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Excess pore water pressure due to ground surface erosion

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Abstract

The Laplace transform is applied to solve the groundwater flow equation with a boundary that is initially fixed but that starts to move at a constant rate after some fixed time. This problem arises in the study of pore water pressures due to erosional unloading where the aquifer lies underneath an unsaturated zone. We derive an analytic solution and examine the predicted pressure profiles and boundary fluxes. We calculate the negative pore water pressure in the aquifer induced by the initial erosion of the unsaturated zone and subsequent erosion of the aquifer.

Keywords: Groundwater flow; erosional unloading; Laplace transform; boost theorem.

1. Introduction

Erosional unloading is the process whereby surface rocks and soil are removed by external processes, resulting in changes to water pressure within the underlying aquifer [1]. [2] used vertical one-dimensional numerical models to investigate abnormal fluid

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pressures in geologic formations caused by gravitational loading or unloading due to deposition or erosion in sedimentary basins.

We consider a mathematical model of changes in excess pore water pressure as a result of erosional unloading. An equivalent porous medium description is used to model the resulting flow [3]. This approach has been shown to be a good model of flow in aquifers [4]. [5] studied this process in the case where the water table initially coincides with the surface. We generalize this case to an ideal aquifer which is initially separated from the ground surface by an unsaturated zone. Rates of erosion are discussed in [6, 7], but in terms of representative values and without addressing temporal variability. In the absence of further information, we consider steady erosion here as a first step.

The problem is solved using the Laplace transform in conjunction with the boost operator derived by [8]. The boost operator is used to boost the solution in the Laplace domain into a frame of reference moving at constant velocity with respect to the original frame. This allows one to solve the the erosional unloading problem in which one boundary moves.

We use our solution to analyze the evolution of the pressure during erosion of the aquifer for small and large erosion rates. We examine the flux at the boundaries a function of time and derive a quasi-steady approximation valid for very small erosion rates in the appendix.

2. Problem formulation

The model studied by [5] consists of a single layer of saturated aquifer where the water table is near the surface. This layer is bounded at the bottom by an impermeable layer.

Our model does not assume that the water table is next to the surface; instead we take the unsaturated zone to have non-negligible thickness (see Figure 1). The capillary and soilwater zones are taken to have negligible thickness and are not considered. The underlying layer below is taken to be impermeable [1]. While both the permeable and impermeable cases are mentioned in [5] and both can be treated using the present approach, the latter is more relevant to applications and is hence considered here.

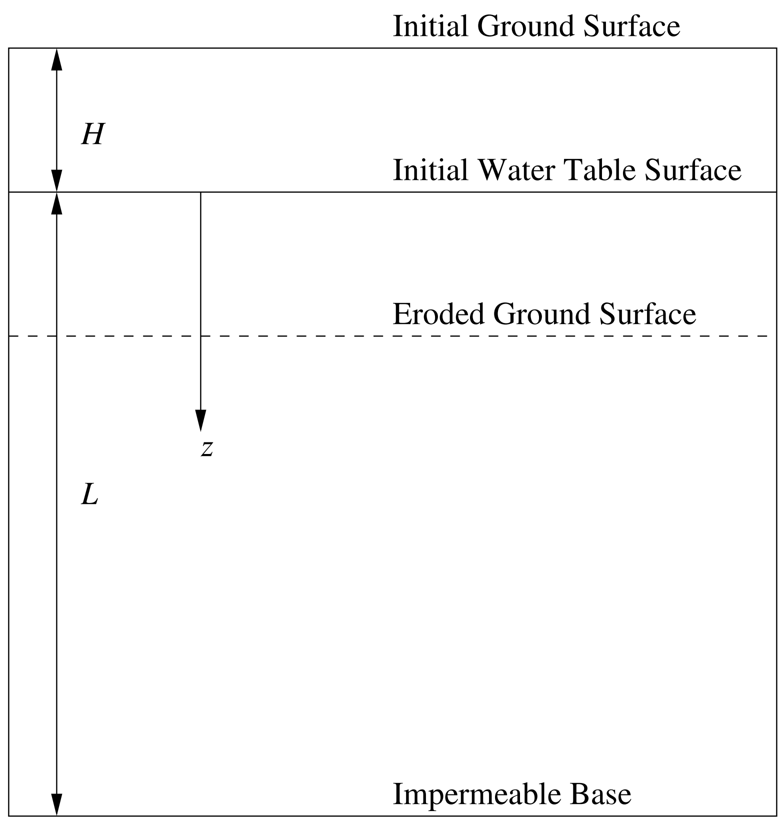

Figure 1: System configuration.

[5] analyzed the following inhomogeneous equation for groundwater flow:

$$c\frac{\partial^2 p'}{\partial z^2} = \frac{\partial p'}{\partial t} - \rho_s g\frac{\partial l}{\partial t}. \tag{1}$$

This equation comes from Darcy's Law and conservation of mass applied to volume elements within the aquifer. Our source term differs from that of Neuzil and Pollock in the time interval before erosion and in the time interval during erosion. The rate of erosion $\partial l/\partial t = b$ will be assumed to be constant, and the aquifer is homogeneous.

