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Inequalities for connectivity events in Bernoulli percolation

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Inequalities for connectivity events in Bernoulli percolation

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by

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ABSTRACT OF THE DISSERTATION

Inequalities for connectivity events
in Bernoulli percolation

by

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Professor Igor Pak, Chair

This thesis consists of six chapters based on papers on probabilities of connectivity events in percolation theory.

- Chapter 1 is based on paper [GP24b] written with Igor Pak. There we study colored percolation, a generalization of the classical percolation model.
- Chapter 2 is based on paper [GZ24] written with Aleksandr Zimin. There we show that bond percolation does not simulate site percolation.
- Chapter 3 is based on paper [G24b]. There we study percolation inequalities and decision trees.
- Chapter 4 is based on paper [GPZ24] with Igor Pak and Aleksandr Zimin. There we disprove the bunkbed conjecture. The appendix proves the bunkbed conjecture for the case of 2 transversal vertices and wasn't published before.
- Chapter 5 is based on an unpublished manuscript written with Aleksandr Zimin. There

we prove multiple inequalities of the form similar to the Harris–Kleitman inequality. This work generalizes and supercedes a previous paper [G24a] by the author.

- Chapter 6 contains an unpublished result related to the main body of work in the thesis.

The dissertation of Nikita Gladkov is approved.

Anton Bernshteyn

Pavel Galashin

Bruce L. Rothschild

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2025

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CHAPTER 1

Positive dependence for colored percolation

1.1 Introduction

The study of *percolation* goes back to the 1957 paper by Broadbent and Hammersley [BH57] and has been incredibly popular in the last few decades across the sciences. It remains one of the most applied statistical models, reaching far corners of statistical physics and probability, and fields as disparate a materials science, network theory and seismology, see e.g. [G18, SCA05, Sah23].

Despite remarkable recent advances, many problems remain open and continued to be actively pursued, see e.g. [1, D18a, Gri23, Mor17]. Note that specific models of percolation wary greatly depending on the scientific context and applications. Here we consider the *colored* bond (site) percolation, where each graph edge (vertex) takes random color, see e.g. [KM17, SCA05, Zal77].

As one studies random events, one is naturally concerned about their correlations. This led to correlation inequalities, the first of which, *Harris–Kleitman inequality* [H60, Kle66] was discovered independently in probability and graph theory. It shows that every two increasing (or two decreasing) random events on the same probability space are positively correlated. On the other hand, when one event is increasing and another is decreasing, such events are negatively correlated. Outside of its fundamental applications to statistical physics and probability, this result has numerous applications in graph theory [Cha17, JLR00], order theory [Fish92, She82] and algebraic combinatorics [CP23].

There are many generalizations and variations on the Harris–Kleitman inequality, see e.g. [DS12, New80, Wer09], including intensely studied but largely mysterious generalizations to multiple functions [G24a, LS22, S08]. In this paper we consider k events $\mathcal{U}_i$ such that every $(k-1)$ of them are mutually independent. To quantify correlations we study the ratio

$$\mu := \frac{\mathbf{P}(\mathcal{U}_1 \cap \cdots \cap \mathcal{U}_k)}{\mathbf{P}(\mathcal{U}_1) \cdots \mathbf{P}(\mathcal{U}_k)}$$

which, we call *mutual dependence*. We prove a general result extending the Harris–Kleitman inequality from $k = 2$ to all k. We concentrate on the case $k = 3$, which is the first nontrivial example and is of independent interest.

Our main application is to 4-colored percolation on infinite graphs and graphs with symmetry. We show that $\mu \geq 1$ or that $\mu \leq 1$ depending on a situation, and in some cases conjecture that our bounds are asymptotically tight. Additionally, we introduce a new colored critical probability for infinite graphs which turns out to be closely related to the usual critical probability.

1.2 Positive correlation in percolation

We first illustrate the Harris–Kleitman inequality. Let $G = (V, E)$ be a simple graph, which can be finite or infinite. Consider a *p-percolation* defined by independently at random deleting edges of G with probability $(1 - p)$. We write $\mathbf{P}_p(x \leftrightarrow y)$ for the probability that vertices $x, y \in V$ are connected.

