-g}}\,\epsilon^{abcd}\,F_{ab}F_{cd} = &\frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}}\,\left(1 - \frac{\epsilon}{2}\,\overline{g}^{cd}h_{cd} + \frac{3\,\epsilon^2}{8}\,\left(\overline{g}^{cd}h_{cd}\right)^2\right)\,\epsilon^{abcd}\,\left(\overline{F}_{ab} + \alpha F^{(1)}_{ab}\right)\left(\overline{F}_{cd} + \alpha F^{(1)}_{cd}\right) \\
= &\frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}}\,\left(1 - \frac{\epsilon}{2}\,\overline{g}^{cd}h_{cd} + \frac{3\,\epsilon^2}{8}\,\left(\overline{g}^{cd}h_{cd}\right)^2\right)\,\epsilon^{abcd}\,\Big(\overline{F}_{ab}\overline{F}_{cd} \\
&+ 2\,\alpha\overline{F}_{ab}F^{(1)}_{cd} + \alpha^2 F^{(1)}_{ab}F^{(1)}_{cd}\Big) \\
= &\frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}}\left(\epsilon^{abcd}\overline{F}_{ab}\overline{F}_{cd}\right) - \frac{\tilde{\lambda}\,\epsilon}{16\sqrt{-\overline{g}}}\left(\overline{g}^{ef}h_{ef}\right)\left(\epsilon^{abcd}\overline{F}_{ab}\overline{F}_{cd}\right) \\
&+ \frac{\tilde{\lambda}\,\alpha}{4\sqrt{-\overline{g}}}\,\epsilon^{abcd}\,\overline{F}_{ab}F^{(1)}_{cd} \\
&- \frac{\tilde{\lambda}\,\alpha\,\epsilon}{8\sqrt{-\overline{g}}}\,\left(\overline{g}^{cd}h_{cd}\right)\epsilon^{abcd}\,\overline{F}_{ab}F^{(1)}_{cd}
\end{aligned}
\tag{4.148}
$$

$$
= \overline{j} - \frac{\epsilon}{2}\left(\overline{g}^{ef}h_{ef}\right)\,\overline{j} + \frac{\tilde{\lambda}\,\alpha}{4\sqrt{-\overline{g}}}\,\epsilon^{abcd}\,\overline{F}_{ab}F^{(1)}_{cd} - \frac{\tilde{\lambda}\,\alpha\,\epsilon}{8\sqrt{-\overline{g}}}\,\left(\overline{g}^{ef}h_{ef}\right)\epsilon^{abcd}\,\overline{F}_{ab}F^{(1)}_{cd}
\tag{4.149}
$$

where

$$\overline{j} = \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}} \left(\epsilon^{abcd}\overline{F}_{ab}\overline{F}_{cd}\right) \,. \tag{4.150}$$

Putting it all together, the background

$$\left(\overline{g}^{ab}\,\partial_a\partial_b - \overline{g}^{ab}\,\overline{\Gamma}^{c}{}_{ab}\,\partial_c - m^2\right)\overline{\phi} = \overline{j} \tag{4.151}$$

and propagating equations of motions are

$$\begin{aligned}\left(\overline{g}^{ab}\,\partial_a\partial_b - \overline{g}^{ab}\,\overline{\Gamma}^{c}{}_{ab}\,\partial_c - m^2\right)\phi^{(1)} = &-\frac{\epsilon}{\alpha}\Big(-h^{ab}\,\partial_a\partial_b + h^{ab}\overline{\Gamma}^{c}{}_{ab}\,\partial_c\\ &-\frac{1}{2}\,\overline{g}^{ab}\overline{g}^{cm}\left(\overline{\nabla}_b\,h_{ma} + \overline{\nabla}_a\,h_{mb} - \overline{\nabla}_m\,h_{ab}\right)\partial_c\Big)\overline{\phi}\\ &-\epsilon\Big(-h^{ab}\,\partial_a\partial_b + h^{ab}\overline{\Gamma}^{c}{}_{ab}\,\partial_c\\ &-\frac{1}{2}\,\overline{g}^{ab}\,\overline{g}^{cm}\left(\overline{\nabla}_b\,h_{ma} + \overline{\nabla}_a\,h_{mb} - \overline{\nabla}_m\,h_{ab}\right)\,\partial_c\Big)\phi^{(1)}\\ &-\frac{\epsilon}{2}\left(\overline{g}^{ef}h_{ef}\right)\,\overline{j} + \frac{\tilde{\lambda}\,\alpha}{4\sqrt{-\overline{g}}}\,\epsilon^{abcd}\,\overline{F}_{ab}F^{(1)}_{cd}\\ &-\frac{\tilde{\lambda}\,\alpha\,\epsilon}{8\sqrt{-\overline{g}}}\,\left(\overline{g}^{ef}h_{ef}\right)\epsilon^{abcd}\,\overline{F}_{ab}F^{(1)}_{cd}\,.\end{aligned} \tag{4.152}$$

In the Minkowski space limit with no axion background, this equation reduces to

$$\left(\Box - m^2\right)\phi^{(1)} = -\epsilon\left(-h^{ab}\,\partial_a\partial_b - \eta^{ab}\,\eta^{cm}\left(\partial_b\,h_{ma}\right)\partial_c\right)\phi^{(1)} + \frac{\tilde{\lambda}\,\alpha}{4}\,\epsilon^{abcd}\,\overline{F}_{ab}F^{(1)}_{cd}\,. \tag{4.153}$$

4.2.1 Expanded Electromagnetic Wave Equation

The expanded electromagnetic equations of motion is

$$\partial_a \left(g^{ac} g^{bd} F_{cd}\right) + g^{ac} g^{bd}\, \Gamma^{e}{}_{ae}\, F_{cd} = j^b + \nabla_a\, P^{ab}\,. \tag{4.154}$$

The LHS equation is expanded as

$$\begin{aligned}
\partial_a \left(g^{ac} g^{bd} F_{cd}\right) + g^{ac} g^{bd}\, \Gamma^{e}{}_{ae}\, F_{cd} &= \partial_a \left(\overline{g}^{ac}\, \overline{g}^{bd}\right) \overline{F}_{cd} + \overline{g}^{ac}\, \overline{g}^{bd}\, \partial_a \overline{F}_{cd} + \overline{g}^{ac}\, \overline{g}^{bd}\, \overline{\Gamma}^{e}{}_{ae} \overline{F}_{cd} \\
&+ \alpha \left(\partial_a \left(\overline{g}^{ac}\, \overline{g}^{bd}\right) F^{(1)}_{cd} + \overline{g}^{ac}\, \overline{g}^{bd}\, \partial_a F^{(1)}_{cd} + \overline{g}^{ac}\, \overline{g}^{bd}\, \overline{\Gamma}^{e}{}_{ae}\, F^{(1)}_{cd} \right) \\
&- \epsilon \left(\partial_a \left(\left(\overline{g}^{ac}\, h^{bd} + \overline{g}^{bd}\, h^{ac}\right) \overline{F}_{cd} \right) + \left(\overline{g}^{ac}\, h^{bd} + h^{ac}\, \overline{g}^{bd}\right) \overline{\Gamma}^{e}{}_{ae} \overline{F}_{cd} \right. \\
&\left. - \frac{1}{2}\, \overline{g}^{em}\, \overline{g}^{ac}\, \overline{g}^{bd} \left(\overline{\nabla}_e\, h_{ma} + \overline{\nabla}_a\, h_{me} - \overline{\nabla}_m\, h_{ae} \right) \overline{F}_{cd} \right) \\
&- \alpha\, \epsilon \left(\partial_a \left(\left(\overline{g}^{ac}\, h^{bd} + \overline{g}^{bd}\, h^{ac}\right) F^{(1)}_{cd} \right) \right. \\
&\left. - \frac{1}{2}\, \overline{g}^{em}\, \overline{g}^{ac}\, \overline{g}^{bd} \left(\overline{\nabla}_e\, h_{ma} + \overline{\nabla}_a\, h_{me} - \overline{\nabla}_m\, h_{ae} \right) F^{(1)}_{cd} \right)
\end{aligned} \tag{4.155}$$

Putting it all together, we have the following background equation of motion

$$\partial_a \left(\overline{g}^{ac}\, \overline{g}^{bd}\right) \overline{F}_{cd} + \overline{g}^{ac}\, \overline{g}^{bd}\, \partial_a \overline{F}_{cd} + \overline{g}^{ac}\, \overline{g}^{bd}\, \overline{\Gamma}^{e}{}_{ae} \overline{F}_{cd} = \overline{j}^{b} + \overline{\nabla}_a \overline{P}^{ab} \tag{4.156}$$

and propagating equation of motions

$$
\begin{aligned}
\overline{g}^{ac}\,\overline{g}^{bd}\left(\partial_a F^{(1)}_{cd} + \overline{\Gamma}^{e}{}_{ae}\,F^{(1)}_{cd}\right) = & -\partial_a\left(\overline{g}^{ac}\,\overline{g}^{bd}\right)F^{(1)}_{cd} + \frac{\epsilon}{\alpha}\Bigg(\,\partial_a\Big(\left(\overline{g}^{ac}\,h^{bd} + \overline{g}^{bd}\,h^{ac}\right)\overline{F}_{cd}\Big) \\
& + \left(\overline{g}^{ac}\,h^{bd} + h^{ac}\,\overline{g}^{bd}\right)\overline{\Gamma}^{e}{}_{ae}\overline{F}_{cd} \\
& - \frac{1}{2}\,\overline{g}^{em}\,\overline{g}^{ac}\,\overline{g}^{bd}\left(\overline{\nabla}_e\,h_{ma} + \overline{\nabla}_a\,h_{me} - \overline{\nabla}_m\,h_{ae}\right)\overline{F}_{cd}\Bigg) \\
& + \epsilon\Bigg(\,\partial_a\Big(\left(\overline{g}^{ac}\,h^{bd} + \overline{g}^{bd}\,h^{ac}\right)F^{(1)}_{cd}\Big) \\
& - \frac{1}{2}\,\overline{g}^{em}\,\overline{g}^{ac}\,\overline{g}^{bd}\left(\overline{\nabla}_e\,h_{ma} + \overline{\nabla}_a\,h_{me} - \overline{\nabla}_m\,h_{ae}\right)F^{(1)}_{cd}\Bigg) \\
& + j^{(1)\,b} + \frac{\epsilon}{\alpha}\,j' + \overline{\nabla}_a P^{(1)\,ab} + \frac{\epsilon}{\alpha}\,\overline{\nabla}_a P'^{ab}
\end{aligned}
\tag{4.157}
$$

where we have used the currents,

$$\overline{j}^{b} = \frac{1}{\sqrt{-\overline{g}}}\,\frac{\tilde{\lambda}}{2}\,\epsilon^{abcd}\partial_a\left(\overline{\phi}\,\overline{F}_{cd}\right) \tag{4.158}$$

$$\alpha\,j^{(1)\,b} = \frac{\alpha}{\sqrt{-\overline{g}}}\,\frac{\tilde{\lambda}}{2}\,\epsilon^{abcd}\partial_a\left(\overline{\phi}\,F^{(1)}_{cd} + \phi^{(1)}\,\overline{F}_{cd}\right) \tag{4.159}$$

$$\epsilon\,j' = -\frac{\epsilon}{\sqrt{-\overline{g}}}\,\frac{\tilde{\lambda}}{4}\left(\overline{g}^{cd}h_{cd}\right)\epsilon^{abcd}\partial_a\left(\,\overline{\phi}\,\overline{F}_{cd}\,\right)\;. \tag{4.160}$$

We also include the contributions from the Euler-Heisenberg lagrangian

$$\begin{aligned}
\alpha\,\overline{\nabla}_a P^{(1)\,ab} &= \frac{\alpha}{\sqrt{-\overline{g}}}\left(\frac{4}{45 m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \partial_a\left(\sqrt{-\overline{g}}\,g^{mn}g^{op}\left(\overline{F}_{mo}\,\overline{F}_{np}\,F^{ba\,(1)}\right.\right.\right.\\
&+ \overline{F}_{mo}\,F^{(1)}_{np}\,\overline{F}^{ba} + F^{(1)}_{mo}\,\overline{F}_{np}\,\overline{F}^{ba}\\
&+ \left.\left.\left.\frac{7}{180\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\epsilon^{baef}\epsilon^{cdgh}\;\left(\overline{F}_{ef}\overline{F}_{cd}F^{(1)}_{gh} + \overline{F}_{ef}F^{(1)}_{cd}\overline{F}_{gh} + F^{(1)}_{ef}\overline{F}_{cd}\overline{F}_{gh}\right)\right)\right)\right)
\end{aligned} \tag{4.161}$$

$$\overline{\nabla}_a\overline{P}^{ab} = \frac{1}{\sqrt{-\overline{g}}}\left(\frac{4}{45 m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \partial_a\left(\sqrt{-\overline{g}}\,\overline{g}^{mn}\overline{g}^{op}\,z\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}^{ba}\right)\right. \tag{4.162}$$

$$\left.+\frac{7}{180\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\epsilon^{baef}\epsilon^{cdgh}\,\overline{F}_{ef}\,\overline{F}_{cd}\,\overline{F}_{gh}\right)\right),$$

$$\epsilon\,\overline{\nabla}_a P'^{ab} = \frac{\epsilon}{\sqrt{-\overline{g}}}\left(\frac{2}{45 m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \partial_a\left(\sqrt{-\overline{g}}\;\left(\overline{g}^{cd}h_{cd}\right)\;\overline{g}^{mn}\overline{g}^{op}\,\overline{F}^{ba}\,\overline{F}_{mo}\,\overline{F}_{np}\right)\right. \tag{4.163}$$

$$\begin{aligned}
&- \frac{7}{360\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \left(\overline{g}^{cd}h_{cd}\right)\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\epsilon^{baef}\epsilon^{cdgh}\,\overline{F}_{ef}\,\overline{F}_{cd}\,\overline{F}_{gh}\right)\\
&- \left.\frac{7}{180\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \frac{1}{\sqrt{-\overline{g}}}\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\left(\overline{g}^{cd}h_{cd}\right)\epsilon^{baef}\epsilon^{cdgh}\overline{F}_{ef}\overline{F}_{cd}\overline{F}_{gh}\right)\right)
\end{aligned}$$

Minkowski Limit: Take the Minkowski limit and using the gauge gauge fixing conditions,

$$\partial^a h_{ab} = 0 \qquad\qquad h = 0 \tag{4.164}$$

the high-frequency wave equation is therefore

$$\begin{aligned}\partial_a F^{ab\,(1)} &= \partial_a\Big(\big(\eta^{ac}\,h^{bd} + \eta^{bd}\,h^{ac}\big)\,\overline{F}_{cd}\Big) + \partial_a\Big(\big(\eta^{ac}\,h^{bd} + \eta^{bd}\,h^{ac}\big)\,F^{(1)}_{cd}\Big) \\ &- \frac{1}{2}\,\eta^{em}\,\eta^{ac}\,\eta^{bd}\,\Big(\partial_e\,h_{ma} + \partial_a\,h_{me} - \partial_m\,h_{ae}\Big)\overline{F}_{cd} \\ &- \frac{1}{2}\,\eta^{em}\,\eta^{ac}\,\eta^{bd}\,\Big(\partial_e\,h_{ma} + \partial_a\,h_{me} - \partial_m\,h_{ae}\Big)\,F^{(1)}_{cd} \\ &= \partial_a\Big(\big(\eta^{ac}\,h^{bd} + \eta^{bd}\,h^{ac}\big)\,F^{(1)}_{cd}\Big)\end{aligned} \tag{4.165}$$

where we have eliminated the Euler-Heisenberg corrections, $\alpha = \epsilon = 1$ and axion terms for comparison with [55]. This matches [55] if the background electromagnetic fields are removed.

4.2.2 Effective Equations of Motion

4.2.2 Low-Frequency Equations of Motion

To summarize, the low-frequency equations of motion for the axion, electromagnetic and gravity waves are respectively,

$$\big(\overline{g}^{ab}\,\partial_a\partial_b - \overline{g}^{ab}\,\overline{\Gamma}^c{}_{ab}\,\partial_c - m^2\big)\,\overline{\phi} = \overline{j} \tag{4.166}$$

$$\partial_a\overline{F}^{ab} + \overline{\Gamma}^c{}_{ac}\,\overline{F}^{ab} = \overline{j}^b + \overline{\nabla}_a\overline{P}^{ab} \tag{4.167}$$

$$\overline{R}_{ab} - \frac{1}{2}\,\overline{g}_{ab}\,\overline{R} = \kappa\,\big(\overline{T}_{ab} + \epsilon^2\,\overline{t}_{ab}\big) \tag{4.168}$$

where $\overline{\Gamma}^{m}_{\ ij} = \frac{1}{2}\,\overline{g}^{mk}\left(\partial_j \overline{g}_{ki} + \partial_j \overline{g}_{ki} - \partial_j \overline{g}_{ki}\right)$ and the currents are

$$\overline{j} = \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}}\,\epsilon^{abcd}\overline{F}_{ab}\overline{F}_{cd} \tag{4.169}$$

$$\overline{j}^{b} = \frac{1}{\sqrt{-\overline{g}}}\,\frac{\tilde{\lambda}}{2}\,\epsilon^{abcd}\partial_a\left(\overline{\phi}\,\overline{F}_{cd}\right)\ . \tag{4.170}$$

The Euler-Heisenberg corrections are

$$\begin{aligned}\overline{\nabla}_a \overline{P}^{ab} = \frac{1}{\sqrt{-\overline{g}}}\,\frac{4}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \Bigg(&\partial_a\left(\sqrt{-\overline{g}}\,\overline{g}^{mn}\overline{g}^{op}\,\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}^{ba}\right) \\ &+ \frac{7}{16}\,\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\epsilon^{baef}\epsilon^{cdgh}\,\overline{F}_{ef}\,\overline{F}_{cd}\,\overline{F}_{gh}\right)\Bigg)\,.\end{aligned}$$

The energy-momentum tensor contributions are

$$\begin{aligned}\overline{T}_{ab} &= \overline{g}^{ef}\overline{F}_{ae}\overline{F}_{bf} + \partial_a\overline{\phi}\,\partial_b\overline{\phi} - \frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \overline{g}^{ef}\overline{F}^2\,\overline{F}_{ae}\overline{F}_{bf} \\ &\quad - \overline{g}_{ab}\Bigg(\frac{1}{4}\,\overline{F}^2 + \frac{1}{2}\left(\overline{g}^{mn}\,\partial_m\overline{\phi}\,\partial_n\overline{\phi} + m^2\,\overline{\phi}^2\right) - \frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\overline{F}^4 - \frac{7}{16\,\overline{g}}\left(\epsilon^{mnop}\overline{F}_{mn}\overline{F}_{op}\right)^2\right)\Bigg) \\ \bar{t}_{ab} &= -\frac{1}{\kappa}\langle R^{(2)}_{ab} - \frac{1}{2}\overline{g}_{ab}\,R^{(2)}\rangle\end{aligned} \tag{4.171}$$

where latter is the contributions from the long-wavelength gravitational waves. Importantly, it is clear the background curvature is sourced by all of the background fields.

4.2.2 High-Frequency Equations of Motion and Gauge Fixing

In summary, the high-frequency equations of motion for the gravitational, axion and electromagnetic waves are

$$-\frac{1}{2}\,\overline{\Box}\,h_{bd} = -\frac{1}{2}\left(\overline{\nabla}_b\overline{\nabla}_a\,h^a{}_d + \overline{\nabla}_d\overline{\nabla}_a\,h^a{}_b - \overline{\nabla}_d\overline{\nabla}_b\,h\right) - \frac{1}{2}\left(-\overline{R}^s{}_{dab}\,h^a{}_s - \overline{R}^s{}_{bad}\,h^a{}_s\right) \tag{4.172}$$
$$+\,\kappa\left(\frac{\alpha}{\epsilon}\,T^{(1,0)}_{ab} + T^{(0,1)}_{ab} + \alpha\,T^{(1,1)}_{ab} - \frac{1}{2}\,g_{ab}\left(\frac{\alpha}{\epsilon}\,T^{(1,0)} + T^{(0,1)} + \alpha\,T^{(1,1)}\right)\right)$$

$$\left(\overline{g}^{ab}\,\partial_a\partial_b - \overline{g}^{ab}\,\overline{\Gamma}^c{}_{ab}\,\partial_c - m^2\right)\phi^{(1)} = -\tfrac{\epsilon}{\alpha}\left(-h^{ab}\,\partial_a\partial_b + h^{ab}\overline{\Gamma}^c{}_{ab}\,\partial_c\right)\overline{\phi}$$
$$-\,\tfrac{\epsilon}{\alpha}\left(-\tfrac{1}{2}\,\overline{g}^{ab}\overline{g}^{cm}\left(\overline{\nabla}_b\,h_{ma} + \overline{\nabla}_a\,h_{mb} - \overline{\nabla}_m\,h_{ab}\right)\partial_c\right)\overline{\phi}$$
$$-\,\epsilon\left(-h^{ab}\,\partial_a\partial_b + h^{ab}\overline{\Gamma}^c{}_{ab}\,\partial_c\right.$$
$$\left.-\,\tfrac{1}{2}\,\overline{g}^{ab}\,\overline{g}^{cm}\left(\overline{\nabla}_b\,h_{ma} + \overline{\nabla}_a\,h_{mb} - \overline{\nabla}_m\,h_{ab}\right)\,\partial_c\right)\phi^{(1)}$$
$$-\,\tfrac{\epsilon}{2}\left(\overline{g}^{ef}h_{ef}\right)\,\overline{j} + \tfrac{\tilde{\lambda}\,\alpha}{4\sqrt{-\overline{g}}}\,\epsilon^{abcd}\,\overline{F}_{ab}F^{(1)}_{cd}$$
$$-\,\tfrac{\tilde{\lambda}\,\alpha\,\epsilon}{8\sqrt{-\overline{g}}}\,\left(\overline{g}^{ef}h_{ef}\right)\epsilon^{abcd}\,\overline{F}_{ab}F^{(1)}_{cd}$$

$$\partial_a F^{(1)\ ab} + \overline{\Gamma}^e{}_{ae}\, F^{(1)\ ab} = \frac{\epsilon}{\alpha} \Bigg(\partial_a\Big(\big(\overline{g}^{ac}\, h^{bd} + \overline{g}^{bd}\, h^{ac}\big)\, \overline{F}_{cd}\Big) + \big(\overline{g}^{ac}\, h^{bd} + h^{ac}\, \overline{g}^{bd}\big)\, \overline{\Gamma}^e{}_{ae} \overline{F}_{cd} \tag{4.173}$$

$$- \frac{1}{2}\, \overline{g}^{em}\, \overline{g}^{ac}\, \overline{g}^{bd} \Big(\overline{\nabla}_e\, h_{ma} + \overline{\nabla}_a\, h_{me} - \overline{\nabla}_m\, h_{ae} \Big) \overline{F}_{cd} \Bigg)$$

$$+ \epsilon \Bigg(\partial_a\Big(\big(\overline{g}^{ac}\, h^{bd} + \overline{g}^{bd}\, h^{ac}\big)\, F^{(1)}_{cd}\Big)$$

$$- \frac{1}{2}\, \overline{g}^{em}\, \overline{g}^{ac}\, \overline{g}^{bd} \Big(\overline{\nabla}_e\, h_{ma} + \overline{\nabla}_a\, h_{me} - \overline{\nabla}_m\, h_{ae} \Big)\, F^{(1)}_{cd} \Bigg)$$

$$+ j^{(1)\, b} + \frac{\epsilon}{\alpha}\, j' + \overline{\nabla}_a P^{(1)\, ab} + \frac{\epsilon}{\alpha}\, \overline{\nabla}_a P'^{ab}$$

where the energy-momentum tensor and currents are defined in equations (4.121)- (4.255), equations (4.138)- (4.140) and equations (4.158)- (4.163). To fix the gauge, we first make the redefinition

$$\hat{h}_{ab} = h_{ab} - \frac{1}{2}\, \overline{g}_{ab}\, h \tag{4.174}$$

where $\hat{h} = \overline{g}^{ab}\, \hat{h}_{ab} = -h$. Thus, $\hat{h}_{ab} - \frac{1}{2}\, \overline{g}_{ab}\, \hat{h} = h_{ab}$. After the shift, the high-frequency gravitational wave equation of motion is

$$- \frac{1}{2} \overline{\Box} \left(\hat{h}_{bd} - \frac{1}{2}\, \overline{g}_{bd}\, \hat{h} \right) \tag{4.175}$$

$$= -\frac{1}{2} \bigg(\overline{\nabla}_b \overline{\nabla}_a \Big(\hat{h}^a{}_d - \frac{1}{2}\, \overline{g}^a{}_d\, \hat{h} \Big) + \overline{\nabla}_d \overline{\nabla}_a \Big(\hat{h}^a{}_d - \frac{1}{2}\, \overline{g}^a{}_d\, \hat{h} \Big) + \overline{\nabla}_d \overline{\nabla}_b\, \hat{h} \bigg) \tag{4.176}$$

$$+ \kappa \bigg(\frac{\alpha}{\epsilon}\, T^{(1,0)}_{ab} + T^{(0,1)}_{ab} + \alpha\, T^{(1,1)}_{ab} - \frac{1}{2}\, g_{ab} \left(\frac{\alpha}{\epsilon}\, T^{(1,0)} + T^{(0,1)} + \alpha\, T^{(1,1)} \right) \bigg) \bigg|_{h \to \hat{h}, h_{ab} \to \hat{h}_{ab}}$$

$$- \frac{1}{2} \bigg(-\overline{R}^s{}_{dab} \left(\hat{h}^a{}_s - \frac{1}{2}\, \overline{g}^a{}_s\, \hat{h} \right) - \overline{R}^s{}_{bad} \left(\hat{h}^a{}_s - \frac{1}{2}\, \overline{g}^a{}_s\, \hat{h} \right) \bigg)$$

We can now make the shift

$$h_{ab} \to h'_{ab} = h_{ab} - 2\,\overline{\nabla}_{(a}\xi_{b)} \qquad\qquad h \to h' = h - 2\,\overline{g}^{ab}\,\overline{\nabla}_a\,\xi_b\,. \tag{4.177}$$

which results in the following transformation for $\hat{h}_{ab}$ and $\hat{h}$,

$$\hat{h}_{ab} \to \hat{h}'_{ab} = h'_{ab} - \frac{1}{2}\,\overline{g}_{ab}\,h' = \hat{h}_{ab} - 2\,\overline{\nabla}_{(a}\xi_{b)} + \overline{g}_{ab}\,\overline{g}^{cd}\,\overline{\nabla}_c\,\xi_d \tag{4.178}$$

$$\hat{h} \to \hat{h}' = \overline{g}^{ab}\,\hat{h}'_{ab} = \left(\overline{g}^{ab}\,h'_{ab} - \frac{1}{2}\,\overline{g}^{ab}\overline{g}_{ab}\,h'\right) = \overline{g}^{ab}\hat{h}_{ab} - 2\,\overline{g}^{ab}\,\overline{\nabla}_{(a}\xi_{b)} + \overline{g}^{cd}\overline{g}_{cd}\,\overline{g}^{ab}\,\overline{\nabla}_a\,\xi_b \tag{4.179}$$

$$= \hat{h} - 2\,\overline{g}^{ab}\,\overline{\nabla}_{(a}\xi_{b)} + 4\,\overline{g}^{ab}\,\overline{\nabla}_a\,\xi_b$$

We can fix the ξ's to apply the standard gauge conditions[65], [68],

$$\overline{\nabla}^a\,\hat{h}_{ab} = 0 \qquad\qquad \hat{h} = 0, \tag{4.180}$$

which reduces the total degrees of freedom of $\hat{h}_{ab}$ to five degrees of freedom.

4.2.2 Diffeomorphism Gauged Fixed Equations of Motion

After gauge fixing and relabeling $\hat{h}_{ab} \to h_{ab}$, the high-frequency equations of motion become

$$-\frac{1}{2}\,\overline{\Box}\,h_{bd} = \overline{R}^{s}{}_{dab}\,h^{a}{}_{s} + \kappa\Bigg(\frac{\alpha}{\epsilon}\,T^{(1,0)}_{ab} + T^{(0,1)}_{ab} + \alpha\,T^{(1,1)}_{ab} - \frac{1}{2}\,g_{ab}\left(\frac{\alpha}{\epsilon}\,T^{(1,0)} + T^{(0,1)} + \alpha\,T^{(1,1)}\right)\Bigg) \tag{4.181}$$

$$\left(\overline{g}^{ab}\,\partial_a\partial_b - \overline{g}^{ab}\,\overline{\Gamma}^{c}{}_{ab}\,\partial_c - m^2\right)\phi^{(1)} = -\frac{\epsilon}{\alpha}\left(-\partial_a\partial_b\,\overline{\phi} + \overline{\Gamma}^{c}{}_{ab}\,\partial_c\,\overline{\phi}\right)h^{ab} - \epsilon\,h^{ab}\left(-\partial_a\partial_b + \overline{\Gamma}^{c}{}_{ab}\,\partial_c\right)\phi^{(1)} + \frac{\tilde{\lambda}\,\alpha}{4\sqrt{-\overline{g}}}\,\epsilon^{abcd}\,\overline{F}_{ab}F^{(1)}_{cd} \tag{4.182}$$

$$\left(\partial_a + \overline{\Gamma}^{e}{}_{ae}\right)F^{(1)\,ab} = \frac{\epsilon}{\alpha}\left(\partial_a\Big(\left(\overline{g}^{ac}\,h^{bd} + \overline{g}^{bd}\,h^{ac}\right)\overline{F}_{cd}\Big) + \left(\overline{g}^{ac}\,h^{bd} + h^{ac}\,\overline{g}^{bd}\right)\overline{\Gamma}^{e}{}_{ae}\overline{F}_{cd}\right) + \epsilon\left(\partial_a\Big(\left(\overline{g}^{ac}\,h^{bd} + \overline{g}^{bd}\,h^{ac}\right)F^{(1)}_{cd}\Big)\right) + j^{(1)\,b} + \frac{\epsilon}{\alpha}\,j' + \overline{\nabla}_a P^{(1)\,ab} + \frac{\epsilon}{\alpha}\,\overline{\nabla}_a P'^{ab} \tag{4.183}$$

where we have used

$$\mathrm{T}^{(1,1)}_{ab} = -h^{ef}\left(F^{(1)}_{ae}\,\overline{F}_{bf} + \overline{F}_{ae}\,F^{(1)}_{bf}\right) + \tfrac{2}{45\,m_e^4}\left(\tfrac{e^2}{4\pi}\right)^2\Bigg(2\,h^{ef}\overline{g}^{mn}\overline{g}^{op}\,\overline{F}^{(1)}_{np}\,\overline{F}_{mo}\,\overline{F}_{ae}\,\overline{F}_{bf} + h^{ef}\,\overline{F}^2\left(\overline{F}^{(1)}_{ae}\,\overline{F}_{bf} + \overline{F}_{ae}\,\overline{F}^{(1)}_{bf}\right) + 2\,h^{mn}\,\overline{g}^{ef}\,\overline{g}^{op}\left(F^{(1)}_{np}\,\overline{F}_{mo} + \overline{F}_{np}\,\overline{F}^{(1)}_{mo}\right)\overline{F}_{ae}\,\overline{F}_{bf} + 2\,h^{mn}\,\overline{g}^{ef}\,\overline{g}^{op}\,\overline{F}_{np}\,\overline{F}_{mo}\left(F^{(1)}_{ae}\,\overline{F}_{bf} + \overline{F}_{ae}\,F^{(1)}_{bf}\right)\Bigg) - h_{ab}\Bigg(\tfrac{1}{4}\,\overline{g}^{mn}\overline{g}^{op}\left(\overline{F}_{np}\,F^{(1)}_{mo} + \overline{F}_{mo}\,F^{(1)}_{np}\right)$$

$$+\,\overline{g}^{mn}\,\partial_m\overline{\phi}\,\partial_n\phi^{(1)} + m^2\,\overline{\phi}\,\phi^{(1)} - \tfrac{2}{45\,m_e^4}\left(\tfrac{e^2}{4\pi}\right)^2\Bigg(\overline{g}^{mn}\,\overline{g}^{op}\,\overline{g}^{qr}\,\overline{g}^{st}\,F^{(1)}_{rt}\,\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}_{qs}$$

$$-\,\tfrac{7}{16\,\overline{g}}\,\epsilon^{hijk}\epsilon^{defg}\,F^{(1)}_{jk}\,\overline{F}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi}\Bigg) - \overline{g}_{ab}\Bigg(-\tfrac{1}{2}\,h^{mn}g^{op}\left(\overline{F}_{np}\,F^{(1)}_{mo} + \overline{F}_{mo}\,F^{(1)}_{np}\right) - h^{mn}\,\partial_m\overline{\phi}\,\partial_n\phi^{(1)}$$

$$+\,\tfrac{2}{45\,m_e^4}\left(\tfrac{e^2}{4\pi}\right)^2\,h^{mn}\,g^{op}\,g^{qr}\,g^{st}\,\left(F^{(1)}_{rt}\,\overline{F}_{mo} + \overline{F}_{rt}\,F^{(1)}_{mo}\right)\,\overline{F}_{np}\,\overline{F}_{qs}\Bigg)$$

$$T^{(1,0)} = -2\,\overline{g}^{ab}\,\partial_a\overline{\phi}\,\partial_b\phi^{(1)} - 4\,m^2\,\overline{\phi}\,\phi^{(1)} - \tfrac{14}{45\,m_e^4}\left(\tfrac{e^2}{4\pi}\right)^2\Bigg(\tfrac{1}{\overline{g}}\,\epsilon^{hijk}\epsilon^{defg}\,F^{(1)}_{jk}\,\overline{F}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi}\Bigg)$$

$$T^{(0,1)} = \overline{g}^{mn}h^{op}\,\overline{F}_{np}\,\overline{F}_{mo} + 2\,h^{mn}\,\partial_m\overline{\phi}\,\partial_n\overline{\phi} + \tfrac{4}{45\,m_e^4}\left(\tfrac{e^2}{4\pi}\right)^2 h^{cd}\,\overline{g}^{ab}\,\overline{F}_{ac}\,\overline{F}_{bd}\,\overline{F}^2$$

$$T^{(1,1)} = -2\,h^{ef}\,\overline{g}^{ab}F^{(1)}_{ae}\,\overline{F}_{bf} + \tfrac{12}{45\,m_e^4}\left(\tfrac{e^2}{4\pi}\right)^2\,h^{ef}\,\overline{g}^{ab}\,\overline{g}^{mn}\overline{g}^{op}\,\left(F^{(1)}_{np}\,\overline{F}_{mo}\,\overline{F}_{ea}\,\overline{F}_{fb} + F^{(1)}_{em}\,\overline{F}_{fn}\,\overline{F}_{oa}\,\overline{F}_{pb}\right)$$

$$-\,4\Bigg(-\tfrac{1}{2}\,h^{mn}g^{op}\left(\overline{F}_{np}\,F^{(1)}_{mo} + \overline{F}_{mo}\,F^{(1)}_{np}\right)$$

$$-\,h^{mn}\,\partial_m\overline{\phi}\,\partial_n\phi^{(1)} + \tfrac{1}{45\,m_e^4}\left(\tfrac{e^2}{4\pi}\right)^2\Bigg(h^{mn}\,g^{op}\,g^{qr}\,g^{st}\Big(F^{(1)}_{rt}\,\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}_{qs} + \overline{F}_{rt}\,F^{(1)}_{mo}\,\overline{F}_{np}\,\overline{F}_{qs}$$

$$+\,\overline{F}_{rt}\,\overline{F}_{mo}\,F^{(1)}_{np}\,\overline{F}_{qs} + \overline{F}_{rt}\,\overline{F}_{mo}\,\overline{F}_{np}\,F^{(1)}_{qs}\Big)\Bigg) + \tfrac{1}{45\,m_e^4}\left(\tfrac{e^2}{4\pi}\right)^2\Bigg(g^{mn}\,g^{op}\,h^{qr}\,g^{st}\Big(F^{(1)}_{rt}\,\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}_{qs}$$

$$+\,\overline{F}_{rt}\,F^{(1)}_{mo}\,\overline{F}_{np}\,\overline{F}_{qs} + \overline{F}_{rt}\,\overline{F}_{mo}\,F^{(1)}_{np}\,\overline{F}_{qs} + \overline{F}_{rt}\,\overline{F}_{mo}\,\overline{F}_{np}\,F^{(1)}_{qs}\Big)\Bigg)$$

The currents and Euler-Heisenberg corrections are $j' = 0$ and

$$j^{(1)\,b} = \frac{1}{\sqrt{-\overline{g}}}\,\frac{\tilde{\lambda}}{2}\left(\epsilon^{abcd}\partial_a\left(\overline{\phi}\,F^{(1)}_{cd}\right) + \epsilon^{abcd}\partial_a\left(\phi^{(1)}\,\overline{F}_{cd}\right)\right) \tag{4.184}$$

$$\begin{aligned}\overline{\nabla}_a P^{(1)\,ab} = \frac{1}{\sqrt{-\overline{g}}}&\left(\frac{8}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2\partial_a\left(\sqrt{-\overline{g}}\,\overline{g}^{mn}\,\overline{g}^{op}\,\overline{g}^{bq}\,\overline{g}^{ar}\,F^{(1)}_{mo}\,\overline{F}_{np}\,\overline{F}_{qr}\right)\right.\\ &+\frac{4}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2\partial_a\left(\sqrt{-\overline{g}}\,\overline{g}^{mn}\,\overline{g}^{op}\,\overline{g}^{bq}\,\overline{g}^{ar}\,\overline{F}_{mo}\,\overline{F}_{np}\,F^{(1)}_{qr}\right)\\ &+\frac{14}{180\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\epsilon^{baef}\epsilon^{cdgh}\,\overline{F}_{ef}\,\overline{F}_{cd}\,F^{(1)}_{gh}\right)\\ &\left.+\frac{7}{180\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\epsilon^{baef}\epsilon^{cdgh}\,F^{(1)}_{ef}\,\overline{F}_{cd}\,\overline{F}_{gh}\right)\right)\end{aligned} \tag{4.185}$$

$$\begin{aligned}\overline{\nabla}_a P'^{ab} = \frac{1}{\sqrt{-\overline{g}}}&\left(\frac{2}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2\partial_a\left(\sqrt{-\overline{g}}\,\left(\overline{g}^{cd}h_{cd}\right)\overline{F}^{ba}\,\overline{F}^2\right)\right.\\ &-\frac{4}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2\partial_a\left(\sqrt{-\overline{g}}\,\left(\overline{g}^{mn}\overline{g}^{op}\left(\overline{g}^{bq}\,h^{ar}+\overline{g}^{ar}\,h^{bq}\right)+2\,h^{mn}\overline{g}^{op}\,\overline{g}^{bq}\,\overline{g}^{ar}\right)\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}_{qr}\right)\\ &-\frac{7}{360\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\overline{g}^{cd}h_{cd}\right)\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\epsilon^{baef}\epsilon^{cdgh}\overline{F}_{ef}\overline{F}_{cd}\overline{F}_{gh}\right)\\ &\left.-\frac{7}{180\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\frac{1}{\sqrt{-\overline{g}}}\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\left(\overline{g}^{cd}h_{cd}\right)\epsilon^{baef}\epsilon^{cdgh}\overline{F}_{ef}\overline{F}_{cd}\overline{F}_{gh}\right)\right)\end{aligned} \tag{4.186}$$

The above equations mix photon-scalar dark matter through the $F\tilde{F}$ term (or terms proportional $\tilde{\lambda}$. The photon-graviton mixing should be symmetric like Dolgov and Ejili. What about graviton scalar mixing?