Let the unsaturated zone and the aquifer have initial thicknesses of H and L respectively. It follows that the permeable layer is at an initial depth of $H+L$ from the ground surface. The coordinate system is chosen so that the origin coincides with the initial depth of the water table. The z-coordinate will be taken to point down (see Figure 1).

2.1. *Governing Equations*

For the period before erosion, the governing equation is

$$\frac{\partial p}{\partial t} - c\frac{\partial^2 p}{\partial z^2} = -\gamma\rho_m g b. \tag{2}$$

Here p is the excess pore water pressure, $c = K/S$ with hydraulic conductivity K and specific storage S, ρ_m is the moist density of the unsaturated zone, γ is the loading efficiency and g is gravity.

The initial condition is $p = 0$ at $t = 0$, while the boundary conditions are $p = 0$ at $z = 0$ and $\partial p/\partial z = 0$ at $z = L$. Erosion starts at $t = H/b$, and during erosion the field equation is

$$\frac{\partial p}{\partial t} - c\frac{\partial^2 p}{\partial z^2} = -\gamma(\rho_s - \rho_f)gb. \tag{3}$$

Here ρ_s is the saturated density of the aquifer and ρ_f is the groundwater density. The boundary condition at the bottom of the aquifer remains $p = 0$, but now the upper boundary moves, so that $p = 0$ at $z = bt - H$.

We non-dimensionalize using L for length, $c^{-1}L^2$ for time and $\gamma\rho_m gbc^{-1}L^2$ for pressure. The non-dimensional equations are then

$$\frac{\partial p}{\partial t} - \frac{\partial^2 p}{\partial z^2} = \begin{cases} -1 & \text{for } t < t_0, \\ -r & \text{for } t > t_0, \end{cases} \tag{4}$$

where $t_0 = Hc/bL^2$ and $r = (\rho_s - \rho_f)/\rho_m$. We keep the same variable names as before. The initial condition is $p = 0$ at $t = 0$. The fixed boundary condition is $\partial p/\partial z = 0$ at $z = 1$. The other boundary condition is $p = 0$ at $z = 0$ for $t < t_0$ and at $z = \beta(t - t_0)$ for $t > t_0$ with $\beta = bL/c$. The model is thus completely characterized by three parameters r, β, and $t_0 = H/\beta L$.

The erosion of the unsaturated zone and the aquifer ends after a time $t_m = \beta^{-1} + t_0$. In particular, erosion of the aquifer takes place in the time interval $t_0 \le t \le t_m$.

3. Solution

3.1. *Boost Theorem*

We first review the Laplace transform boost and present a theorem derived by [8]. Let $p(z, t)$ be the solution in a frame O, and let $q(z - \beta t, t)$ be the solution in a frame O'

which is moving at a constant speed β with respect to O. Their Laplace transforms are related by the boost operator:

$$\tilde{p}(z,s) = \exp\left(\beta \frac{\partial}{\partial z}\frac{\partial}{\partial s}\right)\tilde{q}(z,s) \tag{5}$$

or equivalently

$$\exp\left(-\beta \frac{\partial}{\partial z}\frac{\partial}{\partial s}\right)\tilde{p}(z,s) = \tilde{q}(z,s). \tag{6}$$

Theorem 1. *[8] If $\tilde{p}(z,s) = A(\sqrt{s})e^{-z\sqrt{s}} + B(\sqrt{s})e^{z\sqrt{s}}$ tends to zero sufficiently fast for large s, and the two functions $A(\sqrt{s})$ and $B(\sqrt{s})$ are analytic in the complex plane with at most a countable number of singularities, then*

$$\exp\left(-\beta \frac{\partial}{\partial z}\frac{\partial}{\partial s}\right)A(\sqrt{s})e^{-z\sqrt{s}} = \left[1 + \frac{\beta}{\sqrt{\beta^2+4s}}\right]A\left(\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4}+s}\right)e^{-(\beta/2+\sqrt{\beta^2/4+s})z}$$

and

$$\exp\left(-\beta \frac{\partial}{\partial z}\frac{\partial}{\partial s}\right)B(\sqrt{s})e^{z\sqrt{s}} = \left[1 - \frac{\beta}{\sqrt{\beta^2+4s}}\right]B\left(-\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4}+s}\right)e^{(-\beta/2+\sqrt{\beta^2/4+s})z}.$$

3.2. Before Erosion

By expanding in a Fourier series, we find the solution for $t < t_0$ in the form

$$p_0(z,t) = -\frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(n-1/2)^3}\left(1 - e^{-(n-1/2)^2\pi^2 t}\right)\sin\left[(n-1/2)\pi z\right]. \tag{7}$$

Define $\lambda_n = -2/[\pi(n-1/2)]^3(1 - e^{-(n-1/2)^2\pi^2 t_0})$. The pressure at $t = t_0$ can hence be written as

$$p_0(z,t_0) = \sum_{n=1}^{\infty}\lambda_n \sin \mu\pi z \tag{8}$$

where $\mu = n - 1/2$.