In its basic application, the Harris–Kleitman inequality proves a positive correlation of connectivity of two pairs of vertices:

$$\mathbf{P}_p(x \leftrightarrow y, u \leftrightarrow v) \geq \mathbf{P}_p(x \leftrightarrow y)\, \mathbf{P}_p(u \leftrightarrow v), \tag{1.1}$$

for all $x, y, u, v \in V$. Equivalently, this says that the probability that two vertices are connected increases if *some other* two vertices are connected, even if these two vertices are far apart in the graph: $\mathbf{P}_p(x \leftrightarrow y \,|\, u \leftrightarrow v) \geq \mathbf{P}_p(x \leftrightarrow y)$. This implies that the *critical* *probability* $p_c := \sup\{p : \mathbf{P}_p(x \leftrightarrow \infty) = 0\}$ is independent on the vertex x in every connected graph, see e.g. [BR06, G18]. The idea is that for two vertices x, y, the ratio $\frac{\mathbf{P}_p(x\leftrightarrow\infty)}{\mathbf{P}_p(y\leftrightarrow\infty)}$ can not go below $\mathbf{P}_p(x \leftrightarrow y)$. For the case when $G = \mathbb{Z}^2$ is a square lattice, Harris used the inequality to prove that $p_c \geq \frac{1}{2}$ [H60]. Famously, Kesten [Kes80] established the equality $p_c = \frac{1}{2}$ twenty years later.

Denote by 2^E the collection of all subsets of E. A subcollection $\mathcal{A} \subseteq 2^E$ is called *closed upward*, if $A + e \in \mathcal{A}$ for every $A \in \mathcal{A}$ and $e \in E \smallsetminus A$. Similarly, $\mathcal{A}$ is *closed downward*, if $A - e \in \mathcal{A}$ for every $A \in \mathcal{A}$ and $e \in A$. We think of $\mathcal{A}$ as *graph property*, and write $\mathbf{P}_p(\mathcal{A})$ for the probability that the property holds for a p-percolation. In this notation, the *Harris–Kleitman inequality* states:

$$\mathbf{P}_p(\mathcal{A} \cap \mathcal{B}) \geq \mathbf{P}_p(\mathcal{A})\,\mathbf{P}_p(\mathcal{B}), \tag{1.2}$$

for every two closed upward subcollections $\mathcal{A}, \mathcal{B}$. For $\mathcal{A} = \{H : x \leftrightarrow y\}$ and $\mathcal{B} = \{H : u \leftrightarrow v\}$ we obtain (1.1). Note that (1.2) holds also for every two closed downward $\mathcal{A}, \mathcal{B}$. Indeed, their complements $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ will be closed upwards and

$$\begin{aligned}\mathbf{P}_p(\mathcal{A} \cap \mathcal{B}) &= 1 - \mathbf{P}_p(\bar{\mathcal{A}}) - \mathbf{P}_p(\bar{\mathcal{B}}) + \mathbf{P}_p(\bar{\mathcal{A}} \cap \bar{\mathcal{B}}) \\ &\geq 1 - \mathbf{P}_p(\bar{\mathcal{A}}) - \mathbf{P}_p(\bar{\mathcal{B}}) + \mathbf{P}_p(\bar{\mathcal{A}})\mathbf{P}_p(\bar{\mathcal{B}}) = \mathbf{P}_p(\mathcal{A})\mathbf{P}_p(\mathcal{B}).\end{aligned}$$

When $\mathcal{A}$ is closed upward and $\mathcal{B}$ is closed downward, the negative correlation follows by the same argument.

Now, let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be pairwise independent events. We say that they have *positive mutual dependence* if $\mathbf{P}(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}) \geq \mathbf{P}(\mathcal{U})\,\mathbf{P}(\mathcal{V})\,\mathbf{P}(\mathcal{W})$. Similarly, we say that they have *negative mutual dependence* if $\mathbf{P}(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}) \leq \mathbf{P}(\mathcal{U})\,\mathbf{P}(\mathcal{V})\,\mathbf{P}(\mathcal{W})$.

