Title

The Algebra And Arithmetic Of Vector-Valued Modular Forms On $\Gamma_{0}(2)$

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THE ALGEBRA AND ARITHMETIC OF VECTOR-VALUED MODULAR FORMS ON $\Gamma_0(2)$

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in

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by

Richard Gottesman

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Abstract

The algebra and arithmetic of vector-valued modular forms on $\Gamma_0(2)$

by

Richard Gottesman

In this thesis, we investigate the module structure and the arithmetic of vector-valued modular forms. We show that for certain subgroups H of the modular group, the module $M(\rho)$ of vector-valued modular forms for a representation ρ of H is a free module of dimension dim ρ. In the case when ρ is an irreducible two-dimensional representation of $\Gamma_0(2)$, we compute a basis for $M(\rho)$ using the modular derivative. We then express the component functions of an element F of $M(\rho)$ of minimal weight in terms of the Gaussian hypergeometric series, a Hauptmodul of $\Gamma_0(2)$, and the Dedekind η-function. This allows us to obtain explicit formulas for the Fourier coefficients of F. We say that a function f whose Fourier coefficients are algebraic numbers has unbounded denominators if the sequence of the denominators of the Fourier coefficients of f is unbounded. We show that if ρ has certain properties then the Fourier coefficients of a normalization of each of the component functions of F are algebraic numbers. Moreover, we show that both component functions of this normalization have unbounded denominators. We then prove that if X is any vector-valued modular form for ρ whose component functions have Fourier coefficients that are algebraic numbers then both of the component functions of X have unbounded denominators.

Dedication

I dedicate this thesis to the memory of Bernie and Bernice Carson.

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I have been very fortunate to have had Geoff Mason as my doctoral advisor. I have greatly enjoyed and profited from my weekly meetings with Geoff, which would often last five to six hours. These meetings were quite lively and I think that half of the math department knew when we were meeting. It is possible that they may have heard some of our bad jokes. During these meetings, Geoff's questions and ideas helped me to understand the essence of many mathematical issues. These meetings inspired me to think more deeply about mathematics and I will miss them. Geoff's encouragement and interest in my research has made my time in Santa Cruz rewarding and a lot of fun. Geoff has been an extraordinary mentor, a great teacher, and a true friend. To Geoff: 196883 + 1 thank you's is simply not enough.

I thank Geoff and Chris Mason for being so incredibly welcoming during my time in Santa Cruz. I appreciate their hospitality, their kindness, and their friendship.

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My Mom and Dad have always supported my passion for mathematics and believed in me. I especially appreciate their encouragement during my final year of graduate school, which was particularly intense and challenging. Their support has made all the difference.

I know my grandmother, the late Bernice Carson, would have been thrilled to read my thesis. Bernice defied society's expectations to be one of the few women (at the time) to complete a master's degree in mathematics from Columbia University. Bernice was my teacher, my mentor, and one of my closest friends.

My grandfather, the late Bernie Carson, was a true mensch. He had an inexhaustible supply of kindness. He was a creative person and he loved poetry, puppetry, woodworking, and painting. Even though Bernie was not a math person, he was always very excited to see me doing mathematics because he knew it was my passion.

I have dedicated my thesis to the memory of Bernie and Bernice Carson. I know that they would be filled with joy and pride to see my thesis.

Chapter 1

Introduction

1.1 Motivation

Vector-valued modular forms play a fundamental role in number theory. Two examples include the theory of Jacobi forms [8] and the work of Borcherds [4]. Vector-valued modular forms have been effectively studied from different perspectives including algebraic geometry ([5], [6]), vector-valued Poincaré series ([16], [17]), and a Riemann-Hilbert approach ([12]). The arithmetic of the Fourier coefficients of vector-valued modular forms for a representation of the modular group Γ have been intensively studied by Cameron Franc, Chris Marks, and Geoff Mason ([9], [11], [19]). One of the motivations for their work is the unbounded denominator conjecture of Atkin and Swinnerton-Dyer ([1]). Atkin and Swinnerton-Dyer gave examples of modular forms on noncongruence subgroups whose Fourier coefficients have unbounded denominators. Franc and Mason have shown in [9] that if ρ is a two-dimensional irreducible representation of Γ such that $\ker \rho$ is a noncongruence subgroup then any vector-valued modular form for ρ whose Fourier coefficients are algebraic numbers has the property that the denominators of the Fourier coefficients of each of its component functions is unbounded. Their technique involves showing that a minimal weight vector-valued modular form for ρ satisfies a second order differential equation whose coefficients are modular forms on Γ. This differential equation can be described using a single parameter that depends on ρ.

The focus of this thesis is the study of the module structure together with the arithmetic properties of vector-valued modular forms for a two-dimensional irreducible representation ρ of $\Gamma_0(2)$. In this thesis, we show that the module of vector-valued modular forms for such a ρ is a free module. We then study the modular linear differential equation that is satisfied by a vector-valued modular form of minimal weight. In contrast to the case when ρ is a representation of Γ, this differential equation is dependent on three parameters, instead of one. These additional parameters present an interesting and important challenge when studying the arithmetic of two-dimensional vector-valued modular forms on $\Gamma_0(2)$. In this work, we make progress towards proving that if ρ is a two-dimensional irreducible representation of $\Gamma_0(2)$ such that ker ρ is a noncongruence subgroup then any vector-valued modular form for ρ whose Fourier coefficients are algebraic numbers has the property that the denominators of the Fourier coefficients of each of its component functions is unbounded. Indeed, we prove this conjecture is true for a certain class of representations.

1.2 Overview of the thesis

In chapter two, we define vector-valued modular forms and study the module structure of vector-valued modular forms. We use ideas from commutative algebra to show in Theorem 2.2.2 that the module of vector-valued modular forms is Cohen-Macaulay. We then show in Theorem 2.2.3 that for certain subgroups H of the modular group Γ, the module of vector-valued modular forms $M(\rho)$ for a representation ρ of H is free of rank dim ρ. In particular, we show that the module of vector-valued modular forms for a representation ρ of $\Gamma_0(2)$ is a free module. We note that the results we obtained in chapter one were also obtained using other methods by Cameron Franc and Luca Candelori. (see [5], [6]).

In chapter three, we show how to use the modular derivative to compute a basis for the module of vector-valued modular forms $M(\rho)$ with respect to a two-dimensional irreducible representation ρ of $\Gamma_0(2)$. Let k_0 denote the least integer for which $M_{k_0}(\rho) \neq 0$ and let F denote a nonzero element in $M_{k_0}(\rho)$. We prove in Theorem 3.0.1 that F and $D_{k_0}F := q\frac{d}{dq}(F) - \frac{k_0}{12}E_2F$ form a basis for $M_{k_0}(\rho)$.

In chapter four, we show that F satisfies an ordinary differential equation on the complex upper half-plane whose coefficients are modular forms on $\Gamma_0(2)$. We then use a Hauptmodul of $\Gamma_0(2)$, which we denote by $\mathfrak{J}$, to transform this differential equation into a second order ordinary differential equation on $\mathbf{P}^1(\mathbf{C})$. The singularities of this differential equation occur at $0, 1$, and ∞ and they are all regular. A differential equation of this form can be solved explicitly using the Gaussian hypergeometric function. In Theorem 4.1.7, we express the two component functions of F in terms of the Dedekind η-function, the Hauptmodul $\mathfrak{J}$, and the Gaussian hypergeometric function evaluated at $\mathfrak{J}^{-1}$. In the appendix, we prove that the function $\mathfrak{J}$ is a Hauptmodul and compute its first and second derivatives. These properties are used in chapter four. We also establish a certain integrality property related to $\mathfrak{J}$. This property is used in chapter five.

In chapter five, we study the arithmetic properties of the Fourier coefficients of vector-valued modular forms with respect to ρ. To do so, we put some stipulations on ρ to ensure that a certain normalization F' of the component functions of F have Fourier coefficients which are algebraic numbers. In Theorem 5.2.14, we show that if two sets of prime numbers S and $\widetilde{S}$ that are determined by ρ are infinite then the denominators of the Fourier coefficients of each of the component functions of F' are unbounded. In Theorem 5.2.17, we prove that for a certain class of representations ρ, S and $\widetilde{S}$ are infinite. Consequently, each of the sequences of denominators of the Fourier coefficients of the component functions of F' are unbounded. We show in Theorem 5.3.7 that if S and $\widetilde{S}$ are infinite and if X is any vector-valued modular form for ρ whose component functions have Fourier coefficients that are algebraic numbers then the sequence of the denominators of the Fourier coefficients of each of these component functions are unbounded. In Theorem 5.3.8, we prove that for a certain class of representations ρ, every vector-valued modular form for ρ whose Fourier coefficients are algebraic numbers has the property the sequence of the the denominators of the Fourier coefficients of each of its component functions are unbounded.

Chapter 2

The module of vector-valued modular forms

2.1 Preliminaries on vector-valued modular forms

Let $\mathfrak{H}$ denote the complex upper-half plane, let $k \in \mathbf{Z}$, and let $\Gamma = \mathrm{SL}_2(\mathbf{Z})$. Let $T := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

If $F : \mathfrak{H} \to \mathbf{C}^t$ is a holomorphic function and if $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$ then

$$F|_k\gamma(\tau) := (c\tau + d)^{-k} F\left(\frac{a\tau + b}{c\tau + d}\right).$$

In this way, Γ acts on holomorphic functions on $\mathfrak{H}$. Indeed, if $\gamma_1, \gamma_2 \in \Gamma$ then $F|_k\gamma_1\gamma_2 = (F|_k\gamma_1)|_k\gamma_2$. Let H denote a finite index subgroup of Γ and let ρ denote a finite-dimensional complex representation of H. Let d denote the dimension of ρ.

Definition 2.1.1. *A **vector-valued modular form** F of weight k with respect to ρ is a holomorphic function $F : \mathfrak{H} \to \boldsymbol{C}^d$ which is also **holomorphic at all of the cusps of** $H\backslash(\mathfrak{H} \bigcup \mathbb{P}^1(\boldsymbol{Q}))$ and such that for all $\tau \in \mathfrak{H}$ and for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H$,*

$$F\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \rho\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) F(\tau). \quad (1)$$

The above equation can also be expressed as $F|_k\gamma = \rho(\gamma)F$ for all $\gamma \in H$. We now describe what it means for F to be holomorphic at a cusp of $H\backslash(\mathfrak{H} \bigcup \mathbb{P}^1(\mathbf{Q}))$. Our exposition closely follows [10]. As H is a finite index subgroup of Γ, the subgroup $\bigcap_{\gamma\in\Gamma} \gamma^{-1}H\gamma$ is a finite index subgroup of Γ. Therefore there exists a smallest positive integer N for which $T^N \in \bigcap_{\gamma\in\Gamma} \gamma^{-1}H\gamma$. We now fix some $\gamma \in \Gamma$ and we will explain what it means for F to be holomorphic at the cusp $\gamma \cdot \infty$. Let $h \in H$ such that $\gamma T^N = h\gamma$. Then

$$(F|_k\gamma)|_k T^N = F|_k(\gamma T^N) = F|_k(h\gamma) = (F|_k h)|_k\gamma = (\rho(h)F)|_k\gamma = \rho(h)(F|_k\gamma).$$

Let A be an invertible matrix such that $A\rho(h)A^{-1}$ is in *modified* Jordan canonical form. A matrix is in *modified* Jordan canonical form if it is a block diagonal matrix whose blocks are of the form

$$\begin{bmatrix} \lambda & & & \\ \lambda & \ddots & & \\ & \ddots & \ddots & \\ & & \lambda & \lambda \end{bmatrix}.$$

The number λ is an eigenvalue of $\rho(h)$. We note that

$$((AF)|_k\gamma)|_k T^N = A(F|_k\gamma T^N) = A\rho(h)(F|_k\gamma) = A\rho(h)A^{-1}(AF)|_k\gamma.$$

Mason and Knopp have proven (see Theorem 2.2 in [15]) that the component functions of $(AF)|_k\gamma$ corresponding to the block above (whose row and column size we denote by m) have the form

$$\begin{bmatrix} h_1 \\ h_2 + \tau h_1 \\ h_3 + \tau h_2 + \binom{\tau}{2} h_1 \\ \vdots \\ h_m + \tau h_{m-1} + \binom{\tau}{2} h_{m-2} + \cdots + \binom{\tau}{m-1} h_1 \end{bmatrix}$$

where

$$h_i(\tau) = \sum_{n\in\mathbf{Z}} a_n(i) q_N^{n+\mu}, \lambda = e^{2\pi i\mu}, q_N = e^{\frac{2\pi i\tau}{N}}.$$

We say that F (or equivalently AF) is **meromorphic, holomorphic, or cuspidal** at the cusp $\gamma\cdot\infty$ if the q_N-expansion of each h_i has respectively only finitely many nonzero coefficients a_n for which $\mathrm{Re}(n+\mu) < 0$, no nonzero coefficients a_n for which $\mathrm{Re}(n+\mu) < 0$, and no nonzero coefficients a_n for which $\mathrm{Re}(n+\mu) \leq 0$. We note that this definition is independent of the choice of μ. If at least one of the component functions of AF contains a term with a nonzero power of τ then we say that AF and F are **logarithmic vector valued modular forms**.

2.2 The module of vector-valued modular forms

We denote the collection of all weight k vector-valued modular forms with respect to ρ by $M_k(\rho)$. We let $M(\rho) := \bigoplus_{k\in\mathbf{Z}} M_k(\rho)$. We emphasize that every vector-valued modular form for ρ has a weight. Therefore the homogeneous elements of $M(\rho)$ are exactly the vector-valued modular forms for ρ. Let $M_t(H)$ denote the collection of all classical weight t modular forms on H and let $M(H) := \bigoplus_{t\in\mathbf{Z}} M_t(H)$. The ring of classical modular forms on H acts on $M(\rho)$. In fact, if $m \in M_t(H)$ and if $F \in M_k(\rho)$ then for any $\gamma \in H$, $(mF)|_{k+t}\gamma = m|_t\gamma F|_k\gamma = m\rho(\gamma)F = \rho(\gamma)(mF)$. Thus $mF \in M_{k+t}(\rho)$. In this way, $M(\rho)$ has the structure of a $\mathbf{Z}$-graded $M(H)$-module. If $H = \Gamma$ then the $M(H)$-module structure of $M(\rho)$ is completely understood:

Theorem 2.2.1. *(Marks-Mason, Gannon, Franc-Candelori) Let ρ denote a representation of Γ. Then $M(\rho)$ is a free $M(\Gamma)$-module of rank equal to the dimension of ρ.*

Theorem 2.2.1 was proven by Chris Marks and Geoff Mason using vector-valued Poincaré series ([20]), by Terry Gannon using a Riemann-Hilbert perspective ([12]), and by Cameron Franc and Luca Candelori using an algebro-geometric approach ([6]). In general, $M(\rho)$ is not free as a $M(H)$-module. The purpose of the rest of this chapter is to prove the following two theorems concerning the $M(H)$-module structure of $M(\rho)$. We note that one may also obtain these theorems from the works of Franc and Candelori (see [6], [5]) but using different methods than those in this thesis.

Theorem 2.2.2. *Let H denote a finite index subgroup of $SL_2(\mathbf{Z})$ and let ρ denote a representation of H. Then $M(\rho)$ is Cohen-Macaulay as a $M(H)$-module.*

Theorem 2.2.3. *Let H denote a finite index subgroup of $SL_2(\mathbf{Z})$ and let ρ denote a representation of H. Suppose that there exist homogenous elements X and Y in $M(H)$ which are algebraically independent such that $M(H) = \mathbf{C}[X,Y]$. Then $M(\rho)$ is a free $M(H)$-module whose rank equals the dimension of ρ. Moreover, there exists a $M(H)$-basis for $M(\rho)$ which consists of homogeneous elements of $M(\rho)$.*

The hypothesis of Theorem 2.2.3 is satisfied for only a few subgroups of Γ. In [21], Wagreich classified those finitely-generated Fuchsian subgroups for which the graded ring of modular forms is generated as an algebra by two or three elements. He then determines the algebra structure of all such graded rings of modular forms. Examples of subgroups H which satisfy the hypothesis of Theorem 2.2.3 are $\Gamma, \Gamma_0(2), \Gamma(2)$, and $\Gamma(4)$.

We will use the following Lemma in the proof of Theorem 2.2.2 and Theorem 2.2.3.

Lemma 2.2.4. *Let $Ind_H^\Gamma(\rho)$ denote the induction of the representation ρ from H to Γ. Then $M(\rho)$ and $M(Ind_H^\Gamma(\rho))$ are isomorphic as $\mathbf{Z}$-graded $M(\Gamma)$-modules. In particular, for all $k \in \mathbf{Z}$, $M_k(\rho) \cong M_k(Ind_H^\Gamma(\rho))$.*

We postpone the proof of Lemma 2.2.4 until the end of the chapter. It was proven by Geoff Mason and Marvin Knopp in [17] that if $H = \Gamma$ and $\rho(T)$ has finite order then $M_k(\rho)$ has finite dimension, $M_k(\rho) = 0$ if $k << 0$, and $M(\rho) \neq 0$. These results of Mason and Knopp combined with Lemma 2.2.4 imply that for any representation ρ of H for which $\rho(T^N)$ has finite order, we have that $M_k(\rho)$ has finite dimension, $M_k(\rho) = 0$ if $k << 0$, and $M(\rho) \neq 0$. Luca Candelori and Cameron Franc (see [6], [5]) describe how to interpret $M_k(\rho)$ as the space of global sections of a certain holomorphic vector bundle. This interpretation allows them to show that for any representation ρ, $M_k(\rho)$ has finite dimension, $M_k(\rho) = 0$ if $k << 0$, and $M(\rho) \neq 0$. In particular, there exists a unique integer k such that $M_k(\rho) \neq 0$ and $M_j(\rho) = 0$ if $j < k$.

We now recall some definitions and theorems from commutative algebra that we will use in our proofs of Theorem 2.2.2 and Theorem 2.2.3. Our reference is Benson's text (see [2].)

Let A denote a commutative Noetherian ring and let M denote a finitely generated A-module.

Definition 2.2.5. *An element $a \in A$ is a **non-zero-divisor** for M if $am = 0$ for $m \in M$ implies $m = 0$.*

Definition 2.2.6. *An element $a \in A$ is **regular** for M provided that $0 \neq M$, $M \neq aM$, and a is a non-zero-divisor for M.*

Definition 2.2.7. *A sequence $x_1, ..., x_r \in A$ is a **regular sequence** for M if x_1 is regular for M and if for all i with $2 \leq i \leq r$, x_i is regular for $M/(x_1M + ... + x_{i-1}M)$.*

Definition 2.2.8. *The **depth of the module** M is the length of the longest regular sequence for M. The **depth of the ring** A is its depth as an A-module.*

Definition 2.2.9. *The **Krull dimension of a commutative ring** A is the maximum length n of a chain of proper inclusions of prime ideals $\mathfrak{p}_n \subset \mathfrak{p}_{n-1} \subset \cdots \subset \mathfrak{p}_1 \subset \mathfrak{p}_0$ or ∞ if there are such chains of unbounded length.*

Definition 2.2.10. *If M is an A-module, the **Krull dimension of the module** M is defined to be the Krull dimension of the ring $A/Ann_A(M)$ where $Ann_A(M) := \{a \in A : aM = 0\}$.*

Definition 2.2.11. *The ring A or the module M is **Cohen-Macaulay** if its depth is equal to its Krull dimension.*

Definition 2.2.12. *Let A and B denote commutative rings such that $A \subset B$. We say that B is an **integral extension** of A if every element of B is integral over A. If B is an integral extension of A and finitely generated over A as a ring then we say that B is a **finite extension** of A.*

We shall use the following result from commutative algebra. For a proof, see Corollary 1.4.5 in Benson [2].

Theorem 2.2.13. *If B is a finite extension of A then the Krull dimensions of A and B are equal.*

Let $E_4 = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n \in M_4(\Gamma)$ and let $\Delta = q(1-q^n)^{24} \in S_{12}(\Gamma)$.

Lemma 2.2.14. *Let H denote a finite index subgroup of $SL_2(\mathbf{Z})$ and let ρ denote a representation of H. The sequence Δ, E_4 is a regular sequence for the $M(H)$-module $M(\rho)$.*

Proof. It is clear that Δ is a non-zero-divisor for $M(\rho)$ since Δ has no zeros in $\mathfrak{H}$. To prove that Δ is regular for $M(\rho)$, it suffices to show that $M(\rho) \neq \Delta M(\rho)$. Suppose that $M(\rho) = \Delta M(\rho)$. Let X denote a non-zero element in $M(\rho)$ of minimal weight. Then $X = \Delta V$ for some $V \in M(\rho)$. The weight of V is less than weight of X. This is a contradiction because of the minimality of the weight of X. Hence $M(\rho) \neq \Delta M(\rho)$. We have thus shown that Δ is regular for $M(\rho)$. We will now show that E_4 is regular for $M(\rho)/\Delta M(\rho)$. We have already shown that $M(\rho)/\Delta M(\rho) \neq 0$. We now argue that E_4 is non-zero-divisor for the module $M(\rho)/\Delta M(\rho)$. Suppose that $Y \in M(\rho)$ and $E_4(Y + \Delta M(\rho)) = \Delta M(\rho)$. Then $E_4 Y \in \Delta M(\rho)$. We write $E_4 Y = \Delta Z$ for some $Z \in M(\rho)$. We wish to show that $Y \in \Delta M(\rho)$ and it suffices to show that this is true when Y is homogeneous. Let k denote the weight of Y. Let y_i denote the i-th component function of Y, let z_i denote the i-th component function of Z, and let $\gamma \in \Gamma$. Therefore $E_4 y_i = \Delta z_i$ and $E_4(y_i|_k\gamma) = \Delta(z_i|_{k-8}\gamma)$. As $\Delta = q + O(q^2)$ and $E_4 = 1 + O(q)$, all the powers of q in $y_i|_k\gamma$ occur to at least the first power. We have thus shown that $\Delta^{-1}(y_i|_k\gamma)$ contains no negative powers of q and is therefore holomorphic at the cusp $\gamma\cdot\infty$. Hence $\frac{Y}{\Delta}$ is holomorphic at all of the cusps. As Δ does not vanish in $\mathfrak{H}$, we have that $\frac{Y}{\Delta}$ is holomorphic in $\mathfrak{H}$. Hence $\frac{Y}{\Delta} \in M(\rho)$. Therefore $Y \in \Delta M(\rho)$ and $Y + \Delta M(\rho) = \Delta M(\rho)$.

We have now proven that E_4 is a non-zero-divisor for the module $M(\rho)/\Delta M(\rho)$. Finally, we need to show that $E_4(M(\rho)/\Delta M(\rho)) \neq M(\rho)/\Delta M(\rho)$. Let X denote a nonzero element in $M(\rho)$ of minimal weight, which we denote by w. If $M(\rho)/\Delta M(\rho) = E_4(M(\rho)/\Delta M(\rho))$ then there exists some $F \in M(\rho)$ such that $X + \Delta M(\rho) = E_4 F + \Delta M(\rho)$. Let $G \in M(\rho)$ such that $X = E_4 F + \Delta G$. We may write F and G uniquely as a sum of their homogeneous components. Let F_{w-4} and G_{w-12} denote the weight $w-4$ and the weight $w-12$ homogeneous components of F and G. Then $X = E_4 F_{w-4} + \Delta G_{w-12}$. We must have that $F_{w-4} \neq 0$ or $G_{w-12} \neq 0$ since $X \neq 0$. Thus we have found a nonzero element of $M(\rho)$ (namely, F_{w-4} or G_{w-12}) whose weight is less than the weight of X. This is a contradiction. Thus $M(\rho)/\Delta M(\rho) \neq E_4(M(\rho)/\Delta M(\rho))$. We have shown that E_4 is regular for $M(\rho)/\Delta M(\rho)$ and our proof is complete. $\square$

The proofs of Theorem 2.2.2 and Theorem 2.2.3 require some results about $M(H)$. We prove them now.

Lemma 2.2.15. *If H is a finite index subgroup of Γ then $M(H)$ is a free $M(\Gamma)$-module whose rank equals $[\Gamma : H]$.*

Proof. Let α denote the trivial representation of H. Lemma 2.2.4 implies that $M(\alpha) = M(H)$ and $M(\mathrm{Ind}_H^\Gamma \alpha)$ are isomorphic $M(\Gamma)$-modules. Moreover, Theorem 2.2.1 implies that $M(\mathrm{Ind}_H^\Gamma \alpha)$ is a free $M(\Gamma)$-module of rank equal to $\dim \mathrm{Ind}_H^\Gamma \alpha = [\Gamma : H]$. Thus $M(H)$ is a free $M(\Gamma)$-module of rank $[\Gamma : H]$. $\square$

Lemma 2.2.16. *If H is a finite index subgroup of Γ then $M(H)$ is a Noetherian ring.*

Proof. Lemma 2.2.15 states that there exist $b_1, \ldots, b_{[\Gamma:H]} \in M(H)$ such that $M(H) = \bigoplus_{i=1}^{[\Gamma:H]} M(\Gamma) b_i = M(\Gamma)[b_1, \ldots, b_{[\Gamma:H]}] = \mathbf{C}[E_4, E_6][b_1, \ldots, b_{[\Gamma:H]}]$. Let $X_1, \ldots, X_{[\Gamma:H]}$ denote indeterminates.