3.3. During Erosion

For $t > t_0$, define the time variable $\tau = t - t_0$, and the Laplace Transform

$$\tilde{p}(z,s) = \int_0^{\infty} p(z,\tau)e^{-s\tau}\mathrm{d}\tau. \tag{9}$$

The Laplace transform of (4) is

$$s\tilde{p} - \frac{\partial^2\tilde{p}}{\partial z^2} = \sum_{n=1}^{\infty}\lambda_n \sin\mu\pi z - \frac{r}{s}. \tag{10}$$

This has solution

$$\tilde{p} = A(\sqrt{s})e^{-z\sqrt{s}} + B(\sqrt{s})e^{z\sqrt{s}} - \frac{r}{s^2} + \sum_{n=1}^{\infty} \frac{\lambda_n}{s + (\mu\pi)^2} \sin \mu\pi z. \tag{11}$$

The boundary condition at $z = 1$ gives

$$-A(\sqrt{s})e^{-\sqrt{s}} + B(\sqrt{s})e^{\sqrt{s}} = 0. \tag{12}$$

To account for the moving boundary, we consider the solution in a frame in which this boundary is at rest. Consider the boost to the variable $y = z - \beta\tau$, so that the second boundary condition is at $y = 0$. Writing $p(z, \tau) = q(y, \tau)$ leads to the governing equation

$$\frac{\partial q}{\partial \tau} - \frac{\partial^2 q}{\partial y^2} - \beta\frac{\partial q}{\partial y} = -r. \tag{13}$$

For this initial condition, we have $q(y, 0) = p(y, 0) = p_0(y, t_0)$. Hence the Laplace transform of (13) gives

$$s\tilde{q} - \frac{\partial^2 \tilde{q}}{\partial y^2} - \beta\frac{\partial \tilde{q}}{\partial y} = \sum_{n=1}^{\infty} \lambda_n \sin \mu\pi y - \frac{r}{s}, \tag{14}$$

since at $\tau = 0$ we have $y = z$. We obtain

$$\tilde{q} = Ce^{-(\beta/2+\sqrt{\beta^2/4+s})y} + De^{(-\beta/2+\sqrt{\beta^2/4+s})y} - \frac{r}{s^2} + \sum_{n=1}^{\infty}[s_n \sin \mu\pi y + c_n \cos \mu\pi y]. \tag{15}$$

The coefficients s_n and c_n are given by

$$s_n = \frac{s + (\mu\pi)^2}{[s + (\mu\pi)^2]^2 + (\beta\mu\pi)^2}\lambda_n, \qquad c_n = \frac{\beta\mu\pi}{[s + (\mu\pi)^2]^2 + (\beta\mu\pi)^2}\lambda_n. \tag{16}$$

The boundary condition at $y = 0$ leads to

$$C + D - \frac{r}{s^2} + \sum_{n=1}^{\infty} c_n = 0. \tag{17}$$

We now apply Theorem 1 to write C and D in terms of A and B. This gives

$$C = \left[1 + \frac{\beta}{\sqrt{\beta^2 + 4s}}\right] A \left[\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} + s}\right], \qquad D = \left[1 - \frac{\beta}{\sqrt{\beta^2 + 4s}}\right] B \left[-\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} + s}\right]. \tag{18}$$

Define the new variable σ by

$$\sqrt{\sigma} = \frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} + s}. \tag{19}$$

Then $s = \sigma - \beta\sqrt{\sigma}$. The two boundary conditions (with σ in (12) rather than s) become

$$-A(\sqrt{\sigma})e^{-\sqrt{\sigma}} + B(\sqrt{\sigma})e^{\sqrt{\sigma}} = 0, \tag{20}$$

$$\frac{\sqrt{\sigma}}{\sqrt{\sigma}-\beta/2}A(\sqrt{\sigma}) + \frac{\sqrt{\sigma}-\beta}{\sqrt{\sigma}-\beta/2}B(\sqrt{\sigma}-\beta) - \frac{r}{(\sigma-\beta\sqrt{\sigma})^2} + \sum_{n=1}^{\infty} c_n = 0, \tag{21}$$

where c_n is expressed in terms of σ as

$$c_n = \frac{\beta\mu\pi}{[\sigma-\beta\sqrt{\sigma}+(\mu\pi)^2]^2+(\beta\mu\pi)^2}\lambda_n. \tag{22}$$

Following [8], we define $\xi = \sqrt{\sigma}$ and $A(\xi) = \xi^{-1}H(\xi)$. Then the lower boundary condition gives

$$B(\xi) = \xi^{-1}H(\xi)e^{-2\xi}. \tag{23}$$

The upper (moving) boundary condition leads to

$$H(\xi)+e^{-2(\xi-\beta)}H(\xi-\beta)+(\xi-\beta/2)\left[-\frac{r}{\xi^2(\xi-\beta)^2}+\sum_{n=1}^{\infty}\frac{\lambda_n\beta\mu\pi}{[\xi^2-\beta\xi+(\mu\pi)^2]^2+(\beta\mu\pi)^2}\right] = 0. \tag{24}$$

