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Structural and Statistical Consequences of the Closed Point Sieve

A dissertation submitted in partial satisfaction of the
requirements for the degree
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in

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by

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Committee in charge:

Professor Kiran Kedlaya, Chair
Professor Alina Bucur
Professor Shachar Lovett
Professor Cristian Popescu
Professor Steven Sam

2022

The dissertation of Thomas Grubb is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

2022

DEDICATION

To my family.

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criterion for overconvergence of F-isocrystals, which is joint work with Kiran Kedlaya and James Upton.

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ABSTRACT OF THE DISSERTATION

Structural and Statistical Consequences of the Closed Point Sieve

by

Thomas Grubb

Doctor of Philosophy in Mathematics

University of California San Diego, 2022

Professor Kiran Kedlaya, Chair

Poonen's Closed Point Sieve has proven to be a powerful technique for producing structural and combinatorial results for varieties over finite fields. In this thesis we will discuss three results which come, in part, as a consequence of this technique. First we will discuss semiample Bertini Theorems over finite fields, wherein we examine the probability with which a semiample complete intersection is smooth. In doing so we generalize work of Bucur and Kedlaya to the semiample setting of Erman and Wood. In the next chapter

we apply the Closed Point Sieve to compute the probability with which a high degree projective hypersurface over $\mathbb{F}_2$ is locally Frobenius split (a characteristic p analog of log canonical singularities). This probability approaches 1 as the dimension of the ambient projective space grows, showing that "most" projective hypersurfaces over $\mathbb{F}_2$ are only mildly singular. The final chapter, which is based on joint work with Kiran Kedlaya and James Upton, discusses an application of Bertini Theorems over finite fields to the topic of p-adic coefficient objects in rigid cohomology. Namely, we show (under a geometric tameness hypothesis) that the overconvergence of a Frobenius isocrystal can be detected by the restriction of that isocrystal to the collection of smooth curves on a variety.

Chapter 1

Introduction

This thesis concerns itself with various transversality theorems in arithmetic geometry. In particular, we study questions of the form: given a geometric object X with a "nice" property P, under what conditions will a sub-object $Y \hookrightarrow X$ also have this property. Examples of properties of interest include smoothness, connectivity, or irreducibility. Such questions can lead to structural results which can be leveraged in, for example, proofs by induction on dimension. When carried out in the arithmetic setting, they also lead to interesting enumerative results in the realm of arithmetic statistics. To motivate what follows, we will start by reviewing the classical Bertini Smoothness Theorem.

1.1 Bertini Smoothness over an Infinite Field

In this section we will review a form of the classical Bertini Smoothness Theorem. For the purposes of comparison later on, we will provide a sketch of the proof in the clas-

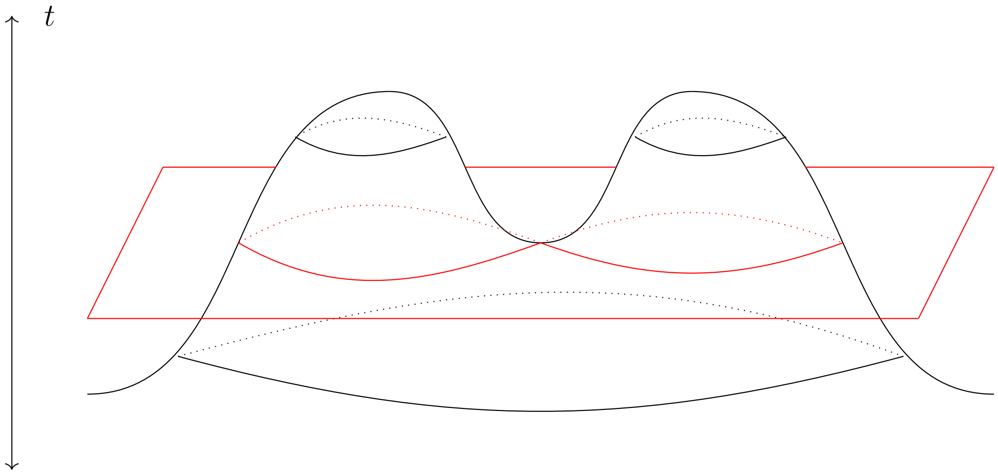

Figure 1.1: A local depiction of the Bertini Smoothness Theorem. Here the t coordinate provides a one-parameter family of hyperplanes which are perpendicular to the t axis. We see that all such hyperplanes define smooth intersections with the surface, away from the saddle point.

sical setting. A local depiction of the Bertini Smoothness Theorem is given in Figure 1.1 for reference.

Theorem 1.1.1. *Let k be a field. Let X be a smooth projective variety embedded in projective space $\mathbb{P}^n_k$ over k. Consider the dual space $(\mathbb{P}^n_k)^*$ parametrizing hyperplanes inside $\mathbb{P}^n_k$. Define*

$$U_{good} := \{H \in (\mathbb{P}^n_k)^* : X \cap H \text{ is smooth, and } X \text{ is not contained in } H\}.$$

Then U_{good} is a dense open subscheme of $(\mathbb{P}^n_k)^*$.

Proof. The method of proof is to consider an incidence locus inside

$$I \subset X \times (\mathbb{P}^n_k)^*$$

and then examine the corresponding projection map $I \to (\mathbb{P}^n_k)^*$.

For $x \in X$ consider the set of "bad at x" hyperplanes

$$B_x := \{H : X \subset H \text{ or } H \cap X \text{ is not smooth at } x\}.$$

We may characterize these as follows: any hyperplane H is determined by a global section f of $H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$. Given x as above, we may also choose $f_x \in H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$ which does not vanish at x. Let $\mathfrak{m}_x$ be the maximal ideal associated to x in X, so that $x = \mathrm{Spec}(\mathcal{O}_{X,x}/\mathfrak{m}_x)$. The first order infinitesimal neighborhood of x in X is then given by $\mathrm{Spec}(\mathcal{O}_{X,x}/\mathfrak{m}_x^2)$. We obtain the following diagram:

$$\begin{tikzcd}
H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \arrow[r, dashed, "\phi_x"] \arrow[d] & H^0(x^{(2)}, \mathcal{O}_{x^{(2)}}) \\
H^0(X, \mathcal{O}_X(1)) \arrow[r] & H^0(x^{(2)}, \mathcal{O}_{x^{(2)}}(1)) \arrow[u]
\end{tikzcd}$$

The dashed horizontal arrow ϕ_x is the composition of the remaining three maps. The three solid arrows are (given in counterclockwise order from top left to top right): two restriction maps and a (noncanonical) untwisting defined locally by $f \to f/f_x$.

Now by the Jacobian criterion, $H \cap X$ is singular at x if the defining equation of H vanishes with vanishing partial derivatives at x. Additionally, if H contains X then its defining equation is identically zero on X, and hence at x as well. Thus the bad at x hyperplanes are given precisely as the kernel of ϕ_x.

The map ϕ_x is a composition of surjective maps, and hence is surjective. As

k-vector spaces, the domain and image of ϕ_x have the following dimension counts:

$$\dim H^0(\mathbb{P}^n_k, \mathcal{O}(1)) = n + 1,$$

$$\dim H^0(x^{(2)}, \mathcal{O}_{x^{(2)}}) = d + 1.$$

Thus $B_x \to x$ is a linear system of dimension $n - d - 1$ (we drop a dimension to account for scaling).

Now consider the incidence locus

$$I := \{(x, H) \in X \times (\mathbb{P}^n_k)^* : H \in B_x\};$$

this is a closed subset of the product $X \times (\mathbb{P}^n_k)^*$, which we can endow with a reduced scheme structure. The arguments above show that I is a fibration over X with fibers isomorphic to $\mathbb{P}^{n-r-1}_k$; in particular, I has dimension $n - 1$.

To conclude, we note that the second projection $\pi : X \times (\mathbb{P}^n_k)^* \to (\mathbb{P}^n_k)^*$ is proper, since X is projective. In particular, $\pi(I)$ is a closed subset of $(\mathbb{P}^n_k)^*$ of dimension at most $n-1$. This implies that the space of "globally good" hyperplanes, $(\mathbb{P}^n_k)^* \setminus \pi(I)$, is an open dense subscheme of $(\mathbb{P}^n_k)^*$ as desired. □

The theorem above has the following important consequence in the case that k is an infinite field:

Corollary 1.1.2. *Maintain the notation as in Theorem 1.1.1, with the additional as-*

sumption that k is infinite. Then X has a hyperplane section X_H which is defined over k and is smooth of dimension $d-1$.

Proof. Let U_{good} be the open subscheme described in Theorem 1.1.1. As k is an infinite field, such a subscheme contains a k-rational point; this point leads to the desired hyperplane section of X. □

Results such as Theorem 1.1.1 and Corollary 1.1.2 provide important structural results in geometry, in part because they allow for dimensionality reduction. To give a sense of the utility of such results, we give a very quick sketch of the fact that all abelian varieties over infinite fields arise as quotients of a Jacobian. For a full proof, we refer to [Mil08].

Corollary 1.1.3. *Let k be an infinite field and let A be an abelian variety over k. There exists a curve C, also defined over k, whose Jacobian realizes A as a quotient. That is to say, there is a surjection*

$$\mathrm{Jac}(C) \twoheadrightarrow A.$$

Sketch of Proof. If A has dimension 1 then A is an elliptic curve and hence isomorphic to its own Jacobian. Thus we may assume $\dim(A) > 1$.

In this case, one applies Corollary 1.1.2 iteratively $\dim(A) - 1$ times. This results in a curve C embedded in A which is:

- smooth, and

- defined as a complete intersection of sections taken from a very ample divisor on A.

The embedding $C \to A$ gives rise to a map $\mathrm{Jac}(C) \to A$; by standard techniques in the yoga of abelian varieties, one shows that this map is surjective, giving the desired claim. □

Given the structural power that Bertini Theorems provide, it would be desirable to have an analog of Corollary 1.1.2 which applied to a *finite* field k. We will discuss this notion in the next section.

1.2 Failure of Bertini Smoothness over a Finite Field and a Question of Katz

Note that the proof of Corollary 1.1.2 relies on the fact that a dense open subset of projective space over an infinite field k contains a k-rational point. But, if k is finite, then $\mathbb{P}^n_k$ contains only finitely many k-rational points, and hence it is possible for a dense open subset of $\mathbb{P}^n_k$ to avoid all such points. The following example of Katz shows that this is a failure of the result, and not solely a failure of the method of proof:

Example 1.2.1 ([Kat99]). *Let $k = \mathbb{F}_q$ and take homogenous coordinates $X_1, \ldots, X_{n+1}$, $Y_1, \ldots, Y_{n+1}$ on $\mathbb{P}^{2n+1}_k$. Consider the hypersurface $Hyp(2n+1, q)$ defined by*

$$\sum_{i=1}^{n+1} (X_i Y_i^q - Y_i X_i^q) = 0.$$

Then there is no hyperplane H in $\mathbb{P}_k^{2n+1}$ defined over k which intersects transversely with $Hyp(2n+1, q)$.

Proof. A hyperplane H in $\mathbb{P}_k^{2n+1}$ defined over k is given by a linear form

$$(a_1, \ldots, a_{n+1}, b_1, \ldots, b_{n+1}) \cdot (X_1, \ldots, X_{n+1}, Y_1, \ldots, Y_{n+1}) = 0$$

with $a_i, b_j \in k$. The intersection $Hyp(2n+1, q) \cap H$ contains the point

$$P = [-b_1 : \cdots : -b_{n+1} : a_1 : \cdots : a_{n+1}].$$

At this point, the gradient of the defining equation of $Hyp(2n+1, q)$ is given by

$$\nabla = (a_1^q, \ldots, a_{n+1}^q, -(-b_1)^q, \ldots, -(-b_{n+1})^q).$$

As the a_i and b_j are k-rational, and as $|k| = q$, we have $a_i^q = a_i$ and $(-b_j)^q = -b_j$. Hence

$$\nabla = (a_1, \ldots, a_{n+1}, b_1, \ldots, b_{n+1}).$$

We see that the tangent space of $Hyp(2n+1, q)$ at P equals the hyperplane H. Thus H is tangent to $Hyp(2n+1, q)$ at P, and in particular the intersection is singular at P. □

Remark 1.2.2. *The previous example can be thought of more conceptually using the idea of a* dual variety*. Given X in $\mathbb{P}_k^n$, the dual variety of X is a variety X^* in the dual* *space* $(\mathbb{P}^n_k)^*$ *which parameterizes hyperplanes* H *in* $\mathbb{P}^n_k$ *which have a tangency with* X*. The space* $Hyp(2n+1,q)$ *is isomorphic to its dual variety; since* $Hyp(2n+1,q)$ *contains all k-rational points in* $\mathbb{P}^n_k$*, the analogous statement must hold in the dual variety. Hence there is no k-rational hyperplane which intersects smoothly with* $Hyp(2n+1,q)$*.*

After producing Example 1.2.1 in [Kat99], Katz raised the question of whether or not a Bertini type theorem could be recovered for finite fields by allowing *high degree* hypersurface sections, instead of working solely with hyperplanes. In [Gab01], Gabber was able to find high degree hypersurfaces which intersect transversely with a given variety, so long as the degree is divisible by the underlying characteristic p. Contemporaneously, Katz's question was put to rest conclusively with Poonen's introduction of the Closed Point Sieve [Poo04]. Namely, Poonen shows the following:

Theorem 1.2.3 ([Poo04], Theorem 1.1)**.** *Let* k *be a finite field of cardinality* q *and let* $S_{homog} \subset k[x_0, \ldots, x_n]$ *be the homogenous polynomial ring over* $\mathbb{P}^n_k$*. For* $d \geq 1$ *let* $S_d \subset S_{homog}$ *be the degree* d *component. Let* X *be an* m*-dimensional quasiprojective subscheme of* $\mathbb{P}^n_k$ *with zeta function*

$$\zeta_X(s) = \prod_{\textit{closed points } P \in X} (1 - q^{-s\deg(P)})^{-1}.$$

Given $f \in S_{homog}$*, let* H_f *denote the hypersurface of* X *cut out by* f*. Then*

$$\lim_{d\to\infty} \frac{\#\{f \in S_d : H_f \textit{ is smooth of dimension } m-1\}}{\#S_d} = \zeta_X(m+1)^{-1}.$$

q	2	3	4	5	7	8	9	11
$\approx \zeta_{\mathbb{P}^2_{\mathbb{F}_q}}(3)^{-1}$	32.8%	57.1%	69.2%	76.2%	83.7%	86.0%	87.8%	90.0%

Figure 1.2: The probability with which high degree projective plane curves are smooth.

In other words, not only do good hypersurface sections exist, but one can specify the limiting probability that such a hypersurface section is smooth. For example, applying Poonen's result allows one to conclude that a high degree plane curve over $\mathbb{F}_q$ is smooth with probability

$$\zeta_{\mathbb{P}^2_{\mathbb{F}_q}}(3)^{-1} = (1 - q^{-1})(1 - q^{-2})(1 - q^{-3}).$$

One can see sample calculations for this in Figure 1.2.

This result verifies local heuristics: for a hypersurface to intersect with X singularly at a point P, the defining polynomial and all of its gradients must simultaneously vanish. This constitutes $m + 1$ linear evaluations, all of which take place in the residue field of P. Thus the local probability of singularity should equal $1 - q^{-(m+1)\deg(P)}$. If one can show that evaluation behaves "independently" across points, one would naturally arrive at Theorem 1.2.3.

In fact, Poonen is able to prove stronger results by imposing local Taylor conditions that the hypersurface must satisfy at a certain locus on X. We will take a quick detour to motivate Poonen's proof technique before stating the more general result.

1.3 Motivation for the Closed Point Sieve

Having discussed the classical Bertini Theorem as well as its failure over finite fields, we now move to a short motivation to discuss how we may recover Bertini-type results in arithmetic settings. For this purpose we will loosely describe an "arithmetic" object X to be either a scheme flat and of finite type over $\mathbb{Z}$ or a variety over a finite field $\mathbb{F}_p$. The first observation is that in both of these settings the following hold:

- Global properties of interest are often defined pointwise locally; for example, a scheme over $\mathbb{Z}$ is regular if the local rings at all closed points are regular.
- If X is arithmetic and C is an integer, then the number of closed points $x \hookrightarrow X$ for which $\#(\mathcal{O}_{X,x}/\mathfrak{m}_x) < C$ is finite. In particular, there are at most countably many closed points in an arithmetic object.

Thus if we wish to show that a section of some bundle, $f \in H^0(X, \mathcal{L}^d)$, defines a hypersurface in X with a desirable property, we may try the following strategy:

1. Show that, as $d \to \infty$, the hypersurface defined by f locally satisfies the desired property, and that satisfying this property is *independent across (a finite number of) points.*
2. Collect points according to the size of their residue field, and show, using a sieving argument, that the analysis is dominated by the behaviour of f at the finite set of points with bounded residue field.

To motivate why this strategy has legs, we give two classical examples of these techniques.

1.3.1 Lagrange and Hermite Interpolation

In this section we remind the reader of Lagrange and Hermite Interpolation. Our purpose in doing so is to give an "easy" example of the independence across points phenomena.

Let $f \in \mathbb{C}[x]$ be a degree d polynomial,

$$f(x) = c_0 + c_1 x + \cdots + c_n x^d.$$

It is natural to ask the following: for how many distinct points t_i must we evaluate $f(t_i)$ in order to uniquely determine f? As f is determined by $d+1$ coefficients, it is natural to propose that the answer is $d+1$. Indeed, this is the case; a straightforward calculation shows that given $d+1$ evaluations, $(t_i, f(t_i))$, one can uniquely reconstruct f as a sum of *Lagrange polynomials*,

$$f(x) = \sum_{i=1}^{d+1} f(t_j) \prod_{\substack{1 \leq m \leq d+1 \\ m != i}} \frac{x - t_m}{t_i - t_m}.$$

To phrase this in terms of independence across points, we will reformulate Lagrange Interpolation as follows. Let $\mathbb{C}[x]_{\leq d}$ be the space of polynomials of degree at most d. Given $t \in \mathbb{C}$ we obtain an evaluation map $ev_t : \mathbb{C}[x]_{\leq d} \to \mathbb{C}$ which sends f to $f(t)$. Then Lagrange Interpolation may be restated as follows:

Theorem 1.3.1. *Let $T = \{t_1, \ldots, t_k\}$ be a set of complex numbers. For all $d > k$, the* *maps*

$$ev_T : \mathbb{C}[x]_{\leq d} \to (\mathbb{C})^k$$

$$ev_T(f) = (f(t_1), \ldots, f(t_k))$$

are surjective. □

In particular, as long as the degree of f is high enough, the vanishing or nonvanishing of f at a finite collection of points tells you nothing of the vanishing or nonvanishing of f at another finite collection of points.

To discuss smoothness we need knowledge not just of the evaluation of f but of the evaluation of partial derivatives of f as well. For this we generalize to Hermite Interpolation:

Theorem 1.3.2. *Let $T = \{t_1, \ldots, t_k\}$ be a set of complex numbers and let m be a positive integer. Given $f \in \mathbb{C}[x]$ let $f^{(m)}$ be the mth derivative of f. For all $d > k(m+1)$, the evaluation map*

$$ev_{T,m} : \mathbb{C}[x]_{\leq d} \to (\mathbb{C})^{k,m+1}$$

$$ev_{T,m}(f) = \begin{pmatrix} f(t_1) & \ldots & f(t_k) \\ f^{(1)}(t_1) & \ldots & f^{(1)}(t_k) \\ \vdots & & \vdots \\ f^{(m)}(t_1) & \ldots & f^{(m)}(t_k) \end{pmatrix}$$

is surjective. □

As with Lagrange Interpolation, we can interpret Hermite Interpolation as a statement of independence across a finite collection of points. So long as the total degree of your polynomial is allowed to grow, the behaviour of your polynomial at $k-1$ points tells you nothing of its behaviour at a kth point. Both Lagrange and Hermite Interpolation admit natural multivariate extensions, and for a modern introduction we recommend [Lor00].

It is important to note that in both notions of interpolation above, the evaluation maps in question are *linear maps between vector spaces.* This directly motivates the proof techniques for Bertini Theorems over finite fields. We will establish linear evaluation maps between vector spaces over a finite field; the domain will be the space of global sections of a bundle, and the range will be the coordinate ring of a collection of points. The theorems of interest largely follow by showing surjectivity of that evaluation map, and then counting the size of the kernel.

1.3.2 The Probability that an Integer is Squarefree

The previous subsection gave heuristics for why one might expect "independence" among a finite collection of points. This section gives motivation for how to turn a statement regarding a finite set of points into a statement regarding an infinite number of points by using a sieve. The motivation will be to compute the probability with which an integer is square free. We have the following:

Theorem 1.3.3. *Let*

$$SF_n = \{m \in \mathbb{Z} : 0 \leq m \leq n \textit{ and } p^2 \nmid m \textit{ for all primes } p\}.$$

Then

$$\lim_{n\to\infty} \frac{\#SF_n}{n} = \prod_{p \textit{ prime}} (1 - p^{-2}) = \frac{6}{\pi^2}.$$

Proof. For a single prime p it is straightforward to see that

$$\lim_{n\to\infty} \frac{\#\{m \in \mathbb{Z} : 0 \leq m \leq n \text{ and } p^2 \nmid m\}}{n} = 1 - \frac{1}{p^2}.$$

Moreover, for a finite set of primes B, the Chinese Remainder Theorem assures us that being "squarefree at B" can be computed prime by prime:

$$\lim_{n\to\infty} \frac{\#\{m \in \mathbb{Z} : 0 \leq m \leq n \text{ and } p^2 \nmid m \text{ for all } p \in B\}}{n} = \prod_{p\in B} \left(1 - \frac{1}{p^2}\right).$$

Unfortunately the Chinese Remainder Theorem does not directly apply to an infinite collection of primes.