Let $\phi : \mathbf{C}[E_4,E_6][X_1,...,X_{[\Gamma:H]}] \to \mathbf{C}[E_4,E_6][b_1,...,b_{[\Gamma:H]}]$ be the map that sends each X_i to b_i. The surjectivity of ϕ implies that $\mathbf{C}[E_4,E_6][b_1,...,b_{[\Gamma:H]}] = M(H)$ is a quotient of $\mathbf{C}[E_4,E_6][X_1,...,X_{[\Gamma:H]}]$. The ring $\mathbf{C}[E_4,E_6][X_1,...,X_{[\Gamma:H]}]$ is Noetherian and therefore any quotient of it is also Noetherian. We have thus shown that $M(H)$ is Noetherian. $\square$

Lemma 2.2.17. *$M(H)$ is a finite extension of $M(\Gamma)$. Moreover, $M(H)_0 = \mathbf{C}$ and $M(H)$ is finitely generated as a $\mathbf{C}$-algebra.*

Proof. We need to show that $M(H)$ is an integral extension of $M(\Gamma)$ and that $M(H)$ is a finite ring extension of $M(\Gamma)$. We see that $M(H)$ is a finite ring extension of $M(\Gamma)$ because $M(H) = \mathbf{C}[E_4,E_6][b_1,...,b_{[\Gamma:H]}] = \mathbf{C}[E_4,E_6,b_1,...,b_{[\Gamma:H]}]$. We now show that $M(H)$ is an integral extension of $M(\Gamma)$. Let $\{\gamma_i : 1 \le i \le [\Gamma : H]\}$ denote a complete set of right coset representatives of H in $\mathrm{SL}_2(\mathbf{Z})$ where γ_1 denotes the identity element in $\mathrm{SL}_2(\mathbf{Z})$. If $f \in M_k(H)$ then f is a root of the polynomial $P(z) := \prod_{i=1}^{[\Gamma:H]}(z - f|_k\gamma_i)$. The polynomial $P(z)$ is monic and we will show that it is an element of $M(\mathrm{SL}_2(\mathbf{Z}))[z]$. Let $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ and let $\gamma_{\sigma(i)}$ denote the unique element in the set $\{\gamma_i : 1 \le i \le [\Gamma : H]\}$ such that $\gamma\gamma_i \in H\gamma_{\sigma(i)}$. Let $P(z)|_k\gamma$ denote the polynomial obtained by acting via $|_k\gamma$ on each of the coefficients of $P(z)$. We have that $P(z)|_k\gamma = \prod_{i=1}^{[\Gamma:H]}(z - f|_k\gamma_i\gamma) = \prod_{i=1}^{[\Gamma:H]}(z - f|_k\gamma_{\sigma(i)}) = P(z)$. Thus $P(z) \in M(\mathrm{SL}_2(\mathbf{Z}))[z]$ since its coefficients are invariant under $|_k\gamma$ for any $\gamma \in \mathrm{SL}_2(\mathbf{Z})$. We have thus shown that $M(H)$ is a finite extension of $M(\mathrm{SL}_2(\mathbf{Z}))$.

The holomorphic modular forms of weight zero for H give maps from the compact Riemann surface $H\backslash(\mathfrak{H}\bigcup\mathbb{P}^1(\mathbf{Q}))$ to $\mathbf{C}$. Any such map is bounded and is therefore constant.

Thus $M(H)_0 = \mathbf{C}$. The fact that $M(H) = \mathbf{C}[E_4, E_6, b_1, ..., b_{[\Gamma:H]}]$ demonstrates that $M(H)$ is finitely generated as a $\mathbf{C}$-algebra. □

Lemma 2.2.18. *The Krull dimension of the $M(H)$-module $M(\rho)$ is equal to two.*

Proof. The Krull dimension of the $M(H)$-module $M(\rho)$ is defined to be the Krull dimension of the ring $M(H)/\mathrm{Ann}_{M(H)}(M(\rho))$. As the zeros of a nonzero holomorphic function are isolated, $\mathrm{Ann}_{M(H)}M(\rho) = 0$. Therefore the Krull dimension of $M(\rho)$ is equal to the Krull dimension of $M(H)$. We have shown in Lemma 2.2.17 that $M(H)$ is a finite extension of $M(\mathrm{SL}_2(\mathbf{Z}))$. It now follows from Theorem 2.2.13 that the Krull dimension of $M(H)$ is equal to the Krull dimension of $M(\mathrm{SL}_2(\mathbf{Z}))$. We conclude our proof by noting that the Krull dimension of $M(\mathrm{SL}_2(\mathbf{Z})) = \mathbf{C}[E_4, E_6]$, is equal to two since the Krull dimension of a polynomial ring in two variables is equal to two. □

We now prove that $M(\rho)$ is Cohen-Macaulay as a $M(H)$-module (Theorem 2.2.2).

Proof. (**Proof of Theorem 2.2.2.**)

We have shown that the Krull dimension of the $M(H)$-module $M(\rho)$ is equal to two and that $M(\rho)$ has a regular sequence of length two. Therefore the depth of $M(\rho)$ is at least two. Moreover, the depth is at most the Krull dimension (see page 50 in [2]), which is equal to two. Hence the depth and the Krull dimension of $M(\rho)$ are both equal to two. □

We need the following result from commutative algebra to prove Theorem 2.2.3. This result is stated and proven in Benson's book [2].

Theorem 2.2.19. *(Theorem 4.3.5. in [2]) Let A denote a commutative Noetherian ring and let M denote a finitely generated A-module. Assume that* $A = \bigoplus_{j=0}^{\infty} A_j$ *and* $M = \bigoplus_{j=-\infty}^{\infty} M_j$ *are graded,* $A_0 = K$ *is a field, and A is finitely generated over K by elements of positive degree. Then the following statements are equivalent:*

(i): M is Cohen-Macaulay.

(ii): If $x_1, \ldots x_n \in A$ *are homogenous elements generating a polynomial subring* $K[x_1, \ldots, x_n] \subset A/Ann_A(M)$*, over which M is finitely generated, then M is a free* $K[x_1, \ldots, x_n]$*-module. Moreover, there exists a* $K[x_1, \ldots, x_n]$*-basis for M consisting of homogeneous elements of M.*

Remark 1: We first note that if j is sufficiently negative then $M_j = 0$ since M is finitely generated as an A-module. The fact that there exists a $K[x_1, \ldots, x_n]$-basis for M consisting of *homogeneous* elements of M follows from the proof of Theorem 4.3.5 in [2] but this fact does not appear (at least explicitly) in the statement of Theorem 4.3.5 in [2]. In the proof that (i) implies (ii) in Theorem 4.3.5, Benson shows that if $y_1, \ldots, y_t$ are homogeneous elements of M whose images form a K-vector space basis for $M/(x_1M + \cdots + x_nM)$ then $y_1, \ldots, y_t$ form a basis for M as a $K[x_1, \ldots, x_n]$-module. We now explain why there exist homogeneous elements $y_1, \ldots, y_t$ of M whose images form a K-vector space basis for $M/(x_1M + \cdots + x_nM)$. Benson shows in his proof of Theorem 4.3.5 that $M/(x_1M + \cdots + x_nM)$ has Krull dimension zero. We note that $M/(x_1M + \cdots + x_nM)$ is a finitely generated A-module since M is a finitely generated A-module. We also note that $M/(x_1M + \cdots + x_nM)$ is a graded A-module since M is a graded A-module and each of the x_i are homogeneous. It follows from the graded form of Noether normalization (Theorem 2.2.7 in Benson [2]) and the fact that $M/(x_1M + \cdots + x_nM)$ has Krull dimension zero that $K \subset A/\mathrm{Ann}_A(M/(x_1M+\cdots+x_nM))$ and $M/(x_1M+\cdots+x_nM)$ is a finitely generated K-module. Because K is a field, $M/(x_1M+\cdots+x_nM)$ is a K-vector space of finite dimension. The fact that $M_0 = K \subset A/\mathrm{Ann}_A(M/(x_1M+\cdots+x_nM))$ together with the fact that $M/(x_1M+\cdots+x_nM)$ is a graded A-module imply that $M/(x_1M+\cdots+x_nM)$ is a graded K-module. Thus $M/(x_1M+\cdots+x_nM)$ is a graded K-vector space of finite dimension. Therefore there exist a finite number of homogeneous elements in M whose images form a K-vector space basis for $M/(x_1M+\cdots+x_nM)$. This finite collection of homogenous elements in M form a $K[x_1,...,x_n]$-basis for M.

Remark 2**:** If $x_1,...,x_n \in A$ are homogenous elements generating a polynomial subring $K[x_1,...,x_n] \subset A/\mathrm{Ann}_A(M)$ over which M is finitely generated then n is equal to the Krull dimension of M. This is explained in the graded form of Noether normalization (Theorem 2.2.7 in Benson [2]).

We will apply Theorem 2.2.19 by taking $A = M(H)$ and $M = M(\rho)$. We proceed with the proof of Theorem 2.2.3.

Proof. (**Proof of Theorem 2.2.3**) We begin by showing that the hypotheses of Theorem 2.2.19 are satisfied if we take $A := M(H)$ and $M := M(\rho)$. It was shown in Lemma 2.2.16 that $M(H)$ is a Noetherian ring. We need to show that $M(\rho)$ is finitely generated as a $M(H)$-module. Theorem 2.2.1 implies that $M(\mathrm{Ind}_H^\Gamma\rho)$ is a free $M(\Gamma)$-module whose rank equals the dimension of $\mathrm{Ind}_H^\Gamma\rho$. Lemma 2.2.4 states that $M(\rho)$ and $M(\mathrm{Ind}_H^\Gamma\rho)$ are isomorphic as $M(\Gamma)$-modules. Thus $M(\rho)$ is a free $M(\Gamma)$-module of rank $\dim(\mathrm{Ind}_H^\Gamma\rho)$. As $M(H)$ contains $M(\Gamma)$ and the index of H in $\mathrm{SL}_2(\mathbf{Z})$ is finite, $M(\rho)$ is finitely generated as a $M(H)$-module. We have shown in Lemma

2.2.17 that $M(H)_0 = \mathbf{C}$ and that $M(H)$ is finitely generated as a $\mathbf{C}$-algebra. In fact, the hypothesis of Theorem 2.2.3 assumes that there exist homogeneous elements X and Y in $M(H)$ which are algebraically independent such that $M(H) = \mathbf{C}[X,Y]$. It is also well-known that there are no nonzero modular forms of negative weight on a finite index subgroup of Γ. We have thus shown that the hypotheses of Theorem 2.2.19 are satisfied when A is taken to be $M(H)$ and M is taken to be $M(\rho)$.

We established in Theorem 2.2.2 that $M(\rho)$ is Cohen-Macaulay as a $M(H)$-module. We now apply Theorem 2.2.19 to conclude that statement (ii) in Theorem 2.2.19 is true since we have shown that statement (i) in Theorem 2.2.19 is true. We have explained in the previous paragraph that $M(\rho)$ is finitely generated over $M(H) = \mathbf{C}[X,Y]$. Thus the hypothesis of statement (ii) in Theorem 2.2.19 is satisfied by X and Y. Therefore the conclusion of statement (ii) is also true since we have proven that statement (ii) is true. Thus $M(\rho)$ is free as a $\mathbf{C}[X,Y] = M(H)$-module.

We seek to compute the rank of $M(\rho)$ as a $M(H)$-module, which we denote by r. We have proven in Lemma 2.2.15 that $M(H)$ is a free $M(\Gamma)$-module of rank $[\Gamma : H]$. This fact together with the fact that $M(\rho)$ is a free $M(H)$-module of rank r implies that $M(\rho)$ is a free $M(\Gamma)$-module of rank $r[\Gamma : H]$. However, we already explained in the beginning of this proof why $M(\rho)$ is a free $M(\Gamma)$-module of rank equal to $\dim \mathrm{Ind}_H^\Gamma \rho = [\Gamma : H] \dim \rho$. Hence $[\Gamma : H] \dim \rho = [\Gamma : H] r$ and $r = \dim \rho$. We have thus shown that $M(\rho)$ is free as a $M(H)$-module of rank $\dim \rho$ if $M(H)$ is a polynomial ring in two variables. □

It remains to prove Lemma 2.2.4. Let n denote the index of H in Γ and let $g_1,...,g_n$ denote a complete set of left coset representatives of H in Γ. We extend ρ to a function on Γ, which we also denote by ρ, by setting $\rho(\gamma)=0$ if $\gamma\notin H$. With respect to our choice of left coset representatives, we recall that for any $g\in H$,

$$\mathrm{Ind}_H^\Gamma\rho(g)=\begin{bmatrix}\rho(g_1^{-1}gg_1) & \rho(g_1^{-1}gg_2) & \cdots & \rho(g_1^{-1}gg_n)\\ \rho(g_2^{-1}gg_1) & \rho(g_2^{-1}gg_2) & \cdots & \rho(g_2^{-1}gg_n)\\ \vdots & \vdots & \ddots & \vdots\\ \rho(g_n^{-1}gg_1) & \rho(g_n^{-1}gg_2) & \cdots & \rho(g_n^{-1}gg_n)\end{bmatrix}.$$

In this section, we sometimes write Ind ρ for $\mathrm{Ind}_H^\Gamma\rho$.

Definition 2.2.20. *Let $k\in\mathbf{Z}$ and let $F\in M_k(\rho)$. We define*

$$\mathit{Ind}\,F:=\begin{bmatrix}F|_k g_1^{-1}\\ F|_k g_2^{-1}\\ \cdot\\ \cdot\\ \cdot\\ F_k|g_n^{-1}\end{bmatrix}$$

In what follows, it will be convenient to take g_1 to be the identity element and we take $g_1=e$

from this point forwards. We define a map $\pi : M_k(\mathrm{Ind}\,\rho) \to M_k(\rho)$ as follows:

$$\pi\left(\begin{bmatrix} G_1 \\ G_2 \\ \cdot \\ \cdot \\ \cdot \\ G_n \end{bmatrix}\right) := G_1.$$

where each G_i is a function from $\mathfrak{H}$ to $\mathbf{C}^d$. We recall that $d = \dim \rho$. We extend the map Ind to all of $M(\rho)$ and the map π to all of $M(\mathrm{Ind}\,\rho)$ by linearity. We shall prove the following theorem about the map Ind.

Theorem 2.2.21. *If $k \in \mathbf{Z}$ then the the map Ind $: M_k(\rho) \to M_k(\mathit{Ind}\,\rho)$ is a vector space isomorphism. The map Ind extends to $M(\rho)$ and is a $\mathbf{Z}$-graded $M(\Gamma)$-module isomorphism from $M(\rho)$ to $M(\mathit{Ind}\,\rho)$.*

We will first prove some results about the maps π and Ind before we prove Theorem 2.2.21.

Proposition 2.2.22. *If $F \in M_k(\rho)$ then $\mathit{Ind}_H^\Gamma F \in M_k(\mathit{Ind}_H^\Gamma\,\rho)$.*

Proof. We note that as F is holomorphic in $\mathfrak{H}$ then for each $\gamma \in \Gamma$, $F|_k\gamma$ is holomorphic in $\mathfrak{H}$ and therefore $\mathrm{Ind}_H^\Gamma F$ is holomorphic in $\mathfrak{H}$. Let N denote the smallest positive integer for which $T^N \in \bigcap_{\gamma\in\Gamma} \gamma^{-1}H\gamma$ and let $q_N = e^{\frac{2\pi i\tau}{N}}$. As F is holomorphic at all the cusps of $H/(\mathfrak{H}\bigcup\mathbb{P}^1(\mathbf{Q}))$, we have that for each $\gamma \in \Gamma$, no negative powers of q_N occur in the component functions of $F|_k\gamma$. Thus $\mathrm{Ind}_H^\Gamma F$ is holomorphic at the cusp $\mathrm{SL}_2(\mathbf{Z})/(\mathfrak{H}\bigcup\mathbb{P}^1(\mathbf{Q}))$. It now suffices to prove that for each $g \in \Gamma$, $(\mathrm{Ind}_H^\Gamma F)|_k g = (\mathrm{Ind}_H^\Gamma \rho)(g)\mathrm{Ind}_H^\Gamma F$. Equivalently, we will show that for each integer i with $1 \le i \le n$,

$$(F|_k g_i^{-1})|_k g = \sum_{t=1}^{n} \rho(g_i^{-1} g g_t) F|_k g_t^{-1}.$$

Fix an index i with $1 \le i \le n$. Then there exists a unique index j for which $g_i^{-1} g g_j \in H$. We then have that $F|_k g_i^{-1} g g_j = \rho(g_i^{-1} g g_j) F$. Therefore

$$F|_k g_i^{-1} g = (F|_k g_i^{-1} g g_j)|_k g_j^{-1} = (\rho(g_i^{-1} g g_j) F)|_k g_j^{-1} = \rho(g_i^{-1} g g_j)(F|_k g_j^{-1}).$$

We now have that

$$\sum_{t=1}^{n} \rho(g_i^{-1} g g_t) F|_k g_t^{-1} = \rho(g_i^{-1} g g_j) F|_k g_j^{-1} = F|_k g_i^{-1} g = (F|_k g_i^{-1})|_k g.$$

We have thus proven that $(\mathrm{Ind}_H^\Gamma F)|_k g = (\mathrm{Ind}_H^\Gamma \rho)(g)\mathrm{Ind}_H^\Gamma F$.

□

Proposition 2.2.23. *If* $F \in M_k(\mathit{Ind}\,\rho)$ *then* $\pi(F) \in M_k(\rho)$.

Proof. Let $F \in M_k(\mathrm{Ind}\,\rho)$ and let $F_1, \ldots, F_n$ denote the functions from $\mathfrak{H}$ to $\mathbf{C}^d$ such that

$$F = \begin{bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_n \end{bmatrix}.$$

We note that $\pi(F) = F_1$. Let $g \in H$. There exists a unique index j for which $g_1^{-1} g g_j \in H$. As $g_1 = e$ and $g \in H$, we must have that $g_j = e$. The assumption that $F \in M_k(\mathrm{Ind}\,\rho)$ implies that

$$F_1|_k g = \sum_{t=1}^{n} \rho(g_1^{-1} g g_t) F_t = \rho(e^{-1} g e) F_1 = \rho(g) F_1.$$

As F is a holomorphic function in $\mathfrak{H}$, F_1 is also holomorphic in $\mathfrak{H}$. Let N denote the smallest positive integer for which $T^N \in \bigcap_{\gamma \in \Gamma} \gamma^{-1} H \gamma$ and let $q_N = e^{\frac{2\pi i \tau}{N}}$. As F is holomorphic at the cusp $\mathrm{SL}_2(\mathbf{Z})/(\mathbb{P}^1(\mathbf{Q}) \bigcup \mathfrak{H})$, there are no negative powers of q_N in any of the component functions of F_1. Let $\gamma \in \Gamma$. Then the component functions of $F_1|_k \gamma$ contains no negative powers of q_N since $F_1|_k \gamma = \sum_{t=1}^{n} \rho(\gamma)_{1,t} F_t$ and each of the component functions F_t contain no negative powers of q_N. We have thus shown that F_1 is holomorphic at all of the cusps of $H \backslash (\mathfrak{H} \bigcup \mathbb{P}^1(\mathbf{Q}))$ and conclude that $F_1 \in M_k(\rho)$. $\square$

We observe that $\pi \circ \mathrm{Ind} = \mathrm{id}$. Thus π is surjective.

Proposition 2.2.24. *The map π is a bijection.*

Proof. As π is surjective, it suffices to show that π is injective. We first show that the restriction of π to $M_k(\mathrm{Ind}\,\rho)$ is injective. Let $F \in M_k(\mathrm{Ind}\,\rho)$ such that $\pi(F) = 0$. We write

$$F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix}$$

where each F_i is a function from $\mathfrak{H}$ to $\mathbf{C}^{\dim \rho}$. We claim that $F = 0$. Suppose not. Then there exists some index i with $F_i \neq 0$. Let $g \in g_i H g_1^{-1}$. Then $g_i^{-1} g g_j \in H$ if and only if $g_1 = g_j$. Thus

$F_i|_k g = \sum_{t=1}^n \rho(g_i^{-1} g g_t) F_t = \rho(g_i^{-1} g g_1) F_1$. As $\pi(F) = F_1 = 0$, we have that $F_i|_k g = 0$. Hence $F_i = (F_i|_k g)|_k g^{-1} = 0$, a contradiction. We have thus shown that the restriction of π to $M_k(\mathrm{Ind}\,\rho)$ is injective.

Now, let $F \in M(\mathrm{Ind}\,\rho)$ for which $\pi(F) = 0$. We may write F uniquely as a sum of its homogeneous components: $F = \sum_i G_i, G_i \in M_i(\mathrm{Ind}\,\rho)$. As $\pi(F) = \sum_i \pi(G_i) = 0$, $\pi(G_i) \in M_i(\rho)$, we have that for each i, $\pi(G_i) = 0$. Hence $G_i = 0$ as the restriction of π to each $M_i(\mathrm{Ind}\,\rho)$ is injective. Thus $F = 0$. □

Corollary 2.2.25. *The maps π and Ind are $\mathbf{C}$-vector space isomorphisms and are inverse to each other.*

Proof. The maps π and Ind are linear. We have shown that π is a bijection and it is therefore an isomorphism. As $\pi \circ \mathrm{Ind}$ is the identity map, Ind is the inverse of π and it is therefore an isomorphism. □

We now proceed with the proof of Theorem 2.2.21.

Proof. (**Proof of Theorem 2.2.21**). All that we have left to check is that Ind is a graded $M(\Gamma)$-module map. Let $l, k \in \mathbf{Z}$. If $f \in M_l(\Gamma)$ and $F \in M_k(\rho)$ then we must show that

$$\mathrm{Ind}(fF) = f\mathrm{Ind}\,F.$$

We note that if i is any index then

$$(fF)|_{k+l} g_i^{-1} = f|_l g_i^{-1} F|_k g_i^{-1} = f(F|_k g_i^{-1}).$$

We have that

$$\mathrm{Ind}(fF) = \begin{bmatrix} (fF)|_{k+l}g_1^{-1} \\ (fF)|_{k+l}g_2^{-1} \\ . \\ . \\ . \\ (fF)_{k+l}|g_n^{-1} \end{bmatrix} = \begin{bmatrix} f(F|_k g_1^{-1}) \\ f(F|_k g_2^{-1}) \\ . \\ . \\ . \\ f(F_k|g_n^{-1}) \end{bmatrix} = f\mathrm{Ind}F.$$

□

Chapter 3

The modular derivative and a basis for vector-valued modular forms

We proved in Theorem 2.2.3 that if $M(H)$ is a polynomial ring in two variables then $M(\rho)$ is a free $M(H)$-module whose rank equals the dimension of ρ. It is proven in the appendix that the hypotheses of Theorem 2.2.3 are satisfied when $H = \Gamma_0(2)$. The purpose of this section to use the modular derivative D_k to describe a basis for $M(\rho)$ as a $M(\Gamma_0(2))$-module when ρ is a two-dimensional irreducible representation of $\Gamma_0(2)$.

Let $E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma(n)q^n$. If $k \in \mathbf{Z}$ then D_k acts on holomorphic functions and meromorphic functions from $\mathfrak{H}$ to $\mathbf{C}^n$ as follows:

$$D_k A := \frac{1}{2\pi i}\frac{dA}{d\tau} - \frac{k}{12}E_2 A = q\frac{d}{dq}A - \frac{k}{12}E_2 A$$

The modular derivative D_k has the lovely property (section 10.5 in [18]) that for all $\gamma \in \Gamma$,

$$D_k(A|_k\gamma) = (D_kA)|_{k+2}\gamma.$$

If $F \in M_k(\rho)$ and if $\gamma \in H$ then $(D_kF)|_{k+2}\gamma = D_k(F|_k\gamma) = D_k(\rho(\gamma)F) = \rho(\gamma)D_kF$. Thus $D_kF \in M_{k+2}(\rho)$. Similarly, if $m \in M_k(H)$ then $D_km \in M_{k+2}(H)$. The linear maps $D_k : M_k(\rho) \to M_{k+2}(\rho)$ and $D_k : M_k(H) \to M_{k+2}(H)$ are quite useful. We shall use the notation θ to denote $D_0 = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$. The goal of this section is to prove the following theorem.

Theorem 3.0.1. *Let $\rho : \Gamma_0(2) \to GL_2(\boldsymbol{C})$ be an irreducible representation. Let k denote the least integer for which $M_k(\rho) \neq 0$ and let F denote a nonzero element in $M_k(\rho)$. Then F and D_kF form a basis for $M(\rho)$ as a $M(\Gamma_0(2))$-module.*

We will use the following two results to prove Theorem 3.0.1. We give their proofs immediately after the proof of Theorem 3.0.1.

Theorem 3.0.2. *Let ρ be an irreducible representation of a finite index subgroup H of Γ. Let F be a nonzero vector valued modular form of weight $k \in \mathbf{Z}$ with respect to ρ. Then the component functions of F are linearly independent over $\boldsymbol{C}$.*

Theorem 3.0.3. *If ρ is an irreducible representation of a finite index subgroup H of Γ for which $-I \in H$ then there exists an integer k such that $\rho(-I) = (-1)^kI$ and the weights of all the homogeneous elements in $M(\rho)$ are congruent to k modulo two.*

Let $G(\tau) := -E_2(\tau) + 2E_2(2\tau)$. Then $G \in M_2(\Gamma_0(2))$ (see Example 4 in Chapter IX in [14]) and $E_4 \in M_4(\Gamma_0(2))$. The modular forms G and E_4 are algebraically independent and $M(\Gamma_0(2)) = \mathbf{C}[G, E_4]$. This fact is well-known and we provide a proof of it in the appendix. In particular,

$M_2(\Gamma_0(2)) = \mathbf{C}G$ and $M_4(\Gamma_0(2)) = \mathbf{C}G^2 \bigoplus \mathbf{C}E_4$. Therefore Theorem 2.2.3 implies that $M(\rho)$ is a free $M(\Gamma_0(2))$-module. We now proceed with the proof of Theorem 3.0.1.