Decomposing into partial fractions, this may be rewritten as

$$H(\xi)+e^{-2(\xi-\beta)}H(\xi-\beta) = \frac{r}{2\beta(\xi-\beta)^2} - \frac{r}{2\beta\xi^2} - \sum_{n=1}^{\infty}\sum_{j=1}^{4}\frac{s_j}{\xi-b_j}, \tag{25}$$

where the s_j and b_j depend implicitly on n and are defined by

$$b_j = i\mu\pi, -i\mu\pi, \beta+i\mu\pi, \beta-i\mu\pi, \qquad s_j = i\lambda_n/4, -i\lambda_n/4, -i\lambda_n/4, i\lambda_n/4. \tag{26}$$

The relation (25) is a linear functional equation for $H(\xi)$. We can treat the terms on the right-hand side separately. It can be verified that the solution to

$$H(\xi)+e^{-2(\xi-\beta)}H(\xi-\beta) = \frac{1}{(\xi-\alpha)^\nu} \tag{27}$$

is

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{(\xi-m\beta-\alpha)^\nu}e^{-2m\xi+m(m+1)\beta} = F(\nu,\alpha,\xi). \tag{28}$$

While ν is arbitrary, we will only require $\nu = 1$ and 2, . Hence

$$H(\xi) = \frac{r}{2\beta}F(2,\beta,\xi) - \frac{r}{2\beta}F(2,0,\xi) - \sum_{n=1}^{\infty}\sum_{j=1}^{4}s_jF(1,b_j,\xi). \tag{29}$$

Note that the sum in (29) does not converge for $\beta > 0$, as in [8]. However, the resulting solution in the time variable does converge.

The solution $\tilde{p}$, with B written in terms of A from (11) and (12), is

$$\tilde{p}(z,s) = A(\sqrt{s})e^{-z\sqrt{s}} + A(\sqrt{s})e^{-(2-z)\sqrt{s}} - \frac{r}{s^2} + \sum_{n=1}^{\infty} \frac{\lambda_n}{s+(\mu\pi)^2} \sin\mu\pi z. \tag{30}$$

The third and fourth terms in (30) have inverse transform

$$-r\tau + \sum_{n=1}^{\infty} \lambda_n e^{-\mu^2\pi^2\tau} \sin\mu\pi z. \tag{31}$$

The terms in A in (30) are

$$\frac{1}{\sqrt{s}}\left(+ \frac{r}{2\beta}F(2,\beta,\sqrt{s}) - \frac{r}{2\beta}F(2,0,\sqrt{s}) - \sum_{n=1}^{\infty}\sum_{j=1}^{4} s_j F(1,b_j,\sqrt{s})\right)e^{-\zeta\sqrt{s}}, \tag{32}$$

with $\zeta = z$ and $\zeta = 2 - z$ successively. We define the inverse transform of each term to be a sum of functions $f_m^{\nu}(d,\alpha,t,\zeta)$. Expressions for these functions, including the simple forms for $\nu = 1$ and 2 , are given in Appendix A.

We arrive at the inverse Laplace transform of $\tilde{p}(z,s)$ in the form

$$\begin{aligned} p(z,\tau) = -r\tau + \sum \lambda_n e^{-n^2\pi^2\tau} \sin\mu\pi z + \sum_{m=0}^{\infty}\Bigg(\frac{r}{2\beta}[f_m^2(\beta,\tau,z) + f_m^2(\beta,\tau,2-z)] \\ - \frac{r}{2\beta}[f_m^2(0,\tau,z) + f_m^2(0,\tau,2-z)] - \sum_{n=1}^{\infty}\sum_{j=1}^{4} s_j[f_m^1(b_j,\tau,z) + f_m^1(b_j,\tau,2-z)]\Bigg). \end{aligned} \tag{33}$$

Its derivative with respect to z is

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}z} = \sum_{n=1}^{\infty} \mu\pi\lambda_n e^{-n^2\pi^2\tau} \cos\mu\pi z + \sum_{m=0}^{\infty}\Bigg(\frac{r}{2\beta}[f_m^{'2}(\beta,\tau,z) - f_m^{'2}(\beta,\tau,2-z)] \\ - \frac{r}{2\beta}[f_m^{'2}(0,\tau,z) - f_m^{'2}(0,\tau,2-z)] - \sum_{n=1}^{\infty}\sum_{j=1}^{4} s_j[f_m^{'1}(b_j,\tau,z) - f_m^{'1}(b_j,\tau,2-z)]\Bigg), \end{aligned} \tag{34}$$

where the derivatives $f_m^{'\nu}$ are given in Appendix A.