To get around this, for any bound P define

$$SF_n^P = \{m \in \mathbb{Z} : m \leq n \text{ and } p^2 \nmid m \text{ for all primes } p \leq P\}.$$

Note the chain of inclusions

$$SF_n \subset SF_n^P \subset SF_n \cup \{m \in \mathbb{Z} : 0 \leq m \leq n \text{ and } p \mid m \text{ for some } p > P\}.$$

Let us examine

$$\lim_{P\to\infty} \lim_{n\to\infty} \frac{\#\{m \in \mathbb{Z} : 0 \leq m \leq n \text{ and } p \mid m \text{ for some } p > P\}}{n}.$$

By counting the multiples of a given prime in the interval $[0, n]$ (occasionally *overcounting*), we have

$$\lim_{P\to\infty} \lim_{n\to\infty} \frac{\#\{m \in \mathbb{Z} : 0 \leq m \leq n \text{ and } p \mid m \text{ for some } p > P\}}{n} \leq \lim_{P\to\infty} \lim_{n\to\infty} \frac{1}{n} \sum_{p>P} \frac{1+n}{p^2}.$$

The summation above is absolutely convergent. Bounding the sum and taking limits gives

$$\lim_{P\to\infty} \lim_{n\to\infty} \frac{\#\{m \in \mathbb{Z} : 0 \leq m \leq n \text{ and } p \mid m \text{ for some } p > P\}}{n} = 0.$$

Returning to

$$SF_n \subset SF_n^P \subset SF_n \cup \{m \in \mathbb{Z} : 0 \leq m \leq n \text{ and } p \mid m \text{ for some } p > P\},$$

we see that

$$\lim_{P\to\infty} \lim_{n\to\infty} \frac{\#SF_n^P}{n} = \lim_{n\to\infty} \frac{\#SF_n}{n}.$$

Using the Chinese Remainder Theorem on the first limit, we obtain the desired result

$$\lim_{n\to\infty}\frac{\#SF_n}{n}=\lim_{P\to\infty}\prod_{p<P}\left(1-p^{-2}\right)=\frac{6}{\pi^2}.$$

$\square$

We note in passing that $\frac{6}{\pi^2}$ *also* appears in the study of coprime integers. Namely, we have:

Theorem 1.3.4. *Let*

$$CP_n=\{(m_1,m_2)\in\mathbb{Z}^2:0\leq m_i\leq n \text{ and } \gcd(m_1,m_2)\}=1.$$

Then

$$\lim_{n\to\infty}\frac{\#CP_n}{n^2}=\frac{6}{\pi^2}.$$

$\square$

One may prove this statement using a similar sieve as above; the fact that the limits in Theorems 1.3.3 and 1.3.4 are the same foreshadows the following nice fact, which we will discuss in more detail later:

Theorem 1.3.5 ([Poo04, Theorem 1.1] and [BK12, Theorem 1.2])**.** *Maintain the notation in Theorem 1.2.3, i.e. X is an m-dimensional quasiprojective subscheme of $\mathbb{P}^n_k$. Given k sections in S_d, $\mathbf{f}=(f_1,\ldots,f_k)$, let $H_{\mathbf{f}}$ denote the complete intersection these sections* *define in* X*. Then* both

$$\lim_{d\to\infty} \frac{\#\{f \in S_d : H_f \text{ is smooth of dimension } m-1\}}{\#S_d} = \zeta_X(m+1)^{-1}$$

and

$$\lim_{d\to\infty} \frac{\#\{\mathbf{f} \in (S_d)^{m+1} : H_{\mathbf{f}} \text{ is empty }\}}{\#(S_d)^{m+1}} = \zeta_X(m+1)^{-1}.$$

□

This theorem can be seen as a polynomial analog of the fact that squarefree and coprime integers can be counted in terms of the Riemann zeta function.

1.4 The Closed Point Sieve and Preliminary Consequences

With Section 1.3 as motivation, we can now return to Poonen's work to discuss the strategy of proof for Theorem 1.2.3; we do this as this strategy will be directly adapted in our Chapters 2 and 3. In the end of this section we will outline several results which come thanks to adaptations of the Closed Point Sieve or from structural consequences of the Closed Point Sieve.

We start by stating the more general form of Poonen's result, which allows specified Taylor conditions on the hypersurfaces in question:

Theorem 1.4.1 ([Poo04, Theorem 1.2]). *Let X be a quasiprojective subscheme of $\mathbb{P}^n_{\mathbb{F}_q}$. Let Z be a finite subscheme of $\mathbb{P}^n_{\mathbb{F}_q}$, and assume that $U := X \setminus (Z \cap X)$ is smooth of dimension m. Fix a subset*

$$T \subset H^0(Z, \mathcal{O}_Z).$$

Given $f \in S_d$, let $f|_Z$ be the element of $H^0(Z, \mathcal{O}_Z)$ that on each connected component Z_i equals the restriction of $x_j^{-d} f$ to Z_i, where $j = j(i)$ is the smallest $j \in 0, 1, \ldots, n$ such that the coordinate x_j is invertible on Z_i. Then

$$\lim_{d\to\infty} \frac{\#\{f \in S_d : H_f \cap U \text{ is smooth of dimension } m-1 \text{ and } f|_Z \in T\}}{\#S_d}$$

equals

$$\frac{\#T}{\#H^0(Z, \mathcal{O}_Z)} \zeta_U(m+1)^{-1}.$$

Sketch of Proof. We will outline the proof of this result in analogy with Section 1.3. Poonen orders the closed points of $X \cup Z$ by degree. For any finite subscheme $Y \subset \mathbb{P}^n_{\mathbb{F}_q}$, one examines the restriction maps

$$\phi_d : S_d \to H^0(Y, \mathcal{O}_Y)$$

and shows that ϕ_d is surjective for $d \geq \dim_{\mathbb{F}_q} H^0(Y, \mathcal{O}_Y) - 1$. This is a consequence of Serre's Vanishing Theorem, as the cokernel of the ϕ_d lies in $H^1(Y, \mathcal{I}_Y(d))$, for $\mathcal{I}_Y$ the ideal sheaf cutting out Y. By choosing Y to be an infinitesimal neighborhood around a finite collection of points, this lemma provides a scheme theoretic analog of Hermite Interpolation, Theorem 1.3.2.

Now set $U_{<r}$ to be the finite scheme given by the closed points in U of degree $< r$, $U_{<r} = \{P_1, \ldots, P_s\}$. Let $P_i^{(2)}$ denote the first order infinitesimal thickening of P_i, and let

$$Y = Z \cup P_1^{(2)} \cup \cdots \cup P_s^{(2)}.$$

Note that for f to be singular at P_i it must vanish on $P_i^{(2)}$. Examining the restriction maps

$$\phi_d : S_d \to H^0(Y, \mathcal{O}_Y),$$

and noting that

$$H^0(Y, \mathcal{O}_Y) \cong H^0(Z, \mathcal{O}_Z) \times H^0(P_i^{(2)}, \mathcal{O}_{P_1^{(2)}}) \times \cdots \times H^0(P_i^{(2)}, \mathcal{O}_{P_s^{(2)}})$$

we can rewrite the polynomials f in S_d which are smooth at U_r and which restrict to T on Z as the inverse image

$$\phi_d^{-1}\left(T \times \left(H^0(P_1^{(2)}, \mathcal{O}_{P_1^{(2)}}) \setminus \{0\}\right) \times \cdots \times \left(H^0(P_s^{(2)}, \mathcal{O}_{P_1^{(2)}}) \setminus \{0\}\right)\right).$$

Since ϕ_d is eventually surjective, we conclude that

$$\lim_{d\to\infty} \frac{\#\{f \in S_d : f \text{ is smooth at } U_{<r} \text{ and } f|_Z \in T\}}{\#S_d}$$

is equal to

$$\frac{\#T}{\#H^0(Z, \mathcal{O}_Z)} \prod_{P \in U_{<r}} (1 - q^{-(m+1)\deg(P)}).$$

Now the product on the right hand side above converges to the zeta function of U as $r \to \infty$. So the goal now is, as in Subsection 1.3.2, to show that high degree points do not impact the probability of smoothness.

The sieving argument here is much more subtle than in Theorem 1.3.3. Poonen rewrites $f \in S_d$ in such a way that the partial derivatives of f become decoupled; by invoking Bezout's Theorem, he is then able to bound the probability with which both f and all of its partial derivatives vanish at a high degree point. In doing so, Poonen shows that the asymptotics of smoothness are well approximated by the low degree points of U; taking a limit as $r \to \infty$ provides the desired result. □

With Theorem 1.4.1 in hand, we get the specialization of Theorem 1.2.3 by setting $Z = \emptyset$.

Following Poonen's work there has been an influx of "nonclassical" Bertini type theorems. Over finite fields we have seen Bertini irreducibility theorems [CP16], Bertini smoothness theorems in semiample and toric settings [EW15], [Lin17], and Bertini theorems on containing a specified closed subscheme [Gun17], [Poo08], [Wut16], [Wut17]. Additionally one has Bertini type results over discrete valuation rings [JS12] as well as motivic analogs of results over finite fields [VW15], [How16], [BH21]. Erman and Wood's Bertini theorem in the semiample setting will be one of the central focuses of this paper, and will be discussed in more detail below.

The work listed above was followed by a flurry of structural and statistical results for schemes over finite fields. Structural results are usually obtained via the probabilistic method; namely, one shows the existence of a desirable object by showing a random object has the desired property with positive probability. This has led to Whitney embedding theorems over finite fields [Ngu05], results on extending self maps of schemes to maps of projective space [Poo13], or the construction of algebraic cycles with various conditions imposed upon them [Dri12],[van18].

Instead of simply asking for a probability to be nonzero, one can examine local probabilities more closely to obtain results on the arithmetic statistics of such schemes. This has led to results on expected point counts for smooth plane curves [BDFL10], complete intersections in projective space [BK12], and Hirzebruch surfaces [EW15]. These results fall into a growing literature of point counting; we will not give an exhaustive list, but refer to [AEK+15], [BDFL16], [Ho14], [KW11], [Woo12] and references therein for examples.

For the remainder of this thesis we will examine three additional results which come, in part, thanks to the introduction of the Closed Point Sieve. In Chapter 2 we will adapt the work of Bucur and Kedlaya [BK12] on complete intersections to the semiample setting of Erman and Wood [EW15]. Chapter 3 will focus on the concept of Frobenius singularities; we will introduce the concept of Frobenius splittings and then use the closed point sieve to count projective hypersuraces over $\mathbb{F}_2$ which are locally F-split. In doing so we show that most of these varieties are only mildly singular. Finally, we end the thesis with Chapter 4 (based on joint work with Kiran Kedlaya and James Upton [GKU22]), which shows a structural application of the Closed Point Sieve. Namely, we use Theorem 1.4.1 to produce families of space filling curves on varieties. We then use these curves to prove a cut by curves criteria for p-adic coefficient objects on X; namely, under a suitable tameness hypothesis, that a convergent Frobenius isocrystal is in fact *overconvergent* on X if its restriction on all curves in X is overconvergent.

Chapter 2

Semiample Complete Intersections

2.1 Introduction

The purpose of this chapter is to examine smooth complete intersections over finite fields in Erman and Wood's semiample setting [EW15]. We start by recalling the work of Bucur and Kedlaya [BK12]. By applying Poonen's Bertini Theorem [Poo04, Theorem 1.1] iteratively, one obtains the existence of smooth complete intersections of arbitrary dimension over finite fields. However by doing this inductively one loses precision on the probability that a random complete intersection is smooth. Bucur and Kedlaya remedy this with Theorem 1.2 of [BK12]. Namely, let X be a smooth projective variety of dimension m over $\mathbb{F}_q$ and $d_1 \leq \cdots \leq d_k$ positive integers. Let X° be the closed points of X. Take k sections $f_1, \ldots, f_k$ with $f_i \in H^0(X, \mathcal{O}_X(d_i))$. Then Bucur and Kedlaya show (under a mild growth assumption for the d_i) that the limiting probability with which the complete intersection defined by the f_i is smooth equals

$$\prod_{P \in X^\circ} \left(1 - q^{-\deg(P)k} + q^{-\deg(P)k} \prod_{j=0}^{k-1} (1 - q^{-\deg(P)(m-j)}) \right).$$

Analogously with Poonen's result, this verifies local heuristics. Note that the product $\prod_{j=0}^{k-1}(1 - q^{-\deg(P)(m-j)})$ computes the probability that k vectors chosen randomly from $\mathbb{F}^m_{q^{\deg(P)}}$ are linearly independent. Thus the factor

$$1 - q^{-\deg(P)k} + q^{-\deg(P)k} \prod_{j=0}^{k-1} (1 - q^{-\deg(P)(m-j)})$$

simply computes the local probability with which the f_i do not simultaneously vanish at P or simultaneously vanish at P with linearly independent gradients. As it is impossible for $m+1$ vectors to be linearly independent in $\mathbb{F}^m_{q^{\deg(P)}}$, when $k = m+1$ the only way for the complete intersection to be smooth is to have an empty intersection. We thus obtain the probability

$$\prod_{P \in X^\circ} \left(1 - q^{-\deg(P)(m+1)} \right) = \zeta_X(m+1)^{-1}.$$

This explains our comments at the end of Subsection 1.3.2.

Our goal will be to provide a generalization of the above result to the following setting, formulated by Erman and Wood in [EW15]. As usual, let X be a smooth projective variety over $\mathbb{F}_q$. Let A be a very ample divisor and E a globally generated divisor on X. Let $\pi : X \to \mathbb{P}^M$ denote the map corresponding to the complete linear series of E.

Then Theorem 1.1 of [EW15] states that there exists an n_0 (depending only on X and q) such that, for all $n \geq n_0$ and as $d \to \infty$, a random section $f \in H^0(X, \mathcal{O}_X(nA + dE))$ satisfies

$$\lim_{d\to\infty} \mathrm{P}(H_f \text{ is smooth}) = \prod_{P \in \mathbb{P}^M} \mathrm{P}(H_f \text{ is smooth at all points of } \pi^{-1}(P)).$$

Moreover they show that this product is nonzero if n is sufficiently large. Notably, this probability *does not* factor as a product over points of X, but instead factors over the fibers of π.

With this setup, we present our main theorem, which serves as the analog of [BK12, Theorem 1.2] in the semiample setting.

Theorem 2.1.1. *Let X be a smooth quasiprojective variety over $\mathbb{F}_q$. Let A be an ample divisor on X and E a globally generated divisor on X. Let π be the map given by the linear series of E on X. Take integers $d_1 \leq \cdots \leq d_k$ with d_k subexponential in d_1*[1]*. Then there exists a positive integer n_0 depending only on X and char$(\mathbb{F}_q)$ such that for any $n \geq n_0$, choosing k sections $f_1, \ldots, f_k$ with $f_i \in H^0(X, \mathcal{O}_X(nA + d_iE))$ results in a smooth complete intersection $H_{f_1} \cap \cdots \cap H_{f_k}$ with probability tending to (as $d_1 \to \infty$)*

$$\prod_{P \in \pi(X)} \mathrm{P}(H_{f_1} \cap \cdots \cap H_{f_k} \textit{ is smooth at } \pi^{-1}(P)).$$

[1]This is a technical growth condition; see the proof of Lemma 2.3.4 in Section 2.5. It suffices to have $d_k^{\dim X} q^{-d_1} \to 0$.

The above product converges, is zero if and only if one of its factors is zero, and is always nonzero for sufficiently large n.

Following the style of [EW15], Theorem 2.1.1 will come as a corollary to the more general Theorem 2.3.1, which allows one to prescribe the behavior of the complete intersection at finitely many fibers of π. In Section 7 we will obtain the following more general result, which allows one to impose the condition that the complete intersection intersects a specified closed subscheme $Y \subset X$ transversely, so long as the singular locus of Y has sufficiently small dimension. This provides an analog of [Poo04, Theorem 1.3]; such a result was predicted in Section 4 of [BK12], but to our knowledge has not been written down for complete intersections in either the ample or semiample setting.

Theorem 2.1.2. *With the same assumptions as Theorem 2.1.1, assume additionally that $Y \subset X$ is a closed subscheme with $\dim(Y^{sing}) \leq k-1$. Then the probability that both $H_{f_1} \cap \cdots \cap H_{f_k}$ and $H_{f_1} \cap \cdots \cap H_{f_k} \cap Y$ are smooth tends to*

$$\prod_{P \in \pi(X)} \mathrm{P}(H_{f_1} \cap \cdots \cap H_{f_k} \textit{ and } H_{f_1} \cap \cdots \cap H_{f_k} \cap Y \textit{ are smooth at } \pi^{-1}(P)).$$

The above product converges, is zero if and only if one of its factors is zero, and is always nonzero for sufficiently large n.

We end the introduction with several comments. First, by setting $E = A$ in Theorem 2.1.1 we recover the work of Bucur and Kedlaya in the ample setting, just as the result of Erman and Wood recovers Poonen's result for hypersurfaces.

In the event that E is very ample and if X has dimension m, the product in Theorem 2.1.1 reduces to

$$\prod_{P \in X^\circ} \left(1 - q^{-\deg(P)k} + q^{-\deg(P)k} \prod_{j=0}^{k-1} (1 - q^{-\deg(P)(m-j)})\right).$$

This should be compared to Example 4.3 in [EW15]. Setting $k = m+1$, we recover Theorem 1.3.5 for a wider variety of curves:

Example 2.1.3. *Let $X = \mathbb{P}^i_{\mathbb{F}_q} \times \mathbb{P}^j_{\mathbb{F}_q}$. Fix an integer l. The probability that $i+j+1$ sections from $H^0(\mathbb{P}^i_{\mathbb{F}_q} \times \mathbb{P}^j_{\mathbb{F}_q}, \mathcal{O}(d+l, d))$ have empty intersection approaches $\zeta_{\mathbb{P}^i_{\mathbb{F}_q} \times \mathbb{P}^j_{\mathbb{F}_q}}(i+j+1)^{-1}$ as $d \to \infty$. This is the same probability with which any one of these sections defines a smooth hypersurface in $\mathbb{P}^i_{\mathbb{F}_q} \times \mathbb{P}^j_{\mathbb{F}_q}$.*

Next, we emphasize that while the equality in Theorems 2.1.1 and 2.1.2 hold for any $n \geq n_0$, it may be necessary to increase n to obtain a nonzero probability. Analogous problems exist even for infinite fields of characteristic p, as is discussed in the introduction of Erman and Wood's original paper.

The condition in Theorem 2.1.2 that $\dim(Y^{sing}) \leq k-1$ may be thought of as a mild Altman Kleiman type condition [KA79]; such conditions were used by Gunther [Gun17] and Wutz [Wut16], [Wut17] to prove Bertini theorems for hypersurfaces *containing* a given subscheme. Our requirement is weaker than the condition used by Gunther and by Wutz, since our complete intersection is only required to intersect smoothly with Y. In particular the condition $\dim(Y^{sing}) \leq k-1$ is nothing other than the statement that the complete intersection will avoid Y^{sing} with positive probability.

2.2 Notation

We will largely follow the notation set up in [EW15]. Take X a quasiprojective variety of dimension m over $\mathbb{F}_q$, with $\mathrm{char}(\mathbb{F}_q) = p$. Let A be a very ample divisor on X and E a globally generated divisor on X. To it we associate the map

$$\pi : X \to \mathbb{P}^M$$

given by the complete linear series on E. Set $B = \pi(X)$, with $\dim(B) = b$. Since A is very ample, it gives rise to an embedding $X \hookrightarrow \mathbb{P}^N$ and hence an embedding $X \hookrightarrow \mathbb{P}^N \times \mathbb{P}^M$. For a subvariety $Y \subset X$, we will set $\deg_A(Y)$ to be the degree of Y under this embedding.

When $b > 0$ and W is a 0-dimensional subscheme of $\mathbb{P}^M$, $\mathcal{O}_W(1)$ is trivial on W and hence via pulling back $\mathcal{O}_{\pi^{-1}(W)}(E)$ is trivial on $\pi^{-1}(W)$. The choice of trivialization of $\mathcal{O}_W(1)$ is noncanonical but can be done by dividing by nonvanishing coordinates on the connected components of W; we will do so implicitly for the remainder of the paper.

We employ big O notation for functions $\mathbb{N}^k \to \mathbb{R}$; the constants will depend solely on the choice of X, A, E, and the underlying characteristic p. Explicitly, we say $F = O(G)$ if there is a positive constant C for which $F \leq C \cdot G$ on all of $\mathbb{N}^k$. Suppose we have a family of sets $\mathcal{A} = (A_n)$ graded by a natural number n. If we have subsets $\mathcal{B} = (B_n)$ with $B_n \subset A_n$, then we define *the natural density* of $\mathcal{B}$ in $\mathcal{A}$ to be the following limit, if it exists:

$$\mu_{\mathcal{A}}(\mathcal{B}) = \lim_{n\to\infty} \frac{|B_1 \cup \cdots \cup B_n|}{|A_1 \cup \cdots \cup A_n|}.$$

In particular we are interested in the natural density of smooth complete intersections inside the family of all possible complete intersections; we will abuse notation and simply consider this as a "probability," i.e. we will denote this as $\mathrm{P}(H_{\mathbf{f}} \text{ is smooth})$. We hope that our meaning is clear from context and that no confusion will occur.

Recall from the introduction the notation

$$L(q, m, k) = \prod_{j=0}^{k-1} (1 - q^{-(m-j)}),$$

which computes the probability that k vectors chosen uniformly at random from $\mathbb{F}_q^m$ are linearly independent.

To simplify notation we will use vector notation for sheaves on X. Given sequences of positive integers $\mathbf{n} = (n_1, \ldots, n_k)$ and $\mathbf{d} = (d_1, \ldots, d_k)$ we set

$$\mathcal{O}_X(\mathbf{n}A + \mathbf{d}E) := \bigoplus_{i=1}^{k} \mathcal{O}_X(n_i A + d_i E).$$

By rearranging the summands we may always assume $d_1 \leq \cdots \leq d_k$. Given a global section f on X we let H_f denote the corresponding hypersurface in X. Given k sections $\mathbf{f} = (f_1, \ldots, f_k)$ with $f_i \in H^0(X, \mathcal{O}_X(n_i A + d_i E))$, we will be interested in studying the complete intersection $H_{\mathbf{f}} := H_{f_1} \cap \cdots \cap H_{f_k}$.

We maintain the convention that the empty scheme is smooth of any dimension. Thus given a choice of sections $\mathbf{f} = (f_1, \ldots, f_k)$ the corresponding complete intersection $H_{\mathbf{f}}$ will be smooth at a closed point $P \in X$ if at least one of the f_i do not vanish at P, or if the f_i simultaneously vanish at P with linearly independent gradient vectors. To examine this we examine the restriction of the f_i to the first order infinitesimal neighborhood of P, denoted $P^{(2)}$. Explicitly, if $P \in X$ is a closed point then $P^{(2)} := \operatorname{Spec}(\mathcal{O}_{X,P}/\mathfrak{m}_P^2)$. Given such a point we may decompose $H^0(P^{(2)}, \mathcal{O}_{P^{(2)}})$ as a direct sum

$$H^0(P^{(2)}, \mathcal{O}_{P^{(2)}}) \cong H^0(P, \mathcal{O}_P) \oplus V$$

where V is the m dimensional tangent space. We may extend these notions to zero dimensional subschemes W with connected components $P_1, \ldots, P_s$ by setting $W^{(2)} := \bigsqcup_{i=1}^{s} P^{(2)}$. Finally, given a closed subscheme $W \subset Y$ we let X_W denote the fibered product $X_W = X \times_B W$.

2.3 Main Theorem

In this section we will prove Theorem 2.1.1 as a corollary to the following.

Theorem 2.3.1. *Let X be a quasiprojective variety over $\mathbb{F}_q$ with a very ample divisor A and a globally generated divisor E. Let $\mathbf{n} = (n, \ldots, n)$ and $\mathbf{d} = (d_1, \ldots, d_k)$ be positive integers with $d_1 \leq \cdots \leq d_k$ with d_k subexponential in d_1[2]. Fix a finite subscheme $Z \subsetneqq$*

[2]This is a technical growth condition; see the proof of Lemma 2.3.4 in Section 2.5. It suffices to have

$\pi(X)$. *Assume* $X \setminus \pi^{-1}(Z)$ *is smooth, and set* $n_0 = \max(b(m+1)-1, bp+1)$.