Proof. (Proof of Theorem 3.0.1) Let F_1, F_2 be a homogeneous basis for $M(\rho)$. The crux of our proof is to show that the weights of F_1 and F_2 are not equal. We proceed by contradiction and suppose that the weights of F_1 and F_2 are equal. Then there exist $a,b,c,d \in \mathbf{C}$ such that $D_kF_1 = aGF_1 + bGF_2$ and $D_kF_2 = cGF_1 + dGF_2$. We rewrite this pair of equations as follows:

$$\begin{bmatrix} D_kF_1 \\ D_kF_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} G \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.$$

If P an invertible matrix then we have that

$$P \begin{bmatrix} D_kF_1 \\ D_kF_2 \end{bmatrix} = P \begin{bmatrix} a & b \\ c & d \end{bmatrix} G \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = P \begin{bmatrix} a & b \\ c & d \end{bmatrix} P^{-1}GP \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.$$

We may put the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in Jordan canonical form and we now choose P so that $P \begin{bmatrix} a & b \\ c & d \end{bmatrix} P^{-1} = \begin{bmatrix} * & * \\ 0 & \lambda \end{bmatrix}$ for some $\lambda \in \mathbf{C}$. We define the functions A_1 and A_2 by

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = P \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.$$

As P is invertible, the functions A_1 and A_2 are a basis for $M(\rho)$. We now have that

$$\begin{bmatrix} D_kA_1 \\ D_kA_2 \end{bmatrix} = P \begin{bmatrix} D_kF_1 \\ D_kF_2 \end{bmatrix}$$

$$= P \begin{bmatrix} a & b \\ c & d \end{bmatrix} P^{-1} G P \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$= P \begin{bmatrix} a & b \\ c & d \end{bmatrix} P^{-1} G \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

$$= \begin{bmatrix} * & * \\ 0 & \lambda \end{bmatrix} G \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

$$= \begin{bmatrix} * \\ \lambda G A_2 \end{bmatrix}.$$

Thus $D_k A_2 = \lambda G A_2$. Therefore the two component functions of the vector-valued function A_2 satisfy an ordinary differential equation of order one and must be linearly dependent. As A_2 is part of a basis for $M(\rho)$, $A_2 \neq 0$. Theorem 3.0.2 states the component functions of any nonzero vector-valued modular form with respect to an irreducible representation are linearly independent. We have thus shown that the components of A_2 are both linearly dependent and independent, a contradiction. We conclude that the weights of F_1 and F_2 are not equal.

Let k denote the least integer for which $M_k(\rho) \neq 0$ and let $F \in M_k(\rho)$ such that $F \neq 0$. We have shown that the weights of a $M(\Gamma_0(2))$-basis for $M(\rho)$ cannot be equal and therefore $M_k(\rho) = \mathbf{C}F$. We may therefore take F to be an element of a basis for $M(\rho)$. Let B denote a homogenous element in $M(\rho)$ such that F and B form a basis for $M(\rho)$. We claim that the weight of B, which we denote by w, is equal to $k+2$. It follows from Theorem 3.0.3 that $M_{k+1}(\rho) = 0$. Thus $w \geq k+2$. If $w > k+2$ then $D_k F = mF$ for some $m \in M_2(\Gamma_0(2))$. But then the two component functions of F would satisfy an ordinary differential equation of order one and therefore be linearly dependent. This would contradict Theorem 3.0.2 as $F \neq 0$ and ρ is irreducible. Thus the weight of B is $k+2$. We then have that $D_k F = \alpha F + \gamma B$ where $\alpha, \gamma \in M(\Gamma_0(2))$. We observe that $\gamma \in M_0(\Gamma_0(2)) = \mathbf{C}$. If $\gamma = 0$ then $D_k F = \alpha F$ and so the two component functions of F are linearly dependent. The irreducibility of ρ together with Theorem 3.0.2 ensure that the component functions of F are linearly independent. Hence $\gamma \neq 0$. We thus have that $B \in \mathrm{span}_{M(\Gamma_0(2))}(F, D_k F)$. As $M(\rho)$ is spanned by F and B, it is also spanned by F and $D_k F$. Finally, as $M(\rho)$ is a free module of rank two over $M(\Gamma_0(2))$, an integral domain, and F and $D_k F$ span $M(\rho)$, we conclude that F and $D_k F$ form a basis for $M(\rho)$. $\square$

We now give the proof of Theorem 3.0.2.

Proof. (Proof of Theorem 3.0.2) Let $f_1, \ldots, f_n$ denote the components of F and let E denote the $\mathbf{C}$-span of $f_1, \ldots f_n$. We view E as a right H-module via the action: $g \cdot f_i := f_i|_k g$. The fact that E is a H-module is immediate from the fact that F is a vector-valued modular form. Let W denote the right H-module that furnishes ρ. This means that $(w \cdot \gamma_1) \cdot \gamma_2 = w \cdot (\gamma_1 \gamma_2)$ for all $w \in W$ and $\gamma_1, \gamma_2 \in H$ and that there exists a $\mathbf{C}$-basis $e_1, \ldots, e_n$ of W such that for every i, $e_i \cdot \gamma = \sum_{j=1}^{n} \rho(\gamma)_{i,j} e_j$. We define a map $\psi : W \to E$ by setting $\psi(e_i) = f_i$ and extending linearly. We now check that the map ψ is a map of H-modules. Let $g \in H$ and let $g_{i,j}$ denote the i-th row and j-th column entry of $\rho(g)$. We have that $\psi(e_i) \cdot g = f_i \cdot g = f_i|_k g = \sum_{j=1}^{n} \rho(g)_{i,j} f_j = \sum_{j=1}^{n} g_{i,j} \psi(e_i) = \psi(\sum_{j=1}^{n} g_{i,j} e_i) = \psi(e_i \cdot g)$. As ψ is a H-module map, ker ψ is a H-submodule of W. As ρ is irreducible, ker ψ is equal to either 0 or W. As each $f_i = \psi(e_i)$ and $F \neq 0$, we have that $E \neq 0$. Thus ker $\psi \neq W$ and so ψ is injective. It is clear that ψ is surjective and thus ψ

is an isomorphism. Hence the elements $f_1, \ldots, f_n$ are linearly independent over $\mathbf{C}$. $\square$

We end this section with the proof of Theorem 3.0.3.

Proof. (Proof of Theorem 3.0.3) Let W denote the right H-module that furnishes ρ. As $\rho(-I)^2 = 1$, the eigenvalues of $\rho(-I)$ are 1 and -1. Let W^1 denote the $+1$-eigenspace and W^{-1} denote the -1-eigenspace. We note that $\rho(-I)$ is in the center of $\mathrm{Im}\,\rho$ since $-I$ is in the center of H. Hence W^1 and W^{-1} are right H-submodules of W. As ρ is irreducible, every right H-submodule of W is either W or 0. Hence there exists some $k \in \mathbf{Z}$ such that $W^{(-1)^k} = W$ and $W^{(-1)^{k+1}} = 0$. Thus $\rho(-I) = (-1)^k I$. Finally, if B is a nonzero vector-valued modular form of weight j then $(-1)^j B = B|_j - I = \rho(-I)B = (-1)^k B$ and thus j and k have the same parity. $\square$

Chapter 4

Hypergeometric Series

4.1 Differential Equations

From this point forwards, ρ will denote a complex irreducible representation of $\Gamma_0(2)$ of dimension two and k_0 will denote the least integer for which $M_{k_0}(\rho) \neq 0$. Let $F \in M_{k_0}(\rho)$ such that $F \neq 0$. We proved in Theorem 3.0.1 that F and $D_{k_0}F$ form a basis for $M(\rho)$ as a $M(\Gamma_0(2))$-module. In particular, $M_{k_0}(\rho) = \mathbf{C}F$. Hence F is determined by ρ up to multiplication by a nonzero complex number. In this section, we will use Theorem 3.0.1 to compute an ordinary differential equation that F satisfies. We will then solve this differential equation explicitly using the Dedekind η-function, the Gaussian hypergeometric series ${}_2F_1$, and a Hauptmodul of $\Gamma_0(2)$.

We recall that $G(\tau) := -E_2(\tau) + 2E_2(2\tau)$ where $E_2(\tau) := 1 - 24\sum_{n=1}^{\infty} \sigma(n)q^n$. We have that $M(\Gamma_0(2)) = \mathbf{C}[G, E_4]$. As $D_{k_0+2}(D_{k_0}(F)) \in M_{k_0+4}(\rho)$, we may apply Theorem 3.0.1 to write

$D_{k_0+2}(D_{k_0}F) = C_1(D_{k_0}F) + C_2F$ where $C_1 \in M_2(\Gamma_0(2)) = \mathbf{C}G$ and $C_2 \in M_4(\Gamma_0(2)) = \mathbf{C}G^2 \bigoplus \mathbf{C}E_4$. The modular forms C_1 and C_2 are invariants of ρ. Let $a, b, c \in \mathbf{C}$ be the complex numbers such that $C_1 = -aG$ and $C_2 = -(bG^2 + cE_4)$.We have shown that F satisfies the differential equation

$$D_{k_0+2}(D_{k_0}F) + aGD_{k_0}F + (bG^2 + cE_4)F = 0. \qquad (2)$$

We note that one can recover the representation ρ from the differential equation (2) as $\Gamma_0(2)$ acts on the space of solutions to the differential equation

$$D_{k_0+2}(D_{k_0}S) + aGD_{k_0}S + (bG^2 + cE_4)S = 0 \qquad (3)$$

by sending a solution S of (3) to $S|_{k_0}\gamma$ for any $\gamma \in \Gamma_0(2)$. It would be interesting to determine which tuples (k_0, a, b, c) correspond to irreducible ρ.

We sometimes use the notation D_k^2 to denote $D_{k+2} \circ D_k$. We recall that $\theta := D_0 = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$. We make use of the Dedekind η-function to solve the differential equation (2). A good reference for the Dedekind η-function and its properties is [7]. The function η^2 is holomorphic in $\mathfrak{H}$ and it does not vanish in $\mathfrak{H}$. Let ω denote the character of Γ for which $\eta^2|_1 g = \omega(g)\eta^2$. Let $F_0 := \frac{F}{\eta^{2k_0}}$. We observe that for all $g \in \Gamma_0(2)$,

$$F_0|_0 g = (\eta^{-2k_0}|_{k_0} g)(F|_{k_0} g) = \omega^{-k_0}(g)\eta^{-2k_0}\rho(g)F = (\rho \otimes \omega^{-k_0})(g)\eta^{-2k_0}F = (\rho \otimes \omega^{-k_0})(g)F_0.$$

Thus F_0 is a meromorphic vector-valued modular form of weight zero with respect to the representation $\rho_0 := \rho \otimes \omega^{-k_0}$. We note that F_0 is holomorphic in $\mathfrak{H}$ since η is holomorphic in $\mathfrak{H}$ and η never vanishes in $\mathfrak{H}$. We now compute a differential equation that F_0 satisfies.

Lemma 4.1.1. *Let* $F_0 := \frac{F}{\eta^{2k_0}}$. *Then*

$$D_2(D_0(F_0)) + aGD_0(F_0) + (bG^2 + cE_4)F_0 = 0. \qquad (4).$$

Proof. Let f denote a function on $\mathfrak{H}$, let $k \in \mathbf{Z}$, and let $g = \frac{f}{\eta^{2k}}$. To prove the lemma, we observe that because η never vanishes in $\mathfrak{H}$, it suffices to show that

$$\eta^{2k}(D_0^2(g) + aGD_0(g) + (bG^2 + cE_4)g) = D_k^2(f) + aGD_k(f) + (bG^2 + cE_4)f.$$

To show that the equation above holds, it suffices to prove that $D_k(f) = \theta(\frac{f}{\eta^{2k}})\eta^{2k}$ and that $D_k^2(f) = \eta^{2k}D_0^2(\frac{f}{\eta^{2k}})$. We recall that $\theta(\eta) = \frac{1}{2\pi i}\eta' = \frac{1}{24}E_2\eta$ (Section 5.8 in [7]). Thus $\theta(\eta^{2k}) = 2k\eta^{2k-1}\theta(\eta) = \frac{kE_2\eta^{2k}}{12}$. Hence $D_k(\eta^{2k}) = \theta(\eta^{2k}) - \frac{k}{12}E_2\eta^{2k} = 0$. If $t, l \in \mathbf{Z}$ and if α, β are holomorphic functions then

$$\begin{aligned}
D_{t+l}(\alpha\beta) &= \theta(\alpha\beta) - \frac{t+l}{12}E_2\alpha\beta \\
&= \alpha\theta(\beta) + \beta\theta(\alpha) - \frac{t+l}{12}E_2\alpha\beta \\
&= \alpha(\theta(\beta) - \frac{t}{12}E_2\beta) + \beta(\theta(\alpha) - \frac{l}{12}E_2\alpha) \\
&= \alpha D_t\beta + \beta D_l\alpha.
\end{aligned}$$

We now have that

$$\begin{aligned}
D_k(f) &= D_k\left(\eta^{2k} \cdot \frac{f}{\eta^{2k}}\right) \\
&= \frac{f}{\eta^{2k}}D_k(\eta^{2k}) + \eta^{2k}\theta\left(\frac{f}{\eta^{2k}}\right) \\
&= \eta^{2k}\theta\left(\frac{f}{\eta^{2k}}\right).
\end{aligned}$$

Thus

$$D_k^2(f) = D_{k+2}(D_k(f))$$

$$
\begin{aligned}
&= D_{k+2}\left(\eta^{2k}\theta\left(\frac{f}{\eta^{2k}}\right)\right) \\
&= \theta\left(\frac{f}{\eta^{2k}}\right)D_k(\eta^{2k}) + \eta^{2k}D_2\left(\theta\left(\frac{f}{\eta^{2k}}\right)\right) \\
&= \eta^{2k}D_2\left(\theta\left(\frac{f}{\eta^{2k}}\right)\right) \\
&= \eta^{2k}D_0^2\left(\frac{f}{\eta^{2k}}\right).
\end{aligned}
$$

□

We now explain how to make a change of variables to solve the differential equation (4). This technique can be found in a paper of Kaneko and Zagier [13]. The modular curve $\Gamma_0(2)\backslash(\mathfrak{H}\bigcup\mathbb{P}^1(\mathbf{Q}))$ is a compact Riemann surface of genus zero. In a fundamental region, its cusps are 0 and ∞ and its elliptic point is $\frac{1+i}{2}$. There exists a complex-analytic isomorphism from $\Gamma_0(2)\backslash(\mathfrak{H}\bigcup\mathbb{P}^1(\mathbf{Q}))$ to $\mathbb{P}^1(\mathbf{C})$. A function with this property is called a *Hauptmodul* of $\Gamma_0(2)$. We define the function $\mathfrak{J}$ by setting $\mathfrak{J}(\tau) := 3\frac{G(\tau)^2}{E_4(\tau)-G(\tau)^2}$. The function $\mathfrak{J}$ is a modular function and it induces a complex-analytic isomorphism from $\Gamma_0(2)\backslash(\mathfrak{H}\bigcup\mathbb{P}^1(\mathbf{Q}))$ to $\mathbb{P}^1(\mathbf{C})$, which we sometimes also denote by $\mathfrak{J}$. A proof that $\mathfrak{J}$ is a Hauptmodul is given in the appendix.

We will solve the differential equation (4) that F_0 satisfies by writing F_0 locally as a function of $\mathfrak{J}$. The function $\mathfrak{J}$ enjoys the property that $\mathfrak{J}(\tau_1) = \mathfrak{J}(\tau_2)$ if and only if $\Gamma_0(2)\cdot\tau_1 = \Gamma_0(2)\cdot\tau_2$. This fact implies that if $\tau\in\mathfrak{H}$ such that τ is not an elliptic point of $\Gamma_0(2)$ then there exists a connected and simply connected open set U_τ containing τ for which $U_\tau\subset\mathfrak{H}$ and the restriction of $\mathfrak{J}$ to U_τ is injective. In particular, U_τ does not contain an elliptic point. Consequently, if $\tau\in\mathfrak{H}$ which is not an elliptic point then there exists a unique function H such that $F_0|_{U_\tau} = H\circ\mathfrak{J}$.

We note that H is holomorphic since $\mathfrak{J}|_{U_\tau}$ is biholomorphic and F_0 is holomorphic. The next theorem computes the differential equation that H satisfies.

Theorem 4.1.2. *Let $\tau_0 \in \mathfrak{H}$ such that τ_0 is not an elliptic point of $\Gamma_0(2)$. Let U_{τ_0} denote a connected and simply connected open set containing τ_0 for which $U_{\tau_0} \subset \mathfrak{H}$ and the restriction of $\mathfrak{J}$ to U_{τ_0} is injective. Let $F_0 = \frac{F}{\eta^{2k_0}}$ and let H denote the function for which $F_0|_{U_{\tau_0}} = H \circ \mathfrak{J}$. We have that for all $\tau \in U_{\tau_0}$,*

$$H^{''}(\mathfrak{J}(\tau)) + \frac{7\mathfrak{J}(\tau) - 6a\mathfrak{J}(\tau) - 3}{6\mathfrak{J}(\tau)(\mathfrak{J}(\tau)-1)} H^{'}(\mathfrak{J}(\tau)) + \frac{(b+c)\mathfrak{J}(\tau)+3c}{\mathfrak{J}(\tau)(\mathfrak{J}(\tau)-1)^2} H(\mathfrak{J}(\tau)) = 0 \qquad (5)$$

The proof of Theorem 4.1.2 uses the following propositions whose proofs are given in the appendix of this thesis.

Proposition 4.1.3.

$$\theta(\mathfrak{J}) = (1-\mathfrak{J})G.$$

Proposition 4.1.4.

$$\theta^2(\mathfrak{J}) = G^2(1-\mathfrak{J})(\frac{3-7\mathfrak{J}}{6\mathfrak{J}}) + \frac{1}{6}E_2\theta(\mathfrak{J}).$$

We will also need to use the fact that $\frac{E_4}{G^2} = \frac{\mathfrak{J}+3}{\mathfrak{J}}$ in our proof of Theorem 4.1.2. This fact can be derived using elementary algebra (or from the Riemann-Roch Theorem) since $\mathfrak{J} := \frac{3G^2}{E_4 - G^2}$. We now proceed with the proof of Theorem 4.1.2.

Proof. (**Proof of Theorem 4.1.2**) We have that $D_0 F_0 = \theta F_0$ and $D_0^2 F_0 = (\theta - \frac{1}{6}E_2)(\theta F_0) = \theta^2 F_0 - \frac{1}{6}E_2\theta F_0$. Therefore

$$D_0^2 F_0 + aGD_0F_0 + (bG^2 + cE_4)F_0 = \theta^2 F_0 + (aG - \frac{1}{6}E_2)\theta F_0 + (bG^2 + cE_4)F_0 = 0.$$

We have that

$$\begin{aligned}\theta(F_0)(\tau) &= \theta(H\circ\mathfrak{J})(\tau)\\ &= \frac{1}{2\pi i}(H\circ\mathfrak{J})^{'}(\tau)\\ &= H^{'}(\mathfrak{J}(\tau))\cdot\frac{1}{2\pi i}\mathfrak{J}^{'}(\tau)\\ &= H^{'}(\mathfrak{J}(\tau))\cdot(\theta\mathfrak{J})(\tau)\\ &= H^{'}(\mathfrak{J}(\tau))\cdot G(\tau)(1-\mathfrak{J}(\tau)).\end{aligned}$$

Therefore

$$\begin{aligned}&\theta^2(F_0)(\tau)\\ &= \theta((H^{'}\circ\mathfrak{J})\cdot\theta\mathfrak{J})(\tau)\\ &= \theta(H^{'}\circ\mathfrak{J})(\tau)\cdot(\theta\mathfrak{J})(\tau)+(H^{'}\circ\mathfrak{J})(\tau)\cdot(\theta^2\mathfrak{J})(\tau)\\ &= H^{''}(\mathfrak{J}(\tau))\cdot\theta(\mathfrak{J})(\tau)\cdot\theta(\mathfrak{J})(\tau)+(H^{'}\circ\mathfrak{J})(\tau)\cdot(\theta^2\mathfrak{J})(\tau)\\ &= H^{''}(\mathfrak{J}(\tau))\cdot(1-\mathfrak{J}(\tau))^2G^2(\tau)\\ &\qquad +H^{'}(\mathfrak{J}(\tau))\cdot G^2(\tau)(1-\mathfrak{J}(\tau))\left(\frac{3-7\mathfrak{J}(\tau)}{6\mathfrak{J}(\tau)}\right)\\ &\qquad +\frac{1}{6}E_2(\tau)H^{'}(\mathfrak{J}(\tau))\theta(\mathfrak{J})(\tau)\\ &= G^2(\tau)\left(H^{''}(\mathfrak{J}(\tau))(1-\mathfrak{J}(\tau))^2+(H^{'}(\mathfrak{J}(\tau)))\cdot(1-\mathfrak{J}(\tau))\cdot\frac{3-7\mathfrak{J}(\tau)}{6\mathfrak{J}(\tau)}\right)\\ &\qquad +\frac{1}{6}E_2(\tau)\theta(\mathfrak{J})(\tau)H^{'}(\mathfrak{J}(\tau)).\end{aligned}$$

Thus

$$D_0^2F_0(\tau)+aG(\tau)D_0F_0(\tau)+\Big(bG^2(\tau)+cE_4(\tau)\Big)F_0(\tau)$$

$$
\begin{aligned}
&= \theta^2 F_0(\tau) + \left(aG(\tau) - \frac{1}{6}E_2(\tau)\right)\theta F_0(\tau) + \left(bG^2(\tau) + cE_4(\tau)\right)F_0(\tau) \\
&= G^2(\tau)\left(H^{''}(\mathfrak{J}(\tau))(1-\mathfrak{J}(\tau))^2 + H^{'}(\mathfrak{J}(\tau))\cdot(1-\mathfrak{J}(\tau))\Big(\frac{3-7\mathfrak{J}(\tau)}{6\mathfrak{J}(\tau)}\Big)\right) \\
&\quad + \frac{1}{6}E_2(\tau)\theta(\mathfrak{J})(\tau)H^{'}(\mathfrak{J}(\tau)) \\
&\qquad + \left(aG(\tau) - \frac{1}{6}E_2(\tau)\right)H^{'}(\mathfrak{J}(\tau))\cdot G(\tau)(1-\mathfrak{J}(\tau)) \\
&\qquad\quad + G^2(\tau)\left(b + c\frac{E_4(\tau)}{G^2(\tau)}\right)H(\mathfrak{J}(\tau)) \\
&= G^2(\tau)\left(H^{''}(\mathfrak{J}(\tau))(1-\mathfrak{J}(\tau))^2 + H^{'}(\mathfrak{J}(\tau))\cdot(1-\mathfrak{J}(\tau))\Big(\frac{3-7\mathfrak{J}(\tau)}{6\mathfrak{J}(\tau)}\Big)\right) \\
&\qquad + aG(\tau)H^{'}(\mathfrak{J}(\tau))\cdot G(\tau)(1-\mathfrak{J}(\tau)) \\
&\qquad\quad + G^2(\tau)\left(b + c\frac{E_4(\tau)}{G^2(\tau)}\right)H(\mathfrak{J}(\tau)) \\
&= G^2(\tau)H^{''}(\mathfrak{J}(\tau))(1-\mathfrak{J}(\tau))^2 \\
&\qquad + G^2(\tau)H^{'}(\mathfrak{J}(\tau))((1-\mathfrak{J}(\tau))\left(a + \frac{3-7\mathfrak{J}(\tau)}{6\mathfrak{J}(\tau)}\right) \\
&\qquad\quad + G^2(\tau)\left(b + c\frac{E_4(\tau)}{G^2(\tau)}\right)H(\mathfrak{J}(\tau)) \\
&= G^2(\tau)H^{''}(\mathfrak{J}(\tau))(1-\mathfrak{J}(\tau))^2 \\
&\qquad + G^2(\tau)H^{'}(\mathfrak{J}(\tau))(1-\mathfrak{J}(\tau))\left(a + \frac{3-7\mathfrak{J}(\tau)}{6\mathfrak{J}(\tau)}\right) \\
&\qquad\quad + G^2(\tau)\left(b + c\frac{\mathfrak{J}(\tau)+3}{\mathfrak{J}(\tau)}\right)H(\mathfrak{J}(\tau)).
\end{aligned}
$$

We have shown that

$$
\begin{aligned}
0 = G^2(\tau)\Bigg(&H^{''}(\mathfrak{J}(\tau))(1-\mathfrak{J}(\tau))^2 + H^{'}(\mathfrak{J}(\tau))(1-\mathfrak{J}(\tau))\left(\frac{(6a-7)\mathfrak{J}(\tau)+3}{6\mathfrak{J}(\tau)}\right) \\
&+ \left(b + c\frac{\mathfrak{J}(\tau)+3}{\mathfrak{J}(\tau)}\right)H(\mathfrak{J}(\tau))\Bigg).
\end{aligned}
$$

It is shown in the appendix that $G^2(\tau) = 0$ if and only if τ is an elliptic point. Thus if τ is not an elliptic point then

$$0 = H''(\mathfrak{J}(\tau))(1-\mathfrak{J}(\tau))^2 + H'(\mathfrak{J}(\tau))(1-\mathfrak{J}(\tau))\left(\frac{(6a-7)\mathfrak{J}(\tau)+3}{6\mathfrak{J}(\tau)}\right)$$
$$+\left(\frac{(b+c)\mathfrak{J}(\tau)+3c}{\mathfrak{J}(\tau)}\right)H(\mathfrak{J}(\tau)).$$

As U_{τ_0} does not contain any elliptic points, the above equation holds for all τ in U_{τ_0}. Finally, we divide the above equation by $(1-\mathfrak{J}(\tau))^2$ and we obtain the differential equation (5). $\square$

Let $V_{\tau_0} := \mathfrak{J}(U_{\tau_0})$ and let $Y := \mathfrak{J}(\tau)$. We have shown that

$$D_0^2 F_0(\tau) + aG(\tau)D_0F_0(\tau) + \Big(bG^2(\tau) + cE_4(\tau)\Big)F_0(\tau) = 0 \text{ for all } \tau \in U_{\tau_0}$$

if and only if every $Y \in V_{\tau_0}$ satisfies the differential equation

$$H''(Y) + \frac{7Y - 6aY - 3}{6Y(Y-1)}H'(Y) + \frac{(b+c)Y+3c}{Y(Y-1)^2}H(Y) = 0 \quad (6).$$

Our goal is now to solve the differential equation (6). We will need some background on second order ordinary differential equations and we closely follow Chapter 6 of [3]. It is immediate from examining the equation (6) that its singularities occur at $Y = 0, 1, \infty$.

Definition 4.1.5. *(Sections 6 and 12 in [3]) Let $z_0 \in \boldsymbol{C}$. A second-order differential equation $w''(z) + p(z)w'(z) + q(z)w(z) = 0$ for which $p(z)$ and $q(z)$ are analytic in a deleted neighborhood of z_0 has a regular singular point at z_0 if $p(z)$ has at worst a simple pole at $z = z_0$ and $q(z)$ has at worst a double pole at $z = z_0$. We say that ∞ is a regular singular point of $w''(z) + p(z)w'(z) + q(z)w(z) = 0$ if and only if $t = 0$ is a regular singular point of the differential equation in t*

obtained by making the substitution $z = \frac{1}{t}$ *for the differential equation* $w''(z) + p(z)w'(z) + q(z)w(z) = 0$.