1.3 Positive dependence in colored percolation

We are now ready to formalize the approach above to state the result in full generality.

Let $f : E \to \{a, b, c, d\}$ be a uniform random coloring of the edges of G, where each edge is colored uniformly and independently. As before, denote by E_s, $s \in \{a, b, c, d\}$, a subset of edges of the corresponding color. Similarly, for every two distinct colors $s, t \in \{a, b, c, d\}$, let $E_{st} := E_s \cup E_t$. One can think of E_{st} as either a $\frac{1}{2}$-percolation or a uniformly random subset of edges of G, so that $G_{st} = (V, E_{st})$ is a uniform random subgraph of G.

Theorem 1. *Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be closed upward graph properties. Denote by $\mathcal{U}_{ab}$, $\mathcal{V}_{ac}$ and $\mathcal{W}_{bc}$ the corresponding properties of G_{ab}, G_{ac} and G_{bc}, respectively. Then the events $\mathcal{U}_{ab}$, $\mathcal{V}_{ac}$ and $\mathcal{W}_{bc}$ are pairwise independent, but have negative mutual dependence:*

$$\mathbf{P}(\mathcal{U}_{ab} \cap \mathcal{V}_{ac} \cap \mathcal{W}_{bc}) \le \mathbf{P}(\mathcal{U}_{ab})\,\mathbf{P}(\mathcal{V}_{ac})\,\mathbf{P}(\mathcal{W}_{bc}), \tag{1.3}$$

where the probability is over uniform random colorings $f : E \to \{a, b, c, d\}$. Similarly, events $\mathcal{U}_{ab}$, $\mathcal{V}_{ac}$ and $\mathcal{W}_{ad}$ are pairwise independent, but have positive mutual dependence:

$$\mathbf{P}(\mathcal{U}_{ab} \cap \mathcal{V}_{ac} \cap \mathcal{W}_{ad}) \ge \mathbf{P}(\mathcal{U}_{ab})\,\mathbf{P}(\mathcal{V}_{ac})\,\mathbf{P}(\mathcal{W}_{ad}), \tag{1.4}$$

where $\mathcal{W}_{ad}$ is the property of E_{ad}. Additionally, for $\mathcal{U}, \mathcal{V}, \mathcal{W}$ closed downward graph properties, the inequalities in both (1.3) and (1.4) are reversed.

Since all E_{st} are $\frac{1}{2}$-percolations, we can rewrite the RHS of both (1.3) and (1.4) as a more symmetric product:

$$\mathbf{P}_{\frac{1}{2}}(\mathcal{U})\,\mathbf{P}_{\frac{1}{2}}(\mathcal{V})\,\mathbf{P}_{\frac{1}{2}}(\mathcal{W}). \tag{1.5}$$

The proof of the theorem is given in the appendix. After a quick argument proving pairwise independence, we now proceed to a number of applications of the theorem to many percolation examples.

1.4 Proof of Theorem 1

Since E_{ab} and E_{ac} are independent $\frac{1}{2}$-percolations, this implies that events $\mathcal{U}_{ab}$ and $\mathcal{V}_{ac}$ are also independent. This proves the pairwise independence part.

We prove (1.3) by induction on the number of edges in E. For $E = \varnothing$, the inequality is trivial. Fix an edge $e \in E$. Consider the probability space of colorings of $E - e$. For an event $\mathcal{X}_{ab} \subseteq 2^E$, denote by $\mathcal{X}_{ab}^+$ the subset of $\mathcal{X}_{ab}$ such that $f(e) \in \{a, b\}$. Similarly, denote by $\mathcal{X}_{ab}^-$ the subset of $\mathcal{X}_{ab}$ such that $f(e) \in \{c, d\}$.

