Title

When to Block versus Interleave Practice?Evidence Against Teaching Fraction Addition before Fraction Multiplication

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When to Block versus Interleave Practice? Evidence Against Teaching Fraction Addition before Fraction Multiplication

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Abstract

In practice, mathematics education is blocked (i.e., teaching one topic at a time; CCSS, 2010), but research generally promotes interleaving (i.e., teaching multiple topics together; Rohrer & Taylor, 2007). For example, fraction arithmetic is blocked with students being taught fraction addition before fraction multiplication. Since students often confuse fraction operations to produce arithmetic errors, interleaved fraction arithmetic instruction might be more productive than blocked instruction to teach students to discriminate between the operations. Additionally, a cognitive task analysis suggests that fraction multiplication may be a prerequisite to fraction addition and thus reversing the blocking order may enhance learning. Two experiments with fraction addition and fraction multiplication were run. Experiments 1 and 2 show that interleaved instruction is generally better than the current blocked instruction. Experiment 2 provides evidence that blocking that reverses the standard order -- providing practice on fraction multiplication before fraction addition -- produces better learning.

Keywords: blocking; interleaving; fractions

Introduction

A number of researchers recommend interleaving of practice, whereby multiple topics are taught and practiced together (Kornell & Bjork, 2008; Rohrer & Taylor, 2007). Others have pointed to circumstances where blocking, in which instruction/practice on each type of problem is grouped together and is much more typical in school instruction, is better for learning (Carvalho & Goldstone, 2014). The Knowledge, Learning, and Instruction Framework (Koedinger, Corbett, & Perfetti, 2012) suggests, more generally, that effectiveness of an instructional recommendation depends on a fit with the specifics of the nature of knowledge to be acquired and what learning processes are needed to acquire such knowledge. For example, when the target knowledge involves a need to differentiate similar situations to respond appropriately, interleaving can help the induction of appropriate cues (e.g., as might be represented in the if-parts of production rules) for such differentiation. Fraction addition tasks (e.g., 1/3 + 1/2 = ?) and fraction multiplication tasks (e.g., 1/3 * 1/2 = ?) are a good example in that the tasks look quite similar, only differing the operator (+ vs. *), but the appropriate responses are quite different (converting to common denominators vs. multiplying the numerators and denominators). Thus, this domain provides a fitting context in which to explore theoretical conditions that can explain and predict when interleaving will be better than blocking. We present two experimental studies contrasting interleaved versus blocked practice of these two topics within fraction arithmetic.

We also illustrate how a cognitive task analysis of this domain suggests a reasonable alternative to the typical and recommended approach of teaching fraction addition before fraction multiplication. The analysis not only suggests that fraction multiplication is simpler than fraction addition, but that it may be a prerequisite to the extent fraction multiplication is used to implement the fraction conversion steps required in the fraction addition procedure. The second of the two studies explores this hypothesis.

Practitioners Block, Researchers Interleave

In practice, mathematics is taught in a blocked fashion. The Common Core State Standards (CCSS, 2010) have been adopted by 42 of the 50 states, suggesting the CCSS are representative of the standards for when mathematics topics are generally taught in the United States. Based on these recommendations, fractions should be taught across 3 years, with different topics in different years. Note that these standards do not force blocking or blocking in this order, but the textbooks aligned with the Common Core follow the blocked fraction addition to fraction multiplication sequence. In contrast, research in interleaving and blocking mathematics knowledge shows advantages of interleaving over blocking (Rohrer & Taylor, 2007). Interleaving may aid learning of strategy selection because it provides negative feedback on over-generalized induction of when to perform an operation that blocking does not (cf., Li, Cohen, & Koedinger, 2012).

Sequencing Fraction Arithmetic Instruction

In 2008, the National Mathematics Advisory Panel concluded, “The most important foundational skill not presently developed, appears to be proficiency with fractions” (p. 18). Fraction arithmetic errors are commonly the product of incorrect strategy choices, not computational errors or fact recall errors (Siegler & Pyke, 2013). A common incorrect strategy choice students make on fraction addition problems is to independently add the numerators and denominators of the addends, for example, getting 2/5, when adding 1/2 and 1/3 (Ni & Zhou, 2005). To make this issue more complicated, this independent whole number strategy is the correct method for solving fraction multiplication problems (e.g., 1/3 times 2/5 does equal (1*2)/(3*5)). Learning fraction multiplication might encourage students to make this error once again in fraction addition problems even if they did learn the correct fraction addition strategy. In addition, fraction addition knowledge may also interfere with fraction multiplication knowledge (Siegler et al., 2011). For example, students sometimes find a common denominator before solving a fraction multiplication problem and then only multiply the numerators resulting in an incorrect answer (e.g., 1/2 * 1/3 = 3/6 * 2/6 = 6/6 = 1).