4.3 Riemann-Normal Equations of Motion

To do the mixing we consider a locally flat patch (local Minkowski space). A locally flat patch is needed as the axion-photon-gravity wave mixing requires knowing the orientation, e.g. the background electromagnetic fields [56]. Thus, a local frame of reference is required. To ensure local flatness (with a vanishing connection) [65] and account for curvature corrections in the patch, we consider Riemann-normal coordinates. We denote the local Minkowski coordinates with Greek indices in parenthesis.

4.3.1 Preliminaries: Kerr Geometry

The Kerr metric is defined as

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{r_s r}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2 + \left(r^2 + a^2 + \frac{r_s r\, a^2}{\Sigma}\sin^2\theta\right)\sin^2\theta\, d\phi^2 \\
&\quad - \frac{2\, r_s r a\, \sin^2\theta}{\Sigma}\, dt\, d\phi \\
&= \begin{pmatrix} dt & dr & d\theta' & d\phi' \end{pmatrix}
\begin{pmatrix}
-\left(1 - \frac{r_s r}{\Sigma}\right) & 0 & 0 & -\frac{r_s r a\, \sin^2\theta}{r_s\,\Sigma} \\
0 & \frac{\Sigma}{\Delta} & 0 & 0 \\
0 & 0 & \frac{\Sigma}{r_s^2} & 0 \\
-\frac{r_s r a\, \sin^2\theta}{r_a\,\Sigma} & 0 & 0 & \frac{1}{r_s^2}\left(r^2 + a^2 + \frac{r_s r\, a^2}{\Sigma}\sin^2\theta\right)\sin^2\theta
\end{pmatrix}
\begin{pmatrix} dt \\ dr \\ d\theta' \\ d\phi' \end{pmatrix}
\end{aligned}
\tag{4.187}
$$

where

$$r_s = 2\,G\,M \qquad\qquad \Sigma = r^2 + a^2\cos^2\theta \tag{4.188}$$

$$a = J/M \qquad\qquad \Delta = r^2 - r_s\,r + a^2\,, \tag{4.189}$$

$\hbar = c = 1$ and we rescaled the coordinates as

$$\theta = \frac{1}{r_s}(r_s\theta) = \frac{1}{r_s}\theta' \qquad\qquad \phi = \frac{1}{r_s}(r_s\phi) = \frac{1}{r_s}\phi' \tag{4.190}$$

so all of the coordinates have units of length.

4.3.2 Preliminaries: Riemann-Normal Coordinates

The Kerr metric is not locally flat metric, i.e. around a generic point $x_0 = \{r_0, \theta_0, \phi_0\}$ the metric does not reduce to the Minkowski metric. In the next subsection, we describe how to construct a locally flat metric from Kerr geometry. In this section, we verify the important computations of Riemann-Normal coordinates. Consider a local point x_0 and a nearby point x. We can define a coordinate system around a point x_0 so that

$$\xi^\mu = sa^\mu + \xi_0^\mu\,. \tag{4.191}$$

Here ξ^μ are the Riemann-Normal coordinates. a^μ and ξ_0 is just numbers. Notice we use Greek indices to denote the local Minkowski space. ξ_0 is the original local point in Riemann-Normal coordinates. It is clear from the geodesic equation

$$\left(\frac{d^2\xi^\mu}{ds^2} + \Gamma^\mu{}_{\alpha\beta}\frac{d\xi^\alpha}{ds}\frac{d\xi^\beta}{ds}\right)\bigg|_{\xi=\xi_0} = 0\,. \tag{4.192}$$

$$\Gamma^\mu{}_{\alpha\beta}\bigg|_{\xi=\xi_0}\frac{d\xi^\alpha}{ds}\frac{d\xi^\beta}{ds} = 0 \tag{4.193}$$

$\dot{\xi}^\alpha = d\xi^\alpha/ds$ are chosen to be linearly independent thereby implying

$$\Gamma^\mu{}_{\alpha\beta}\bigg|_{\xi=\xi_0} = 0. \tag{4.194}$$

As we describe below, it is easy to show that ξ^α can be chosen so that $g_{\mu\nu}(\xi_0) = \eta_{\mu\nu}$. Now consider the geodesic equation,

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{\ bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0\,. \tag{4.195}$$

where we are using general covariant coordinates (latin indices). Geodesics passing through the point x_0 (with $s = 0$) and initial four-velocity

$$x^a\big|_{s=0} = x_0^a \qquad\qquad \frac{dx^a}{ds}\bigg|_{s=0} = u_0^a \tag{4.196}$$

The second and third derivatives of the geodesic are

$$\begin{aligned}
\frac{d^2x^a}{ds^2}\bigg|_{s=0} &= \Gamma^a_{\ bc}\bigg|_{s=0} u_0^b u_0^c \\
\frac{d^3x^a}{ds^3}\bigg|_{s=0} &= -\left(\frac{d}{ds}\Gamma^a_{\ bc}\right)\frac{dx^b}{ds}\frac{dx^c}{ds}\bigg|_{s=0} - 2\Gamma^a_{\ bc}\frac{d^2x^b}{ds^2}\frac{dx^c}{ds}\bigg|_{s=0} \\
&= -\left(\partial_d\Gamma^a{}_{ef} - 2\Gamma^a{}_{gf}\Gamma^g{}_{de}\right)\bigg|_{s=0} u_0^d u_0^e u_0^f
\end{aligned} \tag{4.197}$$

where on the second line we used the second derivative equation on the first line. We have also assumed the standard symmetric connection. It is clear x is a function of the geodesic arc length, s. We can also expand x around $s = 0$ to get a solution to the geodesic equation. We can write

$$x^m(s) = x_0^m + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k x^m(s)}{\partial s^k}\bigg|_{s=0} s^k \tag{4.198}$$

$$= x_0^m + \frac{\partial x^m(s)}{\partial s}\bigg|_{s=0} s + \frac{1}{2}\frac{\partial^2 x^m(s)}{\partial s^2}\bigg|_{s=0} s^2 + \frac{1}{3!}\frac{\partial^3 x^m(s)}{\partial s^3}\bigg|_{s=0} s^3 + \dots \tag{4.199}$$

$$= x_0^m + s\, u_0^m - \frac{1}{2}\, s^2\, \Gamma^m_{\ ab}\bigg|_{s=0} u_0^a\, u_0^b - \frac{1}{3!}\, s^3\, \left(\partial_d \Gamma^m_{\ gb} - 2\,\Gamma^m_{\ ab}\Gamma^a_{\ dg}\right)\bigg|_{s=0} u_0^d\, u_0^g\, u_0^b + \dots$$

where at second and higher order we have the curvature corrections. The initial 4-velocity, u_0^a, are four linearly independent and orthonormal vectors at x_0^a so that

$$u_0^m = a^\mu e^m_{\ \ \mu} \qquad\qquad g_{ab}(x_0)\, e^a_{\ \alpha}\, e^b_{\ \beta} = \eta_{\alpha\beta} \tag{4.200}$$

where $e^\alpha_{\ a}$ is the tetrad (velbein) and the greek indices are the local Minkowski indices. The general covariant indices are the latin indices. We can rewrite the expansion the generic coordinates in terms of the Riemann-Normal coordinates. Using equation (4.191), we find

$$\begin{aligned} x^m(s) &= x_0^m + s\, a^\mu e^m_\mu - \frac{1}{2}\, s^2\, \Gamma^m_{\ ab}\bigg|_{s=0} a^\mu a^\nu e^a_\mu e^b_\nu \\ &\quad - \frac{1}{3!}\, s^3\, \left(\partial_d \Gamma^m_{\ gb} - 2\,\Gamma^m_{\ ab}\Gamma^a_{\ dg}\right)\bigg|_{s=0} a^\mu\, a^\nu\, a^\beta\, e^d_\mu\, e^g_\nu\, e^b_\beta + \dots \\ &= x_0^m + e^m_\mu\,(\xi^\mu - \xi_0^\mu) - \frac{1}{2}\,\Gamma^m_{\ ab}\bigg|_{s=0} e^a_\mu\, e^b_\nu\, (\xi^\mu - \xi_0^\mu)\,(\xi^\nu - \xi_0^\nu) \\ &\quad - \frac{1}{3!}\, \left(\partial_d \Gamma^m_{\ gb} - 2\,\Gamma^m_{\ ab}\Gamma^a_{\ dg}\right)\bigg|_{s=0} e^d_\mu\, e^g_\nu\, e^b_\beta\,(\xi^\mu - \xi_0^\mu)(\xi^\nu - \xi_0^\nu)(\xi^\beta - \xi_0^\beta) + \dots \end{aligned} \tag{4.201}$$

We can do a similar computation for a *locally flat* metric. The metric has the following Taylor expansion in Riemann-Normal coordinates

$$g_{\mu\nu}(\xi) = \eta_{\mu\nu} + \frac{1}{2}\frac{\partial^2 g_{\mu\nu}(\xi)}{\partial\xi^\alpha\partial\xi^\beta}\bigg|_{\xi=\xi_0}(\xi-\xi_0)^\alpha(\xi-\xi_0)^\beta + \ldots \tag{4.202}$$

Again, the Greek indices are local Minkowski indices. At the origin, ξ_0, we have Minkowski geometry and the first derivative term,

$$g_{\mu\nu}(\xi_0) = \eta_{\mu\nu} \qquad\qquad \frac{\partial g_{\mu\nu}(\xi)}{\partial\xi^\alpha}\bigg|_{\xi=\xi_0} = 0 \tag{4.203}$$

vanishes as one would expect for a locally flat metric. We construct the *locally flat* Kerr metric in the next section. We now work out the details of the second derivative term. We can rewrite the third derivative of the geodesic equation, in Riemann-Normal coordinates and we find

$$\frac{d^3\xi^\mu}{ds^3}\bigg|_{s=0} = -\left(\frac{d}{ds}\Gamma^\mu{}_{\alpha\beta}\right)\bigg|_{s=0}\frac{d\xi^\alpha}{ds}\frac{d\xi^\beta}{ds} - 2\,\Gamma^\mu{}_{\alpha\beta}\bigg|_{s=0}\frac{d^2\xi^\alpha}{ds^2}\frac{d\xi^\beta}{ds} \tag{4.204}$$

$$0 = \left(\partial_\delta\Gamma^\mu{}_{\beta\gamma}\right)\bigg|_{\xi=\xi_0}\frac{d\xi^\delta}{ds}\frac{d\xi^\beta}{ds}\frac{d\xi^\gamma}{ds}\,. \tag{4.205}$$

Equivalently, we have

$$\left(\partial_\delta\Gamma^\alpha{}_{\beta\gamma} + \partial_\beta\Gamma^\alpha{}_{\gamma\delta} + \partial_\gamma\Gamma^\alpha{}_{\delta\beta}\right)\bigg|_{\xi=\xi_0} = 0 \qquad \text{or} \qquad \partial_{(\delta}\Gamma^\alpha{}_{\beta\gamma)}\bigg|_{\xi=\xi_0} = 0 \tag{4.206}$$

For the RH equation, the indices are symmetric under exchange. Given the definition of the

Riemann tensor and knowing $\Gamma^{\alpha}{}_{\beta\gamma}(\xi_0) = 0$, we can write

$$\begin{aligned}
\left(R^{\alpha}{}_{\beta\gamma\delta} + R^{\alpha}{}_{\gamma\beta\delta)}\right)\Big|_{\xi=\xi_0} &= \left(\partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}{}_{\beta\gamma} + \partial_\beta \Gamma^{\alpha}{}_{\gamma\delta} - \partial_\delta \Gamma^{\alpha}{}_{\gamma\beta}\right)\Big|_{\xi=\xi_0} \\
&= \left(\partial_\gamma \Gamma^{\alpha}{}_{\beta\delta} + \partial_\beta \Gamma^{\alpha}{}_{\gamma\delta} - 2\,\partial_\delta \Gamma^{\alpha}{}_{\gamma\beta}\right)\Big|_{\xi=\xi_0} \\
&= -3\,\partial_\delta \Gamma^{\alpha}{}_{\gamma\beta}\Big|_{\xi=\xi_0} \\
-\frac{1}{3}\left(R^{\alpha}{}_{\beta\gamma\delta} + R^{\alpha}{}_{\gamma\beta\delta)}\right)\Big|_{\xi=\xi_0} &= \partial_\delta \Gamma^{\alpha}{}_{\gamma\beta}\Big|_{\xi=\xi_0}
\end{aligned} \tag{4.207}$$

where in the second to last step we use equation (4.206). We can also write for affine connections

$$\Gamma_{abc}(\xi) + \Gamma_{bac}(\xi) = \frac{1}{2}\left(g_{ab,c} + g_{ac,b} - g_{bc,a}\right) + \frac{1}{2}\left(g_{ba,c} + g_{bc,a} - g_{ac,b}\right) = g_{ab,c} \tag{4.208}$$

(4.209)

and

$$\partial_d \Gamma_{abc}(\xi) + \partial_d \Gamma_{bac}(\xi) = g_{ab,cd}\,. \tag{4.210}$$

Using equation (4.207), we can write

$$g_{\alpha\beta,\gamma\delta}\big|_{\xi=\xi_0} = -\frac{1}{3}\Big(R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\beta\delta} + R_{\beta\alpha\gamma\delta} + R_{\beta\gamma\alpha\delta}\Big)\big|_{\xi=\xi_0} \tag{4.211}$$

$$= -\frac{1}{3}\Big(R_{\alpha\gamma\beta\delta} + R_{\beta\gamma\alpha\delta}\Big)\big|_{\xi=\xi_0} \tag{4.212}$$

$$= -\frac{1}{3}\Big(R_{\alpha\gamma\beta\delta} + R_{\alpha\delta\beta\gamma}\Big)\big|_{\xi=\xi_0} \tag{4.213}$$

Thus, the expansion of the metric

$$g_{\mu\nu}(\xi) = \eta_{\mu\nu} - \frac{1}{6}\Big(R_{\mu\alpha\nu\beta}(\xi_0) + R_{\mu\beta\nu\alpha}(\xi_0)\Big)(\xi-\xi_0)^\alpha(\xi-\xi_0)^\beta + \dots \tag{4.214}$$

$$= \eta_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}(\xi_0)\ (\xi-\xi_0)^\alpha(\xi-\xi_0)^\beta + \dots \tag{4.215}$$

As similar computation generates the equivalent contravariant computation

$$g^{\mu\nu}(\xi) = \eta^{\mu\nu} + \frac{1}{3}g^{\nu\zeta}R^{\mu}{}_{\alpha\zeta\beta}(\xi_0)\ (\xi-\xi_0)^\alpha(\xi-\xi_0)^\beta + \dots \tag{4.216}$$

In the next subsection, we compute the locally flat metric in Kerr geometry around a point x_0. Other important equations include

$$\Gamma^{\mu}{}_{\alpha\beta,\nu} = -\frac{1}{3}\left(R^{\mu}{}_{\alpha\beta\nu} + R^{\mu}{}_{\beta\alpha\nu}\right) \tag{4.217}$$

which is derived from the equations in (4.206). We can also write

$$\Gamma^{\mu}{}_{\alpha\beta}(\xi) = -\frac{1}{3}\left(R^{\mu}{}_{\alpha\beta\nu} + R^{\mu}{}_{\beta\alpha\nu}\right)(\xi-\xi_0)^\nu + \dots \tag{4.218}$$

4.3.2 Constructing a Locally Flat Metric from Kerr Geometry

To find $g_{\mu\nu}(\xi)$ for Kerr geometry, we need to compute $R_{\beta\gamma\alpha\delta}(\xi_0)$. We can write

$$R_{\alpha\beta\delta\gamma}(\xi_0) = \left(\frac{\partial x^a}{\partial\xi^\alpha}\frac{\partial x^b}{\partial\xi^\beta}\frac{\partial x^c}{\partial\xi^\gamma}\frac{\partial x^d}{\partial\xi^\delta}\right)\Bigg|_{s=0} R_{acbd}(x_0) \tag{4.219}$$

$$= \left(\left(\frac{ds}{\partial\xi^\alpha}\frac{\partial x^a}{ds}\right)\left(\frac{ds}{\partial\xi^\beta}\frac{\partial x^b}{ds}\right)\left(\frac{ds}{\partial\xi^\gamma}\frac{\partial x^c}{ds}\right)\left(\frac{ds}{\partial\xi^\delta}\frac{\partial x^d}{ds}\right)\right)\Bigg|_{s=0} R_{acbd}(x_0) \tag{4.220}$$

$$= e^a{}_{(\alpha)} e^b{}_{(\beta)} e^c{}_{(\gamma)} e^d{}_{(\delta)}\, R_{acbd}(x_0) \tag{4.221}$$

where we have used equation (4.200). For Kerr geometry, we have the following for the tetrad vector $e^{(\lambda)}_{\ m}$

$$e^{(0)}_{\ m}\, dx^m = \sqrt{\frac{\Delta}{\Sigma}}\,\left(dt - a\sin^2\theta\, d\phi\right) \tag{4.222}$$

$$e^{(1)}_{\ m}\, dx^m = \sqrt{\frac{\Sigma}{\Delta}}\, dr \tag{4.223}$$

$$e^{(2)}_{\ m}\, dx^m = \sqrt{\Sigma}\, d\theta \tag{4.224}$$

$$e^{(3)}_{\ m}\, dx^m = \frac{\sin\theta}{\sqrt{\Sigma}}\,\left(-a\, dt + (r^2+a^2)\, d\phi\right) \tag{4.225}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta \tag{4.226}$$

$$\Delta = r^2 + a^2 - 2Mr \tag{4.227}$$

We used a simple Mathematica program to compute

$$R_{\alpha\beta\delta\gamma}(\xi_0) = e^a{}_{(\alpha)} e^b{}_{(\beta)} e^c{}_{(\gamma)} e^d{}_{(\delta)}\, g_{ae} R^e{}_{cbd}(x_0) \tag{4.228}$$

$$= e^a{}_{(\alpha)} e^b{}_{(\beta)} e^c{}_{(\gamma)} e^d{}_{(\delta)}\, e_{a(\sigma)} e_e{}^{(\sigma)}\, R^e{}_{cbd}(x_0) \tag{4.229}$$

As an example, we can choose $x_0 = \{t_0, r_0, \theta_0 = \pi/2, \phi_0\}$ as well as a small but non-trivial, we get the following components of $R^e{}_{cbd}(x_0)$,

$$R^1{}_{221}(x_0) = R^1{}_{331}(x_0) = -R^0{}_{202}(x_0) = -R^0{}_{303}(x_0) = \frac{M}{r_0} \tag{4.230}$$

$$R^1{}_{301}(x_0) = R^1{}_{031}(x_0) = \frac{2aM(4M - 3\,r_0)}{r_0^4} \tag{4.231}$$

$$R^2{}_{302}(x_0) = R^2{}_{032}(x_0) = \frac{2aM(-2M + r_0)}{r_0^4} \tag{4.232}$$

$$R^3{}_{101}(x_0) = -R^0{}_{131}(x_0) = \frac{6aM}{r_0^4(r_0 - 2M)} \tag{4.233}$$

$$R^2{}_{121}(x_0) = R^3{}_{131}(x_0) = -\frac{M}{r_0^2(r_0 - 2M)} \tag{4.234}$$

$$R^3{}_{303}(x_0) = -R^0{}_{003}(x_0) = -\frac{4aM^2}{r_0^4} \tag{4.235}$$

$$R^2{}_{332}(x_0) = -R^3{}_{232}(x_0) = \frac{-2M + r_0}{r_0} \tag{4.236}$$

$$R^2{}_{002}(x_0) = -R^3{}_{003}(x_0) = -\frac{M(-2M + r_0)}{r_0^4} \tag{4.237}$$

$$R^1{}_{001}(x_0) = \frac{2M(-2M + r_0)}{r_0^4} \tag{4.238}$$

$$R^3{}_{202}(x_0) = -\frac{2aM}{r_0^3} \tag{4.239}$$

$$R^0{}_{101}(x_0) = \frac{2M}{r_0^2(-2M + r_0)} \tag{4.240}$$

$$R^3{}_{202}(x_0) = -\frac{6aM}{r_0} \tag{4.241}$$

Recall, a has units of r and the Schwartzschild radius is $r_s = 2M$ in these units. The curvature corrections go as $\mathcal{O}(1/n)$ where n is the number of Schwartzchild radii x_0 is from the event horizon. The key point is as the particle goes from patch to patch the curvature corrections become more important and alter the local Minkowski space computations.

4.3.2 Other Important Equations

The gravitational wave equation of motion in Riemann-Normal coordinates is

$$-\frac{1}{2}\overline{\Box}\, h_{\beta\delta} = \overline{R}^{\sigma}{}_{\delta\alpha\beta}\, h^{\alpha}_{\sigma} + \kappa\bigg(\frac{\alpha}{\epsilon}\, T^{(1,0)}_{\alpha\beta} + T^{(0,1)}_{\alpha\beta} + \alpha\, T^{(1,1)}_{\alpha\beta} - \frac{1}{2}\, g_{ab}\left(\frac{\alpha}{\epsilon}\, T^{(1,0)} + T^{(0,1)} + \alpha\, T^{(1,1)} \right)\bigg) \tag{4.242}$$

where

$$\overline{\Box}\, h_{\beta\delta} = \left(\overline{g}^{\xi\kappa}\partial_{\xi}\partial_{\kappa} - \overline{g}^{\xi\kappa}\overline{\Gamma}^{\lambda}{}_{\xi\kappa}\partial_{\lambda}\right) h_{\beta\delta} \tag{4.243}$$

$$= \eta^{\xi\kappa}\, \partial_{\xi}\partial_{\kappa} h_{\beta\delta} + \frac{1}{3}\, g^{\kappa\zeta}\, R^{\xi}{}_{\alpha'\zeta\beta'}(\xi_0)\; (\xi-\xi_0)^{\alpha'}(\xi-\xi_0)^{\beta'}\, (\partial_{\xi}\partial_{\kappa} h_{\beta\delta}) + \ldots \tag{4.244}$$

$$- \left(\eta^{\xi\kappa} + \frac{1}{3}\, g^{\kappa\zeta}\, R^{\xi}{}_{\alpha\zeta\beta}(\xi_0)\; (\xi-\xi_0)^{\alpha}(\xi-\xi_0)^{\beta} + \ldots\right)\left(-\frac{1}{3}\left(\overline{R}^{\lambda}{}_{\xi\kappa\nu}(\xi_0)\right.\right.$$

$$+ \overline{R}^{\lambda}{}_{\kappa\xi\nu}(\xi_0)(\xi-\xi_0)^{\nu} + \ldots\; \partial_{\lambda} h_{\beta\delta}$$

$$= \eta^{\xi\kappa}\, \partial_{\xi}\partial_{\kappa} h_{\beta\delta} + \frac{1}{3}\, \eta^{\xi\kappa}\big(\overline{R}^{\lambda}{}_{\xi\kappa\nu}(\xi_0)\; + \overline{R}^{\lambda}{}_{\kappa\xi\nu}(\xi_0)\big)\big(\,\partial_{\lambda} h_{\beta\delta}\,\big)\, (\xi-\xi_0)^{\nu} \tag{4.245}$$

$$+ \frac{1}{3}\, g^{\kappa\zeta}\, R^{\xi}{}_{\alpha'\zeta\beta'}(\xi_0)\big(\partial_{\xi}\partial_{\kappa} h_{\beta\delta}\big)\; (\xi-\xi_0)^{\alpha'}(\xi-\xi_0)^{\beta'}$$

$$+ \frac{1}{9}\, g^{\kappa\zeta}\, R^{\xi}{}_{\alpha\zeta\beta}(\xi_0)\,\big(\,\overline{R}^{\lambda}{}_{\xi\kappa\nu}(\xi_0)\; + \overline{R}^{\lambda}{}_{\kappa\xi\nu}(\xi_0)\,\big)\big(\,\partial_{\lambda} h_{\beta\delta}\,\big)\, (\xi-\xi_0)^{\alpha}(\xi-\xi_0)^{\beta}(\xi-\xi_0)^{\nu} + \ldots$$

$$
\begin{aligned}
\overline{R}^{\sigma}{}_{\delta\alpha\beta}(\xi) &= \partial_\alpha \Gamma^{\sigma}{}_{\beta\delta}(\xi) - \partial_\delta \Gamma^{\sigma}{}_{\beta\alpha}(\xi) + \Gamma^{\iota}{}_{\beta\delta}(\xi)\Gamma^{\sigma}{}_{\iota\alpha}(\xi) - \Gamma^{\iota}{}_{\beta\alpha}(\xi)\Gamma^{\sigma}{}_{\iota\delta}(\xi) && (4.246)\\
&= -\frac{1}{3}\,\partial_\alpha\Big(\Big(R^{\sigma}{}_{\delta\beta\nu}(\xi_0) + R^{\sigma}{}_{\beta\delta\nu}(\xi_0)\Big)\,(\xi-\xi_0)^\nu\Big)\\
&+ \frac{1}{3}\,\partial_\delta\Big(\Big(R^{\sigma}{}_{\alpha\beta\nu}(\xi_0) + R^{\sigma}{}_{\beta\alpha\nu}(\xi_0)\Big)\,(\xi-\xi_0)^\nu\Big) && (4.247)\\
&+ \frac{1}{9}\left(R^{\iota}{}_{\delta\beta\nu} + R^{\iota}{}_{\beta\delta\nu}\right)\left(R^{\sigma}{}_{\iota\alpha\mu} + R^{\sigma}{}_{\alpha\iota\mu}\right)(\xi-\xi_0)^\nu\,(\xi-\xi_0)^\mu\\
&- \frac{1}{9}\left(\,R^{\iota}{}_{\alpha\beta\nu} + R^{\iota}{}_{\beta\alpha\nu}\,\right)\left(\,R^{\sigma}{}_{\iota\delta\mu} + R^{\sigma}{}_{\delta\iota\mu}\,\right)(\xi-\xi_0)^\mu\,(\xi-\xi_0)^\nu\\
&= \frac{1}{3}\Big(R^{\sigma}{}_{\alpha\beta\delta}(\xi_0) + R^{\sigma}{}_{\beta\alpha\delta}(\xi_0) - R^{\sigma}{}_{\delta\beta\alpha}(\xi_0) - R^{\sigma}{}_{\beta\delta\alpha}(\xi_0)\Big) && (4.248)\\
&+ \frac{1}{9}\Big(\left(\,R^{\iota}{}_{\delta\beta\nu}(\xi_0) + R^{\iota}{}_{\beta\delta\nu}(\xi_0)\,\right)\left(\,R^{\sigma}{}_{\iota\alpha\mu}(\xi_0) + R^{\sigma}{}_{\alpha\iota\mu}(\xi_0)\,\right)\\
&- \left(R^{\iota}{}_{\alpha\beta\nu}(\xi_0) + R^{\iota}{}_{\beta\alpha\nu}(\xi_0)\right)\left(R^{\sigma}{}_{\iota\delta\mu}(\xi_0) + R^{\sigma}{}_{\delta\iota\mu}(\xi_0)\right)\Big)(\xi-\xi_0)^\nu\,(\xi-\xi_0)^\mu + \ldots\\
&= R^{\sigma}{}_{\beta\alpha\delta}(\xi_0) + \frac{1}{9}\Big(\left(\,R^{\iota}{}_{\delta\beta\nu}(\xi_0) + R^{\iota}{}_{\beta\delta\nu}(\xi_0)\,\right)\left(\,R^{\sigma}{}_{\iota\alpha\mu}(\xi_0) + R^{\sigma}{}_{\alpha\iota\mu}(\xi_0)\,\right) && (4.249)\\
&- \left(R^{\iota}{}_{\alpha\beta\nu}(\xi_0) + R^{\iota}{}_{\beta\alpha\nu}(\xi_0)\right)\left(R^{\sigma}{}_{\iota\delta\mu}(\xi_0) + R^{\sigma}{}_{\delta\iota\mu}(\xi_0)\right)\Big)(\xi-\xi_0)^\nu\,(\xi-\xi_0)^\mu + \ldots
\end{aligned}
$$

Before moving on, we note the leading term is just $R^{\sigma}{}_{\beta\alpha\delta}(\xi_0) = \partial_\alpha \Gamma^{\sigma}{}_{\beta\delta}(\xi_0) - \partial_\delta \Gamma^{\sigma}{}_{\beta\alpha}(\xi_0)$.

From now on we keep all the term to first order in $(\xi - \xi_0)^\alpha$. We can write out the equations of motion

$$
\begin{aligned}
-\frac{1}{2}\,\eta^{\xi\kappa}\,\partial_\xi\partial_\kappa h_{\beta\delta} &= \frac{1}{6}\,\eta^{\xi\kappa}\big(\,\overline{R}^{\lambda}{}_{\xi\kappa\nu}(\xi_0)\;+\overline{R}^{\lambda}{}_{\kappa\xi\nu}(\xi_0)\,\big)\big(\,\partial_\lambda h_{\beta\delta}\,\big)\,(\xi-\xi_0)^\nu + \overline{R}^{\sigma}{}_{\delta\alpha\beta}(\xi_0)\,h^{\alpha}_{\ \sigma} \\
&\quad + \kappa\bigg(\,\frac{\alpha}{\epsilon}\,T^{(1,0)}_{\beta\delta} + T^{(0,1)}_{\beta\delta} + \alpha\,T^{(1,1)}_{\beta\delta} - \frac{1}{2}\,\eta_{\beta\delta}\Big(\,\frac{\alpha}{\epsilon}\,T^{(1,0)} + T^{(0,1)} + \alpha\,T^{(1,1)}\,\Big)\bigg)
\end{aligned}
\tag{4.250}
$$

$$
\begin{aligned}
\big(\eta^{\alpha\beta}\partial_\alpha\partial_\beta - m^2\big)\,\phi^{(1)} &= \frac{\epsilon}{\alpha}\,\big(\,\partial_\alpha\partial_\beta\,\overline{\phi}\,\big)\,h^{\alpha\beta} + \epsilon\,h^{\alpha\beta}\big(\partial_\alpha\partial_\beta\,\phi^{(1)}\big) \\
&\quad - \frac{1}{3}\eta^{\alpha\beta}\,\big(\overline{R}^{\mu}{}_{\alpha\beta\nu}(\xi_0) + \overline{R}^{\mu}{}_{\beta\alpha\nu}(\xi_0)\big)\,\big(\partial_\mu\phi^{(1)}\big)\,(\xi-\xi_0)^\nu \\
&\quad + \frac{\epsilon}{\alpha}\,\frac{1}{3}\,\big(\overline{R}^{\mu}{}_{\alpha\beta\nu}(\xi_0) + \overline{R}^{\mu}{}_{\beta\alpha\nu}(\xi_0)\big)\,\big(\,\partial_\mu\overline{\phi}\,\big)\,h^{\alpha\beta}\,(\xi-\xi_0)^\nu \\
&\quad + \frac{1}{3}\epsilon\,h^{\alpha\beta}\,\big(\overline{R}^{\mu}{}_{\alpha\beta\nu}(\xi_0) + \overline{R}^{\mu}{}_{\beta\alpha\nu}(\xi_0)\big)\,(\xi-\xi_0)^\nu\,\big(\partial_\mu\phi^{(1)}\big) \\
&\quad + \frac{\tilde{\lambda}\,\alpha}{4\sqrt{-\overline{g}}}\,\epsilon^{abcd}\,\overline{F}_{ab}F^{(1)}_{cd}
\end{aligned}
\tag{4.251}
$$

$$
\begin{aligned}
\big(\partial_a + \overline{\Gamma}^{e}{}_{ae}\big)\,F^{(1)\,ab} &= \frac{\epsilon}{\alpha}\,\bigg(\,\partial_a\Big(\big(\overline{g}^{ac}\,h^{bd} + \overline{g}^{bd}\,h^{ac}\big)\,\overline{F}_{cd}\Big) + \big(\overline{g}^{ac}\,h^{bd} + h^{ac}\,\overline{g}^{bd}\big)\,\overline{\Gamma}^{e}{}_{ae}\overline{F}_{cd}\bigg) \\
&\quad + \epsilon\,\bigg(\,\partial_a\Big(\big(\overline{g}^{ac}\,h^{bd} + \overline{g}^{bd}\,h^{ac}\big)\,F^{(1)}_{cd}\Big)\bigg) + j^{(1)\,b} + \frac{\epsilon}{\alpha}\,j' \\
&\quad + \overline{\nabla}_a P^{(1)\,ab} + \frac{\epsilon}{\alpha}\,\overline{\nabla}_a P'^{ab}
\end{aligned}
\tag{4.252}
$$