4. Results

The values used in this section are $l = H = 10$, $c = 30$, $\gamma\rho_m g = 1$, and $\gamma(\rho_s - \rho_f)g = 0.5$. We vary b and hence t_0. Since pressure is nondimensionalized with $\gamma\rho_m g b c^{-1} L^2$,

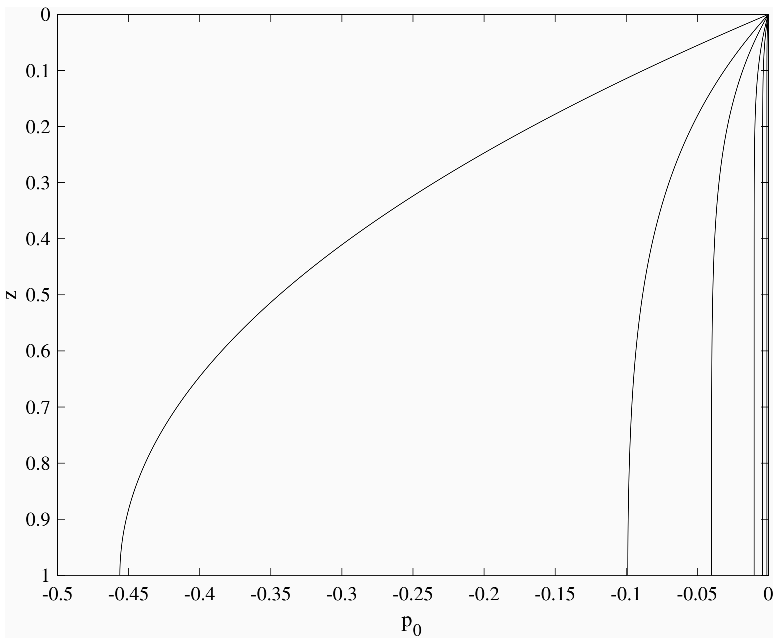

Figure 2: Pressure at t_0. From right to left, the erosion rates are $\beta = 1, 10, 25, 100, 250, 1000$.

it suffices to scale by this number to recover the physical values of pressure. For this choice of values, this is equivalent to scaling by 10β.

Erosion of the unsaturated zone induces an initial increase in negative pore water pressure throughout the aquifer. The magnitude of the pressure change peaks at the base of the aquifer. Large erosion rates cause a greater increase in negative pressure at the base of the aquifer . This is shown in Figure 2. Since l and H are fixed, increasing β corresponds to decreasing $t_0 = H/\beta L$.

A plot of pressure vs. depth (with the origin set at the top of the aquifer) is given in Figure 3. Each curve represents a different stage of erosion. The first curve at time $\tau = 0$ is the initial pressure at the beginning of aquifer erosion. Figure 4 shows a similar plot with a higher erosion rate.

For low erosion rates, the greatest pressure remains localized at the base of the aquifer. In addition, the magnitude of the pressure at any given depth decreases during erosion. For large erosion rates, the pressure increases most rapidly at the base, leading to a rapid variation in pressure near the surface and more uniform pressure near the base

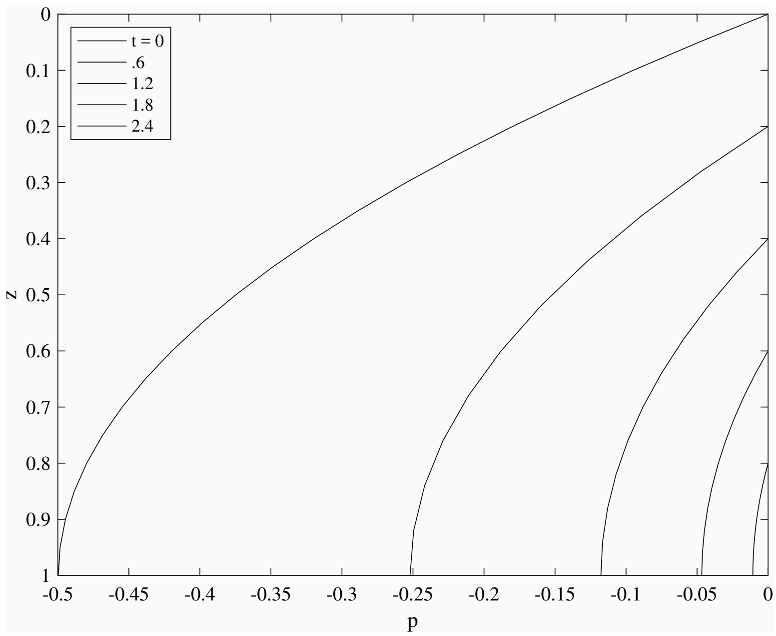

Figure 3: Pressure vs. depth for an erosion rate of $\beta = 1/3$.