Then so long as $n \geq n_0$, *for any* $T \subset H^0(\pi^{-1}(Z), \mathcal{O}_{\pi^{-1}(Z)}(\mathbf{n}A))$ *and for* $\mathbf{f} \in H^0(X, \mathcal{O}(\mathbf{n}A + \mathbf{d}E))$ *chosen uniformly at random as* $d_1 \to \infty$, *we have*

$$\begin{aligned} &\mathrm{P}(H_{\mathbf{f}} \cap (X \backslash \pi^{-1}(Z)) \text{ is smooth and } \mathbf{f}|_{\pi^{-1}(Z)} \in T) \\ &\qquad = \mathrm{P}(\mathbf{f}|_{\pi^{-1}(T)} \in T) \prod_{P \in (B \backslash Z)} \mathrm{P}(H_{\mathbf{f}} \text{ is smooth at all points of } X_P). \end{aligned}$$

The product over $P \in B \setminus Z$ *converges, is zero if and only if one of the factors is zero, and is always nonzero so long as* n *is sufficiently large.*

The proof of the above theorem will follow by adapting the sieve techniques introduced by Poonen and extended to the semiample setting by Erman and Wood. Maintaining the notation of the latter authors, we define for some e_0

$$\mathcal{P}^{low}_{e_0,\mathbf{n},T} := \bigcup_{\mathbf{d}} \left\{ \mathbf{f} \in H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E)) : \begin{array}{c} H_{\mathbf{f}} \text{ smooth at all } Q \in X \setminus \pi^{-1}(Z) \\ \text{with } \deg(\pi(Q)) < e_0, \text{ and } \mathbf{f}|_{\pi^{-1}(Z)} \in T \end{array} \right\},$$

$$\mathcal{Q}^{med}_{e_0,\mathbf{n}} := \bigcup_{\mathbf{d}} \left\{ \mathbf{f} \in H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E)) : \begin{array}{c} H_{\mathbf{f}} \text{ singular at a point } Q \in X \setminus \pi^{-1}(Z) \\ \text{with } \deg(\pi(Q)) \in [e_0, \frac{d_1}{\max\{M,p\}}] \end{array} \right\},$$

$$\mathcal{Q}^{high}_{\mathbf{n}} := \bigcup_{\mathbf{d}} \left\{ \mathbf{f} \in H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E)) : \begin{array}{c} H_{\mathbf{f}} \text{ singular at a point } Q \in X \setminus \pi^{-1}(Z) \\ \text{with } \deg(\pi(Q)) \in (\frac{d_1}{\max\{M,p\}}, \infty) \end{array} \right\}.$$

As e_0 will be taken to infinity, we will always choose e_0 larger than $\deg(\pi^{-1}(Z))$, which

$d_k^{\dim X} q^{-d_1} \to 0$.

ensures that points in the medium and high degree fibers of π lie in X_{sm}.

The goal of our argument will be to show that singularities in medium and high degree fibers are so unlikely that the asymptotics of interest in Theorem 2.3.1 are dominated by $\mathcal{P}^{low}_{e_0,\mathbf{n},T}$. To that end, we have the following three lemmas, whose proofs will come in Section 2.5. Recall that all calculations below should be interpreted as limiting densities as $d_1, \ldots, d_k \to \infty$.

Lemma 2.3.2. *For any $e_0 > 0$, $\mathbf{n}$, and for $\mathbf{f}$ chosen uniformly at random from $H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E))$, we have*

$$\mathrm{P}(\mathbf{f} \in \mathcal{P}^{low}_{e_0,\mathbf{n},T}) = \mathrm{P}(\mathbf{f} \in T) \prod_{\substack{P \in B \setminus Z \\ \deg(P) < e_0}} \mathrm{P}(H_{\mathbf{f}} \textit{ is smooth at all points of } \pi^{-1}(P)).$$

Lemma 2.3.3. *For $n \geq n_0$ and for $\mathbf{f}$ chosen uniformly at random from $H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E))$, we have*

$$\lim_{e_0 \to \infty} \mathrm{P}(\mathbf{f} \in \mathcal{Q}^{med}_{e_0,\mathbf{n}}) = 0.$$

Lemma 2.3.4. *For any n and for $\mathbf{f}$ chosen uniformly at random from $H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E))$, we have*

$$\mathrm{P}(\mathbf{f} \in \mathcal{Q}^{high}_{\mathbf{n}}) = 0.$$

Assuming the lemmas above, as well as the convergence properties exhibited in

Section 2.6, we may prove two of our main theorems.

Proof of Theorem 2.3.1. Note that $\mathbf{f}$ lying in $\mathcal{P}^{low}_{e_0,\mathbf{n},T}$ is a strictly weaker assumption than $H_{\mathbf{f}} \cap X \setminus \pi^{-1}(Z)$ being smooth and having $\mathbf{f}|_{\pi^{-1}}(Z) \in T$. We may bound the difference between the two computations by accounting for $\mathcal{Q}^{med}_{e_0,\mathbf{n}}$ and $\mathcal{Q}^{high}_{\mathbf{n}}$, giving

$$\begin{aligned} \mathrm{P}(\mathbf{f} \in \mathcal{P}^{low}_{e_0,\mathbf{n},T}) &\geq \mathrm{P}(H_{\mathbf{f}} \cap (X \setminus \pi^{-1}(Z)) \text{ is smooth and } \mathbf{f}|_{\pi^{-1}(Z)} \in T) \\ &\geq \mathrm{P}(\mathbf{f} \in \mathcal{P}^{low}_{e_0,\mathbf{n},T}) - \mathrm{P}(\mathbf{f} \in \mathcal{Q}^{med}_{e_0,\mathbf{n}}) - \mathrm{P}(\mathbf{f} \in \mathcal{Q}^{high}_{\mathbf{n}}). \end{aligned}$$

Taking e_0 to infinity, Lemmas 2.3.2, 2.3.3, and 2.3.4 guarantee the equality of interest so long as $n \geq n_0$. Once the equality has been established, it is not hard to show that the infinite product converges and is zero if and only if one of its factors is zero. This is the content of Section 2.6. $\square$

From Theorem 2.3.1 we may deduce Theorem 2.1.1, by setting $Z = \emptyset$.

Proof of Theorem 2.1.1. If $\dim(\pi(X)) = 0$, then $\pi(X)$ consists of finitely many closed points, and hence Lemma 2.3.2 gives the desired result. If $\dim \pi(X) > 0$ we apply Theorem 2.3.1 with $Z = \emptyset$, giving

$$\mathrm{P}(H_{\mathbf{f}} \text{ is smooth}) = \prod_{P \in B} \mathrm{P}(H_{\mathbf{f}} \text{ is smooth at all points of } X_P),$$

with the convergence being implied by Theorem 2.3.1. $\square$

We will see in Section 2.7 that we can generalize Theorem 2.3.1 further still by requiring that the complete intersection $H_{\mathbf{f}}$ intersect smoothly with a given closed subscheme $Y \subset X$, so long as Y satisfies a weak Altman Kleiman type condition. Since the proof technique is only slightly different in this setting, we will exhibit the proof in the simpler context first.

2.4 Preparatory Lemmas

In this section we will present several lemmas needed to compute the asymptotic densities of $\mathcal{P}^{low}_{e_0,\mathbf{n},T}$, $\mathcal{Q}^{med}_{e_0,\mathbf{n}}$, and $\mathcal{Q}^{high}_{\mathbf{n}}$. We first recall Lemma 5.1 of [EW15]; it provides a uniform bound for the degree of a fiber X_P in terms of the degree of $P \in B$, and will be used frequently in our applications of Bezout's Lemma and the Lang-Weil bound.

Lemma 2.4.1 ([EW15, Lemma 5.1]]). *There exists an integer t (depending only on X, A, and E) such that for any closed point $P \in B$, $t \cdot \deg(P)$ is at least the sum of the A degrees of the irreducible components of X_P.* $\square$

Our next result is an analog of Lemma 5.2 in [EW15]. It allows us to control behavior of the complete intersection at finitely many fibers; in particular, it will allow us to control fibers X_P for $P \in B$ of bounded degree.

Lemma 2.4.2. *Suppose $b > 0$, and thus $M > 0$. Let $W \subset \mathbb{P}^M$ be a zero dimensional* *subscheme. Then as* $d_1, \ldots, d_k \to \infty$, *The image of*

$$H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E)) \to H^0(X_W, \mathcal{O}_{X_W}(\mathbf{n}A))$$

stabilizes to the image of

$$H^0(W, \pi_*(\mathcal{O}_X(\mathbf{n}A)) \otimes \mathcal{O}_W) \to H^0(X_W, \mathcal{O}_{X_W}(\mathbf{n}A)).$$

If the connected components of W *are* $Q_1, \ldots, Q_s$, *then the stable image above equals the stable image of*

$$\bigoplus_{j=1}^{s} \mathrm{Im}\left(H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E)) \to H^0(X_{Q_j}, \mathcal{O}_{X_{Q_j}}(\mathbf{n}A))\right).$$

Proof. We have a commutative diagram

$$\begin{tikzcd}
H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E)) \arrow[rr, "g"'] \arrow[d, "(1)"] & & H^0(X_W, \mathcal{O}_{X_W}(\mathbf{n}A)) \\
H^0(B, \pi_*(\mathcal{O}_X(\mathbf{n}A + \mathbf{d}E))) \arrow[d, "(2)"] & & H^0(W, \pi_*(\mathcal{O}_{X_W}(\mathbf{n}A))) \arrow[u, "(4)"'] \\
H^0(B, \bigoplus_{i=1}^k \pi_*(\mathcal{O}_X(nA)) \otimes \mathcal{O}_B(d_i)) \arrow[dr, "h"'] & & H^0(W, \pi_*(\mathcal{O}_X(\mathbf{n}A)) \otimes \mathcal{O}_W) \arrow[u, "\alpha"'] \\
& H^0(W, \bigoplus_{i=1}^k \pi_*(\mathcal{O}_X(nA)) \otimes \mathcal{O}_W(d_i)) \arrow[ur, "(3)"'] &
\end{tikzcd}$$

We first note that (1), (2), (3), and (4) are isomorphisms. Indeed (1) and (4) are isomorphisms since pushforward behaves nicely with global sections, (2) is an isomorphism by the projection formula, and (3) is an isomorphism as W is zero dimensional.

Next, note that the map h is surjective so long as the d_i are all sufficiently large. Indeed h factors as a product of maps

$$H^0(B, \pi_*(\mathcal{O}_X(nA)) \otimes \mathcal{O}_B(d_i)) \to H^0(W, \pi_*(\mathcal{O}_X(nA)) \otimes \mathcal{O}_W(d_i)),$$

which are surjective for d_i sufficiently large as in [EW15, Lemma 5.2]. It suffices, for instance, to take d_i larger than the Castelnuovo-Mumford regularity of $\pi_*(\mathcal{O}_X(nA)) \otimes \mathcal{O}_W$ on $\mathbb{P}^M$. As a result of surjectivity of h and commutativity of the diagram, the image of g must equal the image of α as desired.

For the second claim, the map

$$H^0(W, \pi_*(\mathcal{O}_X(\mathbf{n}A)) \otimes \mathcal{O}_W) \to H^0(X_W, \mathcal{O}_{X_W}(\mathbf{n}A))$$

is nothing more than the product map corresponding to the maps

$$H^0(Q_i, \pi_*(\mathcal{O}_X(\mathbf{n}A)) \otimes \mathcal{O}_{Q_i}) \to H^0(X_{Q_i}, \mathcal{O}_{X_{Q_i}}(\mathbf{n}A)).$$

Thus by the first claim, the second claim follows. $\square$

Next we will move to controlling points in medium degree fibers. Following [EW15], we will split the points of medium degree fibers into low relative degree and high relative degree. The low relative degree points are easy to control with the following surjectivity result of [EW15] and the subsequent Lang-Weil bound.

Lemma 2.4.3 ([EW15, Lemma 5.3])**.** *Let $W \subset \mathbb{A}^N \times \mathbb{A}^M$ be a closed subscheme and let π_1, π_2 denote the projection from $\mathbb{A}^N \times \mathbb{A}^M$ onto its first and second factors, respectively. Suppose π_1 is an isomorphism on W and that $\pi_2(W)$ is supported at a closed point P of degree e. Let $w := \dim_{\mathbb{F}_q} H^0(W, \mathcal{O}_W)$ (allowing the possibility of $w = \infty$ if W is not zero dimensional). Let $r := \deg(\pi_2(W))$, and consider the restriction map*

$$\phi_{W,n,d} : H^0(\mathbb{P}^N \times \mathbb{P}^M, \mathcal{O}(n,d)) \to H^0(W, \mathcal{O}_W).$$

Then for $n, d \geq 0$,

- $\# \operatorname{Im}(\phi_{W,n,d}) \geq q^{\min(d+1,r)}$
- *for $d + 1 \geq r$, we have $\# \operatorname{Im}(\phi_{W,n,d}) \geq q^{\min(en+r,w)}$.*

Lemma 2.4.4 ([LW54, Lemma 1])**.** *For all positive integers e,*

$$\#X(\mathbb{F}_{q^e}) \leq 2^m \deg(\overline{X}) q^{me}.$$

Let us stratify B as a disjoint union $B = \bigsqcup_s B_s$, where B_s denotes the locus of points with an $m - s$ dimensional fiber. For a point Q with low degree compared to $\pi(Q)$ we will obtain a surjection of sections from Lemma 2.4.3; this will allow us to compute the probability of singularity at Q directly. The paucity of such points Q (as implied by the Lang-Weil bound) will then allow us to conclude that they do not matter to the asymptotics.

Lemma 2.4.5. *Let $P \in B_S$. Suppose P has degree e, and $d_1 \geq e(M+1)$. Then*

$$\frac{\#\{\mathbf{f} \in H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E)) : \mathbf{f} \textit{ is singular at some } Q \in X_P \cap X_{sm} \textit{ with } \frac{\deg(Q)}{e} \leq \frac{n+1}{m+1}\}}{\#H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E))}$$

is $O(keq^{-e(s+1)})$.

Proof. Suppose $Q \in X_{sm}$ has degree ef with $\pi(Q) = P$ and with $f \leq \frac{(n+1)}{m+1}$. By Lemma 2.4.3 we obtain, for all i, surjections

$$H^0(\mathbb{P}^N \times \mathbb{P}^M, \mathcal{O}(n, d_i)) \to H^0(Q^{(2)}, \mathcal{O}_{Q^{(2)}}).$$

In particular we have a surjection

$$\bigoplus_{i=1}^{k} H^0(\mathbb{P}^N \times \mathbb{P}^M, \mathcal{O}(n, d_i)) \to \bigoplus_{i=1}^{k} H^0(Q^{(2)}, \mathcal{O}_{Q^{(2)}})$$

which factors thru $H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E))$. Since the map factors we may apply the rank nullity theorem to show that the probability of singularity weakly increases when we work with $\bigoplus_{i=1}^{k} H^0(\mathbb{P}^N \times \mathbb{P}^M, \mathcal{O}(n, d_i))$ instead of $H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E))$, hence we will do so with no further comment.

Now $H_{\mathbf{f}}$ will be singular at Q if and only if the f_i simultaneously vanish at Q and the gradients of the f_i are linearly dependent. As Q is a degree ef point, the probability that $H_{\mathbf{f}}$ is singular at Q is

$$(q^{-ef})^k \left(1 - L(q^{ef}, k, m)\right).$$

Now we want to add these probabilities over all points Q. By the Lang-Weil bound and Lemma 2.4.1, we have

$$\#X_P(\mathbb{F}_{q^{ef}}) = O(eq^{ef(m-s)}).$$

Utilizing the bound

$$\begin{aligned} L(q^{ef}, m, k) &= \prod_{j=0}^{k-1}(1 - q^{-ef(m-j)}) \\ &\geq 1 - \sum_{j=0}^{k-1} q^{-ef(m-j)} \\ &\geq 1 - kq^{-ef(m-k+1)} \end{aligned}$$

one obtains

$$\begin{aligned} \sum_{f=1}^{\left\lfloor \frac{n+1}{m+1} \right\rfloor} \#X_p(\mathbb{F}_{q^{ef}})q^{-kef}(1 - L(q^{ef}, k, m)) &= \sum_{f=1}^{\left\lfloor \frac{n+1}{m+1} \right\rfloor} \mathcal{O}(eq^{ef(m-s)})q^{-kef}kq^{-ef(m-k+1)} \\ &\preccurlyeq \sum_{f=1}^{\left\lfloor \frac{n+1}{m+1} \right\rfloor} keq^{-ef(s+1)} \\ &= O(keq^{-e(s+1)}). \end{aligned}$$

$\square$

The final result we will need is Lemma 5.4 of [EW15], which we will apply iteratively to control high relative degree points in medium degree fibers as well as to points in high degree fibers. We restate the lemma below, with the slight modification that we have not absorbed the implicit dependence on the degree of X and the degree of a fiber X_P into the constant of the big O notation. We do this because we will be applying the following lemma iteratively on the intersections $H_{f_1} \cap \cdots \cap H_{f_i}$, and thus we will need to see how $\deg(X \cap H_{f_1} \cap \cdots \cap H_{f_i})$ and $\deg(X_P \cap H_{f_1} \cap \cdots \cap H_{f_i})$ depend on the integers n and $d_1, \ldots, d_k$.

Lemma 2.4.6 ([EW15, Lemma 5.4])**.** *Let j and J be integers. Fix $n \geq 1$ and $d \geq 0$, and let $f \in H^0(\mathbb{P}^N \times \mathbb{P}^M, \mathcal{O}(n,d))$ be chosen uniformly at random.*

1. *The probability that H_f has a singularity at a closed point $Q \in X_{sm}$ with $\deg(\pi(Q)) \in [j, \infty)$ is at most*

$$O(\deg(\overline{X})(n^m + d^m)q^{-\min(\lfloor d/p \rfloor + 1, j)}).$$

2. *The probability that H_f has a singularity at a closed point $Q \in X_{sm}$ with $\deg(\pi(Q)) \in [j, \lfloor d/p \rfloor + 1]$ and $\deg(Q)/\deg(\pi(Q)) \geq J$ is at most*

$$O(\deg(\overline{X})(n^m + d^m)q^{-(\lfloor d/p \rfloor + 1)} + \sum_{\substack{P \in B \\ j \leq \deg(P) \leq \lfloor d/p \rfloor + 1}} \deg(\overline{X_P}) n^m q^{-e \min(J, \lfloor (n-1)/p \rfloor + 1)}).$$

3. *Let P be a closed point of B with $\deg(P) = e$. For $\lfloor d/p \rfloor + 1 > e$, the probability that*

H_f has a singularity at a closed point $Q \in X_{sm}$ with $\pi(Q) = P$ and $\deg(Q)/e \geq J$ *is at most*

$$O(\deg(\overline{X})(n^m + d^m)q^{-(\lfloor d/p \rfloor + 1)} + \deg(\overline{X_P})e(n^m + e^m)q^{-e\min(J, \lfloor (n-1)/p \rfloor + 1)}).$$

With this in hand we will present the last prerequisite lemma. The proof technique is analogous to that used in [BK12] to adapt Poonen's high degree argument for hypersurfaces to the complete intersection setting.

Lemma 2.4.7. *Let j and J be integers. Fix integers n and $d_1 \leq \cdots \leq d_k$, and for $1 \leq i \leq k$ choose f_i uniformly at random from $H^0(\mathbb{P}^N \times \mathbb{P}^M, \mathcal{O}(n, d_i))$. Then*

1. *The probability that $H_{\mathbf{f}}$ has a singularity at a closed point $Q \in X_{sm}$ with* $\deg(\pi(Q)) \in [j, \infty)$ *is at most*
$$O\left((n^m + d_k^m)q^{-\min(\lfloor d_1/p \rfloor + 1, j)}\right).$$

2. *The probability that $H_{\mathbf{f}}$ has a singularity at a closed point $Q \in X_{sm}$ with* $\deg(\pi(Q)) \in [j, \lfloor d_1/p \rfloor + 1]$ *and* $\deg(Q)/\deg(\pi(Q)) \geq J$ *is at most*
$$O\left((n^m + d_k^m)q^{-\min(\lfloor d_1/p \rfloor + 1, j)} + \sum_{e=j}^{\lfloor d_1/p \rfloor + 1} n^m q^{e(b - \min(J, \lfloor (n-1)/p \rfloor + 1))}\right).$$

3. *Let P be a closed point of B with* $\deg(P) = e$. *For $\lfloor d_1/p \rfloor + 1 > e$, the probability that $H_{\mathbf{f}}$ has a singularity at a closed point $Q \in X_{sm}$ with $\pi(Q) = P$ and* $\deg(Q)/e \geq J$

is at most

$$O\left((n^m + d_k^m)q^{-(\lfloor d_1/p \rfloor + 1)} + n^m q^{-e\min(J, \lfloor (n-1)/p \rfloor + 1)}\right).$$

Proof. Following the style of [EW15] we will prove all claims simultaneously. To do so, we say a closed point $Q \in X_{sm}$ is *admissible* in the first case if $\deg(\pi(Q)) \in [j, \infty)$, in the second case if $\deg(\pi(Q)) \in [j, \lfloor d_1/p \rfloor + 1]$ and $\deg(Q)/\deg(\pi(Q)) \geq J$, and in the third case if $\pi(Q) = P$ and $\deg(Q)/e \geq J$.

Suppose that for some i, $f_1, \ldots, f_{i-1}$ have been chosen so that $X_{sm} \cap H_{f_1} \cap \ldots H_{f_{i-1}}$ is smooth of dimension $m-i+1$ at every admissible point of $X_{sm} \cap H_{f_1} \cap \ldots H_{f_{i-1}}$. Let Y be the open subscheme obtained by removing the locus of points at which $X_{sm} \cap H_{f_1} \cap \ldots H_{f_{i-1}}$ is not smooth of dimension $m-i+1$. By considering $\mathbb{P}^N \times \mathbb{P}^M$ as a subscheme of $\mathbb{P}^{MN+M+N}$ under the Segre embedding, we see that $\deg(\overline{Y}) = \deg(\overline{X})(n+d_1)\ldots(n+d_{i-1})$. Crucially, the same reasoning does not apply directly in going from a fiber X_P to a fiber Y_P. Suppose $P \in B$ has degree e; then over the residue field $\kappa(P)$, the polynomials $f_1, \ldots, f_{i-1}$ are defined by equations of degree $O(n+e)$. The degree of X_P over $\kappa(P)$ is bounded above by t as in Lemma 2.4.1, and hence we conclude $\deg(\overline{Y_P}) = O(\deg(\overline{X_P})(n+e)^{i-1})$.

We now apply Lemma 2.4.5 by considering $X_{sm} \cap H_{f_1} \cap \ldots H_{f_i}$ as a hypersurface inside $X_{sm} \cap H_{f_1} \cap \ldots H_{f_{i-1}}$. In particular, $X_{sm} \cap H_{f_1} \cap \ldots H_{f_i}$ has a singularity at an admissible point with probability

$$O((n+d_1)\ldots(n+d_{i-1})(n^{m-i+1} + d_i^{m-i+1})q^{-\min(\lfloor d_i/p \rfloor + 1, j)})$$

in the first case, with probability

$$O((n+d_1)\dots(n+d_{i-1})(n^{m-i+1}+d_i^{m-i+1})q^{-(\lfloor d_i/p\rfloor+1)}) \\ +O\left(\sum_{\substack{P\in B\\ j\le\deg(P)\le\lfloor d/p\rfloor+1}} n^m q^{-e\min(J,\lfloor(n-1)/p\rfloor+1)})\right)$$

in the second case, and with probability

$$O\left((n^{m-i+1}+d_i^{m-i+1})q^{-(\lfloor d_1/p\rfloor+1)}+n^{m-i+1}q^{-e\min(J,\lfloor(n-1)/p\rfloor+1)}\right)$$

in the third case. Using the Lang-Weil bound reduces the second summand in case 2 to

$$\sum_{\substack{P\in B\\ j\le\deg(P)\le\lfloor d/p\rfloor+1}} n^m q^{-e\min(J,\lfloor(n-1)/p\rfloor+1)}=O\left(\sum_{e=j}^{\lfloor d_i/p\rfloor+1} n^m q^{e(b-\min(J,\lfloor(n-1)/p\rfloor+1))}\right),$$

from which we obtain the desired bounds of the proposition in all cases by summing over i and applying the obvious bounds. □

2.5 The Sieving Argument

We are now ready to compute the asymptotic densities of $\mathcal{P}^{low}_{e_0,\mathbf{n},T}$, $\mathcal{Q}^{med}_{e_0,\mathbf{n}}$, and $\mathcal{Q}^{high}_{\mathbf{n}}$.