If we average the background fields by a suitable averaging mechanism and if the background fields are indeed static on the length scale of interest, we have:

$$\frac{1}{2}\,\eta^{\xi\kappa}\,\partial_\xi\partial_\kappa h_{\beta\delta} = -\Big\langle\,\overline{R}^{\sigma}{}_{\delta\alpha\beta}(\xi_0)\,\Big\rangle\,h^{\alpha}{}_{\sigma} - \frac{1}{6}\Big\langle\,\eta^{\xi\kappa}\big(\,\overline{R}^{\lambda}{}_{\xi\kappa\nu}(\xi_0)\;+\overline{R}^{\lambda}{}_{\kappa\xi\nu}(\xi_0)\,\big)\,(\xi-\xi_0)^{\nu}\,\Big\rangle\,\partial_\lambda h_{\beta\delta}$$

$$- \kappa\bigg(\,\frac{\alpha}{\epsilon}\,T^{(1,0)}_{\beta\delta} + T^{(0,1)}_{\beta\delta} - \frac{1}{2}\,\eta_{\beta\delta}\Big(\,\frac{\alpha}{\epsilon}\,T^{(1,0)} + T^{(0,1)}\,\Big)\bigg) \tag{4.253}$$

where

$$\begin{aligned}
T^{(1,0)}_{\alpha\beta} &= \Big\langle\,\overline{F}_{\beta}{}^{\mu}\,\Big\rangle\,F^{(1)}_{\alpha\mu} + \Big\langle\,\overline{F}_{\alpha}{}^{\mu}\,\Big\rangle\,F^{(1)}_{\beta\mu} - \frac{1}{2}\,\eta_{\alpha\beta}\Big\langle\,\overline{F}_{np}\,\Big\rangle\,F^{(1)\,np} - \eta_{\alpha\beta}\,m^2\,\Big\langle\,\overline{\phi}\,\Big\rangle\,\phi^{(1)} \\
&- \frac{4}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\bigg(\Big\langle\,\overline{F}^{np}\,\overline{F}_{\alpha e}\,\overline{F}_{\beta}{}^{e}\,\Big\rangle\,F^{(1)}_{np} + \Big\langle\,\overline{F}_{np}\,\overline{F}^{np}\,\overline{F}_{\beta}{}^{e}\,\Big\rangle\,F^{(1)}_{\alpha e} \\
&- \frac{1}{2}\,\eta_{ab}\,\Big(\Big\langle\,\overline{F}_{qs}\,\overline{F}^{2}\,\Big\rangle\,F^{(1)\,qs} - \frac{7}{4}\,\epsilon^{hijk}\epsilon^{defg}\,\Big\langle\,\overline{F}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi}\,\Big\rangle\,F^{(1)}_{jk}\,\Big)\bigg)
\end{aligned}$$

$$\begin{aligned}
T^{(0,1)}_{\alpha\beta} &= -h^{cd}\,\Big\langle\,\overline{F}_{\alpha c}\overline{F}_{\beta d}\,\Big\rangle - h_{ab}\Big\langle\,\frac{1}{4}\,\overline{F}^{2} + \frac{1}{2}m^2\,\overline{\phi}^{2}\,\Big\rangle + \eta_{\alpha\beta}\Big\langle\,\frac{1}{2}\,\eta^{op}\,\overline{F}_{np}\,\overline{F}_{mo}\,\Big\rangle\,h^{mn} \\
&+ h_{ab}\bigg(\,\frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\Big\langle\,\overline{F}^{4} - \frac{7}{16}\,\big(\epsilon^{mnop}\overline{F}_{mn}\overline{F}_{op}\big)^2\,\Big\rangle\bigg) \\
&- \frac{4\,\eta_{ab}}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\,h^{mn}\,\eta^{op}\,\eta^{qr}\,\eta^{st}\,\Big\langle\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}^{2}\Big\rangle \\
&+ \frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\Big(h^{cd}\Big\langle\,\overline{F}_{ac}\,\overline{F}_{bd}\,\overline{F}^{2}\Big\rangle + 2\,h^{mn}\,\eta^{cd}\,\eta^{op}\,\Big\langle\overline{F}_{ac}\,\overline{F}_{bd}\,\overline{F}_{mo}\,\overline{F}_{np}\Big\rangle\Big)
\end{aligned}$$

Notice the energy-momentum tensor features transitions between graviton-photon and graviton-dark matter.

$$\begin{aligned}\left(\eta^{\alpha\beta}\partial_\alpha\partial_\beta - m^2\right)\phi^{(1)} &= -\frac{1}{3}\left\langle \eta^{\alpha\beta}\ \left(\overline{R}^{\mu}{}_{\alpha\beta\nu}(\xi_0) + \overline{R}^{\mu}{}_{\beta\alpha\nu}(\xi_0)\right)(\xi-\xi_0)^\nu \right\rangle \partial_\mu\phi^{(1)} \\ &+ \frac{\epsilon}{\alpha}\,\frac{1}{3}\left(\overline{R}^{\mu}{}_{\alpha\beta\nu}(\xi_0) + \overline{R}^{\mu}{}_{\beta\alpha\nu}(\xi_0)\right)\left(\partial_\mu\overline{\phi}\right) h^{\alpha\beta}\,(\xi-\xi_0)^\nu \\ &+ \frac{1}{3}\epsilon\, h^{\alpha\beta}\ \left(\overline{R}^{\mu}{}_{\alpha\beta\nu}(\xi_0) + \overline{R}^{\mu}{}_{\beta\alpha\nu}(\xi_0)\right)(\xi-\xi_0)^\nu\ \left(\partial_\mu\phi^{(1)}\right) \\ &+ \frac{\tilde{\lambda}\,\alpha}{4\sqrt{-\overline{g}}}\,\epsilon^{abcd}\,\overline{F}_{ab}F^{(1)}_{cd}\end{aligned} \tag{4.254}$$

where

$$T^{(1,1)}_{\alpha\beta} = -h^{ef}\left(F^{(1)}_{ae}\,\overline{F}_{bf} + \overline{F}_{ae}\,F^{(1)}_{bf}\right) + \frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\Bigg(2\,h^{ef}\eta^{mn}\eta^{op}\,\overline{F}^{(1)}_{np}\,\overline{F}_{mo}\,\overline{F}_{ae}\,\overline{F}_{bf} \tag{4.255}$$

$$\begin{aligned}&+ h^{ef}\,\overline{F}^2\left(\overline{F}^{(1)}_{ae}\,\overline{F}_{bf} + \overline{F}_{ae}\,\overline{F}^{(1)}_{bf}\right) + 2\,h^{mn}\,\eta^{ef}\,\eta^{op}\ \left(F^{(1)}_{np}\,\overline{F}_{mo} + \overline{F}_{np}\,\overline{F}^{(1)}_{mo}\right)\overline{F}_{ae}\,\overline{F}_{bf} \\ &+ 2\,h^{mn}\,\eta^{ef}\,\eta^{op}\,\overline{F}_{np}\,\overline{F}_{mo}\ \left(F^{(1)}_{ae}\,\overline{F}_{bf} + \overline{F}_{ae}\,F^{(1)}_{bf}\right)\Bigg) - h_{ab}\Bigg(\frac{1}{4}\,\eta^{mn}\eta^{op}\left(\overline{F}_{np}\,F^{(1)}_{mo} + \overline{F}_{mo}\,F^{(1)}_{np}\right) \\ &+ \eta^{mn}\,\partial_m\overline{\phi}\,\partial_n\phi^{(1)} + m^2\,\overline{\phi}\,\phi^{(1)} - \frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\Bigg(\eta^{mn}\,\eta^{op}\,\eta^{qr}\,\eta^{st}\,F^{(1)}_{rt}\,\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}_{qs} \\ &- \frac{7}{16}\,\epsilon^{hijk}\epsilon^{defg}\,F^{(1)}_{jk}\,\overline{F}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi}\Bigg) - \eta_{ab}\Bigg(-\frac{1}{2}\,h^{mn}g^{op}\left(\overline{F}_{np}\,F^{(1)}_{mo} + \overline{F}_{mo}\,F^{(1)}_{np}\right) \\ &- h^{mn}\,\partial_m\overline{\phi}\,\partial_n\phi^{(1)} + \frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 h^{mn}\,g^{op}\,g^{qr}\,g^{st}\ \left(F^{(1)}_{rt}\,\overline{F}_{mo} + \overline{F}_{rt}\,F^{(1)}_{mo}\right)\overline{F}_{np}\,\overline{F}_{qs}\Bigg)\end{aligned}$$

and

$$j^{(1)\,b} = \frac{\tilde{\lambda}}{2}\left(\epsilon^{abcd}\partial_a\left(\overline{\phi}\,F^{(1)}_{cd}\right) + \epsilon^{abcd}\partial_a\left(\phi^{(1)}\,\overline{F}_{cd}\right)\right) \tag{4.256}$$

$$\begin{aligned}\overline{\nabla}_a P^{(1)\,ab} &= \frac{8}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \partial_a\left(\eta^{mn}\,\eta^{op}\,\eta^{bq}\,\eta^{ar}\,F^{(1)}_{mo}\,\overline{F}_{np}\,\overline{F}_{qr}\right) \\ &+ \frac{4}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \partial_a\left(\eta^{mn}\,\eta^{op}\,\eta^{bq}\,\eta^{ar}\overline{F}_{mo}\,\overline{F}_{np}\,F^{(1)}_{qr}\right) \\ &+ \frac{14}{180\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \partial_a\left(\epsilon^{baef}\epsilon^{cdgh}\,\overline{F}_{ef}\,\overline{F}_{cd}\,F^{(1)}_{gh}\right) \\ &+ \frac{7}{180\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \partial_a\left(\epsilon^{baef}\epsilon^{cdgh}\,F^{(1)}_{ef}\,\overline{F}_{cd}\,\overline{F}_{gh}\right)\end{aligned} \tag{4.257}$$

$$\begin{aligned}\overline{\nabla}_a P'^{ab} &= \frac{1}{\sqrt{-\eta}}\left(\frac{2}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \partial_a\left(\left(\eta^{cd}h_{cd}\right)\overline{F}^{ba}\,\overline{F}^2\right)\right. \\ &- \frac{4}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \partial_a\left(\left(\eta^{mn}\eta^{op}\left(\eta^{bq}\,h^{ar} + \eta^{ar}\,h^{bq}\right) + 2\,h^{mn}\eta^{op}\,\eta^{bq}\,\eta^{ar}\right)\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}_{qr}\right) \\ &- \frac{7}{360\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \left(\eta^{cd}h_{cd}\right)\partial_a\left(\epsilon^{baef}\epsilon^{cdgh}\overline{F}_{ef}\overline{F}_{cd}\overline{F}_{gh}\right) \\ &\left.- \frac{7}{180\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \partial_a\left(\left(\eta^{cd}h_{cd}\right)\epsilon^{baef}\epsilon^{cdgh}\overline{F}_{ef}\overline{F}_{cd}\overline{F}_{gh}\right)\right)\end{aligned} \tag{4.258}$$

We can now set $F^{(1)}_{\alpha\beta} = F_{\alpha\beta}$ and $\phi^{(1)} = \phi$. After averaging, the high-frequency equation of motions are

$$\frac{1}{2}\,\eta^{\xi\kappa}\,\partial_\xi\partial_\kappa h_{\beta\delta} = -\Big\langle\,\overline{R}^{\sigma}{}_{\delta\alpha\beta}(\xi_0)\,x\Big\rangle\,h^{\alpha}{}_{\sigma} - \frac{1}{6}\Big\langle\eta^{\xi\kappa}\big(\,\overline{R}^{\lambda}{}_{\xi\kappa\nu}(\xi_0)\;+\overline{R}^{\lambda}{}_{\kappa\xi\nu}(\xi_0)\,\big)(\xi-\xi_0)^{\nu}\Big\rangle\,\partial_\lambda h_{\beta\delta} \tag{4.259}$$

$$\begin{aligned}
&- \frac{\kappa\,\alpha}{\epsilon}\,\Bigg(\langle\overline{F}_{\delta}{}^{\mu}\rangle\,F^{(1)}_{\beta\mu} + \langle\overline{F}_{\beta}{}^{\mu}\rangle\,F^{(1)}_{\delta\mu} + \langle\,\partial_\delta\overline{\phi}\,\rangle\,\partial_\beta\phi^{(1)} + \langle\,\partial_\beta\overline{\phi}\,\rangle\partial_\delta\phi^{(1)} \\
&- \eta_{\beta\gamma}\bigg(\,\frac{1}{2}\,\langle\,\overline{F}_{\alpha\beta}\,\rangle\,F^{(1)\,\alpha\beta} + \langle\,\partial_m\overline{\phi}\,\rangle\,\partial^m\phi^{(1)} + m^2\,\langle\,\overline{\phi}\,\rangle\,\phi^{(1)}\bigg) \\
&- \frac{4}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\bigg(\Big\langle\overline{F}^{np}\,\overline{F}_{\alpha e}\,\overline{F}_{\beta}{}^{e}\Big\rangle\,F^{(1)}_{np} + \Big\langle\overline{F}_{np}\,\overline{F}^{np}\,\overline{F}_{\beta}{}^{e}\Big\rangle\,F^{(1)}_{\alpha e} \\
&- \frac{1}{2}\,\eta_{ab}\,\Big(\Big\langle\overline{F}_{qs}\,\overline{F}^{2}\Big\rangle F^{(1)\,qs} - \frac{7}{4}\,\epsilon^{hijk}\epsilon^{defg}\,\Big\langle\overline{F}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi}\Big\rangle\,F^{(1)}_{jk}\,\Big)\,\bigg)\Bigg) \\
&- \kappa\,T^{(0,1)}_{\beta\delta}\bigg(-h^{cd}\,\overline{F}_{ac}\overline{F}_{bd} - h_{ab}\bigg(\,\frac{1}{4}\,\overline{F}^2 + \frac{1}{2}\,\Big(\eta^{mn}\,\partial_m\overline{\phi}\,\partial_n\overline{\phi} + m^2\,\overline{\phi}^2\Big)\bigg) \\
&- \eta_{ab}\bigg(-\frac{1}{2}\,h^{mn}\eta^{op}\,\overline{F}_{np}\,\overline{F}_{mo} - \frac{1}{2}\,h^{mn}\,\partial_m\overline{\phi}\,\partial_n\overline{\phi}\bigg) \\
&+ h_{ab}\bigg(\,\frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\Big(\,\overline{F}^4 - \frac{7}{16}\,\big(\epsilon^{mnop}\overline{F}_{mn}\overline{F}_{op}\big)^2\,\Big)\bigg) \\
&- \eta_{ab}\bigg(\,\frac{4}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\,h^{mn}\,\eta^{op}\,\eta^{qr}\,\eta^{st}\,\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}^2\bigg) \\
&+ \frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\Big(h^{cd}\,\overline{F}_{ac}\,\overline{F}_{bd}\,\overline{F}^2 + 2\,\eta^{cd}\,h^{mn}\eta^{op}\,\overline{F}_{ac}\,\overline{F}_{bd}\,\overline{F}_{mo}\,\overline{F}_{np}\,\Big)\bigg) \\
&+ \frac{\kappa}{2}\,\eta_{\beta\delta}\,\frac{\alpha}{\epsilon}\,T^{(1,0)} \\
&+ \frac{\kappa}{2}\,\eta_{\beta\delta}T^{(0,1)}
\end{aligned} \tag{4.260}$$

$$= \tag{4.261}$$

$$\left(\eta^{\alpha\beta}\partial_\alpha\partial_\beta - m^2\right)\phi^{(1)} = \epsilon\, h^{\alpha\beta}\left(\partial_\alpha\partial_\beta\,\phi^{(1)}\right) - \frac{1}{3}\eta^{\alpha\beta}\,\left(\overline{R}^{\mu}{}_{\alpha\beta\nu}(\xi_0) + \overline{R}^{\mu}{}_{\beta\alpha\nu}(\xi_0)\right)\,\left(\partial_\mu\phi^{(1)}\right)(\xi-\xi_0)^\nu$$
$$+\frac{1}{3}\epsilon\left(\overline{R}^{\mu}{}_{\alpha\beta\nu}(\xi_0) + \overline{R}^{\mu}{}_{\beta\alpha\nu}(\xi_0)\right)\left(\partial_\mu\phi^{(1)}\right)h^{\alpha\beta}\,(\xi-\xi_0)^\nu$$
$$+\frac{\tilde{\lambda}\,\alpha}{4}\,\epsilon^{\alpha\beta\nu\delta}\,\overline{F}_{\alpha\beta}F^{(1)}_{\nu\delta}$$

$$\partial_\alpha F^{(1)\,\alpha\beta} = \frac{\epsilon}{\alpha}\left(\eta^{\alpha\gamma}\,\partial_\alpha h^{\beta\delta} + \eta^{\beta\delta}\,\partial_\alpha h^{\alpha\gamma}\right)\overline{F}_{\gamma\delta} + \epsilon\left(\eta^{\alpha\gamma}\,\partial_\alpha h^{\beta\delta} + \eta^{\beta\delta}\,\partial_\alpha h^{\alpha\gamma}\right)F^{(1)}_{\gamma\delta} \tag{4.262}$$
$$+\epsilon\left(\eta^{\alpha\gamma}\,h^{\beta\delta} + \eta^{\beta\delta}\,h^{\alpha\gamma}\right)\partial_\alpha F^{(1)}_{\gamma\delta} + \frac{1}{3}\left(\overline{R}^{\mu}{}_{\alpha\mu\nu}(\xi_0) + \overline{R}^{\mu}{}_{\mu\alpha\nu}(\xi_0)\right)F^{(1)\,\alpha\beta}\,(\xi-\xi_0)^\nu$$
$$-\frac{\epsilon}{\alpha}\frac{1}{3}\left(\eta^{\alpha\gamma}\,h^{\beta\delta} + h^{\alpha\gamma}\,\eta^{\beta\delta}\right)\left(\overline{R}^{\mu}{}_{\alpha\mu\nu}(\xi_0) + \overline{R}^{\mu}{}_{\mu\alpha\nu}(\xi_0)\right)\overline{F}_{\gamma\delta}\,(\xi-\xi_0)^\nu$$
$$+ j^{(1)\,b} + \frac{\epsilon}{\alpha}\,j' + \overline{\nabla}_a P^{(1)\,ab} + \frac{\epsilon}{\alpha}\,\overline{\nabla}_a P'^{ab}$$

4.3.3 Riemann-Normal Equations of Motion

The equations of motion are in Riemann-Normal coordinates are:

4.3.3 Other Important Equations

For a point x_0 in the patch, the low-frequency, background metric in equation (4.1) can be expanded as

$$\overline{g}_{ab}(x) = \overline{g}_{ab}(x_0) + \frac{1}{2}\,\overline{g}_{ab,cd}\,(x-x_0)^c(x-x_0)^d + \dots \tag{4.263}$$

Given the above expansion, we can write the background connection and background Riemann tensor as

$$\overline{R}_{mnab} = \overline{g}_{an,mb} - \overline{g}_{am,nb} \tag{4.264}$$

$$\overline{\Gamma}^{m}{}_{ab,n} = -\frac{1}{3}\left(\overline{R}^{m}{}_{abn} + \overline{R}^{m}{}_{ban}\right) . \tag{4.265}$$

In the next section, we expand around the origin to derive important relations.

4.3.3 Important Equations

Given this expansion, we can write the expansion of the background connection as

$$\Gamma^{m}{}_{ab}(x) = \Gamma^{m}{}_{ab}(x_0) + \Gamma^{m}{}_{ab,p}(x)\Big|_{x=x_0}(x-x_0)^p + \frac{1}{2}\,\Gamma^{m}{}_{ab,pt}\Big|_{x=x_0}(x-x_0)^p(x-x_0)^t + \ldots \tag{4.266}$$

where $\Gamma^{m}{}_{ab}(x_0) = 0$ by definition of the Riemann-Normal coordinates. We can substitute in for the background connection is and background Riemann tensor to get the following expanded metric

$$\overline{g}_{ab}(x) = \eta_{ab} - \frac{1}{3}\,\beta^2\,\overline{R}_{acbd}\Big|_{x=x_0} y^c\,y^d + \mathcal{O}\left(\beta^3\right) \tag{4.267}$$

$$\overline{g}^{ab}(x) = \eta^{ab} + \frac{1}{3}\,\beta^2\,\overline{g}^{be}\,\overline{R}^{a}{}_{ced}\Big|_{x=x_0} y^c\,y^d + \mathcal{O}\left(\beta^3\right) . \tag{4.268}$$

where x is a general point in the patch, $x = x_0 + \beta\,y$ and β is a dimensionless, perturbative parameter. η_{ab} is the Minkowski metric. We have assumed a Lorentzian manifold. The radius of curvature of the background metric is β; $\beta\,y$ sets the length scale of the patch. Our analysis is valid for $\beta\,y < \mathcal{L}$. The modified background connection is now

$$\overline{\Gamma}^{m}{}_{ab}(x) = -\frac{1}{3}\,\beta\,\left(\overline{R}^{m}{}_{abn} + \overline{R}^{m}{}_{ban}\right)\Big|_{x=x_0} y^n + \mathcal{O}\left(\beta^3\right) . \tag{4.269}$$

We can substitute in to find the expanded background Riemann tensor as

$$
\begin{aligned}
\overline{R}^{s}{}_{dab} &= \overline{\Gamma}^{s}_{db,a} - \overline{\Gamma}^{s}_{da,b} + \overline{\Gamma}^{m}_{db}\,\overline{\Gamma}^{s}_{ma} - \overline{\Gamma}^{m}_{da}\,\overline{\Gamma}^{s}_{mb} \\
&= -\frac{1}{3}\,\beta\,\left(\overline{R}^{s}{}_{dbn,a} + \overline{R}^{s}{}_{bdn,a}\right)\Big|_{x=x_0} y^n + \frac{1}{3}\,\beta\,\left(\overline{R}^{s}{}_{dan,b} + \overline{R}^{s}{}_{adn,b}\right)\Big|_{x=x_0} y^n \\
&+ \frac{1}{9}\,\beta^2\left(\left(\overline{R}^{m}{}_{dbn} + \overline{R}^{m}{}_{bdn}\right)\left(\overline{R}^{s}{}_{map} + \overline{R}^{s}{}_{amp}\right)\right)\Big|_{x=x_0} y^n\,y^p \\
&- \frac{1}{9}\,\beta^2\left(\left(\overline{R}^{m}{}_{dan} + \overline{R}^{m}{}_{adn}\right)\left(\overline{R}^{s}{}_{mbp} + \overline{R}^{s}{}_{bmp}\right)\right)\Big|_{x=x_0} y^n\,y^p + \ldots
\end{aligned}
\tag{4.270}
$$

We now choose a coordinate system where $\overline{g}_{ab}(x_0) = \eta_{ab}$ and average the background quantities to find

$$
\begin{aligned}
\left\langle\,\overline{g}_{ab}\,\right\rangle &= \eta_{ab} - \frac{1}{3}\,\beta^2\left\langle\,\overline{R}_{acbd}\Big|_{x=x_0} y^c\,y^d\,\right\rangle + \mathcal{O}\left(\beta^3\right) \\
&= \eta_{ab} - \frac{1}{3}\,\beta^2\,\Delta_{ab} + \mathcal{O}\left(\beta^3\right) && (4.271) \\
\left\langle\,\overline{g}^{ab}\,\right\rangle &= \eta^{ab} + \frac{1}{3}\,\beta^2\left\langle\,\overline{g}^{be}\,\overline{R}^{a}{}_{ced}\Big|_{x=x_0} y^c\,y^d\,\right\rangle + \mathcal{O}\left(\beta^3\right) \\
&= \eta^{ab} + \frac{1}{3}\,\beta^2\,\Delta^{ab} + \mathcal{O}\left(\beta^3\right) && (4.272) \\
\left\langle\,\overline{\Gamma}^{m}_{ab}\,\right\rangle &= -\frac{1}{3}\,\beta\left\langle\left(\overline{R}^{m}{}_{abn} + \overline{R}^{m}{}_{ban}\right)\Big|_{x=x_0} y^n\,\right\rangle + \mathcal{O}\left(\beta^3\right) \\
&= -\frac{1}{3}\,\beta\,\tilde{\Delta}^{m}_{ab} + \mathcal{O}\left(\beta^3\right) && (4.273) \\
\left\langle\,\overline{\Gamma}^{m}_{ab,c}\,\right\rangle &= -\frac{1}{3}\,\beta\left\langle\left(\overline{R}^{m}{}_{abn,c} + \overline{R}^{m}{}_{ban,c}\right)\Big|_{x=x_0} y^n\,\right\rangle + \mathcal{O}\left(\beta^3\right) \\
&= -\frac{1}{3}\,\beta\,\tilde{\Delta}^{m}_{ab,c} + \mathcal{O}\left(\beta^3\right). && (4.274)
\end{aligned}
$$

where

$$\Delta_{ab} = \Big\langle \overline{R}_{acbd}\Big|_{x=x_0} y^c\, y^d \Big\rangle \tag{4.275}$$

$$\tilde{\Delta}^m_{ab} = \Big\langle \big(\overline{R}^m{}_{abn} + \overline{R}^m{}_{ban}\big)\Big|_{x=x_0} y^n \Big\rangle \tag{4.276}$$

$$\tilde{\Delta}^m_{ab,c} = \Big\langle \big(\overline{R}^m{}_{abn} + \overline{R}^m{}_{ban}\big)\Big|_{x=x_0} y^n \Big\rangle \tag{4.277}$$

and Δs retain information on Schwarzschild radius. The indices are Minkowski indices in the local patch. We assume the external fields are homogeneous over the patch. The averaged Riemann tensor is

$$\begin{aligned}
\Big\langle \overline{R}^s{}_{dab} \Big\rangle = &-\frac{1}{3}\,\beta \Big\langle \big(\overline{R}^s{}_{dbn,a} + \overline{R}^s{}_{bdn,a}\big)\Big|_{x=x_0} y^n \Big\rangle + \frac{1}{3}\,\beta \Big\langle \big(\overline{R}^s{}_{dan,b} + \overline{R}^s{}_{adn,b}\big)\Big|_{x=x_0} y^n \Big\rangle \\
&+ \frac{1}{9}\,\beta^2 \Big\langle \Big(\big(\overline{R}^m{}_{dbn} + \overline{R}^m{}_{bdn}\big)\big(\overline{R}^s{}_{map} + \overline{R}^s{}_{amp}\big)\Big)\Big|_{x=x_0} y^n\, y^p \Big\rangle \\
&- \frac{1}{9}\,\beta^2 \Big\langle \Big(\big(\overline{R}^m{}_{dan} + \overline{R}^m{}_{adn}\big)\big(\overline{R}^s{}_{mbp} + \overline{R}^s{}_{bmp}\big)\Big)\Big|_{x=x_0} y^n\, y^p \Big\rangle + \ldots \\
= &-\frac{1}{3}\,\beta \left(\tilde{\Delta}^s{}_{db,a} - \tilde{\Delta}^s{}_{da,b} \right) + \frac{1}{9}\,\beta^2 \left(\tilde{\Delta}^m{}_{db}\, \tilde{\Delta}^s{}_{ma} - \tilde{\Delta}^m{}_{da}\, \tilde{\Delta}^s{}_{mb} \right) + \ldots
\end{aligned} \tag{4.278}$$

We combine these quantities to re-express the wave equations.

4.3.4 An Estimate

We can estimate the values of the quantities listed in this section. We consider, e.g. the Kerr-Newman metric for the background geometry

$$ds^2 = -\left(1 + \frac{r_q^2 - r_s r}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2 + \left(r^2 + a^2 + \frac{a^2\left(r_s r - r_q^2\right)}{\Sigma} \sin^2\theta\right) \sin^2\theta\, d\phi^2 \tag{4.279}$$

$$- \frac{2\, a\left(r_s r - r_q^2\right) \sin^2\theta}{\Sigma}\, dt\, d\phi$$

where $r_s = 2\, G\, M$, $a = J/M$, $\Sigma = r^2 + a^2\, \cos^2\theta$, $\Delta = r^2 - r_s\, r + a^2 + r_q^2$ and $r_q^2 Q^2$ and $r_q = Q^2/$ Please note, astrophysical black holes are known to have a minuscule charge. We consider Q to be effectively zero throughout these notes. We seek to show this background metric is locally flat for a local neighborhood of points around point P. For this example background metric, we will reproduce equations (4.267) and (4.268). For there we will be able to reproduce the results of the previous sections for this example.

Locally Flat Metric: We can make the transformation on the background metric in equation (4.279)

$$g'_{ls}(x') = g_{mn}(x)\, \frac{\partial x^m}{\partial x'^l}\, \frac{\partial x^n}{\partial x'^s} \tag{4.280}$$

where x is point P. We henceforth set $x = 0$. We can Taylor expand the metric

$$g_{mn}(x) = g_{mn}(0) + \left.\frac{\partial g_{mn}(x)}{\partial x'^l}\right|_{x=0} x'^l + \frac{1}{2} \left.\frac{\partial g_{mn}(x)}{\partial x'^l \partial x'^s}\right|_{x=0} x'^l x'^s + \ldots \tag{4.281}$$

$$= g_{mn}(0) + A_{mn,l}\, x'^l + B_{mn,ls}\, x'^l x'^s + \ldots \tag{4.282}$$

and around point P

$$x^m = x^m(0) + \left.\frac{\partial x'^m}{\partial x^l}\right|_{x=0} x'^l + \frac{1}{2}\left.\frac{\partial x^m}{\partial x'^l \partial x'^s}\right|_{x=0} x'^l x'^s + \dots \tag{4.283}$$

$$= K^m{}_l\, x'^l + L^m{}_{ls}\, x'^l x'^s + M^m{}_{vls}\, x'^n x'^l x'^s + \dots \tag{4.284}$$

If we require $x' = 0$, the transformation on the background metric at the point P is now

$$g'_{ls}(0) = g_{mn}(0)\, K^m{}_l\, K^n{}_s \tag{4.285}$$

where K is a matrix which diagonalizes the background metric. In general, K has n^2 degrees of freedom (dof). $g_{mn}(0)$ is a symmetric matrix with $n(n+1)/2$. Thus, $n(n+1)/2$ dof of K are needed to diagonalize the matrix leaving the anti-symmetric components $n(n-1)/2$. We use these degrees of fix $g'_{ls}(0) = \eta_{ls}$, the Minkowski metric.

4.3.5 High-Frequency Gravitational Wave Equation in Riemann-Normal Coordinates

The gravitational wave equations is

$$-\frac{1}{2}\overline{\Box}\, h_{bd} = \overline{R}^s{}_{dab}\, h^a{}_s + \kappa\left(\frac{\alpha}{\epsilon}\, T^{(1,0)}_{ab} + T^{(0,1)}_{ab} + \alpha\, T^{(1,1)}_{ab} - \frac{1}{2}\, g_{ab}\left(\frac{\alpha}{\epsilon}\, T^{(1,0)} + T^{(0,1)} + \alpha\, T^{(1,1)}\right)\right) \tag{4.286}$$

where we can write

$$\begin{aligned}
\overline{g}^{ac}\,\overline{\nabla}_a\overline{\nabla}_c\,h_{bd} &= \overline{g}^{ac}\,\overline{\nabla}_a\overline{\nabla}_c\,h_{bd} && (4.287)\\
&= \overline{g}^{ac}\Big(\partial_a\left(\partial_c\,h_{bd} - \overline{\Gamma}^e{}_{cb}\,h_{ed} - \overline{\Gamma}^e{}_{cd}\,h_{be}\right) - \overline{\Gamma}^e{}_{ac}\left(\partial_e\,h_{bd} - \overline{\Gamma}^f{}_{eb}\,h_{fd} - \overline{\Gamma}^f{}_{ed}\,h_{bf}\right)\\
&\quad - \overline{\Gamma}^e{}_{ab}\left(\partial_c\,h_{ed} - \overline{\Gamma}^f{}_{ce}\,h_{fd} - \overline{\Gamma}^f{}_{cd}\,h_{ef}\right) - \overline{\Gamma}^e{}_{ad}\left(\partial_c\,h_{be} - \overline{\Gamma}^f{}_{cb}\,h_{fe} - \overline{\Gamma}^f{}_{ce}\,h_{bf}\right)\Big)\\
&= \langle\overline{g}^{ac}\rangle\Big(\partial_a\partial_c\,h_{bd} - \langle\partial_a\,\overline{\Gamma}^e{}_{cb}\rangle\,h_{ed} - \langle\overline{\Gamma}^e{}_{cb}\rangle\,\partial_a h_{ed} - \langle\partial_a\overline{\Gamma}^e{}_{cd}\rangle\,h_{be} - \langle\overline{\Gamma}^e{}_{cd}\rangle\,\partial_a h_{be}\\
&\quad - \langle\overline{\Gamma}^e{}_{ac}\rangle\,\partial_e\,h_{bd} && (4.288)\\
&\quad + \langle\overline{\Gamma}^e{}_{ac}\,\overline{\Gamma}^f{}_{eb}\rangle\,h_{fd} + \langle\overline{\Gamma}^e{}_{ac}\,\overline{\Gamma}^f{}_{ed}\rangle\,h_{bf} - \langle\overline{\Gamma}^e{}_{ab}\rangle\,\partial_c\,h_{ed} + \langle\overline{\Gamma}^e{}_{ab}\,\overline{\Gamma}^f{}_{ce}\rangle\,h_{fd} + \langle\overline{\Gamma}^e{}_{ab}\,\overline{\Gamma}^f{}_{cd}\rangle\,h_{ef}\\
&\quad - \langle\overline{\Gamma}^e{}_{ad}\rangle\,\partial_c\,h_{be} + \langle\overline{\Gamma}^e{}_{ad}\,\overline{\Gamma}^f{}_{cb}\rangle\,h_{fe} + \langle\overline{\Gamma}^e{}_{ad}\,\overline{\Gamma}^f{}_{ce}\rangle\,h_{bf}\Big)\\
&= \eta^{ac}\partial_a\partial_c\,h_{bd} + \eta^{ac}\Big(-\langle\partial_a\,\overline{\Gamma}^e{}_{cb}\rangle\,h_{ed} - \langle\partial_a\,\overline{\Gamma}^e{}_{cd}\rangle\,h_{be} - 2\,\langle\overline{\Gamma}^e{}_{cb}\rangle\,\partial_a\,h_{ed} - 2\,\langle\overline{\Gamma}^e{}_{cd}\rangle\,\partial_a h_{be} && (4.289)\\
&\quad - \langle\overline{\Gamma}^e{}_{ac}\rangle\,\partial_e\,h_{bd} + \langle\overline{\Gamma}^e{}_{ac}\,\overline{\Gamma}^f{}_{eb}\rangle\,h_{fd} + \langle\overline{\Gamma}^e{}_{ac}\,\overline{\Gamma}^f{}_{ed}\rangle\,h_{bf} + \langle\overline{\Gamma}^e{}_{ab}\,\overline{\Gamma}^f{}_{ce}\rangle\,h_{fd}\\
&\quad + \langle\overline{\Gamma}^e{}_{ab}\,\overline{\Gamma}^f{}_{cd}\rangle\,h_{ef} + \langle\overline{\Gamma}^e{}_{ad}\,\overline{\Gamma}^f{}_{cb}\rangle\,h_{fe} + \langle\overline{\Gamma}^e{}_{ad}\,\overline{\Gamma}^f{}_{ce}\rangle\,h_{bf}\Big)\\
&= \eta^{ac}\partial_a\partial_c\,h_{bd} + \eta^{ac}\Big(-\langle\,\overline{\Gamma}^e{}_{cb,a}\rangle\,h_{ed} - \langle\,\overline{\Gamma}^e{}_{cd,a}\rangle\,h_{be} - 2\,\langle\overline{\Gamma}^e{}_{cb}\rangle\,\partial_a\,h_{ed} - 2\,\langle\overline{\Gamma}^e{}_{cd}\rangle\,\partial_a h_{be} && (4.290)\\
&\quad - \langle\overline{\Gamma}^e{}_{ac}\rangle\,\partial_e\,h_{bd} + \langle\overline{\Gamma}^e{}_{ac}\,\overline{\Gamma}^f{}_{eb}\rangle\,h_{fd} + \langle\overline{\Gamma}^e{}_{ac}\,\overline{\Gamma}^f{}_{ed}\rangle\,h_{bf} + \langle\overline{\Gamma}^e{}_{ab}\,\overline{\Gamma}^f{}_{ce}\rangle\,h_{fd}\\
&\quad + \langle\overline{\Gamma}^e{}_{ab}\,\overline{\Gamma}^f{}_{cd}\rangle\,h_{ef} + \langle\overline{\Gamma}^e{}_{ad}\,\overline{\Gamma}^f{}_{cb}\rangle\,h_{fe} + \langle\overline{\Gamma}^e{}_{ad}\,\overline{\Gamma}^f{}_{ce}\rangle\,h_{bf}\Big)\\
&= \eta^{ac}\partial_a\partial_c\,h_{bd} + \frac{1}{3}\,\beta\,\eta^{ac}\Big(\,\tilde{\Delta}^e{}_{cb,a}h_{ed} + \tilde{\Delta}^e{}_{cd,a}h_{be} + 2\,\tilde{\Delta}^e{}_{cb}\,\partial_a h_{ed} + 2\,\tilde{\Delta}^e{}_{cd}\,\partial_a h_{be} && (4.291)\\
&\quad + \tilde{\Delta}^e{}_{ac}\,\partial_e\,h_{bd}\Big) + \ldots
\end{aligned}$$

In addition, we have the new components of the energy-momentum tensor

$$\begin{aligned}
T_{ab}^{(1,0)} &= \langle \overline{g}^{ef} \rangle \langle \overline{F}_{bf} \rangle F_{ae}^{(1)} + \langle \overline{g}^{ef} \rangle \langle \overline{F}_{ae} \rangle F_{bf}^{(1)} + \langle \partial_a \overline{\phi} \rangle \partial_b \phi^{(1)} + \langle \partial_b \overline{\phi} \rangle \partial_a \phi^{(1)} \qquad (4.292) \\
&- \frac{4}{45\, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \langle \overline{g}^{ef}\, \overline{g}^{mn}\, \overline{g}^{op} \rangle \Big(\langle \overline{F}_{mo}\, \overline{F}_{ae}\, \overline{F}_{bf} \rangle F_{np}^{(1)} + \langle \overline{F}_{np}\, \overline{F}_{mo}\, \overline{F}_{bf} \rangle F_{ae}^{(1)} \Big) \\
&- \langle \overline{g}_{ab} \rangle \Bigg(\frac{1}{2} \langle \overline{g}^{mn} \overline{g}^{op}\, \overline{F}_{np} \rangle F_{mo}^{(1)} + \langle \overline{g}^{mn} \rangle \langle \partial_m \overline{\phi} \rangle \partial_n \phi^{(1)} + m^2 \langle \overline{\phi} \rangle \phi^{(1)} \\
&- \frac{4}{90\, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \Big(\langle \overline{g}^{mn}\, \overline{g}^{op}\, \overline{g}^{qr}\, \overline{g}^{st} \rangle \langle \overline{F}_{mo}\, \overline{F}_{np}\, \overline{F}_{qs} \rangle F_{rt}^{(1)} - \epsilon^{hijk} \epsilon^{defg} \langle \frac{7}{4\, \overline{g}}\, \overline{F}_{de}\, \overline{F}_{fg}\, \overline{F}_{hi} \rangle F_{jk}^{(1)} \Big) \Bigg) \\
&= \eta^{ef} \Big(\langle \overline{F}_{bf} \rangle F_{ae}^{(1)} + \langle \overline{F}_{ae} \rangle F_{bf}^{(1)} \Big) + \langle \partial_a \overline{\phi} \rangle \partial_b \phi^{(1)} + \langle \partial_b \overline{\phi} \rangle \partial_a \phi^{(1)} \qquad (4.293) \\
&- \frac{4}{45\, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \eta^{ef} \eta^{mn} \eta^{op} \Big(\langle \overline{F}_{mo}\, \overline{F}_{ae}\, \overline{F}_{bf} \rangle F_{np}^{(1)} + \langle \overline{F}_{np}\, \overline{F}_{mo}\, \overline{F}_{bf} \rangle F_{ae}^{(1)} \Big) \\
&- \eta_{ab} \Bigg(\frac{1}{2} \eta^{mn} \eta^{op} \langle \overline{F}_{np} \rangle F_{mo}^{(1)} + \eta^{mn} \langle \partial_m \overline{\phi} \rangle \partial_n \phi^{(1)} + m^2 \langle \overline{\phi} \rangle \phi^{(1)} \\
&- \frac{4}{90\, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \Big(\eta^{qr}\, \eta^{st} \langle \overline{F}^2\, \overline{F}_{qs} \rangle F_{rt}^{(1)} - \frac{7}{4} \epsilon^{hijk} \epsilon^{defg} \langle \overline{F}_{de}\, \overline{F}_{fg}\, \overline{F}_{hi} \rangle F_{jk}^{(1)} \Big) \Bigg) + \ldots
\end{aligned}$$

$$\begin{aligned}
T_{ab}^{(0,1)} &= -h^{cd} \langle \overline{F}_{ac} \overline{F}_{bd} \rangle + \frac{2}{45\, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \Big(h^{cd} \langle \overline{F}_{ac}\, \overline{F}_{bd}\, \overline{F}^2 \rangle + 2\, \overline{g}^{cd}\, \overline{g}^{op}\, h^{mn} \langle \overline{F}_{ac}\, \overline{F}_{bd}\, \overline{F}_{mo}\, \overline{F}_{np} \rangle \Big) \qquad (4.294) \\
&- h_{ab} \Bigg(\frac{1}{4} \overline{F}^2 + \frac{1}{2} \Big(\overline{g}^{mn}\, \partial_m \overline{\phi}\, \partial_n \overline{\phi} + m^2\, \overline{\phi}^2 \Big) - \frac{1}{90\, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \Big(\overline{F}^4 - \frac{7}{16\, \overline{g}} \left(\epsilon^{mnop} \overline{F}_{mn} \overline{F}_{op} \right)^2 \Big) \Bigg) \\
&- \overline{g}_{ab} \Bigg(-\frac{1}{2} h^{mn} \overline{g}^{op}\, \overline{F}_{np}\, \overline{F}_{mo} - \frac{1}{2} h^{mn}\, \partial_m \overline{\phi}\, \partial_n \overline{\phi} + \frac{4}{90\, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 h^{mn}\, \overline{g}^{op}\, \overline{g}^{qr}\, \overline{g}^{st}\, \overline{F}_{mo}\, \overline{F}_{np}\, \overline{F}^2 \Bigg)
\end{aligned}$$

We can account for this by requiring the external electromagnetic fields to numerically vary slowly over the size of the patch, perhaps with a phenomenologically modeled profile. We will likely explore any additional experimental consequences of this in future work.