Given the potential sources of confusion and the presence of interference errors, interleaving fraction arithmetic operations might be beneficial relative to blocking operations, especially for learning good strategy choices. Blocking can have the benefit, however, of strengthening prerequisite subskills before moving on to more complex skills (VanLehn, 1987). Contrary to typical practice, fraction multiplication may be a prerequisite to fraction addition since a variant of fraction multiplication is needed to get equivalent fractions in fraction addition problems with different denominators Further, fraction multiplication provides whole number multiplication practice that may support finding common denominators in fraction addition.

Current Studies

Experiment 1 compares blocked fraction addition problems before fraction multiplication problems to interleaving both problem types to see if interleaving alleviates interference errors between the two types of problems and if blocked instruction causes them. Experiment 2 compares blocked fraction addition to fraction multiplication, blocked fraction multiplication to fraction addition, and interleaving both problem types to see if progressive skill building (i.e., blocked fraction multiplication to fraction addition) outperforms the other blocked condition and reduces the interference amongst the fraction operations.

Experiment 1

Methods

Participants. Participants were 70 sixth grade students in a school in the Greater Pittsburgh Area. 35 students were randomly assigned to each of the two conditions in the study. These students had fraction instruction, but were far from mastery.

Procedure and Design. All participants were run during their regularly scheduled math class with the rest of their classmates. The study took approximately 80 minutes (i.e., two classroom periods). Students worked independently on a computer, using an online intelligent tutoring system. All students had a pretest, midtest, and posttest with two instruction periods between the assessment periods. The inclusion of the midtest allowed us to track changes over time.

Conditions. *Blocked Condition (Fraction Addition before Fraction Multiplication):* Students see 24 fraction addition problems with correctness feedback at every step in the first instruction period followed by 24 fraction multiplication problems with correctness feedback in the second instruction period.

Interleaved Condition: Students are presented the same 24 fraction addition problems and 24 fraction multiplication problems with correctness feedback from the blocked condition in random order. This design was chosen as opposed to perfectly interleaving to avoid students from learning that they simply needed to use the alternating strategies with every other problem. Twelve addition and 12 multiplication problems are randomly presented in each instruction block.

Instruction. All students were presented with a tutor interface for practicing fraction addition problems with same denominators, fraction addition problems with different denominators, and fraction multiplication problems with different denominators (created with CTAT; Aleven, McClarren,, Sewall, & Koedinger, 2006). For all three problem types, students were provided a problem and provided a space to input a numerator and denominator answer. Students were also provided a checkbox they could use to indicate whether the fractions in the problem need to be converted before the problem could be solved (for fraction addition problems with different denominators). See Figure 1 below for example problems. If students were incorrect completing any particular step, the system would mark that step red. Students were required to input the correct answer for any given step before moving on to the next step. If he or she was stuck, the system provided up to three hints per step. The first hint was always abstract (e.g., for fraction multiplication problems, multiply the denominators to get the denominator answer). The second hint provided the concrete instantiation (e.g., in this case, that means multiplying 2 and 3). The final hint provided the answer (e.g., please input 6 into highlighted cell). Once students provided the correct answer, the step would turn green to acknowledge a correct answer.

Assessment Tasks. *Fraction Addition:* Participants were presented 4 fraction addition problems (2 same denominator and 2 different denominators). Students were scored on accuracy and incorrect strategy choice. Accuracy was based on students having the correct magnitude (i.e., if the answer was 1/2, all equivalent fractions, even if they are not reduced, count as a valid answer). In this case, an incorrect strategy choice reflects a strategy choice that would be successful for a fraction multiplication problem. This includes the independent whole number strategy (adding the numerators and denominators independently) or multiplying either the two numerators or the two denominators while adding the others.