It is now immediate that 0 and 1 are regular singularities of the differential equation (6). We state a useful criterion (Theorem 9 in [3]) for determining when ∞ is a regular singular point of $w'' + p(z)w' + q(z)w = 0$.

Theorem 4.1.6. *(Theorem 9 in [3]) Infinity is a regular singular point for* $w'' + p(z)w' + q(z)w = 0$ *if and only if the coefficients of p and q have power series expansions, convergent for sufficiently large* $|z|$*, of the form* $p(z) = \sum_{n=1}^{\infty} \frac{p_n}{z^n}$, $q(z) = \sum_{n=2}^{\infty} \frac{q_n}{z^n}$*. Equivalently,* ∞ *is a regular singular point for* $w'' + p(z)w' + q(z)w = 0$ *if and only if p has a zero of at least the first order at* ∞ *and q has a zero of at least the second order at* ∞.

Hence ∞ is a regular singularity of the differential equation (6). The singularities of (6) are $0, 1, \infty$ and these singularities are all regular. The method of Frobenius provides a way to find a power series solution to the differential equation $w'' + p(z)w' + q(z)w = 0$ in a neighborhood of a regular singular point z_0. One writes $w = (z - z_0)^v (1 + \sum_{n=1}^{\infty} c_n (z - z_0)^n)$ and then one solves for v and recursively computes the coefficients c_n. One often rewrites the differential equation $w'' + p(z)w' + q(z)w = 0$ as $(z - z_0)^2 w'' + (z - z_0)P(z)w' + Q(z)w = 0$ where $P(z) = (z - z_0)p(z) = \sum_{k=0}^{\infty} P_k (z - z_0)^k$ and $Q(z) = (z - z_0)^2 q(z) = \sum_{k=0}^{\infty} Q_k (z - z_0)^k$ are convergent in a neighborhood of z_0. Then

$$0 = (z - z_0)^2 w''(z) + (z - z_0)P(z)w'(z) + Q(z)w(z) = (z - z_0)^v (v(v-1) + P_0 v + Q_0 + O(z - z_0)).$$

Therefore v satisfies the *indicial equation* at z_0:

$$v(v-1) + P_0 v + Q_0 = 0.$$

It is well-known that if the difference of the roots of the indicial equation is not an integer then one can find a basis consisting of power series for the space of solutions to $w'' + p(z)w' + q(z)w = 0$ in a neighborhood of z_0. We remark that $P_0 = \lim_{z \to z_0} P(z) = \lim_{z \to z_0} (z - z_0)p(z)$ and $Q_0 = \lim_{z \to z_0} Q(z) = \lim_{z \to z_0} (z - z_0)^2 q(z)$. In the differential equation (6), $p(Y) = \frac{7Y - 6aY - 3}{6Y(Y-1)}$ and $q(Y) = \frac{(b+c)Y + 3c}{Y(Y-1)^2}$. At the regular singular point 0, $P_0 = \lim_{Y \to 0} Y p(Y) = \frac{1}{2}$ and $Q_0 = \lim_{Y \to 0} Y^2 q(Y) = 0$. The indicial equation at 0 is $Y(Y-1) + \frac{1}{2}Y = 0$. Therefore the indicial roots at 0 are 0 and $\frac{1}{2}$. At the singular point 1, $P_0 = \lim_{Y \to 1} (Y-1)p(Y) = \frac{2-3a}{3}$, and
$Q_0 = \lim_{Y \to 1} (Y-1)^2 q(Y) = b + 4c$. Therefore the indicial equation at 1 is $Y(Y-1) + \frac{2-3a}{3}Y + b + 4c = 0$.

We have shown that the differential equation (6) is Fuchsian since all of its singularities are regular. In fact, this type of differential equation is a Riemann differential equation since it is a second order differential equation with exactly three singularities, all of which are regular. The standard technique to solve a Riemann differential equation of order two which is not in Gauss normal form is to make a change of variables to obtain a differential equation in Gauss normal form. We proceed in this manner and define the function $W(Y)$ via the equation $H(Y) = Y^{\lambda}(Y-1)^r W(Y)$ where λ is an indicial root of (6) at 0 and r is an indicial root of (6) at 1. We recall that the indicial roots at 0 are 0 and $\frac{1}{2}$ and we make the choice of setting $\lambda = 0$. We also recall that the indicial roots at 1 are the roots of the quadratic equation:

$$r(r-1) + \left(\frac{2-3a}{3}\right) r + (b + 4c) = 0.$$

We have that

$$H(Y) = (Y-1)^r W(Y)$$

$$H^{'}(Y) = r(Y-1)^{r-1}W(Y) + (Y-1)^r W^{'}(Y)$$

$$H^{''}(Y) = r(r-1)(Y-1)^{r-2}W(Y) + 2r(Y-1)^{r-1}W^{'}(Y) + (Y-1)^r W^{''}(Y)$$

Thus

$$\begin{aligned}
0 &= H^{''}(Y) + \frac{7Y-6aY-3}{6Y(Y-1)}H^{'}(Y) + \frac{(b+c)Y+3c}{Y(Y-1)^2}H(Y) \\
&= r(r-1)(Y-1)^{r-2}W(Y) + 2r(Y-1)^{r-1}W^{'}(Y) + (Y-1)^r W^{''}(Y) \\
&\quad + \frac{7Y-6aY-3}{6Y(Y-1)}\Big(r(Y-1)^{r-1}W(Y) + (Y-1)^r W^{'}(Y)\Big) \\
&\quad\quad + \frac{(b+c)Y+3c}{Y(Y-1)^2}(Y-1)^r W(Y) \\
&= (Y-1)^r W^{''}(Y) + W^{'}(Y)\left(2r(Y-1)^{r-1} + \frac{7Y-6aY-3}{6Y(Y-1)}(Y-1)^r\right) \\
&\quad + W(Y)\left((Y-1)^r\frac{(b+c)Y+3c}{Y(Y-1)^2} + r(Y-1)^{r-1}\frac{7Y-6aY-3}{6Y(Y-1)} + r(r-1)(Y-1)^{r-2}\right).
\end{aligned}$$

In the computation below, we will use the fact that r satisfies the equation $r(r-1) + (\frac{2-3a}{3})r + (b+4c) = 0$ to get that $6b+6c+7r-6ar+6r(r-1) = 6b+6c+7r-6ar-6(b+4c)-2r(2-3a) = -18c+3r$. We have that

$$\begin{aligned}
0 &= Y(Y-1)W^{''}(Y) + W^{'}(Y)\left(2rY + \frac{7Y-6aY-3}{6}\right) \\
&\quad + W(Y)\left(\frac{(b+c)Y+3c}{Y-1} + \frac{r(7Y-6aY-3)}{6(Y-1)} + \frac{r(r-1)Y}{Y-1}\right) \\
&= Y(Y-1)W^{''}(Y) + W^{'}(Y)\left(Y\left(2r+\frac{7-6a}{6}\right) - \frac{1}{2}\right) \\
&\quad + W(Y)\left(\frac{(Y(6b+6c+7r-6ar+6r(r-1))+18c-3r}{6(Y-1)}\right)
\end{aligned}$$

$$= Y(Y-1)W''(Y) + W'(Y)\left(Y\left(2r + \frac{7-6a}{6}\right) - \frac{1}{2}\right)$$
$$+ W(Y)\left(\frac{Y(-18c+3r)+18c-3r}{6(Y-1)}\right)$$
$$= Y(Y-1)W''(Y) + W'(Y)\left(Y\left(2r + \frac{7-6a}{6}\right) - \frac{1}{2}\right) + W(Y)\left(\frac{r-6c}{2}\right).$$

We conclude that

$$0 = Y(Y-1)W''(Y) + W'(Y)\left(Y(2r + \frac{7-6a}{6}) - \frac{1}{2}\right) + W(Y)\left(\frac{r-6c}{2}\right). \qquad (7)$$

The differential equation (7) is an example of a differential equation in Gauss normal form:

$$Y(Y-1)W''(Y) + W'(Y)((A+B+1)Y - C) + ABW(Y) = 0 \qquad (8)$$

If $A - B \notin \mathbf{Z}$ then a basis for the space of solutions to the differential equation (8) in a neighborhood of ∞ (see Section 12 in [3]) is

$$Y^{-A}\,{}_2F_1(A, 1+A-C, 1+A-B; Y^{-1}), \text{and } Y^{-B}\,{}_2F_1(B, 1+B-C, 1+B-A; Y^{-1})$$

where we recall that

$${}_2F_1(\alpha, \beta, \gamma; z) := 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \cdot \frac{z^n}{n!}, (\alpha)_n := \prod_{i=0}^{n-1}(\alpha + i)$$

In our case,

$$A + B + 1 = 2r + \frac{7-6a}{6},\ AB = \frac{r-6c}{2},\ C = \frac{1}{2}.$$

We note that A and B are the roots of the quadratic polynomial $x^2 - x(2r + \frac{1-6a}{6}) + \frac{r-6c}{2}$. We recall that $H(Y) = (Y-1)^r W(Y)$ and conclude that if $A - B \notin \mathbf{Z}$ then

$(Y-1)^r Y^{-A}\,{}_2F_1(A, 1+A-C, 1+A-B; Y^{-1})$ and $(Y-1)^r Y^{-B}\,{}_2F_1(B, 1+B-C, 1+B-A; Y^{-1})$

form a basis for the space of solutions to the differential equation (6). As $Y = \mathfrak{J}(\tau)$, we have that if $A-B \notin \mathbf{Z}$ then a basis for the space of solutions in a neighborhood of ∞ to the differential equation

$$D_0^2 f + aGD_0 f + (bG^2 + cE_4)f = 0$$

is

$$(\mathfrak{J}(\tau)-1)^r \mathfrak{J}(\tau)^{-A}\ {}_2F_1(A, 1+A-C, 1+A-B; \mathfrak{J}(\tau)^{-1})$$

and

$$(\mathfrak{J}(\tau)-1)^r \mathfrak{J}(\tau)^{-B}\ {}_2F_1(B, 1+B-C, 1+B-A; \mathfrak{J}(\tau)^{-1}).$$

Finally, we have that if $A-B \notin \mathbf{Z}$ then a basis for the space of solutions in a neighborhood of ∞ to the differential equation

$$D_{k_0}^2 f + aGD_{k_0} f + (bG^2 + cE_4)f = 0$$

is $\eta^{2k_0}(\tau)(\mathfrak{J}(\tau)-1)^r \mathfrak{J}(\tau)^{-A}\ {}_2F_1(A, 1+A-C, 1+A-B; \mathfrak{J}(\tau)^{-1})$ and $\eta^{2k_0}(\tau)(\mathfrak{J}(\tau)-1)^r \mathfrak{J}(\tau)^{-B}\ {}_2F_1(B, 1+B-C, 1+B-A; \mathfrak{J}(\tau)^{-1})$. We have thus found a basis of solutions to the differential equation (2) that the component functions of F satisfy.

The main focus of this thesis is the study of vector-valued valued modular forms which are not logarithmic. To avoid logarithmic vector-valued modular forms, we must assume that $\rho(T)$ is diagonalizable. **Henceforth, ρ will always denote a two-dimensional irreducible complex representation of $\Gamma_0(2)$ for which $\rho(T)$ is diagonalizable.** If $\rho(T)$ is diagonalizable then its eigenvalues are distinct. To see this, we use the fact that $\Gamma_0(2)$ is generated by T and $V := \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$. Indeed, if K is any group such that there exist $M, N \in K$ for which $K = \langle M, N \rangle$

and if $\alpha : K \to \mathrm{GL}(W)$ is a two-dimensional irreducible complex representation then the eigenvalues of $\alpha(M)$ must be distinct if $\alpha(M)$ is diagonalizable. We argue by contradiction and suppose that the eigenvalues of $\alpha(M)$ are not distinct. Then $\alpha(M)$ is a scalar matrix and it commutes with $\alpha(N)$. Let $v \in W$ such that $v \neq 0$ and v is an eigenvector for $\alpha(M)$. Then the 1-dimensional space $\mathbf{C}v$ is a K-invariant subspace of W. But this is a contradiction as α is irreducible and has dimension two. Thus the eigenvalues of $\alpha(M)$ are distinct.

We now use the fact that the eigenvalues of $\rho(T)$ are distinct to show that $A - B \notin \mathbf{Z}$. Let m_1 and m_2 denote complex numbers with $|m_1| \leq |m_2|$ for which the eigenvalues of $\rho(T)$ are $e^{2\pi i m_1}$ and $e^{2\pi i m_2}$. Let $X \in \mathrm{GL}_2(\mathbf{C})$ such that

$$X\rho(T)X^{-1} = \begin{bmatrix} e^{2\pi i m_1} & 0 \\ 0 & e^{2\pi i m_2} \end{bmatrix}.$$

We recall that χ denotes the character associated to the modular form η^2. As $\chi(T) = e^{\frac{2\pi i}{6}}$ and $\rho_0 = \rho \otimes \chi^{-k_0}$,

$$X\rho_0(T)X^{-1} = \begin{bmatrix} e^{2\pi i(m_1 - \frac{k_0}{6})} & 0 \\ 0 & e^{2\pi i(m_2 - \frac{k_0}{6})} \end{bmatrix}.$$

We note that the eigenvalues of $\rho(T)$ are distinct if and only if the eigenvalues of $\rho_0(T)$ are distinct, which is the case exactly when $m_1 - m_2 \notin \mathbf{Z}$. The function XF_0 is a vector-valued modular form with respect to $X\rho_0X^{-1}$ since F_0 is a vector-valued modular form with respect to ρ_0. Let h_1 denote the first and h_2 denote the second component function of XF_0. We have that

$$\begin{bmatrix} e^{2\pi i(m_1 - \frac{k_0}{6})} & 0 \\ 0 & e^{2\pi i(m_2 - \frac{k_0}{6})} \end{bmatrix} \begin{bmatrix} h_1(\tau) \\ h_2(\tau) \end{bmatrix} = \begin{bmatrix} h_1(\tau+1) \\ h_2(\tau+1) \end{bmatrix}.$$

Let $\widehat{h_1}(\tau) := h_1(\tau)e^{-2\pi i(m_1-\frac{k_0}{6})\tau}$ and let $\widehat{h_2}(\tau) := h_2(\tau)e^{-2\pi i(m_2-\frac{k_0}{6})\tau}$. Then $\widehat{h_1}(\tau+1) = \widehat{h_1}(\tau)$ and $\widehat{h_2}(\tau+1) = \widehat{h_2}(\tau)$. Therefore $\widehat{h_1}(\tau) = \sum_{n\in\mathbf{Z}} a_n q^n$ and $\widehat{h_2}(\tau) = \sum_{n\in\mathbf{Z}} b_n q^n$.

Thus $h_1(\tau) = q^{(m_1-\frac{k_0}{6})}\sum_{n\in\mathbf{Z}} a_n q^n$ and $h_2(\tau) = q^{(m_2-\frac{k_0}{6})}\sum_{n\in\mathbf{Z}} b_n q^n$. As F is holomorphic at ∞, F_0 is meromorphic at ∞ and therefore $a_n = 0$ if $n << 0$ and $b_n = 0$ if $n << 0$. Let l_1 and l_2 denote the unique complex numbers such that $h_1(\tau) = q^{l_1}\sum_{n=0}^{\infty} c_n q^n$ with $c_0 \neq 0$ and $h_2(\tau) = q^{l_2}\sum_{n=0}^{\infty} d_n q^n$ with $d_0 \neq 0$. We note that $l_1 - (m_1 - \frac{k_0}{6}) \in \mathbf{Z}$ and $l_2 - (m_2 - \frac{k_0}{6}) \in \mathbf{Z}$. Hence $l_1 - l_2 \notin \mathbf{Z}$ since $m_1 - m_2 \notin \mathbf{Z}$. The component functions h_1 and h_2 of XF_0 are solutions of the differential equation that the component functions of F_0 are solutions of. Therefore h_1 and h_2 are solutions of the differential equation

$$D_2(D_0 g) + aGD_0 g + (bG^2 + cE_4)g = 0.$$

We note that h_1 and h_2 cannot be linearly dependent because $l_1 - l_2 \notin \mathbf{Z}$. Thus the functions h_1 and h_2 form a basis for the space of solutions to the above differential equation. If we substitute $h_1 = q^{l_1}\sum_{n=0}^{\infty} c_n q^n$ into the above differential equation then we get that $0 = D_2(D_0 h_1) + aGD_0 h_1 + (bG^2 + cE_4)h_1 = q^{l_1}(l_1^2 + (a - \frac{1}{6})l_1 + b + c + O(q))$. Hence $l_1^2 + (a - \frac{1}{6})l_1 + b + c = 0$. Similarly, $l_2^2 + (a - \frac{1}{6})l_2 + b + c = 0$. Thus $l_1 l_2 = b + c$ and $l_1 + l_2 = \frac{1}{6} - a$.

The functions h_1 and h_2 both have what we call a *pure q-expansion.* We say that a function has a *pure q-expansion* if the function can be written in the form $q^{\nu}\sum_{n\in\mathbf{Z}} \alpha_n q^n$ for some complex number ν. The only functions in the set $\{g : D_2(D_0 g) + aGD_0 g + (bG^2 + cE_4)g = 0\} = \{z_1 h_1 + z_2 h_2 : z_1, z_2 \in \mathbf{C}\} = \{z_1 q^{l_1}\sum_{n=0}^{\infty} c_n q^n + z_2 q^{l_2}\sum_{n=0}^{\infty} d_n q^n : z_1, z_2 \in \mathbf{C}\}$ which have a pure q-expansion are those which are scalar multiples of h_1 or scalar multiples of h_2.

We now relate the numbers A and B to l_1 and l_2 in order to establish that $A-B \notin \mathbf{Z}$. We recall that A and B are the roots of the polynomial $x^2 - x(2r + \frac{1-6a}{6}) + \frac{r-6c}{2}$. Let D denote the discriminant of this polynomial. We recall that $l_1 l_2 = b + c$ and $l_1 + l_2 = \frac{1}{6} - a$. We have that

$$\begin{aligned}
D &= (2r + \frac{1-6a}{6})^2 - 2(r - 6c) \\
&= 4r^2 + \frac{1}{36} + a^2 - 4ar - \frac{a}{3} - \frac{4r}{3} + 12c \\
&= 4(r(a + \frac{1}{3}) - (b + 4c)) + \frac{1}{36} + a^2 - 4ar - \frac{a}{3} - \frac{4r}{3} + 12c \\
&= -4(b+c) + (a - \frac{1}{6})^2 \\
&= -4l_1 l_2 + (-l_1 - l_2)^2 \\
&= l_1^2 - 2l_1 l_2 + l_2^2 \\
&= (l_1 - l_2)^2.
\end{aligned}$$

We use the quadratic formula to see that A and B are the numbers $r + \frac{1}{2}(\frac{1}{6} - a \pm \sqrt{D}) = r + \frac{1}{2}(l_1 + l_2 \pm (l_1 - l_2))$. Thus $\{A, B\} = \{r + l_1, r + l_2\}$. We now fix the values of A and B by choosing to set $A = r + l_1$ and $B = r + l_2$. Thus $A - B = l_1 - l_2 \equiv m_1 - m_2 \pmod{\mathbf{Z}}$ since $l_1 \equiv m_1 + \frac{k_0}{6} \pmod{\mathbf{Z}}$ and $l_2 \equiv m_2 + \frac{k_0}{6} \pmod{\mathbf{Z}}$. Hence $A - B \notin \mathbf{Z}$ since $m_1 - m_2 \notin \mathbf{Z}$.

We know that $(\mathfrak{J}(\tau) - 1)^r \mathfrak{J}(\tau)^{-A} \, {}_2F_1(A, 1 + A - C, 1 + A - B; \mathfrak{J}(\tau)^{-1})$ and $(\mathfrak{J}(\tau) - 1)^r \mathfrak{J}(\tau)^{-B} \, {}_2F_1(B, 1 + B - C, 1 + B - A; \mathfrak{J}(\tau)^{-1})$ form a basis for the space of solutions to the differential equation

$$D_2(D_0 g) + aGD_0 g + (bG^2 + cE_4)g = 0$$

since we have previously shown that to be the case if $A-B \notin \mathbf{Z}$. We also know that h_1 and h_2 form a basis for the space of solutions to this differential equation. The leading power of q for h_1 is $l_1 = A-r$ and the leading power of q for h_2 is $l_2 = B-r$.

We now show that the functions $(\mathfrak{J}(\tau)-1)^r \mathfrak{J}(\tau)^{-A}\,{}_2F_1(A, 1+A-C, 1+A-B; \mathfrak{J}(\tau)^{-1})$ and $(\mathfrak{J}(\tau)-1)^r \mathfrak{J}(\tau)^{-B}\,{}_2F_1(B, 1+B-C, 1+B-A; \mathfrak{J}(\tau)^{-1})$ have pure q-expansions and we compute their leading power of q. To do so, we shall apply Newton's binomial theorem which states that if $\alpha \in \mathbf{C}$ and if $|x| < 1$ then $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$. We note that $|q| < 1$ because $\tau \in \mathfrak{H}$. This observation will justify our application of Newton's binomial theorem.

In the appendix, we show that $\mathfrak{J}(q) = \frac{1}{64q}(1+O(q))$. We now apply Newton's binomial theorem to get that for each integer n, $\mathfrak{J}(\tau)^{-n} = (64q)^n(1+O(q))$. Thus

$$ {}_2F_1(A, 1+A-C, 1+A-B; \mathfrak{J}(\tau)^{-1}) = 1 + \sum_{n \geq 1} \frac{(A)_n (1+A-C)_n}{(1+A-B)_n n!} \mathfrak{J}(\tau)^{-n} = 1 + O(q) $$

and

$$ {}_2F_1(B, 1+B-C, 1+B-A; \mathfrak{J}(\tau)^{-1}) = 1 + \sum_{n \geq 1} \frac{(B)_n (1+B-C)_n}{(1+B-A)_n n!} \mathfrak{J}(\tau)^{-n} = 1 + O(q). $$

We again apply Newton's binomial theorem to get that

$$ \mathfrak{J}^{-A}(q) = (64q)^A(1+O(q)), \mathfrak{J}^{-B}(q) = (64q)^B(1+O(q)), (\mathfrak{J}-1)^r = (64q)^{-r}(1+O(q)). $$

It now follows that

$$ (\mathfrak{J}(\tau)-1)^r \mathfrak{J}(\tau)^{-A}\,{}_2F_1(A, 1+A-C, 1+A-B; \mathfrak{J}(\tau)^{-1}) = (64q)^{A-r}(1+O(q)) = (64q)^{l_1}(1+O(q)) $$

has a pure q-expansion and that

$$(\mathfrak{J}(\tau)-1)^r\mathfrak{J}(\tau)^{-B}\,{}_2F_1(B,1+B-C,1+B-A;\mathfrak{J}(\tau)^{-1}) = (64q)^{B-r}(1+O(q)) = (64q)^{l_2}(1+O(q))$$

has a pure q-expansion. We previously explained why any function which has a pure q-expansion and which is a solution of the differential equation $D_2(D_0g)+aGD_0g+(bG^2+cE_4)g=0$ must be a scalar multiple of h_1 or h_2. Hence h_1 must be a scalar multiple of $(\mathfrak{J}(\tau)-1)^r\mathfrak{J}(\tau)^{-A}\,{}_2F_1(A,1+A-C,1+A-B;\mathfrak{J}(\tau)^{-1})$ since both functions have leading exponent l_1 and $(\mathfrak{J}(\tau)-1)^r\mathfrak{J}(\tau)^{-B}\,{}_2F_1(B,1+B-C,1+B-A;\mathfrak{J}(\tau)^{-1})$ must be a scalar multiple of h_2 since they both have the same leading exponent l_2. Thus there exist unique nonzero complex numbers κ_1 and κ_2 such that

$$XF_0 = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} \kappa_1(\mathfrak{J}(\tau)-1)^r\mathfrak{J}(\tau)^{-A}\,{}_2F_1(A,1+A-C,1+A-B;\mathfrak{J}(\tau)^{-1}) \\ \kappa_2(\mathfrak{J}(\tau)-1)^r\mathfrak{J}(\tau)^{-B}\,{}_2F_1(B,1+B-C,1+B-A;\mathfrak{J}(\tau)^{-1}) \end{bmatrix}.$$

We substitue $C=\frac{1}{2}$ and we get that

$$\begin{aligned} F(\tau) &= \eta^{2k_0}(\tau)F_0(\tau) \\ &= \eta^{2k_0}(\tau)X^{-1}(XF_0)(\tau) \\ &= X^{-1}\begin{bmatrix} \kappa_1\eta^{2k_0}(\tau)(\mathfrak{J}(\tau)-1)^r\mathfrak{J}(\tau)^{-A}\,{}_2F_1(A,\frac{1}{2}+A,1+A-B;\mathfrak{J}(\tau)^{-1}) \\ \kappa_2\eta^{2k_0}(\tau)(\mathfrak{J}(\tau)-1)^r\mathfrak{J}(\tau)^{-B}\,{}_2F_1(B,\frac{1}{2}+B,1+B-A;\mathfrak{J}(\tau)^{-1}) \end{bmatrix}. \end{aligned}$$

We now record what we've proven below.