Proof of Lemma 2.3.2. Applying Lemma 2.4.1 to the disjoint union

$$Z \bigsqcup \left(\bigsqcup_{P \in B \setminus Z, \deg(P) < e_0} P^{(2)} \right)$$

and using the resulting factorization of the image given in part (b) gives the desired result. $\square$

The medium degree fibers have been partitioned into points of low and high relative degree. To prove Lemma 2.3.3, we will bound the probability of singularity for these two cases independently. Lemma 2.4.2 allows us to deal with the low relative degree case, and Lemma 2.4.6 gives the high relative degree case.

Proof of Lemma 2.3.3. Consider the set of complete intersections

$$M_{low} := \bigcup_{\substack{P \in B \\ e_0 \le \deg(P) \le \frac{d_1+1}{M+1}}} \left\{ \mathbf{f} \in H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E)) : \begin{array}{c} H_{\mathbf{f}} \text{ singular at some} \\ Q \in \pi^{-1}(P) \text{ , } \frac{\deg(Q)}{\deg(P)} \le \left\lfloor \frac{n+1}{m+1} \right\rfloor \end{array} \right\}.$$

Lemma 2.4.4 gives us that

$$\begin{aligned} \frac{\# M_{low}}{\# H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E))} &= \sum_{s=0}^{b} \sum_{e=e_0}^{\frac{d_1+1}{M+1}} \# B_s(\mathbb{F}_{q^e}) O(keq^{-e(s+1)}) \\ &= \sum_{s=0}^{b} \sum_{e=e_0}^{\frac{d_1+1}{M+1}} O(q^{se}) O(keq^{-e(s+1)}) \\ &\le \sum_{e \ge e_0} O(keq^{-e}). \end{aligned}$$

The final sum converges and hence as $e_0 \to \infty$ we obtain that

$$\frac{\# M_{low}}{\# H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E))} \to 0.$$

It remains to control the high relative degree case. Define

$$M_{high} := \bigcup_{\substack{P \in B \\ e_0 \leq \deg(P) \leq \frac{d_1}{p}+1}} \left\{ \mathbf{f} \in H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E) : \begin{array}{c} H_{\mathbf{f}} \text{ singular at some} \\ Q \in \pi^{-1}(P), \ \frac{\deg(Q)}{\deg(P)} > \left\lfloor \frac{n+1}{m+1} \right\rfloor \end{array} \right\}.$$

We apply Lemma 2.4.6 with $j = e_0$ and $J = \left\lfloor \frac{n+1}{m+1} \right\rfloor + 1$ to obtain that $\frac{\# M_{high}}{\# H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E))}$ is bounded above by

$$O\left((n^m + d_k^m) q^{-\min(\lfloor d_1/p \rfloor + 1, j)} + \sum_{e=e_0}^{\lfloor d_1/p \rfloor + 1} n^m q^{e(b - \min(\lfloor (n+1)/(m+1) \rfloor, \lfloor (n-1)/p \rfloor + 1))} \right).$$

The term $(n^m + d_k^m) q^{-\min(\lfloor d_1/p \rfloor + 1, j)}$ converges to zero as $d_1 \to \infty$, so it suffices to deal with the second term. For this part, so long as

$$n \geq \max\{b(m+1)+1, bp+1\}$$

we will have $b - \min(\left\lfloor \frac{n+1}{m+1} \right\rfloor, \lfloor \frac{n-1}{p} \rfloor + 1) < 0$ for all i, and thus the sum

$$\sum_{e \geq e_0} n^m q^{e(b - \min(\lfloor (n+1)/(m+1) \rfloor, \lfloor (n-1)/p \rfloor + 1)}$$

converges.

In particular as we take the d_i to infinity, we obtain

$$\lim_{e_0 \to \infty} \lim_{d_1 \to \infty} \sum_{e \geq e_0} n^m q^{e(b - \min(\lfloor (n+1)/(m+1) \rfloor, \lfloor (n-1)/p \rfloor + 1)} = 0,$$

giving the desired result. $\square$

Finally we deal with the asymptotics of points in high degree fibers.

Proof of Lemma 2.3.4. By Lemma 2.4.7 with $j = \frac{d_1}{\max(M+1,p)}$ we have

$$\frac{\#\mathcal{Q}_{\mathbf{n}}^{high}}{\#H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E))} = O((n^m + d_k^m) q^{-\min(\lfloor d_1/p \rfloor + 1, d_1/(\max(M+1,p))}).$$

So long as d_k is subexponential in d_1 (as we are assuming), this goes to zero as $d_1 \to \infty$. The claim follows. $\square$

2.6 Convergence of the Product

Proposition 2.6.1. *Maintaining the notation of Theorem 2.3.1, for $n \geq n_0$, the product*

$$\prod_{P \in (B \setminus Z)} \mathrm{P}(H_{\mathbf{f}} \textit{ is smooth at all points of } \pi^{-1}(P))$$

is zero if and only if one of its factors is zeros.

Proof. It is known that a product $\prod(1 - a_i)$ converges in the sense of our proposition if and only if the sum $\sum a_i$ converges, so it suffices to show

$$\sum_{P \in B \setminus Z} \mathrm{P}(H_{\mathbf{f}} \text{ has a singularity at a point of } \pi^{-1}(P))$$

converges. Recall that this probability is a limit as $d_1 \to \infty$. Restricting our attention to the fiber of a point P of degree e, Lemma 2.4.5 tells us that the probability of $H_{\mathbf{f}}$ having a singularity at a point $Q \in \pi^{-1}(P)$ with $\frac{\deg(Q)}{e} \leq \left\lfloor \frac{n+1}{m+1} \right\rfloor$ is $O(keq^{-e(s+1)})$. Alternatively, if $Q \in X_P$ has $\frac{\deg(Q)}{e} \geq \left\lfloor \frac{n+1}{m+1} \right\rfloor + 1$, Lemma 2.4.7 tells us that the probability of $H_{\mathbf{f}}$ being singular at Q is

$$O((n^m + d_k^m)q^{-(\lfloor d_1/p \rfloor + 1)} + e(n^m + e^m)q^{-e \min(\lfloor (n+1)/(m+1) \rfloor, \lfloor (n-1)/p \rfloor + 1)}).$$

Using the Lang-Weil bound, we obtain that the sum

$$\sum_{P \in B_s \setminus Z} \mathrm{P}(H_{\mathbf{f}} \text{ has a singularity in } \pi^{-1}(P))$$

is

$$\sum_{e=1}^{\infty} O(q^{es} keq^{-e(s+1)} + q^{es} e(n^m + e^m) q^{-e \min(\lfloor (n+1)/(m+1) \rfloor, \lfloor (n-1)/p \rfloor + 1)}).$$

This converges for $n \geq n_0$. Summing over the stratification $B = \bigsqcup B_s$ gives the desired result. □

Our remaining task is to show that the probability of smoothness on a given fiber is nonzero for n sufficiently large.

Proposition 2.6.2. *Given X,A, and E, the product*

$$\prod_{P\in(B\setminus Z)} \mathrm{P}(H_{\mathbf{f}} \textit{ is smooth at all points of } \pi^{-1}(P))$$

is nonzero for n sufficiently large.

Proof. We now turn our attention to showing that the product in Theorem 2.3.1 is nonzero. Fixing $P \in (B \setminus Z)$ with degree e, our goal is to show

$$\mathrm{P}(H_{\mathbf{f}} \text{ is singular at a point in } \pi^{-1}(P)) > 0.$$

To do this, we perform another sieve over the closed points of $\pi^{-1}(P)$ depending on a parameter $r > 0$. Define a point Q of X_P to be low degree if $\deg(Q) \leq r$, medium degree if $\deg(Q) \in (r, e\frac{n+1}{m+1}]$, and high degree if $\deg(Q) > e\frac{n+1}{m+1}$. While we are considering the preimage of a point of fixed degree e, note that in the arguments that follow the bounds we obtain will be made independent of e which will suffice to show the product is nonzero.

Set

$$\delta = \prod_{Q\in X_P} (1 - q^{-k\deg(Q)} + q^{-k\deg(Q)} L(q^{\deg(Q)}, k, m)),$$

with $0 < \delta < 1$. The choice of this δ comes from the main result of [BK12].

If $Q \in \pi^{-1}(P)$ has high degree, then setting $J = \frac{n+1}{m+1} + 1$ in Lemma 2.4.7, Part 3 gives

$$\begin{aligned}&\mathrm{P}(X_P \text{ is singular at a high degree point })\\ &\quad = O\left((n^m + d_k^m)q^{-(\lfloor d_1/p\rfloor+1)} + n^m q^{-e\min(J,\lfloor (n-1)/p\rfloor+1)}\right),\end{aligned}$$

which reduces to

$$O\left(n^m q^{-e\min(J,\lfloor (n-1)/p\rfloor+1)}\right)$$

after we pass to the limit in d_1. Choose n large enough so that

$$\frac{(m+1)\log(e) - \frac{\delta}{2}}{e\ln(q)} < \min(\left\lfloor \tfrac{n+1}{m+1}\right\rfloor + 1, \tfrac{n-1}{p} + 1).$$

This allows one to bound the probability of having a high degree singularity by $\frac{\delta}{2}$.

For the medium degree case, suppose $Q \in \pi^{-1}(P)$ has degree ef. Then Lemma 2.4.3 applied to $Q^{(2)}$ gives that the probability of Q being a singularity is given by $q^{-efk(m+1)}L(q^{ef}, k, m)$. Summing over such points and applying the Lang-Weil bound gives the probability of having any medium degree singularity at all to be

$$O\left(\sum_{f=r/e+1}^{\left\lfloor \frac{n+1}{m+1}\right\rfloor} q^{efm}q^{-efk}(1 - L(q^{ef}, k, m))\right)$$

and again applying $L(q^{ef}, m, k) \geq 1 - kq^{-ef(m-k+1)}$ gives that this is

$$O\left(\sum_{f=r/e+1}^{\left\lfloor\frac{n+1}{m+1}\right\rfloor} kq^{-ef}\right)$$

We may choose r large enough to ensure that this sum is also bounded above by $\delta/2$.

Finally, let

$$W = \bigsqcup_{\substack{Q \in \pi^{-1}(P) \\ \deg(Q) \leq r}} .$$

By [Poo04, Lemma 2.1] we may choose n large enough so that

$$H^0(\mathbb{P}^N \times \mathbb{P}^M, \mathcal{O}(n, 0)) \to H^0(W^{(2)}, \mathcal{O}_{W^{(2)})})$$

is surjective, which implies surjectivity of

$$H^0(X, \mathcal{O}_X(nA)) \to H^0(W^{(2)}, \mathcal{O}_{W^{(2)})})$$

and hence of

$$H^0(X, \mathcal{O}_X(\mathbf{n}A + \mathbf{d}E)) \to \bigoplus_{i=1}^{k} H^0(W^{(2)}, \mathcal{O}_{W^{(2)})}).$$

This shows that the probability of smoothness at low degree points factors as a product

$$\prod_{\substack{Q \in \pi^{-1}(P) \\ \deg(Q) \leq r}} (1 - q^{-k\deg(Q)} + q^{-k\deg(Q)} L(q^{\deg(Q)}, k, m)) < \delta,$$

so that the probability of singularity at a low degree point is less than $1-\delta$. Adding the behaviour at low, medium, and high degree points gives

$$\mathrm{P}(H_{\mathbf{f}} \text{ has a singularity in } \pi^{-1}(P)) < \delta,$$

so that the complementary probability is nonzero, independent of P. $\square$

2.7 Smooth Intersection with a Subscheme

We now provide a modified version of Theorem 2.3.1 which allows the requirement that a complete intersection intersects transversely with a specified closed subscheme of $Y \subset X$, so long as the singular locus of Y has small enough dimension. The condition on $\dim(Y^{sing})$ is similar to the Altman Kleiman condition used in work of Gunther [Gun17] and Wutz [Wut16],[Wut17] but is a weaker assumption; this comes from the fact that we are not requiring our complete intersection to contain Y entirely.

Theorem 2.7.1. *Maintaining the notation of Theorem 2.3.1, assume additionally that $Y \subset X$ is a closed subscheme with $\dim(Y) = y$ and $\dim(Y_{sing}) \leq k-1$. Then*

$$\begin{aligned}\mathrm{P}(H_{\mathbf{f}} \cap (X \backslash \pi^{-1}(Z)) \textit{ and } H_{\mathbf{f}} \cap X \setminus \pi^{-1}(Z) \cap Y \textit{ are smooth, and } \mathbf{f}|_{\pi^{-1}(Z)} \in T) \\ = \mathrm{P}(\mathbf{f}|_{\pi^{-1}(Z)} \in T) \prod_{P \in (B \backslash Z)} \mathrm{P}(H_{\mathbf{f}} \textit{ is smooth at all points of } X_P \textit{ and } X_P \cap Y).\end{aligned}$$

The product over $P \in B \setminus Z$ converges, is zero if and only if one of the factors is zero, *and is always nonzero so long as the n is sufficiently large.*

Proof. The proof uses the same sieve technique that is used for Theorem 2.3.1, which passes through without difficulty since the requirement $\dim(Y_{sing}) \leq k-1$ allows the complete intersection to avoid the singular locus of Y with positive probability. Lemma 2.4.2 readily adapts to this setting, so it suffices to control the probability of singularities in medium and high degree fibers. We briefly outline the arguments below.

For a given integer e_0, suppose $P \in B$ has degree $e_0 \leq \deg(P) \leq d_1/(\max(M,p))$. If $Q \in X_P$ is of degree ef with $f \leq \frac{e(n+1)}{m+1}$, then by Lemma 2.4.3 we have a surjection

$$\bigoplus_{i=1}^{k} H^0(\mathbb{P}^N \times \mathbb{P}^M, \mathcal{O}(n, d_i)) \to \bigoplus_{i=1}^{k} H^0(Q^{(2)}, \mathcal{O}_{Q^{(2)}}).$$

This leads to the calculation

$$\mathrm{P}(H_{\mathbf{f}} \text{ or } H_{\mathbf{f}} \cap Y \text{ is singular at } Q) = \begin{cases} (q^{-ef})^k \left(1 - L(q^{ef}, m, k)\right), Q \notin Y \\ (q^{-ef})^k \left(1 - L(q^{ef}, y, k)\right), Q \in Y_{sm} \\ (q^{-ef})^k, Q \in Y_{sing}. \end{cases}$$

Summing over the medium degree points $P \in B$ and applying the Lang-Weil bound shows that the probability of $H_{\mathbf{f}}$ or $H_{\mathbf{f}} \cap Z$ being singular in the medium degree, low relative degree case is asymptotically zero.

For the remaining two cases, let us partition X as a disjoint union $X = Y_{sing} \cup Y_{sm} \cup X \setminus Y$. We have seen that $H_{\mathbf{f}}$ has a singularity at a closed point Q with $\deg(Q) \in$

$[e_0, \lfloor d_1/p \rfloor + 1]$ and $\deg(Q)/\deg(\pi(Q)) \geq \left\lfloor \frac{n+1}{m+1} \right\rfloor$ with probability

$$O\left((n^m + d_k^m) q^{-\min(\lfloor d_1/p \rfloor + 1, e_0)} + \sum_{e=e_0}^{\lfloor d_1/p \rfloor + 1} n^m q^{e(b - \min(\lfloor (n+1)/(m+1) \rfloor, \lfloor (n-1)/p \rfloor + 1))} \right).$$

A simple adaptation of Lemma 2.4.7 shows that $H_{\mathbf{f}} \cap Y$ has a singularity at a closed point Q with $\deg(Q) \in [e_0, \lfloor d_1/p \rfloor + 1]$ and $\deg(Q)/\deg(\pi(Q)) \geq \frac{n+1}{m+1}$ with probability

$$O\left((n^y + d_k^y) q^{-\min(\lfloor d_1/p \rfloor + 1, e_0)} + \sum_{e=e_0}^{\lfloor d_1/p \rfloor + 1} n^y q^{e(b - \min(\lfloor (n+1)/(m+1) \rfloor, \lfloor (n-1)/p \rfloor + 1))} \right).$$

Finally, Lemma 2.4.3 asserts $H_{\mathbf{f}}$ intersects the singular locus of Y at such a point with probability bounded above by

$$\sum_{e=e_0}^{\lfloor d_1/p \rfloor + 1} O(q^{(k-1)e} q^{-ek}) = \sum_{e=e_0}^{\lfloor d_1/p \rfloor + 1} O(q^{-e}).$$

All three of the above calculations vanish as e_0 grows so long as $n > n_0$.

In the high degree case we again utilize Lemma 2.4.3 and an adaptation of Lemma 2.4.7 to obtain

$$\begin{aligned} &\mathrm{P}(H_{\mathbf{f}} \text{ or } H_{\mathbf{f}} \cap Y \text{ is singular at a point } Q, \deg(\pi(Q)) > \lfloor d_1/p \rfloor + 1) = \\ &\quad O\left((n^m + d_k^m) q^{-\min(\lfloor d_1/p \rfloor + 1, j)}) + (n^y + d_k^y) q^{-\min(\lfloor d_1/p \rfloor + 1, j)} + \sum_{e=\lfloor d_1/p \rfloor + 1}^{\infty} q^{-e} \right). \end{aligned}$$

This is also asymptotically vanishing, establishing the desired product formula for our probability calculation. That the product converges and is nonzero follows via the same arguments as in Section 2.6. □

Chapter 3

On F-Splittings

3.1 Introduction

In this chapter we will discuss an application of Poonen's sieve technique to Frobenius singularities. To start, we will recall the necessary definitions and motivation.

In classical complex algebraic geometry a wealth of notions exist to describe "mildly singular" varieties. Such categories include *rational* singularities, *canonical* and *log-canonical* singularities, or *terminal* and *log-terminal* singularities. Such categorizations prove crucial in the birational geometry of singular varieties and the minimal model program. For more information, one may examine [KMM87].

Subsequently, work has been done to translate notions of "mild singularities" to characteristic p. Let p be a prime and X a variety of finite type over $\mathbb{F}_p$. Associated to

X we have the Frobenius morphism,

$$F : X \to X,$$

which acts as the identity on the closed points of X but which acts as the p-power map on $\mathcal{O}_X$. It turns out that the behaviour of this morphism provides useful insight into singularities of X. For example, a fundamental singularity type can be characterized by injectivity of F:

Proposition 3.1.1. *The scheme X is reduced if and only if the p-power map $\mathcal{O}_X \to \mathcal{O}_X$ is injective.*

Proof. The statement is local, so it suffices to show that a ring R of characteristic p has no nilpotents if and only if the map $r \to r^p$ is injective. The kernel of the Frobenius morphism is clearly contained in the nilradical of R, so it suffices to show that if F is injective then there are no nontrivial nilpotents in R. Suppose $r^n = 0$ for some $r \in R$ and $n \geq 1$. Choose e so that $p^e > n$. Then $(r^{p^{e-1}})^p = r^{p^e} = (r^n)^{p^e - n} = 0$. Injectivity of Frobenius implies $r^{p^{e-1}} = 0$. But this is an iterate of the Frobenius morphism applied to r, so injectivity also implies $r = 0$ as desired. $\square$

A more subtle result comes from Kunz [Kun69], which restates regularity in terms of F.

Theorem 3.1.2 ([Kun69] Theorem 2.1)**.** *X is smooth if and only if it is reduced and $F_*\mathcal{O}_X$ is a flat $\mathcal{O}_X$ module..*

Since this result there has been much work in studying X through the behaviour of F; for a modern survey we recommend [TW14]. The most relevant notion for us will be that of F-split singularities.

Given X and F as above, we will say X is globally F-split if the map

$$F : \mathcal{O}_X \to F_*\mathcal{O}_X$$

splits as a map of $\mathcal{O}_X$ modules. We will say X is locally F-split if

$$F : \mathcal{O}_{X,x} \to F_*\mathcal{O}_{X,x}$$

is split for all closed points x in X. In the commutative algebra literature, one may also see this referred to as F-purity.

Locally F-split varieties form a characteristic p-analog of complex varieties with log canonical singularities; they include the class of smooth varieties, as well as "mildly singular" varieties. For example, a consequence of Fedder's criterion, which we will introduce below, is that the union of the coordinate hyperplanes in an affine space,

$$\operatorname{Spec}\Big(k[x_1, \ldots, x_n]/(x_1 \cdot \cdots \cdot x_n)\Big),$$

is locally F-split. Global F-splitting is of course a stronger condition; for example, all abelian varieties are smooth, but only the *ordinary* abelian varieties are globally F-split

[MS87, Lemma 1.1]. In the case of affine varieties however, these notions are the same:

Theorem 3.1.3 ([SZ15, Lemma 1.8])**.** *Let X be an affine variety over k. Then X is globally F-split if and only if it is locally F-split.*

For the remainder of this chapter we will exclusively use the term F-split to refer to *locally* F-split varieties. Given that F-split varieties form a p-adic analog of log canonical singularities, we may expect the two categories of singularities to behave similarly under hyperplane restrictions. In [Kol97] Kollár presents several Bertini Theorems regarding log canonical singularities; it is thus natural to look for similar results for F-split varieties or more generally for other F-singularities.

To date there have been three advances in this area, all under the assumption that the base field is infinite. Schwede and Zhang established the first Bertini results for F-splitting in [SZ13] (among other things). Carvajal-Rojas, Schwede, and Tucker later provided Bertini results for the notion of *Hilbert-Kunz multiplicity* in [CRST21], and these results were later extended by Datta and Simpson in [DS22]. All three of these efforts rely on an axiomatic framework for proving Bertini theorems provided by Cumini, Greco, and Manaresi [CGM86]; the reliance on this tool restricts them to the setting of algebraically closed fields[1] by the same reasoning as in Section 1.2.

Our goal in the remainder of this chapter is to explore how Poonen's sieve technique may be utilized to examine statistics of F-singularities over finite fields. While we cannot recover strong structural results, such as an analog of the work of Schwede and

[1] "algebraically closed" can be replaced by "infinite" thanks to work of Spreafico's generalization of the Cumini-Greco-Manaresi technique [Spr98].

Zhang, we will end up providing an interesting result on the arithmetic statistics of F-split hypersurfaces in Grassmannians over $\mathbb{F}_{2^l}$. We hope that this may provide a starting point for others to explore Bertini theorems for F-singularities over finite fields.

3.2 Preliminary Results

The remainder of this chapter will be dedicated to proving the following result.

Theorem 3.2.1. *Let q be a power of 2 and let X be a smooth projective variety of dimension m over $\mathbb{F}_q$. Then*

$$\lim_{d\to\infty} \frac{\#\{f \in H^0(X, \mathcal{O}_X(d)) : H_f \text{ is locally } F\text{-split}\}}{\#H^0(X, \mathcal{O}_X(d))} = \zeta_X(2^m)^{-1}.$$

The formula in Theorem 3.2.1 is a product of local probabilities taken over the closed point of X. As with Poonen's original Bertini Theorem for smoothness, the local F-splitting probabilities are independent across closed points.

The proof to Theorem 3.2.1 will come in Section 3.3, after we set up several preliminary results in this section. The results in this section do not depend on being in characteristic 2, and we hope that they may be used to generalize Theorem 3.2.1 in future work.

After proving Theorem 3.2.1, we will analyze the resulting formula for hypersurfaces of Grassmannian varieties (in particular, hypersurfaces in projective space). In a suitable sense, we will show that locally F-split hypersurfaces in Grassmannians are very common, despite the fact that local F-splittings only ensure “mild” singularities. These calculations will be carried out hand in hand with smoothness calculations, using Poonen’s work; our main result is to show that for fixed k and growing n, 100% of high degree hypersurfaces of $\mathrm{Gr}(k, n)$ are locally F-split, whereas strictly less than 100% are smooth. Precise definitions and statements will come in Section 3.4.