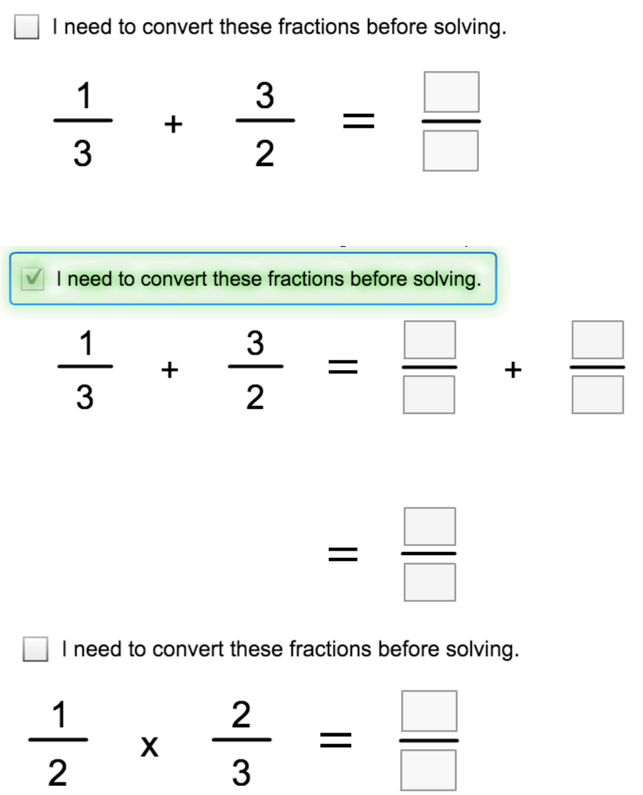

Figure 1. Example instructional interfaces of (a) a fraction addition problem with different denominators before converting, (b) a fraction addition problem with different denominators after the student requests to convert denominators, and (c) a fraction multiplication problem. Students were prompted to solve the following fraction arithmetic problem for every problem provided.

Fraction Multiplication: Participants were presented 4 fraction multiplication problems (2 same denominator and 2 different denominators). Students were scored on accuracy and incorrect strategy choice. Accuracy was based on students having the correct magnitude. In this case, an incorrect strategy choice reflects a strategy choice that would be successful for a fraction multiplication problem. This includes finding a common denominator and only multiplying the numerators (assuming a common denominator was not present already), multiplying the numerators (or denominators) and adding the denominators (or numerators).

Fraction Division: Participants were presented 4 fraction division problems (2 same denominator and 2 different denominators) at posttest only. Students were scored on accuracy.

Results

Blocked Addition-to-Multiplication VS. Interleaved. In a regression, both pretest proportion correct and condition (interleaved = 1) were used to predict posttest proportion correct (R^2 = 32.54%, $F(2,67) = 16.16$, $p < .01$). Pretest was a significant predictor (i.e., students with higher scores pretest scores did better on the posttest; $B = 0.50$ (0.10), $t = 5.06$, $p < 0.01$). Condition was a marginally significant predictor (i.e., interleaved marginally outperformed blocking at posttest; $B = 0.08$ (0.05), $t = 1.80$, $p = 0.08$). The interleaved condition had a posttest accuracy of 79% compared to the blocked addition-to-multiplication posttest accuracy of 68%.

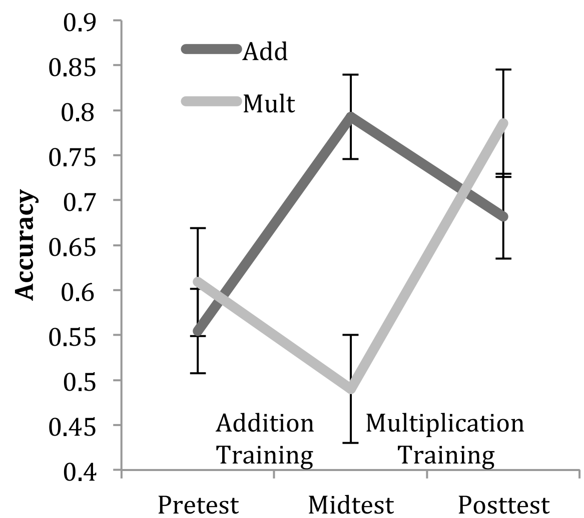

Figure 2. Students in the blocked addition-to-multiplication condition in Experiment 1 performed better on fraction addition after fraction addition training (between the pretest and the midtest), while doing worse on fraction multiplication. After fraction multiplication training (between midtest and posttest), students performed better on fraction multiplication and worse on fraction addition. Error bars reflect +/- 1 SE.

Transfer to Fraction Division. An independent t-test was conducted to compare posttest division proportion correct between conditions. The interleaved group significantly outperformed the blocked addition-to-multiplication group (70% vs. 57%; $t = -2.27$, $df = 66$, $p = 0.03$).