Theorem 4.1.7. *Let ρ denote an irreducible complex representation of $\Gamma_0(2)$ of dimension two such that $\rho(T)$ is diagonalizable. Let k_0 denote the least integer for which $M_{k_0}(\rho)\neq 0$ and let F*

denote a non-zero element in $M_{k_0}(\rho)$*. Let* $e^{2\pi i m_1}$ *and* $e^{2\pi i m_2}$ *denote the eigenvalues of the matrix* $\rho(T)$ *with* $|m_1| \leq |m_2|$*. Let* $X \in GL_2(\boldsymbol{C})$ *such that* $X\rho(T)X^{-1} = \begin{bmatrix} e^{2\pi i m_1} & 0 \\ 0 & e^{2\pi i m_2} \end{bmatrix}$*. Let* a, b*, and* c *denote the unique complex numbers such that*

$$D_{k_0}^2 F + aGD_{k_0}F + (bG^2 + cE_4)F = 0.$$

Let r *denote a complex number such that* $r(r-1) + (\frac{2-3a}{3})r + (b+4c) = 0$*. Let* A *and* B *denote the roots of the quadratic polynomial* $x^2 - x(2r + \frac{1-6a}{6}) + \frac{r-6c}{2}$*. Then there exist unique nonzero complex numbers* κ_1 *and* κ_2 *such that*

$$F(\tau) = X^{-1} \begin{bmatrix} \kappa_1 \eta^{2k_0}(\tau)(\mathfrak{J}(\tau)-1)^r \mathfrak{J}(\tau)^{-A} \,{}_2F_1(A, \frac{1}{2}+A, 1+A-B; \mathfrak{J}(\tau)^{-1}) \\ \\ \kappa_2 \eta^{2k_0}(\tau)(\mathfrak{J}(\tau)-1)^r \mathfrak{J}(\tau)^{-B} \,{}_2F_1(B, \frac{1}{2}+B, 1+B-A; \mathfrak{J}(\tau)^{-1}) \end{bmatrix}.$$

Chapter 5

The arithmetic of vector-valued modular forms

5.1 The Fourier expansions of the component functions of F

In the previous chapter, we proved that there exist unique nonzero complex numbers κ_1 and κ_2 such that

$$F(\tau) = X^{-1} \begin{bmatrix} \kappa_1 \eta^{2k_0}(\tau)(\mathfrak{J}(\tau)-1)^r \mathfrak{J}(\tau)^{-A}\, {}_2F_1(A, \tfrac{1}{2}+A, 1+A-B; \mathfrak{J}(\tau)^{-1}) \\ \\ \kappa_2 \eta^{2k_0}(\tau)(\mathfrak{J}(\tau)-1)^r \mathfrak{J}(\tau)^{-B}\, {}_2F_1(B, \tfrac{1}{2}+B, 1+B-A; \mathfrak{J}(\tau)^{-1}) \end{bmatrix}.$$

We wish to study the arithmetic properties of the Fourier coefficients of the component functions of F. It is too much to ask that these Fourier coefficients be algebraic numbers since κ_1 and κ_2 need not be algebraic numbers. Nevertheless, we will study the q-series expansions of the functions:

$$\eta^{2k_0}(\mathfrak{J}(\tau)-1)^r\mathfrak{J}(\tau)^{-A}\ {}_2F_1(A,\frac{1}{2}+A,1+A-B;\mathfrak{J}(\tau)^{-1})$$

$$\eta^{2k_0}(\mathfrak{J}(\tau)-1)^r\mathfrak{J}(\tau)^{-B}\ {}_2F_1(B,\frac{1}{2}+B,1+B-A;\mathfrak{J}(\tau)^{-1}).$$

We will show that if ρ has certain properties then the q-series coefficients of these two functions are algebraic numbers.

Definition 5.1.1. *We let* $\{h(K)\}_{K=1}^{\infty}$ *and* $\{\widetilde{h}(K)\}_{K=1}^{\infty}$ *denote the sequences for which*

$$\mathfrak{J}^{-A}(\mathfrak{J}-1)^r\ {}_2F_1(A,\frac{1}{2}+A,1+A-B;\mathfrak{J}(\tau)^{-1}) = 64^{A-r}q^{A-r}\left(1+\sum_{K=1}^{\infty}h(K)q^K\right)$$

and

$$\mathfrak{J}^{-B}(\mathfrak{J}-1)^r\ {}_2F_1(B,\frac{1}{2}+B,1+B-A;\mathfrak{J}(\tau)^{-1}) = 64^{B-r}q^{B-r}\left(1+\sum_{K=1}^{\infty}\widetilde{h}(K)q^K\right).$$

Remark: The fact that there exist such sequences $\{h(K)\}_{K=1}^{\infty}$ and $\{\widetilde{h}(K)\}_{K=1}^{\infty}$ will be justified in this section.

Definition 5.1.2. *Let* $F' := \begin{bmatrix} \eta^{2k_0}q^{A-r}(1+\sum_{K=1}^{\infty}h(K)q^K) \\ \eta^{2k_0}q^{B-r}(1+\sum_{K=1}^{\infty}\widetilde{h}(K)q^K) \end{bmatrix}.$

The vector-valued function F' may be obtained from XF by normalizing both of the component functions of XF to have their leading Fourier coefficients equal one. In fact,

$$F' = \begin{bmatrix} 64^{r-A}\kappa_1^{-1} & 0 \\ 0 & 64^{r-B}\kappa_2^{-1} \end{bmatrix} XF.$$

Definition 5.1.3. *Let* $E := \begin{bmatrix} 64^{r-A}\kappa_1^{-1} & 0 \\ 0 & 64^{r-B}\kappa_2^{-1} \end{bmatrix} X$ *and let* $\rho' = E\rho E^{-1}$.

For each $k \in \mathbf{Z}$, the map $Z \mapsto EZ$ gives an isomorphism from $M_k(\rho)$ to $M_k(\rho')$ and an isomorphism from $M(\rho)$ to $M(\rho')$. Thus $F' \in M_{k_0}(\rho')$. It is convenient (although not necessary) to phrase our results in terms of vector-valued modular forms for ρ'. We will show in this section that if ρ has certain properties then for each integer k, there is a basis for $M_k(\rho')$ whose component functions have the property that all of their Fourier coefficients are algebraic numbers.

We are particularly interested in the arithmetic properties of the Fourier coefficients of F'.

Definition 5.1.4. *Let $\{d(K)\}_{K=1}^{\infty}$ and $\{\widetilde{d}(K)\}_{K=1}^{\infty}$ denote the sequences of numbers for which*

$$F' = \begin{bmatrix} q^{\frac{k_0}{12}+A-r}(1+\sum_{K=1}^{\infty} d(K)q^K) \\ q^{\frac{k_0}{12}+B-r}(1+\sum_{K=1}^{\infty} \widetilde{d}(K)q^K) \end{bmatrix}.$$

Remark: The fact that the sequences $\{d(K)\}_{K=1}^{\infty}$ and $\{\widetilde{d}(K)\}_{K=1}^{\infty}$ exist will be justified in this chapter.

To effectively study the Fourier coefficients of F', we will give formulas for $h(K)$ and $\widetilde{h}(K)$ in Theorem 5.1.9. In section two of this chapter, we will use the formulas in Theorem 5.1.9 to study the denominators of the Fourier coefficients of the component functions of F'. In particular, we will show that the sequence of the denominators of the Fourier coefficients of each of the component functions of F' is unbounded provided ρ satisfies a certain hypothesis. In the last section of this chapter, we show that if ρ satisfies a certain hypothesis then the sequence of the denominators of the Fourier coefficients of the component functions of every vector-valued modular form for ρ' is unbounded provided the Fourier coefficients are algebraic numbers.

To give formulas for $h(K)$ and $\widetilde{h}(K)$, it will also be important to use the Hauptmodul $\mathfrak{K} := 64\mathfrak{J}$ because $\mathfrak{K} \in \frac{1}{q}\mathbf{Z}[[q]]^{\times}$. A proof of this fact is given in Lemma 6.2.1 in the appendix.

We will express $h(K)$ and $\widetilde{h}(K)$ in terms of several sequences, which we will now define. Lemma 6.2.1 implies that for each integer $k \geq 0$, $\mathfrak{K}^{-k} \in q^k\mathbf{Z}[[q]]^{\times}$. We also show in the appendix that $\mathfrak{K} = q^{-1}(1+O(q))$. This fact together with Lemma 6.2.1 imply that for each positive integer t, $(q^{-1}\mathfrak{K}^{-1}-1)^t \in q^t\mathbf{Z}[[q]]$.

Definition 5.1.5. *For each integer $k \geq 0$, let $\{D(s,k)\}_{s=0}^{\infty}$ denote the sequence of integers such that*

$$\mathfrak{K}^{-k} = \sum_{s=0}^{\infty} D(s,k)q^s = q^k + \sum_{s=k+1}^{\infty} D(s,k)q^s.$$

Definition 5.1.6. *For each integer $t > 0$, let $\{C(t,d)\}_{d=0}^{\infty}$ denote the sequence of integers for which*

$$(q^{-1}\mathfrak{K}^{-1}-1)^t = \sum_{d=0}^{\infty} C(t,d)q^d = \sum_{d=t}^{\infty} C(t,d)q^d.$$

Definition 5.1.7. *We define*

$$g(m,n) := \binom{r}{n}\frac{(-1)^n 2^{4m+6n}(2A)_{2m}}{(1+A-B)_m m!}, \quad \widetilde{g}(m,n) := \binom{r}{n}\frac{(-1)^n 2^{4m+6n}(2B)_{2m}}{(1+B-A)_m m!}.$$

Definition 5.1.8. *We define*

$$f(k) := \sum_{\substack{n,m\geq 0\\ n+m=k}} g(m,n), \quad \widetilde{f}(k) := \sum_{\substack{n,m\geq 0\\ n+m=k}} \widetilde{g}(m,n).$$

Theorem 5.1.9. *There exist sequences $\{h(K)\}_{K=1}^{\infty}$ and $\{\widetilde{h}(K)\}_{K=1}^{\infty}$for which*

$$\mathfrak{J}^{-A}(\mathfrak{J}-1)^r\, {}_2F_1(A, \frac{1}{2}+A, 1+A-B; \mathfrak{J}(\tau)^{-1}) = 64^{A-r}q^{A-r}\left(1+\sum_{K=1}^{\infty} h(K)q^K\right)$$

and

$$\mathfrak{J}^{-B}(\mathfrak{J}-1)^r\,{}_2F_1(B,\frac{1}{2}+B,1+B-A;\mathfrak{J}(\tau)^{-1}) = 64^{B-r}q^{B-r}\left(1+\sum_{K=1}^{\infty}\widetilde{h}(K)q^K\right).$$

Moreover,

$$\begin{aligned} h(K) &= \sum_{\substack{d,s\geq 0\\ d+s=K}}\left(\sum_{t=0}^{d}C(t,d)\binom{A-r}{t}\right)\left(\sum_{k=0}^{s}f(k)D(s,k)\right)\\ &= f(K)+\sum_{k=0}^{K-1}f(k)D(s,k)+\sum_{\substack{d,s\geq 0\\ d+s=K\\ s<K}}\left(\sum_{t=0}^{d}C(t,d)\binom{A-r}{t}\right)\left(\sum_{k=0}^{s}f(k)D(s,k)\right)\end{aligned}$$

and

$$\begin{aligned} \widetilde{h}(K) &= \sum_{\substack{d,s\geq 0\\ d+s=K}}\left(\sum_{t=0}^{d}C(t,d)\binom{B-r}{t}\right)\left(\sum_{k=0}^{s}\widetilde{f}(k)D(s,k)\right)\\ &= \widetilde{f}(K)+\sum_{k=0}^{K-1}\widetilde{f}(k)D(s,k)+\sum_{\substack{d,s\geq 0\\ d+s=K\\ s<K}}\left(\sum_{t=0}^{d}C(t,d)\binom{B-r}{t}\right)\left(\sum_{k=0}^{s}\widetilde{f}(k)D(s,k)\right).\end{aligned}$$

Proof. We have that

$${}_2F_1(A,\frac{1}{2}+A,1+A-B;\mathfrak{J}^{-1}) = 1+\sum_{m=1}^{\infty}\frac{(A)_m(\frac{1}{2}+A)_m}{(1+A-B)_m m!}\mathfrak{J}^{-m} = 1+\sum_{m=1}^{\infty}\frac{2^{6m}(A)_m(\frac{1}{2}+A)_m}{(1+A-B)_m m!}\mathfrak{K}^{-m}$$

We note that $(A)_m(A+\frac{1}{2})_m = \left(2^{-m}\prod_{j=0}^{m-1}(2A+2j)\right)\left(2^{-m}\prod_{j=0}^{m-1}(2A+1+2j)\right) = 2^{-2m}(2A)_{2m}$.

Therefore

$${}_2F_1(A,\frac{1}{2}+A,1+A-B;\mathfrak{J}^{-1}) = 1+\sum_{m=1}^{\infty}\frac{2^{4m}(2A)_{2m}}{(1+A-B)_m m!}\mathfrak{K}^{-m}.$$

Similarly,

$${}_2F_1(B,\frac{1}{2}+B,1+B-A;\mathfrak{J}(\tau)^{-1}) = 1+\sum_{m=1}^{\infty}\frac{2^{4m}(2B)_{2m}}{(1+B-A)_m m!}\mathfrak{K}^{-m}.$$

As $\mathfrak{J} = \frac{1}{64q}(1+O(q))$, $\mathfrak{J}^{-1} = 64q(1+O(q))$. We may therefore apply Newton's binomial theorem and we have that:

$$\mathfrak{J}^{-A}(\mathfrak{J}-1)^r = \mathfrak{J}^{-A}\mathfrak{J}^r(1-\mathfrak{J}^{-1})^r = \mathfrak{J}^{r-A}(1-\mathfrak{J}^{-1})^r$$

$$= \mathfrak{J}^{r-A}\left(1+\sum_{n=1}^{\infty}\binom{r}{n}(-1)^n\mathfrak{J}^{-n}\right)$$
$$= 64^{A-r}\mathfrak{K}^{r-A}\left(1+\sum_{n=1}^{\infty}\binom{r}{n}(-1)^n 2^{6n}\mathfrak{K}^{-n}\right).$$

Thus

$$\begin{aligned}
&\mathfrak{J}^{-A}(\mathfrak{J}-1)^r\, {}_2F_1(A,\frac{1}{2}+A,1+A-B;\mathfrak{J}(\tau)^{-1}) \\
&= 64^{A-r}\mathfrak{K}^{r-A}\left(1+\sum_{n=1}^{\infty}\binom{r}{n}(-1)^n 2^{6n}\mathfrak{K}^{-n}\right)\left(1+\sum_{m=1}^{\infty}\frac{2^{4m}(2A)_{2m}}{(1+A-B)_m m!}\mathfrak{K}^{-m}\right) \\
&= 64^{A-r}\mathfrak{K}^{r-A}\left(1+\sum_{k=1}^{\infty}\left(\sum_{\substack{n,m\geq 0\\ n+m=k}}\binom{r}{n}\frac{(-1)^n 2^{4m+6n}(2A)_{2m}}{(1+A-B)_m m!}\right)\mathfrak{K}^{-k}\right)
\end{aligned}$$

We recall that

$$\begin{aligned}
g(m,n) := \binom{r}{n}\frac{(-1)^n 2^{4m+6n}(2A)_{2m}}{(1+A-B)_m m!} &= \frac{(-1)^n(-r)_n}{n!}\cdot\frac{(-1)^n 2^{4m+6n}(2A)_{2m}}{(1+A-B)_m m!} \\
&= \frac{2^{4m+6n}(-r)_n(2A)_{2m}}{(1+A-B)_m m! n!}.
\end{aligned}$$

We also recall that

$$f(k) := \sum_{\substack{n,m\geq 0\\ n+m=k}} g(m,n).$$

We have thus shown that

$$\mathfrak{J}^{-A}(\mathfrak{J}-1)^r\, {}_2F_1(A,\frac{1}{2}+A,1+A-B;\mathfrak{J}(\tau)^{-1}) = 64^{A-r}\mathfrak{K}^{r-A}\left(1+\sum_{k=1}^{\infty}f(k)\mathfrak{K}^{-k}\right).$$

Similarly,

$$\begin{aligned}
&\mathfrak{J}^{-B}(\mathfrak{J}-1)^r\, {}_2F_1(B,\frac{1}{2}+B,1+B-A;\mathfrak{J}(\tau)^{-1}) \\
&= 64^{B-r}\mathfrak{K}^{r-B}\left(1+\sum_{n=1}^{\infty}\binom{r}{n}(-1)^n 2^{6n}\mathfrak{K}^{-n}\right)\left(1+\sum_{m=1}^{\infty}\frac{2^{4m}(2B)_{2m}}{(1+B-A)_m m!}\mathfrak{K}^{-m}\right)
\end{aligned}$$

$$= 64^{B-r}\mathfrak{K}^{r-B}\left(1+\sum_{k=1}^{\infty}\left(\sum_{\substack{n,m\geq 0\\ n+m=k}}\binom{r}{n}\frac{(-1)^n 2^{4m+6n}(2B)_{2m}}{(1+B-A)_m m!}\right)\mathfrak{K}^{-k}\right).$$

We recall that

$$\widetilde{g}(m,n) := \binom{r}{n}\frac{(-1)^n 2^{4m+6n}(2B)_{2m}}{(1+B-A)_m m!} = \frac{2^{4m+6n}(-r)_n(2B)_{2m}}{(1+B-A)_m m! n!}.$$

We also recall that

$$\widetilde{f}(k) := \sum_{\substack{n,m\geq 0\\ n+m=k}} \widetilde{g}(m,n).$$

We have thus shown that

$$\mathfrak{J}^{-B}(\mathfrak{J}-1)^r\, {}_2F_1(B,\frac{1}{2}+B,1+B-A;\mathfrak{J}(\tau)^{-1}) = 64^{B-r}\mathfrak{K}^{r-B}\left(1+\sum_{k=1}^{\infty}\widetilde{f}(k)\mathfrak{K}^{-k}\right).$$

For each integer $k \geq 0$, we recall that $\{D(s,k)\}_{s=0}^{\infty}$ denotes the sequence of integers such that

$$\mathfrak{K}^{-k} = \sum_{s=0}^{\infty} D(s,k)q^s = q^k + \sum_{s=k+1}^{\infty} D(s,k)q^s.$$

Therefore

$$\begin{aligned}
&\mathfrak{J}^{-A}(\mathfrak{J}-1)^r\, {}_2F_1(A,\frac{1}{2}+A,1+A-B;\mathfrak{J}(\tau)^{-1})\\
&= 64^{A-r}\mathfrak{K}^{r-A}\left(1+\sum_{k=1}^{\infty} f(k)\mathfrak{K}^{-k}\right)\\
&= 64^{A-r}\mathfrak{K}^{r-A}\left(1+\sum_{k=1}^{\infty} f(k)\left(q^k+\sum_{s=k+1}^{\infty}D(s,k)q^s\right)\right)\\
&= 64^{A-r}\mathfrak{K}^{r-A}\left(1+\sum_{s=1}^{\infty} q^s\left(f(s)+\sum_{k=0}^{s-1}D(s,k)f(k)\right)\right).
\end{aligned}$$

Similarly,

$$\mathfrak{J}^{-B}(\mathfrak{J}-1)^r\, {}_2F_1(B,\frac{1}{2}+B,1+B-A;\mathfrak{J}(\tau)^{-1})$$

$$
\begin{aligned}
&= 64^{B-r}\mathfrak{K}^{r-B}\left(1+\sum_{k=1}^{\infty}\widetilde{f}(k)\mathfrak{K}^{-k}\right)\\
&= 64^{B-r}\mathfrak{K}^{r-B}\left(1+\sum_{k=1}^{\infty}\widetilde{f}(k)\left(q^k+\sum_{s=k+1}^{\infty}D(s,k)q^s\right)\right)\\
&= 64^{B-r}\mathfrak{K}^{r-B}\left(1+\sum_{s=1}^{\infty}q^s\left(\widetilde{f}(s)+\sum_{k=0}^{s-1}D(s,k)\widetilde{f}(k)\right)\right).
\end{aligned}
$$

We now compute the q-expansions of $\mathfrak{K}^{r-A}$ and $\mathfrak{K}^{r-B}$. We may use Newton's binomial theorem because $\mathfrak{K}^{-1} = q(1+O(q))$. We let $X(q)$ be the function for which $\mathfrak{K}^{-1} = q(1+X(q))$. It follows from Lemma 6.2.1 that $X(q) \in q\mathbf{Z}[[q]]$. We have that

$$
\mathfrak{K}^{r-A} = (q(1+X))^{A-r} = q^{A-r}(1+X)^{A-r} = q^{A-r}\left(1+\sum_{t=1}^{\infty}\binom{A-r}{t}X^t\right)
$$

and

$$
\mathfrak{K}^{r-B} = (q(1+X))^{B-r} = q^{B-r}(1+X)^{B-r} = q^{B-r}\left(1+\sum_{t=1}^{\infty}\binom{B-r}{t}X^t\right).
$$

For each positive integer t, we recall that $\{C(t,d)\}_{d=0}^{\infty}$ denotes the sequence of integers for which

$$
(q^{-1}\mathfrak{K}^{-1}-1)^t = X^t = \sum_{d=0}^{\infty}C(t,d)q^d = \sum_{d=t}^{\infty}C(t,d)q^d.
$$

Hence

$$
\begin{aligned}
\mathfrak{K}^{r-A} &= q^{A-r}\left(1+\sum_{t=1}^{\infty}\binom{A-r}{t}X^t\right)\\
&= q^{A-r}\left(1+\sum_{t=1}^{\infty}\binom{A-r}{t}\left(\sum_{d=t}^{\infty}C(t,d)q^d\right)\right)\\
&= q^{A-r}\left(1+\sum_{d=1}^{\infty}q^d\left(\sum_{t=0}^{d}C(t,d)\binom{A-r}{t}\right)\right).
\end{aligned}
$$

Similarly,

$$
\mathfrak{K}^{r-B} = q^{B-r}\left(1+\sum_{d=1}^{\infty}q^d\left(\sum_{t=0}^{d}C(t,d)\binom{B-r}{t}\right)\right).
$$

We have that

$$\begin{aligned}
&\mathfrak{J}^{-A}(\mathfrak{J}-1)^r\, {}_2F_1(A,\frac{1}{2}+A,1+A-B;\mathfrak{J}(\tau)^{-1})\\
&=64^{A-r}\mathfrak{K}^{r-A}\left(1+\sum_{s=1}^{\infty}q^s\left(f(s)+\sum_{k=0}^{s-1}D(s,k)f(k)\right)\right)\\
&=64^{A-r}q^{A-r}\left(1+\sum_{d=1}^{\infty}q^d\left(\sum_{t=0}^{d}C(t,d)\binom{A-r}{t}\right)\right)\left(1+\sum_{s=1}^{\infty}q^s\left(f(s)+\sum_{k=0}^{s-1}D(s,k)f(k)\right)\right).
\end{aligned}$$

We also have that

$$\begin{aligned}
&\mathfrak{J}^{-B}(\mathfrak{J}-1)^r\, {}_2F_1(B,\frac{1}{2}+B,1+B-A;\mathfrak{J}(\tau)^{-1})\\
&=64^{B-r}\mathfrak{K}^{r-B}\left(1+\sum_{s=1}^{\infty}q^s\left(\widetilde{f}(s)+\sum_{k=0}^{s-1}D(s,k)\widetilde{f}(k)\right)\right)\\
&=64^{B-r}q^{B-r}\left(1+\sum_{d=1}^{\infty}q^d\left(\sum_{t=0}^{d}C(t,d)\binom{B-r}{t}\right)\right)\left(1+\sum_{s=1}^{\infty}q^s\left(\widetilde{f}(s)+\sum_{k=0}^{s-1}D(s,k)\widetilde{f}(k)\right)\right).
\end{aligned}$$

We have thus shown that there exist sequences $\{h(K)\}_{K=1}^{\infty}$ and $\{\widetilde{h}(K)\}_{K=1}^{\infty}$ for which

$$\mathfrak{J}^{-A}(\mathfrak{J}-1)^r\, {}_2F_1(A,\frac{1}{2}+A,1+A-B;\mathfrak{J}(\tau)^{-1})=64^{A-r}q^{A-r}\left(1+\sum_{K=1}^{\infty}h(K)q^K\right)$$

and

$$\mathfrak{J}^{-B}(\mathfrak{J}-1)^r\, {}_2F_1(B,\frac{1}{2}+B,1+B-A;\mathfrak{J}(\tau)^{-1})=64^{B-r}q^{B-r}\left(1+\sum_{K=1}^{\infty}\widetilde{h}(K)q^K\right).$$

Moreover, we have proven that

$$\begin{aligned}
h(K)&=\sum_{\substack{d,s\geq 0\\ d+s=K}}\left(\sum_{t=0}^{d}C(t,d)\binom{A-r}{t}\right)\left(f(s)+\sum_{k=0}^{s-1}f(k)D(s,k)\right)\\
&=f(K)+\sum_{k=0}^{K-1}f(k)D(s,k)+\sum_{\substack{d,s\geq 0\\ d+s=K\\ s<K}}\left(\sum_{t=0}^{d}C(t,d)\binom{A-r}{t}\right)\left(f(s)+\sum_{k=0}^{s-1}f(k)D(s,k)\right).
\end{aligned}$$

We have also proven that

$$\widetilde{h}(K) = \sum_{\substack{d,s\geq 0\\ d+s=K}} \left(\sum_{t=0}^{d} C(t,d)\binom{B-r}{t}\right)\left(\widetilde{f}(s) + \sum_{k=0}^{s-1}\widetilde{f}(k)D(s,k)\right)$$
$$= \widetilde{f}(K) + \sum_{k=0}^{K-1}\widetilde{f}(k)D(s,k) + \sum_{\substack{d,s\geq 0\\ d+s=K\\ s<K}} \left(\sum_{t=0}^{d} C(t,d)\binom{B-r}{t}\right)\left(\widetilde{f}(s) + \sum_{k=0}^{s-1}\widetilde{f}(k)D(s,k)\right).$$

□

The following result about F' will be quite useful.