Our first result of this section shows that the probability of a hypersurface being locally F-split is always well approximated by being locally F-split at low degree points.

Lemma 3.2.2. *Let X be a smooth projective variety over a finite field. The limit superior*

$$\limsup_{d\to\infty} \frac{\#\{f \in H^0(X, \mathcal{O}_X(d)) : H_f \cap X \text{ is locally } F\text{-split}\}}{H^0(X, \mathcal{O}_X(d))}$$

equals the following:

$$\limsup_{r\to\infty} \limsup_{d\to\infty} \frac{\#\{f \in H^0(X, \mathcal{O}_X(d)) : H_f \cap X \text{ is locally } F\text{-split at all points in } X^\circ_{\leq r}\}}{\#H^0(X, \mathcal{O}_X(d))}.$$

The same is true with “superior” replaced by “inferior;” the limit

$$\liminf_{d\to\infty} \frac{\#\{f \in H^0(X, \mathcal{O}_X(d)) : H_f \cap X \text{ is locally } F\text{-split}\}}{\#H^0(X, \mathcal{O}_X(d))}$$

equals the double limit

$$\liminf_{r\to\infty} \liminf_{d\to\infty} \frac{\#\{f \in H^0(X, \mathcal{O}_X(d)) : H_f \cap X \text{ is locally } F\text{-split at all points in } X^\circ_{\leq r}\}}{\#H^0(X, \mathcal{O}_X(d))}.$$

Proof. For any fixed r and d we may consider the following sets of global sections on X:

$$\mathrm{Nonsplit}(r,d) := \{f \in H^0(X, \mathcal{O}_X(d)) : H_f \cap X \text{ is not locally } F\text{-split at a point in } X^\circ_{>r}\}$$

$$\mathrm{Split}(r,d) := \{f \in H^0(X, \mathcal{O}_X(d)) : H_f \cap X \text{ is locally } F\text{-split at all points in } X^\circ_{\leq r}\}$$

$$\mathrm{Split}(\infty,d) := \{f \in H^0(X, \mathcal{O}_X(d)) : H_f \cap X \text{ is locally } F\text{-split}\}$$

$$\mathrm{Singular}(r,d) := \{f \in H^0(X, \mathcal{O}_X(d)) : H_f \cap X \text{ is not smooth at a point in } X^\circ_{>r}\}.$$

Note the following inclusions:

$$\mathrm{Split}(\infty,d) \subset \mathrm{Split}(r,d) \subset \mathrm{Split}(\infty,d) \cup \mathrm{Nonsplit}(r,d).$$

From this we obtain the following inequality:

$$0 \leq \frac{\#\mathrm{Split}(r,d)}{\#H^0(X, \mathcal{O}_X(d))} - \frac{\#\mathrm{Split}(\infty,d)}{\#H^0(X, \mathcal{O}_X(d))} \leq \frac{\#\mathrm{Nonsplit}(r,d)}{\#H^0(X, \mathcal{O}_X(d))}.$$

Now we invoke the fact that

$$\mathrm{Nonsplit}(r,d) \subset \mathrm{Singular}(r,d).$$

Indeed by Kunz's theorem, being smooth at $P \in X^\circ$ implies $F_*\mathcal{O}_X$ is a vector bundle near P, and hence locally split at P. This gives

$$0 \leq \frac{\#\mathrm{Split}(r,d)}{\#H^0(X,\mathcal{O}_X(d))} - \frac{\#\mathrm{Split}(\infty,d)}{\#H^0(X,\mathcal{O}_X(d))} \leq \frac{\#\mathrm{Singular}(r,d)}{\#H^0(X,\mathcal{O}_X(d))}.$$

Appealing to Lemmas 2.4 and 2.6 of [Poo04], we see that $\#\mathrm{Singular}(r,d)$ is asymptotically unimportant. Taking double limit superiors across the inequality, first with respect to d and then with respect to r, we obtain the following:

$$\limsup_{r\to\infty}\limsup_{d\to\infty}\frac{\#\mathrm{Split}(r,d)}{\#\mathcal{O}_X(d)} = \limsup_{r\to\infty}\limsup_{d\to\infty}\frac{\#\mathrm{Split}(\infty,d)}{\#\mathcal{O}_X(d)}.$$

The right hand side is now independent of r, so we may simplify to

$$\limsup_{r\to\infty}\limsup_{d\to\infty}\frac{\#\mathrm{Split}(r,d)}{\#\mathcal{O}_X(d)} = \limsup_{d\to\infty}\frac{\#\mathrm{Split}(\infty,d)}{\#\mathcal{O}_X(d)}.$$

The calculation for limit inferiors is identical. $\square$

Lemma 3.2.2 sets the stage for a sieving argument, as in Poonen's work. To obtain a meaningful result we now need a local calculation. For this, we turn to Fedder's criterion for local F-splitting [Fed83]. We start by recalling several notions from commutative algebra.

Let R be a ring of characteristic p. Let I be an ideal of R. Define $I^{[p]}$ to be the ideal generated by the pth powers of I,

$$I^{[p]} := (a^p : a \in I).$$

The *ideal quotient* $(I^{[p]} : I)$ is given by

$$(I^{[p]} : I) := \{r \in R : rI \subset I^{[p]}\}.$$

With this, we can state the following key result:

Theorem 3.2.3 ([Fed83, Theorem 1.12])**.** *Let $(S, \mathfrak{m})$ be a regular local ring of characteristic p and let $R = S/I$. Then R is F-split if and only if*

$$(I^{[p]} : I) \not\subset \mathfrak{m}^{[p]}.$$

Fedder's criterion is particularly useful in the case of hypersurfaces and complete intersections, since for principal ideals $I = (f)$ we have $(I^{[p]} : I) = (f^{p-1})$.

Corollary 3.2.4 ([Fed83], Proposition 2.1)**.** *Let $(S, \mathfrak{m})$ be a regular local ring of characteristic p. Let $f_1, \ldots, f_k$ be a regular sequence in S. Let $f := f_1 \cdot \cdots \cdot f_k$ and $I = (f_1, \ldots, f_k)$. Then S/I is F-pure if and only if $f^{p-1} \notin \mathfrak{m}^{[p]}$.*

Proof. This follows from Theorem 3.2.3 and from the computation

$$(I^{[p]} : I) = (f^{p-1}) + (f_1^p, \ldots, f_k^p).$$

□

The following examples display the power of Fedder's criterion:

Example 3.2.5. *Let κ be a finite field of characteristic p. All of the following affine varieties are singular at the origin; some are F-split by Fedder's criterion:*

- *The normal crossings singularity,* $\mathrm{Spec}(\kappa[x_1, \ldots, x_n]/(x_1 \cdots x_n))$*, is singular but F-split.*

- *Generalizing the previous example, if I is a monomial ideal generated by squarefree monomials, then the Stanley-Reisner ring $\kappa[x_1, \ldots, x_n]/I$ is F-split. This is because $x_1 \cdots x_n$ is a multiple of all squarefree monomials; hence $(x_1 \cdots x_n)^{p-1}$ will be in $(I^{[p]} : I)$ but not in $(x_1, \ldots, x_n)^{[p]}$.*

- *The variety* $\mathrm{Spec}(\kappa[x, y, z]/(x^3 + y^3 + z^3))$ *is F-split at the origin if $p \equiv 1 \mod 3$ and is not F-split if $p \equiv 2 \mod 3$.*

In general Fedder's criterion is much easier to verify than finding an explicit splitting of Frobenius; this will be the key tool in proving our Theorem 3.2.1. However, even with this simpler criterion, it can still be intractable to deal with in the general case. Hence at this point we will restrict ourself to $p = 2$.

3.3 Computations in Characteristic 2

We now specialize to the case of finite fields of characteristic 2. This case makes testing the condition of Lemma 3.2.4 particularly tractable:

Observation 3.3.1. *Let $(S, \mathfrak{m})$ be a regular local ring of characteristic 2. Let $f_1, \ldots, f_k$ be a regular sequence in S. Let $f := f_1 \cdot \cdots \cdot f_k$ and $I = (f_1, \ldots, f_k)$. Then S/I is F-pure if and only if $f \notin \mathfrak{m}^{[2]}$.*

As with Poonen's Bertini Theorem, our first result will be a form of Hermite Interpolation for finitely many closed points. For a point P in a variety X cut out by $\mathfrak{m}_P$, let $P^{[2]}$ be the subscheme cut out by $\mathfrak{m}_P^{[2]}$. For C a finite collection of closed points, let $C^{[2]}$ be the disjoint union

$$C^{[2]} := \bigcup_{P \in C} P^{[2]}.$$

Lemma 3.3.2. *Let q be a power of 2. Take a smooth projective variety $X \hookrightarrow \mathbb{P}^n_{\mathbb{F}_q}$ of dimension m. Let C be a finite collection of closed points of X. There exists an integer $d(C)$ for which the restriction map*

$$H^0(X, \mathcal{O}_X(d)) \to H^0(C^{[2]}, \mathcal{O}_{C^{[2]}})$$

is surjective for all $d > d(C)$.

Proof. Let $\mathcal{I}_{C^{[2]}}$ be the ideal sheaf cutting out $C^{[2]}$. Starting with the short exact sequence

$$0 \to \mathcal{I}_{C^{[2]}} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_{C^{[2]}} \to 0,$$

we see that the cokernel of the restriction map

$$H^0(X, \mathcal{O}_X(d)) \to H^0(C^{[2]}, \mathcal{O}_{C^{[2]}})$$

lies in

$$H^1(X, \mathcal{I}_{C^{[2]}}(d)).$$

The claim follows from Serre Vanishing. □

Our next two lemmas will be a dimension count and an application of the Chinese Remainder Theorem.

Lemma 3.3.3. *For q and X as above and for P a closed point of X, we have*

$$\dim_{\mathbb{F}_q} H^0(P^{[2]}, \mathcal{O}_{P^{[2]}}) = 2^m \deg(P).$$

Proof. The question is local, so we may assume that $X = \operatorname{Spec}(R)$ for R an $\mathbb{F}_q$ algebra. Choose a regular system of parameters $x_1, \ldots, x_m$ generating the maximal ideal $\mathfrak{m}_P$ of P.

Let $\overline{x_i}$ be the image of x_i under the quotient map

$$R_{\mathfrak{m}_P} \to R_{\mathfrak{m}_P}/(x_1^2, \dots, x_m^2).$$

For a subset $S \subset \{1, 2, \dots, m\}$, let $\overline{x}_S := \prod_{i \in S} \overline{x_i}$. Choose a total ordering

$$\{1, 2, \dots, m\} = S_1 < S_2 < \cdots < S_{2^m} = \emptyset$$

of the subsets of $\{1, 2, \dots, m\}$ which extends the partial order

$$S \leq T \text{ if } |S| \geq |T|.$$

As a module over the local ring $R_{\mathfrak{m}_P}$, the quotient $R_{\mathfrak{m}_P}/(x_1^2, \dots, x_m^2)$ has length 2^m, as is witnessed by the chain of submodules

$$R_{\mathfrak{m}_P} \cdot (S_1) \subset R_{\mathfrak{m}_P} \cdot (S_1, S_2) \subset \cdots \subset R_{\mathfrak{m}_P} \cdot (S_1, S_2 \dots, S_{2^m}).$$

Multiplying by the size of the residue field gives the desired formula,

$$\begin{aligned} \dim_{\mathbb{F}_q} H^0(P^{[2]}, \mathcal{O}_{P^{[2]}}) &= 2^m [\mathbb{F}_q(P) : \mathbb{F}_q] \\ &= 2^m \deg(P). \end{aligned}$$

$\square$

Lemma 3.3.4. *Let C be a collection of closed points of X, with $C^{[2]}$ as defined above. The restriction maps*

$$S_{n,d} \to H^0(C^{[2]}, \mathcal{O}_{C^{[2]}})$$

factor as a product map

$$S_{n,d} \to \prod_{P \in C} H^0(P^{[2]}, \mathcal{O}_{P^{[2]}}).$$

Proof. Zero dimensional subschemes of projective varieties are affine, so we may assume we have $X = \operatorname{Spec}(R)$. Let P and Q be closed points of X with maximal ideals $\mathfrak{m}_P$, $\mathfrak{m}_Q$. The ideals $\mathfrak{m}_P$, $\mathfrak{m}_Q$ are coprime, and hence so are $\mathfrak{m}_P^{[2]}$ and $\mathfrak{m}_Q^{[2]}$. The result follows from the Chinese Remainder Theorem. □

We can now compute the probability with which a hypersurface is F-split at a finite collection of points.

Corollary 3.3.5. *Fix an integer r, and take X as above. Let C be a finite collection of closed points in X. Then*

$$\lim_{d \to \infty} \frac{\#\{f \in H^0(X, \mathcal{O}_X(d)) : H_f \textit{ is } F\textit{-split at all points in } C\}}{\#H^0(X, \mathcal{O}_X(d))} = \prod_{P \in C} (1 - q^{-2^m \deg(P)}).$$

Proof. Consider the restriction map

$$H^0(X, \mathcal{O}_X(d)) \to H^0(C^{[2]}, \mathcal{O}_C).$$

By Lemma 3.3.2, this map is surjective for d sufficiently large. By Lemma 3.3.4, this map factors as

$$H^0(X, \mathcal{O}_X(d)) \to \prod_{P \in C} H^0(P^{[2]}, \mathcal{O}_{P^{[2]}}).$$

Fedder's criterion at $p = 2$ ensures us that H_f is locally F-split at C as long as the image of f under the projections

$$H^0(X, \mathcal{O}_X(d)) \to \prod_{P \in C} H^0(P^{[2]}, \mathcal{O}_{P^{[2]}}) \to H^0(P^{[2]}, \mathcal{O}_{P^{[2]}})$$

are nonzero for all $P \in C$. Thus the set of good hypersurface sections is the preimage of

$$\prod_{P \in C} \left(H^0(P^{[2]}, \mathcal{O}_{P^{[2]}}) \setminus \{0\} \right).$$

The fact that the restriction is a surjective linear map, combined with the dimension count of Lemma 3.3.3, gives the desired result. $\square$

We can now end this section with a proof of Theorem 3.2.1.

Proof of Theorem 3.2.1. Applying Corollary 3.3.5 to the set $X^{\circ}_{\leq r}$ of points of bounded degree allows us to simplify

$$\lim_{r \to \infty} \lim_{d \to \infty} \frac{\#\{f \in H^0(X, \mathcal{O}_X(d)) : H_f \cap X \text{ is locally } F\text{-split at all points in } X^{\circ}_{\leq r}\}}{\# H^0(X, \mathcal{O}_X(d))}$$

to the limit

$$\lim_{r\to\infty}\prod_{P\in X^{\circ}_{\leq r}}(1-q^{-2^m\deg(P)}).$$

Since $2^m > m$, the limit above converges to $\zeta_X(2^m)^{-1}$. Applying Lemma 3.2.2 gives the desired result. $\square$

3.4 Consequences for Grassmannians

Recall that the *Grassmannian* variety $\mathrm{Gr}_{\mathbb{F}_q}(k,n)$ over $\mathbb{F}_q$ is the projective variety of dimension $k(n-k)$ whose κ rational points parametrize k dimensional subspaces of κ^r. As an homage to the last paragraph of [Wei49], we will apply Theorem 3.2.1 to Grassmannian varieties and obtain the following theorem.

Theorem 3.4.1. *Let k be a fixed integer. Let q be a power of 2. Then*

$$\lim_{n\to\infty}\lim_{d\to\infty}\frac{\#\{f\in H^0(\mathrm{Gr}_{\mathbb{F}_q}(k,n),\mathcal{O}(d)) : H_f \textit{ is locally F-split}\}}{\#H^0(\mathrm{Gr}_{\mathbb{F}_q}(k,n),\mathcal{O}(d))}=1.$$

The same is true if dimension is replaced with codimension, i.e. $\mathrm{Gr}_{\mathbb{F}_q}(k,n)$ *is replaced by* $\mathrm{Gr}_{\mathbb{F}_q}(n-k,n)$.

To prove Theorem 3.4.1 we first need to recall several elementary facts regarding Grassmannians. We start with a simple point count:

Lemma 3.4.2. *For κ an extension of $\mathbb{F}_q$ of size q^e, we have*

$$\#\operatorname{Gr}_{\mathbb{F}_q}(k,n)(\kappa) = \frac{(q^{ne}-1)(q^{(n-1)e}-1)\dots(q^{(n-k+1)e}-1)}{(q^e-1)(q^{2e}-1)\dots(q^{ke}-1)}.$$

Proof. To count the k dimensional subspaces of κ^n it suffices to count sets of linearly independent vectors and then account for overcounting. Let $LI_\kappa(k,n)$ be the set

$$LI_\kappa(k,n) := \{S = \{v_1,\dots,v_k\} \subset \kappa^n : v_1,\dots,v_k \text{ are linearly independent }\}.$$

A set in $LI_\kappa(k,n)$ may be formed iteratively by first selecting a nonzero vector. At each step, one then selects a vector from κ^n which is not in the κ-span of the previously chosen vectors. This yields

$$\#LI_\kappa(k,n) = (q^{ne}-1)(q^{ne}-q^e)\dots(q^{ne}-q^{(k-1)e}).$$

But now for any fixed subspace $V \subset \kappa^n$ of dimension k, the exact same calculation shows that there are $(q^{ke}-1)(q^{ke}-q^e)\dots(q^{ke}-q^{(k-1)e})$ sets in $\#LI_\kappa(k,n)$ which span V. This yields the desired calculation. □

The counts appearing in Lemma 3.4.2 are the *Gaussian binomial coefficients*; for any integers n and k,

$$\binom{n}{k}_q = \frac{(q^n-1)(q^{n-1}-1)\dots(q^{n-k+1}-1)}{(q-1)(q^2-1)\dots(q^k-1)}$$

is a polynomial in q of degree $k(n-k)$. These polynomials arise as Poincaré polynomials of real Grassmannians, and admit beautiful combinatorial interpretations; see, for instance, [Sag20, Section 3.2]. More importantly, the fact that this polynomial is independent of q allows us to compute the zeta function of Grassmannians with ease:

Lemma 3.4.3. *For a fixed k, n let us expand the Gaussian binomial coefficient*

$$\binom{n}{k}_q = \sum_{i=0}^{k(n-k)} b_i q^i.$$

Then the zeta function of $\mathrm{Gr}_{\mathbb{F}_q}(k,n)$ equals

$$\zeta_{\mathrm{Gr}_{\mathbb{F}_q}(k,n)}(s) = \frac{1}{(1-q^{-s})^{b_0}(1-q^{1-s})^{b_1}\dots(1-q^{k(n-k)-s})^{b_{k(n-k)}}}.$$

As a sanity check, recall that $\mathbb{P}^n_{\mathbb{F}_q} = \mathrm{Gr}_{\mathbb{F}_q}(1, n+1)$. In this case,

$$\binom{n+1}{1}_q = \frac{q^{n+1}-1}{q-1} = 1 + q + \dots + q^n.$$

Lemma 3.4.3 is thus nothing more than a generalization of the fact that

$$\zeta_{\mathbb{P}^n_{\mathbb{F}_q}}(s) = \frac{1}{(1-q^{-s})(1-q^{1-s})\dots(1-q^{n-s})}.$$

Proof. By Lemma 3.4.2, we have

$$\begin{aligned}\sum_{r=1}^{\infty} \# \operatorname{Gr}_{\mathbb{F}_q}(k,n)(\mathbb{F}_{q^r}) \frac{q^{-rs}}{r} &= \sum_{r=1}^{\infty} \binom{n}{k}_{q^r} \frac{q^{-rs}}{r} \\ &= \sum_{r=1}^{\infty} \left(\sum_{i=0}^{k(n-k)} b_i q^{ri} \right) \frac{q^{-rs}}{r}.\end{aligned}$$

Exchanging the order of summation gives

$$\begin{aligned}\sum_{r=1}^{\infty} \# \operatorname{Gr}_{\mathbb{F}_q}(k,n)(\mathbb{F}_{q^r}) \frac{q^{-rs}}{r} &= \sum_{i=0}^{k(n-k)} b_i \sum_{r=1}^{\infty} \frac{q^{r(i-s)}}{r} \\ &= \sum_{i=0}^{k(n-k)} -b_i \log(1-q^{i-s}) \\ &= \log\left(\frac{1}{(1-q^{-s})^{b_0}(1-q^{1-s})^{b_1} \dots (1-q^{d(n-d)-s})^{b_{d(n-d)}}} \right).\end{aligned}$$

Exponentiating gives the desired result. □

We need one final result before proving Theorem 3.4.1, which calculates the evaluation of Gaussian binomial coefficients at 1.

Lemma 3.4.4. *For a fixed n and k, let*

$$\binom{n}{k}_q = b_0 + b_1 q + \cdots + b_{k(n-k)} q^{k(n-k)}$$

as before. Then the evaluation at 1 of this polynomial returns the usual *binomial coeffi-*

cient,

$$b_0 + b_1 + \cdots + b_{k(n-k)} = \binom{n}{k}.$$

Proof. The standard proof is to show that Gaussian binomial coefficients obey a q-twisted binomial recurrence which collapses to the standard binomial recurrence at $q = 1$:

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q.$$

See [Sag20, Section 3.2] for more on this. □

With this in hand, we can now return to Theorem 3.4.1 to provide a proof.

Proof of Theorem 3.4.1. From Theorem 3.2.1, it suffices to study the asymptotics of $\zeta_{\mathrm{Gr}_{\mathbb{F}_q}(k,n)}(2^{k(n-k)})^{-1}$ for k fixed and $n \to \infty$. By Lemma 3.4.3, we have

$$\zeta_{\mathrm{Gr}_{\mathbb{F}_q}(k,n)}(2^{k(n-k)})^{-1} = \prod_{i=0}^{k(n-k)} (1 - q^{i-2^{k(n-k)}})^{b_i}.$$

Replacing each term in the product with the minimal factor, we get

$$\begin{aligned} \zeta_{\mathrm{Gr}_{\mathbb{F}_q}(k,n)}(2^{k(n-k)})^{-1} &\geq \prod_{i=0}^{k(n-k)} (1 - q^{k(n-k)-2^{k(n-k)}})^{b_i} \\ &= (1 - q^{k(n-k)-2^{k(n-k)}})^{b_0+b_1+\cdots+b_{k(n-k)}}. \end{aligned}$$

We finish by combining Lemma 3.4.3 with two crude bounds:

$$\begin{aligned}
(1-q^{k(n-k)-2^{k(n-k)}})^{b_0+b_1+\cdots+b_{k(n-k)}} &\geq 1-(b_0+b_1+\cdots+b_{k(n-k)})q^{k(n-k)-2^{k(n-k)}} \\
&= 1-\binom{n}{k}q^{k(n-k)-2^{k(n-k)}} \\
&\geq 1-2^n q^{k(n-k)-2^{k(n-k)}} \\
&\geq 1-q^{n+k(n-k)-2^{k(n-k)}}.
\end{aligned}$$

In total, we obtain

$$1 \geq \lim_{n\to\infty} \zeta_{\mathrm{Gr}_{\mathbb{F}_q}(k,n)}(2^{k(n-k)})^{-1} \geq \lim_{n\to\infty} 1-q^{n+k(n-k)-2^{k(n-k)}},$$

completing the calculation. The calculation for $\mathrm{Gr}_{\mathbb{F}_q}(n-k,n)$ is identical. □

Note that the proportion of these hypersurfaces which are actually *smooth* is always strictly less than 1:

Observation 3.4.5. *Let k be a fixed integer. Then*

$$\lim_{n\to\infty}\lim_{d\to\infty} \frac{\#\{f \in H^0(\mathrm{Gr}_{\mathbb{F}_q}(k,n),\mathcal{O}(d)) : H_f \textit{ is smooth}\}}{\#H^0(\mathrm{Gr}_{\mathbb{F}_q}(k,n),\mathcal{O}(d))} < 1-q^{-1}.$$

Proof. This is a direct application of Poonen's smoothness computation [Poo04, Theorem

1.1]. The product

$$\zeta_{\mathrm{Gr}_{\mathbb{F}_q}(k,n)}(k(n-k)+1)^{-1} = \prod_{i=0}^{k(n-k)} (1 - q^{i-k(n-k)-1})^{b_i}$$

is smaller than the factor appearing at $i = k(n-k)$, and this factor is simply $1 - q^{-1}$, independent of k and n (it is not hard to show that the Gaussian binomial coefficients are monic, i.e. $b_{k(n-k)} = 1$). $\square$

The key difference between local F-splitting and smoothness in this computation is that the point at which the zeta function is evaluated grows exponentially with the dimension for F-splitting, and grows linearly with the dimension for smoothness.