4.3.6 Averaging

The high-frequency waves described in the previous section have a wavelength that is much smaller than the characteristic length-scales of the background curvature. This is the geometric optics regime. If we consider some suitable covariant averaging [59]–[64], the background quantities become $\overline{g}_{ab} \equiv \langle g_{ab} \rangle$, $\overline{R}_{abcd} \equiv \langle R_{abcd} \rangle$ and $\overline{R}_{ab} \equiv \langle R_{ab} \rangle$. As well, I consider $\overline{\phi} \equiv \langle \phi \rangle$ and $\overline{F}_{ab} \equiv \langle F_{ab} \rangle$ as background quantities. Before I leave the averaging procedures to the appropriate graduate student, we note the Brill-Hartle average of a tensor field, A_{ab}, is defined by

$$\langle A_{ab} \rangle = \int d^4x' \sqrt{-g(x')}\, g_a{}^c(x,x')\, g_b{}^d(x,x'))\, A_{cd}\, f(x,x') \tag{4.295}$$

where $f(x,x')$ is a weighted function that decreases to zero when the difference of x to x' is greater than some scale d. Thus the value is finite. Understand all averaging schemes have a weighting function that goes to zero when the difference is greater than a scale.

4.4 Eikonal Approximation wave mixing

$$R_{\,bd} = \overline{R}_{\,bd} + \frac{\epsilon}{2}\Bigg(-\Box\, h_{bd} - \nabla_d \nabla_b\, h + \nabla_q \nabla_b\, h^q{}_d + \nabla_q \nabla_d\, h^q{}_b\Bigg) + \frac{\epsilon^2}{4}\Bigg(-\Box\,(h_{br}\, h^r{}_d) - \nabla_d\, \nabla_b\,(h^a{}_r\, h^r{}_a) \tag{4.296}$$

$$+ \nabla_q \nabla_b\,(h^q{}_r\, h^r{}_d) + \nabla_q \nabla_d\,(h^q{}_r\, h^r{}_b)$$

$$- 2\,\nabla_a\Big(h^{am}\big(\nabla_d\, h_{mb} + \nabla_b\, h_{md} - \nabla_m\, h_{bd}\big)\Big) + 2\,\nabla_d\Big(\, h^{am}\big(\nabla_a\, h_{mb} + \nabla_b\, h_{ma} - \nabla_m\, h_{ba}\big)\Big)$$

$$+ \big(\nabla_e\, h^a{}_a + \nabla_a\, h^a{}_e - \overline{g}^{am} \nabla_m\, h_{ae}\big)\big(\nabla_d\, h^e{}_b + \nabla_b\, h^e{}_d - \overline{g}^{eq}\,\nabla_q\, h_{bd}\big)$$

$$- \big(\nabla_e\, h^a{}_d + \nabla_d\, h^a{}_e - \overline{g}^{aq} \nabla_q\, h_{de}\big)\big(\nabla_a\, h^e{}_b + \nabla_b\, h^e{}_a - \overline{g}^{em}\,\nabla_m\, h_{ba}\big)\Bigg) + \mathcal{O}(\epsilon^3)$$

$$= \overline{R}_{\,bd} + \frac{\epsilon}{2}\Bigg(-\Box\, h_{bd} - \nabla_d \nabla_b\, h + \nabla_q \nabla_b\, h^q{}_d + \nabla_q \nabla_d\, h^q{}_b\Bigg) + \frac{\epsilon^2}{4}\Bigg(-\Box\,(h_{br}\, h^r{}_d) - \nabla_d\, \nabla_b\,(h^a{}_r\, h^r{}_a) \tag{4.297}$$

$$+ \nabla_q \nabla_b\,(h^q{}_r\, h^r{}_d) + \nabla_q \nabla_d\,(h^q{}_r\, h^r{}_b) - 2\,\Big(h^{am} \nabla_a \nabla_d\, h_{mb} + h^{am} \nabla_a \nabla_b\, h_{md} - h^{am} \nabla_a \nabla_m\, h_{bd}\Big)$$

$$+ 2\,\Big(\,(\nabla_d h^{am})(\nabla_a\, h_{mb}) + (\nabla_d h^{am})(\nabla_b\, h_{ma}) - (\nabla_d h^{am})(\nabla_m\, h_{ba})\,\Big) \tag{4.298}$$

$$+ 2\,\Big(\, h^{am} \nabla_d \nabla_a\, h_{mb} + h^{am} \nabla_d \nabla_b\, h_{ma} - h^{am} \nabla_d \nabla_m\, h_{ba}\Big)$$

$$+ (\nabla_a\, h^a{}_e)(\nabla_d\, h^e{}_b) + (\nabla_a\, h^a{}_e)(\nabla_b\, h^e{}_d) - (\nabla_a\, h^a{}_e)(\overline{g}^{eq}\,\nabla_q\, h_{bd})$$

$$- (\overline{g}^{am} \nabla_m\, h_{ae}) \nabla_d\, h^e{}_b - (\overline{g}^{am} \nabla_m\, h_{ae}) \nabla_b\, h^e{}_d + (\overline{g}^{am} \nabla_m\, h_{ae})(\overline{g}^{eq}\,\nabla_q\, h_{bd})$$

$$- \big(\nabla_e\, h^a{}_d + \nabla_d\, h^a{}_e - \overline{g}^{aq} \nabla_q\, h_{de}\big)\big(\nabla_a\, h^e{}_b\big)$$

$$+ \big(\nabla_e\, h^a{}_d + \nabla_d\, h^a{}_e - \overline{g}^{aq} \nabla_q\, h_{de}\big)\big(\nabla_b\, h^e{}_a\big)$$

$$- \big(\nabla_e\, h^a{}_d + \nabla_d\, h^a{}_e - \overline{g}^{aq} \nabla_q\, h_{de}\big)\big(\overline{g}^{em}\,\nabla_m\, h_{ba}\big)\Bigg) + \mathcal{O}(\epsilon^3)$$

In the lagrangian, we can use integration by parts

$$R_{\ bd} = \overline{R}_{\ bd} + \frac{\epsilon}{2}\left(-\Box\, h_{bd} - \nabla_d \nabla_b\, h + \nabla_q \nabla_b\, h^q_{\ d} + \nabla_q \nabla_d\, h^q_{\ b}\right) \tag{4.299}$$

$$+ \frac{\epsilon^2}{4}\Bigg(-\Box\,(h_{br}\, h^r_{\ d}) - \nabla_d\, \nabla_b\,(h^a_{\ r}\, h^r_{\ a}) + \nabla_q \nabla_b\,(h^q_{\ r}\, h^r_{\ d}) + \nabla_q \nabla_d\,(h^q_{\ r}\, h^r_{\ b}) \tag{4.300}$$

$$- 2\left(h^{am}\nabla_a \nabla_d\, h_{mb} + h^{am}\nabla_a \nabla_b\, h_{md} - h^{am}\nabla_a \nabla_m\, h_{bd}\right)$$

$$+ 2\left(-h^{am}\nabla_d \nabla_a\, h_{mb} - h^{am}\, \nabla_d \nabla_b\, h_{ma} + h^{am}\nabla_d \nabla_m\, h_{ba}\right) \tag{4.301}$$

$$+ 2\left(h^{am}\nabla_d \nabla_a\, h_{mb} + h^{am}\nabla_d \nabla_b\, h_{ma} - h^{am}\nabla_d \nabla_m\, h_{ba}\right)$$

$$- h^a_{\ e}\nabla_a\, \nabla_d\, h^e_{\ b} - h^a_{\ e}\, \nabla_a \nabla_b\, h^e_{\ d} - \overline{g}^{eq}\, h^a_{\ e}\nabla_a \nabla_q\, h_{bd}$$

$$- \overline{g}^{am}\, h_{ae}\nabla_m \nabla_d\, h^e_{\ b} - (\overline{g}^{am}\nabla_m\, h_{ae})\nabla_b\, h^e_{\ d} - \overline{g}^{am}\, \overline{g}^{eq}\, h_{ae}\nabla_m \nabla_q\, h_{bd}$$

$$+ h^a_{\ d}\nabla_e \nabla_a\, h^e_{\ b} + h^a_{\ e}\nabla_d\, \nabla_a\, h^e_{\ b} - \overline{g}^{aq} h_{de}\nabla_q \nabla_a\, h^e_{\ b}$$

$$- h^a_{\ d}\nabla_e \nabla_b\, h^e_{\ a} - h^a_{\ e}\nabla_d \nabla_b\, h^e_{\ a} + \overline{g}^{aq}\, h_{de}\nabla_q \nabla_b\, h^e_{\ a}$$

$$- \left(\left(\overline{g}^{em}\, \nabla_m\, h_{ba}\right)\nabla_e\, h^a_{\ d} + \left(\overline{g}^{em}\, \nabla_m\, h_{ba}\right)\nabla_d\, h^a_{\ e} - \left(\overline{g}^{em}\, \nabla_m\, h_{ba}\right)\overline{g}^{aq}\nabla_q\, h_{de}\right)\Bigg) + \mathcal{O}(\epsilon^3)$$

$$R_{\ bd} = \overline{R}_{\ bd} + \frac{\epsilon}{2}\left(-\Box\, h_{bd} - \nabla_d \nabla_b\, h + \nabla_q \nabla_b\, h^q_{\ d} + \nabla_q \nabla_d\, h^q_{\ b} \right) \tag{4.302}$$

$$+ \frac{\epsilon^2}{4}\Bigg(-\Box\,(h_{br}\, h^r_{\ d}) - \nabla_d\, \nabla_b\,(h^a_{\ r}\, h^r_{\ a}) + \nabla_q \nabla_b\,(h^q_{\ r}\, h^r_{\ d}) + \nabla_q \nabla_d\,(h^q_{\ r}\, h^r_{\ b}) \tag{4.303}$$

$$- 2\, h^{am}\Big(\nabla_a \nabla_d\, h_{mb} + \nabla_a \nabla_b\, h_{md} - \nabla_a \nabla_m\, h_{bd} \Big)$$

$$- h^a_{\ e}\Big(\nabla_a\, \nabla_d\, h^e_{\ b} + \nabla_a \nabla_b\, h^e_{\ d} + \overline{g}^{eq} \nabla_a \nabla_q\, h_{bd} \Big)$$

$$- \overline{g}^{am}\, h_{ae} \nabla_m \nabla_d\, h^e_{\ b} + \overline{g}^{am} h_{ae} \nabla_m\, \nabla_b\, h^e_{\ d} - \overline{g}^{am}\, \overline{g}^{eq}\, h_{ae} \nabla_m \nabla_q\, h_{bd}$$

$$+ h^a_{\ d} \nabla_e \nabla_a\, h^e_{\ b} + h^a_{\ e} \nabla_d\, \nabla_a\, h^e_{\ b} - h_{de} \Box\, h^e_{\ b}$$

$$- h^a_{\ d} \nabla_e \nabla_b\, h^e_{\ a} - h^a_{\ e} \nabla_d \nabla_b\, h^e_{\ a} + \overline{g}^{aq}\, h_{de} \nabla_q \nabla_b\, h^e_{\ a}$$

$$+ h_{ba} \Box\, h^a_{\ d} + h_{ba} \nabla_m \nabla_d\, h^{am} - h_b^{\ q} \nabla_m \nabla_q\, h_d^{\ m} \Bigg) + \mathcal{O}(\epsilon^3)$$

Denoting $h = h^a_{\ a}$, the Ricci tensor and scalar are

$$R_{\ bd} = \overline{R}_{\ bd} + \frac{\epsilon}{2}\left(-\Box\, h_{bd} - \nabla_d \nabla_b\, h + \nabla_q \nabla_b\, h^q_{\ d} + \nabla_q \nabla_d\, h^q_{\ b}\right) + \frac{\epsilon^2}{4}\left(-\Box\,(h_{br}\, h^r_{\ d}) - \nabla_d\, \nabla_b\,(h^a_{\ r}\, h^r_{\ a}) \right. \tag{4.304}$$

$$+ \nabla_q \nabla_b\,(h^q_{\ r}\, h^r_{\ d}) + \nabla_q \nabla_d\,(h^q_{\ r}\, h^r_{\ b})$$

$$- 2\,\nabla_a\Big(h^{am}\big(\nabla_d\, h_{mb} + \nabla_b\, h_{md} - \nabla_m\, h_{bd}\big)\Big) + 2\,\nabla_d\Big(\, h^{am}\big(\nabla_a\, h_{mb} + \nabla_b\, h_{ma} - \nabla_m\, h_{ba}\big)\Big)$$

$$+ \big(\nabla_e\, h^a_{\ a} + \nabla_a\, h^a_{\ e} - \overline{g}^{am} \nabla_m\, h_{ae}\big)\big(\nabla_d\, h^e_{\ b} + \nabla_b\, h^e_{\ d} - \overline{g}^{eq}\, \nabla_q\, h_{bd}\big)$$

$$\left. - \big(\nabla_e\, h^a_{\ d} + \nabla_d\, h^a_{\ e} - \overline{g}^{aq} \nabla_q\, h_{de}\big)\big(\nabla_a\, h^e_{\ b} + \nabla_b\, h^e_{\ a} - \overline{g}^{em}\, \nabla_m\, h_{ba}\big)\right) + \mathcal{O}(\epsilon^3)$$

$$= \overline{R}_{\ bd} + \frac{\epsilon}{2}\left(-\Box\, h_{bd} - \nabla_d \nabla_b\, h + \nabla_q \nabla_b\, h^q_{\ d} + \nabla_q \nabla_d\, h^q_{\ b}\right) + \frac{\epsilon^2}{4}\left(-\Box\,(h_{br}\, h^r_{\ d}) - \nabla_d\, \nabla_b\,(h^a_{\ r}\, h^r_{\ a}) \right. \tag{4.305}$$

$$+ \nabla_q \nabla_b\,(h^q_{\ r}\, h^r_{\ d}) + \nabla_q \nabla_d\,(h^q_{\ r}\, h^r_{\ b})$$

$$- 2\,\nabla_a\Big(h^{am}\big(\nabla_d\, h_{mb} + \nabla_b\, h_{md} - \nabla_m\, h_{bd}\big)\Big) + 2\,\nabla_d\Big(\, h^{am}\big(\nabla_a\, h_{mb} + \nabla_b\, h_{ma} - \nabla_m\, h_{ba}\big)\Big)$$

$$+ (\nabla_a\, h^a_{\ e})\big(\nabla_d\, h^e_{\ b} + \nabla_b\, h^e_{\ d} - \overline{g}^{eq}\, \nabla_q\, h_{bd}\big) - (\overline{g}^{am} \nabla_m\, h_{ae})\big(\nabla_d\, h^e_{\ b} + \nabla_b\, h^e_{\ d} - \overline{g}^{eq}\, \nabla_q\, h_{bd}\big)$$

$$\left. - \big(\nabla_e\, h^a_{\ d} + \nabla_d\, h^a_{\ e} - \overline{g}^{aq} \nabla_q\, h_{de}\big)\big(\nabla_a\, h^e_{\ b} + \nabla_b\, h^e_{\ a} - \overline{g}^{em}\, \nabla_m\, h_{ba}\big)\right) + \mathcal{O}(\epsilon^3)$$

The ϵ^2 terms can further simplified to find

$$
\begin{aligned}
-g^{gh}\nabla_g \nabla_h \left(h_{br}\, h^r{}_d\right) &= -g^{gh}\nabla_g \left(h_{br}\, \nabla_h\, h^r{}_d\right) - g^{gh}\nabla_g \left(h^r{}_d\, \nabla_h\, h_{br}\right) \\
&= -g^{gh} \left(\nabla_g\, h_{br}\right) \left(\nabla_h\, h^r{}_d\right) - g^{gh} \left(h_{br}\, \nabla_g \nabla_h\, h^r{}_d\right) - g^{gh} \left(\nabla_g h^r{}_d\right) \left(\nabla_h\, h_{br}\right) \\
&- g^{gh} \left(h^r{}_d\, \nabla_g \nabla_h\, h_{br}\right) \\
&= -2\, g^{gh} \left(\nabla_g\, h_{br}\right) \left(\nabla_h\, h^r{}_d\right) - g^{gh} \left(h_{br}\, \nabla_g \nabla_h\, h^r{}_d\right) - g^{gh} \left(h^r{}_d\, \nabla_g \nabla_h\, h_{br}\right)
\end{aligned} \tag{4.306}
$$

$$
\begin{aligned}
-\nabla_d\, \nabla_b \left(h^a{}_r\, h^r{}_a\right) &= -\nabla_d \left(h^a{}_r\, \nabla_b\, h^r{}_a\right) - \nabla_d \left(h^r{}_a\, \nabla_b\, h^a{}_r\right) \\
&= -\left(\nabla_d h^a{}_r\right) \left(\nabla_b\, h^r{}_a\right) - h^a{}_r\, \nabla_d \nabla_b\, h^r{}_a - \left(\nabla_d h^r{}_a\right) \left(\nabla_b\, h^a{}_r\right) - h^r{}_a \nabla_d \nabla_b\, h^a{}_r \\
&= -2\, \left(\nabla_d h^a{}_r\right) \left(\nabla_b\, h^r{}_a\right) - 2\, h^a{}_r\, \nabla_d \nabla_b\, h^r{}_a
\end{aligned} \tag{4.307}
$$

$$
\begin{aligned}
\nabla_q \nabla_b \left(h^q{}_r\, h^r{}_d\right) &= \nabla_q \left(h^q{}_r\, \nabla_b\, h^r{}_d\right) + \nabla_q \left(h^r{}_d\, \nabla_b\, h^q{}_r\right) \\
&= \left(\nabla_q h^q{}_r\right) \left(\nabla_b\, h^r{}_d\right) + h^q{}_r \nabla_q \nabla_b h^r{}_d + \left(\nabla_q h^r{}_d\right) \left(\nabla_b\, h^q{}_r\right) + h^r{}_d\, \nabla_q \nabla_b\, h^q{}_r
\end{aligned} \tag{4.308}
$$

$$
\begin{aligned}
\nabla_q \nabla_d \left(h^q{}_r\, h^r{}_b\right) &= \nabla_q \left(h^q{}_r \nabla_d h^r{}_b\right) + \nabla_q \left(h^r{}_b \nabla_d h^q{}_r\right) \\
&= \left(\nabla_q h^q{}_r\right) \left(\nabla_d h^r{}_b\right) + h^q{}_r \nabla_q \nabla_d h^r{}_b + \left(\nabla_q h^r{}_b\right) \left(\nabla_d h^q{}_r\right) + h^r{}_b \nabla_q \nabla_d h^q{}_r
\end{aligned} \tag{4.309}
$$

as well as

$$-2\,\nabla_a\Big(h^{am}\big(\nabla_d\,h_{mb}+\nabla_b\,h_{md}-\nabla_m\,h_{bd}\big)\Big) = -2\,\nabla_a\big(h^{am}\nabla_d\,h_{mb}\big)-2\,\nabla_a\big(h^{am}\nabla_b\,h_{md}\big) + 2\,\nabla_a\big(h^{am}\nabla_m\,h_{bd}\big) \tag{4.310}$$

$$= -2\big(\nabla_a h^{am}\big)\big(\nabla_d\,h_{mb}\big)-2h^{am}\nabla_a\nabla_d\,h_{mb} - 2\big(\nabla_a h^{am}\big)\big(\nabla_b\,h_{md}\big) - 2h^{am}\nabla_a\nabla_b\,h_{md}+2\big(\nabla_a h^{am}\big)\big(\nabla_m\,h_{bd}\big) + 2h^{am}\nabla_a\nabla_m\,h_{bd} \tag{4.311}$$

$$= -2\big(\nabla_a h^{am}\big)\big(\nabla_d\,h_{mb}\big)-2h^{am}\nabla_a\nabla_d\,h_{mb} - 2\big(\nabla_a h^{am}\big)\big(\nabla_b\,h_{md}\big) - 2h^{am}\nabla_a\nabla_b\,h_{md}+2\big(\nabla_a h^{am}\big)\big(\nabla_m\,h_{bd}\big) + 2h^{am}\nabla_a\nabla_m\,h_{bd} \tag{4.312}$$

$$2\,\nabla_d\Big(h^{am}\big(\nabla_a\,h_{mb}+\nabla_b\,h_{ma}-\nabla_m\,h_{ba}\big)\Big) = 2\,\nabla_d\big(h^{am}\nabla_a\,h_{mb}\big)+2\,\nabla_d\big(h^{am}\nabla_b\,h_{ma}\big) - 2\,\nabla_d\big(h^{am}\nabla_m\,h_{ba}\big) \tag{4.313}$$

$$= 2\,\big(\nabla_d h^{am}\big)\big(\nabla_a\,h_{mb}\big)+2\,h^{am}\nabla_d\nabla_a\,h_{mb} + 2\,\big(\nabla_d h^{am}\big)\big(\nabla_b\,h_{ma}\big) + 2\,h^{am}\nabla_d\nabla_b\,h_{ma}-2\,\big(\nabla_d h^{am}\big)\big(\nabla_m h_{ba}\big) - 2\,h^{am}\nabla_d\nabla_m\,h_{ba}$$

$$= 2\,\big(\nabla_d h^{am}\big)\big(\nabla_a\,h_{mb}\big)+2\,\big(\nabla_d h^{am}\big)\big(\nabla_b\,h_{ma}\big) + 2\,h^{am}\nabla_d\nabla_b\,h_{ma}-2\,\big(\nabla_d h^{am}\big)\big(\nabla_m h_{ba}\big)$$

and

$$\left(\nabla_e h^a{}_a + \nabla_a h^a{}_e - \overline{g}^{am}\nabla_m h_{ae}\right)\left(\nabla_d h^e{}_b + \nabla_b h^e{}_d - \overline{g}^{eq}\nabla_q h_{bd}\right) \tag{4.314}$$

$$= (\nabla_e h^a{}_a)(\nabla_d h^e{}_b) + (\nabla_e h^a{}_a)(\nabla_b h^e{}_d) - \overline{g}^{eq}\,(\nabla_e h^a{}_a)(\nabla_q h_{bd})$$

$$+ (\nabla_a h^a{}_e)(\nabla_d h^e{}_b) + (\nabla_a h^a{}_e)(\nabla_b h^e{}_d)$$

$$- \overline{g}^{eq}\,(\nabla_a h^a{}_e)(\nabla_q h_{bd}) - \overline{g}^{am}\,(\nabla_m h_{ae})(\nabla_d h^e{}_b) - \overline{g}^{am}\,(\nabla_m h_{ae})(\nabla_b h^e{}_d)$$

$$+ \overline{g}^{am}\,(\nabla_m h_{ae})(\overline{g}^{eq}\nabla_q h_{bd})$$

$$-\left(\nabla_e h^a{}_d + \nabla_d h^a{}_e - \overline{g}^{aq}\nabla_q h_{de}\right)\left(\nabla_a h^e{}_b + \nabla_b h^e{}_a - \overline{g}^{em}\nabla_m h_{ba}\right)$$

$$= (\nabla_e h^a{}_d)(\nabla_a h^e{}_b) + (\nabla_e h^a{}_d)(\nabla_b h^e{}_a) - \overline{g}^{eq}\,(\nabla_e h^a{}_d)(\nabla_q h_{ba})$$

$$+ (\nabla_d h^a{}_e)(\nabla_a h^e{}_b) + (\nabla_d h^a{}_e)(\nabla_b h^e{}_a)$$

$$- \overline{g}^{eq}(\nabla_d h^a{}_e)(\nabla_q h_{ba}) - \overline{g}^{am}(\nabla_m h_{de})(\nabla_a h^e{}_b) - \overline{g}^{am}(\nabla_m h_{de})(\nabla_b h^e{}_a)$$

$$+ \overline{g}^{am}(\nabla_m h_{de})(\overline{g}^{eq}\nabla_q h_{ba})$$

We can sum everything up to find

$$\text{terms} = (\nabla_d h_{ar})(\nabla_b h^{ar}) + 2h^{am}\nabla_a\nabla_m h_{bd} \tag{4.315}$$

$$\text{terms} = \tag{4.316}$$

$$
\begin{aligned}
&- 2\, g^{gh} \left(\nabla_g\, h_{br}\right) \left(\nabla_h\, h^r{}_d\right) - g^{gh} \left(h_{br}\, \nabla_g \nabla_h\, h^r{}_d\right) - g^{gh} \left(h^r{}_d\, \nabla_g \nabla_h\, h_{br}\right)\\
&- 2\, \left(\nabla_d h^a{}_r\right) \left(\nabla_b\, h^r{}_a\right) - 2\, h^a{}_r\, \nabla_d \nabla_b\, h^r{}_a\\
&+ \left(\nabla_q h^q{}_r\right) \left(\nabla_b\, h^r{}_d\right) + h^q{}_r \nabla_q \nabla_b h^r{}_d + \left(\nabla_q h^r{}_d\right) \left(\nabla_b\, h^q{}_r\right)\\
&+ h^r{}_d\, \nabla_q \nabla_b\, h^q{}_r \left(\nabla_q h^q{}_r\right) \left(\nabla_d h^r{}_b\right) + h^q{}_r \nabla_q \nabla_d h^r{}_b\\
&+ \left(\nabla_q h^r{}_b\right) \left(\nabla_d h^q{}_r\right) + h^r{}_b \nabla_q \nabla_d h^q{}_r - 2\big(\nabla_a h^{am}\big)\big(\nabla_d\, h_{mb}\big) - 2h^{am} \nabla_a \nabla_d\, h_{mb}\\
&- 2\big(\nabla_a h^{am}\big)\big(\nabla_b\, h_{md}\big) - 2h^{am} \nabla_a \nabla_b\, h_{md} + 2\big(\nabla_a h^{am}\big)\big(\nabla_m\, h_{bd}\big) + 2h^{am} \nabla_a \nabla_m\, h_{bd}\\
&+ 2\,\big(\nabla_d h^{am}\big)\big(\nabla_a\, h_{mb}\big) + 2\,\big(\nabla_d h^{am}\big)\big(\nabla_b\, h_{ma}\big)\\
&+ 2\, h^{am} \nabla_d \nabla_b\, h_{ma} - 2\,\big(\nabla_d h^{am}\big)\big(\nabla_m h_{ba}\big) + \left(\nabla_e\, h^a{}_a\right) \left(\nabla_d\, h^e{}_b\right)\\
&+ \left(\nabla_e\, h^a{}_a\right) \left(\nabla_b\, h^e{}_d\right) - \overline{g}^{eq}\, \left(\nabla_e\, h^a{}_a\right) \left(\nabla_q\, h_{bd}\right)\\
&+ \left(\nabla_a\, h^a{}_e\right) \left(\nabla_d\, h^e{}_b\right) + \left(\nabla_a\, h^a{}_e\right) \left(\nabla_b\, h^e{}_d\right) - \overline{g}^{eq}\, \left(\nabla_a\, h^a{}_e\right) \left(\nabla_q\, h_{bd}\right) - \overline{g}^{am}\, \left(\nabla_m\, h_{ae}\right) \left(\nabla_d\, h^e{}_b\right)\\
&- \overline{g}^{am}\, \left(\nabla_m\, h_{ae}\right) \left(\nabla_b\, h^e{}_d\right) + \overline{g}^{am}\, \left(\nabla_m\, h_{ae}\right) \left(\overline{g}^{eq}\, \nabla_q\, h_{bd}\right) \left(\nabla_e\, h^a{}_d\right) \left(\nabla_a\, h^e{}_b\right)\\
&+ \left(\nabla_e\, h^a{}_d\right) \left(\nabla_b\, h^e{}_a\right) - \overline{g}^{eq}\, \left(\nabla_e\, h^a{}_d\right) \left(\nabla_q\, h_{ba}\right) + \left(\nabla_d\, h^a{}_e\right) \left(\nabla_a\, h^e{}_b\right) + \left(\nabla_d\, h^a{}_e\right) \left(\nabla_b\, h^e{}_a\right)\\
&- \overline{g}^{eq}\, \left(\nabla_d\, h^a{}_e\right) \left(\nabla_q\, h_{ba}\right) - \overline{g}^{am}\, \left(\nabla_m\, h_{de}\right) \left(\nabla_a\, h^e{}_b\right) - \overline{g}^{am}\, \left(\nabla_m\, h_{de}\right) \left(\nabla_b\, h^e{}_a\right)\\
&+ \overline{g}^{am}\, \left(\nabla_m\, h_{de}\right) \left(\overline{g}^{eq}\, \nabla_q\, h_{ba}\right)
\end{aligned}
$$

$$
\begin{aligned}
R = \overline{R} + \epsilon \left(-\Box\, h + \nabla_q \nabla_b\, h^{qb} \right) + \frac{\epsilon^2}{4} \Bigg(& -2\,\Box \left(h_{br}\, h^{rb} \right) + 2\,\nabla_q \nabla_b \left(h^q_{\ r}\, h^{rb} \right) \\
& - 2\,\nabla_a \big(h^{am} \big(2\,\nabla_d\, h^d_{\ m} - \nabla_m\, h \big) \big) \\
& + 2\,\nabla_d \Big(h^{am} \big(\nabla_a\, h^d_{\ m} + \overline{g}^{bd} \nabla_b\, h_{ma} - \nabla_m\, h^d_{\ a} \big) \Big) + \big(\nabla_e\, h \big) \big(2\,\nabla_d\, h^{ed} - \overline{g}^{eq}\, \nabla_q\, h \big) \\
& - \big(\nabla_e\, h^{ab} + \overline{g}^{bd} \nabla_d\, h^a_{\ e} - \overline{g}^{aq} \nabla_q\, h^b_{\ e} \big) \big(\nabla_a\, h^e_{\ b} + \nabla_b\, h^e_{\ a} - \overline{g}^{em}\, \nabla_m\, h_{ba} \big) \Bigg) + \mathcal{O}(\epsilon^3)
\end{aligned}
$$

4.5 Appendix: Explicit Calculations

4.5.1 Einstein Tensor Computations

The covariant and contravariant perturbative expansions for the metric are

$$
g_{ab} = \overline{g}_{ab} + \epsilon\, h_{ab} + \frac{1}{2}\, \epsilon^2\, h_{ac}\, h^c_{\ b} + \mathcal{O}(\epsilon^3) \tag{4.317}
$$

$$
g^{ab} = \overline{g}^{ab} - \epsilon\, h^{ab} + \frac{1}{2}\, \epsilon^2\, h^{ac}\, h_c^{\ b} + \mathcal{O}(\epsilon^3) \tag{4.318}
$$