Evidence for Interference. Figure 2 shows how blocked fraction addition practice (before the midtest) changes performance on fraction addition and fraction multiplication and how fraction multiplication practice (between mid and posttest) produces further changes. Notice how there is positive transfer within the two task types, but negative transfer or interference between them. Paired t-tests demonstrate that fraction addition practice between the pretest and midtest produces improvement on fraction addition (56% vs. 82%, $t = -4.61$, $df = 37$, $p < 0.01$), but decline on fraction multiplication (62% vs. 48%, $t = 2.58$, $df = 37$, $p = 0.01$). Similarly, practice on fraction multiplication between the midtest and the posttest produces improvement on fraction multiplication (48% vs. 82%, $t = -3.34$, $df = 34$, $p < 0.01$), but decline on fraction addition (82% vs. 70%, $t = 2.63$, $df = 34$, $p = 0.01$).

Insights from Process Data. We analyzed the learning process data to further investigate why the interleaved practice may have yielded better learning. To do so, we fit a logistic regression model predicting first-attempt correctness on each step of each problem. The predictors included an intercept parameter for each student, an intercept parameter for each problem step (e.g., entering the correct numerator on a multiplication problem) for each condition (blocked vs. interleaved), and a slope parameter for each type of problem step for each condition. Examining the beta estimates revealed striking condition-driven differences in one particular type of skill reflecting the very initial step in which students must check a box to indicate whether to convert the fractions. In particular, the slope estimates for this skill were lower in the interleaved than in the blocked condition, suggesting a slower progression in mastering this skill during learning (Figure 3). Yet, the slower progression on this skill resulted in better pre-post gains.

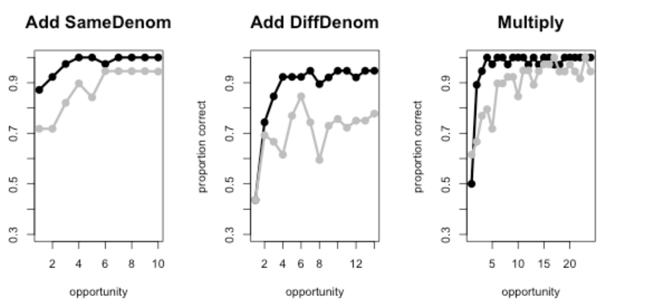

Figure 3. Process data learning curves for the skill of deciding whether to convert fractions. Severe interference when switching problem types is apparent in the blocked condition (black) but is more incremental in the interleaved condition (grey).

We interpreted this as evidence of local trial-to-trial interference between problem types, in the step where students must decide the appropriate procedure to carry out given the problem (i.e., they had to pay attention to the operator to see if it was a '+' or 'x'). Despite evidence of more local interference and slower progression in this procedure selection step, students get much more explicit practice handling the effects of such interference. We hypothesize that this "procedure selection" practice underlies the superior pre-post gains observed in the interleaved condition.

Figure 3 also illustrates the striking interference that occurs for this procedure selection skill when there are shifts between problem types in the blocked addition-to-multiplication condition. This interference is apparent at the sharp learning curve dips that occurs in the fraction addition block at the point where students switch from adding same-denominator fractions (which do not require checking the box to convert denominators) to adding different-denominator fractions (which do require checking the box), and again at the point where students switch to multiplying fractions (which do not require checking the box).

Discussion and Motivation for Experiment 2

We saw support for the hypothesis that interleaved fraction addition and multiplication practice produces better learning than does conventional blocked ordering, particularly on the measure of transfer to fraction division. We found clear evidence of interference that we have previously modeled as over-generalized induction of the if-part of production rules in blocked practice (Li et al., 2012). Interleaved practice provides frequent negative feedback that can be used to refine the if-part to prevent over-generalization errors.

In Experiment 2, we replicated the two conditions from Experiment 1 and included fraction division at all assessment points. Further, based on cognitive task analysis supported by the tutor process data, we also added a third condition that provides blocked instruction in a non-conventional ordering, going from fraction multiplication to fraction addition. In the tutor process data, we observed interference on specific steps within fraction arithmetic procedures such as students incorrectly trying to apply the independent whole number strategy to a fraction addition problem (Liu, Patel, & Koedinger, 2016). These errors were more likely if there had been practice on fraction multiplication problems prior to the fraction addition problem. We also observed student difficulties with finding a common denominator, but also with converting to equivalent fractions. Since this step involves fraction multiplication, at least in principle, we wondered whether there may be positive transfer from fraction multiplication to fraction addition.