Chapter 4

Cut by Curves Criteria and Overconvergent F-Isocrystals

4.1 Introduction

To end this thesis we will discuss several structural results that may be obtained in part using Bertini theorems over finite fields. This will be done using a *cut by curves* technique which, loosely speaking, studies a space X by studying the collection of one dimensional objects lying in X. The material in this chapter is based upon joint work with Kiran Kedlaya and James Upton [GKU22].

Throughout this chapter, let X be a smooth, geometrically irreducible scheme over a finite field k of characteristic p. The arithmetic and geometry of X are elucidated through the use of *Weil cohomology theories*; for each prime $\ell \neq p$ we have ℓ-adic étale cohomology, and at p we have rigid cohomology. As with standard topological Betti cohomology, these theories are further enhanced through the use of coefficient objects on X. In étale cohomology we have lisse Weil sheaves and in rigid cohomology we have two flavours of *Frobenius isocrystals*: the convergent category and the overconvergent category, which is a fully faithful subcategory of the former.

The overconvergent isocrystals are those of "geometric origin," being connected to the ℓ-adic Weil sheaves through a largely conjectural theory of motives. From this perspective, the overconvergent category is desirable to work in. In practice the category of convergent isocrystals cannot be forgotten; in this larger category one obtains the crucial computational tool of slope filtrations. As a result it is important to understand the relationship between convergent and overconvergent isocrystals. The remainder of this chapter will outline a test for measuring when a convergent isocrystal on X is in fact overconvergent, by studying the restriction of that coefficient object to the curves lying on X.

We will start in the next section by briefly defining the categories of interest. Subsequently we will recall the theory of *companions* in the context of Weil cohomology. Finally we will state a cut by curves criterion for overconvergence. After providing a heuristic proof using companions, we will fill in the details to provide a full proof.

$$\begin{tikzcd}
P \arrow[r, "\sigma"] & P \arrow[r] & \mathrm{Spf}(W(k)) \\
X \arrow[u, hook, "\iota"] \arrow[r, "F"'] & X \arrow[u, hook, "\iota"'] \arrow[r] & \mathrm{Spec}(k) \arrow[u]
\end{tikzcd}$$

Figure 4.1: A lift of X to $W(k)$.

4.2 F-Isocrystals

First we construct the category of convergent and overconvergent F-isocrystals for affine schemes over k. Let $F : X \to X$ denote the absolute Frobenius map on X, and let $W(k)$ denote the p-typical Witt vectors over k. A *lift* of X over $W(k)$ will be the data of

- a formal scheme P over $W(k)$ with special fiber of $W(k)$ is X and
- a map $\sigma : P \to P$ which lifts F.

From a theorem of Elkik, such lifts are known to exist (and are "unique up to homotopy" in a suitable sense) [Elk73]. When no confusion will occur, we will often speak solely of P as the lift of X without referring to σ explicitly.

Let K be the fraction field of $W(k)$. Given a lift P of X, let P_K denote its Raynaud generic fiber. A convergent F-isocrystal on X is a vector bundle $\mathcal{E}$ over P_K equipped with

- an integrable connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega$ and
- a horizontal isomorphism $\sigma^*\mathcal{E} \to \mathcal{E}$.

We will refer to the category of such objects as F-Isoc(X).

In the setting wherein X is not affine, one uses a more functorial construction of F-Isoc to patch together F-Isoc(X) locally; we refer to [LS07] for full details.

To construct the category of *overconvergent* isocrystals on X, one selects the isocrystals in F-Isoc(X) which extend in a small neighborhood of X. Formally, we choose an immersion $X \hookrightarrow Y$ of X into a proper k-scheme. We lift the pair (X, Y) to formal schemes (P, Q) over $W(k)$ with generic fibers P_K and Q_K.

$$\begin{tikzcd} P \arrow[r] & Q \\ X \arrow[u, hook, "\iota"] \arrow[r, hook] & Y \arrow[u, hook, "\iota'"'] \end{tikzcd}$$

Figure 4.2: Lifting (X, Y) to (P, Q).

This gives an inclusion $P_K \hookrightarrow Q_K$. Inside Q_K we may take a strict neighborhood U of P_K. An overconvergent F-iscrystal on X relative to U is a vector bundle $\mathcal{E}$ on U, again equipped with integrable connection and a flat isomorphism with its Frobenius pullback.

One shows that the choice of U and Y make no impact on the resulting category, and hence one obtains the category F-Isoc$^\dagger(X)$ of overconvergent F-isocrystals on X; again, we refer to [LS07] for details.

Note that in the construction of the overconvergent category, a vector bundle on U may always be restricted to P_K. This leads to the following observations:

- There is a restriction functor F-Isoc$^\dagger(X) \to$ F-Isoc(X) which forgets overconvergence.
- When X is itself proper, F-Isoc(X) and F-Isoc$^\dagger(X)$ are the same category.

In general the restriction functor F-Isoc$^\dagger(X) \to$ F-Isoc(X) is not an equivalence of cateogories, although it is known to be fully faithful [Ked04]. The following example illustrates F-Isoc(X), F-Isoc$^\dagger(X)$, and the relationship between them, when $X = \mathbb{A}^n_k$.

For this example, suppose $k = \mathbb{F}_p$ for p prime. In this case the Witt vector ring $W(k)$ is the ring of p-adic integers $\mathbb{Z}_p$, with fraction field $\mathbb{Q}_p$. We have $X = \mathrm{Spec}(\mathbb{F}_p[x_1, \ldots, x_n])$, which admits the lift $P = \mathrm{Spf}(\mathbb{Z}_p\{x_1, \ldots, x_n\})$, where $\mathbb{Z}_p\{x_1, \ldots, x_n\}$ is the ring of restricted formal power series over $\mathbb{Z}_p$.

The Raynaud generic fiber functor constructs from $\mathrm{Spf}(\mathbb{Z}_p\{x_1, \ldots, x_n\})$ a rigid analytic space over $\mathbb{Q}_p$. The rigid analytic space in this example comes from the *Tate algebra*,

$$T_n(\mathbb{Q}_p) := \left\{ \sum_I c_I x^I \in \mathbb{Q}_p[[x_1, \ldots, x_n]] : c_I \to 0 \text{ as } \max(I) \to \infty \right\}.$$

Explictly, in this example the Raynaud generic fiber is $\mathrm{MaxSpec}(T_n)$. This space plays the role of an n-dimensional rigid analytic unit ball; it is related to, but smaller than, the analytification $\mathbb{A}^{n,an}_{\mathbb{Q}_p}$ of affine space.

From this, we may describe an isocrystal in F-Isoc$(\mathbb{A}^n_k)$ as a vector bundle on a rigid analytic unit ball (with the additional data of a connection and a Frobenius action), whereas an overconvergent isocrystal in F-Isoc$^\dagger(\mathbb{A}^n_k)$ is a vector bundle on a rigid analytic ball of radius $1+\epsilon$ for some $\epsilon > 0$ (again with additional data of a connection and a Frobenius action). It is worth noting that in this setting a rigid analytic Quillen-Suslin Theorem guarantees that all such vector bundles are trivial [Lüt77], and hence the data of interest is the connection and the Frobenius action.

With the categories F-Isoc(X) and F-Isoc$^\dagger(X)$ constructed, we may now formally state the main question of this chapter.

Question 4.2.1. *Let $\mathcal{E}$ be an isocrystal in* F-Isoc(X). *Under what conditions can we conclude that $\mathcal{E}$ is in the essential image of the restriction functor* F-Isoc$^\dagger(X) \to$ F-Isoc(X)*?*

With motivation from work of Shiho [Shi10], [Shi11], as well as from the study of regular connections in characteristic zero [AB01, Theorem 5.7], Kedlaya has proposed the following specialization of Question 4.2.1:

Conjecture 4.2.2 ([Ked22a, Conjecture 5.17])**.** *Let $\mathcal{E} \in$* F-Isoc(X). *Suppose that for every curve $C \hookrightarrow X$, the restriction $\mathcal{E}|_C$ is in the essential image of the restriction functor* F-Isoc$^\dagger(X) \to$ F-Isoc(X). *Can we conclude that $\mathcal{E}$ is also in the essential image of the restriction functor* F-Isoc$^\dagger(X) \to$ F-Isoc(X)*?*

In [GKU22] we provide an affirmative answer to Question 4.2.2 under an additional tameness hypothesis. To discuss the proof, we will first need to recall the theory of *companions.*

4.3 Companions and Skeleton Sheaves

Throughout this section, define a *coefficient object* on X to be either an ℓ-adic lisse Weil sheaf, for $\ell \neq p$, or an overconvergent F-isocrystal. Given a coefficient object

$\mathcal{E}$ and a closed point x of X there is an associated Weil polynomial, given by taking the characteristic polynomial of Frobenius acting on the fiber of $\mathcal{E}$ at x:

$$\mathrm{F}(\mathcal{E}, x) := \det(1 - t\,\mathrm{F}\,|\mathcal{E}_x).$$

We will call two coefficient objects $\mathcal{E}$ and $\mathcal{E}'$ *companions* if these polynomials always agree; that is, $\mathrm{F}(\mathcal{E}, x) = \mathrm{F}(\mathcal{E}', x)$ for all closed points x in X.

In his work on the Weil conjectures, Deligne made a series of conjectures regarding companions of coefficient objects which "come from geometry" [Del80]. We state a modern version of some of these conjectures below; for a full treatment, we recommend [Cad20].

Conjecture 4.3.1 ([Del80])**.** *Let $\mathcal{E}$ be a coefficient object on X. Suppose $\mathcal{E}$ is irreducible and that* $\det(\mathcal{E})$ *has finite order*[1]*. Then:*

1. *Algebraicity: There exists a number field E over $\mathbb{Q}$ that contains* $\mathrm{F}(\mathcal{E}, x)$ *for all closed points x in X.*

2. *Étale - étale companions: if $\mathcal{E}$ is an ℓ-adic lisse Weil sheaf, then $\mathcal{E}$ has an ℓ'-adic companion for all $\ell' \neq p$.*

3. *Étale - crystalline companions: if $\mathcal{E}$ is an ℓ-adic lisse Weil sheaf, then $\mathcal{E}$ has a p-adic overconvergent companion.*

[1] Recall that the determinant of $\mathcal{E}$ is the line bundle given by the top exterior power, $\det(\mathcal{E}) = \bigwedge^{\mathrm{rank}(\mathcal{E})} \mathcal{E}$. As $\det(E)$ is a line bundle we may examine its tensor powers; to say $\mathcal{E}$ has finite determinant is to say that one such tensor power is trivial.

4. *Crystalline - étale companions: if $\mathcal{E}$ is an overconvergent isocrystal, then $\mathcal{E}$ has an ℓ-adic companion for all $\ell \neq p$.*

This conjecture is now known to be true, in large part thanks to the understanding of, and ability to reduce to, the case of curves. The first breakthrough came via Drinfeld's [Dri80] and L. Lafforgue's [Laf02] work on the Langland's correspondence for the general linear group over function fields. This gave a geometric understanding of ℓ-adic coefficient objects on curves via the theory of chtoucas, and hence proved Parts 1, 2, and 3 of Conjecture 4.3.1 in the case where $\mathcal{E}$ is an étale coefficient object and $\dim(X) = 1$.

Part 1 for étale coefficient objects was extended to arbitrary dimension by Deligne in [Del12]. Deligne used a Bertini theorem to cover X with a suitable collection of curves to which previous work applied; a uniformity argument allowed him to recover the statement on X from the restriction of the coefficient object to these curves.

Building on this idea, Drinfeld proved Parts 2 and 3 of Conjecture 4.3.1 for arbitrary X in [Dri12]. Translating the question into one of tame Galois representations and using an arithmetic covering theorem of Wiesand [Wie06],[Wie08] in place of a Bertini theorem, Drinfeld covers X with a suitable collection of curves and is able to recover the global result based on this (given the theme of this thesis, we note in passing that Wiesand's Theorem may be replaced by Poonen's finite field Bertini theorem; see the appendix of [Dri12]).

The relative lag in development of p-adic cohomology theories compared to that of étale cohomology led to the p-adic version of these results to be delayed; but once the necessary foundations were in place the evolution was fairly similar. In [Abe18] Abe was able to reproduce Lafforgue's work p-adically, hence proving Part 1 and Part 4 of Conjecture 4.3.1 in the overconvergent setting. The general case of Part 1 was finished off contemporaneously by Abe and Esnault [AE19] and by Kedlaya [Ked18], both of which followed the spirit of [Del12] although using different methods. Finally, the general case of Part 4 came from Kedlaya in [Ked20], who patched together companions on curves in order to construct p-adic companions globally.

Critical to the development of these results, and to our partial answer to Conjecture 4.2.2, is Drinfeld's work in [Dri12]. Let E be an algebraic extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}$. Let P_r denote the set of polynomials in $\mathcal{O}[t]$ given by

$$P_r(\mathcal{O}) := \{1 + c_1 t + \cdots + c_{r-1}t^{r-1} + c_r t^r : c_r \in \mathcal{O}^\times\}.$$

Letting X° denote the closed points of X, we will say that a (set-theoretic) map

$$\rho : X^\circ \to P_r(\mathcal{O})$$

is *induced by* a lisse E-sheaf $\mathcal{F}$ if, for all $x \in X^\circ$,

$$\rho(x) = \det(1 - t\,\mathrm{F}\,|\mathcal{F}_x).$$

Drinfeld's main result is as follows:

Theorem 4.3.2 ([Dri12]). *Let X be a regular scheme of finite type over $\mathbb{Z}[\ell^{-1}]$. A map $\rho : X^\circ \to P_r(\mathcal{O})$ is induced by a lisse E-sheaf $\mathcal{F}$ if and only if:*

- *for every regular curve C and every map $\phi : C \to X$, the pullback map $\phi^*(\rho)$ is induced by a lisse E-sheaf on C, and,*
- *there exists a dominant étale morphism $X' \to X$ which simultaneously kills the wild ramification of the lisse sheaves on the curves described above.*

Rephrasing this theorem allows the following recipe for constructing coefficient objects on X in the situation we are interested in:

Corollary 4.3.3 ([Dri12]). *Let X be a smooth scheme of finite type over k. Let*

$$\mathcal{F}_{curve} = \{\mathcal{F}_C\}_{\phi:C\to X}$$

be a family of lisse E-sheaves indexed by smooth curves $\phi : C \to X$. Suppose that for all diagrams

$$\begin{array}{ccc} C_1 \times_X C_2 & \longrightarrow & C_2 \\ \downarrow & & \downarrow{\scriptstyle \phi_2} \\ C_1 & \xrightarrow[\phi_1]{} & X \end{array}$$

we have

$$\mathcal{F}_{C_1}|_{C_1\times_X C_2} = \mathcal{F}_{C_2}|_{C_1\times_X C_2}.$$

Suppose further that all Frobenius polynomials have coefficients in a single number field

K. If there is a dominant étale morphism which kills the wild ramification simultaneously on all $\mathcal{F}_C$, then there exists a lisse E-sheaf $\mathcal{F}$ on X such that for all indexing curves $\phi : C \to X$ we have

$$\phi^*(\mathcal{F}) = \mathcal{F}_C.$$

The previous corollary tells us that, to a large extent, the curves on X understand the étale coefficient objects on X. So long as we are careful, the theory of companions will allow us to transfer much of this understanding to the overconvergent context. This forms the basis of our partial proof to Conjecture 4.2.2, which we will explain in the next section.

4.4 Towards a Proof

We now return to the question of when a convergent F-isocrystal is in fact overconvergent. For the remainder of this section, we will always decorate elements of $\text{F-Isoc}^\dagger(X)$ with a dagger: $\mathcal{E}^\dagger$. The image of this object under the restriction functor $\text{F-Isoc}^\dagger(X) \to \text{F-Isoc}(X)$ will be written by removing the dagger. The main question of this chapter can now be restated as: given $\mathcal{E} \in \text{F-Isoc}(X)$, does there exist $\mathcal{F}^\dagger$ in $\text{F-Isoc}^\dagger(X)$ with $\mathcal{F} \cong \mathcal{E}$?

To this end, we present the following partial answer to Conjecture 4.2.2. With the existence of companions, the proof is *almost* a formality.

Theorem 4.4.1. *Let $\mathcal{E} \in \text{F-Isoc}(X)$. Suppose that for every smooth curve $\phi : C \to X$*

there exists an object $\mathcal{F}_C^\dagger$ with $\mathcal{F}_C^\dagger \cong \phi^*(\mathcal{E})$. Suppose further that there exists a dominant morphism $f : X' \to X$ such that for every smooth curve C, $f^*(\mathcal{F}_C^\dagger)$ is tame. Then there exists $\mathcal{F}^\dagger \in \text{F-Isoc}^\dagger(X)$ with $\mathcal{F} \cong \mathcal{E}$.

Heuristic Proof. The fundamental tool in the proof of Theorem 4.4.1 is the following argument. Given $\mathcal{E} \in \text{F-Isoc}(X)$ we obtain a map ρ as in the statement of Theorem 4.3.2 by setting

$$\rho(x) = \det(1 - t\,\mathrm{F} : \mathcal{E}_x).$$

By assumption, on each curve $\phi : C \to X$ we have an overconvergent $\mathcal{F}_C^\dagger$ which restricts to $\phi^*(\mathcal{E})$. On each such curve the overconvergent $\mathcal{F}_C^\dagger$ has an ℓ-adic companion, $\mathcal{G}_C$.

By the definition of a companion the map ρ is induced by these sheaves $\mathcal{G}_C$. Further, by our tameness hypothesis, the wild ramification of $\mathcal{G}_C$ is killed by the dominant morphism $X' \to X$. Hence Theorem 4.3.2 allows us to patch together the $\mathcal{G}_C$ to a global ℓ-adic coefficient object $\mathcal{G}$ on X.

We can now find a p-adic companion $\mathcal{F}^\dagger \in \text{F-Isoc}^\dagger(X)$ to $\mathcal{G}$. By construction the restriction of $\mathcal{F}^\dagger$ has the property that

$$\det(1 - t\,\mathrm{F}\,|\mathcal{F}_x) = \det(1 - t\,\mathrm{F}\,|\mathcal{E}_x)$$

for all $x \in X^\circ$. From this we would like to conclude that in fact $\mathcal{F} \cong \mathcal{E}$, giving the desired theorem statement. Unfortunately we cannot make this conclusion yet. $\square$

The chief difficulty in using the above argument is the following result of Tsuzuki, which says that we have only identified an isocrystal up to semisimplification. The following appears in Appendix A.4 of [Abe18].

Theorem 4.4.2 (Tsuzuki)**.** *Let $\mathcal{E}^\dagger$ and $\mathcal{F}^\dagger$ be overconvergent isocrystals in* $\text{F-Isoc}^\dagger(X)$*. Suppose $\mathcal{E}^\dagger$ and $\mathcal{F}^\dagger$ are companions in the sense of Conjecture 4.3.1. Then $\mathcal{E}^\dagger$ and $\mathcal{F}^\dagger$ have the same semisimplification.*

To get around this difficulty, we will need to first recall several facts regarding isocrystals and their cohomology.

4.5 More on F-Isocrystals

4.5.1 Basic Facts

In this section we will extend the introduction to isocrystals given in Section 4.2. We refer to [Ked22a] and references therein for a more detailed exposition of this theory. Recall the setup: for X over k, we will set K to be the fraction field of the Witt ring $W(k)$. The category $\text{F-Isoc}(X)$ is the category of convergent F-isocrystals on X. It is a K-linear abelian tensor category, with unit object $\mathcal{O}$. The category of overconvergent F-isocrystals, $\text{F-Isoc}^\dagger(X)$, is also a K-linear abelian tensor category, with unit object $\mathcal{O}^\dagger$. We will start with an important lemma regarding base changing under a dominant morphism.

Lemma 4.5.1 ([Ked22a, Corollary 5.9]). *Let $Y \to X$ be a dominant morphism of smooth, geometrically irreducible schemes over k. The map*

$$\text{F-Isoc}^\dagger(X) \to \text{F-Isoc} \times_{\text{F-Isoc}(Y)} \text{F-Isoc}^\dagger(Y)$$

is an equivalence of categories.

In the next subsection we will recall basic facts on the cohomology of isocrystals.

4.5.2 Cohomology of F-Isocrystals

Given $\mathcal{E}$ an isocrystal in F-Isoc(X) we obtain cohomology groups

$$H^i(X, \mathcal{E}).$$

These groups are K-vector spaces equipped with a Frobenius action. In the affine setting, $H^i(X, \mathcal{E})$ is computed as the de Rham cohomology of $\mathcal{E}$ on a lift of X. We will further define

$$H^0_F(X, \mathcal{E}) = \text{Hom}_{\text{F-Isoc}(X)}(\mathcal{O}, \mathcal{E})$$

$$H^1_F(X, \mathcal{E}) = \text{Ext}^1_{\text{F-Isoc}(X)}(\mathcal{O}, \mathcal{E}).$$

We have the alternate description

$$H^0_F(X,\mathcal{E}) \cong H^0(X,\mathcal{E})^{\varphi}$$

and $H^1_F(X,\mathcal{E})$ fits into the Hochschild-Serre exact sequence

$$H^0(X,\mathcal{E})_{\varphi} \to H^1_F(X,\mathcal{E}) \to H^1(X,\mathcal{E})^{\varphi}.$$

Let U be a dense open subscheme of X. We need the following injectivity results:

Lemma 4.5.2. *[Ked22a, Theorem 5.1 and Remark 5.2] For $\mathcal{E}$ in* F-Isoc(X), *the map*

$$H^0_F(X,\mathcal{E}) \to H^0_F(U,\mathcal{E}|_U)$$

is an isomorphism. In particular, the restriction functor F-Isoc$(X) \to$ F-Isoc(U) *is fully faithful.*

Corollary 4.5.3. *The map $H^1_F(X,\mathcal{E}) \to H^1_F(U,\mathcal{E}|_U)$ is injective.*

Proof. Viewing an element of $H^1_F(\cdot,\cdot)$ as an extension of vector bundles, this statement translates to the statement that an exact sequence in F-Isoc(X) splits if and only if the restriction of that sequence splits in F-Isoc(U). This follows from the previous lemma. $\square$

For $\mathcal{E}^\dagger$ an overconvergent isocrystal in F-Isoc$^\dagger(X)$, we define $H^i(X,\mathcal{E}^\dagger), H^0_F(X,\mathcal{E}^\dagger)$, and $H^1_F(X,\mathcal{E}^\dagger)$ as in the convergent setting. We need an additional restriction theorem and an overconvergent analogy of Corollary 4.5.3:

Lemma 4.5.4 ([Ked22a, Theorem5.3])**.** *The restriction $H^0_F(X, \mathcal{E}^\dagger) \to H^0_F(X, \mathcal{E})$ is an isomorphism, and the restriction functor* F-Isoc$^\dagger(X) \to$ F-Isoc(X) *is fully faithful.*

Corollary 4.5.5. *For $\mathcal{E}^\dagger \in$* F-Isoc$^\dagger(X)$*, the map $H^1_F(X, \mathcal{E}^\dagger) \to H^1_F(X, \mathcal{E})$ is injective.*

Proof. The proof is identical to Corollary 4.5.3. □

We end this subsection with a result on filtrations.