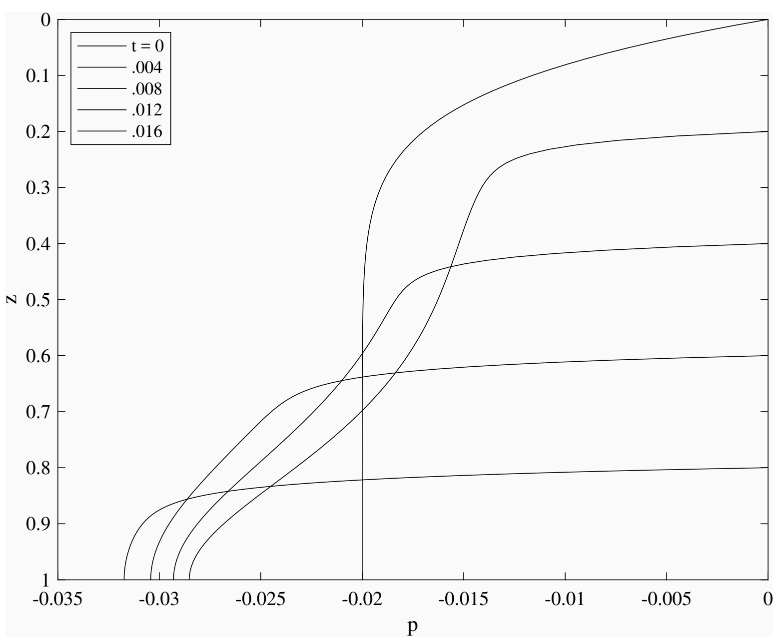

Figure 4: Pressure vs. depth for an erosion rate of $\beta = 50$.

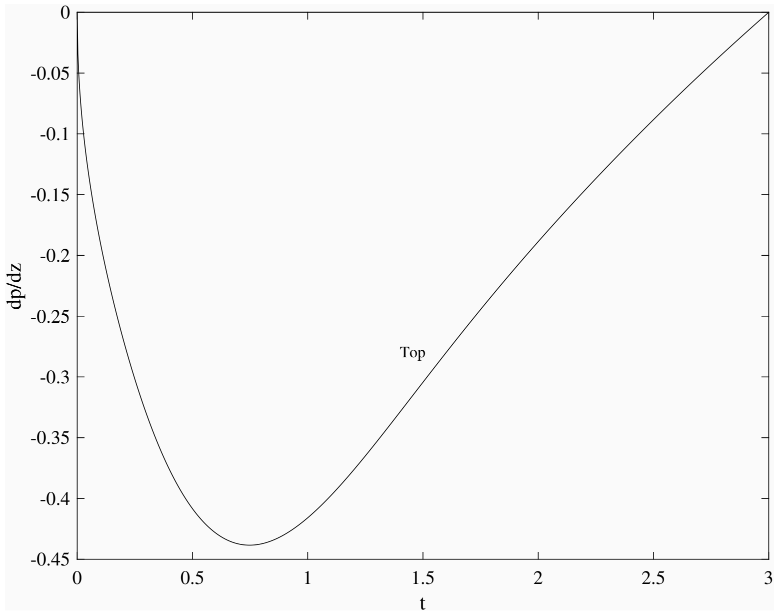

Figure 5: Boundary fluxes for $\beta = 1/3$.

for times close to the end of erosion. In contrast with the case of low erosion rates, the magnitude of the pressure increases for a short time at certain depths. Figures 5 and 6 show the flux at each boundary as a function of time. Appendix B presents an approximate calculation for the fluxes.

Next, we examine pressure as a function of the erosion rate. Figure 7 shows the pressure halfway through the remaining depth of the aquifer after 25, 50 and 75% erosion. The dependence is strikingly linear.

5. Conclusion

The analysis of the erosional unloading problem studied by [5] has been extended to the case of a subsurface boundary. An analytic solution was obtained through use of the Laplace transform. While standard Laplace transform methods are insufficient for handling most moving boundary value problems, we showed that the Laplace Transform boost theorem derived by King may be readily applied to such problems.

The initial pressure profile as a function of erosion rate was calculated. The subsequent time evolution of pressure in the aquifer was obtained and its dependence on the

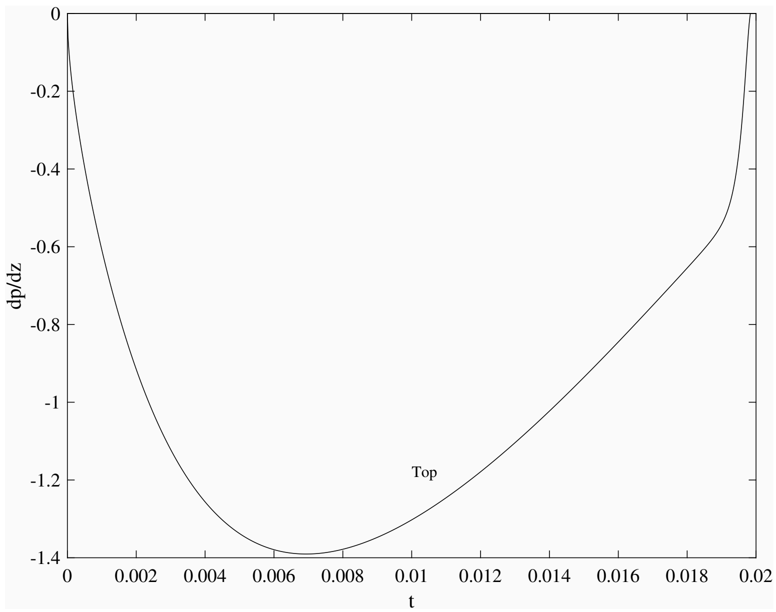

Figure 6: Boundary fluxes for $\beta = 50$.