By the symmetry, we have:

$$\mathbf{P}\big(\mathcal{X}_{ab} : f(e) = a\big) = \mathbf{P}(\mathcal{X}_{ab} : f(e) = b) = 2\mathbf{P}_{\frac{1}{2}}(\mathcal{X}^+),$$
$$\mathbf{P}\big(\mathcal{X}_{ab} : f(e) = c\big) = \mathbf{P}(\mathcal{X}_{ab} : f(e) = d) = 2\mathbf{P}_{\frac{1}{2}}(\mathcal{X}^-).$$

Clearly, $\mathbf{P}_{\frac{1}{2}}(\mathcal{X}) = \mathbf{P}_{\frac{1}{2}}(\mathcal{X}^-) + \mathbf{P}_{\frac{1}{2}}(\mathcal{X}^+)$. When $\mathcal{X}$ is closed upward, we also have $\mathbf{P}_{\frac{1}{2}}(\mathcal{X}^-) \leq \mathbf{P}_{\frac{1}{2}}(\mathcal{X}^+)$. We use this notation for $\mathcal{X} \in \{\mathcal{U}, \mathcal{V}, \mathcal{W}\}$ and all pairs of colors.

Considering all possible colors of e and using the induction hypothesis, we have:

$$\begin{aligned}
\mathbf{P}(\mathcal{U}_{ab} \cap \mathcal{V}_{ac} \cap \mathcal{W}_{bc}) \ &= \ \mathbf{P}\big(\mathcal{U}_{ab}^+ \cap \mathcal{V}_{ac}^+ \cap \mathcal{W}_{bc}^-\big) + \mathbf{P}\big(\mathcal{U}_{ab}^+ \cap \mathcal{V}_{ac}^- \cap \mathcal{W}_{bc}^+\big) \\
&\qquad + \mathbf{P}\big(\mathcal{U}_{ab}^- \cap \mathcal{V}_{ac}^+ \cap \mathcal{W}_{bc}^+\big) + \mathbf{P}\big(\mathcal{U}_{ab}^- \cap \mathcal{V}_{ac}^- \cap \mathcal{W}_{bc}^-\big) \\
&\leq \ 2\Big(\mathbf{P}\big(\mathcal{U}_{ab}^+\big)\mathbf{P}\big(\mathcal{V}_{ac}^+\big)\mathbf{P}\big(\mathcal{W}_{bc}^-\big) + \mathbf{P}\big(\mathcal{U}_{ab}^+\big)\mathbf{P}\big(\mathcal{V}_{ac}^-\big)\mathbf{P}\big(\mathcal{W}_{bc}^+\big) \\
&\qquad + \mathbf{P}\big(\mathcal{U}_{ab}^-\big)\mathbf{P}\big(\mathcal{V}_{ac}^+\big)\mathbf{P}\big(\mathcal{W}_{bc}^+\big) + \mathbf{P}\big(\mathcal{U}_{ab}^-\big)\mathbf{P}\big(\mathcal{V}_{ac}^-\big)\mathbf{P}\big(\mathcal{W}_{bc}^-\big)\Big).
\end{aligned}$$

Simplifying the notation as above, the RHS is equal to:

$$\begin{aligned}
&2\Big(\mathbf{P}_{\frac{1}{2}}(\mathcal{U}^+)\mathbf{P}_{\frac{1}{2}}(\mathcal{V}^+)\mathbf{P}_{\frac{1}{2}}(\mathcal{W}^-) + \mathbf{P}_{\frac{1}{2}}(\mathcal{U}^+)\mathbf{P}_{\frac{1}{2}}(\mathcal{V}^-)\mathbf{P}_{\frac{1}{2}}(\mathcal{W}^+) \\
&\qquad + \mathbf{P}_{\frac{1}{2}}(\mathcal{U}^-)\mathbf{P}_{\frac{1}{2}}(\mathcal{V}^+)\mathbf{P}_{\frac{1}{2}}(\mathcal{W}^+) + \mathbf{P}_{\frac{1}{2}}(\mathcal{U}^-)\mathbf{P}_{\frac{1}{2}}(\mathcal{V}^-)\mathbf{P}_{\frac{1}{2}}(\mathcal{W}^-)\Big) \\
&= \ \big(\mathbf{P}_{\frac{1}{2}}(\mathcal{U}^+) + \mathbf{P}_{\frac{1}{2}}(\mathcal{U}^-)\big)\big(\mathbf{P}_{\frac{1}{2}}(\mathcal{V}^+) + \mathbf{P}_{\frac{1}{2}}(\mathcal{V}^-)\big)\big(\mathbf{P}_{\frac{1}{2}}(\mathcal{W}^+) + \mathbf{P}_{\frac{1}{2}}(\mathcal{W}^-)\big) \\
&\qquad - \big(\mathbf{P}_{\frac{1}{2}}(\mathcal{U}^+) - \mathbf{P}_{\frac{1}{2}}(\mathcal{U}^-)\big)\big(\mathbf{P}_{\frac{1}{2}}(\mathcal{V}^+) - \mathbf{P}_{\frac{1}{2}}(\mathcal{V}^-)\big)\big(\mathbf{P}_{\frac{1}{2}}(\mathcal{W}^+) - \mathbf{P}_{\frac{1}{2}}(\mathcal{W}^-)\big) \\
&\leq \ \mathbf{P}_{\frac{1}{2}}(\mathcal{U})\mathbf{P}_{\frac{1}{2}}(\mathcal{V})\mathbf{P}_{\frac{1}{2}}(\mathcal{W}),
\end{aligned}$$