Corollary 4.5.6. *Let $\mathcal{E}^\dagger \in$* F-Isoc$^\dagger(X)$*. Suppose that the restriction $\mathcal{E}$ admits a filtration in* F-Isoc(X)*,*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E},$$

where for each i, $\mathcal{E}_i/\mathcal{E}_{i-1}$ is the restriction of an object of F-Isoc$^\dagger(X)$*. Then there exists a filtration of $\mathcal{E}^\dagger$ in* F-Isoc$^\dagger(X)$ *lifting the filtration of $\mathcal{E}$.*

Proof. We induct on l. If $l = 1$, there is nothing to prove. In the general case, we note that by hypothesis $\mathcal{E}_1 = \mathcal{E}_1/\mathcal{E}_0$ lifts to an element $\mathcal{E}_1^\dagger$ in F-Isoc$^\dagger(X)$. By Lemma 4.5.4, the inclusion $\mathcal{E}_1 \hookrightarrow \mathcal{E}$ lifts to an inclusion $\mathcal{E}_1^\dagger \hookrightarrow \mathcal{E}^\dagger$. Passing to $\mathcal{E}^\dagger/\mathcal{E}_1^\dagger$, the result follows from the induction hypothesis. □

4.5.3 Slopes and Weights

In the case of $X = \mathrm{Spec}(k)$, the categories F-Isoc$(X) =$ F-Isoc$^\dagger(X)$ are easy to describe thanks to the Dieudonne-Manin classification theorem [Man63]. We recall the construction here. Fix an algebraic closure $\overline{k}$ over k. For coprime integers r and s, define the isocrystal $\mathcal{E}_{r/s}$ to be the K vector space spanned by $e_1, e_2, \dots, e_s$, with Frobenius action given by

$$\varphi(e_1) = e_2, \dots, \varphi(e_{s-1}) = e_s, \varphi(e_s) = p^r e_1.$$

We have

Theorem 4.5.7 ([Man63]). *Every element $\mathcal{F}$ in* F-Isoc(Spec($\overline{k}$)) *is isomorphic to a direct sum*

$$\bigoplus_{\frac{r}{s} \in \mathbb{Q}} \left(\oplus_{i=0}^{m_{r/s}} \mathcal{E}_{r/s}\right)$$

for nonnegative multiplicities $m_{r/s}$.

Returning to the case of nonalgebraically closed fields, suppose $\mathcal{E}$ is an object of F-Isoc(Spec(k)). We may pull back $\mathcal{E}$ to $\mathcal{E}' \in$ F-Isoc(Spec($\overline{k}$)) and apply the above theorem to $\mathcal{E}'$. This direct sum decomposition descends to $\mathcal{E}$, allowing us to define the *slope multiset* and *slope polygon* of $\mathcal{E}$. This generalizes to a coefficient object on X; given $\mathcal{E} \in$ F-Isoc(X) and x is a (not necessarily closed) point in X, we obtain a slope multiset and slope polygon of $\mathcal{E}|_x$. See Definition 3.3 and the surrounding discussion in [Ked22a] for more.

If the slope multiset of $\mathcal{E} \in$ F-Isoc(X) is constant and contains only a single slope μ, we say that $\mathcal{E}$ is *isoclinic* of slope μ. If $\mathcal{E}$ is isoclinic of slope 0 we will call it *unit root*. We recall here two facts on slopes of isocrystals; first, that the slope polygon of any F-isocrystal is locally constant, and second that any F-isocrystal admits a slope filtration.

Lemma 4.5.8 ([Ked22a, Theorem 3.12]). *For $\mathcal{E} \in$ F-Isoc(X), there exists a dense open subscheme U in X on which $\mathcal{E}$ has a constant Newton polygon.*

Lemma 4.5.9 ([Ked22a, Corollary 4.2]). *Suppose $\mathcal{E} \in$ F-Isoc(X) has a constant Newton polygon. There exists a unique filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

for which $\mathcal{E}_i/\mathcal{E}_{i-1}$ is isoclinic of some slope μ_i and $\mu_1 < \cdots < \mu_l$.

We now transition to the theory of *weights*. As discussed in Section 4.4, two p-adic coefficient objects which are companions are only unique up to semisimplification. The use of weights allows us to get around this complication.

As before, fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Let $\overline{\mathbb{Q}}$ be the integral closure of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_p$, and fix an embedding

$$\iota : \overline{\mathbb{Q}} \to \mathbb{C}.$$

We will say $\mathcal{E} \in$ F-Isoc$(X) \otimes \overline{\mathbb{Q}}_p$ is *ι-pure of weight w* if, for all $x \in X^\circ$, the eigenvalues of Frobenius acting on $\mathcal{E}_x$ all have absolute value $|\mathcal{O}_{X,x}/\mathfrak{m}_x|^{w/2}$ under the embedding ι. As in the étale setting, p-adic coefficient objects always admit weight filtrations:

Theorem 4.5.10 ([Ked20, Theorem 3.1.10]). *Suppose $\mathcal{E}^\dagger \in$ F-Isoc$^\dagger(X) \otimes \overline{\mathbb{Q}}_p$ is algebraic. Fix ι as above. The following hold:*

- *If $\mathcal{E}^\dagger$ is irreducible, then it is ι-pure.*

- *In any case, there exists a unique filtration*

$$0 = \mathcal{E}_0^\dagger \subset \mathcal{E}_1^\dagger \subset \cdots \subset \mathcal{E}_l^\dagger = \mathcal{E}^\dagger$$

  in $\text{F-Isoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$ *and an increasing sequence $w_1 < \cdots < w_l$ for which $\mathcal{E}_i^\dagger/\mathcal{E}_{i-1}^\dagger$ is ι-pure of weight w_i.*

4.6 Exhaustive Families of Curves

Our first goal is to construct a sequence of space-filling curves on X which "understand" the tangent space of X. For the remainder of this chapter, a *smooth curve* will be a geometrically irreducible, locally closed subscheme of X, smooth of dimension 1. We use $T_X \to X$ to refer to the tangent bundle of X (also viewed as a k-scheme).

Definition 4.6.1. *A sequence of curves $C_1, C_2, \ldots, C_n, \ldots$ in X will be called* exhaustive *if the following hold:*

- *For each finite set of closed points $S \subset X^\circ$, there exists an $N \geq 0$ such that $S \subset C_i^\circ$ for all $i \geq N$, and*

- *For each finite set of closed points $T \subset T_X^\circ$ which map injectively to X°, there exists* *an infinite subsequence of curves $C_{j_1}, C_{j_2}, \ldots, C_{j_n}, \ldots$ for which*

$$T \subset T_{C_{j_i}}$$

for all i.

A family of curves which satisfies the first bullet in the above definition will be called simply *space-filling*. A key consequence of Bertini theorems with Taylor coefficients [Poo04, Theorem 1.2] is that exhaustive sequences of curves always exist.

Lemma 4.6.2. *For X quasiprojective, there exists an exhaustive sequence of curves on X.*

Proof. Call a subset $T \subset T_X^\circ$ *admissible* if it projects injectively to X°. Since X is a scheme of finite type over k, both X° and T_X° are countable. This also implies that the set of finite admissible subsets of T_X° is countable. Choose a sequence $T_\infty = (T_1, T_2, \ldots, T_n, \ldots)$ in which each finite admissible subset of T_X° appears infinitely often.

Let $x_1, \ldots, x_n, \ldots$ be an arbitrary ordering of X°. By iteratively applying [Poo04, Theorem 1.2], for each $n \geq 1$ we may find a complete intersection curve C_n in X satisfying both of the following:

- C_n contains each closed point in $\{x_1, \ldots, x_n\}$ and each closed point in the projection of T_n to X°.

- At each closed point P in the projection of T_n to X°, the tangent data of C_n at P

is given by T_n.

By construction, the family of curves $(C_1, \dots, C_n, \dots)$ is exhaustive on X. $\square$

We now provide several sample properties that an exhaustive sequence of curves can detect.

Lemma 4.6.3. *Let $(C_1, \dots, C_n, \dots,)$ be an exhaustive sequence of curves in an affine scheme* $\mathrm{Spec}(R)$ *(with R of characteristic p). Then f is a pth power if and only if f restricts to a pth power on all curves C_i.*

Proof. The pth powers in R can be identified via derivations:

$$d(f^p) = pf^{p-1}df = 0.$$

The vanishing of $d(f^p)$ can be detected by pairing $d(f^p)$ with tangent vectors; as the sequence of curves is exhaustive, the result follows. $\square$

Lemma 4.6.4. *Let $(C_1, \dots, C_n, \dots)$ be a space-filling sequence of curves and let $f : Y \to X$ be a finite étale covering. Then f admits a section if and only if $f_{C_i} : C_i \times_X Y \to C_i$ admits a section for all i.*

Proof. By passing to connected components we may assume that Y is connected and not isomorphic to X. In this case our goal is to find an i for which $f_{C_i} : C_i \times_X Y \to C_i$ admits no section.

Let $\pi_1(X, \overline{x})$ be the étale fundamental group of X based at a geometric point $\overline{x} \to X$. The covering f corresponds to a transitive representation $\pi_1(X, \overline{x}) \to S_n$ for some $n > 1$. Let $(C'_1, \dots, C'_n, \dots)$ be the subsequence of $(C_1, \dots, C_n, \dots)$ consisting of the curves which pass thru the image of $\overline{x}$. For each such C'_i, the restriction

$$Y \times_X C'_i \to C'_i$$

admits a section if and only if the image of $\pi_1(C'_i, \overline{x})$ under ρ fixes an element of $\{1, \dots, n\}$. However, as the curves are space filling, the Chebotarëv density theorem [Mea18] assures us that the images of $\pi_1(C'_i, \overline{x})$ will stabilize to the image of ρ. All transitive subgroups of S_n contain fixed point free elements by Burnside's Lemma, so the conclusion follows. □

4.7 Restricting Isocrystals to Curves

In this section we fix an exhaustive sequence of curves on X, $\mathfrak{C} = (C_1, \dots, C_n, \dots)$ and examine properties of isocrystals that are preserved upon restriction to $\mathfrak{C}$. We start with a fact about H^1_F.

Lemma 4.7.1. *Suppose $\mathcal{E}$ in* F-Isoc(X) *is an isocrystal whose Newton polygon has negative slopes at all points x of X. Then the map*

$$H^1_F(X, \mathcal{E}) \to \prod_{C_i \in \mathfrak{C}} H^1_F(C_i, \mathcal{E}|_{C_i})$$

is injective.

Proof. By Corollary 4.5.3 we may freely pass to an intermediate dense open subscheme U. We will thus work affine locally, and by applying Lemmas 4.5.8 and 4.5.9 we may assume $\mathcal{E}$ is isoclinic on X with negative slope $\mu < 0$. As we are working affine locally, we may take a lift P of X and represent $\mathcal{E}$ as a finite projective $\Gamma(P, \mathcal{O}_P)[p^{-1}]$ module; passing to a dense open subscheme once more, we may assume this module is in fact free.

Fix a basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of this module. Let A be the coefficient matrix of the semi-linear Frobenius action with respect to this basis, i.e. on *basis vectors,* $\sigma(\mathbf{e}_i) = A\mathbf{e}_i$. A direct calculation shows that for positive integers n, σ^n acts on these same basis vectors via the matrix

$$A_n := A\sigma(A)\ldots\sigma^{n-1}(A).$$

By the Sharp Slope Estimate of Katz [Kat79, Theorem 1.5.1], there is an $n \geq 1$ for which A_n^{-1} has entries in $p\Gamma(P, \mathcal{O}_P)$.

Given $\mathbf{v} \in H^1_F(X, \mathcal{E})$, we may represent it by a pair

$$(\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{E} \times \left(\mathcal{E} \otimes \Omega^1\right)$$

satisfying the relation $\nabla\mathbf{v}_1 = (\sigma - 1)\mathbf{v}_2$. The element $\mathbf{v}$ is zero if and only if there is $\mathbf{w} \in \mathcal{E}$ for which $\mathbf{v_1} = (\sigma - 1)\mathbf{w}$ and $\mathbf{v}_2 = \nabla\mathbf{w}$.

Let us examine the first half of the above condition, $\mathbf{v_1} = (\sigma - 1)\mathbf{w}$. Define

$$\mathbf{v}_{1,n} := \mathbf{v}_1 + \sigma \mathbf{v}_1 + \cdots + \sigma^{n-1}\mathbf{v}_1.$$

Let us further write $\hat{v}_1, \hat{v}_{1,n}$ as column coefficient vectors in the basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$, so that

$$\mathbf{v}_1 = \hat{v}_1 \cdot (\mathbf{e}_1, \ldots, \mathbf{e}_n)$$

and

$$\mathbf{v}_{1,n} = \hat{v}_{1,n} \cdot (\mathbf{e}_1, \ldots, \mathbf{e}_n).$$

By clearing denominators we may assume $\mathbf{v}_{1,n}$ is p-integral.

Let $\hat{w}$ be a column of indeterminates. If there exists a solution to $\mathbf{v}_1 = (\sigma - 1)\mathbf{w}$, then we would obtain a solution to $\mathbf{v}_{1,n} = (\sigma^n - 1)\mathbf{w}$, which in turn (by looking at the $\mathbf{e}_i$-coefficients of $\mathbf{w}$) implies a solution to the $\Gamma(P, \mathcal{O}_P)$ equation

$$\hat{v}_{1,n} = (A_n \sigma^n - 1) \cdot \hat{w},$$

or

$$A_n^{-1} \hat{v}_{1,n} = \sigma^n(\hat{w}) - A_n^{-1}\hat{w}.$$

This would imply $\hat{v}_{1,n}$ reduces to a p^n-th power mod p. By Lemma 4.6.3, this is testable by our exhaustive family $\mathfrak{C}$; in other words, if this equation is not solvable on X, then there exists a curve C_i on which it is not solvable, so that the restriction of $(\mathbf{v}_1, \mathbf{v}_2)$ to $H^1_F(C_i, \mathcal{E})$ is nonzero as desired.

If this equation is solvable, then we may translate the original pair $(\mathbf{v}_1, \mathbf{v}_2)$ to a new pair, $(\mathbf{v}'_1, \mathbf{v}'_2)$, representing the *same class in cohomology.* But now repeating the argument for $\mathbf{v}'_1$, we obtain elements of $\Gamma(P, \mathcal{O}_P)$ vanishing modulo arbitrarily large powers of p. In other words, we reduce to the case $\mathbf{v}_{1,n} = 0$. This reduces the equation $\nabla \mathbf{w} = \mathbf{v}_2$ to $\mathbf{v}_2 = 0$; the vanishing or nonvanishing of $\mathbf{v}_2$ can again be tested on curves, as desired. □

With this in hand we now move to controlling H^0_F.

Lemma 4.7.2. *For* $\mathcal{E} \in$ F-Isoc(X) *with* $\mathcal{E}$ *unit-root, there exists an exhaustive subsequence* $\mathfrak{C}'$ *of* $\mathfrak{C}$ *for which*

$$H^0_F(X, \mathcal{E}) \cong H^0_F(C, \mathcal{E}|_C)$$

for all $C \in \mathfrak{C}'$.

Proof. If $\mathcal{E}$ is spanned by global sections then the result is immediate. If not, we may put $\mathcal{E}$ in a short exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$, apply induction on rank and the five lemma (in view of Lemma 4.7.1) to reduce to the case $H^0_F(X, \mathcal{E}) = 0$.

In this setting, $\mathcal{E}$ corresponds to an étale $\mathbb{Q}_p$-local system on X admitting no sections [Ked22a, Theorem 3.7]. Choosing a lattice, we obtain a $\mathbb{Z}_p$-local system whose reduction modulo a power of p admits no section. Applying Lemma 4.6.4, we see $H^0_F(C_i, \mathcal{E}|_{C_i}) = 0$ as well. As we only require curves be space filling for Lemma 4.6.4 to apply, we may specify tangency restrictions on the curves as desired to obtain an exhaustive sequence. □

We now promote the previous lemma to F-isocrystals in general.

Lemma 4.7.3. *For any $\mathcal{E} \in$ F-Isoc(X), there exists an exhaustive subsequence $\mathfrak{C}'$ of $\mathfrak{C}$ for which*

$$H^0_F(X, \mathcal{E}) \cong H^0_F(C, \mathcal{E}|_C)$$

for all $C \in \mathfrak{C}'$.

Proof. As before, we may pass to an intermediate open subscheme on which $\mathcal{E}$ admits a slope filtration,

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{l-1} \subset \mathcal{E}_l = \mathcal{E}.$$

Let i be the largest index with $\mu_l \leq 0$ (setting $i = 0$ if no such index exists). Then $H^0_F(X, \mathcal{E}) = H^0_F(X, \mathcal{E}_i)$. If $\mu_i < 0$, then $H^0_F(X, \mathcal{E}) = H^0_F(C, \mathcal{E}|_C) = 0$ for all smooth curves C. We reduce to $\mu_i = 0$. In this case, by Lemma 4.7.2, we may find an exhaustive subsequence for which the desired conclusion holds for $\mathcal{E}_i/\mathcal{E}_{i-1}$. Consider the diagram

$$\begin{array}{ccc} H^0(X, \mathcal{E}_i) & \xrightarrow{(1)} & H^0_F(X, \mathcal{E}_i/\mathcal{E}_{i-1}) \\ \downarrow{\scriptstyle (4)} & & \downarrow{\scriptstyle (2)} \\ H^0_F(C_i, \mathcal{E}|_{C_i}) & \xrightarrow{(3)} & H^0_F(C_i, (\mathcal{E}_i/\mathcal{E}_{i-1})|_{C_i}). \end{array}$$

The maps (1) is an isomorphism by Lemma 4.7.1, and (2) is an isomorphism from our choice of subsequence. The map (3) is injective, since any map from $\mathcal{O}$ to $\mathcal{E}_i$ is determined by the unit root piece of $\mathcal{E}_i$. As (1) and (2) are isomorphisms, (3) is also surjective. Hence (3) is an isomorphism. This implies (4) is an isomorphism as desired. $\square$

We end this section with a brief detour to discuss algebraicity of coefficient objects.

Fix an algebraic closure $\mathbb{Q}_p \subset \overline{\mathbb{Q}}_p$. For each intermediate extension L which is finite over $\mathbb{Q}_p$, define

$$\text{F-Isoc}(X) \otimes L$$

to be pairs $(\mathcal{E}, \tau_L)$ with $\mathcal{E}$ an object of F-Isoc and τ_L a $\mathbb{Q}_p$ linear action of L on $\mathcal{E}$. Let $\text{F-Isoc}(X) \otimes \overline{\mathbb{Q}}_p$ be the categorical 2-colimit taken over all such intermediate extensions L. Similarly define $\text{F-Isoc}^\dagger(X) \otimes L$ and $\text{F-Isoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$.

Given $\mathcal{E}$ an object of $\text{F-Isoc}(X) \otimes \overline{\mathbb{Q}}_p$ and a closed point $x \in X^\circ$, let $c(\mathcal{E}, x)$ count the number of eigenvalues of Frobenius (with multiplicity) acting on the fiber $\mathcal{E}_x$ which are algebraic over $\mathbb{Q}$. We will call $\mathcal{E}$ *algebraic* if $c(\mathcal{E}, \cdot)$ is constant and equal to the rank of $\mathcal{E}$.

Theorem 4.7.4. *Let $C_1, C_2, \ldots$ be an exhaustive sequence of smooth curves on X. Let $\mathcal{E} \in \text{F-Isoc}(X) \otimes \overline{\mathbb{Q}}_p$. Suppose that for every j, $\mathcal{E}|_{C_j}$ is the restriction of an overconvergent object $\mathcal{E}_j^\dagger$ of $\text{F-Isoc}^\dagger(C_j) \otimes \overline{\mathbb{Q}}_p$. Then both of the following hold:*

1. *There exists a unique direct sum decomposition*

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$$

   with $\mathcal{E}_2$ algebraic and with $c(\mathcal{E}, x) = c(\mathcal{E}_2, x)$ for all x.

2. *For every j, both $\mathcal{E}_1|_{C_j}$ and $\mathcal{E}_2|_{C_j}$ arise as restrictions of overconvergent objects $\mathcal{E}_{1,j}^\dagger$ and $\mathcal{E}_{2,j}^\dagger$ from $\text{F-Isoc}^\dagger(C_j) \otimes \overline{\mathbb{Q}}_p$.*

Proof. It suffices to work affine locally. If X is a curve, then this is the content of [Ked20, Corollary 8.4.4]. We may reduce the general case to this using Lemmas 4.7.3 and 4.5.4. Indeed on each curve C in our exhaustive family we have a decomposition

$$\mathcal{E}|_C \cong \mathcal{E}_{C,1} \oplus \mathcal{E}_{C,2}$$

where $\mathcal{E}_{C,1}, \mathcal{E}_{C,2}$ are as in Part 1. In particular, we obtain projection morphisms $\mathcal{E}|_C \to \mathcal{E}_{C,1}$ on each curve, which we may promote to a map $\mathcal{E}|_C \to \mathcal{E}|_C$. By Lemma 4.7.3, we may patch these morphisms together to form a map $\mathcal{E} \to \mathcal{E}$. The image and cokernel of this morphism provide the desired decomposition. $\square$

4.8 Logarithmic Isocrystals and a Local Calculation

In this section we will briefly recall "nice" compactifications of geometric objects, and a resulting category of isocrystals that can be defined relative to such a compactification. We will then work locally to give a cut-by-curves calculation for determining when an extension of two logarithmic isocrystals is again logarithmic.

4.8.1 Compactifications and Logarithmic Isocrystals

Given a scheme $\overline{X}$, a *strict normal crossings divisor* on $\overline{X}$ is an effective Cartier divisor D such that

- For every $p \in D$, the local ring $\mathcal{O}_{\overline{X},p}$ is regular

- For every $p \in D$, there exist local parameters $x_1, \ldots, x_d$ and an integer r, $1 \leq r \leq d$, for which the defining equation of D in $\mathcal{O}_{\overline{X},p}$ is the product $x_1 \cdot x_2 \cdot \cdots \cdot x_r$.

In other words, a strict normal crossings divisor locally looks like a union of coordinate hyperplanes.

We now return to the setting in which X is a smooth geometrically irreducible scheme over k. We will say that X admits a *good compactification* if we have a quasicompact open immersion $j : X \hookrightarrow \overline{X}$ with

- $\overline{X}$ smooth and projective
- The complement $Z := \overline{X} \setminus X$ is a strict normal crossings divisor.

Good compactifications of X exist up to an alteration [dJ96], and we will use that implicitly throughout the remainder of the paper.

Good compactifications are useful in that they allow one to study non-proper spaces via an embedding into relatively easier to understand proper spaces. This is achieved through the concept of a *log structure*; in the context of a good compactification $j : X \to \overline{X}$, the scheme $\overline{X}$ admits a log structure given by the data of the inclusion

$$j_* : \mathcal{O}_{\overline{X}} \cap \mathcal{O}_X^\times \to \mathcal{O}_{\overline{X}}.$$

In other words, the log structure on X is the multiplicative data of the sections which are nonvanishing along Z. We will use the notation $\overline{X}_{\log}$ when we need to remember this log structure. For a deeper introduction to log geometry, we recommend [Ogu18].