Experiment 2

Methods

Participants were 118 sixth grade students in a school in the Greater Pittsburgh Area. 59 students were in the interleaved condition, while 59 students were in the blocked condition. In the blocked condition, 29 students were given fraction addition training before fraction multiplication training, and the other 30 students were given fraction multiplication training before fraction addition training.

The main difference from Experiment 1 is that this experiment included an additional blocked condition, moving from fraction multiplication to fraction addition. In this condition, students saw 24 fraction multiplication problems followed by 24 fraction addition problems with correctness feedback on every step of all problems. This experiment also employed a delayed posttest 3 days after the study concluded, and included fraction division problems at all time points. All remaining methods were the same as in Experiment 1.

Results

Interleaved vs. both Blocked Conditions. A regression was run with pretest proportion correct and condition (1 for interleaved and 0 for both blocked conditions) predicting proportion correct on the delayed posttest (R^2 = 55.44%, $F(2,105) = 65.32$, $p < 0.01$). Both pretest (B = 0.73 (0.06), t = 11.40, $p < 0.01$) and condition (B = 0.06 (0.03), t = 2.28, p = 0.02) were significant predictors. The interleaved condition had a posttest accuracy of 79% compared to the blocked posttest accuracy of 74%.

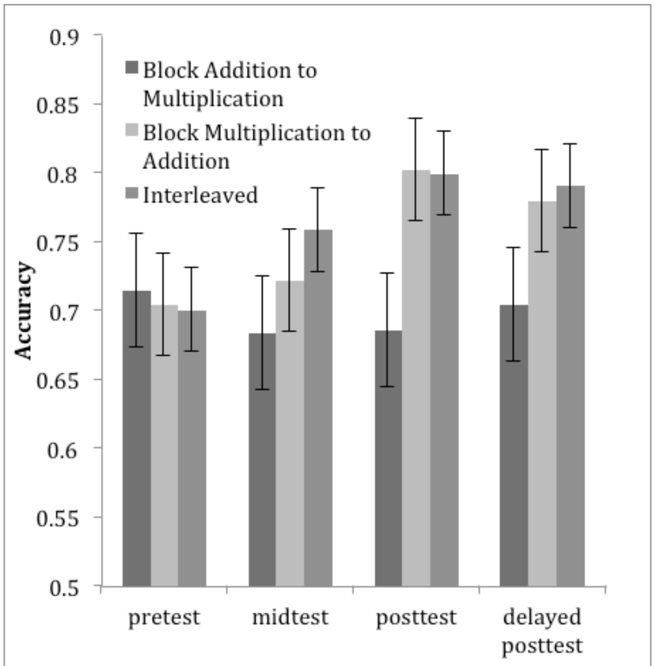

Figure 4. The three sequencing groups in Experiment 2 started with similar accuracies at pretest, but the interleaved and blocked multiplication-to-addition conditions outperformed the standard blocked addition-to-multiplication on both the posttest and delayed posttest.

Comparing Blocking Orders Individually. Separating the blocked conditions by order type reveals a more interesting story (see Figure 4). In a regression, pretest and condition as a factor (all conditions were compared to the blocked addition-to-multiplication condition) were used to predict the delayed posttest (R^2 = 57.13%, $F(3,104) = 47.34$, $p <$ 0.01). Pretest was a significant predictor (B = 0.73 (0.06), t = 11.59, $p < 0.01$). On average, the students in the blocked multiplication-to-addition condition outperformed the students in the blocked addition-to-multiplication condition by 8% (78% vs. 70%; B = 0.07 (0.04), t = 3.57, p = 0.04). On average, the students in the interleaved condition outperformed the students in the blocked addition-to-multiplication condition by 9% (79% vs. 70%; B = 0.10 (0.03), t = 3.12, $p < 0.01$). Note: The interleaved condition did not outperform the blocked multiplication-to-addition condition (79% vs. 78%; B = 0.02 (0.03), t = 0.88, p = 0.38).