as desired. The proof of (1.4) goes along the same lines. Finally, the closed downward version follows the inclusion exclusion argument earlier in the paper. □

1.5 Variations and generalizations

First, note that we never use the graph structure, and the theorem can be viewed as a result about abstract set systems, cf. [SA94]. Second, the pairwise independent $\frac{1}{2}$-percolation argument that we discussed after the theorem can be generalized in several ways. Notably, it can be extended to the p-percolation for all $0 \leq p \leq 1$, but the resulting coupling of percolations then require seven colors and have somewhat inelegant probabilities.

Next, the theorem can be extended to a larger number of events. Start by taking $k-1$ independent $\frac{1}{2}$-percolations $E_1, \ldots, E_{k-1}$ on the same graph. Define a new $\frac{1}{2}$-percolation

$$E_k := \bigoplus_{i=1}^{k-1} E_i \mod 2,$$

where the edge e is present if and only if it is present in an odd number of E_i's. Observe that every $k-1$ of $E_1, \ldots, E_k$ are mutually independent.

Then, for every closed downward properties $\mathcal{X}_1, \ldots, \mathcal{X}_k$ we have:

$$\mathbf{P}\big(\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_k\big) \;\geq\; \mathbf{P}(\mathcal{X}_1) \cdots \mathbf{P}(\mathcal{X}_k). \tag{1.6}$$

Once again, the proof follows verbatim the proof of the theorem. Note that for $k=2$, we have $E_1 = E_2$ and (1.6) is the Harris–Kleitman inequality (1.2). For $k=3$, the inequality (1.6) gives (1.4).

Finally, one can easily obtain a colored version with $m = 2^{k-1}$ colors. For example, for $k=4$, take a uniform random edge coloring $f : E \to \{1, \ldots, 8\}$. Consider four pairwise independent $\frac{1}{2}$-percolations E_{1234}, E_{1256}, E_{1357} and E_{1467} with natural labeling. Note that every three of these are mutually independent. Then, for closed downward properties $\mathcal{U}, \mathcal{V}, \mathcal{W}$ and $\mathcal{X}$, the inequality (1.6) gives:

$$\mathbf{P}(\mathcal{U}_{1234} \cap \mathcal{V}_{1256} \cap \mathcal{W}_{1357} \cap \mathcal{X}_{1467}) \\ \geq\; \mathbf{P}(\mathcal{U}_{1234})\,\mathbf{P}(\mathcal{V}_{1256})\,\mathbf{P}(\mathcal{W}_{1357})\,\mathbf{P}(\mathcal{X}_{1467}).$$

1.6 Probability of the majority

Let E_1 and E_2 be two independent $\frac{1}{2}$-percolations, and let $E_3 = E_1 \oplus E_2$ to be the new $\frac{1}{2}$-percolation where every edge is open if it is open in exactly one of E_1, E_2. Consider the coloring

$$f(e) := \begin{cases} a & \text{if } \ e \in E_1 \cap E_2 \\ b & \text{if } \ e \in E_1, e \notin E_2 \\ c & \text{if } \ e \in E_2, e \notin E_1 \\ d & \text{if } \ e \notin E_1, e \notin E_2 \end{cases}$$

Then $E_{ab} = E_1$, $E_{ac} = E_2$, $E_{bc} = E_3$, which implies the pairwise independence. This observation is motivational and generalizes to $k \geq 2$ mutually independent $\frac{1}{2}$-percolations (see the appendix).