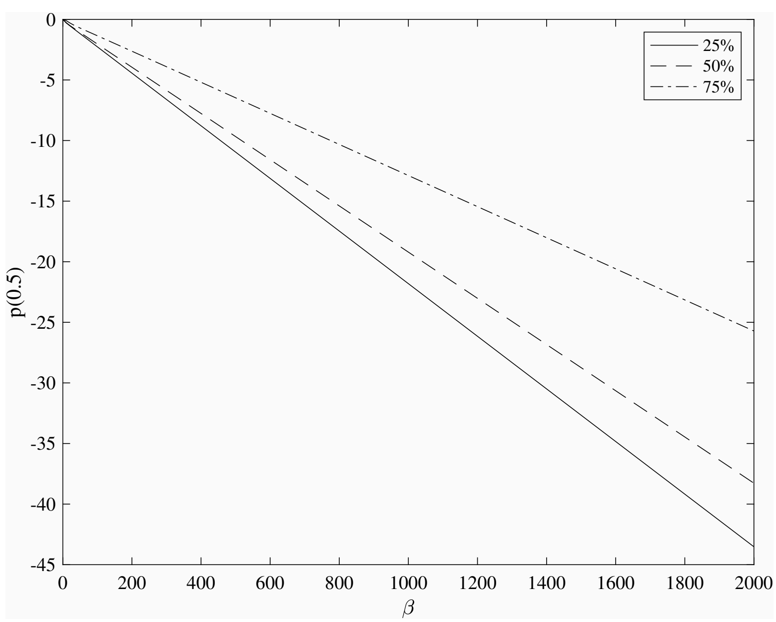

Figure 7: Pressure vs. erosion rate.

parameters of the problem was examined. It is possible to obtain approximate solutions for the boundary fluxes for small erosion rates.

These results can be used to compare to more complex calculations of erosional unloading. The boost theorem can be used when both boundaries move, although it is limited to the case when the boundaries move with constant speed.

Appendix A. Calculation of $f_m^\nu(\alpha, \tau, \zeta)$

The function $f_m^\nu(\alpha, \tau, \zeta)$ is the inverse Laplace transform of

$$(-1)^m \frac{e^{-(2m+\zeta)\sqrt{s}+m(m+1)\beta}}{\sqrt{s}(\sqrt{s}-m\beta-\alpha)^\nu} = \frac{\Delta e^{-k\sqrt{s}}}{\sqrt{s}(\sqrt{s}+a)^\nu}, \tag{A.1}$$

where we have defined $\Delta = (-1)^m e^{m(m+1)\beta}$, $k = 2m + \zeta$ and $a = -m\beta - \alpha$. Note that ν is an integer. For $\nu = 0$, the inverse transform is

$$f_m^0 = \frac{\Delta e^{-k^2/4\tau}}{\sqrt{\pi\tau}}. \tag{A.2}$$

For other ν, we see that

$$\left(a - \frac{\mathrm{d}}{\mathrm{d}k}\right)^\nu f_m^\nu = f_m^0, \tag{A.3}$$

with $f_m^\nu \to 0$ as $k \to \infty$. This ordinary differential equation has the solution

$$\begin{aligned} f_m^\nu &= \frac{\Delta}{(\nu-1)!\sqrt{\pi\tau}} \int_k^\infty (w-k)^{\nu-1} e^{a(k-w)} e^{-w^2/4\tau} \mathrm{d}w \\ &= \frac{\Delta e^{ak+a^2\tau}}{(\nu-1)!\sqrt{\pi}} \int_{\eta^2}^\infty (2\sqrt{u\tau}-k-2at)^{\nu-1} e^{-u} \frac{\mathrm{d}u}{\sqrt{u}} \\ &= \frac{\Delta e^{ak+a^2\tau}}{(\nu-1)!\sqrt{\pi}} \sum_{l=0}^{\nu-1} \binom{\nu-1}{l} (2\sqrt{\tau})^l (-k-2a\tau)^{\nu-1-l} \Gamma\left(\frac{l+1}{2}, \eta^2\right), \end{aligned} \tag{A.4}$$

where $\eta = a\sqrt{\tau} + k/(2\sqrt{\tau})$. Here $\Gamma(a, x)$ is the incomplete Gamma function, which satisfies $\Gamma(a+1, x) = x^a e^{-x} + a\Gamma(a, x)$. In addition $\Gamma(\frac{1}{2}, x^2) = \sqrt{\pi}\,\mathrm{erfc}\,x$, $\Gamma(1, x^2) = e^{-x^2}$, and $\Gamma(\frac{3}{2}, x^2) = xe^{-x^2} + \frac{1}{2}\sqrt{\pi}\,\mathrm{erfc}\,x$.