We compute up to $\mathcal{O}(\epsilon^3)$ in order to better understand any ϵ dependence of the mixing matrix. Given the metric expansion, the perturbative expansion for the connection is

$$
\begin{aligned}
\Gamma^a{}_{bc} &= \frac{1}{2}\, g^{am}\Big(\partial_c\, g_{mb} + \partial_b\, g_{mc} - \partial_m\, g_{bc}\Big)\\
&= \frac{1}{2}\,\left(\overline{g}^{am} - \epsilon\, h^{am} + \frac{1}{2}\,\epsilon^2\, h^{ac}\, h_c{}^m\right)\left(\partial_c\, \overline{g}_{mb} + \partial_b\, \overline{g}_{mc} - \partial_m\, \overline{g}_{bc} + \epsilon\,(\partial_c\, h_{mb} + \partial_b\, h_{mc} - \partial_m\, h_{bc})\right.
\end{aligned}
\tag{4.319}
$$

$$
\begin{aligned}
&\left. + \frac{\epsilon^2}{2}\,\Big(\partial_c\,(h_{mc}\, h^c{}_b) + \partial_b\,\big(h_{md}\, h^d{}_c\big) - \partial_m\,\big(h_{bd}\, h^d{}_c\big)\Big)\right) + \mathcal{O}(\epsilon^3)\\
&= \frac{1}{2}\,\overline{g}^{am}\,(\partial_c\, \overline{g}_{mb} + \partial_b\, \overline{g}_{mc} - \partial_m\, \overline{g}_{bc}) + \frac{\epsilon}{2}\,\Big(\overline{g}^{am}\,(\partial_c\, h_{mb} + \partial_b\, h_{mc} - \partial_m\, h_{bc})\\
&- h^{am}\,(\partial_c\, \overline{g}_{mb} + \partial_b\, \overline{g}_{mc} - \partial_m\, \overline{g}_{bc})\Big)\\
&+ \frac{\epsilon^2}{4}\,\Big(\overline{g}^{am}\,\big(\partial_c\,(h_{mc}\, h^c{}_b) + \partial_b\,\big(h_{md}\, h^d{}_c\big) - \partial_m\,\big(h_{bd}\, h^d{}_c\big)\big) - 2\, h^{am}\,(\partial_c\, h_{mb} + \partial_b\, h_{mc} - \partial_m\, h_{bc})\\
&+ h^{ac}\, h_c{}^m\,(\partial_c\, \overline{g}_{mb} + \partial_b\, \overline{g}_{mc} - \partial_m\, \overline{g}_{bc})\Big) + \mathcal{O}(\epsilon^3)
\end{aligned}
$$

$$
= \overline{\Gamma}^a{}_{bc} + \frac{\epsilon}{2}\Big(\overline{g}^{am}\,(\partial_c\, h_{mb} + \partial_b\, h_{mc} - \partial_m\, h_{bc}) - h^{am}\,(\partial_c\, \overline{g}_{mb} + \partial_b\, \overline{g}_{mc} - \partial_m\, \overline{g}_{bc})\Big) \tag{4.320}
$$

$$
\begin{aligned}
&+ \frac{\epsilon^2}{4}\,\Big(\overline{g}^{am}\,\big(\partial_c\,(h_{mc}\, h^c{}_b) + \partial_b\,\big(h_{md}\, h^d{}_c\big) - \partial_m\,\big(h_{bd}\, h^d{}_c\big)\big) - 2\, h^{am}\,(\partial_c\, h_{mb} + \partial_b\, h_{mc} - \partial_m\, h_{bc})\\
&\qquad + h^{ac}\, h_c{}^m\,(\partial_c\, \overline{g}_{mb} + \partial_b\, \overline{g}_{mc} - \partial_m\, \overline{g}_{bc})\Big) + \mathcal{O}(\epsilon^3)
\end{aligned}
$$

$$
= \overline{\Gamma}^a{}_{bc} + \frac{\epsilon}{2}\,\overline{g}^{am}\Big(\partial_c\, h_{mb} + \partial_b\, h_{mc} - \partial_m\, h_{bc}\Big) - \epsilon\, h^{am}\,\overline{g}_{me}\overline{\Gamma}^e{}_{bc} \tag{4.321}
$$

$$
\begin{aligned}
&+ \frac{\epsilon^2}{4}\Big(\overline{g}^{am}\Big(\partial_c\,(h_{mc}\, h^c{}_b) + \partial_b\,\big(h_{md}\, h^d{}_c\big) - \partial_m\,\big(h_{bd}\, h^d{}_c\big)\Big)\\
&- 2\, h^{am}\Big(\partial_c\, h_{mb} + \partial_b\, h_{mc} - \partial_m\, h_{bc}\Big) + 2\, h^{ac}\, h_c{}^m\,\overline{\Gamma}_{mbc}\Big)
\end{aligned}
$$

$$
\Gamma^a{}_{bc} = \overline{\Gamma}^a{}_{bc} + \frac{\epsilon}{2}\,\overline{g}^{am}\Big(\nabla_c\, h_{mb} + \nabla_b\, h_{mc} - \nabla_m\, h_{bc}\Big) \tag{4.322}
$$

$$
\begin{aligned}
&+ \frac{\epsilon^2}{4}\Big(\overline{g}^{am}\Big(\nabla_c\,(h_{mq}\, h^q{}_b) + \nabla_b\,(h_{mq}\, h^q{}_c) - \nabla_m\,(h_{bq}\, h^q{}_c)\Big)\\
&- 2\, h^{am}\Big(\nabla_c\, h_{mb} + \nabla_b\, h_{mc} - \nabla_m\, h_{bc}\Big)\Big) + \mathcal{O}(\epsilon^3)\,.
\end{aligned}
$$

For the $\epsilon/2$ term, we calculate

$$\overline{g}^{am}\Big(\nabla_c\, h_{mb} + \nabla_b\, h_{mc} - \nabla_m\, h_{bc}\Big) = \tag{4.323}$$

$$\overline{g}^{am}\Big(\partial_c\, h_{mb} - \overline{\Gamma}^d{}_{mc}\, h_{db} - \overline{\Gamma}^d{}_{bc}\, h_{md} + \partial_b\, h_{mc} - \overline{\Gamma}^d{}_{mb}\, h_{dc} - \overline{\Gamma}^d{}_{bc}\, h_{md} - \partial_m\, h_{bc} + \overline{\Gamma}^d{}_{bm}\, h_{dc}$$

$$+ \overline{\Gamma}^d{}_{cm}\, h_{bd}\Big) =$$

$$\overline{g}^{am}\Big(\partial_c\, h_{mb} + \partial_b\, h_{mc} - \partial_m\, h_{bc} - \overline{\Gamma}^d{}_{mc}\, h_{db} - \overline{\Gamma}^d{}_{bc}\, h_{md} - \overline{\Gamma}^d{}_{mb}\, h_{dc} - \overline{\Gamma}^d{}_{bc}\, h_{md} + \overline{\Gamma}^d{}_{bm}\, h_{dc}$$

$$+ \overline{\Gamma}^d{}_{cm}\, h_{bd}\Big) =$$

$$\overline{g}^{am}\Big(\partial_c\, h_{mb} + \partial_b\, h_{mc} - \partial_m\, h_{bc}\Big) - \overline{g}^{am}\left(\overline{\Gamma}^d{}_{bc}\, h_{md} + \overline{\Gamma}^d{}_{bc}\, h_{md}\right) =$$

$$\overline{g}^{am}\Big(\partial_c\, h_{mb} + \partial_b\, h_{mc} - \partial_m\, h_{bc}\Big) - 2\, \overline{g}^{am}\overline{\Gamma}^d{}_{bc}\, h_{md}\,.$$

We compute the ϵ^2 terms,

$$\overline{g}^{am}\left(\nabla_c\,(h_{me}\,h^{e}{}_{b}) + \nabla_b\,\left(h_{md}\,h^{d}{}_{c}\right) - \nabla_m\,\left(h_{bd}\,h^{d}{}_{c}\right)\right) - 2\,h^{am}\left(\nabla_c\,h_{mb} + \nabla_b\,h_{mc} - \nabla_m\,h_{bc}\right) = \tag{4.324}$$

$$\overline{g}^{am}\Bigg(\partial_c\,(h_{me}\,h^{e}{}_{b}) - \Gamma^{f}{}_{mc}\,(h_{fe}\,h^{e}{}_{b}) - \Gamma^{f}{}_{bc}\,\left(h_{me}\,h^{e}{}_{f}\right) + \partial_b\,(h_{me}\,h^{e}{}_{c}) - \Gamma^{f}{}_{mb}\,(h_{fe}\,h^{e}{}_{c}) - \Gamma^{f}{}_{cb}\,\left(h_{me}\,h^{e}{}_{f}\right)$$

$$- \partial_m\,\left(h_{bd}\,h^{d}{}_{c}\right) + \Gamma^{f}{}_{bm}\,\left(h_{fd}\,h^{d}{}_{c}\right) + \Gamma^{f}{}_{cm}\,\left(h_{bd}\,h^{d}{}_{f}\right)\Bigg)$$

$$- 2\,h^{am}\Bigg(\partial_c\,h_{mb} - \Gamma^{f}{}_{mc}\,h_{fb} - \Gamma^{f}{}_{cb}\,h_{mf} + \partial_b\,h_{mc}$$

$$- \Gamma^{f}{}_{mb}\,h_{fc} - \Gamma^{f}{}_{cb}\,h_{mf} - \partial_m\,h_{bc} + \Gamma^{g}{}_{bm}\,h_{gc} + \Gamma^{g}{}_{cm}\,h_{bg}\Bigg) =$$

$$\overline{g}^{am}\Bigg(\partial_c\,(h_{me}\,h^{e}{}_{b}) - \cancel{\Gamma^{f}{}_{mc}\,(h_{fe}\,h^{e}{}_{b})} - \Gamma^{f}{}_{bc}\,\left(h_{me}\,h^{e}{}_{f}\right) + \partial_b\,(h_{me}\,h^{e}{}_{c})$$

$$- \cancel{\Gamma^{f}{}_{mb}\,(h_{fe}\,h^{e}{}_{c})} - \Gamma^{f}{}_{cb}\,\left(h_{me}\,h^{e}{}_{f}\right) - \partial_m\,\left(h_{bd}\,h^{d}{}_{c}\right)$$

$$+ \cancel{\Gamma^{f}{}_{bm}\,\left(h_{fd}\,h^{d}{}_{c}\right)} + \cancel{\Gamma^{f}{}_{cm}\,\left(h_{bd}\,h^{d}{}_{f}\right)}\Bigg) - 2\,h^{am}\Bigg(\partial_c\,h_{mb} - \cancel{\Gamma^{f}{}_{mc}\,h_{fb}}$$

$$- \Gamma^{f}{}_{cb}\,h_{mf} + \partial_b\,h_{mc} - \cancel{\Gamma^{f}{}_{mb}\,h_{fc}} - \Gamma^{f}{}_{cb}\,h_{mf}$$

$$- \partial_m\,h_{bc} + \cancel{\Gamma^{g}{}_{bm}\,h_{gc}} + \cancel{\Gamma^{g}{}_{cm}\,h_{bg}}\Bigg) =$$

$$\overline{g}^{am}\left(\partial_c\,(h_{me}\,h^{e}{}_{b}) + \partial_b\,(h_{me}\,h^{e}{}_{c}) - \partial_m\,\left(h_{bd}\,h^{d}{}_{c}\right)\right) - 2\,h^{am}\left(\partial_c\,h_{mb} + \partial_b\,h_{mc} - \partial_m\,h_{bc}\right)$$

$$- 2\,\overline{g}^{am}\,\Gamma^{f}{}_{bc}\,\left(h_{me}\,h^{e}{}_{f}\right) + 4\,h^{am}\,\Gamma^{f}{}_{cb}\,h_{mf} =$$

$$\overline{g}^{am}\left(\partial_c\,(h_{me}\,h^{e}{}_{b}) + \partial_b\,(h_{me}\,h^{e}{}_{c}) - \partial_m\,\left(h_{bd}\,h^{d}{}_{c}\right)\right) - 2\,h^{am}\left(\partial_c\,h_{mb} + \partial_b\,h_{mc} - \partial_m\,h_{bc}\right)$$

$$+ 2\,h^{ac}\,h_{c}{}^{m}\,\overline{\Gamma}_{mbc}$$

We now compute the Riemann tensor, $R^a_{\ bcd} = \partial_c \,\Gamma^a_{\ bd} - \partial_d \,\Gamma^a_{\ bc} + \Gamma^a_{\ ce}\,\Gamma^e_{\ bd} - \Gamma^a_{\ de}\,\Gamma^e_{\ bc}$. Each of the terms in this equation are

$$\begin{aligned}\partial_c \,\Gamma^a_{\ bd} = \partial_c \,\overline{\Gamma}^a_{\ bd} &+ \frac{\epsilon}{2}\,\partial_c\Big(\overline{g}^{am}\big(\nabla_d\, h_{mb} + \nabla_b\, h_{md} - \nabla_m\, h_{bd}\big)\Big) \\ &+ \frac{\epsilon^2}{4}\,\partial_c\Big(\overline{g}^{am}\Big(\nabla_d\,(h_{mq}\, h^q_{\ b}) + \nabla_b\,(h_{mq}\, h^q_{\ d}) - \nabla_m\,(h_{bq}\, h^q_{\ d})\Big) \\ &- 2\, h^{am}\Big(\nabla_d\, h_{mb} + \nabla_b\, h_{md} - \nabla_m\, h_{bd}\Big)\Big) + \mathcal{O}(\epsilon^3)\end{aligned} \tag{4.325}$$

$$\begin{aligned}-\partial_d \,\Gamma^a_{\ bc} = -\partial_d \,\overline{\Gamma}^a_{\ bc} &- \frac{\epsilon}{2}\,\partial_d\Big(\overline{g}^{am}\big(\nabla_c\, h_{mb} + \nabla_b\, h_{mc} - \nabla_m\, h_{bc}\big)\Big) \\ &- \frac{\epsilon^2}{4}\,\partial_d\,\Big(\overline{g}^{am}\Big(\nabla_c\,(h_{mg}\, h^g_{\ b}) + \nabla_b\,(h_{mg}\, h^g_{\ c}) - \nabla_m\,(h_{bg}\, h^g_{\ c})\Big) \\ &- 2\, h^{am}\Big(\nabla_c\, h_{mb} + \nabla_b\, h_{mc} - \nabla_m\, h_{bc}\Big)\Big) + \mathcal{O}(\epsilon^3)\end{aligned} \tag{4.326}$$

$$
\begin{aligned}
\Gamma^{a}{}_{ce}\,\Gamma^{e}{}_{bd} = \overline{\Gamma}^{a}{}_{ce}\overline{\Gamma}^{e}{}_{bd} + \frac{\epsilon}{2}\Bigg(&\overline{g}^{em}\,\overline{\Gamma}^{a}{}_{ce}\big(\nabla_d\,h_{mb} + \nabla_b\,h_{md} - \nabla_m\,h_{bd}\big)\\
&+ \overline{g}^{am}\,\overline{\Gamma}^{e}{}_{bd}\left(\nabla_e\,h_{mc} + \nabla_c\,h_{me} - \nabla_m\,h_{ce}\right)\Bigg)\\
&+ \frac{\epsilon^2}{4}\Bigg(\overline{g}^{em}\,\overline{\Gamma}^{a}{}_{ce}\left(\nabla_d\left(h_{mq}\,h^{q}{}_{b}\right) + \nabla_b\left(h_{mg}\,h^{g}{}_{d}\right) - \nabla_m\left(h_{bg}\,h^{g}{}_{d}\right)\right)\\
&- 2\,h^{em}\,\overline{\Gamma}^{a}{}_{ce}\left(\nabla_d\,h_{mb} + \nabla_b\,h_{md} - \nabla_m\,h_{bd}\right)\\
&+ \overline{g}^{eq}\,\overline{g}^{am}\big(\nabla_e\,h_{mc} + \nabla_c\,h_{me} - \nabla_m\,h_{ce}\big)\big(\nabla_d\,h_{qb} + \nabla_b\,h_{qd} - \nabla_q\,h_{bd}\big)\\
&+ \overline{g}^{am}\,\overline{\Gamma}^{e}{}_{bd}\left(\nabla_e\left(h_{mn}\,h^{n}{}_{c}\right) + \nabla_c\left(h_{mn}\,h^{n}{}_{e}\right) - \nabla_m\left(h_{cm}\,h^{m}{}_{e}\right)\right)\\
&- 2\,h^{am}\,\overline{\Gamma}^{e}{}_{bd}\left(\nabla_e\,h_{mc} + \nabla_c\,h_{me} - \nabla_m\,h_{ce}\right)\Bigg)\\
&+ \mathcal{O}(\epsilon^3)
\end{aligned}
$$

$$
\begin{aligned}
-\Gamma^{a}{}_{de}\,\Gamma^{e}{}_{bc} = -\overline{\Gamma}^{a}{}_{de}\overline{\Gamma}^{e}{}_{bc} - \frac{\epsilon}{2}\Bigg(&\overline{g}^{em}\,\overline{\Gamma}^{a}{}_{de}\big(\nabla_c\,h_{mb} + \nabla_b\,h_{mc} - \nabla_m\,h_{bc}\big)\\
&+ \overline{g}^{am}\,\overline{\Gamma}^{e}{}_{bc}\big(\nabla_e\,h_{md} + \nabla_d\,h_{me} - \nabla_m\,h_{de}\big)\Bigg)\\
&- \frac{\epsilon^2}{4}\Bigg(\overline{g}^{em}\,\overline{\Gamma}^{a}{}_{de}\left(\nabla_c\left(h_{mg}\,h^{g}{}_{b}\right) + \nabla_b\left(h_{mg}\,h^{g}{}_{c}\right) - \nabla_m\left(h_{bg}\,h^{g}{}_{c}\right)\right)\\
&- 2\,h^{em}\,\overline{\Gamma}^{a}{}_{de}\left(\nabla_c\,h_{mb} + \nabla_b\,h_{mc} - \nabla_m\,h_{bc}\right)\\
&+ \overline{g}^{em}\,\overline{g}^{aq}\big(\nabla_e\,h_{qd} + \nabla_d\,h_{qe} - \nabla_q\,h_{de}\big)\big(\nabla_c\,h_{mb} + \nabla_b\,h_{mc} - \nabla_m\,h_{bc}\big)\\
&+ \overline{g}^{am}\,\overline{\Gamma}^{e}{}_{bc}\left(\nabla_e\left(h_{mn}\,h^{n}{}_{d}\right) + \nabla_d\left(h_{mn}\,h^{n}{}_{e}\right) - \nabla_m\left(h_{dm}\,h^{m}{}_{e}\right)\right)\\
&- 2\,h^{am}\,\overline{\Gamma}^{e}{}_{bc}\left(\nabla_e\,h_{md} + \nabla_d\,h_{me} - \nabla_m\,h_{de}\right)\Bigg)\\
&+ \mathcal{O}(\epsilon^3)
\end{aligned}
$$

This yields

$$
\begin{aligned}
R^a{}_{bcd} &= \partial_c \overline{\Gamma}^a{}_{bd} - \partial_d \overline{\Gamma}^a{}_{bc} + \overline{\Gamma}^a{}_{ce}\overline{\Gamma}^e{}_{bd} - \overline{\Gamma}^a{}_{de}\overline{\Gamma}^e{}_{bc} \qquad (4.327)\\
&+ \frac{\epsilon}{2}\Bigg(\partial_c\big(\overline{g}^{am}\big(\nabla_d h_{mb} + \nabla_b h_{md} - \nabla_m h_{bd}\big)\big) - \partial_d\big(\overline{g}^{am}\big(\nabla_c h_{mb} + \nabla_b h_{mc} - \nabla_m h_{bc}\big)\big)\\
&+ \overline{g}^{em}\,\overline{\Gamma}^a{}_{ce}\big(\nabla_d h_{mb} + \nabla_b h_{md} - \nabla_m h_{bd}\big) + \overline{g}^{am}\,\overline{\Gamma}^e{}_{bd}\big(\nabla_e h_{mc} + \nabla_c h_{me} - \nabla_m h_{ce}\big)\\
&- \overline{g}^{em}\,\overline{\Gamma}^a{}_{de}\big(\nabla_c h_{mb} + \nabla_b h_{mc} - \nabla_m h_{bc}\big) - \overline{g}^{am}\,\overline{\Gamma}^e{}_{bc}\big(\nabla_e h_{md} + \nabla_d h_{me} - \nabla_m h_{de}\big)\Bigg)\\
&+ \frac{\epsilon^2}{4}\Bigg(\partial_c\Big(\overline{g}^{am}\Big(\nabla_d\,(h_{mq}\,h^q{}_b) + \nabla_b\,(h_{mq}\,h^q{}_d) - \nabla_m\,(h_{bq}\,h^q{}_d)\Big)\\
&- 2\,h^{am}\Big(\nabla_d h_{mb} + \nabla_b h_{md} - \nabla_m h_{bd}\Big)\Big)\\
&- \partial_d\Bigg(\overline{g}^{am}\Big(\nabla_c\,(h_{mg}\,h^g{}_b) + \nabla_b\,(h_{mg}\,h^g{}_c) - \nabla_m\,(h_{bg}\,h^g{}_c)\Big)\\
&- 2\,h^{am}\Big(\nabla_c h_{mb} + \nabla_b h_{mc} - \nabla_m h_{bc}\Big)\Bigg)\\
&+ \overline{g}^{em}\,\overline{\Gamma}^a{}_{ce}\,\big(\nabla_d\,(h_{mc}\,h^c{}_b) + \nabla_b\,(h_{mg}\,h^g{}_d) - \nabla_m\,(h_{bg}\,h^g{}_d)\big)\\
&- 2\,h^{em}\,\overline{\Gamma}^a{}_{ce}\,\big(\nabla_d h_{mb} + \nabla_b h_{md} - \nabla_m h_{bd}\big)\\
&+ \overline{g}^{em}\,\overline{g}^{am}\big(\nabla_e h_{mc} + \nabla_c h_{me} - \nabla_m h_{ce}\big)\big(\nabla_d h_{mb} + \nabla_b h_{md} - \nabla_m h_{bd}\big)\\
&+ \overline{g}^{am}\,\overline{\Gamma}^e{}_{bd}\,\big(\nabla_e\,(h_{mn}\,h^n{}_c) + \nabla_c\,(h_{mn}\,h^n{}_e) - \nabla_m\,(h_{cm}\,h^m{}_e)\big)\\
&- 2\,h^{am}\,\overline{\Gamma}^e{}_{bd}\,\big(\nabla_e h_{mc} + \nabla_c h_{me} - \nabla_m h_{ce}\big)\\
&- \overline{g}^{em}\,\overline{\Gamma}^a{}_{de}\,\big(\nabla_c\,(h_{mg}\,h^g{}_b) + \nabla_b\,(h_{mg}\,h^g{}_c) - \nabla_m\,(h_{bg}\,h^g{}_c)\big)\\
&+ 2\,h^{em}\,\overline{\Gamma}^a{}_{de}\,\big(\nabla_c h_{mb} + \nabla_b h_{mc} - \nabla_m h_{bc}\big)\\
&- \overline{g}^{em}\,\overline{g}^{aq}\big(\nabla_e h_{qd} + \nabla_d h_{qe} - \nabla_q h_{de}\big)\big(\nabla_c h_{mb} + \nabla_b h_{mc} - \nabla_m h_{bc}\big)\\
&- \overline{g}^{am}\,\overline{\Gamma}^e{}_{bc}\,\big(\nabla_e\,(h_{mn}\,h^n{}_d) + \nabla_d\,(h_{mn}\,h^n{}_e) - \nabla_m\,(h_{dm}\,h^m{}_e)\big)\\
&+ 2\,h^{am}\,\overline{\Gamma}^e{}_{bc}\,\big(\nabla_e h_{md} + \nabla_d h_{me} - \nabla_m h_{de}\big)\Bigg)\\
&+ \mathcal{O}(\epsilon^3)
\end{aligned}
$$

Like before, we can promote the partial derivatives to covariant derivatives and cancel terms.

$$\begin{aligned}\nabla_c F^a_{\ bd} - \nabla_d F^a_{\ bc} &= \partial_c F^a_{\ bd} - \overline{\Gamma}^q_{\ bc} F^a_{\ qd} - \cancel{\overline{\Gamma}^q_{\ cd} F^a_{\ bq}} + \overline{\Gamma}^a_{\ cq} F^q_{\ bd} - \partial_d F^a_{\ bc} + \overline{\Gamma}^q_{\ bd} F^a_{\ qc} + \cancel{\overline{\Gamma}^q_{\ cd} F^a_{\ bq}} - \overline{\Gamma}^a_{\ dq} F^q_{\ bc} \\ &= \partial_c F^a_{\ bd} - \partial_d F^a_{\ bc} - \overline{\Gamma}^q_{\ bc} F^a_{\ qd} + \overline{\Gamma}^a_{\ cq} F^q_{\ bd} + \overline{\Gamma}^q_{\ bd} F^a_{\ qc} - \overline{\Gamma}^a_{\ dq} F^q_{\ bc}\end{aligned}$$

Here F is a generic tensor.

$$\begin{aligned}R^a_{\ bcd} &= \overline{R}^a_{\ bcd} \\ &+ \frac{\epsilon}{2}\left(\nabla_c\big(\overline{g}^{am}\big(\nabla_d\, h_{mb} + \nabla_b\, h_{md} - \nabla_m\, h_{bd}\big)\big) - \nabla_d\big(\overline{g}^{am}\big(\nabla_c\, h_{mb} + \nabla_b\, h_{mc} - \nabla_m\, h_{bc}\big)\big)\right) \\ &+ \frac{\epsilon^2}{4}\Bigg(\nabla_c\Big(\overline{g}^{am}\big(\nabla_d\left(h_{mr}\, h^r_{\ b}\right) + \nabla_b\left(h_{mr}\, h^r_{\ d}\right) - \nabla_m\left(h_{br}\, h^r_{\ d}\right)\big) \\ &- 2\, h^{am}\big(\nabla_d\, h_{mb} + \nabla_b\, h_{md} - \nabla_m\, h_{bd}\big)\Big) \\ &\qquad - \nabla_d\Big(\overline{g}^{am}\big(\nabla_c\left(h_{mg}\, h^g_{\ b}\right) + \nabla_b\left(h_{mg}\, h^g_{\ c}\right) - \nabla_m\left(h_{bg}\, h^g_{\ c}\right)\big) \\ &- 2\, h^{am}\big(\nabla_c\, h_{mb} + \nabla_b\, h_{mc} - \nabla_m\, h_{bc}\big)\Big) \\ &\qquad + \overline{g}^{eq}\, \overline{g}^{am}\big(\nabla_e\, h_{mc} + \nabla_c\, h_{me} - \nabla_m\, h_{ce}\big)\big(\nabla_d\, h_{qb} + \nabla_b\, h_{qd} - \nabla_q\, h_{bd}\big) \\ &\qquad - \overline{g}^{em}\, \overline{g}^{aq}\big(\nabla_e\, h_{qd} + \nabla_d\, h_{qe} - \nabla_q\, h_{de}\big)\big(\nabla_c\, h_{mb} + \nabla_b\, h_{mc} - \nabla_m\, h_{bc}\big)\Bigg) + \mathcal{O}(\epsilon^3)\end{aligned} \tag{4.328}$$

$$
\begin{aligned}
R^a{}_{bcd} = \overline{R}^a{}_{bcd} &+ \frac{\epsilon}{2}\Bigg(\overline{g}^{am}\big(\nabla_c\nabla_d\, h_{mb} + \nabla_c\nabla_b\, h_{md} - \nabla_c\nabla_m\, h_{bd}\big) \\
&- \overline{g}^{am}\big(\nabla_d\nabla_c\, h_{mb} + \nabla_d\nabla_b\, h_{mc} - \nabla_d\nabla_m\, h_{bc}\big)\Bigg) \\
&+ \frac{\epsilon^2}{4}\Bigg(\overline{g}^{am}\big(\nabla_c\nabla_d\,(h_{mr}\, h^r{}_b) + \nabla_c\nabla_b\,(h_{mr}\, h^r{}_d) - \nabla_c\nabla_m\,(h_{br}\, h^r{}_d)\big) \\
&- 2\,\nabla_c\big(h^{am}\big(\nabla_d\, h_{mb} + \nabla_b\, h_{md} - \nabla_m\, h_{bd}\big)\big) \\
&\quad - \overline{g}^{am}\big(\nabla_d\nabla_c\,(h_{mg}\, h^g{}_b) + \nabla_d\nabla_b\,(h_{mg}\, h^g{}_c) - \nabla_d\nabla_m\,(h_{bg}\, h^g{}_c)\big) \\
&- 2\,\nabla_d\big(h^{am}\big(\nabla_c\, h_{mb} + \nabla_b\, h_{mc} - \nabla_m\, h_{bc}\big)\big) \\
&\quad + \overline{g}^{eq}\,\overline{g}^{am}\big(\nabla_e\, h_{mc} + \nabla_c\, h_{me} - \nabla_m\, h_{ce}\big)\big(\nabla_d\, h_{qb} + \nabla_b\, h_{qd} - \nabla_q\, h_{bd}\big) \\
&\quad - \overline{g}^{em}\,\overline{g}^{aq}\big(\nabla_e\, h_{qd} + \nabla_d\, h_{qe} - \nabla_q\, h_{de}\big)\big(\nabla_c\, h_{mb} + \nabla_b\, h_{mc} - \nabla_m\, h_{bc}\big)\Bigg) + \mathcal{O}(\epsilon^3)
\end{aligned}
\tag{4.329}
$$

We can simplify a bit more to find a final form for the Riemann tensor

$$
\begin{aligned}
R^a{}_{bcd} = \overline{R}^a{}_{bcd} &+ \frac{\epsilon}{2}\Bigg(\nabla_c\nabla_d\, h^a{}_b + \nabla_c\nabla_b\, h^a{}_d - \overline{g}^{am}\nabla_c\nabla_m\, h_{bd} - \nabla_d\nabla_c\, h^a{}_b - \nabla_d\nabla_b\, h^a{}_c + \overline{g}^{am}\nabla_d\nabla_m\, h_{bc}\Bigg) \\
&+ \frac{\epsilon^2}{4}\Bigg(\nabla_c\nabla_d\,(h^a{}_r\, h^r{}_b) + \nabla_c\nabla_b\,(h^a{}_r\, h^r{}_d) - \overline{g}^{am}\nabla_c\nabla_m\,(h_{br}\, h^r{}_d) \\
&- \nabla_d\nabla_c\,\big(h^a{}_g\, h^g{}_b\big) - \nabla_d\,\nabla_b\,(h^a{}_r\, h^r{}_c) + \overline{g}^{am}\nabla_d\,\nabla_m\,(h_{br}\, h^r{}_c) \\
&- 2\,\nabla_c\,\Big(h^{am}\big(\nabla_d\, h_{mb} + \nabla_b\, h_{md} - \nabla_m\, h_{bd}\big)\Big) + 2\,\nabla_d\Big(h^{am}\big(\nabla_c\, h_{mb} + \nabla_b\, h_{mc} - \nabla_m\, h_{bc}\big)\Big) \\
&+ \big(\nabla_e\, h^a{}_c + \nabla_c\, h^a{}_e - \overline{g}^{am}\nabla_m\, h_{ce}\big)\big(\nabla_d\, h^e{}_b + \nabla_b\, h^e{}_d - \overline{g}^{eq}\,\nabla_q\, h_{bd}\big) \\
&- \big(\nabla_e\, h^a{}_d + \nabla_d\, h^a{}_e - \overline{g}^{aq}\nabla_q\, h_{de}\big)\big(\nabla_c\, h^e{}_b + \nabla_b\, h^e{}_c - \overline{g}^{em}\,\nabla_m\, h_{bc}\big)\Bigg) + \mathcal{O}(\epsilon^3)\,.
\end{aligned}
\tag{4.330}
$$

Denoting $h = h^a_{\ a}$, the Ricci tensor and scalar are

$$R_{\ bd} = \overline{R}_{\ bd} + \frac{\epsilon}{2}\left(-\Box h_{bd} - \nabla_d \nabla_b h + \nabla_q \nabla_b h^q_{\ d} + \nabla_q \nabla_d h^q_{\ b}\right) + \frac{\epsilon^2}{4}\Bigg(-\Box\left(h_{br} h^r_{\ d}\right) - \nabla_d \nabla_b \left(h^a_{\ r} h^r_{\ a}\right) \tag{4.331}$$

$$+ \nabla_q \nabla_b \left(h^q_{\ r} h^r_{\ d}\right) + \nabla_q \nabla_d \left(h^q_{\ r} h^r_{\ b}\right)$$

$$- 2\nabla_a\Big(h^{am}\big(\nabla_d h_{mb} + \nabla_b h_{md} - \nabla_m h_{bd}\big)\Big) + 2\nabla_d\Big(h^{am}\big(\nabla_a h_{mb} + \nabla_b h_{ma} - \nabla_m h_{ba}\big)\Big)$$

$$+ \big(\nabla_e h^a_{\ a} + \nabla_a h^a_{\ e} - \overline{g}^{am} \nabla_m h_{ae}\big)\big(\nabla_d h^e_{\ b} + \nabla_b h^e_{\ d} - \overline{g}^{eq} \nabla_q h_{bd}\big)$$

$$- \big(\nabla_e h^a_{\ d} + \nabla_d h^a_{\ e} - \overline{g}^{aq} \nabla_q h_{de}\big)\big(\nabla_a h^e_{\ b} + \nabla_b h^e_{\ a} - \overline{g}^{em} \nabla_m h_{ba}\big)\Bigg) + \mathcal{O}(\epsilon^3)$$

$$= \overline{R}_{\ bd} + \frac{\epsilon}{2}\left(-\Box h_{bd} - \nabla_d \nabla_b h + \nabla_q \nabla_b h^q_{\ d} + \nabla_q \nabla_d h^q_{\ b}\right) + \frac{\epsilon^2}{4}\Bigg(-\Box\left(h_{br} h^r_{\ d}\right) - \nabla_d \nabla_b \left(h^a_{\ r} h^r_{\ a}\right) \tag{4.332}$$

$$+ \nabla_q \nabla_b \left(h^q_{\ r} h^r_{\ d}\right) + \nabla_q \nabla_d \left(h^q_{\ r} h^r_{\ b}\right)$$

$$- 2\nabla_a\Big(h^{am}\big(\nabla_d h_{mb} + \nabla_b h_{md} - \nabla_m h_{bd}\big)\Big) + 2\nabla_d\Big(h^{am}\big(\nabla_a h_{mb} + \nabla_b h_{ma} - \nabla_m h_{ba}\big)\Big)$$

$$+ (\nabla_a h^a_{\ e})\big(\nabla_d h^e_{\ b} + \nabla_b h^e_{\ d} - \overline{g}^{eq} \nabla_q h_{bd}\big) - (\overline{g}^{am} \nabla_m h_{ae})\big(\nabla_d h^e_{\ b} + \nabla_b h^e_{\ d} - \overline{g}^{eq} \nabla_q h_{bd}\big)$$

$$- \big(\nabla_e h^a_{\ d} + \nabla_d h^a_{\ e} - \overline{g}^{aq} \nabla_q h_{de}\big)\big(\nabla_a h^e_{\ b} + \nabla_b h^e_{\ a} - \overline{g}^{em} \nabla_m h_{ba}\big)\Bigg) + \mathcal{O}(\epsilon^3)$$