1.7 Crossing probabilities in a rectangle

Let $G = (V, E)$ be a $n \times (n+1)$ rectangle as in Figure 1. Consider a uniform random edge coloring $f : E \to \{a, b, c, d\}$. Note that E_{ab}, E_{ac} and E_{ad} are pairwise independent bond $\frac{1}{2}$-percolations with free boundary conditions (BC). Let $\mathcal{U} = \{12 \leftrightarrow 34\}$ be the connectivity property of the opposite sides of G, and recall that $\mathbf{P}_{\frac{1}{2}}(\mathcal{U}_{ab}) = \frac{1}{2}$, see e.g. [BR06]. Then (1.4) gives:

$$\mathbf{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{ad}\big) \geq \mathbf{P}_{\frac{1}{2}}(\mathcal{U})^3 = \tfrac{1}{8}, \tag{1.7}$$

for all $n \geq 1$. On the other hand, by the pairwise independence we have:

$$\mathbf{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{ad}\big) \leq \mathbf{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac}\big) = \mathbf{P}_{\frac{1}{2}}(\mathcal{U})^2 = \tfrac{1}{4}.$$

Note that as a function of p the crossing probability in a rhombus under p-percolation has a sharp threshold [BR06], so the trivial lower bound is unhelpful:

$$\mathbf{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{ad}\big) \geq \mathbf{P}(\mathcal{U}_a) = \mathbf{P}_{\frac{1}{4}}(\mathcal{U}) \xrightarrow[n \to \infty]{} 0$$

For $n = 30$, the sampling of $N = 4 \cdot 10^7$ trials gives an approximation $\mathbf{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{ad}\big) = 0.125098 \pm 0.000052$. We conjecture that this probability is $\frac{1}{8}$ in the limit $n \to \infty$.

1.8 Crossing probabilities in a rhombus

Let $G = (V, E)$ be a m-rhombus on the triangular lattice, see Figure 1. Consider a uniform random vertex coloring $f : V \to \{a, b, c, d\}$. Note that V_{ab}, V_{ac} and V_{ad} are pairwise independent site $\frac{1}{2}$-percolations with free BC. Let $\mathcal{U} = \{12 \leftrightarrow 34\}$ and $\mathcal{U}' = \{14 \leftrightarrow 23\}$ be connectivity properties of the opposite sides of G. Recall that $\mathbf{P}_{\frac{1}{2}}(\mathcal{U}_{ab}) + \mathbf{P}_{\frac{1}{2}}(\mathcal{U}'_{cd}) = 1$ by a topological argument, so $\mathbf{P}_{\frac{1}{2}}(\mathcal{U}) = \mathbf{P}_{\frac{1}{2}}(\mathcal{U}') = \frac{1}{2}$ by the symmetry. Then (1.3) and (1.4) give:

$$\begin{aligned} \mathbf{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{bc}\big) \ &\le \ \mathbf{P}_{\frac{1}{2}}(\mathcal{U})^3 \ = \ \tfrac{1}{8}, \\ \mathbf{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{ad}\big) \ &\ge \ \mathbf{P}_{\frac{1}{2}}(\mathcal{U})^3 \ = \ \tfrac{1}{8}, \end{aligned} \tag{1.8}$$

for all $m \ge 1$. We conjecture that

$$\mathbf{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{bc}\big) \text{ and } \mathbf{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{ad}\big) \to \tfrac{1}{8}$$

as $m \to \infty$. If this holds, we also have other similar limits, e.g.

$$\mathbf{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}'_{bc}\big) = \mathbf{P}_{\frac{1}{2}}(\mathcal{U})^2 - \mathbf{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{ad}\big) \ \to \ \tfrac{1}{8}.$$

This is in contrast with limits such as $\mathbf{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{bc} \cap \mathcal{U}_{cd}\big)$ which can be computed using *Watts' formula* [Watts96] (see also [Dub06, SW11]).