Proposition 5.1.10.

$$F' = \begin{bmatrix} q^{\frac{k_0}{12}+A-r}(1+O(q)) \\ q^{\frac{k_0}{12}+B-r}(1+O(q)) \end{bmatrix}.$$

Proof. We have that

$$F' = \begin{bmatrix} 64^{r-A}\kappa_1^{-1} & 0 \\ 0 & 64^{r-B}\kappa_2^{-1} \end{bmatrix} XF$$
$$= \begin{bmatrix} 64^{r-A}\kappa_1^{-1} & 0 \\ 0 & 64^{r-B}\kappa_2^{-1} \end{bmatrix} \begin{bmatrix} \kappa_1\eta^{2k_0}(\mathfrak{J}-1)^r\mathfrak{J}^{-A}\ {}_2F_1(A,\frac{1}{2}+A,1+A-B;\mathfrak{J}^{-1}) \\ \kappa_2\eta^{2k_0}(\mathfrak{J}-1)^r\mathfrak{J}^{-B}\ {}_2F_1(B,\frac{1}{2}+B,1+B-A;\mathfrak{J}^{-1}) \end{bmatrix}$$
$$= \begin{bmatrix} 64^{r-A}\kappa_1^{-1} & 0 \\ 0 & 64^{r-B}\kappa_2^{-1} \end{bmatrix} \begin{bmatrix} \kappa_1\eta^{2k_0}64^{A-r}q^{A-r}(1+\sum_{K=1}^{\infty}h(K)q^K) \\ \kappa_2\eta^{2k_0}64^{B-r}q^{B-r}(1+\sum_{K=1}^{\infty}\widetilde{h}(K)q^K) \end{bmatrix}$$
$$= \begin{bmatrix} \eta^{2k_0}q^{A-r}(1+\sum_{K=1}^{\infty}h(K)q^K) \\ \eta^{2k_0}q^{B-r}(1+\sum_{K=1}^{\infty}\widetilde{h}(K)q^K) \end{bmatrix}$$
$$= \begin{bmatrix} q^{\frac{k_0}{12}+A-r}(1+O(q)) \\ q^{\frac{k_0}{12}+B-r}(1+O(q)) \end{bmatrix}.$$

□

We remark that we have thus shown that there exist sequences $\{d(K)\}_{K=1}^{\infty}$ and $\{\widetilde{d}(K)\}_{K=1}^{\infty}$ for which

$$F' = \begin{bmatrix} q^{\frac{k_0}{12}+A-r}(1+\sum_{K=1}^{\infty} d(K)q^K) \\ q^{\frac{k_0}{12}+B-r}(1+\sum_{K=1}^{\infty} \widetilde{d}(K)q^K) \end{bmatrix}.$$

We shall now place some assumptions on ρ to ensure that all of the Fourier coefficients of F' are algebraic numbers. One way to proceed is to study those representations ρ for which $\rho(T)$ has finite order. **Henceforth, we shall always assume that $\rho(T)$ has finite order.** This assumption implies that $\rho(T)$ is diagonalizable. We recall that $A = r + l_1$, $B = r + l_2$, $l_1 \equiv m_1 + \frac{k_0}{6} \pmod{\mathbf{Z}}$, and $l_2 \equiv m_2 + \frac{k_0}{6} \pmod{\mathbf{Z}}$. Therefore $A - B = l_1 - l_2 \equiv m_1 - m_2 \pmod{\mathbf{Z}}$. We have previously shown that the irreducibility of ρ implies that $m_1 - m_2 \notin \mathbf{Z}$. Thus $A - B \notin \mathbf{Z}$. The assumption that $\rho(T)$ has finite order implies that the eigenvalues $e^{2\pi i m_1}$ and $e^{2\pi i m_2}$ of $\rho(T)$ are roots of unity and that $m_1, m_2 \in \mathbf{Q}$. Because $m_1, m_2 \in \mathbf{Q}$ and $k_0 \in \mathbf{Z}$, we have that $l_1, l_2 \in \mathbf{Q}$. Thus $A - B = l_1 - l_2 \in \mathbf{Q} \setminus \mathbf{Z}$. The fact that $l_1, l_2 \in \mathbf{Q}$ also implies that $\mathbf{Q}(A) = \mathbf{Q}(r) = \mathbf{Q}(B)$.

Theorem 5.1.11. *If $\rho(T)$ has finite order and if $r \in \overline{\mathbf{Q}}$ then all of the Fourier coefficients of both of the component functions of F' are elements of $\mathbf{Q}(r)$ and are therefore algebraic numbers. Moreover, for each $k \in \mathbf{Z}$, there exists a basis of $M_k(\rho')$ consisting of vector-valued modular forms whose component functions have Fourier coefficients which are elements of $\mathbf{Q}(r)$ and thus are algebraic numbers.*

Proof. We recall that $\eta = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \in q^{\frac{1}{24}}\mathbf{Z}[[q]]^{\times}$. Therefore the Fourier coefficients of the component functions of F' are algebraic numbers if and only if for all K, $h(K)$ and

$\widetilde{h}(K)$ are algebraic numbers. The formulas for $h(K)$ and $\widetilde{h}(K)$ in Theorem 5.1.9 show that $h(K) \in \mathbf{Q}(A,r) = \mathbf{Q}(r)$ and $\widetilde{h}(K) \in \mathbf{Q}(B,r) = \mathbf{Q}(r)$. Thus if $r \in \overline{\mathbf{Q}}$ then all of the Fourier coefficients of both components of F' are elements of $\mathbf{Q}(r)$ and are therefore algebraic numbers. For each integer k, the map $Z \mapsto EZ$ gives an isomorphism from $M_k(\rho)$ to $M_k(\rho')$. Thus $M(\rho') = M(\Gamma_0(2))F' \bigoplus M(\Gamma_0(2))D_{k_0}F'$. The fact that the Fourier coefficients of the component functions of F' are elements of $\mathbf{Q}(r)$ together with the fact that $E_2 \in \mathbf{Z}[[q]]$ imply that all of the Fourier coefficients of the component functions of $D_{k_0}F'$ are elements of $\mathbf{Q}(r)$ and are therefore algebraic numbers. Finally, for each integer k, there exists a basis of $M_k(\Gamma_0(2))$ consisting of modular forms with integral Fourier coefficients since $M(\Gamma_0(2)) = \mathbf{C}[E_4, G]$ and $E_4, G \in \mathbf{Z}[[q]]$. In fact, a basis for $M_k(\Gamma_0(2))$ consisting of vector-valued modular forms whose component functions have Fourier coefficients which are algebraic numbers is $\{G^a E_4^b F' : 2a+4b = k - k_0, a, b \geq 0, a, b \in \mathbf{Z}\} \bigcup \{G^a E_4^b D_{k_0} F' : 2a+4b = k-k_0-2, a, b \geq 0, a, b \in \mathbf{Z}\}$.

□

5.2 Unbounded Denominators: The Minimal Weight Case

In this section, we study the arithmetic of the Fourier coefficients of the component functions of F'. These Fourier coefficients are algebraic numbers but they need not be rational numbers. We therefore need to define the numerator and the denominator of an algebraic number. Let $\overline{\mathbf{Z}}$ denote the ring of algebraic integers. It is well-known that if ζ is an algebraic number then there exists a positive integer N such that $N\zeta \in \overline{\mathbf{Z}}$.

Definition 5.2.1. *If ζ is a nonzero algebraic number then **the denominator of** ζ is the smallest* *positive integer Z such that $Z\zeta \in \overline{\mathbf{Z}}$ and **the numerator of** ζ is defined to be the algebraic integer $Z\zeta$.*

We say that an integer Z is **a denominator of** ζ if $Z\zeta \in \overline{\mathbf{Z}}$. The collection of denominators of ζ form a non-zero ideal of $\mathbf{Z}$ and is therefore generated by a smallest positive integer, which is **the denominator of** ζ. We observe that there does not exist an integer $j > 1$ which divides both the denominator and numerator of ζ in the ring $\overline{\mathbf{Z}}$. To see why, we notice that if there exists some integer $j > 1$ which divides the denominator N of ζ and which also divides $N\zeta$ in the ring $\overline{\mathbf{Z}}$ then $\frac{N}{j}\zeta \in \overline{\mathbf{Z}}$, which contradicts the minimality of N.

Definition 5.2.2. *Let p denote a prime number. We say that an algebraic number ζ is **p-integral** if p does not divide the denominator of ζ.*

We shall have occasion to use the following lemma.

Lemma 5.2.3. *Let p denote a prime number. The collection of all algebraic numbers which are p-integral form a ring.*

Proof. Let ζ_1 and ζ_2 denote algebraic numbers which are p-integral. Let n_1 denote the denominator of ζ_1 and let n_2 denote the denominator of ζ_2. Then $n_1 n_2(\zeta_1\zeta_2) = (n_1\zeta_1)(n_2\zeta_2) \in \overline{\mathbf{Z}}$ and $n_1 n_2(\zeta_1+\zeta_2) = n_2(n_1\zeta_1)+n_1(n_2\zeta_2) \in \overline{\mathbf{Z}}$. Thus both the denominator of $\zeta_1\zeta_2$ and the denominator of $\zeta_1+\zeta_2$ divide $n_1 n_2$. We note that $p \nmid n_1$ and $p \nmid n_2$ since ζ_1 and ζ_2 are p-integral. Therefore $p \nmid n_1 n_2$. Thus p does not divide the denominator of $\zeta_1\zeta_2$ and p does not divide the denominator of $\zeta_1+\zeta_2$ since the denominator of $\zeta_1\zeta_2$ and the denominator of $\zeta_1+\zeta_2$ both divide $n_1 n_2$. We conclude that $\zeta_1\zeta_2$ and $\zeta_1+\zeta_2$ are p-integral.

□

The following Lemma will be quite useful when studying the denominators of the Fourier coefficients of F'.

Lemma 5.2.4. *Let M denote a square-free integer. Let p denote an odd prime number for which M is not a quadratic residue mod p. Let $X \in \boldsymbol{Q}(\sqrt{M})$ such that $X \notin \boldsymbol{Q}$. Let Z denote the smallest positive integer such that ZX is an algebraic integer and let $Y := ZX$. Let y and z denote the integers for which $Y = \frac{x+y\sqrt{M}}{2}$. Let $R \in \boldsymbol{Q}$. If $p \nmid y$ then p does not divide the numerator of any element in the set $\{(X+R)_t : t \geq 1\}$.*

Remark: We note that $y \neq 0$ since $Y \notin \mathbf{Q}$. We also note that y and z have the same parity since Y is an algebraic integer.

Proof. Let σ denote the non-trivial element in $\mathrm{Gal}(\mathbf{Q}(\sqrt{M})/\mathbf{Q})$ and let N denote the norm map from $\mathbf{Q}(\sqrt{M})$ from $\mathbf{Q}$. Let $O_{\mathbf{Q}(\sqrt{M})}$ denote the ring of integers of $\mathbf{Q}(\sqrt{M})$. We proceed by contradiction and suppose that there exists some positive integer t such that p divides the numerator of $(X+R)_t = (\frac{Y+RZ}{Z})_t = Z^{-t}\prod_{i=0}^{t-1}(Y+RZ+iZ)$ in the ring $O_{\mathbf{Q}(\sqrt{M})}$. Then $p \mid \prod_{i=0}^{t-1}(Y+RZ+iZ)$ in the ring $O_{\mathbf{Q}(\sqrt{M})}$ and $p \nmid Z^t$ in the ring $O_{\mathbf{Q}(\sqrt{M})}$. Thus $p \nmid Z$. We have that $N(p) = p^2 \mid \prod_{i=0}^{t-1} N(Y+RZ+iZ)$ in the ring $\mathbf{Z}$. Thus $p \mid N(Y+RZ+jZ)$ for some integer j with $0 \leq j \leq t-1$. Therefore

$$0 \equiv 4N(Y+RZ+jZ) = 4N\left(\frac{x}{2}+RZ+jZ+\frac{y}{2}\sqrt{M}\right) = (x+2(R+j)Z)^2 - My^2 \pmod{p}.$$

As $p \nmid y$, M is a quadratic residue mod p. This is a contradiction and our proof is now complete. $\square$

We recall that $\mathbf{Q}(A) = \mathbf{Q}(B) = \mathbf{Q}(r)$ and $l_1, l_2 \in \mathbf{Q}$ since $\rho(T)$ has finite order. We note that

$a, b+c \in \mathbf{Q}$ since $l_1 l_2 = b+c$ and $l_1 + l_2 = \frac{1}{6} - a$. We recall that r satisfies the quadratic equation $r^2 + (\frac{-1-3a}{3})r + b + 4c = 0$. We have that $\mathbf{Q}(\frac{-1-3a}{3}, b+4c) = \mathbf{Q}(b+4c) = \mathbf{Q}(c)$. Thus $[\mathbf{Q}(c)(r) : \mathbf{Q}(c)] \leq 2$. We are most interested in the case when $c \in \mathbf{Q}$. If $c \in \mathbf{Q}$ and if $[\mathbf{Q}(r) : \mathbf{Q}] = 2$ then we will be able to apply Lemma 5.2.4 to analyze the denominators of the Fourier coefficients of F'.

Hypothesis 5.2.5. ***Throughout the rest of this paper, we shall assume that*** $\rho(T)$ ***has finite order,*** $c \in \boldsymbol{Q}$ ***and that*** $[\boldsymbol{Q}(r) : \boldsymbol{Q}] = 2$.

Definition 5.2.6. *Let M denote the square-free integer for which $\boldsymbol{Q}(r) = \boldsymbol{Q}(\sqrt{M})$.*

We have previously shown that $A - B \in \mathbf{Q} \setminus \mathbf{Z}$. We therefore make the following definition.

Definition 5.2.7. *Let $u, v \in \mathbf{Z}$ with $v > 1$, $gcd(u,v) = 1$ such that $A - B = \frac{u}{v}$.*

Definition 5.2.8. *Let S denote the set of odd prime numbers p for which M is not a quadratic residue mod p and $p \equiv u \pmod{v}$.*

Definition 5.2.9. *Let $\widetilde{S}$ denote the set of odd prime numbers p for which M is not a quadratic residue mod p and $p \equiv -u \pmod{v}$.*

It follows from the quadratic reciprocity law and Dirichlet's theorem on primes in arithmetic progressions that if S is infinite then S has positive density in the set of prime numbers and if $\widetilde{S}$ is infinite then $\widetilde{S}$ has positive density in the set of prime numbers.

We will show that if S is infinite then every sufficiently large element in S divides the denominator of at least one Fourier coefficient of the first component of F'. We will also show that if $\widetilde{S}$ is infinite then every sufficiently large element in $\widetilde{S}$ divides the denominator of at least one Fourier coefficient of the second component of F'. At the end of this section, we will give examples of representations for which we can prove that S and $\widetilde{S}$ are infinite. We begin with the following proposition.

Proposition 5.2.10. *Assume that S is an infinite set. Let K denote an integer such that $p_K := u+Kv \in S$. If $m+n \leq K$ and if $m \neq K$ then for all sufficiently large K, $g(m,n)$ is p_K-integral and p_K does not divide the numerator of $g(m,n)$. Consequently, for all sufficiently large K, $f(k)$ is p_K-integral provided $k < K$.*

Proof. We recall that

$$g(m,n) := \frac{2^{4m+6n}(-r)_n(2A)_{2m}}{(1+A-B)_m m!n!} = \frac{2^{4m+6n}(-r)_n(2A)_{2m}}{(1+\frac{u}{v})_m m!n!} = \frac{v^m 2^{4m+6n}(-r)_n(2A)_{2m}}{\prod_{j=1}^{m}(u+jv)m!n!}.$$

We will show that if K is sufficiently large then p_K does not divide the numerator of $(-r)_n$, p_K does not divide the numerator of $(2A)_{2m}$, and p_K does not divide any of the integers v^m, $2^{4m+6n}, \prod_{j=1}^{m}(u+jv), m!, n!$. Lemma 5.2.3 will then imply that $g(m,n)$ is p_K-integral.

The stipulations $m+n \leq K$ and $m \neq K$ imply $m \leq K-1$ and $n \leq K$. In particular, $p_K = u+Kv > u+mv \geq u+jv$ for any j with $1 \leq j \leq m$. Thus p_K does not divide any of the positive elements in the set $\{u+jv : 1 \leq j \leq m\}$. We note that $0 \notin \{u+jv : 1 \leq j \leq m\}$ since $\gcd(u,v) = 1$ and $v > 1$. It is possible that some element(s) in the set $\{u+jv : 1 \leq j \leq m\}$ are negative since u might be negative. Nevertheless, only finitely many elements in the set $\{u+jv : 1 \leq j \leq m\}$ are negative since $v > 0$. Because u and v are fixed, we may choose a sufficiently large K such that p_K does not divide any of the negative elements in the set $\{u+jv : 1 \leq j \leq m\}$. For such a K, p_K does not divide any element in the set $\{u+jv : 1 \le j \le m\}$. Because p_K is prime, $p_K \nmid \prod_{j=1}^{m}(u+jv)$. In particular, we have shown that $p_K \nmid \prod_{j=1}^{K-1}(u+jv)$.

If K is sufficiently large then $p_K > K$ and thus $p_K > m$ and $p_K > n$. Because p_K is prime, $p_K \nmid m!$ and $p_K \nmid n!$. If K is sufficiently large then $p_K \nmid v$ and so $p_K \nmid v^m$ for any m. We recall that $\mathbf{Q}(A) = \mathbf{Q}(r) = \mathbf{Q}(\sqrt{M})$. Let $x_1, y_1, x_2, y_2 \in \mathbf{Z}$ such that $2A = \frac{x_1+y_1\sqrt{M}}{2}$ and $-r = \frac{x_2+y_2\sqrt{M}}{2}$. We note that $y_1 \neq 0$ and $y_2 \neq 0$ since $A, r \notin \mathbf{Q}$. If K is sufficiently large then $p_K \nmid y_1$ and $p_K \nmid y_2$ and it then follows from Lemma 5.2.4 that p_K does not divide the numerator of $(2A)_{2m}$ for any m and p_K does not divide the numerator of $(-r)_n$ for any n. Moreover, if K is sufficiently large then p_K also does not divide the denominators of $2A$ and $-r$. Hence p_K does not divide the denominators of $(2A)_{2m}$ and $(-r)_n$ for any n and m. Finally, $p_K \nmid 2^{4m+6n}$ since p_K is an odd prime. We have shown that p_K does not divide the numerator of $(-r)_n(2A)_{2m}$ and that p_K does not divide any of the integers v^m, 2^{4m+6n}, $\prod_{j=1}^{m}(u+jv)$, $m!$, and $n!$. We conclude that $g(m,n)$ is p_K-integral by applying Lemma 5.2.3.

If $k < K$ and if $m+n=k$ then we have shown that $g(m,n)$ is p_K-integral. Hence if $k < K$ then $f(k) := \sum_{\substack{n,m\ge 0\\ n+m=k}} g(m,n)$ is a sum of p_K-integral numbers and is therefore p_K -integral. □

Theorem 5.2.11. *Assume that* ρ *satisfies Hypothesis 5.2.5. Assume that* S *is an infinite set. If* K *is sufficiently large then* $f(K)$ *is not* p_K*-integral and* $p_K f(K)$ *is* p_K*-integral.*

Proof. We have that $f(K) = g(K,0) + \sum_{\substack{m+n\le K\\ m\neq K}} g(m,n)$. We have shown in Proposition 5.2.10 that $g(m,n)$ is p_K-integral if $m+n \le K$ and if $m \neq K$. We apply Lemma 5.2.3 to get that

$\sum_{\substack{m+n\leq K\\ m\neq K}} g(m,n)$ is p_K-integral. It now suffices to show that $g(K,0)$ is not p_K-integral. We note that $g(K,0) = \frac{v^K 2^{4K}(2A)_{2K}}{\prod_{j=1}^{K}(u+jv)K!}$. We have previously shown that if K is sufficiently large then $v^K, 2^{4K}$, $K!$, and $\prod_{j=1}^{K-1}(u+jv)$ are not divisible by p_K and that p_K does not divide the numerator nor the denominator of $(2A)_{2K}$. Therefore $p_K \parallel (u+Kv)\prod_{j=1}^{K-1}(u+jv) = \prod_{j=1}^{K}(u+jv)$. We conclude that p_K but not p_K^2 divides the denominator of $g(K,0)$. Hence $f(K)$ is not p_K-integral and $p_K f(K)$ is p_K-integral. □

Theorem 5.2.12. *Assume that* ρ *satisfies Hypothesis 5.2.5. Assume* S *is infinite. If* K *is sufficiently large then the denominator of* $h(K)$ *is divisible by* p_K*. Moreover,* $h(J)$ *is* p_K*-integral if* $J < K$*. Hence every prime in* S *which is sufficiently large divides the denominator of* $h(K)$ *for some* K*.*

Proof. We recall that:

$$h(K) = f(K) + \sum_{k=0}^{K-1} f(k)D(s,k) + \sum_{\substack{d,s\geq 0\\ d+s=K\\ s<K}} \left(\sum_{t=0}^{d} C(t,d)\binom{A-r}{t}\right)\left(\sum_{k=0}^{s} f(k)D(s,k)\right).$$

We have shown that $f(K)$ is not p_K-integral and that $f(k)$ is p_K-integral if $k < K$. Moreover, the numbers $D(s,k)$ and $C(t,d)$ are integers. Thus for all $s \leq K-1$, $\sum_{k=0}^{s} f(k)D(s,k)$ is p_K-integral. We recall that $A - r = l_1 \in \mathbf{Q}$. Let $y,z \in \mathbf{Z}$ such that $l_1 = \frac{y}{z}$. Then $\binom{A-r}{t} = \binom{l_1}{t} = \frac{\prod_{j=0}^{t-1}(y-jz)}{z^t t!}$. We note that $t \leq d \leq K$ and $K < p_K$ if K is sufficiently large. Therefore $p_K \nmid t!$ if K is sufficiently large. We also note that $p_K \nmid z$ if K is sufficiently large and thus $p_K \nmid z^t$. Thus if K is sufficiently large then $\binom{A-r}{t}$ is p_K-integral for all $t \leq K$. Hence for all $d \leq K$, $\sum_{t=0}^{d} C(t,d)\binom{A-r}{t}$ is p_K-

integral. Thus $h(J)$ is p_K-integral if $J < K$. We have also shown that

$$h(K) - f(K) = \sum_{k=0}^{K-1} f(k)D(s,k) + \sum_{\substack{d,s\geq 0 \\ d+s=K \\ s<K}} \left(\sum_{t=0}^{d} C(t,d) \binom{A-r}{t} \right) \left(\sum_{k=0}^{s} f(k)D(s,k) \right)$$

is p_K-integral. Because $f(K)$ is not p_K-integral, $h(K)$ is not p_K-integral. $\square$

Theorem 5.2.13. *Assume that* ρ *satisfies Hypothesis 5.2.5. Let* $\widetilde{p_K} := -u + Kv$*. Assume that* $\widetilde{S}$ *is infinite. If* K *is sufficiently large then* $\widetilde{p_K}$ *divides the denominator of* $\widetilde{h}(K)$*. Moreover,* $\widetilde{h}(J)$ *is* p_K*-integral if* $J < K$*. Thus the set of primes that divide the denominator of* $\widetilde{h}(K)$ *for some* K *is infinite and has positive density within the set of primes.*

Proof. The proof is completely analogous to the proof of Theorem 5.2.12. We note that $\mathbf{Q}(B) = \mathbf{Q}(A) = \mathbf{Q}(r) = \mathbf{Q}(\sqrt{M})$ and $B - A = \frac{-u}{v}$. We also have that

$$\widetilde{g}(m,n) := \frac{2^{4m+6n}(-r)_n(2B)_{2m}}{(1+B-A)_m m!n!} = \frac{2^{4m+6n}(-r)_n(2B)_{2m}}{(1+\frac{-u}{v})_m m!n!} = \frac{v^m 2^{4m+6n}(-r)_n(2A)_{2m}}{\prod_{j=1}^{m}(-u+jv)m!n!}.$$

$\square$

We recall that $\{d(K)\}_{K=1}^{\infty}$ and $\{\widetilde{d}(K)\}_{K=1}^{\infty}$ denote the sequences for which

$$F' = \begin{bmatrix} q^{\frac{k_0}{12}+A-r}(1+\sum_{K=1}^{\infty} d(K)q^K) \\ q^{\frac{k_0}{12}+B-r}(1+\sum_{K=1}^{\infty} \widetilde{d}(K)q^K) \end{bmatrix}.$$

Theorem 5.2.14. *Assume that* ρ *satisfies Hypothesis 5.2.5. If* S *is infinite then for all sufficiently large* K*,* p_K *divides the denominator of* $d(K)$ *and* $d(i)$ *is* p_K*-integral if* $i < K$*. Thus if* S *is infinite then the set of primes that divide the denominator of at least one Fourier coefficient of the first component function of* F' *is infinite and has positive density within the set of primes. If* $\widetilde{S}$ *is* *infinite then for all sufficiently large K, $\widetilde{p_K}$ divides the denominator of $\widetilde{d}(K)$ and $\widetilde{d}(i)$ is $\widetilde{p_K}$-integral if $i < K$. Thus if $\widetilde{S}$ is infinite then the set of primes that divide the denominator of at least one Fourier coefficient of the second component function of F' is infinite and has positive density within the set of primes.*

Proof. We recall that $\eta = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) = q^{\frac{1}{24}}(1+O(q)) \in q^{\frac{1}{24}}\mathbf{Z}[[q]]^{\times}$. Therefore $\eta^{2k_0} = q^{\frac{k_0}{12}}(1+O(q)) \in q^{\frac{k_0}{12}}\mathbf{Z}[[q]]^{\times}$. We define the sequence of integers $\{e(K)\}_{K=1}^{\infty}$ by setting $\eta^{2k_0} = q^{\frac{k_0}{12}}(1+\sum_{K=1}^{\infty} e(K)q^K)$. We have that

$$\begin{aligned}
&\eta^{2k_0}(\tau)(\mathfrak{J}(\tau)-1)^r\mathfrak{J}(\tau)^{-A}\ {}_2F_1(A, \frac{1}{2}+A, 1+A-B; \mathfrak{J}(\tau)^{-1}) \\
&= q^{\frac{k_0}{12}}(1+\sum_{i=1}^{\infty} e(i)q^i)64^{A-r}q^{A-r}\left(1+\sum_{K=1}^{\infty} h(K)q^K\right).
\end{aligned}$$

Thus $1+\sum_{K=1}^{\infty} d(K)q^k = (1+\sum_{i=1}^{\infty} e(i)q^i)(1+\sum_{K=1}^{\infty} h(K)q^K)$.