The ϵ^2 terms can further simplified to find

$$
\begin{aligned}
-g^{gh}\nabla_g \nabla_h \left(h_{br} \, h^r{}_d\right) &= -g^{gh}\nabla_g \left(h_{br} \nabla_h h^r{}_d\right) - g^{gh}\nabla_g \left(h^r{}_d \nabla_h h_{br}\right) \\
&= -g^{gh} \left(\nabla_g h_{br}\right) \left(\nabla_h h^r{}_d\right) - g^{gh} \left(h_{br} \nabla_g \nabla_h h^r{}_d\right) \\
&\quad - g^{gh} \left(\nabla_g h^r{}_d\right) \left(\nabla_h h_{br}\right) - g^{gh} \left(h^r{}_d \nabla_g \nabla_h h_{br}\right) \\
&= -2\, g^{gh} \left(\nabla_g h_{br}\right) \left(\nabla_h h^r{}_d\right) - g^{gh} \left(h_{br} \nabla_g \nabla_h h^r{}_d\right) - g^{gh} \left(h^r{}_d \nabla_g \nabla_h h_{br}\right)
\end{aligned} \tag{4.333}
$$

$$
\begin{aligned}
-\nabla_d \nabla_b \left(h^a{}_r \, h^r{}_a\right) &= -\nabla_d \left(h^a{}_r \nabla_b h^r{}_a\right) - \nabla_d \left(h^r{}_a \nabla_b h^a{}_r\right) \\
&= -\left(\nabla_d h^a{}_r\right) \left(\nabla_b h^r{}_a\right) - h^a{}_r \nabla_d \nabla_b h^r{}_a - \left(\nabla_d h^r{}_a\right) \left(\nabla_b h^a{}_r\right) - h^r{}_a \nabla_d \nabla_b h^a{}_r \\
&= -2 \left(\nabla_d h^a{}_r\right) \left(\nabla_b h^r{}_a\right) - 2\, h^a{}_r \nabla_d \nabla_b h^r{}_a
\end{aligned} \tag{4.334}
$$

$$
\begin{aligned}
\nabla_q \nabla_b \left(h^q{}_r \, h^r{}_d\right) &= \nabla_q \left(h^q{}_r \nabla_b h^r{}_d\right) + \nabla_q \left(h^r{}_d \nabla_b h^q{}_r\right) \\
&= \left(\nabla_q h^q{}_r\right) \left(\nabla_b h^r{}_d\right) + h^q{}_r \nabla_q \nabla_b h^r{}_d + \left(\nabla_q h^r{}_d\right) \left(\nabla_b h^q{}_r\right) + h^r{}_d \nabla_q \nabla_b h^q{}_r
\end{aligned} \tag{4.335}
$$

$$
\begin{aligned}
\nabla_q \nabla_d \left(h^q{}_r \, h^r{}_b\right) &= \nabla_q \left(h^q{}_r \nabla_d h^r{}_b\right) + \nabla_q \left(h^r{}_b \nabla_d h^q{}_r\right) \\
&= \left(\nabla_q h^q{}_r\right) \left(\nabla_d h^r{}_b\right) + h^q{}_r \nabla_q \nabla_d h^r{}_b + \left(\nabla_q h^r{}_b\right) \left(\nabla_d h^q{}_r\right) + h^r{}_b \nabla_q \nabla_d h^q{}_r
\end{aligned} \tag{4.336}
$$

as well as

$$
\begin{aligned}
-2\,\nabla_a\Big(h^{am}\big(\nabla_d\,h_{mb}+\nabla_b\,h_{md}-\nabla_m\,h_{bd}\big)\Big) &= -2\,\nabla_a\big(h^{am}\nabla_d\,h_{mb}\big)-2\,\nabla_a\big(h^{am}\nabla_b\,h_{md}\big)\\
&+2\,\nabla_a\big(h^{am}\nabla_m\,h_{bd}\big) \qquad (4.337)\\
&= -2\big(\nabla_a h^{am}\big)\big(\nabla_d\,h_{mb}\big)-2h^{am}\nabla_a\nabla_d\,h_{mb}\\
&-2\big(\nabla_a h^{am}\big)\big(\nabla_b\,h_{md}\big)\\
&-2h^{am}\nabla_a\nabla_b\,h_{md}+2\big(\nabla_a h^{am}\big)\big(\nabla_m\,h_{bd}\big)\\
&+2h^{am}\nabla_a\nabla_m\,h_{bd}\\
&= -2\big(\nabla_a h^{am}\big)\big(\nabla_d\,h_{mb}\big)-2h^{am}\nabla_a\nabla_d\,h_{mb}\\
&-2\big(\nabla_a h^{am}\big)\big(\nabla_b\,h_{md}\big) \qquad (4.338)\\
&-2h^{am}\nabla_a\nabla_b\,h_{md}+2\big(\nabla_a h^{am}\big)\big(\nabla_m\,h_{bd}\big)\\
&+2h^{am}\nabla_a\nabla_m\,h_{bd}
\end{aligned}
$$

$$
\begin{aligned}
2\,\nabla_d\Big(h^{am}\big(\nabla_a\,h_{mb}+\nabla_b\,h_{ma}-\nabla_m\,h_{ba}\big)\Big) &= 2\,\nabla_d\big(h^{am}\nabla_a\,h_{mb}\big)+2\,\nabla_d\big(h^{am}\nabla_b\,h_{ma}\big)\\
&-2\,\nabla_d\big(h^{am}\nabla_m\,h_{ba}\big) \qquad (4.339)\\
&= 2\,\big(\nabla_d h^{am}\big)\big(\nabla_a\,h_{mb}\big)+2\,h^{am}\nabla_d\nabla_a\,h_{mb}\\
&+2\,\big(\nabla_d h^{am}\big)\big(\nabla_b\,h_{ma}\big)\\
&+2\,h^{am}\nabla_d\nabla_b\,h_{ma}-2\,\big(\nabla_d h^{am}\big)\big(\nabla_m h_{ba}\big)\\
&-2\,h^{am}\nabla_d\nabla_m\,h_{ba}\\
&= 2\,\big(\nabla_d h^{am}\big)\big(\nabla_a\,h_{mb}\big)+2\,\big(\nabla_d h^{am}\big)\big(\nabla_b\,h_{ma}\big)\\
&+2\,h^{am}\nabla_d\nabla_b\,h_{ma}-2\,\big(\nabla_d h^{am}\big)\big(\nabla_m h_{ba}\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\big(\nabla_e h^a{}_a + \nabla_a h^a{}_e - \overline{g}^{am}\nabla_m h_{ae}\big)\big(\nabla_d h^e{}_b + \nabla_b h^e{}_d - \overline{g}^{eq}\nabla_q h_{bd}\big) \\
&\quad = (\nabla_e h^a{}_a)(\nabla_d h^e{}_b) + (\nabla_e h^a{}_a)(\nabla_b h^e{}_d) - \overline{g}^{eq}\,(\nabla_e h^a{}_a)(\nabla_q h_{bd}) \\
&\quad + (\nabla_a h^a{}_e)(\nabla_d h^e{}_b) + (\nabla_a h^a{}_e)(\nabla_b h^e{}_d) \\
&\quad - \overline{g}^{eq}\,(\nabla_a h^a{}_e)(\nabla_q h_{bd}) - \overline{g}^{am}\,(\nabla_m h_{ae})(\nabla_d h^e{}_b) - \overline{g}^{am}\,(\nabla_m h_{ae})(\nabla_b h^e{}_d) \\
&\quad + \overline{g}^{am}\,(\nabla_m h_{ae})(\overline{g}^{eq}\nabla_q h_{bd})
\end{aligned}
\tag{4.340}
$$

$$
\begin{aligned}
&-\big(\nabla_e h^a{}_d + \nabla_d h^a{}_e - \overline{g}^{aq}\nabla_q h_{de}\big)\big(\nabla_a h^e{}_b + \nabla_b h^e{}_a - \overline{g}^{em}\nabla_m h_{ba}\big) \\
&\quad = (\nabla_e h^a{}_d)(\nabla_a h^e{}_b) + (\nabla_e h^a{}_d)(\nabla_b h^e{}_a) - \overline{g}^{eq}\,(\nabla_e h^a{}_d)(\nabla_q h_{ba}) \\
&\quad + (\nabla_d h^a{}_e)(\nabla_a h^e{}_b) + (\nabla_d h^a{}_e)(\nabla_b h^e{}_a) \\
&\quad - \overline{g}^{eq}(\nabla_d h^a{}_e)(\nabla_q h_{ba}) - \overline{g}^{am}(\nabla_m h_{de})(\nabla_a h^e{}_b) - \overline{g}^{am}(\nabla_m h_{de})(\nabla_b h^e{}_a) \\
&\quad + \overline{g}^{am}(\nabla_m h_{de})(\overline{g}^{eq}\nabla_q h_{ba})
\end{aligned}
$$

We can sum everything up to find

$$
\text{terms} = (\nabla_d h_{ar})(\nabla_b h^{ar}) + 2h^{am}\nabla_a\nabla_m h_{bd}
\tag{4.341}
$$

$$
\begin{aligned}
\text{terms} = & \qquad\qquad (4.342)\\
& -2\, g^{gh} \left(\nabla_g\, h_{br}\right) \left(\nabla_h\, h^r{}_d\right) - g^{gh} \left(h_{br}\, \nabla_g \nabla_h\, h^r{}_d\right) - g^{gh} \left(h^r{}_d\, \nabla_g \nabla_h\, h_{br}\right) \\
& -2\, \left(\nabla_d h^a{}_r\right) \left(\nabla_b\, h^r{}_a\right) - 2\, h^a{}_r\, \nabla_d \nabla_b\, h^r{}_a \\
& + \left(\nabla_q h^q{}_r\right) \left(\nabla_b\, h^r{}_d\right) + h^q{}_r \nabla_q \nabla_b h^r{}_d + \left(\nabla_q h^r{}_d\right) \left(\nabla_b\, h^q{}_r\right) \\
& + h^r{}_d\, \nabla_q \nabla_b\, h^q{}_r \left(\nabla_q h^q{}_r\right) \left(\nabla_d h^r{}_b\right) + h^q{}_r \nabla_q \nabla_d h^r{}_b \\
& + \left(\nabla_q h^r{}_b\right) \left(\nabla_d h^q{}_r\right) + h^r{}_b \nabla_q \nabla_d h^q{}_r - 2\big(\nabla_a h^{am}\big)\big(\nabla_d\, h_{mb}\big) \\
& - 2 h^{am} \nabla_a \nabla_d\, h_{mb} - 2\big(\nabla_a h^{am}\big)\big(\nabla_b\, h_{md}\big) \\
& - 2 h^{am} \nabla_a \nabla_b\, h_{md} + 2\big(\nabla_a h^{am}\big)\big(\nabla_m\, h_{bd}\big) + 2 h^{am} \nabla_a \nabla_m\, h_{bd} \\
& + 2\, \big(\nabla_d h^{am}\big)\big(\nabla_a\, h_{mb}\big) + 2\, \big(\nabla_d h^{am}\big)\big(\nabla_b\, h_{ma}\big) \\
& + 2\, h^{am} \nabla_d \nabla_b\, h_{ma} - 2\, \big(\nabla_d h^{am}\big)\big(\nabla_m h_{ba}\big) + \left(\nabla_e\, h^a{}_a\right) \left(\nabla_d\, h^e{}_b\right) \\
& + \left(\nabla_e\, h^a{}_a\right) \left(\nabla_b\, h^e{}_d\right) - \overline{g}^{eq}\, \left(\nabla_e\, h^a{}_a\right) \left(\nabla_q\, h_{bd}\right) \\
& + \left(\nabla_a\, h^a{}_e\right) \left(\nabla_d\, h^e{}_b\right) + \left(\nabla_a\, h^a{}_e\right) \left(\nabla_b\, h^e{}_d\right) - \overline{g}^{eq}\, \left(\nabla_a\, h^a{}_e\right) \left(\nabla_q\, h_{bd}\right) - \overline{g}^{am}\, \left(\nabla_m\, h_{ae}\right) \left(\nabla_d\, h^e{}_b\right) \\
& - \overline{g}^{am}\, \left(\nabla_m\, h_{ae}\right) \left(\nabla_b\, h^e{}_d\right) + \overline{g}^{am}\, \left(\nabla_m\, h_{ae}\right) \left(\overline{g}^{eq}\, \nabla_q\, h_{bd}\right) \left(\nabla_e\, h^a{}_d\right) \left(\nabla_a\, h^e{}_b\right) \\
& + \left(\nabla_e\, h^a{}_d\right) \left(\nabla_b\, h^e{}_a\right) - \overline{g}^{eq}\, \left(\nabla_e\, h^a{}_d\right) \left(\nabla_q\, h_{ba}\right) + \left(\nabla_d\, h^a{}_e\right) \left(\nabla_a\, h^e{}_b\right) + \left(\nabla_d\, h^a{}_e\right) \left(\nabla_b\, h^e{}_a\right) \\
& - \overline{g}^{eq}\, \left(\nabla_d\, h^a{}_e\right) \left(\nabla_q\, h_{ba}\right) - \overline{g}^{am}\, \left(\nabla_m\, h_{de}\right) \left(\nabla_a\, h^e{}_b\right) - \overline{g}^{am}\, \left(\nabla_m\, h_{de}\right) \left(\nabla_b\, h^e{}_a\right) \\
& + \overline{g}^{am}\, \left(\nabla_m\, h_{de}\right) \left(\overline{g}^{eq}\, \nabla_q\, h_{ba}\right)
\end{aligned}
$$

$$R = \overline{R} + \epsilon \left(-\Box\, h + \nabla_q \nabla_b\, h^{qb} \right) + \frac{\epsilon^2}{4} \left(-2\,\Box \left(h_{br}\, h^{rb} \right) + 2\, \nabla_q \nabla_b \left(h^q_{\ r}\, h^{rb} \right) \right.$$

$$- 2\, \nabla_a \big(h^{am} \big(2\, \nabla_d\, h^d_{\ m} - \nabla_m\, h \big) \big)$$

$$+ 2\, \nabla_d \Big(h^{am} \big(\nabla_a\, h^d_{\ m} + \overline{g}^{bd} \nabla_b\, h_{ma} - \nabla_m\, h^d_{\ a} \big) \Big) + \big(\nabla_e\, h \big) \big(2\, \nabla_d\, h^{ed} - \overline{g}^{eq}\, \nabla_q\, h \big)$$

$$\left. - \big(\nabla_e\, h^{ab} + \overline{g}^{bd} \nabla_d\, h^a_{\ e} - \overline{g}^{aq} \nabla_q\, h^b_{\ e} \big) \big(\nabla_a\, h^e_{\ b} + \nabla_b\, h^e_{\ a} - \overline{g}^{em}\, \nabla_m\, h_{ba} \big) \right) + \mathcal{O}(\epsilon^3)$$

The Einstein tensor is now

$$G_{ab} = \overline{G}_{ab} + \frac{\epsilon}{2} \left(-\Box\, h_{ab} - \nabla_b \nabla_a\, h + \nabla_q \nabla_a\, h^q_{\ b} + \nabla_q \nabla_b\, h^q_{\ a} - g_{ab} \left(-\Box\, h + \nabla_c \nabla_d\, h^{cd} \right) \right) \tag{4.343}$$

$$+ \frac{\epsilon^2}{4} \left(-\Box \left(h_{ar}\, h^r_{\ b} \right) - \nabla_b\, \nabla_a \left(h^q_{\ r}\, h^r_{\ q} \right) + \nabla_q \nabla_a \left(h^q_{\ r}\, h^r_{\ b} \right) + \nabla_q \nabla_b \left(h^q_{\ r}\, h^r_{\ a} \right) \right.$$

$$- g_{ab} \left(-\Box \left(h_{qr}\, h^{qr} \right) + \nabla_q \nabla_u \left(h^q_{\ r}\, h^{ru} \right) \right)$$

$$- 2\, \nabla_q \big(h^{qm} \big(\nabla_b\, h_{ma} + \nabla_a\, h_{mb} - \nabla_m\, h_{ab} \big) \big) + 2\, \nabla_b \big(h^{qm} \big(\nabla_q\, h_{ma} + \nabla_a\, h_{mq} - \nabla_m\, h_{aq} \big) \big)$$

$$+ \big(\nabla_e\, h + \nabla_q\, h^q_{\ e} - \overline{g}^{qm} \nabla_m\, h_{qe} \big) \big(\nabla_b\, h^e_{\ a} + \nabla_a\, h^e_{\ b} - \overline{g}^{eq}\, \nabla_q\, h_{ab} \big)$$

$$- \big(\nabla_e\, h^q_{\ b} + \nabla_b\, h^q_{\ e} - \overline{g}^{qu} \nabla_u\, h_{be} \big) \big(\nabla_q\, h^e_{\ a} + \nabla_a\, h^e_{\ q} - \overline{g}^{em}\, \nabla_m\, h_{aq} \big)$$

$$- g_{ab} \Big(-\nabla_q \big(h^{qm} \big(2\, \nabla_d\, h^d_{\ m} - \nabla_m\, h \big) \big) + \nabla_d \big(h^{qm} \big(\nabla_q\, h^d_{\ m} + \overline{g}^{ud} \nabla_u\, h_{mq} - \nabla_m\, h^d_{\ q} \big) \big)$$

$$+ \frac{1}{2} \big(\nabla_e\, h \big) \big(2\, \nabla_d\, h^{ed} - \overline{g}^{eq}\, \nabla_q\, h \big)$$

$$\left. - \frac{1}{2} \left(\nabla_e\, h^{qp} + \overline{g}^{pd} \nabla_d\, h^q_{\ e} - \overline{g}^{qu} \nabla_u\, h^p_{\ e} \right) \big(\nabla_q\, h^e_{\ p} + \nabla_p\, h^e_{\ q} - \overline{g}^{em}\, \nabla_m\, h_{pq} \big) \Big) \right)$$

For future convenience, we can use the identity,

$$\nabla_q \nabla_x h^v{}_y - \nabla_x \nabla_q h^v{}_y = R^v{}_{rqx} h^r{}_y - R^s{}_{yqx} h^v{}_s \,, \tag{4.344}$$

to alter the order ϵ and ϵ^2 terms

$$\begin{aligned}
\epsilon \text{ term } &= -\Box h_{ab} - \nabla_b \nabla_a h + \nabla_q \nabla_a h^q_b + \nabla_q \nabla_b h^q_a - g_{ab}\left(-\Box h + \nabla_c \nabla_d h^{cd}\right) \\
&= -\Box h_{ab} - \nabla_b \nabla_a h + \nabla_a \nabla_q h^q_b + R^q{}_{sqa} h^s{}_b - R^s{}_{bqa} h^q{}_s + \nabla_b \nabla_q h^q{}_a \\
&\quad + R^q{}_{rqb} h^r{}_a - R^s{}_{aqb} h^q{}_s - g_{ab}\left(-\Box h + \nabla_c \nabla_d h^{cd}\right)
\end{aligned} \tag{4.345}$$

$$\begin{aligned}
\epsilon^2 \text{ term } &= -\Box (h_{ar} h^r{}_b) - \nabla_b \nabla_a \left(h^q{}_r h^r{}_q\right) + \nabla_q \nabla_a (h^q{}_r h^r{}_b) + \nabla_q \nabla_b (h^q{}_r h^r{}_a) \\
&\quad - g_{ab}\left(-\Box (h_{qr} h^{qr})\right) \qquad (4.346) \\
&\quad + \nabla_q \nabla_u (h^q{}_r h^{ru}) - 2\,\nabla_q \Big(h^{qm}\big(\nabla_b h_{ma} + \nabla_a h_{mb} - \nabla_m h_{ab}\big)\Big) \\
&\quad + 2\,\nabla_b \Big(h^{qm}\big(\nabla_q h_{ma} + \nabla_a h_{mq} - \nabla_m h_{aq}\big)\Big) \\
&\quad + \big(\nabla_e h + \nabla_q h^q{}_e - \overline{g}^{qm} \nabla_m h_{qe}\big)\big(\nabla_b h^e{}_a + \nabla_a h^e{}_b - \overline{g}^{eq} \nabla_q h_{ab}\big) \\
&\quad - \big(\nabla_e h^q{}_b + \nabla_b h^q{}_e - \overline{g}^{qu} \nabla_u h_{be}\big)\big(\nabla_q h^e{}_a + \nabla_a h^e{}_q - \overline{g}^{em} \nabla_m h_{aq}\big) \\
&\quad - g_{ab}\Big(-\nabla_q \Big(h^{qm}\big(2\,\nabla_d h^d{}_m - \nabla_m h\big)\Big) + \nabla_d \Big(h^{qm}\big(\nabla_q h^d{}_m + \overline{g}^{ud} \nabla_u h_{mq} - \nabla_m h^d{}_q\big)\Big) \\
&\quad + \frac{1}{2}\,\big(\nabla_e h\big)\big(2\,\nabla_d h^{ed} - \overline{g}^{eq} \nabla_q h\big) \\
&\quad - \frac{1}{2}\,\big(\nabla_e h^{qp} + \overline{g}^{pd} \nabla_d h^q{}_e - \overline{g}^{qu} \nabla_u h^p{}_e\big)\big(\nabla_q h^e{}_p + \nabla_p h^e{}_q - \overline{g}^{em} \nabla_m h_{pq}\big)\Big)
\end{aligned} \tag{4.347}$$

$$
\begin{aligned}
\epsilon^2 \text{ term } =& -\Box\left(h_{ar}\, h^{r}{}_{b}\right) - \nabla_b \nabla_a \left(h^{q}{}_{r}\, h^{r}{}_{q}\right) + g_{ab}\Box\left(h_{qr}\, h^{qr}\right) \\
&+ \nabla_a\nabla_q \left(h^{q}{}_{r}\, h^{r}{}_{b}\right) + R^{q}{}_{sqa}\left(h^{s}{}_{r}\, h^{r}{}_{b}\right) - R^{s}{}_{bqa}\left(h^{q}{}_{r}\, h^{r}{}_{s}\right) \\
&+ \nabla_b\nabla_q \left(h^{q}{}_{r}\, h^{r}{}_{a}\right) + R^{q}{}_{sqb}\left(h^{s}{}_{r}\, h^{r}{}_{a}\right) - R^{s}{}_{aqb}\left(h^{q}{}_{r}\, h^{r}{}_{s}\right) \\
&+ \nabla_u\nabla_q \left(h^{q}{}_{r}\, h^{ru}\right) + R^{q}{}_{squ}\left(h^{s}{}_{r}\, h^{ru}\right) + R^{u}{}_{squ}\left(h^{q}{}_{r}\, h^{rs}\right) \\
&- 2\left(\nabla_q h^{qm}\right)\left(\nabla_b\, h_{ma} + \nabla_a\, h_{mb} - \nabla_m\, h_{ab}\right) \\
&- 2\, h^{qm}\left(\nabla_q\nabla_b\, h_{ma} + \nabla_q\nabla_a\, h_{mb} - \nabla_q\nabla_m\, h_{ab}\right) \\
&+ 2\left(\nabla_b\, h^{qm}\right)\left(\nabla_q\, h_{ma} + \nabla_a\, h_{mq} - \nabla_m\, h_{aq}\right) \\
&+ 2\, h^{qm}\left(\nabla_b\nabla_q\, h_{ma} + \nabla_b\nabla_a\, h_{mq} - \nabla_b\nabla_m\, h_{aq}\right) \\
&+ \left(\nabla_d\, h^{qm}\right)\left(\nabla_q\, h^{d}{}_{m} + \overline{g}^{ud}\nabla_u\, h_{mq} - \nabla_m\, h^{d}{}_{q}\right) \\
&+ h^{qm}\left(\nabla_d\nabla_q\, h^{d}{}_{m} + \overline{g}^{ud}\nabla_d\nabla_u\, h_{mq} - \nabla_d\nabla_m\, h^{d}{}_{q}\right) \\
&+ \left(\nabla_e\, h + \nabla_q\, h^{q}{}_{e} - \overline{g}^{qm}\nabla_m\, h_{qe}\right)\left(\nabla_b\, h^{e}{}_{a} + \nabla_a\, h^{e}{}_{b} - \overline{g}^{eq}\,\nabla_q\, h_{ab}\right) \\
&- \left(\nabla_e\, h^{q}{}_{b} + \nabla_b\, h^{q}{}_{e} - \overline{g}^{qu}\nabla_u\, h_{be}\right)\left(\nabla_q\, h^{e}{}_{a} + \nabla_a\, h^{e}{}_{q} - \overline{g}^{em}\,\nabla_m\, h_{aq}\right) \\
&- g_{ab}\Big(-\left(\nabla_q\, h^{qm}\right)\left(2\,\nabla_d\, h^{d}{}_{m} - \nabla_m\, h\right) - h^{qm}\left(2\,\nabla_q\nabla_d\, h^{d}{}_{m} - \nabla_q\nabla_m\, h\right) \\
&+ \frac{1}{2}\left(\nabla_e\, h\right)\left(2\,\nabla_d\, h^{ed} - \overline{g}^{eq}\,\nabla_q\, h\right) \\
&- \frac{1}{2}\left(\nabla_e\, h^{qp} + \overline{g}^{pd}\nabla_d\, h^{q}{}_{e} - \overline{g}^{qu}\nabla_u\, h^{p}{}_{e}\right)\left(\nabla_q\, h^{e}{}_{p} + \nabla_p\, h^{e}{}_{q} - \overline{g}^{em}\,\nabla_m\, h_{pq}\right)\Big)
\end{aligned}
\tag{4.348}
$$

Similar to the linearized Einstein equation, we used the "traced-reversed" perturbation to simplify the order ϵ terms,

$$
\overline{h}_{ab} = h_{ab} - \frac{1}{2} g_{ab}\, h \qquad\qquad \overline{h} = -h\,. \tag{4.349}
$$

In addition, we make the gauge transformation,

$$\overline{h}_{ab} \rightarrow \overline{h}'_{ab} = \overline{h}_{ab} - 2\left(\overline{\nabla}_a\,\xi_b + \overline{\nabla}_b\,\xi_a\right)\;, \tag{4.350}$$

where we can pick ξ to generate the conditions,

$$\overline{\nabla}^a\,\overline{h}'_{ab} = 0 \qquad\qquad\qquad \overline{h}' = 0\,. \tag{4.351}$$

This requires $\Box\xi_b = \nabla^a\overline{h}_{ab}$. The metric perturbation is now traceless and transverse. In essence we substitute in for

$$h_{ab} = \left(\overline{h}_{ab} - \frac{1}{2}g_{ab}\,\overline{h}\right) \rightarrow \overline{h}'_{ab} \qquad\qquad\qquad h = -\overline{h} \rightarrow 0\,, \tag{4.352}$$

and take $\overline{\nabla}^a \, \overline{h}'_{ab} = 0$. The ϵ and ϵ^2 terms are now

$$\epsilon \text{ term } = -\Box\, h'_{ab} - g_{ab}\, \nabla_c \nabla_d\, h'^{cd} - R^s{}_{bqa}\, h'^q{}_s - R^s{}_{aqb}\, h'^q{}_s + R\,{}_{sa}\, h'^s{}_b + R\,{}_{sb}\, h'^s{}_a$$

$$\begin{aligned}
\epsilon^2 \text{ term } =& -\Box\,(h'_{ar}\, h'^r{}_b) - \nabla_b \nabla_a \left(h'^q{}_r\, h'^r{}_q\right) + g_{ab}\Box\left(h'_{qr}\, h'^{qr}\right) + \nabla_a\,(h'^q{}_r\, \nabla_q\, h'^r{}_b) \\
&+ R_{sa}\, h'^s{}_r\, h'^r{}_b - R^s{}_{bqa}\, h'^q{}_r\, h'^r{}_s \qquad (4.353) \\
&+ \nabla_b\,(h'^q{}_r\, \nabla_q\, h'^r{}_a) + R_{sb}\, h'^s{}_r\, h'^r{}_a - R^s{}_{aqb}\, h'^q{}_r\, h'^r{}_s + \nabla_u\,(h'^q{}_r\, \nabla_q\, h'^{ru}) \\
&+ R_{su}\, h'^s{}_r\, h'^{ru} - R_{sq}\, h'^q{}_r\, h'^{rs} \\
&- 2\, h'^{qm} \nabla_q \nabla_b\, h'_{ma} - 2\, h'^{qm} \nabla_q \nabla_a\, h'_{mb} + 2\, h'^{qm} \nabla_q \nabla_m\, h'_{ab} \\
&+ 2\, \left(\nabla_b\, h'^{qm}\right)\left(\nabla_a\, h'_{mq}\right) \\
&- 2\, \left(\nabla_b\, h'^{qm}\right)\left(\nabla_m\, h'_{aq}\right) + 2\, h'^{qm} \nabla_b \nabla_q\, h'_{ma} + 2\, h'^{qm} \nabla_b \nabla_a\, h'_{mq} \\
&- 2\, h'^{qm} \nabla_b \nabla_m\, h'_{aq} \\
&+ \left(\nabla_d\, h'^{qm}\right)\left(\nabla_q\, h'^d{}_m + \overline{g}^{ud} \nabla_u\, h'_{mq} - \nabla_m\, h'^d{}_q\right) \\
&+ h'^{qm} \nabla_d \nabla_q\, h'^d{}_m + h'^{qm} \Box\, h'_{mq} - R_{rm}\, h^r{}_q\, h'^{qm} + R^s{}_{qdm}\, h'^{qm}\, h^d{}_s \\
&- \left(\nabla_e\, h'^q{}_b + \nabla_b\, h'^q{}_e - \overline{g}^{qu} \nabla_u\, h'_{be}\right)\left(\nabla_q\, h'^e{}_a + \nabla_a\, h'^e{}_q - \overline{g}^{em}\, \nabla_m\, h'_{aq}\right) \\
&+ \frac{1}{2}\, g_{ab}\left(\nabla_e\, h'^{qp} + \overline{g}^{pd} \nabla_d\, h'^q{}_e - \overline{g}^{qu} \nabla_u\, h'^p{}_e\right)\left(\nabla_q\, h'^e{}_p + \nabla_p\, h'^e{}_q - \overline{g}^{em}\, \nabla_m\, h'_{pq}\right)
\end{aligned}$$

From MTW, the terms proportional to the Ricci tensor are suppressed and are be dropped.

The second order correction is now

$$
\begin{aligned}
&-\Box\,(h_{ar}\,h^{r}_{\ b}) - \nabla_b\,\nabla_a\left(h^{q}_{\ r}\,h^{r}_{\ q}\right) + \nabla_q\nabla_a\,(h^{q}_{\ r}\,h^{r}_{\ b}) + \nabla_q\nabla_b\,(h^{q}_{\ r}\,h^{r}_{\ a})\\
&- g_{ab}\left(-\Box\,(h_{qr}\,h^{qr}) + \nabla_q\nabla_u\,(h^{q}_{\ r}\,h^{ru})\right) \qquad (4.354)\\
&- 2\,\nabla_q\big(h^{qm}\big(\nabla_b\,h_{ma} + \nabla_a\,h_{mb} - \nabla_m\,h_{ab}\big)\big) + 2\,\nabla_b\big(\,h^{qm}\big(\nabla_q\,h_{ma} + \nabla_a\,h_{mq} - \nabla_m\,h_{aq}\big)\big)\\
&+ \big(\nabla_e\,h + \nabla_q\,h^{q}_{\ e} - \overline{g}^{qm}\nabla_m\,h_{qe}\big)\big(\nabla_b\,h^{e}_{\ a} + \nabla_a\,h^{e}_{\ b} - \overline{g}^{eq}\,\nabla_q\,h_{ab}\big)\\
&- \big(\nabla_e\,h^{q}_{\ b} + \nabla_b\,h^{q}_{\ e} - \overline{g}^{qu}\nabla_u\,h_{be}\big)\big(\nabla_q\,h^{e}_{\ a} + \nabla_a\,h^{e}_{\ q} - \overline{g}^{em}\,\nabla_m\,h_{aq}\big)\\
&- g_{ab}\Bigg(-\nabla_q\big(h^{qm}\big(2\,\nabla_d\,h^{d}_{\ m} - \nabla_m\,h\big)\big) + \nabla_d\big(\,h^{qm}\big(\nabla_q\,h^{d}_{\ m} + \overline{g}^{ud}\nabla_u\,h_{mq} - \nabla_m\,h^{d}_{\ q}\big)\big)\\
&+ \frac{1}{2}\left(\nabla_e\,h\right)\big(2\,\nabla_d\,h^{ed} - \overline{g}^{eq}\,\nabla_q\,h\big) - \frac{1}{2}\left(\nabla_e\,h^{qp} + \overline{g}^{pd}\nabla_d\,h^{q}_{\ e} - \overline{g}^{qu}\nabla_u\,h^{p}_{\ e}\right)\\
&\times\left(\nabla_q\,h^{e}_{\ p} + \nabla_p\,h^{e}_{\ q} - \overline{g}^{em}\,\nabla_m\,h_{pq}\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&-\Box\left(\overline{h}'_{ar}\,\overline{h}'^{r}_{\ b}\right) - \nabla_b\,\nabla_a\left(\overline{h}'^{q}_{\ r}\,\overline{h}'^{r}_{\ q}\right) + \nabla_q\nabla_a\left(\overline{h}'^{q}_{\ r}\,\overline{h}'^{r}_{\ b}\right) + \nabla_q\nabla_b\left(\overline{h}'^{q}_{\ r}\,\overline{h}'^{r}_{\ a}\right)\\
&- g_{ab}\left(-\Box\left(\overline{h}'_{qr}\,\overline{h}'^{qr}\right) + \nabla_q\nabla_u\left(\overline{h}'^{q}_{\ r}\,\overline{h}'^{ru}\right)\right)\\
&- 2\,\overline{h}'^{qm}\big(\nabla_q\nabla_b\,\overline{h}'_{ma} + \nabla_q\nabla_a\,\overline{h}'_{mb} - \nabla_q\nabla_m\,\overline{h}'_{ab}\big)\\
&+ 2\,\nabla_b\big(\,\overline{h}'^{qm}\big(\nabla_q\,\overline{h}'_{ma} + \nabla_a\,\overline{h}'_{mq} - \nabla_m\,\overline{h}'_{aq}\big)\big)\\
&+ \big(\nabla_e\,\overline{h}' + \nabla_q\,\overline{h}'^{q}_{\ e} - \overline{g}^{qm}\nabla_m\,\overline{h}'_{qe}\big)\big(\nabla_b\,\overline{h}'^{e}_{\ a} + \nabla_a\,\overline{h}'^{e}_{\ b} - \overline{g}^{eq}\,\nabla_q\,\overline{h}'_{ab}\big)\\
&- \big(\nabla_e\,\overline{h}'^{q}_{\ b} + \nabla_b\,\overline{h}'^{q}_{\ e} - \overline{g}^{qu}\nabla_u\,\overline{h}'_{be}\big)\big(\nabla_q\,\overline{h}'^{e}_{\ a} + \nabla_a\,\overline{h}'^{e}_{\ q} - \overline{g}^{em}\,\nabla_m\,\overline{h}'_{aq}\big)\\
&- g_{ab}\Bigg(\nabla_d\big(\,\overline{h}'^{qm}\big(\nabla_q\,\overline{h}'^{d}_{\ m} + \overline{g}^{ud}\nabla_u\,\overline{h}'_{mq} - \nabla_m\,\overline{h}'^{d}_{\ q}\big)\big)\\
&- \frac{1}{2}\left(\nabla_e\,\overline{h}'^{qp} + \overline{g}^{pd}\nabla_d\,\overline{h}'^{q}_{\ e} - \overline{g}^{qu}\nabla_u\,\overline{h}'^{p}_{\ e}\right)\big(\nabla_q\,\overline{h}'^{e}_{\ p} + \nabla_p\,\overline{h}'^{e}_{\ q} - \overline{g}^{em}\,\nabla_m\,\overline{h}'_{pq}\big)\Bigg)
\end{aligned}
$$

The corresponding Ricci scalar is

$$
\begin{aligned}
R_{bd} = \overline{R} + \frac{\epsilon}{2}\left(g^{bd}\,\nabla_a\nabla_{\{b}\,h^a_{\ d\}} - g^{bd}\,\nabla_d\nabla_b\,h - g^{bd}\,\Box\,h_{bd}\right) \qquad (4.355)\\
+ \frac{\epsilon^2}{4}\left(g^{bd}\,\nabla_b\,h_{ce}\nabla_d\,h^{ce} + 2\,h^{ce}\left(g^{bd}\,\nabla_b\nabla_d\,h_{ce} + g^{bd}\,\nabla_c\nabla_e\,h_{bd}\right.\right.\\
- g^{bd}\,\nabla_d\,\nabla_e\,h_{cb} - g^{bd}\,\nabla_b\,\nabla_e\,h_{cd}\\
+ 2\,g^{bd}g^{eg}g^{ch}\,\nabla_g\,h_{dh}\left(\nabla_e\,h_{cb} - \nabla_c\,h_{eb}\right)\\
\left.- g^{bd}\left(\nabla_e\,h^{ce} - \frac{1}{2}\,g^{ce}\,\nabla_e\,h\right)\left(\nabla_d\,h_{cb} + \nabla_b\,h_{cd} - \nabla_c\,h_{bd}\right)\right) + \mathcal{O}(\epsilon^3)
\end{aligned}
$$

$$
\begin{aligned}
= \overline{R} + \epsilon\left(\nabla_a\nabla_b\,h^{ab} - \Box\,h\right) + \frac{\epsilon^2}{4}\left(g^{bd}\,\nabla_b\,h_{ce}\nabla_d\,h^{ce} + 2\,h^{ce}\,\Box\,h_{ce} + 2\,h^{ce}\,\nabla_c\nabla_e\,h\right.\\
- 4\,h^{ce}\,\nabla_d\,\nabla_e\,h_c^{\ d} \qquad (4.356)\\
+ 2\,g^{eg}\left(\nabla_g\,h_{dc}\right)\left(\nabla_e\,h^{cd}\right) - 2\left(\nabla_g\,h_d^{\ c}\right)\left(\nabla_c\,h^{gd}\right) - \left(\nabla_e\,h^{ce}\right)\left(\nabla_d\,h_c^{\ d}\right) - \left(\nabla_e\,h^{ce}\right)\left(\nabla_b\,h_c^{\ b}\right)\\
+ \left(\nabla_e\,h^{ce}\right)\left(\nabla_c\,h\right) + \frac{1}{2}\left(\nabla_e\,h\right)\left(\nabla_d\,h^{ed}\right) + \frac{1}{2}\left(\nabla_e\,h\right)\left(\nabla_b\,h^{be}\right)\\
\left.- \frac{1}{2}\,g^{ce}\left(\nabla_e\,h\right)\left(\nabla_c\,h\right)\right) + \mathcal{O}(\epsilon^3)
\end{aligned}
$$

$$
\begin{aligned}
= \overline{R} + \epsilon\left(\nabla_a\nabla_b\,h^{ab} - \Box\,h\right) + \frac{\epsilon^2}{4}\left(3g^{bd}\,\nabla_b\,h_{ce}\nabla_d\,h^{ce} + 2\,h^{ce}\,\Box\,h_{ce} + 2\,h^{ce}\,\nabla_c\nabla_e\,h\right.\\
- 4\,h^{ce}\,\nabla_d\,\nabla_e\,h_c^{\ d} \qquad (4.357)\\
- 2\left(\nabla_g\,h_d^{\ c}\right)\left(\nabla_c\,h^{gd}\right) - \left(\nabla_e\,h^{ce}\right)\left(\nabla_d\,h_c^{\ d}\right) - \left(\nabla_e\,h^{ce}\right)\left(\nabla_b\,h_c^{\ b}\right)\\
\left.+ 2\left(\nabla_e\,h^{ce}\right)\left(\nabla_c\,h\right) - \frac{1}{2}\,g^{ce}\left(\nabla_e\,h\right)\left(\nabla_c\,h\right)\right) + \mathcal{O}(\epsilon^3)
\end{aligned}
$$