To $\overline{X}_{\log}$ we can associate the category of logarithmic F isocrystals, F-Isoc($\overline{X}_{\log}$). Heuristically speaking, these are the overconvergent isocrystals on X which are "nice" at infinity; for this reason they are referred to as *docile* objects in the language of companions [Ked18]. Locally, an element of F-Isoc($\overline{X}_{\log}$) looks like a vector bundle on a characteristic zero lift of $\overline{X}$ equipped with a *logarithmic* connection and a horizonal isomorphism with its Frobenius pullback. For a global, site theoretic construction, we refer to work of Shiho [Shi00], [Shi02].

Below we will collect several preliminary facts regarding logarithmic isocrystals.

Theorem 4.8.1 ([Ked22a, Section 7]). *Let $j : X \to \overline{X}$ and $\overline{X}_{\log}$ be as above. Then*

1. *The category* F-Isoc($\overline{X}_{\log}$) *is an abelian tensor category, closed under subquotients and extensions.*

2. *There is a fully faithful forgetful functor*

$$\text{F-Isoc}(\overline{X}_{\log}) \to \text{F-Isoc}^{\dagger}(X).$$

Given $\mathcal{E} \in$ F-Isoc($\overline{X}_{\log}$) we may form the corresponding logarithmic de Rham complex; taking hypercohomology we arrive at the de Rham cohomology groups

$$H^i(\overline{X}_{\log}, \mathcal{E}).$$

We will further define

$$H^0_F(\overline{X}_{\log}, \mathcal{E}) = \mathrm{Hom}_{\text{F-Isoc}(\overline{X}_{\log})}(\mathcal{O}, \mathcal{E})$$

$$H^1_F(\overline{X}_{\log}, \mathcal{E}) = \mathrm{Ext}_{\text{F-Isoc}(\overline{X}_{\log})}(\mathcal{O}, \mathcal{E}).$$

The space $H^0_F(\overline{X}_{\log}, \mathcal{E})$ is nothing other than the Frobenius invariants $H^0(\overline{X}_{\log}, \mathcal{E})^{\varphi}$. Further, there is a natural short exact sequence

$$H^0(\overline{X}_{\log}, \mathcal{E})_{\varphi} \to H^1_F(\overline{X}_{\log}, \mathcal{E}) \to H^1(\overline{X}_{\log}, \mathcal{E})^{\varphi}.$$

The last result we will need is as follows.

Theorem 4.8.2 ([Ked07, Theorem 6.4.5])**.** *For* $\mathcal{E} \in \text{F-Isoc}(\overline{X}_{\log})$*, the map*

$$H^0_F(\overline{X}_{\log}, \mathcal{E}) \to H^0_F(X, \mathcal{E})$$

is an isomorphism. The map

$$H^1_F(\overline{X}_{\log}, \mathcal{E}) \to H^1_F(X, \mathcal{E})$$

is injective.

The last definition we will need is that of (σ, ∇) cohomology for modules over a p-adically complete ring. Let R be a p-adically complete ring, with R/pR reduced. Let σ_R

be a Frobenius lift on R. Extend this to a lift σ on $R[[t]]$ with $\sigma(t) = t^p$. Let M be a finite projective module over $R[[t]][p^{-1}]$, equipped with an R-linear logarithmic connection ∇ and a horizontal isomorphism with its σ pullback. As σ and ∇ are compatible we may promote the de Rham complex to a double complex,

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M & \longrightarrow & M \otimes \Omega^1(\log t) & \longrightarrow & M \otimes \Omega^2(\log t) & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M & \longrightarrow & M \otimes \Omega^1(\log t) & \longrightarrow & M \otimes \Omega^2(\log t) & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with the vertical maps given by $\sigma - 1$ and the horizontal maps by ∇. We will define $H^i_F(M)$ to be the ith cohomology of the total complex of this double complex.

4.8.2 Local Calculations

With these definitions in hand, we will now perform several local calculations with logarithmic isocrystals. These calculations will allow us to control extensions of isocrystals by their restrictions to curves.

Lemma 4.8.3. *Let R be a p-adically complete ring with R/pR reduced. Let σ_R be a lift of Frobenius. Extend σ_R to a lift σ on $R[[t]]$ by setting $\sigma(t) = t^p$. Let M be a finite projective $R[[t]][p^{-1}]$ module with an R linear logarithmic connection and a horizontal isomorphism* *with its σ pullback. Then*

$$H^1_F(M) \to H^1_F(M \otimes_{R[[t]]} R((t))^{\wedge}_{(p)})$$

is injective.

Proof. We embed R/pR into a product of fields; in doing so, it suffices to assume that R/pR itself is a field. Let L be the perfect closure $L = (R/pR)^{perf}$; we have a σ-equivariant embedding $R \hookrightarrow W(L)$ and it will suffice to show

$$H^1_F(M) \to H^1_F(M \otimes_{R[[t]]} W(L)[[t]])$$

is injective.

Take an element $\mathbf{v} \in H^1_F(M)$, represented as $\mathbf{v} = \overline{(\mathbf{v}_1, \mathbf{v}_2)}$, with $\mathbf{v}_1 \in M$ and $\mathbf{v}_2 \in M \times \Omega(\log t)$. If $\mathbf{v}$ is zero in $H^1_F(M \otimes_{R[[t]]} W(L)[[t]])$ we can find $\mathbf{w}$ with $t\frac{d}{dt}\mathbf{w} = \mathbf{v}_2$ and $\sigma\mathbf{w} - \mathbf{w} = \mathbf{v}_1$. The condition $t\frac{d}{dt}\mathbf{w} = \mathbf{v}_2$ implies $\mathbf{w} = \mathbf{c_1} + \mathbf{c_2}$, with $\mathbf{c_1} \in H^0(M) \otimes_R W(L)$ and $\mathbf{c_2} \in M$. We reduce to the case $\mathbf{v}_1 = 0$, and finish by noting that $H^0_F(M) \to H^0_F(M \otimes_{R[[t]]} W(L)[[t]])$ is injective. $\square$

Lemma 4.8.4. *Let $S \to S'$ be a faithfully flat morphism of p-complete, p-torsion free rings. Let M be a finite projective $S[[t]][p^{-1}]$ module equipped with an S-linear connection for the derivation $t\frac{d}{dt}$ and a compatible Frobenius structure. Then the square*

$$\begin{array}{ccc} H^1_F(M) & \longrightarrow & H^1_F(M \otimes_{S[[t]]} S((t))^\wedge_{(p)}) \\ \downarrow & & \downarrow \\ H^1_F(M \otimes_{S[[t]]} S'[[t]]) & \longrightarrow & H^1_F(M \otimes_{S[[t]]} S'((t))^\wedge_{(p)}) \end{array}$$

is cartesian.

Proof. We start with the exact Čech-Alexander sequence

$$0 \to S \to S' \to S' \widehat{\otimes} S' \to S' \widehat{\otimes} S' \widehat{\otimes} S' \to \dots$$

For $i \geq 1$ let S^i be the i-fold (completed) tensor product of S',

$$S^i := \widehat{\bigotimes}_{j=1}^{i} S'.$$

For each i, let $\Omega^\bullet_{S^i}$ be the total complex computing $H^\bullet_F(M \otimes_{S[[t]]} S^i[[t]])$ and $\Omega^\bullet_{S^i,1}$ the complex computing $H^\bullet_F(M \otimes_{S[[t]]} S^i((t))^\wedge_{(p)})$. Exactness of the Čech-Alexander sequence implies that for each j we have an exact sequence

$$0 \to \Omega^j_S \to \Omega^j_{S^1} \to \Omega^j_{S^2} \to \dots \tag{4.1}$$

Consider the diagram in the lemma statement, extended by one row:

$$\begin{array}{ccc}
H^1(\Omega^\bullet_S) & \longrightarrow & H^1(\Omega^\bullet_{S,1}) \\
\downarrow & & \downarrow \\
H^1(\Omega^\bullet_{S^1}) & \longrightarrow & H^1(\Omega^\bullet_{S^1,1}) \\
\downarrow & & \downarrow \\
H^1(\Omega^\bullet_{S^2}) & \longrightarrow & H^1(\Omega^\bullet_{S^2,1})
\end{array}$$

The vertical maps compose to zero by exactness in (4.1), and the horizontal maps are injective by Lemma 4.8.3. To show the top square is cartesian, suppose $\mathbf{v} \in H^1(\Omega^\bullet_{S,1})$ amd $\mathbf{w} \in H^1(\Omega^\bullet_{S^1})$ have the same image, $\mathbf{u}$, in $H^1(\Omega^\bullet_{S^1,1})$. As $\mathbf{u}$ is in the image of $H^1(\Omega^\bullet_{S,1})$ it must map to zero in $H^1(\Omega^\bullet_{S^2,1})$. As the bottom row is injective, this implies $\mathbf{w}$ maps to zero in $H^1(\Omega^\bullet_{S^2})$. Thus we may finish by showing the left column

$$H^1(\Omega^\bullet_S) \to H^1(\Omega^\bullet_{S^1}) \to H^1(\Omega^\bullet_{S^2})$$

is exact. By (4.1) again, we may reduce this to exactness of

$$0 \to H^0(\Omega^\bullet_S) \to H^0(\Omega^\bullet_{S^1}) \to H^0(\Omega^\bullet_{S^2}) \to \dots$$

which follows by identifying $H^0(\cdot)$ with the kernel of the residue map. □

The intended use case for the previous lemma is as follows. Let P be a smooth formal scheme over $W(k)$ and set $S = \mathcal{O}(P)$. For each point x in P_k, let S_x be the completion of S along x. Then

$$S \to \prod_x S_x$$

is faithfully flat.

Lemma 4.8.5. *Let P be a smooth formal scheme over $W(k)$ and set $S = \mathcal{O}(P)$. Let M be a finite projective $S[[t]][p^{-1}]$ module equipped with an S-linear connection for the derivation $t\frac{d}{dt}$ and a compatible Frobenius structure. Then*

$$H^0_F(M) = H^0_F(M \otimes_{S[[t]]} S((t))^\wedge_{(p)}).$$

Proof. Set $\mathfrak{o}_L := (S_{(p)})^\wedge_{(p)}$. Working in $\mathfrak{o}_L((t))^\wedge_{(p)}$, we have

$$S((t))^\wedge_{(p)}[p^{-1}] \cap \mathfrak{o}_L[[t]][p^{-1}] = S[[t]][p^{-1}].$$

Tensoring with M (and working in $M \otimes_{S[[t]]} \mathfrak{o}_L((t))^\wedge_{(p)}$) we obtain

$$S((t))^\wedge_{(p)}[p^{-1}] \otimes_{S[[t]]} M \cap \mathfrak{o}_L[[t]][p^{-1}] \otimes_{S[[t]]} M = S[[t]][p^{-1}] \otimes_{S[[t]]} = M.$$

We reduce to showing

$$H^0_F(M \otimes_{S[[t]]} \mathfrak{o}_L[[t]]) = H^0_F(M \otimes_{S[[t]]} \mathfrak{o}_L((t))^\wedge_{(p)}.$$

This follows from [Ked22b, Theorem 20.3.5]. □

Lemma 4.8.6. *Let P be a smooth formal scheme over $W(k)$ and put $S = \mathcal{O}(P)$. Let M be a finite projective $S[[t]][p^{-1}]$ module with an S-linear connection for the derivation $t\frac{d}{dt}$*

and a compatible Frobenius structure. Suppose that the action of $t\frac{d}{dt}$ *on* $M \otimes_{S[[t]]} S((t))^{\wedge}_{(p)}$ *extends to a* $W(k)$*-linear connection compatible with the same Frobenius structure. Then this induces a* $W(k)$*-linear logarithmic connection on* M*.*

Proof. The connection on $M \otimes_{S[[t]]} S((t))^{\wedge}_{(p)}$ provides a morphism

$$M \otimes_{S[[t]]} S((t))^{\wedge}_{(p)} \to M \otimes_{S[[t]]} S((t))^{\wedge}_{(p)} \otimes \Omega_{S/W(k)}(\log(t)).$$

We may view it as an element

$$\hat{\nabla} \in H^0_F\Big(\Big(M \otimes_{S[[t]]} S((t))^{\wedge}_{(p)}\Big)^{\vee} \otimes M \otimes_{S[[t]]} S((t))^{\wedge}_{(p)} \otimes \Omega_{S/W(k)}(\log(t))\Big).$$

By the previous lemma, from this we obtain an element

$$\nabla \in H^0_F(M^{\vee} \otimes M \otimes \Omega_{S/W(k)}(\log(t))).$$

This provides the desired connection. □

The next lemma concludes our local calculation, and provides the last piece of the puzzle for the full proof to come in the next section. We refer to Figure 4.3 for a picture of the proof.

Lemma 4.8.7. *Suppose* X *admits a good compactification* $\overline{X}$ *with boundary* Z*. Let* $(C_1, \ldots, C_n, \ldots)$ *be an exhaustive sequence of curves in* $\overline{X}$*. Each such curve* C_i *admits a*

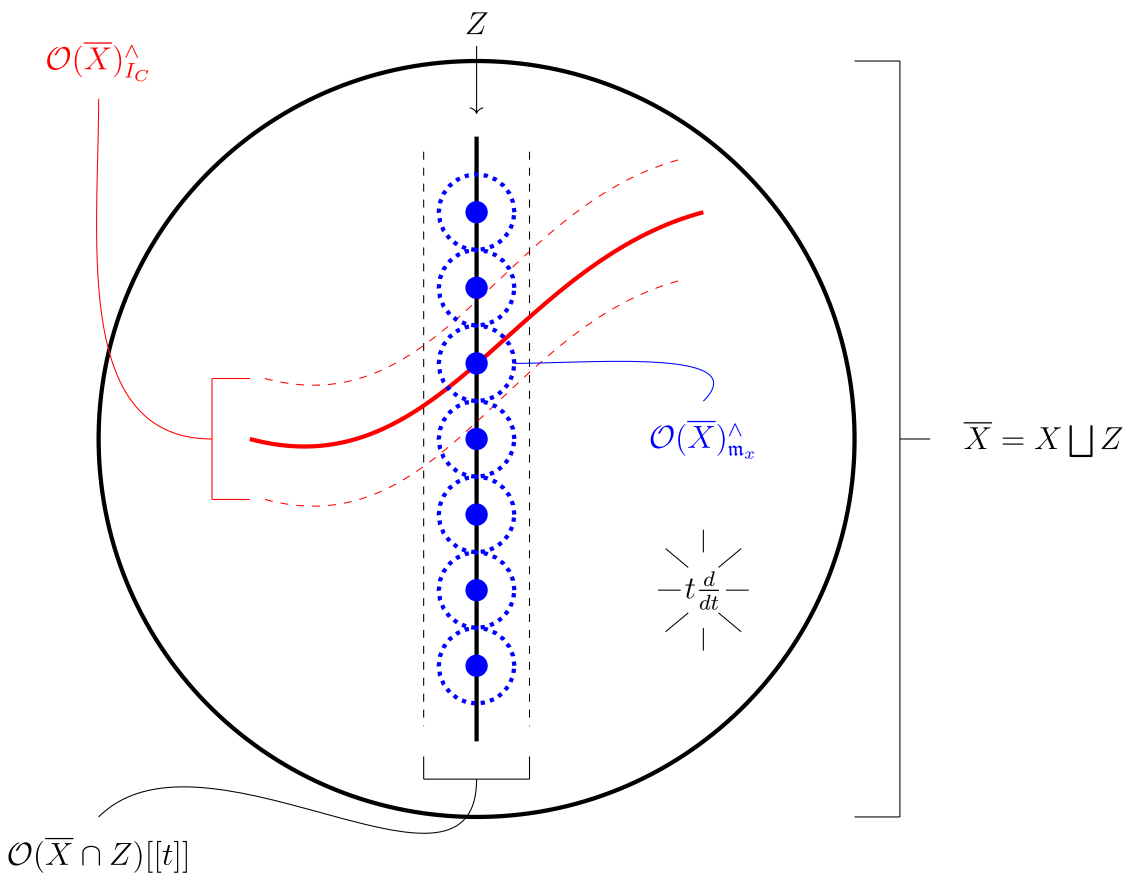

Figure 4.3: Our goal is to extend the $t\frac{d}{dt}$ action across the divisor at infinity (black). By assumption we can extend along formal neighborhoods of curves (red), and hence we can do so through formal neighborhoods of points at infinity (blue). Doing this along all such points allows us to apply Lemma 4.8.4 to conclude the result.

log structure defined by the intersection of Z with C_i, and hence for each curve we obtain a category of logarithmic isocrystals F-Isoc$(C_{i,\log})$ *which are overconvergent on* $C_i \times_{\overline{X}} X$*. Suppose* $\mathcal{E}_1^\dagger, \mathcal{E}_2^\dagger$ *are elements of* F-Isoc$^\dagger(X)$*, and consider an exact sequence in* F-Isoc(X) *of the form:*

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0.$$

Suppose that for all i,

$$\mathcal{E}|_{C_i \times_{\overline{X}} X} \in \text{F-Isoc}(\overline{C}_i).$$

Then

$$\mathcal{E} \in \text{F-Isoc}(\overline{X}_{\log});$$

in particular, $\mathcal{E}$ is overconvergent on X.

Proof. We may assume $\mathcal{E}_2^\dagger = \mathcal{O}^\dagger$ through the use of internal Homs in F-Isoc$^\dagger(X)$. Thus we have an element $\mathbf{v} \in H^1_F(X, \mathcal{E}_1)$ whose image in $H^1_F(C_i \times_{\overline{X}} X, \mathcal{E}_1|_{C_i \times_{\overline{X}} X})$ is docile for all curves C_i. Our goal is to show that this promotes to $\mathbf{v}$ belonging to $H^1_F(\overline{X}_{\log}, \mathcal{E}_1)$.

By Zariski-Nagata Purity for isocrystals [Ked22a, Theorem 5.1], it suffices to work away from the singular locus of Z. Passing to an open subscheme, we may work affine locally and assume that $X \cap Z$ is cut out by a single parameter $t \in \mathcal{O}(X)$. In this setting we will choose an isomorphism $\mathcal{O}(X)^\wedge_{(t)} \cong \mathcal{O}(X \cap Z)[[t]]$.

Let S be a lift of $\mathcal{O}(X \cap Z)$ over $W(k)$. For each point x in $X \cap Z$, let C_x be a curve from the exhaustive family on $\overline{X}$ which passes through x and which meets Z

transversely. The restriction of $\mathcal{E}$ extends logarithmically along C_x by assumption; this extension further promotes to an object over the formal completion of $\overline{X}$ over C_x. In this way we obtain an extension to

$$\prod_{\mathfrak{m}\in \mathrm{MaxSpec}(S)} S_{\mathfrak{m}}^{\wedge}.$$

Finally, by assumption on $\mathcal{E}$ we have an element over the tube $S((t))^{\wedge}_{(p)}$; by Lemma 4.8.4, we conclude that $\mathcal{E}$ belongs to $\text{F-Isoc}^{\dagger}(X)$, and from this we can apply Shiho's work on logarithmic extensions [Shi11] to obtain the desired result.

□

4.9 The Full Proof

We can finally formalize the heuristic proof of our main theorem given in Section 4.4. We start with the following weak version of Theorem 4.4.1.

Lemma 4.9.1. *Let $\mathcal{E} \in \text{F-Isoc}(X)$. Suppose that for every smooth curve C in X, $\mathcal{E}|_C$ is the restriction of an object $\mathcal{E}_C^{\dagger} \in \text{F-Isoc}^{\dagger}(C)$. Suppose there exists a dominant morphism $Y \to X$ such that for every smooth curve C in Y, $(f^*\mathcal{E})|_C$ is a tame object of $\text{F-Isoc}^{\dagger}(C)$. Then there exists $\mathcal{F}^{\dagger} \in \text{F-Isoc}^{\dagger}(X) \otimes \overline{\mathbb{Q}}_p$ such that, for all curves C, $\mathcal{F}^{\dagger}|_C$ has the same semisimplification as $\mathcal{E}_C^{\dagger}$.*

Proof. Immediate from Theorem 4.4.2 and our "heuristic proof" of Theorem 4.4.1 given in Section 4.4. □

We now appeal to the weight filtration (cf. Section 4.5.4) to sort out the semisimplification of the objects given in the previous lemma.

Corollary 4.9.2. *Fix an embedding $\iota : \overline{\mathbb{Q}} \to \mathbb{C}$ For $\mathcal{E}$, $\mathcal{F}^\dagger$ as in Lemma 4.9.1 with $\mathcal{E}$ algebraic, there is a subobject $\mathcal{F}_0^\dagger$ of the first nonzero step of the ι-weight filtration of $\mathcal{F}^\dagger$ whose restriction $\mathcal{F}_0$ is a subobject of $\mathcal{E}$.*

Proof. Let

$$0 = \mathcal{F}_0^\dagger \subset \mathcal{F}_1^\dagger \subset \cdots \subset \mathcal{F}_l^\dagger = \mathcal{F}^\dagger$$

be the ι-weight filtration of $\mathcal{F}^\dagger$. This restricts to the ι-weight filtration of $\mathcal{F}^\dagger|_C$ for any smooth curve C. By Lemma 4.9.1, for every smooth curve C in X there exists a nonzero morphism

$$\mathcal{F}_1^\dagger|_C \to \mathcal{E}_C^\dagger.$$

Forgetting overconvergence, we obtain a nonzero map

$$\mathcal{F}_1|_C \to \mathcal{E}_C;$$

by Lemma 4.7.3, this lifts to a nonzero map $\mathcal{F}_1 \to \mathcal{E}$; the desired subobject of $\mathcal{E}$ is the image of this morphism. $\square$

Corollary 4.9.3. *Fix an embedding $\iota : \overline{\mathbb{Q}} \to \mathbb{C}$. Let $\mathcal{E}$ be as in Lemma 4.9.2. There exists* *a filtration in* $\text{F-Isoc}(X) \otimes \overline{\mathbb{Q}}_p$

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

such that, for $i = 1, 2, \ldots, l$*, the following hold:*

1. *The object* $\mathcal{E}_i/\mathcal{E}_{i-1}$*is the restriction of an object* $\mathcal{F}_i^\dagger$ *in* $\text{F-Isoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$*.*

2. *There exists a sequence* $w_1 \leq \cdots \leq w_l$ *of real numbers, with* $\mathcal{F}_i^\dagger$ *pure of weight* w_i*.*

3. *The restriction of the filtration to any smooth curve* C *lifts to a filtration in the category* $\text{F-Isoc}^\dagger(C) \otimes \overline{\mathbb{Q}}_p$*.*

Proof. The first two claims follow from repeated applications of Lemma 4.9.2. The final claim follows from Lemma 4.5.6. □

We end by proving our main theorem.

Proof of Theorem 4.4.1. By Lemma 4.5.1, we may assume $Y = X$. We also claim it suffices to prove that $\mathcal{E}$ lies in $\text{F-Isoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$. Indeed in $\text{F-Isoc}(X) \otimes \overline{\mathbb{Q}}_p$, $\mathcal{E}$ is isomorphic to each of its Galois conjugates; by Lemma 4.5.4, these isomorphisms extend to the overconvergent category.

By the previous corollary, $\mathcal{E}$ is built as an extension of restrictions of overconvergent objects from $\text{F-Isoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$. By Lemma 4.8.7, we conclude that $\mathcal{E}$ promotes to $\text{F-Isoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$ as desired. □

This chapter of this thesis was based on the article *A cut-by-curves criterion for overconvergence of F-isocrystals*, [GKU22], which is joint work with Kiran Kedlaya and James Upton.

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