Hence $d(K) = h(K)+\sum_{i=0}^{K-1} e(i)h(K-i)$. We note that each $e(K-i)$ is an integer. We have proven that $h(K)$ is not p_K-integral but $h(i)$ is p_K-integral if $i < K$. Therefore $d(i)$ is p_K-integral if $i < K$ and $d(K)$ is not p_K-integral. The proof that if K is sufficiently large then $\widetilde{d}(i)$ is p_K-integral if $i < K$ and $\widetilde{p_K}$ divides the denominator of $\widetilde{d}(K)$ is completely analogous. □

Lemma 5.2.15. *If $m_1 - m_2 \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$ then $S = \widetilde{S}$, S is infinite, and S has density $\frac{1}{2}$ within the set of primes.*

Proof. In this case, $v = 2$ and

$$S = \widetilde{S} = \{p : p \text{ is odd and } M \text{ is not a quadratic residue mod } p\}.$$

The quadratic reciprocity law and Dirichlet's theorem on primes in arithmetic progressions imply that since M is not a perfect square, S is an infinite set and has density $\frac{1}{2}$ in the set of primes. $\square$

Lemma 5.2.16. *If* ρ *is induced from a character of* $\Gamma(2)$ *then* $m_1 - m_2 \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$.

Proof. Assume that ψ is a character on $\Gamma(2)$ such that $\rho = \mathrm{Ind}_{\Gamma(2)}^{\Gamma_0(2)} \psi$. Let $\widetilde{\psi} : \Gamma_0(2) \to \mathbf{C}$ be the function defined by setting $\widetilde{\psi}(g) := \psi(g)$ if $g \in \Gamma(2)$ and $\widetilde{\psi}(g) := 0$ if $g \notin \Gamma(2)$. We note that $\Gamma(2)$ is an index two subgroup of $\Gamma_0(2)$ and $\{I, T\}$ is a basis of left coset representatives for $\Gamma(2)$ in $\Gamma_0(2)$. Then $\rho(T)$ is similar to the matrix $\begin{bmatrix} \widetilde{\psi}(T) & \widetilde{\psi}(T^2) \\ \widetilde{\psi}(I) & \widetilde{\psi}(T) \end{bmatrix} = \begin{bmatrix} 0 & \widetilde{\psi}(T^2) \\ 1 & 0 \end{bmatrix}$. Thus the trace of $\rho(T)$ is equal to zero since the trace of $\rho(T)$ is equal to the trace of the matrix $\begin{bmatrix} 0 & \widetilde{\psi}(T^2) \\ 1 & 0 \end{bmatrix}$. As $\rho(T)$ is a diagonalizable matrix with eigenvalues $e^{2\pi i m_1}$ and $e^{2\pi i m_2}$, we have that $0 = \mathrm{trace}(\rho(T)) = e^{2\pi i m_1} + e^{2\pi i m_2}$. Thus $e^{2\pi i (m_1 - m_2)} = -1$. Hence $m_1 - m_2 \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$. $\square$

Theorem 5.2.17. *Let* ρ *denote a two-dimensional irreducible representation of* $\Gamma_0(2)$ *which is induced from a character of* $\Gamma(2)$. *Assume that* ρ *satisfies Hypothesis 5.2.5. Then S is infinite and S has density* $\frac{1}{2}$ *in the set of prime numbers. Moreover, every sufficiently large prime number in S divides the denominator of at least one Fourier coefficient of the first and of the second component functions of* F'.

Proof. The hypothesis that ρ is induced from a character of $\Gamma(2)$ implies $m_1 - m_2 \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$. Therefore $S = \widetilde{S}$. Lemma 5.2.15 implies that S is infinite and has density one half in the set of primes. The conclusion of the theorem now follows from Theorem 5.2.14. $\square$

5.3 Unbounded Denominators: The General Case

In the previous section, we proved that if ρ satisfies Hypothesis 5.2.5 and if S and $\widetilde{S}$ are infinite then the denominators of the Fourier coefficients of each of the component functions of F' are unbounded. In this section, we will prove the analogous result for vector-valued modular forms of any weight in Theorem 5.3.7. Our method of proof follows very closely Chris Marks' paper [19]. In [19], Marks proved a similar result for three-dimensional vector-valued modular forms with respect to certain three-dimensional representations of Γ. Our proof that the denominators of the Fourier coefficients of each of the component functions of F' are unbounded uses entirely different ideas than those in [19].

Definition 5.3.1. *Let Z denote a vector-valued modular form whose component functions have Fourier coefficients which are algebraic numbers. We say that Z has **bounded denominators** if the sequence of the denominators of the Fourier coefficients of each component function of Z are bounded. If Z does not have bounded denominators, we say that Z has **unbounded denominators**.*

Remark: We proved in Theorem 5.1.11 that for each integer k, there exists a basis for $M_k(\rho')$ consisting of vector-valued modular forms whose component functions have Fourier coefficients in $\overline{\mathbf{Q}}$.

Definition 5.3.2. *Let L denote a number field or $\overline{\mathbf{Q}}$. The notation $M_k(\rho')_L$ denotes those vector-valued modular forms in $M_k(\rho')$ whose component functions have all of their Fourier coefficients in L. The notation $M_k(\Gamma_0(2))_L$ denotes those modular forms in $M_k(\Gamma_0(2))$ whose Fourier* *coefficients all belong to L. We define* $M(\rho)_L := \bigoplus_{k \in \mathbf{Z}} M_k(\rho)_L$ *and* $M(\Gamma_0(2))_L := \bigoplus_{k \in \mathbf{Z}} M_k(\Gamma_0(2))_L$.

We will need to use the fact that if $f \in M(\Gamma_0(2))_L$ then there exists a positive integer N such that $Nf \in \overline{\mathbf{Z}}[[q]]$. To do so, we first prove the following lemma.

Lemma 5.3.3. *Let* $f = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_0(2))$. *Let* $r(k) = dim\, M_k(\Gamma_0(2))$. *Let* R *denote the* $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}]$*-module generated by* $a_0, ..., a_{r(k)-1}$. *Then all* $a_n \in R$ *and* $f = \sum_{2a+4b=k} c_{a,b} G^a E_4^b$ *where each* $c_{a,b} \in R$.

Proof. This proof follows the proof of Theorem 4.2 in Lang's book on modular forms [18]. We proceed by induction on k. The result is clear if $k = 2$ and if $k = 0$ since dim $M_2(\Gamma_0(2)) =$ dim $M_0(\Gamma_0(2)) = 1$. Now, let k denote an even integer for which $k > 2$ and let $f = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_0(2))$. Then $f - a_0 G^{\frac{k}{2}}$ has a zero at ∞. We have shown in the appendix that $E_4 - G^2$ has a simple zero at ∞ and no other zeros. Therefore there exists some $g \in M_{k-4}(\Gamma_0(2))$ such that $f - a_0 G^{\frac{k}{2}} = (E_4 - G^2)g$. We write $g = \sum_{n=0}^{\infty} b_n q^n$. Let R_g denote the $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}]$-module generated by $b_0, ..., b_{r(k)-1}$. We have that $R_g \subset R$ since $E_4 - G^2 = 192q(1 + O(q)) \in 192q\mathbf{Z}[[q]]^{\times}$. We apply the inductive hypothesis to g and have that all $b_n \in R_g \subset R$ and that $g = \sum_{2a+4b=k-4} c_{a,b} G^a E_4^b$ with $c_{a,b} \in R_g \subset R$. Thus $f = a_0 G^{\frac{k}{2}} + (E_4 - G^2)g$ can be written as an R-linear combination of elements in the set $\{G^a E_4^b : a, b \geq 0, 2a + 4b = k\}$. Therefore every $a_n \in R$. This completes the inductive step and our proof is now complete. □

Lemma 5.3.4. *Let L denote a number field or* $\overline{\mathbf{Q}}$. *Let* $M_k(\Gamma_0(2))_L := M_k(\Gamma_0(2)) \cap L[[q]]$. *Then* $\{G^a E_4^b : a, b \geq 0, 2a + 4b = k\}$ *is a basis for the L-vector space* $M_k(\Gamma_0(2))_L$. *If* $f \in M_k(\Gamma_0(2))_L$ *then there exists a positive integer N such that* $Nf \in \overline{\mathbf{Z}}[[q]]$.

Proof. The fact that $\{G^a E_4^b : a,b \geq 0, 2a+4b=k\}$ is a basis for the L-vector space $M_k(\Gamma_0(2))_L$ is immediate from Lemma 5.3.3. Let $f \in M_k(\Gamma_0(2))_L$.

Then $f = \sum_{2a+4b=k} c_{a,b} G^a E_4^b$ where each $c_{a,b} \in \mathbf{C}$. Lemma 5.3.3 and the fact that $f \in M_k(\Gamma_0(2))_L$ imply that each $c_{a,b} \in L$. Therefore there exists a positive integer $N_{a,b}$ such that $N_{a,b} c_{a,b} \in \overline{Z}$. Let $N = \prod_{2a+4b=k} N_{a,b}$. Then each $Nc_{a,b} \in \overline{\mathbf{Z}}$ and $Nf = \sum_{2a+4b=k} Nc_{a,b} G^a E_4^b \in \overline{\mathbf{Z}}[[q]]$. □

We shall need to compute the q-series expansion of $D_{k_0} F'$ in this section and we do so in the following Lemma.

Lemma 5.3.5. *Let* $t_1(K) := d(K)(K+A-r) + \sum_{n=1}^{K-1} 2k_0 \sigma(n) d(K-n)$ *and let* $t_2(K) := \widetilde{d}(K)(K+B-r) + \sum_{n=1}^{K-1} 2k_0 \sigma(n) \widetilde{d}(K-n)$. *Then*

$$D_{k_0} F' = \begin{bmatrix} q^{\frac{k_0}{12}+A-r}(A-r+\sum_{K=1}^{\infty} t_1(K) q^K) \\ q^{\frac{k_0}{12}+B-r}(B-r+\sum_{K=1}^{\infty} t_2(K) q^K) \end{bmatrix}.$$

Moreover, for all integers K, $t_1(K), t_2(K) \in L$. In particular, $D_{k_0} F' \in M_{k_0+2}(\rho')_L$.

Proof. We recall that $E_2 = 1 - 24\sum_{n=1}^{\infty} \sigma(n) q^n$. Let F_1' and F_2' denote the first and second component functions of F'. We have that

$$\begin{aligned} D_{k_0}(F_1') &= D_{k_0}\left(q^{\frac{k_0}{12}+A-r}(1+\sum_{K=1}^{\infty} d(K) q^K)\right) \\ &= \theta\left(q^{\frac{k_0}{12}+A-r}(1+\sum_{K=1}^{\infty} d(K) q^K)\right) \\ &\quad - \frac{k_0}{12} E_2 \left(q^{\frac{k_0}{12}+A-r}(1+\sum_{K=1}^{\infty} d(K) q^K)\right) \\ &= q^{\frac{k_0}{12}+A-r}\theta\left(1+\sum_{K=1}^{\infty} d(K) q^K\right) + \left(1+\sum_{K=1}^{\infty} d(K) q^K\right)\theta(q^{\frac{k_0}{12}+A-r}) \\ &\quad - \frac{k_0}{12}(1-24\sum_{n=1}^{\infty}\sigma(n) q^n)\left(q^{\frac{k_0}{12}+A-r}(1+\sum_{K=1}^{\infty} d(K) q^K)\right) \end{aligned}$$

$$= q^{\frac{k_0}{12}+A-r}(\sum_{K=1}^{\infty} Kd(K)q^K) + (\frac{k_0}{12}+A-r)q^{\frac{k_0}{12}+A-r}\left(1+\sum_{K=1}^{\infty} d(K)q^K\right)$$

$$-\frac{k_0}{12}q^{\frac{k_0}{12}+A-r}(1+\sum_{K=1}^{\infty}(d(K)-\sum_{n=1}^{K-1}24\sigma(n)d(K-n))q^K)$$

$$= q^{\frac{k_0}{12}+A-r}(A-r+\sum_{K=1}^{\infty} t_1(K)q^K)$$

In a similar manner, we have that $D_{k_0}F_2' = q^{\frac{k_0}{12}+B-r}(B-r+\sum_{K=1}^{\infty} t_2(K)q^K)$.

The assumption that $\rho(T)$ has finite order implies that $A-r \in \mathbf{Q}$ and $B-r \in \mathbf{Q}$. We have previously shown that for all integers $K \geq 1$, $d(K) \in L$ and $\widetilde{d}(K) \in L$. It now follows from the formulas for $t_1(K)$ and $t_2(K)$ that for each integer $K \geq 1$, $t_1(K) \in L$ and $t_2(K) \in L$. Hence all of the Fourier coefficients of both of the component functions of $D_{k_0}F'$ belong to L. □

Lemma 5.3.6. *Assume that* ρ *satisfies Hypothesis 5.2.5. Let* M *denote the square-free integer for which* $\boldsymbol{Q}(\sqrt{M}) = \boldsymbol{Q}(r)$ *and let* $L = \boldsymbol{Q}(\sqrt{M})$. *Then* $M(\rho')_L = M(\Gamma_0(2))_L F' \bigoplus M(\Gamma_0(2))_L D_{k_0} F'$. *In particular,* $M(\rho')_L$ *is a free* $M(\Gamma_0(2))_L$*-module of rank two.*

Proof. This proof follows the proof of Lemma 4.1 in Marks' paper [19]. We have shown in Theorem 5.1.11 that $F' \in M_{k_0}(\rho')_L$ and $D_{k_0}F' \in M_{k_0+2}(\rho')_L$.

Hence $M(\Gamma_0(2))_L F' \bigoplus M(\Gamma_0(2))_L D_{k_0} F' \subset M(\rho')_L$. To prove the theorem, we need to show that the reverse inclusion holds. We recall that $M(\rho')_L := \bigoplus_{k\in\mathbf{Z}} M_k(\rho')_L$. Thus it suffices to prove that if $k \in \mathbf{Z}$ then $M_k(\rho')_L \subset M(\Gamma_0(2))_L F' \bigoplus M(\Gamma_0(2))_L D_{k_0} F'$. Let $Z \in M_k(\rho')_L$. Let Z_1 and Z_2 denote the first and second component functions of Z and let F_1' and F_2' denote the first and second component functions of F'. We know that

$M(\rho') = M(\Gamma_0(2))F' \bigoplus M(\Gamma_0(2))D_{k_0}F'$. Therefore $Z = m_1 F' + m_2 D_{k_0} F'$ where

$m_1 \in M_{k-k_0}(\Gamma_0(2)), m_2 \in M_{k-k_0-2}(\Gamma_0(2))$. It suffices to show that $m_1 \in M_{k-k_0}(\Gamma_0(2))_L$ and

$m_2 \in M_{k-k_0-2}(\Gamma_0(2))_L$. We have previously shown that

$$F' = \begin{bmatrix} F_1' \\ F_2' \end{bmatrix} = \begin{bmatrix} q^{\frac{k_0}{12}+A-r}(1+\sum_{K=1}^{\infty} d(K)q^K) \\ q^{\frac{k_0}{12}+B-r}(1+\sum_{K=1}^{\infty} \widetilde{d}(K)q^K) \end{bmatrix}$$

and

$$D_{k_0}(F') = \begin{bmatrix} D_{k_0}F_1' \\ D_{k_0}F_2' \end{bmatrix} = \begin{bmatrix} q^{\frac{k_0}{12}+A-r}(A-r+\sum_{K=1}^{\infty} t_1(K)q^K) \\ q^{\frac{k_0}{12}+B-r}(B-r+\sum_{K=1}^{\infty} t_2(K)q^K) \end{bmatrix}.$$

We write $m_1 = \sum_{n=0}^{\infty} m_1(n)q^n$ and $m_2 = \sum_{n=0}^{\infty} m_2(n)q^n$. The equation $Z = m_1F' + m_2D_{k_0}F'$ implies that there exist sequences $\{Z_1(n)\}_{n=0}^{\infty}$ and $\{Z_2(n)\}_{n=0}^{\infty}$ such that

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} q^{\frac{k_0}{12}+A-r}\sum_{n=0}^{\infty} Z_1(n)q^n \\ q^{\frac{k_0}{12}+B-r}\sum_{n=0}^{\infty} Z_2(n)q^n \end{bmatrix}.$$

In fact,

$$\begin{aligned} Z_1 &= q^{\frac{k_0}{12}+A-r}\sum_{n=0}^{\infty} Z_1(n)q^n \\ &= m_1F_1' + m_2D_{k_0}F_1' \\ &= q^{\frac{k_0}{12}+A-r}\sum_{n=0}^{\infty} m_1(n)q^n(1+\sum_{K=1}^{\infty} d(K)q^K) + q^{\frac{k_0}{12}+A-r}\sum_{n=0}^{\infty} m_2(n)q^n(A-r+\sum_{K=1}^{\infty} t_1(K)q^K). \end{aligned}$$

Thus $Z_1(N) = m_1(N) + \sum_{n=0}^{N-1} m_1(n)d(N-n) + (A-r)m_2(N) + \sum_{n=0}^{N-1} m_2(n)t_1(N-n)$. Similarly, $Z_2(N) = m_1(N) + \sum_{n=0}^{N-1} m_1(n)\widetilde{d}(N-n) + (B-r)m_2(N) + \sum_{n=0}^{N-1} m_2(n)t_2(N-n)$. Hence $\begin{bmatrix} Z_1(0) \\ Z_2(0) \end{bmatrix} = \begin{bmatrix} 1 & A-r \\ 1 & B-r \end{bmatrix} \begin{bmatrix} m_1(0) \\ m_2(0) \end{bmatrix}$ and for all $N \geq 1$, we have that

$$\begin{bmatrix} Z_1(N) \\ Z_2(N) \end{bmatrix} = \begin{bmatrix} 1 & A-r \\ 1 & B-r \end{bmatrix} \begin{bmatrix} m_1(N) \\ m_2(N) \end{bmatrix} + \sum_{n=0}^{N-1} \begin{bmatrix} m_1(n)d(N-n) + m_2(n)t_1(N-n) \\ m_1(n)\widetilde{d}(N-n) + m_2(n)t_2(N-n) \end{bmatrix}.$$

To show that $m_1, m_2 \in M(\Gamma_0(2))_L$, we must show that for all nonnegative integers N, $m_1(N) \in L$ and $m_2(N) \in L$. We proceed by induction on N. Our inductive hypothesis is that for all nonnegative integers $n < N$, $m_1(n), m_2(n) \in L$. We recall that because ρ is irreducible, $B - A \notin \mathbf{Z}$ and thus $B - A \neq 0$. Hence the matrix $\begin{bmatrix} 1 & A-r \\ 1 & B-r \end{bmatrix}$ is invertible. The assumption that $\rho(T)$ has finite order implies that $A - r, B - r \in \mathbf{Q}$. Thus

$$\begin{bmatrix} 1 & A-r \\ 1 & B-r \end{bmatrix}^{-1} = \frac{1}{B-A} \begin{bmatrix} B-r & r-A \\ -1 & 1 \end{bmatrix} \in \mathrm{GL}_2(\mathbf{Q}).$$

The assumption that $Z \in M(\rho)_L$ implies that for all integers $n \geq 0$, $Z_1(n), Z_2(n) \in L$. We have previously shown that $F' \in M(\rho)_L$ and that for all integers $K \geq 1$, $d(K), \widetilde{d}(K) \in L$ We also proved in Lemma 5.3.5 that for all integers K, $t_1(K), t_2(K) \in L$. We now treat the base case where $N = 0$. We have that

$$\begin{bmatrix} m_1(0) \\ m_2(0) \end{bmatrix} = \frac{1}{B-A} \begin{bmatrix} B-r & r-A \\ -1 & 1 \end{bmatrix} \begin{bmatrix} Z_1(0) \\ Z_2(0) \end{bmatrix}.$$

Hence $m_1(0), m_2(0) \in L$ since $Z_1(0), Z_2(0) \in L$, $B - r, r - A, B - A \in \mathbf{Q}$. Let N denote a positive integer. Assume that for all nonnegative integers n with $n < N$, $m_1(n), m_2(n) \in L$.

Then

$$\begin{bmatrix} 1 & A-r \\ 1 & B-r \end{bmatrix} \begin{bmatrix} m_1(N) \\ m_2(N) \end{bmatrix} = \begin{bmatrix} Z_1(N) \\ Z_2(N) \end{bmatrix} - \sum_{n=0}^{N-1} \begin{bmatrix} m_1(n)d(N-n) + m_2(n)t_1(N-n) \\ m_1(n)\widetilde{d}(N-n) + m_2(n)t_2(N-n) \end{bmatrix} \in L^2.$$

Thus $m_1(N), m_2(N) \in L$ since $\begin{bmatrix} 1 & A-r \\ 1 & B-r \end{bmatrix} \in \mathrm{GL}_2(\mathbf{Q})$. This completes the inductive step and our proof is now complete. $\square$

Theorem 5.3.7. *Let ρ denote a two-dimensional irreducible representation of $\Gamma_0(2)$ for which $\rho(T)$ has finite order, $c \in \boldsymbol{Q}$ and $[\boldsymbol{Q}(r):\boldsymbol{Q}] = 2$. Let $k \in \boldsymbol{Z}$ and let Z denote a nonzero element in $M_k(\rho')_L$. Let Z_1 and Z_2 denote the first and second component functions of Z. If S is infinite then the sequence of the denominators of the Fourier coefficients of Z_1 is unbounded. If $\widetilde{S}$ is infinite then the sequence of the denominators of the Fourier coefficients of Z_2 is unbounded.*

Proof. This proof follows closely the proof of Prop 4.3 in Marks' paper [19]. Let Z denote a nonzero element in $M_k(\rho')_L$. Then $D_kZ \in M_{k+2}(\rho')_L$. We have proven in Lemma 5.3.6 that

$$M(\rho')_L = M(\Gamma_0(2))_L F' \bigoplus M(\Gamma_0(2))_L D_{k_0} F'.$$

Therefore there exist $m_1, m_4 \in M_{k-k_0}(\Gamma_0(2))_L$, $m_2 \in M_{k-k_0-2}(\Gamma_0(2))_L$, and $m_3 \in M_{k+2-k_0}(\Gamma_0(2))_L$ such that

$$\begin{bmatrix} Z \\ D_kZ \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} F' \\ D_{k_0}F' \end{bmatrix}.$$

We note that $m_1m_4 - m_2m_3 \in M_{2k-2k_0}(\Gamma_0(2))_L$.

We have that:

$$\begin{aligned} \begin{bmatrix} m_4Z - m_2D_kZ \\ -m_3Z + m_1D_kZ \end{bmatrix} &= \begin{bmatrix} m_4 & -m_2 \\ -m_3 & m_1 \end{bmatrix} \begin{bmatrix} Z \\ D_kZ \end{bmatrix} \\ &= \begin{bmatrix} m_4 & -m_2 \\ -m_3 & m_1 \end{bmatrix} \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} F' \\ D_{k_0}F' \end{bmatrix} \\ &= \begin{bmatrix} (m_1m_4 - m_2m_3)F' \\ (m_1m_4 - m_2m_3)D_{k_0}F' \end{bmatrix}. \end{aligned}$$

Thus $m_4 Z - m_2 D_k Z = (m_1 m_4 - m_2 m_3) F'$. We recall that Z_1 and Z_2 denote the first and second component functions of Z and F'_1 and F'_2 denote the first and second component functions of F'. Thus $m_4 Z_1 - m_2 D_k Z_1 = (m_1 m_4 - m_2 m_3) F'_1$ and $m_4 Z_2 - m_2 D_k Z_2 = (m_1 m_4 - m_2 m_3) F'_2$. Suppose by way of contradiction that the sequence of denominators of the Fourier coefficients of Z_1 is bounded. Then the sequence of denominators of the Fourier coefficients of $D_k Z_1 = q\frac{d}{dq}(Z_1) - \frac{k}{12} E_2 Z_1$ is bounded since $E_2 \in \mathbf{Z}[[q]]$. We have proven in Lemma 5.3.4 that the sequence of the denominators of the Fourier coefficients of any modular form in $M(\Gamma_0(2))_L$ is bounded. Thus the sequence of the denominators of the Fourier coefficients of $m_4 Z_1 - m_2 D_k Z_1$ is bounded since the same is true of the sequence of the denominators of the Fourier coefficients of Z_1, $D_k Z_1$, m_4, and m_2. Let N denote a positive integer for which $m_1 m_4 - m_2 m_3 = \frac{1}{N} q^t (1 + \sum_{n=1}^{\infty} \delta_n q^n)$ with $\delta_n \in \overline{\mathbf{Z}}$ and t a non-negative integer. We have that

$$\begin{aligned}(m_1 m_4 - m_2 m_3) F'_1 &= \frac{1}{N} q^t (1 + \sum_{n=1}^{\infty} \delta_n q^n) q^{A-r+\frac{k_0}{12}} (1 + \sum_{K=1}^{\infty} d(K) q^K) \\ &= \frac{1}{N} q^{t+A-r+\frac{k_0}{12}} \left(1 + \sum_{K=1}^{\infty} q^K \left(d(K) + \delta_K + \sum_{i=1}^{K-1} d(i) \delta_{K-i} \right) \right).\end{aligned}$$

We recall that $d(i)$ is p_K-integral if $i < K$. Thus $\delta_K + \sum_{i=1}^{K-1} d(i)\delta_{K-i}$ is p_K-integral. However, $d(K)$ is not p_K-integral. Thus $d(K) + \delta_K + \sum_{i=1}^{K-1} d(i)\delta_{K-i}$ is not p_K-integral. We have proven that $(m_1 m_4 - m_2 m_3) F'_1$ has unbounded denominators if S is infinite. This is a contradiction since $m_4 Z_1 - m_2 D_k Z_1 = (m_1 m_4 - m_2 m_3) F'_1$ and we have shown that $m_4 Z_1 - m_2 D_k Z_1$ has bounded denominators if Z_1 has bounded denominators. Thus the assumption that Z_1 has bounded denominators is false and we conclude that if S is infinite then the sequence of denominators of the Fourier coefficients of Z_1 is unbounded. A completely analogous argument shows that if $\widetilde{S}$ is infinite then the sequence of denominators of the Fourier coefficients of Z_2 is unbounded. □

Theorem 5.3.8. *Let ρ denote a two-dimensional irreducible representation of $\Gamma_0(2)$ which is induced from a character of $\Gamma(2)$. Assume that $\rho(T)$ has finite order, $c \in \boldsymbol{Q}$ and $[\boldsymbol{Q}(r) : \boldsymbol{Q}] = 2$. Let $k \in \mathbf{Z}$ and $Z \in M_k(\rho')$ whose component functions Z_1 and Z_2 have the property that all of their Fourier coefficients are algebraic numbers. Then the sequence of the denominators of the Fourier coefficients of Z_1 and the sequence of the denominators of the Fourier coefficients of Z_2 are both unbounded.*

Proof. We have shown in Theorem 5.2.17 that if ρ is induced by a character of $\Gamma(2)$ then $S = \widetilde{S}$ and S is an infinite set. This theorem now follows from Theorem 5.3.7. □

We recall that we have proven in Theorem 5.1.11 that any ρ which satisfies the hypotheses of Theorem 5.3.8 (in fact, a weaker set of hypotheses is sufficient) has the property that for every $k \in \mathbf{Z}$, there is a basis for $M_k(\rho')$ consisting of vector-valued modular forms whose component functions have the property that all of their Fourier coefficients are algebraic numbers.