4.5.2 Energy-Momentum Computations

Using the expansion above, the energy-momentum tensor, we get

$$
\begin{aligned}
T_{ab} = & \left(\overline{g}^{ef} - \epsilon\, h^{ef} + \frac{1}{2}\,\epsilon^2\, h^{ec}\, h_c^{\ f}\right) F_{ae}F_{bf} + \partial_a\phi\,\partial_b\phi \\
& - \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}}\left(1 - \frac{\epsilon}{2}\,\mathrm{tr}\left(g^{ef}h_{ef}\right) + \frac{3\,\epsilon^2}{8}\,\mathrm{tr}\left(g^{ef}h_{ef}\right)^2\right) \\
& \times \left(\overline{g}_{ab} + \epsilon\, h_{ab} + \frac{1}{2}\,\epsilon^2\, h_{ac}\, h^c_{\ b}\right)\epsilon^{defg}\,\phi\, F_{de}F_{fg} \\
& - \frac{2}{45\, m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\overline{g}^{ef} - \epsilon\, h^{ef} + \frac{1}{2}\,\epsilon^2\, h^{ec}\, h_c^{\ f}\right) F^2\, F_{ae}F_{bf} \\
& - \left(\overline{g}_{ab} + \epsilon\, h_{ab} + \frac{1}{2}\,\epsilon^2\, h_{ac}\, h^c_{\ b}\right)\left(\frac{1}{4}\,F^2 + \frac{1}{2}\left(\left(\overline{g}^{mn} - \epsilon\, h^{mn} + \frac{1}{2}\,\epsilon^2\, h^{mc}\, h_c^{\ n}\right)\partial_m\phi\,\partial_n\phi + m^2\,\phi^2\right)\right. \\
& - \frac{1}{90\, m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(F^4 - \frac{7}{16}\left(\frac{1}{\overline{g}} - \epsilon\,\frac{\mathrm{tr}\left(g^{ef}h_{ef}\right)}{\overline{g}^2} + \epsilon^2\,\frac{\mathrm{tr}\left(g^{ef}h_{ef}\right)^2}{\overline{g}^3}\right)\left(\epsilon^{mnop}F_{mn}F_{op}\right)^2\right) \\
& - \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}}\left(1 - \frac{\epsilon}{2}\,\mathrm{tr}\,\left(g^{ef}h_{ef}\right) + \frac{3\,\epsilon^2}{8}\,\mathrm{tr}\left(g^{ef}h_{ef}\right)^2\right)\,\epsilon^{defg}\,\phi\, F_{de}F_{fg}\left.\vphantom{\frac{1}{2}}\right) + \ldots
\end{aligned}
\tag{4.358}
$$

Here we have used $\det\left(A + \epsilon\, X\right) = \det A + \epsilon\,\det\left(A\right)\mathrm{tr}\left(A^{-1}X\right) + \mathcal{O}(\epsilon^2)$ to expand the factors of $\sqrt{-g}$. However, the energy momentum tensor is also a function of the vector potential and the axion. Those waves are cohered with the gravitational waves. Knowing this, we can also expand

$$F_{ab} = \overline{F}_{ab} + \epsilon\, F_{ab}^{(1)} + \epsilon^2\, F_{ab}^{(2)} + \ldots \tag{4.359}$$

$$\phi = \overline{\phi} + \epsilon\,\phi^{(1)} + \epsilon^2\,\phi^{(2)} + \ldots \tag{4.360}$$

$$
\begin{aligned}
T_{ab} = & \left(\overline{g}^{ef} - \epsilon\, h^{ef} + \frac{1}{2}\,\epsilon^2\, h^{ec}\, h_c^{\ f}\right) \left(\overline{F}_{ae} + \epsilon\, F^{(1)}_{ae} + \epsilon^2\, F^{(2)}_{ae}\right) \left(\overline{F}_{bf} + \epsilon\, F^{(1)}_{bf} + \epsilon^2\, F^{(2)}_{bf}\right) \\
& + \partial_a \left(\overline{\phi} + \alpha\, \phi^{(1)}\right)\, \partial_b \left(\overline{\phi} + \alpha\, \phi^{(1)}\right) \\
& - \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}} \left(1 - \frac{\epsilon}{2}\,\mathrm{tr}\left(g^{ef} h_{ef}\right) + \frac{3\,\epsilon^2}{8}\,\mathrm{tr}\left(g^{ef} h_{ef}\right)^2\right) \left(\overline{g}_{ab} + \epsilon\, h_{ab} + \frac{1}{2}\,\epsilon^2\, h_{ac}\, h^c_{\ b}\right) \epsilon^{defg}\, \phi\, F_{de} F_{fg} \\
& - \frac{2}{45\, m_e^4} \left(\frac{e^2}{4\pi}\right)^2 \left(\overline{g}^{ef} - \epsilon\, h^{ef} + \frac{1}{2}\,\epsilon^2\, h^{ec}\, h_c^{\ f}\right) F^2\, F_{ae} F_{bf} \\
& - \left(\overline{g}_{ab} + \epsilon\, h_{ab} + \frac{1}{2}\,\epsilon^2\, h_{ac}\, h^c_{\ b}\right) \left(\frac{1}{4}\, F^2 + \frac{1}{2} \left(\left(\overline{g}^{mn} - \epsilon\, h^{mn} + \frac{1}{2}\,\epsilon^2\, h^{mc}\, h_c^{\ n}\right) \partial_m \phi\, \partial_n \phi + m^2\, \phi^2\right)\right. \\
& - \frac{1}{90\, m_e^4} \left(\frac{e^2}{4\pi}\right)^2 \left(F^4 - \frac{7}{16} \left(\frac{1}{\overline{g}} - \epsilon\, \frac{\mathrm{tr}\left(g^{ef} h_{ef}\right)}{\overline{g}^2} + \epsilon^2\, \frac{\mathrm{tr}\left(g^{ef} h_{ef}\right)^2}{\overline{g}^3}\right) \left(\epsilon^{mnop} F_{mn} F_{op}\right)^2\right) \\
& \left. - \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}} \left(1 - \frac{\epsilon}{2}\,\mathrm{tr}\,\left(g^{ef} h_{ef}\right) + \frac{3\,\epsilon^2}{8}\,\mathrm{tr}\left(g^{ef} h_{ef}\right)^2\right)\, \epsilon^{defg}\, \phi\, F_{de} F_{fg}\right) + \ldots
\end{aligned}
\tag{4.361}
$$

$$
\begin{aligned}
T_{ab} = {} & \overline{g}^{ef} F_{ae} F_{bf} + \partial_a \phi \, \partial_b \phi - \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}} \, \overline{g}_{ab} \left(\epsilon^{defg} \, \phi \, F_{de} F_{fg} \right) - \frac{2}{45 \, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \overline{g}^{ef} F^2 \, F_{ae} F_{bf} \\
& - \overline{g}_{ab} \Bigg(\frac{1}{4} F^2 + \frac{1}{2} \left(\overline{g}^{mn} \, \partial_m \phi \, \partial_n \phi + m^2 \, \phi^2 \right) \\
& - \frac{1}{90 \, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \left(F^4 - \frac{7}{16 \, \overline{g}} \left(\epsilon^{mnop} F_{mn} F_{op} \right)^2 \right) \\
& - \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}} \left(\epsilon^{defg} \, \phi \, F_{de} F_{fg} \right) \Bigg) \\
& + \epsilon \Bigg(-h^{ef} F_{ae} F_{bf} - \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}} \, h_{ab} \left(\epsilon^{defg} \, \phi \, F_{de} F_{fg} \right) + \frac{\tilde{\lambda}}{16\sqrt{-\overline{g}}} \, \overline{g}_{ab} \, \mathrm{tr} \left(g^{ef} h_{ef} \right) \left(\epsilon^{defg} \, \phi \, F_{de} F_{fg} \right) \\
& + \frac{2}{45 \, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 h^{ef} F^2 \, F_{ae} F_{bf} - \overline{g}_{ab} \Bigg(-\frac{1}{2} \, h^{mn} \, \partial_m \phi \, \partial_n \phi \\
& - \frac{1}{90 \, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \frac{7}{16 \, \overline{g}^2} \, \mathrm{tr} \left(g^{ef} h_{ef} \right) \left(\epsilon^{mnop} F_{mn} F_{op} \right)^2 \\
& + \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}} \frac{1}{2} \, \mathrm{tr} \, \left(g^{ef} h_{ef} \right) \, \left(\epsilon^{defg} \, \phi \, F_{de} F_{fg} \right) \Bigg) - h_{ab} \left(\frac{1}{4} \, F^2 + \frac{1}{2} \left(\overline{g}^{mn} \, \partial_m \phi \, \partial_n \phi + m^2 \, \phi^2 \right) \right. \\
& \left. - \frac{1}{90 \, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \left(F^4 - \frac{7}{16 \, \overline{g}} \left(\epsilon^{mnop} F_{mn} F_{op} \right)^2 \right) - \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}} \left(\epsilon^{defg} \, \phi \, F_{de} F_{fg} \right) \right) \Bigg) \\
& + \epsilon^2 \Bigg(\frac{1}{2} \, h^{ec} \, h_c{}^f F_{ae} F_{bf} - \frac{\tilde{\lambda}}{16\sqrt{-\overline{g}}} \, h_{ac} \, h^c{}_b \, \phi \left(\epsilon^{defg} \, F_{de} F_{fg} \right) + \frac{\tilde{\lambda}}{16\sqrt{-\overline{g}}} \, \mathrm{tr} \left(g^{ef} h_{ef} \right) h_{ab} \, \phi \left(\epsilon^{defg} \, F_{de} F_{fg} \right) \\
& - \frac{3 \, \tilde{\lambda}}{64\sqrt{-\overline{g}}} \, \mathrm{tr} \left(g^{ef} h_{ef} \right)^2 \phi \, \overline{g}_{ab} \left(\epsilon^{defg} \, F_{de} F_{fg} \right) - \frac{2}{45 \, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \frac{1}{2} \, h^{ec} \, h_c{}^f F^2 \, F_{ae} F_{bf} \\
& - \overline{g}_{ab} \Bigg(\frac{1}{4} \, h^{mc} \, h_c{}^n \, \partial_m \phi \, \partial_n \phi + \frac{1}{90 \, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \frac{7}{16 \, \overline{g}^3} \, \mathrm{tr} \left(g^{ef} h_{ef} \right)^2 \left(\epsilon^{mnop} F_{mn} F_{op} \right)^2 \\
& - \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}} \frac{3}{8} \, \mathrm{tr} \left(g^{ef} h_{ef} \right)^2 \, \left(\epsilon^{defg} \, \phi \, F_{de} F_{fg} \right) \Bigg) \\
& - h_{ab} \left(-\frac{1}{2} \, h^{mn} \, \partial_m \phi \, \partial_n \phi - \frac{1}{90 \, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \frac{7}{16 \, \overline{g}^2} \, \mathrm{tr} \left(g^{ef} h_{ef} \right) \left(\epsilon^{mnop} F_{mn} F_{op} \right)^2 \right. \\
& \left. + \frac{\tilde{\lambda}}{16\sqrt{-\overline{g}}} \, \mathrm{tr} \, \left(g^{ef} h_{ef} \right) \, \left(\epsilon^{defg} \, \phi \, F_{de} F_{fg} \right) \right) \\
& - \frac{1}{2} \, h_{ac} \, h^c{}_b \Bigg(\frac{1}{4} \, F^2 + \frac{1}{2} \left(\overline{g}^{mn} \, \partial_m \phi \, \partial_n \phi + m^2 \, \phi^2 \right) - \frac{1}{90 \, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \left(F^4 - \frac{7}{16 \, \overline{g}} \left(\epsilon^{mnop} F_{mn} F_{op} \right)^2 \right) \\
& - \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}} \left(\epsilon^{defg} \, \phi \, F_{de} F_{fg} \right) \Bigg) \Bigg)
\end{aligned}
\tag{4.362}
$$

We can now expand each term order-by-order. The terms are

$$
\begin{aligned}
\text{first term} = {} & \overline{g}^{ef} F_{ae} F_{bf} + \partial_a \phi \, \partial_b \phi - \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}} \, \overline{g}_{ab} \left(\epsilon^{defg} \, \phi \, F_{de} F_{fg} \right) \\
& - \frac{2}{45 \, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \overline{g}^{ef} F^2 \, F_{ae} F_{bf} \qquad (4.363) \\
& - \overline{g}_{ab} \Bigg(\frac{1}{4} F^2 + \frac{1}{2} \left(\overline{g}^{mn} \, \partial_m \phi \, \partial_n \phi + m^2 \, \phi^2 \right) \\
& - \frac{1}{90 \, m_e^4} \left(\frac{e^2}{4\pi} \right)^2 \left(F^4 - \frac{7}{16 \, \overline{g}} \left(\epsilon^{mnop} F_{mn} F_{op} \right)^2 \right) \\
& - \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}} \left(\epsilon^{defg} \, \phi \, F_{de} F_{fg} \right) \Bigg) \qquad (4.364)
\end{aligned}
$$

$$
\begin{aligned}
\text{first term} = {} & \overline{g}^{ef}\left(\overline{F}_{ae} + \epsilon\, F^{(1)}_{ae} + \epsilon^2\, F^{(2)}_{ae}\right)\left(\overline{F}_{bf} + \epsilon\, F^{(1)}_{bf} + \epsilon^2\, F^{(2)}_{bf}\right) \\
& + \partial_a\left(\overline{\phi} + \epsilon\,\phi^{(1)} + \epsilon^2\,\phi^{(2)}\right)\partial_b\left(\overline{\phi} + \epsilon\,\phi^{(1)} + \epsilon^2\,\phi^{(2)}\right) \\
& - \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}}\,\overline{g}_{ab}\left(\epsilon^{defg}\,\left(\overline{\phi} + \epsilon\,\phi^{(1)} + \epsilon^2\,\phi^{(2)}\right)\left(\overline{F}_{de} + \epsilon\, F^{(1)}_{de} + \epsilon^2\, F^{(2)}_{de}\right)\right. \\
& \times\left(\overline{F}_{fg} + \epsilon\, F^{(1)}_{fg} + \epsilon^2\, F^{(2)}_{fg}\right) \\
& - \frac{2}{45\, m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \overline{g}^{ef} g^{mn} g^{op}\left(\overline{F}_{mo} + \epsilon\, F^{(1)}_{mo} + \epsilon^2\, F^{(2)}_{mo}\right) \\
& \times\left(\overline{F}_{np} + \epsilon\, F^{(1)}_{np} + \epsilon^2\, F^{(2)}_{np}\right) \\
& \times\left(\overline{F}_{ae} + \epsilon\, F^{(1)}_{ae} + \epsilon^2\, F^{(2)}_{ae}\right)\left(\overline{F}_{bf} + \epsilon\, F^{(1)}_{bf} + \epsilon^2\, F^{(2)}_{bf}\right) \\
& - \overline{g}_{ab}\Bigg(\frac{1}{4}\, g^{mn} g^{op}\left(\overline{F}_{mo} + \epsilon\, F^{(1)}_{mo} + \epsilon^2\, F^{(2)}_{mo}\right)\left(\overline{F}_{np} + \epsilon\, F^{(1)}_{np} + \epsilon^2\, F^{(2)}_{np}\right) \\
& + \frac{1}{2}\left(\overline{g}^{mn}\,\partial_m\left(\overline{\phi} + \epsilon\,\phi^{(1)} + \epsilon^2\,\phi^{(2)}\right)\partial_n\left(\overline{\phi} + \epsilon\,\phi^{(1)} + \epsilon^2\,\phi^{(2)}\right)\right. \\
& + m^2\left(\overline{\phi} + \epsilon\,\phi^{(1)} + \epsilon^2\,\phi^{(2)}\right)^2 \\
& - \frac{1}{90\, m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\left(g^{mn} g^{op}\left(\overline{F}_{mo} + \epsilon\, F^{(1)}_{mo} + \epsilon^2\, F^{(2)}_{mo}\right)\right.\right. \\
& \times\left(\overline{F}_{np} + \epsilon\, F^{(1)}_{np} + \epsilon^2\, F^{(2)}_{np}\right)^2 \\
& - \frac{7}{16\,\overline{g}}\left(\epsilon^{mnop}\left(\overline{F}_{mn} + \epsilon\, F^{(1)}_{mn} + \epsilon^2\, F^{(2)}_{no}\right)\left(\overline{F}_{op} + \epsilon\, F^{(1)}_{op} + \epsilon^2\, F^{(2)}_{op}\right)\right)^2\Bigg) \\
& - \frac{\tilde{\lambda}}{8\sqrt{-\overline{g}}}\left(\epsilon^{defg}\,\left(\overline{\phi} + \epsilon\,\phi^{(1)} + \epsilon^2\,\phi^{(2)}\right)\left(\overline{F}_{de} + \epsilon\, F^{(1)}_{de} + \epsilon^2\, F^{(2)}_{de}\right)\right. \\
& times\left.\left(\overline{F}_{fg} + \epsilon\, F^{(1)}_{fg} + \epsilon^2\, F^{(2)}_{fg}\right)\right)
\end{aligned}
\tag{4.365}
$$

(4.366)

$$
\begin{aligned}
T^{(1)}_{ab} &= \overline{g}^{ef}\left(F^{(1)}_{ae}\,\overline{F}_{bf} + \overline{F}_{ae}\,F^{(1)}_{bf}\right) + \partial_a\overline{\phi}\,\partial_b\phi^{(1)} + \partial_a\phi^{(1)}\partial_b\overline{\phi} \\
&- \frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \overline{g}^{ef}\overline{g}^{mn}\overline{g}^{op}\Big(\,F^{(1)}_{np}\,\overline{F}_{mo}\,\overline{F}_{ae}\,\overline{F}_{bf} \qquad (4.367)\\
&+ \overline{F}_{np}\,F^{(1)}_{mo}\,\overline{F}_{ae}\,\overline{F}_{bf} + \overline{F}_{np}\,\overline{F}_{mo}\,F^{(1)}_{ae}\,\overline{F}_{bf} + \overline{F}_{np}\,\overline{F}_{mo}\,\overline{F}_{ae}\,F^{(1)}_{bf}\,\Big) \\
&- g_{ab}\Bigg(\frac{1}{4}\,\overline{g}^{mn}\overline{g}^{op}\left(\overline{F}_{np}\,F^{(1)}_{mo} + \overline{F}_{mo}\,F^{(1)}_{np}\right) \\
&+ \overline{g}^{mn}\,\partial_m\overline{\phi}\,\partial_n\phi^{(1)} + m^2\,\overline{\phi}\,\phi^{(1)} \\
&- \frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\Bigg(\overline{g}^{mn}\,\overline{g}^{op}\,\overline{g}^{qr}\,\overline{g}^{st}\Big(F^{(1)}_{rt}\,\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}_{qs} + \overline{F}_{rt}\,F^{(1)}_{mo}\,\overline{F}_{np}\,\overline{F}_{qs} \\
&+ \overline{F}_{rt}\,\overline{F}_{mo}\,F^{(1)}_{np}\,\overline{F}_{qs} + \overline{F}_{rt}\,\overline{F}_{mo}\,\overline{F}_{np}\,F^{(1)}_{qs}\Big) \\
&- \frac{7}{16\,\overline{g}}\,\epsilon^{hijk}\epsilon^{defg}\Big(F^{(1)}_{jk}\,\overline{F}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi} + \overline{F}_{jk}\,F^{(1)}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi} \\
&+ \overline{F}_{jk}\,\overline{F}_{de}\,F^{(1)}_{fg}\,\overline{F}_{hi} + \overline{F}_{jk}\,\overline{F}_{de}\,\overline{F}_{fg}\,F^{(1)}_{hi}\Big) \\
&+ \frac{7}{16\,\overline{g}^2}\,\epsilon^{hijk}\epsilon^{defg}\,\mathrm{tr}\Big(\overline{g}^{mn}\,h_{mn}\Big)\overline{F}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi}\,\overline{F}_{jk}\Bigg) - \frac{1}{2}h^{mn}\partial_m\overline{\phi}\,\partial_n\overline{\phi}\Bigg) \\
&+ h_{ab}\Bigg(-\frac{1}{4}\overline{g}^{mn}\overline{g}^{op}\overline{F}_{mo}\,\overline{F}_{np} \\
&- \frac{1}{2}\left(\overline{g}^{mn}\,\partial_m\overline{\phi}\,\partial_n\overline{\phi} + m^2\,\overline{\phi}^2\right) - h^{ef}\,\overline{F}_{ae}\,\overline{F}_{bf} \\
&- \frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\left(\overline{F}^4 - \frac{7}{16\,\overline{g}}\left(\epsilon^{mnop}\overline{F}_{mn}\overline{F}_{op}\right)^2\right)\Bigg) \\
&+ \frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\overline{g}^{mn}\overline{g}^{op}\,h^{ef}\,\overline{F}_{ae}\,\overline{F}_{bf}\,\overline{F}_{mo}\,\overline{F}_{np} \qquad (4.368)
\end{aligned}
$$

4.5.3 Axion Equation of Motion

The axion equation of motion is

$$\Big(g^{ab}\,\partial_a\partial_b - g^{ab}\Gamma^c{}_{ab}\,\partial_c - m^2\Big)\phi = \frac{\tilde{\lambda}}{8\sqrt{-g}}\,\epsilon^{abcd}\,F_{ab}F_{cd} \tag{4.369}$$

The LHS is

$$\begin{aligned}\Big(g^{ab}\,\partial_a\partial_b - g^{ab}\Gamma^c{}_{ab}\,\partial_c - m^2\Big)\phi = \Bigg(&\Big(\overline{g}^{ab} - \epsilon\,h^{ab} + \frac{1}{2}\,\epsilon^2\,h^{ac}\,h_c{}^b\Big)\,\partial_a\partial_b \qquad (4.370)\\ &- \Big(\overline{g}^{ab} - \epsilon\,h^{ab} + \frac{1}{2}\,\epsilon^2\,h^{ac}\,h_c{}^b\Big)\Big(\overline{\Gamma}^c{}_{ab}\\ &+ \frac{\epsilon}{2}\,\overline{g}^{cm}\Big(\overline{\nabla}_b\,h_{ma} + \overline{\nabla}_a\,h_{mb} - \overline{\nabla}_m\,h_{ab}\Big)\\ &+ \frac{\epsilon^2}{4}\Big(\overline{g}^{cm}\Big(\overline{\nabla}_b\left(h_{md}\,h^d{}_a\right) + \overline{\nabla}_a\left(h_{md}\,h^d{}_b\right) - \overline{\nabla}_m\left(h_{ad}\,h^d{}_b\right)\Big)\\ &- 2\,h^{cm}\Big(\overline{\nabla}_b\,h_{ma} + \overline{\nabla}_a\,h_{mb} - \overline{\nabla}_m\,h_{ab}\Big)\Big)\Big)\,\partial_c - m^2\Bigg)\phi\end{aligned} \tag{4.371}$$

$$\begin{aligned}
\Big(g^{ab}\,\partial_a\partial_b - g^{ab}\Gamma^c{}_{ab}\,\partial_c - m^2\Big)\phi = \Bigg(&\overline{g}^{ab}\,\partial_a\partial_b - \epsilon\,h^{ab}\,\partial_a\partial_b + \frac{1}{2}\,\epsilon^2\,h^{ac}\,h_c{}^b\,\partial_a\partial_b \\
&- \overline{g}^{ab}\,\overline{\Gamma}^c{}_{ab}\,\partial_c + \epsilon\,h^{ab}\overline{\Gamma}^c{}_{ab}\,\partial_c - \frac{1}{2}\,\epsilon^2\,h^{ac}\,h_c{}^b\,\overline{\Gamma}^c{}_{ab}\,\partial_c \\
&- \Big(\overline{g}^{ab} - \epsilon\,h^{ab}\Big)\frac{\epsilon}{2}\,\overline{g}^{cm}\Big(\overline{\nabla}_b\,h_{ma} + \overline{\nabla}_a\,h_{mb} - \overline{\nabla}_m\,h_{ab}\Big)\,\partial_c \\
&- \frac{\epsilon^2}{4}\,\overline{g}^{ab}\,\overline{g}^{cm}\Big(\overline{\nabla}_b\left(h_{md}\,h^d{}_a\right) + \overline{\nabla}_a\left(h_{md}\,h^d{}_b\right) - \overline{\nabla}_m\left(h_{ad}\,h^d{}_b\right)\Big)\,\partial_c \\
&+ \Big(\overline{g}^{ab} - \epsilon\,h^{ab} + \frac{1}{2}\,\epsilon^2\,h^{ac}\,h_c{}^b\Big)\frac{\epsilon^2}{4}2\,h^{cm}\Big(\overline{\nabla}_b\,h_{ma} \\
&+ \overline{\nabla}_a\,h_{mb} - \overline{\nabla}_m\,h_{ab}\Big)\,\partial_c - m^2\Bigg)\phi
\end{aligned} \tag{4.372}$$

$$\begin{aligned}
&= \left(\overline{g}^{ab}\,\partial_a\partial_b - \overline{g}^{ab}\,\overline{\Gamma}^c{}_{ab}\,\partial_c - m^2\right)\,\phi \\
&+ \epsilon\Big(-h^{ab}\,\partial_a\partial_b + h^{ab}\overline{\Gamma}^c{}_{ab}\,\partial_c - \frac{1}{2}\,\overline{g}^{ab}\overline{g}^{cm}\,\Big(\overline{\nabla}_b\,h_{ma} \\
&+ \overline{\nabla}_a\,h_{mb} - \overline{\nabla}_m\,h_{ab}\partial_c\Big)\phi \\
&+ \epsilon^2\Bigg(\frac{1}{2}\,h^{ac}\,h_c{}^b\,\partial_a\partial_b - \frac{1}{2}\,h^{ac}\,h_c{}^b\,\overline{\Gamma}^c{}_{ab}\,\partial_c + \frac{1}{2}\,h^{ab}\,\overline{g}^{cm}\,\Big(\overline{\nabla}_b\,h_{ma} \\
&+ \overline{\nabla}_a\,h_{mb} - \overline{\nabla}_m\,h_{ab}\partial_c \\
&- \frac{1}{4}\,\overline{g}^{ab}\,\overline{g}^{cm}\Big(\overline{\nabla}_b\left(h_{md}\,h^d{}_a\right) + \overline{\nabla}_a\left(h_{md}\,h^d{}_b\right) - \overline{\nabla}_m\left(h_{ad}\,h^d{}_b\right)\Big)\,\partial_c \\
&+ \frac{1}{2}\,\overline{g}^{ab}\,h^{cm}\Big(\overline{\nabla}_b\,h_{ma} + \overline{\nabla}_a\,h_{mb} - \overline{\nabla}_m\,h_{ab}\Big)\,\partial_c\Bigg)\phi
\end{aligned} \tag{4.373}$$

$$\nabla_a F^{ab} = \tilde{\lambda}\, \nabla_a\big(\phi\, \tilde{F}^{ab}\big) + \frac{4}{45 m_e^4} \left(\frac{e^2}{4\pi}\right)^2 \left(\nabla_a \left(F^{ab}\, F_{cd}\, F^{cd}\right) + \frac{7}{4}\, \nabla_a \left(\tilde{F}^{ab}\, F_{cd}\, \tilde{F}^{cd}\right)\right)$$

$$\begin{aligned}
\nabla_a F^{ab} &= \frac{\tilde{\lambda}}{2\sqrt{-g}}\, \epsilon^{abcd}\, \partial_a \left(\phi\, F_{cd}\right) \\
&+ \frac{4}{45 m_e^4} \left(\frac{e^2}{4\pi}\right)^2 \frac{1}{\sqrt{-g}} \partial_a\Big(\sqrt{-g}\, F^{ba}\, F_{cd}\, F^{cd}\Big) \\
&+ \frac{7}{180\, m_e^4} \left(\frac{e^2}{4\pi}\right)^2 \frac{1}{\sqrt{-g}} \partial_a \left(\frac{1}{\sqrt{-g}} \epsilon^{baef}\, \epsilon^{cdgh} F_{ef}\, F_{cd} F_{gh}\right)
\end{aligned} \tag{4.374}$$

$$\begin{aligned}
\nabla_a F^{ab} &= \frac{\tilde{\lambda}}{2\sqrt{-\overline{g}}} \left(1 - \frac{\epsilon}{2} \left(\overline{g}^{cd} h_{cd}\right) + \frac{3\, \epsilon^2}{8} \left(\overline{g}^{cd} h_{cd}\right)^2\right) \epsilon^{abcd} \partial_a \left(\phi\, F_{cd}\right) \\
&+ \frac{4}{45 m_e^4} \left(\frac{e^2}{4\pi}\right)^2 \frac{1}{\sqrt{-\overline{g}}} \left(1 - \frac{\epsilon}{2} \left(\overline{g}^{cd} h_{cd}\right) + \frac{3\, \epsilon^2}{8} \left(\overline{g}^{cd} h_{cd}\right)^2\right) \\
&\times \partial_a \left(\sqrt{-\overline{g}} \left(1 + \frac{\epsilon}{2} \left(\overline{g}^{cd} h_{cd}\right) - \frac{\epsilon^2}{8} \left(\overline{g}^{cd} h_{cd}\right)^2\right) F^{ba}\, F^2\right) \\
&+ \frac{7}{180\, m_e^4} \left(\frac{e^2}{4\pi}\right)^2 \frac{1}{\sqrt{-\overline{g}}} \left(1 - \frac{\epsilon}{2} \left(\overline{g}^{cd} h_{cd}\right) + \frac{3\, \epsilon^2}{8} \left(\overline{g}^{cd} h_{cd}\right)^2\right) \\
&\times \partial_a \left(\frac{1}{\sqrt{-\overline{g}}} \left(1 - \frac{\epsilon}{2} \left(\overline{g}^{cd} h_{cd}\right) + \frac{3\, \epsilon^2}{8} \left(\overline{g}^{cd} h_{cd}\right)^2\right) \epsilon^{baef} \epsilon^{cdgh} F_{ef} F_{cd} F_{gh}\right)
\end{aligned} \tag{4.375}$$

(4.376)

$$
\begin{aligned}
\nabla_a F^{ab} = &\frac{\tilde{\lambda}}{2\sqrt{-\overline{g}}}\,\epsilon^{abcd}\partial_a\left(\phi\,F_{cd}\right) - \frac{\tilde{\lambda}\,\epsilon}{4\sqrt{-\overline{g}}}\left(\overline{g}^{cd}h_{cd}\right)\epsilon^{abcd}\partial_a\left(\phi\,F_{cd}\right)\\
&+ \frac{3\,\tilde{\lambda}\,\epsilon^2}{16\,\sqrt{-\overline{g}}}\left(\overline{g}^{cd}h_{cd}\right)^2\epsilon^{abcd}\partial_a\left(\phi\,F_{cd}\right)\\
&+ \frac{4}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2\frac{1}{\sqrt{-\overline{g}}}\Bigg(\partial_a\left(\sqrt{-\overline{g}}\,F^{ba}\,F^2\right) + \frac{\epsilon}{2}\,\partial_a\left(\sqrt{-\overline{g}}\,\left(\overline{g}^{cd}h_{cd}\right)F^{ba}\,F^2\right)\\
&- \frac{\epsilon^2}{8}\,\partial_a\left(\sqrt{-\overline{g}}\left(\overline{g}^{cd}h_{cd}\right)^2F^{ba}F^2\right)\Bigg)\\
&- \frac{4}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2\frac{\epsilon}{2\sqrt{-\overline{g}}}\left(\overline{g}^{cd}h_{cd}\right)\left(\partial_a\left(\sqrt{-\overline{g}}\,F^{ba}\,F^2\right) + \frac{\epsilon}{2}\,\partial_a\left(\sqrt{-\overline{g}}\,\left(\overline{g}^{cd}h_{cd}\right)F^{ba}\,F^2\right)\right)\\
&+ \frac{4}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2\frac{3\,\epsilon^2}{8\sqrt{-\overline{g}}}\,\left(\overline{g}^{cd}h_{cd}\right)^2\partial_a\left(\sqrt{-\overline{g}}\,F^{ba}\,F^2\right)\\
&+ \frac{7}{180\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\frac{1}{\sqrt{-\overline{g}}}\Bigg(\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\epsilon^{baef}\epsilon^{cdgh}F_{ef}F_{cd}F_{gh}\right)\\
&- \frac{\epsilon}{2}\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\left(\overline{g}^{cd}h_{cd}\right)\epsilon^{baef}\epsilon^{cdgh}F_{ef}F_{cd}F_{gh}\right)\\
&+ \frac{3\,\epsilon^2}{8}\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\left(\overline{g}^{cd}h_{cd}\right)^2\epsilon^{baef}\epsilon^{cdgh}F_{ef}F_{cd}F_{gh}\right)\Bigg)\\
&- \frac{7}{180\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\frac{1}{\sqrt{-\overline{g}}}\frac{\epsilon}{2}\left(\overline{g}^{cd}h_{cd}\right)\Bigg(\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\epsilon^{baef}\epsilon^{cdgh}F_{ef}F_{cd}F_{gh}\right)\\
&- \frac{\epsilon}{2}\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\left(\overline{g}^{cd}h_{cd}\right)\epsilon^{baef}\epsilon^{cdgh}F_{ef}F_{cd}F_{gh}\right)\\
&+ \frac{3\,\epsilon^2}{8}\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\left(\overline{g}^{cd}h_{cd}\right)^2\epsilon^{baef}\epsilon^{cdgh}F_{ef}F_{cd}F_{gh}\right)\Bigg)\\
&+ \frac{7}{180\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\frac{1}{\sqrt{-\overline{g}}}\frac{3\,\epsilon^2}{8}\,\left(\overline{g}^{cd}h_{cd}\right)^2\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\epsilon^{baef}\epsilon^{cdgh}F_{ef}F_{cd}F_{gh}\right)
\end{aligned}
\tag{4.377}
$$

$$
\begin{aligned}
\nabla_a F^{ab} &= \frac{\tilde{\lambda}}{2\sqrt{-\overline{g}}}\,\epsilon^{abcd}\partial_a\left(\phi\, F_{cd}\right) - \frac{\tilde{\lambda}\,\epsilon}{4\sqrt{-\overline{g}}}\left(\overline{g}^{cd}h_{cd}\right)\epsilon^{abcd}\partial_a\left(\phi\, F_{cd}\right)\\
&+ \frac{3\,\tilde{\lambda}\,\epsilon^2}{16\,\sqrt{-\overline{g}}}\left(\overline{g}^{cd}h_{cd}\right)^2\epsilon^{abcd}\partial_a\left(\phi\, F_{cd}\right)\\
&+ \frac{4}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2\frac{1}{\sqrt{-\overline{g}}}\,\partial_a\left(\sqrt{-\overline{g}}\,F^{ba}\,F^2\right)\\
&+ \frac{4}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2\frac{1}{\sqrt{-\overline{g}}}\,\frac{\epsilon}{2}\,\partial_a\left(\sqrt{-\overline{g}}\,\left(\overline{g}^{cd}h_{cd}\right)F^{ba}\,F^2\right)\\
&- \frac{4}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2\frac{1}{\sqrt{-\overline{g}}}\frac{\epsilon^2}{8}\,\partial_a\left(\sqrt{-\overline{g}}\left(\overline{g}^{cd}h_{cd}\right)^2F^{ba}F^2\right)\\
&- \frac{4}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2\frac{\epsilon}{2\sqrt{-\overline{g}}}\left(\overline{g}^{cd}h_{cd}\right)\Bigg(\partial_a\left(\sqrt{-\overline{g}}\,F^{ba}\,F^2\right)\\
&+ \frac{\epsilon}{2}\,\partial_a\left(\sqrt{-\overline{g}}\,\left(\overline{g}^{cd}h_{cd}\right)F^{ba}\,F^2\right)\Bigg)\\
&+ \frac{4}{45m_e^4}\left(\frac{e^2}{4\pi}\right)^2\frac{3\,\epsilon^2}{8\sqrt{-\overline{g}}}\,\left(\overline{g}^{cd}h_{cd}\right)^2\partial_a\left(\sqrt{-\overline{g}}\,F^{ba}\,F^2\right)\\
&+ \frac{7}{180\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\frac{1}{\sqrt{-\overline{g}}}\Bigg(\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\epsilon^{baef}\epsilon^{cdgh}F_{ef}F_{cd}F_{gh}\right)\\
&- \frac{\epsilon}{2}\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\left(\overline{g}^{cd}h_{cd}\right)\epsilon^{baef}\epsilon^{cdgh}F_{ef}F_{cd}F_{gh}\right)\\
&+ \frac{3\,\epsilon^2}{8}\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\left(\overline{g}^{cd}h_{cd}\right)^2\epsilon^{baef}\epsilon^{cdgh}F_{ef}F_{cd}F_{gh}\right)\Bigg)\\
&- \frac{7}{180\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\frac{1}{\sqrt{-\overline{g}}}\frac{\epsilon}{2}\left(\overline{g}^{cd}h_{cd}\right)\Bigg(\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\epsilon^{baef}\epsilon^{cdgh}F_{ef}F_{cd}F_{gh}\right)\\
&- \frac{\epsilon}{2}\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\left(\overline{g}^{cd}h_{cd}\right)\epsilon^{baef}\epsilon^{cdgh}F_{ef}F_{cd}F_{gh}\right)\\
&+ \frac{3\,\epsilon^2}{8}\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\left(\overline{g}^{cd}h_{cd}\right)^2\epsilon^{baef}\epsilon^{cdgh}F_{ef}F_{cd}F_{gh}\right)\Bigg)\\
&+ \frac{7}{180\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\frac{1}{\sqrt{-\overline{g}}}\frac{3\,\epsilon^2}{8}\left(\overline{g}^{cd}h_{cd}\right)^2\partial_a\left(\frac{1}{\sqrt{-\overline{g}}}\epsilon^{baef}\epsilon^{cdgh}F_{ef}F_{cd}F_{gh}\right)
\end{aligned}
\tag{4.378}
$$