This implies that

$$f_m^1 = \Delta e^{ak+a^2\tau}\,\mathrm{erfc}\,\eta, \tag{A.5}$$

$$f_m^2 = \frac{\Delta e^{ak+a^2\tau}}{\sqrt{\pi}}[(-k-2a\tau)\sqrt{\pi}\,\mathrm{erfc}\,\eta + 2\sqrt{\tau}e^{-\eta^2}]. \tag{A.6}$$

$$\tag{A.7}$$

The convergence of the integral in (A.4) requires that a, and hence η, be non-negative. It can be shown that these three functions are in fact the inverse Laplace transforms for arbitrary a and η by analytic continuation.

The required derivatives with respect to z are

$$f_m^{\prime 1} = \Delta e^{ak+a^2\tau}\left(a \operatorname{erfc}\eta - \frac{1}{\sqrt{\pi\tau}}e^{-\eta^2}\right), \tag{A.8}$$

$$f_m^{\prime 2} = \frac{\Delta e^{ak+a^2\tau}}{\sqrt{\pi}}[2a\sqrt{\tau}e^{-\eta^2} - (1 + ak + 2a^2\tau)\sqrt{\pi}\operatorname{erfc}\eta]. \tag{A.9}$$

(A.10)

Appendix B. Quasi-Steady State Approximation

Suppose that the erosion rate is small ($\beta \ll 1$) and also that the initial time is large ($t_0 \gg 1$). We also assume that the ratio H/L is not too large. Then

$$\lambda_n \approx -\frac{2}{(\pi\mu)^3}. \tag{B.1}$$

The governing equation (4) for $t > t_0$ becomes

$$\frac{\partial^2 p}{\partial z^2} = -r, \tag{B.2}$$

with boundary conditions $p = 0$ at $z = \beta\tau$ and $\partial p/\partial z = 0$ at $z = 1$. This has the solution

$$p(z,t) = \frac{r}{2}(z + \beta\tau - 2)(z - \beta\tau). \tag{B.3}$$

The derivative evaluated at each boundary is

$$\frac{\partial p}{\partial z} = r\begin{cases}(\beta\tau - 1) & \text{at } z = \beta\tau,\\ 0 & \text{at } z = 1.\end{cases} \tag{B.4}$$

Figure Appendix B demonstrates the approximation for small β.

It also shows that the approximation breaks down for very small times. To explain this rapid adjustment, consider the boundaries to be fixed (a good approximation when $\beta \ll 1$) so that the solution can be written as the series

$$p(z,t) = \sum_{n=1}^{\infty} b_n(\tau)\sin\mu\pi z. \tag{B.5}$$

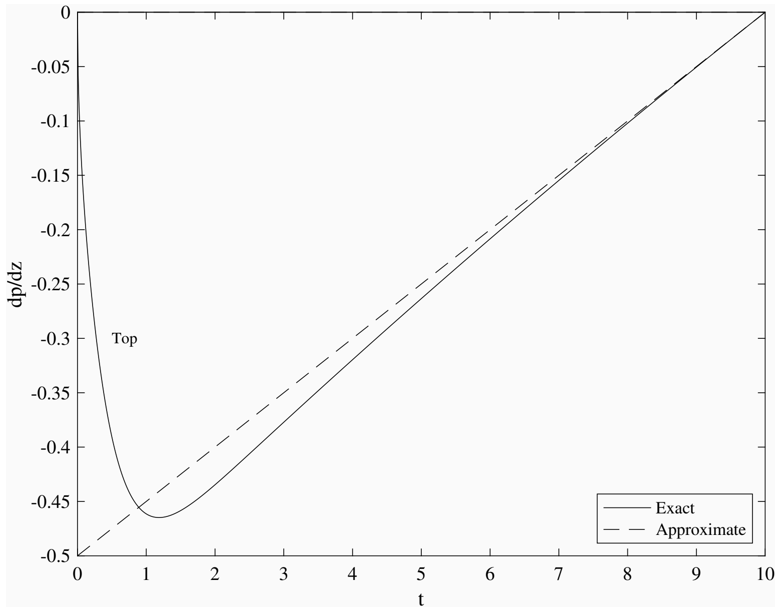

Figure B.8: Flux at each boundary for $\beta = 1/10$.

The governing equation (now keeping the time derivative term) yields the ODEs

$$\dot{b_n} + (\mu\pi)^2 b_n = -\frac{2r}{\mu\pi}. \tag{B.6}$$

With the initial condition $b_n(0) = \lambda_n$ from (B.1), the coefficients are

$$b_n = (1 - r)\lambda_n e^{-(\mu\pi)^2\tau} + r\lambda_n \tag{B.7}$$

for n odd and 0 for n even. The rapid adjustment is explained by the presence of the transient term. For a sufficiently large time (which is still small compared to whole erosion time), the transient term disappears and the series converges to

$$p(z) = \frac{r}{2}z(z - 2), \tag{B.8}$$

which is precisely equation (B.3) with $\tau = 0$.

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