Chapter 6

Appendix

6.1 A Hauptmodul and its first and second derivatives

The purpose of this section is to prove that $\mathfrak{J}(\tau) := 3\frac{G^2(\tau)}{E_4(\tau)-G^2(\tau)}$ is a Hauptmodul of $\Gamma_0(2)$ and to provide proofs of Propositions 4.1.3 and 4.1.4. We shall also show that G and E_4 are algebraically independent and that $M(\Gamma_0(2)) = \mathbf{C}[G, E_4]$. We begin by providing some background on the group $\Gamma_0(2)$ and on modular forms on $\Gamma_0(2)$.

The group $\Gamma_0(2)$ is an index three subgroup of $\mathrm{SL}_2(\mathbf{Z})$. It is generated by $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $V := \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$. As $V^2 = -I$, V^2 acts trivially on $\mathfrak{H}$. We note that V fixes $\frac{1+i}{2}$ and therefore $\frac{1+i}{2}$ is an elliptic point. In fact, $\tau \in \mathfrak{H}$ is an elliptic point for $\Gamma_0(2)$ if and only if $\tau \in \Gamma_0(2) \cdot \frac{1+i}{2}$. In a fundamental region, the cusps of the modular curve $\Gamma_0(2)/(\mathfrak{H} \bigcup \mathbb{P}^1(\mathbf{Q}))$ are 0 and ∞.

We recall that $E_2(\tau) := 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n$ and that $G(\tau) := -E_2(\tau) + 2E_2(2\tau) \in M_2(\Gamma_0(2))$.

We write down the q-expansions for G, E_4, and $E_4 - G^2$:

$$G(\tau) = -(1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n) + 2(1 - 24\sum_{n=1}^{\infty}\sigma(n)q^{2n}) = 1 + 24q + 24q^2 + O(q^3)$$

$$E_4 = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n = 1 + 240q + 2160q^2 + 6720q^3 + O(q^4)$$

$$E_4 - G^2 = 192q + 1536q^2 + O(q^3).$$

The valence formula for $\Gamma_0(2)$ states that if $f \in M_k(\Gamma_0(2))$ then

$$\frac{k}{4} = \frac{k[\mathrm{PSL}_2(\mathbf{Z}) : \overline{\Gamma_0(2)}]}{12} = v_0(f) + v_\infty(f) + \sum_{z \in \Gamma_0(2)\backslash\mathfrak{H}} \frac{v_z(f)}{n_{\Gamma_0(2)}(z)}$$

where $\overline{\Gamma_0(2)} := \Gamma_0(2)/\{I, -I\}$ and $n_{\Gamma_0(2)}(z)$ is equal to one or two or three if z is not an elliptic point or z is Γ-equivalent to i or z is Γ-equivalent to $e^{\frac{2\pi i}{3}}$, respectively. In particular, $n_{\Gamma_0(2)}(z) = 1$ if z is not an elliptic point and $n_{\Gamma_0(2)}(z) = 2$ if z is Γ-equivalent to $\frac{1+i}{2}$. The valence formula for any finite index subgroup of Γ is given in Theorem 5.6.11 in [7].

We shall use the valence formula to computes the zeros of G^2 and $E_4 - G^2$. We see that $E_4 - G^2 = 192q + O(q^2)$ has a simple zero at the cusp ∞. As $E_4 - G^2 \in M_4(\Gamma_0(2))$, the valence formula implies that $(E_4 - G^2)(\tau) = 0$ if and only if $\tau \in \Gamma_0(2)\cdot\infty$. We now use the fact that V fixes $\frac{1+i}{2}$ to compute the zeros of G. We have that

$$G\left(\frac{1+i}{2}\right) = G|_2V\left(\frac{1+i}{2}\right) = \left(2\cdot\frac{1+i}{2} - 1\right)^{-2} G\left(V\left(\frac{1+i}{2}\right)\right) = -G\left(\frac{1+i}{2}\right).$$

Therefore $G(\frac{1+i}{2}) = 0$. The valence formula implies that $G(\tau) = 0$ if and only if $\tau \in \Gamma_0(2) \cdot \frac{1+i}{2}$. The valence formula also shows that any weight two modular form vanishes at $\frac{1+i}{2}$. Thus $M_2(\Gamma_0(2)) = \mathbf{C}G$.

Proposition 6.1.1. *Let k denote a nonnegative even integer. The set $\{G^a E_4^b : 2a+4b = k, a, b \geq 0, a, b \in \mathbf{Z}\}$ is a $\mathbf{C}$-basis for $M_k(\Gamma_0(2))$. Consequently, the modular forms G and E_4 are algebraically independent and $M(\Gamma_0(2)) = \mathbf{C}[E_4, G]$.*

Proof. There are $[\frac{k}{4}]+1$ elements in the set $\{G^a E_4^b : 2a+4b = k, a, b \geq 0, a, b \in \mathbf{Z}\}$. If k is a nonnegative even integer then $\dim M_k(\Gamma_0(2)) = [\frac{k}{4}]+1$ (see page 265 in [7]). Thus to show that the set $\{G^a E_4^b : 2a+4b = k, a, b \geq 0, a, b \in \mathbf{Z}\}$ is a basis for $M_k(\Gamma_0(2))$, it suffices to show that the elements of this set span $M_k(\Gamma_0(2))$. We show that this holds by induction on k. The result is clear for $k = 0$ and $k = 2$ since $\dim M_0(\Gamma_0(2)) = \mathbf{C}$ and $\dim M_2(\Gamma_0(2)) = \mathbf{C}G$. Let k denote an even integer such that $k \geq 4$. We shall assume that the set $\{G^a E_4^b : 2a+4b = t, a, b \geq 0, a, b \in \mathbf{Z}\}$ spans $M_t(\Gamma_0(2))$ for all nonnegative even integers t less than k.

Let $f = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_0(2))$. Then $f - a_0 G^{\frac{k}{2}}$ has a zero at ∞.

Let $g = (E_4 - G^2)^{-1}(f - a_0 G^{\frac{k}{2}})$. Then $g \in M_{k-4}(\Gamma_0(2))$ since $E_4 - G^2$ has a simple zero at ∞ and has no other zeros. By the inductive hypothesis, $g = \sum_{2x+4y=k-4} c_{x,y} G^x E_4^y$ where each of the $c_{x,y}$ are complex numbers. Thus

$$f = a_0 G^{\frac{k}{2}} + (E_4 - G^2)g = a_0 G^{\frac{k}{2}} + \sum_{2x+4(y+1)=k} c_{x,y} G^x E_4^{y+1} - \sum_{2(x+2)+4y=k} c_{x,y} G^{x+2} E_4^y$$

is in the $\mathbf{C}$-span of the set $\{G^a E_4^b : 2a+4b = k, a, b \geq 0, a, b \in \mathbf{Z}\}$. This completes the inductive step. We have thus proven that for each even $k \geq 0$, $\{G^a E_4^b : 2a+4b = k, a, b \geq 0, a, b \in \mathbf{Z}\}$ spans $M_k(\Gamma_0(2))$ and is therefore a $\mathbf{C}$-basis for $M_k(\Gamma_0(2))$.

If $f \in M_k(\Gamma_0(2))$ then we recall that $\frac{k}{4} = \nu_0(f) + \nu_\infty(f) + \sum_{z \in \Gamma_0(2)\backslash\mathfrak{H}} \frac{\nu_z(f)}{n_{\Gamma_0(2)}(z)}$. This equation shows that if f is nonzero then f cannot be expressed as a sum of modular forms of smaller weights. We conclude that E_4 and G are algebraically independent and that $M(\Gamma_0(2)) = \mathbf{C}[G, E_4]$.

□

We shall now prove that $\mathfrak{J}$ is a Hauptmodul.

Proposition 6.1.2. *The modular function* $\mathfrak{J}(\tau) := 3\frac{G^2(\tau)}{E_4(\tau) - G^2(\tau)}$ *is a Hauptmodul of* $\Gamma_0(2)$.

Proof. The modular forms G^2 and $E_4 - G^2$ are both modular forms of weight four and therefore $\mathfrak{J}$ is a modular function on $\Gamma_0(2)$. A modular function on a genus zero subgroup is a Hauptmodul if and only if it has one simple pole and no other poles. This statement is true about the function $\mathfrak{J}$ since G^2 does not vanish at the cusp ∞ and $E_4 - G^2$ has a simple zero at the cusp ∞ and it has no other zeros. □

We now give the proof of Proposition 4.1.3. We will prove a bit more. Namely, we will show that:

$$\theta(\mathfrak{J}) = (1 - \mathfrak{J})G = \frac{G(E_4 - 4G^2)}{E_4 - G^2}$$

Proof. (Proof of Proposition 4.1.3.) The derivative of a modular function is a meromorphic modular form of weight two. The differential operator $\theta = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$ preserves the order of vanishing of a function at ∞. Therefore $\theta(\mathfrak{J})$ has a simple pole at ∞. Moreover, $\theta(\mathfrak{J})$ has no poles elsewhere since $\mathfrak{J}$ has no poles elsewhere. As $E_4 - G^2$ has a simple zero at ∞, the function $(E_4 - G^2)\theta(\mathfrak{J})$ is holomorphic and thus $(E_4 - G^2)\theta(\mathfrak{J}) \in M_6(\Gamma_0(2)) = \mathbf{C}E_4G \bigoplus \mathbf{C}G^3$. We now compute the numbers d_1 and d_2 for which $(E_4-G^2)\theta(\mathfrak{J}) = d_1E_4G + d_2G^3$. We first compute the first two coefficients in the q-expansion of $(E_4-G^2)\theta(\mathfrak{J})$. As $E_4-G^2 = 192q+1536q^2+O(q^3)$, $\frac{3}{E_4-G^2} = \frac{1}{64q}(1-8q+O(q^2))$. We have

$$\mathfrak{J}(\tau) = \frac{3G^2}{E_4-G^2} = \frac{1}{64q}(1-8q+O(q^2))(1+48q+O(q^2)) = \frac{1}{64}q^{-1}(1+40q+O(q^2)).$$

Therefore $\theta(\mathfrak{J}) = q\frac{d}{dq}(\mathfrak{J}) = -\frac{1}{64}q^{-1}+O(q)$. Hence

$$(E_4-G^2)\theta(\mathfrak{J}) = (192q+1536q^2+O(q^3))(-\frac{1}{64}q^{-1}+O(q)) = -3-24q+O(q^2).$$

Thus

$$\begin{aligned}
-3-24q+O(q^2) &= (E_4-G^2)\theta(\mathfrak{J}) \\
&= d_1E_4G+d_2G^3 \\
&= d_1(1+240q+O(q^2))(1+24q+O(q^2))+d_2(1+72q+O(q^2)) \\
&= d_1+d_2+(264d_1+72d_2)q+O(q^2)
\end{aligned}$$

We have that $d_1+d_2 = -3$, $264d_1+72d_2 = -24$, and we obtain that $(d_1,d_2) = (1,-4)$. We have now shown that $(E_4-G^2)\theta(\mathfrak{J}) = G(E_4-4G^2)$. It now suffices to prove that $\frac{E_4-4G^2}{E_4-G^2} = 1-\mathfrak{J}$.

The modular function $\frac{E_4-4G^2}{E_4-G^2}$ has a simple pole at ∞ since $E_4-4G^2 = -3+O(q)$ does not vanish at ∞ and $E_4-G^2 = 192q+O(q^2)$ has a simple zero at ∞. The modular form E_4-G^2 has no other zeros and thus $\frac{E_4-4G^2}{E_4-G^2}$ is holomorphic except at ∞. We have previously shown that $\mathfrak{J}$ also has the property that it has a simple pole at ∞ and it has no other poles. As the Riemann surface $\Gamma_0(2)\backslash(\mathfrak{H}\bigcup\mathbb{P}^1(\mathbf{Q}))$ has genus zero, the Riemann-Roch theorem implies that the dimension of the vector space of meromorphic functions on $\Gamma_0(2)\backslash(\mathfrak{H}\bigcup\mathbb{P}^1(\mathbf{Q}))$ which have at most a simple pole at ∞ and which are holomorphic elsewhere is two. We can take a basis for this space to be the constant function 1 and $\mathfrak{J}$. We have shown that $\frac{E_4-4G^2}{E_4-G^2}$ is in this space. Therefore there exist complex numbers y and z such that $\frac{E_4-4G^2}{E_4-G^2}=y+z\mathfrak{J}$. We compute y and z by comparing the coefficients of q^{-1} and q^0 in the q-expansions of $\frac{E_4-4G^2}{E_4-G^2}$ and $\mathfrak{J}$. We previously computed that $\mathfrak{J}=\frac{1}{64}q^{-1}+\frac{40}{64}+O(q)$. We also have that

$$\begin{aligned}\frac{E_4-4G^2}{E_4-G^2} &= \frac{-3+48q+O(q^2)}{192q+1536q^2+O(q^3)}\\ &= \frac{-3+48q+O(q^2)}{192q(1+8q+O(q^2))}\\ &= \frac{1}{192q}(-3+48q+O(q^2))(1-8q+O(q^2))\\ &= \frac{1}{192q}(-3+72q+O(q^2))\\ &= -\frac{1}{64}q^{-1}+\frac{3}{8}+O(q).\end{aligned}$$

Therefore $z=-1$ and $\frac{3}{8}=y+\frac{40}{64}z=y-\frac{40}{64}$. Thus $y=1$. We conclude that $\frac{E_4-4G^2}{E_4-G^2}=1-\mathfrak{J}$ and our proof of the proposition is now complete. □

Remark: We have shown in the above proof that $\mathfrak{K}=64\mathfrak{J}=q^{-1}(1+O(q))$.

We need the following Proposition before giving the proof of Proposition 4.1.4.

Proposition 6.1.3.

$$\theta(G)=\frac{1}{6}(E_2G+E_4-2G^2)$$

Proof. As $G\in M_2(\Gamma_0(2))$, $D_2(G)=\theta(G)-\frac{1}{6}E_2G\in M_4(\Gamma_0(2))=\mathbf{C}E_4\bigoplus\mathbf{C}G^2$. We now compute the numbers c_1 and c_2 such that $\theta(G) - \frac{1}{6}E_2G = c_1E_4 + c_2G^2$. We have that:

$$\theta(G) = \theta(1 + 24q + 24q^2 + \cdots) = 24q + 48q^2 + O(q^3)$$

$$E_2G = (1 - 24q - O(q^2))(1 + 24q + O(q^2)) = 1 + O(q^2)$$

$$\theta(G) - \frac{1}{6}E_2G = 24q + O(q^2) - \frac{1}{6}(1 + O(q^2)) = -\frac{1}{6} + 24q + O(q^2).$$

We now have that

$$\begin{aligned} -\frac{1}{6} + 24q + O(q^2) &= \theta(G) - \frac{1}{6}E_2G \\ &= c_1E_4 + c_2G^2 \\ &= c_1(1 + 240q + O(q^2)) + c_2(1 + 48q + O(q^2)) \\ &= c_1 + c_2 + (240c_1 + 48c_2)q + O(q^2) \end{aligned}$$

We get that $c_1 + c_2 = -\frac{1}{6}$ and $240c_1 + 48c_2 = 24$. Thus $c_1 = \frac{1}{6}$ and $c_2 = -\frac{1}{3}$ and the proposition is now proven. □

We will also need to use the fact that $\frac{E_4(\tau)}{G^2(\tau)} = \frac{\mathfrak{J}(\tau)+3}{\mathfrak{J}(\tau)}$ in our proof of Proposition 4.1.4. We recall that $\mathfrak{J} = \frac{3G^2}{E_4 - G^2}$. Therefore $\frac{\mathfrak{J}+3}{\mathfrak{J}} = 1 + \frac{3}{\mathfrak{J}} = 1 + \frac{E_4 - G^2}{G^2} = \frac{E_4}{G^2}$. We now give the proof of Proposition 4.1.4. Namely, we show that:

$$\theta^2(\mathfrak{J}) = G^2(1 - \mathfrak{J})(\frac{-7\mathfrak{J} + 3}{6\mathfrak{J}}) + \frac{1}{6}E_2\theta(\mathfrak{J}).$$

Proof. (Proof of Proposition 4.1.4) We will now use the formula $\theta(G) = \frac{1}{6}(E_2G + E_4 - 2G^2)$ to compute $\theta^2(\mathfrak{J})$. We have that

$$\theta^2(\mathfrak{J}) = \theta(G(1 - \mathfrak{J})) = -\theta(\mathfrak{J})G + (1 - \mathfrak{J})\theta(G)$$

$$
\begin{aligned}
&= -G^2(1-\mathfrak{J}) + \frac{1}{6}(E_2G + E_4 - 2G^2)(1-\mathfrak{J}) \\
&= G^2(1-\mathfrak{J})(-\frac{4}{3} + \frac{E_4}{6G^2}) + \frac{1}{6}E_2G(1-\mathfrak{J}) \\
&= G^2(1-\mathfrak{J})(-\frac{4}{3} + \frac{\mathfrak{J}+3}{6\mathfrak{J}}) + \frac{1}{6}E_2\theta(\mathfrak{J}) \\
&= G^2(1-\mathfrak{J})(\frac{-7\mathfrak{J}+3}{6\mathfrak{J}}) + \frac{1}{6}E_2\theta(\mathfrak{J}).
\end{aligned}
$$

□

6.2 An integrality result

Lemma 6.2.1. *Let* $\mathfrak{K} := 64\mathfrak{J} = \frac{192G^2}{E_4 - G^2}$. *Then* $\mathfrak{K} \in \frac{1}{q}\mathbf{Z}[[q]]^\times$.

Proof. We have that

$$
\begin{aligned}
G = -E_2(\tau) + 2E_2(2\tau) &= -(1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n) + 2(1 - 24\sum_{n=1}^{\infty}\sigma(n)q^{2n}) \\
&= 1 + 24\sum_{n=0}^{\infty}\sigma(2n+1)q^{2n+1} + \sum_{n=1}^{\infty}(24\sigma(2n) - 48\sigma(n))q^{2n}.
\end{aligned}
$$

Let $P = \sum_{n=0}^{\infty}\sigma(2n+1)q^{2n+1} + \sum_{n=1}^{\infty}(\sigma(2n) - 2\sigma(n))q^{2n}$. Then $G = 1 + 24P$ and

$$
G^2 = 1 + 48P + 24^2P^2 \equiv 1 + 48P \pmod{192}.
$$

Therefore

$$
\begin{aligned}
G^2 - E_4 &\equiv 48P - 240\sum_{n=1}^{\infty}\sigma_3(n)q^n \\
&\equiv 48\left(P - \sum_{n=1}^{\infty}\sigma_3(n)q^n\right) \pmod{192}.
\end{aligned}
$$

Thus to prove that $G^2 - E_4 \in 192\mathbf{Z}[[q]]$, it suffices to show that

$$
\sum_{n=0}^{\infty}\sigma(2n+1)q^{2n+1} + \sum_{n=1}^{\infty}(\sigma(2n) - 2\sigma(n))q^{2n} = P \equiv \sum_{n=1}^{\infty}\sigma_3(n)q^n \pmod{4}.
$$

Equivalently, we must show that for all $n \geq 0$, $\sigma(2n+1) \equiv \sigma_3(2n+1) \pmod 4$ and that for all $n \geq 1$, $\sigma(2n) - 2\sigma(n) \equiv \sigma_3(2n) \pmod 4$. We observe that since any divisor of an odd integer is odd, $\sigma(2n+1) = \sum_{d|2n+1} d \equiv \sum_{d|2n+1} d^3 = \sigma_3(2n+1) \pmod 4$. To prove that $\sigma(2n) - 2\sigma(n) \equiv \sigma_3(2n) \pmod 4$, we first write $n = 2^e n'$ where n' is odd and $e \geq 0$. We have that

$$\begin{aligned}
\sigma(2n) - 2\sigma(n) &= \sigma(2^{e+1}n') - 2\sigma(2^e n') \\
&= (\sigma(2^{e+1}) - 2\sigma(2^e))\sigma(n') \\
&= (2^{e+2} - 1 - 2(2^{e+1} - 1))\sigma(n') \\
&= \sigma(n').
\end{aligned}$$

We have that $\sigma_3(2n) = \sigma_3(2^{e+1}n') = \sigma_3(2^{e+1})\sigma_3(n') \equiv \sigma_3(n') \pmod 4$. Because n' is odd, $\sigma_3(n') \equiv \sigma(n') \pmod 4$. Thus $\sigma_3(2n) \equiv \sigma(n') \pmod 4$. We have thus proven that for all $n \geq 1$, $\sigma_3(2n) \equiv \sigma(n') \equiv \sigma(2n) - 2\sigma(n) \pmod 4$. Hence $G^2 - E_4 \in 192\mathbf{Z}[[q]]$. Moreover, $G^2 - E_4 = O(q)$ and thus $\frac{G^2 - E_4}{192q} \in \mathbf{Z}[[q]]$. We also have that $\frac{G^2 - E_4}{192q} = 1 + O(q)$. We recall that the elements in $\mathbf{Z}[[q]]^\times$ are exactly those elements of $\mathbf{Z}[[q]]$ whose constant term is equal to 1 or -1. Therefore $\frac{G^2 - E_4}{192q} \in \mathbf{Z}[[q]]^\times$ and hence $\frac{192q}{G^2 - E_4} \in \mathbf{Z}[[q]]^\times$. Finally, $G^2 = 1 + O(q) \in \mathbf{Z}[[q]]^\times$ and so $\frac{192G^2q}{E_4 - G^2} = (64q)\mathfrak{J} \in \mathbf{Z}[[q]]^\times$. Thus $\mathfrak{J} \in \frac{1}{64q}\mathbf{Z}[[q]]^\times$ and $64\mathfrak{J} = \mathfrak{K} \in \frac{1}{q}\mathbf{Z}[[q]]^\times$. $\square$

Bibliography

[1] A. O. L. Atkin and H. P. F. Swinnerton-Dyer. Modular forms on noncongruence subgroups. *Combinatorics*, 19:1–25, 1971.

[2] David J. Benson. *Polynomial invariants of finite groups*, volume 190. Cambridge University Press, 1993.

[3] Garrett Birkhoff and Gian-Carlo Rota. *Ordinary differential equations*. Introductions to Higher Mathematics. Ginn and Company, Boston, Mass.-New York-Toronto, 1962.

[4] Richard E. Borcherds. The Gross-Kohnen-Zagier theorem in higher dimensions. *Duke Math Journal*, 97(2):219–233, 1999.

[5] Luca Candelori and Cameron Franc. Vector bundles and modular forms for Fuchsian groups of genus zero. *arXiv preprint arXiv:1704.01684*, 2017.

[6] Luca Candelori and Cameron Franc. Vector-valued modular forms and the modular orbifold of elliptic curves. *International Journal of Number Theory*, 13(01):39–63, 2017.

[7] Henri Cohen and Fredrik Strömberg. *Modular forms: A classical approach*, volume 179

of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2017.

[8] Martin Eichler and Don Zagier. *The theory of Jacobi forms*, volume 55. Springer, 1985.

[9] Cameron Franc and Geoffrey Mason. Fourier coefficients of vector-valued modular forms of dimension 2. *Canadian Mathematical Bulletin*, 57(3):485–494, 2014.

[10] Cameron Franc and Geoffrey Mason. Hypergeometric series, modular linear differential equations and vector-valued modular forms. *The Ramanujan Journal*, 41(1-3):233–267, 2016.

[11] Cameron Franc and Geoffrey Mason. Three-dimensional imprimitive representations of the modular group and their associated modular forms. *Journal of Number Theory*, 160:186–214, 2016.

[12] Terry Gannon. The theory of vector-valued modular forms for the modular group. In *Conformal Field Theory, Automorphic Forms and Related Topics*, pages 247–286. Springer, 2014.

[13] Masanobu Kaneko and Don Zagier. Supersingular j-invariants, hypergeometric series, and Atkin's orthogonal polynomials. *AMS IP STUDIES IN ADVANCED MATHEMATICS*, 7:97–126, 1998.

[14] Anthony W. Knapp. *Elliptic curves*, volume 40 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1992.

[15] Marvin Knopp and Geoff Mason. Logarithmic vector-valued modular forms. *Acta Arithmetica*, 147(3):261–262, 2011.

[16] Marvin Knopp and Geoffrey Mason. On vector-valued modular forms and their Fourier coefficients. *Acta Arithmetica*, 110:117–124, 2003.

[17] Marvin Knopp and Geoffrey Mason. Vector-valued modular forms and Poincaré series. *Illinois Journal of Mathematics*, 48(4):1345–1366, 2004.

[18] Serge Lang. *Introduction to modular forms*, volume 222. Springer Science & Business Media, 2012.

[19] Christopher Marks. Fourier coefficients of three-dimensional vector-valued modular forms. *Communications in Number Theory and Physics*, 9(2):387–415, 2015.

[20] Christopher Marks and Geoffrey Mason. Structure of the module of vector-valued modular forms. *Journal of the London Mathematical Society*, 82(1):32–48, 2010.

[21] Philip Wagreich. Algebras of automorphic forms with few generators. *Transactions of the American Mathematical Society*, 262(2):367–389, 1980.