4.5.3 *Gauge Fixing*

Similar to the linearized Einstein equation, we use the "traced-reversed" perturbation to simplify the order ϵ terms,

$$\overline{h}_{ab} = h_{ab} - \frac{1}{2} g_{ab}\, h \qquad\qquad \overline{h} = -h\,. \tag{4.379}$$

In addition, we make the gauge transformation,

$$\overline{h}_{ab} \to \overline{h}'_{ab} = \overline{h}_{ab} - 2\left(\overline{\nabla}_a\, \xi_b + \overline{\nabla}_b\, \xi_a\right)\,, \tag{4.380}$$

where we can pick ξ to generate the gauge fixing conditions,

$$\overline{\nabla}^a\, \overline{h}'_{ab} = 0 \qquad\qquad \overline{h}' = 0\,. \tag{4.381}$$

This requires $\Box \xi_b = \nabla^a \overline{h}_{ab}$. The metric perturbation is now traceless and transverse. Overall, we substitute in for

$$h_{ab} = \left(\overline{h}_{ab} - \frac{1}{2} g_{ab}\, \overline{h}\right) \to \overline{h}'_{ab} \qquad\qquad h = -\overline{h} \to 0\,, \tag{4.382}$$

and take $\overline{\nabla}^a \, \overline{h}'_{ab} = 0$. The Einstein tensor is now

$$G_{bd} = \overline{G}_{bd} + \frac{\epsilon}{2}\left(\nabla_a \nabla_{\{b}\, h'^a_{\ d\}} - \Box\, h'_{bd}\right) \tag{4.383}$$
$$+ \frac{\epsilon^2}{4}\Bigg(\nabla_b\, h'_{ce} \nabla_d\, h'^{ce} + 2\, h'^{ce} \left(\nabla_b \nabla_d\, h'_{ce} + \nabla_c \nabla_e\, h'_{bd} - \nabla_d\, \nabla_e\, h'_{cb} - \nabla_b\, \nabla_e\, h'_{cd}\right)$$
$$+ 2\, g^{eg}\, \nabla_g\, h'^{\ c}_{d} \left(\nabla_e\, h'_{cb} - \nabla_c\, h'_{eb}\right)$$
$$- \frac{1}{2}\, g_{bd} \Bigg(g^{mn} \nabla_m\, h'_{ce} \nabla_n\, h'^{ce} + 2\, h'^{ce} \left(\Box\, h'_{ce} - 2\, \nabla_m\, \nabla_e\, h'^{\ m}_{c}\right)$$
$$+ 2\, \nabla_g\, h'^{\ c}_{m} \left(g^{eg}\, \nabla_e\, h'^{\ m}_{c} - \nabla_c\, h'^{gm}\right)\Bigg)\Bigg) + \mathcal{O}(\epsilon^3)\,.$$

Knowing $\nabla_q \nabla_x\, h^v{}_y - \nabla_x \nabla_q\, h^v{}_y = R^v{}_{rqx}\, h^r{}_y - R^s{}_{yqx}\, h^v{}_s$ and terms like $R_{r\{b}\, h'^r_{\ d\}}$ are, according to [65], the same order as the $\mathcal{O}(\epsilon^3)$ corrections, we can place the Einstein tensor in a final form

$$G_{bd} = \overline{G}_{bd} + \frac{\epsilon}{2}\left(-\Box\, h'_{bd} + 2\, R_{abds}\, h'^{as}\right) \tag{4.384}$$
$$+ \frac{\epsilon^2}{4}\Bigg(\nabla_b\, h'_{ce} \nabla_d\, h'^{ce} + 2\, h'^{ce} \left(\nabla_b \nabla_d\, h'_{ce} + \nabla_c \nabla_e\, h'_{bd} - \nabla_d\, \nabla_e\, h'_{cb} - \nabla_b\, \nabla_e\, h'_{cd}\right)$$
$$+ 2\, g^{eg}\, \nabla_g\, h'^{\ c}_{d} \left(\nabla_e\, h'_{cb} - \nabla_c\, h'_{eb}\right)$$
$$- \frac{1}{2}\, g_{bd} \Bigg(g^{mn} \nabla_m\, h'_{ce} \nabla_n\, h'^{ce} + 2\, h'^{ce} \left(\Box\, h'_{ce} - 2\, \nabla_m\, \nabla_e\, h'^{\ m}_{c}\right)$$
$$+ 2\, \nabla_g\, h'^{\ c}_{m} \left(g^{eg}\, \nabla_e\, h'^{\ m}_{c} - \nabla_c\, h'^{gm}\right)\Bigg)\Bigg) + \mathcal{O}(\epsilon^3)\,.$$

Parametrically, the energy-momentum tensor goes as

$$\overline{T}_{ab} = \underbrace{\overline{g}^{ef}\overline{F}_{ae}\overline{F}_{bf}}_{1/\mathcal{L}^2} + \underbrace{\partial_a\overline{\phi}\,\partial_b\overline{\phi}}_{1/\mathcal{L}^2} - \frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2 \overline{g}^{ef}\,\underbrace{\overline{F}^2\,\overline{F}_{ae}\overline{F}_{bf}}_{1/\mathcal{L}^4}$$

$$- \overline{g}_{ab}\Bigg(\frac{1}{4}\,\underbrace{\overline{F}^2}_{1/\mathcal{L}^2} + \frac{1}{2}\Bigg(\underbrace{\overline{g}^{mn}\,\partial_m\overline{\phi}\,\partial_n\overline{\phi}}_{1/\mathcal{L}^2} + \underbrace{m^2\,\overline{\phi}^2}_{m^2}\Bigg) \tag{4.385}$$

$$- \frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\Bigg(\underbrace{\overline{F}^4}_{1/\mathcal{L}^4} - \frac{7}{16\,\overline{g}}\,\underbrace{\left(\epsilon^{mnop}\overline{F}_{mn}\overline{F}_{op}\right)^2}_{1/\mathcal{L}^4}\Bigg)\Bigg)$$

$$= \overline{T}_{ab}^{\,(1/\mathcal{L}^2)} + \overline{T}_{ab}^{\,(m^2)} + \overline{T}_{ab}^{\,(1/\mathcal{L}^4)} \tag{4.386}$$

where we have separated out the $1/\mathcal{L}^n$ terms. It is clear the Euler-Heisenberg terms are less important in terms of the long-wavelength physics as well as the parametric suppression.

The higher order terms are

$$
\begin{aligned}
T^{(1)}_{ab} &= \overline{g}^{ef}\,\underbrace{\left(F^{(1)}_{ae}\,\overline{F}_{bf}+\overline{F}_{ae}\,F^{(1)}_{bf}\right)}_{1/\mathcal{L}}+\underbrace{\partial_a\overline{\phi}\,\partial_b\phi^{(1)}}_{1/\mathcal{L}}+\underbrace{\partial_a\phi^{(1)}\partial_b\overline{\phi}}_{1/\mathcal{L}} \\
&-\frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\overline{g}^{ef}\overline{g}^{mn}\overline{g}^{op}\Big(\underbrace{F^{(1)}_{np}\,\overline{F}_{mo}\,\overline{F}_{ae}\,\overline{F}_{bf}}_{1/\mathcal{L}^3} \qquad (4.387)\\
&+\underbrace{\overline{F}_{np}\,F^{(1)}_{mo}\,\overline{F}_{ae}\,\overline{F}_{bf}}_{1/\mathcal{L}^3}+\underbrace{\overline{F}_{np}\,\overline{F}_{mo}\,F^{(1)}_{ae}\,\overline{F}_{bf}}_{1/\mathcal{L}^3}+\underbrace{\overline{F}_{np}\,\overline{F}_{mo}\,\overline{F}_{ae}\,F^{(1)}_{bf}}_{1/\mathcal{L}^3}\Big) \\
&-g_{ab}\bigg(\frac{1}{4}\,\overline{g}^{mn}\overline{g}^{op}\,\underbrace{\left(\overline{F}_{np}\,F^{(1)}_{mo}+\overline{F}_{mo}\,F^{(1)}_{np}\right)}_{1/\mathcal{L}} \\
&+\overline{g}^{mn}\,\underbrace{\partial_m\overline{\phi}\,\partial_n\phi^{(1)}}_{1/\mathcal{L}}+\underbrace{m^2\,\overline{\phi}\,\phi^{(1)}}_{m^2} \\
&-\frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\bigg(\overline{g}^{mn}\,\overline{g}^{op}\,\overline{g}^{qr}\,\overline{g}^{st}\,\Big(\underbrace{F^{(1)}_{rt}\,\overline{F}_{mo}\,\overline{F}_{np}\,\overline{F}_{qs}}_{1/\mathcal{L}^3}+\underbrace{\overline{F}_{rt}\,F^{(1)}_{mo}\,\overline{F}_{np}\,\overline{F}_{qs}}_{1/\mathcal{L}^3} \\
&+\underbrace{\overline{F}_{rt}\,\overline{F}_{mo}\,F^{(1)}_{np}\,\overline{F}_{qs}}_{1/\mathcal{L}^3}+\underbrace{\overline{F}_{rt}\,\overline{F}_{mo}\,\overline{F}_{np}\,F^{(1)}_{qs}}_{1/\mathcal{L}^3}\Big) \\
&-\frac{7}{16\,\overline{g}}\,\epsilon^{hijk}\epsilon^{defg}\,\Big(\underbrace{F^{(1)}_{jk}\,\overline{F}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi}}_{1/\mathcal{L}^3}+\underbrace{\overline{F}_{jk}\,F^{(1)}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi}}_{1/\mathcal{L}^3} \\
&+\underbrace{\overline{F}_{jk}\,\overline{F}_{de}\,F^{(1)}_{fg}\,\overline{F}_{hi}}_{1/\mathcal{L}^3}+\underbrace{\overline{F}_{jk}\,\overline{F}_{de}\,\overline{F}_{fg}\,F^{(1)}_{hi}}_{1/\mathcal{L}^3}\Big) \\
&+\frac{7}{16\,\overline{g}^2}\,\epsilon^{hijk}\epsilon^{defg}\,\mathrm{tr}\Big(\overline{g}^{mn}\,h_{mn}\Big)\,\underbrace{\overline{F}_{de}\,\overline{F}_{fg}\,\overline{F}_{hi}\,\overline{F}_{jk}}_{1/\mathcal{L}^4}\bigg)-\frac{1}{2}\,\underbrace{h^{mn}\partial_m\overline{\phi}\,\partial_n\overline{\phi}}_{1/\mathcal{L}^2}\bigg) \\
&+h_{ab}\bigg(-\frac{1}{4}\overline{g}^{mn}\overline{g}^{op}\,\underbrace{\overline{F}_{mo}\,\overline{F}_{np}}_{1/\mathcal{L}^2}-\frac{1}{2}\Big(\overline{g}^{mn}\,\underbrace{\partial_m\overline{\phi}\,\partial_n\overline{\phi}}_{1/\mathcal{L}^2}+\underbrace{m^2\,\overline{\phi}^2}_{m^2}\Big)-h^{ef}\,\underbrace{\overline{F}_{ae}\,\overline{F}_{bf}}_{1/\mathcal{L}^2} \\
&-\frac{1}{90\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\bigg(\underbrace{\overline{F}^4}_{1/\mathcal{L}^4}-\frac{7}{16\,\overline{g}}\,\underbrace{\left(\epsilon^{mnop}\overline{F}_{mn}\overline{F}_{op}\right)^2}_{1/\mathcal{L}^4}\bigg)\bigg) \\
&+\frac{2}{45\,m_e^4}\left(\frac{e^2}{4\pi}\right)^2\overline{g}^{mn}\overline{g}^{op}\,h^{ef}\,\underbrace{\overline{F}_{ae}\,\overline{F}_{bf}\,\overline{F}_{mo}\,\overline{F}_{np}}_{1/\mathcal{L}^4} \\
&= T^{(1)\;(1/\mathcal{L})}_{ab}+T^{(1)\;(m^2)}_{ab}+T^{(1)\;(1/\mathcal{L}^2)}_{ab} \qquad (4.388)
\end{aligned}
$$

$$
\begin{aligned}
\epsilon\, T^{(1)}_{ab} &= \overline{g}^{ef}\left(F^{(1)}_{ae}\,\overline{F}_{bf} + \overline{F}_{ae}\,F^{(1)}_{bf}\right) + \partial_a\overline{\phi}\,\partial_b\phi^{(1)} + \\
&\partial_a\phi^{(1)}\partial_b\overline{\phi} - g_{ab}\Bigg(\tfrac{1}{4}\,\overline{g}^{mn}\overline{g}^{op}\left(\overline{F}_{np}\,F^{(1)}_{mo} + \overline{F}_{mo}\,F^{(1)}_{np}\right) \\
&+ \overline{g}^{mn}\,\partial_m\overline{\phi}\,\partial_n\phi^{(1)} + m^2\,\overline{\phi}\,\phi^{(1)} - \tfrac{1}{2}h^{mn}\partial_m\overline{\phi}\,\partial_n\overline{\phi}\Bigg) \\
&+ h_{ab}\Bigg(-\tfrac{1}{4}\overline{g}^{mn}\overline{g}^{op}\overline{F}_{mo}\,\overline{F}_{np} - \tfrac{1}{2}\left(\overline{g}^{mn}\,\partial_m\overline{\phi}\,\partial_n\overline{\phi} + m^2\,\overline{\phi}^2\right) \\
&- h^{ef}\,\overline{F}_{ae}\,\overline{F}_{bf}\Bigg) + \ldots \sim \tfrac{\epsilon}{\lambda\mathcal{L}} + \epsilon\, m^2 + \tfrac{\epsilon}{\mathcal{L}^2} + \text{more supressed terms}
\end{aligned}
$$

Using equations (4.93) and (4.94), we can see the parametrics work

$$
\underbrace{\overline{R}_{ab}}_{1/\mathcal{L}^2} + \underbrace{\epsilon^2\,R^{(2)}_{ab}\Big|_{\text{low}}}_{\epsilon^2/2} = \underbrace{\kappa\left(\overline{T}_{ab} - \frac{1}{2}\,\overline{g}_{ab}\,\overline{T}\right)}_{(1/\mathcal{L}^2)+m^2+\text{more suppressed terms}} + \underbrace{\epsilon^2\,\kappa\left(T^{(2)}_{ab} - \frac{1}{2}\,g_{ab}\,T^{(2)}\right)\Big|_{\text{low}}}_{(\epsilon^2/\lambda^2)+(\epsilon^2/\lambda\mathcal{L})+(\epsilon^2/\mathcal{L}^2)+\epsilon^2\,m^2+\text{more suppressed terms}} \tag{4.389}
$$

$$
\underbrace{\epsilon\,R^{(1)}_{ab}\Big|_{\text{high}}}_{\epsilon/2} + \underbrace{\epsilon^2\,R^{(2)}_{ab}\Big|_{\text{high}}}_{\epsilon^2/2} = \underbrace{\epsilon\,\kappa\left(T^{(1)}_{ab} - \frac{1}{2}\,g_{ab}\,T^{(1)}\right)\Big|_{\text{high}}}_{\epsilon/\mathcal{L}+\epsilon m^2+\epsilon/\mathcal{L}^2} + \underbrace{\epsilon^2\,\kappa\left(T^{(2)}_{ab} - \frac{1}{2}\,g_{ab}\,T^{(2)}\right)\Big|_{\text{high}}}_{(\epsilon^2/\lambda^2)+(\epsilon^2/\lambda\mathcal{L})+(\epsilon^2/\mathcal{L}^2)+\epsilon^2\,m^2+\text{more suppressed terms}} \tag{4.390}
$$

Multiplying both equations by $\mathcal{L}^2$ and keeping the most important terms for the low-frequency equations are

$$
\begin{aligned}
&\overline{R}_{ab} + \epsilon^2 R^{(2)}_{ab}\Big|_{\text{low}} = \kappa \Bigg(\overline{g}^{ef} \overline{F}_{ae} \overline{F}_{bf} + \partial_a \overline{\phi}\, \partial_b \overline{\phi} - \overline{g}_{ab} \Big(\tfrac{1}{4}\, \overline{F}^2 \\
&+ \tfrac{1}{2} \big(\overline{g}^{mn}\, \partial_m \overline{\phi}\, \partial_n \overline{\phi} + m^2\, \overline{\phi}^2 \,\big) \Big) \\
&- \tfrac{1}{2}\, g_{ab} \Big(F^2 + \overline{g}^{cd}\, \partial_c \overline{\phi}\, \partial_d \overline{\phi} - \overline{F}^2 - 2\, \overline{g}^{mn}\, \partial_m \overline{\phi}\, \partial_n \overline{\phi} + 2\, m^2\, \overline{\phi}^2 \Big) \Bigg) \\
&+ \epsilon^2\, \kappa\, \Big(T^{(2)}_{ab} - \tfrac{1}{2}\, g_{ab}\, T^{(2)} \Big) \Big|_{\text{low}} \\
&\epsilon\, R^{(1)}_{ab}\Big|_{\text{high}} + \epsilon^2\, R^{(2)}_{ab}\Big|_{\text{high}} = \kappa\, \big(T_{ab} - \tfrac{1}{2}\, g_{ab}\, T \big) \Big|_{\text{high}} \\
&= \epsilon\, \kappa\, \Big(T^{(1)}_{ab} - \tfrac{1}{2}\, g_{ab}\, T^{(1)} \Big) \Big|_{\text{high}} + \epsilon^2\, \kappa\, \Big(T^{(2)}_{ab} - \tfrac{1}{2}\, g_{ab}\, T^{(2)} \Big) \Big|_{\text{high}}
\end{aligned}
$$

The high-frequency equations of motion for the electromagnetic, axion and gravitational waves are, respectively,

$$
\begin{aligned}
&\overline{g}_{ac'}\, \overline{g}_{bd'}\, \partial_a \left(\overline{g}^{ac}\, \overline{g}^{bd} \right) F^{(1)}_{cd} + \partial_a F^{(1)}_{c'd'} + \overline{\Gamma}^{e}{}_{ae}\, F^{(1)}_{c'd'} \\
&- \frac{\epsilon}{\alpha} \Bigg(\overline{g}_{ac'}\, \overline{g}_{bd'}\, \partial_a \Big(\left(\overline{g}^{ac}\, h^{bd} + \overline{g}^{bd}\, h^{ac} \right) \overline{F}_{cd} \Big) \\
&+ \overline{g}_{ac'}\, \overline{g}_{bd'}\, \left(\overline{g}^{ac}\, h^{bd} + h^{ac}\, \overline{g}^{bd} \right) \overline{\Gamma}^{e}{}_{ae} \overline{F}_{cd} \\
&- \frac{1}{2}\, \overline{g}^{em} \left(\overline{\nabla}_e\, h_{ma} + \overline{\nabla}_a\, h_{me} - \overline{\nabla}_m\, h_{ae} \right) \overline{F}_{c'd'} \Bigg) \\
&- \epsilon \Bigg(\overline{g}_{ac'}\, \overline{g}_{bd'}\, \partial_a \Big(\left(\overline{g}^{ac}\, h^{bd} + \overline{g}^{bd}\, h^{ac} \right) F^{(1)}_{cd} \Big) \\
&- \frac{1}{2}\, \overline{g}^{em} \left(\overline{\nabla}_e\, h_{ma} + \overline{\nabla}_a\, h_{me} - \overline{\nabla}_m\, h_{ae} \right) F^{(1)}_{c'd'} \Bigg) \\
&= \overline{g}_{ac'}\, \overline{g}_{bd'}\, j^{(1)\, b} + \overline{g}_{ac'}\, \overline{g}_{bd'}\, \frac{\epsilon}{\alpha}\, j' + \overline{g}_{ac'}\, \overline{g}_{bd'}\, \overline{\nabla}_a P^{(1)\, ab} + \overline{g}_{ac'}\, \overline{g}_{bd'}\, \frac{\epsilon}{\alpha}\, \overline{\nabla}_a P'^{ab}
\end{aligned}
\tag{4.391}
$$

$$
\begin{aligned}
\partial_a F^{(1)}_{ef} + \overline{\Gamma}^e{}_{ae}\, F^{(1)}_{ef} =& -\overline{g}_{ae}\,\overline{g}_{bf}\,\partial_a\left(\overline{g}^{ac}\,\overline{g}^{bd}\right) F^{(1)}_{cd} \\
&+ \frac{\epsilon}{\alpha}\left(\overline{g}_{ae}\,\overline{g}_{bf}\,\partial_a\Big(\left(\overline{g}^{ac}\,h^{bd} + \overline{g}^{bd}\,h^{ac}\right)\overline{F}_{cd}\Big)\right. \\
&+ \overline{g}_{ae}\,\overline{g}_{bf}\,\left(\overline{g}^{ac}\,h^{bd} + h^{ac}\,\overline{g}^{bd}\right)\overline{\Gamma}^e{}_{ae}\overline{F}_{cd} \\
&\left.- \frac{1}{2}\,\overline{g}^{em}\left(\overline{\nabla}_e\,h_{ma} + \overline{\nabla}_a\,h_{me} - \overline{\nabla}_m\,h_{ae}\right)\overline{F}_{ef}\right) \\
&+ \epsilon\left(\overline{g}_{ae}\,\overline{g}_{bf}\,\partial_a\Big(\left(\overline{g}^{ac}\,h^{bd} + \overline{g}^{bd}\,h^{ac}\right)F^{(1)}_{cd}\Big)\right. \\
&\left.- \frac{1}{2}\,\overline{g}^{em}\left(\overline{\nabla}_e\,h_{ma} + \overline{\nabla}_a\,h_{me} - \overline{\nabla}_m\,h_{ae}\right)F^{(1)}_{ef}\right) \\
&+ \overline{g}_{ae}\,\overline{g}_{bf}\,j^{(1)\,b} + \overline{g}_{ae}\,\overline{g}_{bf}\,\frac{\epsilon}{\alpha}\,j' + \overline{g}_{ae}\,\overline{g}_{bf}\,\overline{\nabla}_a P^{(1)\,ab} + \overline{g}_{ae}\,\overline{g}_{bf}\,\frac{\epsilon}{\alpha}\,\overline{\nabla}_a P'^{ab}
\end{aligned}
\tag{4.392}
$$

In the limit where we go to Minkowski space, we have

$$
\partial_a F^{(1)}_{ef} + \overline{\Gamma}^e{}_{ae}\, F^{(1)}_{ef} = -\overline{g}_{ae}\,\overline{g}_{bf}\,\partial_a\left(\overline{g}^{ac}\,\overline{g}^{bd}\right) F^{(1)}_{cd} + \frac{\epsilon}{\alpha}\left(\overline{g}_{ae}\,\overline{g}_{bf}\,\partial_a\Big(\left(\overline{g}^{ac}\,h^{bd} + \overline{g}^{bd}\,h^{ac}\right)\overline{F}_{cd}\Big)\right. \tag{4.393}
$$

$$
\begin{aligned}
&\left.- \frac{1}{2}\,\overline{g}^{em}\left(\partial_e\,h_{ma} + \partial_a\,h_{me} - \partial_m\,h_{ae}\right)\overline{F}_{ef}\right) \\
&+ \epsilon\left(\overline{g}_{ae}\,\overline{g}_{bf}\,\partial_a\Big(\left(\overline{g}^{ac}\,h^{bd} + \overline{g}^{bd}\,h^{ac}\right)F^{(1)}_{cd}\Big)\right. \\
&\left.- \frac{1}{2}\,\overline{g}^{em}\left(\partial_e\,h_{ma} + \partial_a\,h_{me} - \partial_m\,h_{ae}\right)F^{(1)}_{ef}\right) + \ldots
\end{aligned}
$$

$$
\begin{aligned}
\partial_a F^{(1)}_{ef} =& \frac{\epsilon}{\alpha}\left(\partial_a\Big(\eta_{bf}\,h^{bd}\,\overline{F}_{ed} + \eta_{ae}\,h^{ac}\,\overline{F}_{cf}\Big) - \frac{1}{2}\,\left(\partial_a\,h\right)\overline{F}_{ef}\right) \\
&+ \epsilon\left(\partial_a\Big(h_{fd}F_e{}^{d\,(1)} + h_{ed}F_f{}^{d\,(1)}\Big) - \frac{1}{2}\,\left(\partial_a\,h\right)\,F^{(1)}_{ef}\right) + \ldots
\end{aligned}
\tag{4.394}
$$

Gauge fixing and removing the background electromagnetic field strength yields,

$$\begin{aligned}
\partial_a F^{(1)}_{ef} &= \frac{\epsilon}{\alpha}\,\partial_a\Big(\,h_{fd}\,\overline{F}_e^{\;d} + h_{ec}\,\overline{F}^c_{\;f}\Big) + \epsilon\,\partial_a\Big(\,h_{fd}\,F_e^{\;d\,(1)} + h_{ed}\,F_f^{\;d\,(1)}\Big) + \ldots \\
\eta^{ea}\eta^{fb}\partial_a F^{(1)}_{ef} &= \frac{\epsilon}{\alpha}\,\eta^{ea}\eta^{fb}\,\partial_a\Big(\,h_{fd}\,\overline{F}_e^{\;d} + h_{ec}\,\overline{F}^c_{\;f}\Big) + \epsilon\,\eta^{ea}\eta^{fb}\,\partial_a\Big(\,h_{fd}\,F_e^{\;d\,(1)} + h_{ed}\,F_f^{\;d\,(1)}\Big) + \ldots \\
\partial_a F^{ab\,(1)} &= \frac{\epsilon}{\alpha}\,\partial_a\Big(\,h^{bd}\,\overline{F}^a_{\;d} + h^{ac}\,\overline{F}_c^{\;b}\Big) + \epsilon\,\partial_a\Big(\,h^{bd}\,F^{a\,(1)}_{d} + h^{ac}\,F^{b\,(1)}_{c}\Big) + \ldots
\end{aligned}$$

This reproduces equation 8 in Dolgov and Ejlli.

4.5.4 Local Perturbative Equations of Motion

$$G_{ab} = \overline{G}_{ab} - \frac{1}{2}\,\overline{\Box}\,h_{ab} + \overline{R}_{cbad}\,h^{cd} \tag{4.395}$$

where $\overline{\Box} \equiv \overline{\nabla}_a\overline{\nabla}^a$ and the overlined quantities are composed of the background metric.

$$\overline{R}_{\alpha\beta\gamma\delta} \tag{4.396}$$

$$R^{(w)}_{\alpha\beta\gamma\delta} \tag{4.397}$$

$\epsilon \sim \mathcal{O}(/\mathcal{L}) \ll 1$

$$\left(\omega - i\,\partial_z + \begin{pmatrix} \Delta_\perp & \Delta_M & 0 & 0 & 0 \\ \Delta_M & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_\parallel & \Delta_M & \Delta'_M \\ 0 & 0 & \Delta_M & 0 & 0 \\ 0 & 0 & \Delta'_M & 0 & \Delta_a \end{pmatrix} + \begin{pmatrix} \tilde{\Delta}_\perp & 0 & 0 & 0 & 0 \\ 0 & \tilde{\Delta}_+ & 0 & 0 & 0 \\ 0 & 0 & \tilde{\Delta}_\parallel & 0 & 0 \\ 0 & 0 & 0 & \tilde{\Delta}_\times & 0 \\ 0 & 0 & 0 & 0 & \tilde{\Delta}_a \end{pmatrix} \right) \begin{pmatrix} A_\perp \\ G_+ \\ A_\parallel \\ G_\times \\ \phi \end{pmatrix} = 0$$

$$\tilde{\Delta}_\perp = f_\perp(\beta, \tilde{\Delta}) \tag{4.398}$$

$$\tilde{\Delta}_+ = f_+(\beta, \tilde{\Delta}, \tilde{\Delta}') \tag{4.399}$$

$$\tilde{\Delta}' = h'(\beta, \tilde{\Delta}) \tag{4.400}$$

$$\tilde{\Delta}_\parallel = g_\parallel(\beta, \tilde{\Delta}) \tag{4.401}$$

$$\tilde{\Delta}_\times = g_\times(\beta, \tilde{\Delta}, \tilde{\Delta}') \tag{4.402}$$

$$\tilde{\Delta}_a = h_a(\beta, \tilde{\Delta}) \tag{4.403}$$

Diagonalizing the matrix. The first transformation is

$$V = \begin{pmatrix} \cos\theta & \sin\theta & & & \\ -\sin\theta & \cos\theta & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \tag{4.404}$$

We can do the gravitational perturbative expansion for each term in the action

$$\mathrm{S}_2 = -\tfrac{1}{4}\int d^4x\,\sqrt{-\overline{g}}\,\Bigg(\overline{g}^{ab}\,\overline{g}^{cd}F_{ac}\,F_{bd} - 2\,\epsilon\,h^{cd}\,T^{\rm em}_{cd} + \epsilon^2\left(h^{ab}\,h^{cd}F_{ac}\,F_{bd} - \overline{g}^{ab}\,h^{cd}F_{ac}\,F_{bd}\,h\right)$$
$$+\tfrac{\epsilon^2}{8}\,\overline{g}^{ab}\,\overline{g}^{cd}\,h^2\,F_{ac}\,F_{bd}\Bigg) + \ldots$$

$$S_3 = -\tfrac{1}{2}\int d^4x\,\sqrt{-\overline{g}}\,\Bigg(\overline{g}^{ab}\,\partial_a\phi\,\partial_b\phi + m^2\phi^2 + \epsilon\,h^{cd}\,T^{\rm scalar}_{cd} - \tfrac{\epsilon^2}{2}\Big(h\,h^{ab} - h^{af}\,h_f^{\ b} + \tfrac{1}{4}\,h^2\,\overline{g}^{ab}\Big)\,\partial_a\phi\,\partial_b\phi$$
$$+\tfrac{\epsilon^2}{8}\,h^2\,m^2\phi^2\Bigg)$$

$$S_5 = \tfrac{1}{90\,m_e^4}\left(\tfrac{e^2}{4\pi}\right)^2\int d^4x\,\Bigg(\sqrt{-\overline{g}}\Bigg(\left(\overline{g}^{ab}\,\overline{g}^{cd}\,F_{ac}\,F_{bd}\right)^2 + \tfrac{7}{4}\,\left(\tfrac{1}{\sqrt{-\overline{g}}}\,\epsilon^{abcd}\,F_{ab}\,F_{cd}\right)^2\Bigg)$$
$$+\epsilon\,\sqrt{-\overline{g}}\,\Bigg(\tfrac{1}{2}\,h\,\left(\overline{g}^{ab}\,\overline{g}^{cd}\,F_{ac}\,F_{bd}\right)^2 - 4\,h^{mn}\,\overline{g}^{ab}\,\overline{g}^{ef}\,\overline{g}^{gh}\,F_{eg}\,F_{fh}\,F_{am}\,F_{bn} + \tfrac{7}{8}\,h\left(\tfrac{1}{\sqrt{-\overline{g}}}\,\epsilon^{abcd}\,F_{ab}\,F_{cd}\right)^2$$
$$-\tfrac{7}{4}\,\tfrac{1}{\sqrt{-\overline{g}}}\left(\tfrac{1}{\sqrt{-\overline{g}}}\,\epsilon^{abcd}\,F_{ab}\,F_{cd}\right)\left(h\,\epsilon^{abcd}\,F_{ab}\,F_{cd}\right)\Bigg)\Bigg)$$

CHAPTER 5
CONCLUSION

5.1 Summary of key findings and significance

We live in a time where we can access a wealth of information thanks to the multiple astrophysical probes available to us. However, several questions remain unresolved. The mapping of the Cosmic Microwave Background (CMB) revealed an extremely homogeneous primordial universe which was qualitatively explained by inflationary theories, though the exact mechanism is still unknown. Moreover, it is now believed that only 5% of the universe is made of known particles. The other 95%, of which 27% consists of Dark Matter (DM) and 68% of Dark Energy (DE), remains essentially hidden from our experimental probes and has yet to be under- stood theoretically. These crucial open problems are at the center of my work, which investigates the physics of early universe cosmology, inflationary cosmology and the nature of Dark Matter and Dark Energy. During my time as a PhD student, I have worked on theories of inflation involving multiple scalar fields as well as models of Dark Energy quintessence. In both projects, the Quantum ChromoDynamics (QCD) axion was utilized as a favorable candidate for both the inflaton and Dark Energy. The majority of the work done during my PhD has been focused on theories with observable signatures. For instance, in our work on hybrid inflation I thoroughly analyzed the phenomenology of our theory and verified its consistency with the current bounds on cosmological parameters given by the Planck data. Let us review some of the main projects completed during the course of my PhD.

5.1.1 Hybrid Inflation

One of the main project that I worked on is a theory of hybrid inflation. Inflation is the leading candidate for explaining why the universe is so old and so smooth. Most models realize inflation using a local quantum field theory with a single scalar field in a flat potential. These simple single-field inflation models require both a very flat and super-Planckian field variations and usually predict observable primordial gravitational waves which are in increasing tension with the current observations. Since the experimental data is in conflict with such models, one may be tempted to avoid the issue of Planck-suppressed operators by studying small-field models or more exotic alternatives. With my collaborators Nemanja Kaloper, Albion Lawrence and James Scargill, I revisited two-field hybrid inflation models, where the inflaton field is taken to be an axion, and explored their stability to both quantum field theoretic and quantum gravity corrections. The "classic" model of hybrid inflation of A. Linde, predicts a blue spectrum of CMB perturbations that has been ruled out by observations . However, this does not eliminate hybrid models altogether. In particular, a variant model by E. Stewart does produce a red spectrum and can be made observationally compatible. We demonstrated that an embedding of the latest model that protects it from QFT and quantum gravity corrections exists and is consistent with experimental data. We carefully showed that this model agrees with the current cosmological data from Planck and that it is stable with respect to QFT and quantum gravity corrections. We found that as it stands, our model is unnatural as it suffers from the so-called mass hierarchy problem, and therefore a protection mechanism to make our theory natural is necessary for the consistency of hybrid inflation. The Effective Field Theory of our model is technically natural; in particular, if the scalars are realized as axions dual to a theory with two massive four-forms — which might be realized as an IR limit of string compactifications — we argued that our hybrid model may be protected from quantum gravity corrections. Furthermore, we showed how the ultraviolet safety of the theory can arise in a monodromy construction, which we plan to investigate in future work. To provide additional evidence of the protection mechanism we propose, we gave an explicit dualization procedure for a two-field scalar theory by explicitly carrying out the calculation at weak coupling. I plan to revisit this issue in future work.

5.1.2 Axions as Dark Matter Candidate

In an ongoing project with Professor Devin Walker, Nizar Ezroura and Bradley Shapiro, we found out that axion waves, electromagnetic waves and gravitational waves mixed in curved spacetime. This is the first time that such a mixing has been shown. In previous work, the mixing between gravitational waves and electromagnetic waves in flat spacetime has been studied by Dolgov and Ejlli [55]. In 1988, Raffelt and Stodolsky [56], and later in 2017, Masaki, Aoki and Soda [57] discussed the mixing and probability conversion between axion waves and electromagnetic waves in flat space time. Our works explores for the first time, the mixing term between these three waves.

The next step of that work, is to describe some of the observational consequences of the mixing including the energy and polarization of the waves exiting the compact object.

This could give us a way to indirectly search for axionic dark matter.

5.2 Opportunities for Future Research

5.2.1 Multi-Fields Monodromy Inflation:

The next generation of the CMB experiment, CMB-S4, has recently been approved and is on track for completion by 2029. A major focus of CMB-S4 is to "investigate a spectacular prediction of the inflationary paradigm: primordial gravitational waves." This experiment will be able to improve the current constraints on r, the ratio of gravitational waves to tensor perturbations, by over an order of magnitude. Such a precise measurement of r will eliminate some inflationary theories: CMB-S4 should be designed to rule out or detect the remaining monomial models. It is crucial to develop multi-field models that are in agreement with the observations. Moreover, theories that exhibit a weakly broken shift symmetry are favored. In this light, multi-field inflationary models, which have precisely such a symmetry, should be examined as they appear to be a viable alternative to conventional monomial inflationary theories. The two-field inflationary model that I studied appears as a potential candidate for a viable theory. In addition to investigating the relevance of such multi-field inflationary models to current observational data, I also plan to revisit the open problems regarding their UV completion and naturalness.

5.2.2 Axions as Dark Matter Candidate and Dark Energy Quintessence:

The direct detection of gravitational wave by LIGO opened the door to a new era of detection in cosmology. It is now possible to use gravitational waves to experimentally probe theories of dark matter. String theory predicts the existence of axions in the universe in high abundance. Those particles - theoretically useful as a solution to the strong CP problem - have been extensively studied as a candidate for dark matter. I plan to investigate gravitational waves propagating in an axion background and look for possible gravitational wave signatures. Models of Dark energy quintessence and k-essence involving axion fields with dynamical attractor solutions are increasing in popularity due to the oscillating nature of the potential, and offer a possible explanation of the fine-tuning and coincidence problems which is not anthropic. I plan on studying the observable signatures of these models which can be used to distinguish quintessence from the cosmological constant.

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