Blocked Multiplication-to-Addition Does Not Cause Interference Errors. Four paired t-tests were conducted to test changes in fraction multiplication and fraction addition from pretest to midtest and midtest to posttest in the blocked multiplication-to-addition condition (the blocked addition-to-multiplication followed similar interference patterns as experiment 1). Specifically, these t-tests were designed to test if learning fraction multiplication causes worse performance on fraction addition (and vice versa) due to overgeneralization errors. After being taught fraction multiplication between the pretest and midtest, students got better at fraction multiplication (79% vs. 91%, t = -1.93, df = 29, p = 0.06) and stayed the same at fraction addition (55% vs. 58%, t = -0.43, df = 29, p = 0.66). After being taught fraction addition between the midtest and the posttest, students got better at fraction addition (58% vs. 78%, t = -3.33, df = 27, $p < 0.01$) and stayed the same at fraction multiplication (91% vs. 90%, t = -0.07, df = 27, p = 0.95).

Blocked Addition-to-Multiplication Practice Interferes with Fraction Division. The previous study showed an advantage of interleaving to blocking in terms of fraction division accuracy at posttest. This advantage appears to be due to interference in the blocked addition-to-multiplication condition. Changes in division accuracy were only observed in the blocked addition-to-multiplication condition between the pre- and mid-test (72% vs. 50%, t = 2.68, df = 26, p = 0.01). This decline in performance appears to result from over-generalization of fraction addition procedures.

General Discussion

Perhaps the most interesting result suggests that the U.S. curriculum teaches fraction addition and fraction multiplication in the least effective way out of the three alternatives we explored. Across both experiments, blocked fraction addition-to-multiplication practice produced significantly less learning than both interleaved practice and blocked fraction multiplication-to-addition practice.

Consistent with theory (Li et al, 2012; Rohrer & Taylor, 2007), we observed evidence of interference between practice trials of different kinds of fraction operations. That interference was particularly pronounced for fraction multiplication and division after an initial block of fraction addition practice and for fraction addition after a second block of fraction multiplication practice (see interference sections under the results). The generally slower learning rate in the interleaved condition in Experiment 1 is consistent with interference from over-generalization during interleaved practice, but the better posttest performance suggests these over-generalizations were at least somewhat remediated by error feedback during interleaved practice.

We did not find clear evidence of interference in the blocked multiplication-to-addition condition. Perhaps the lower cognitive load of starting with a simpler task, namely fraction multiplication, left students with capacity for more deliberate and accurate reasoning about strategy selection. Thus, they may have been better able to avoid over-generalizing the fraction multiplication strategy (independent whole number operations) to fraction addition. Further, when moving to fraction addition, the prior fraction multiplication practice may have aided performance of key steps, particularly finding a common denominator (which requires whole number multiplication) and converting the numerators to maintain equivalent fractions (which arguably requires faction multiplication, e.g., 1/2*2/2 = 2/4). With better prior preparation, students may have had more cognitive headroom to more deliberately avoid over-generalizing similar to above, but in this case from fraction addition to fraction multiplication or division. While this argument is, in broad strokes, consistent with VanLehn (1987), a precise application of the SimStudent model in Li et al (2012) would still predict interference due over-generalization. Thus, more work is warranted. A replication of this reverse blocking effect is desirable.

Applications to Education

In the current U.S. standards, fraction arithmetic training follows the same order as whole number arithmetic training (addition, subtraction, multiplication, and finally, division). While there may be some appeal to ordering the operations the same way, our results imply that interleaving practice among different operations yields superior learning outcomes. Interleaved practice forces students to explicitly practice recognizing *when* to carry out which procedure, an aspect that is important for overcoming interference. An alternative implication is that re-structuring blocked instruction to progress from simpler subskills to more complex skills (i.e., practicing fraction and whole number multiplication before adding fractions with different denominators), could be more optimal than what is currently implemented in classrooms.

Generalizing to Other Domains

Our studies targeted the fraction arithmetic domain and past experience and theory give us pause in suggesting cross-domain generalizations (Koedinger et al, 2012). We specifically chose fraction addition and multiplication because they present highly similar tasks features that require different strategic responses. Other task type pairs that have this characteristic are good candidates for applying interleaving. For example, tasks types of finding the area versus perimeter of figures are highly similar (with the words "area" or "perimeter" sometimes being the only difference) and thus interleaving practice of these task types should be effective at reducing over-generalization errors (i.e., computing the area on a perimeter problem). On the other hand, we also found evidence consistent with some forms of blocking being as effective as interleaving. The benefits of blocking may be particularly enhanced when subskills accumulate into more complex tasks. Thus, we recommend careful cognitive task analysis to support the decision of when to block or interleave.

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