1.9 Crossing probabilities in a hexagon

Consider a regular hexagon $G = (V, E)$ on the triangular lattice with side lengths ℓ, see Figure 1. Consider a site $\frac{1}{2}$-percolations with free BC as above. Let $\mathcal{U} := \big\{\exists x \in V : x \leftrightarrow 12, x \leftrightarrow 34, x \leftrightarrow 56\big\}$ be the joint connectivity property of the percolation graph. It was computed by Simmons [Sim13] (see also [FZS15]), that $\mathbf{P}_{\frac{1}{2}}(\mathcal{U}) = 0.2556897...$ in the limit $\ell \to \infty$. Consider a uniform random vertex coloring $f : V \to \{a, b, c, d\}$. Then (1.4) gives:

$$\mathbf{P}_{\frac{1}{2}}(\mathcal{U})^2 = 0.0653772... \geq \mathbf{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{ad}\big) \geq \mathbf{P}_{\frac{1}{2}}(\mathcal{U})^3 = 0.0167162...$$

in the limit $\ell \to \infty$. Similarly, the inequality (1.3) gives:

$$\mathbf{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{bc}\big) \leq \mathbf{P}_{\frac{1}{2}}(\mathcal{U})^3 = 0.0167162...$$

in the limit $\ell \to \infty$. For $\ell = 30$, the sampling of $N = 64000$ trials gives $\mathbf{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{ad}\big) = 0.0172 \pm 0.0005$ and $\mathbf{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{bc}\big) = 0.0166 \pm 0.0005$. We conjecture that both probabilities are $\mathbf{P}_{\frac{1}{2}}(\mathcal{U})^3 = 0.0167162...$ in the limit $\ell \to \infty$.

1.10 New critical probability

Recall the setting we discussed earlier. Let $G = (V, E)$ be an infinite connected graph. Consider a uniform random coloring $f : E \to \{a, b, c, d\}$. For a vertex $x \in V$, consider

$$P(x) := \mathbf{P}\big(x \leftrightarrow_{ab} \infty, x \leftrightarrow_{ac} \infty, x \leftrightarrow_{ad} \infty\big), \tag{1.9}$$

where $x \leftrightarrow_{st} \infty$ means that x belongs to an infinite cluster of st-colored edges. Now (1.4) gives:

$$\mathbf{P}_{\frac{1}{2}}\big(x \leftrightarrow \infty\big)^2 \;\geq\; P(x) \;\geq\; \mathbf{P}_{\frac{1}{2}}\big(x \leftrightarrow \infty\big)^3. \tag{1.10}$$

1.11 Conclusions

The subject of positive dependence for colored percolation is largely unexplored and can be viewed as a special case of algebraic inequalities for cumulants of positive functions. The latter has been actively studied (see [G24a, LS22] for recent references), but the type of inequalities we consider are new.

In full generality, our results extend the Harris–Kleitman inequality (1.2) to multiple pairwise independent events. This allows us to give lower and upper bounds on the mutual dependence of these events, which are asymptotically tight for the (conjectured) crossing probabilities of the colored percolation on lattices, exhibiting the same phenomenon as the majority property.

CHAPTER 2

Bond percolation does not simulate site percolation

2.1 Introduction

For a graph $G = (V, E)$ we consider three independent opening models. The *(inhomogeneous) Bernoulli bond percolation* μ opens each edge $e \in E$ with probability p_e, whereas the *(inhomogeneous) Bernoulli site percolation* σ opens each vertex $v \in V$ with probability p_v. The parameters p_e and p_v may vary from edge to edge and from vertex to vertex throughout the paper.

Finally, we shall occasionally need a third model. Given a hypergraph $H = (V, E)$ – so each $e \in E$ is a finite subset of V – the *(inhomogeneous) Bernoulli hyperedge percolation* η opens each hyperedge e with probability $p_e \in [0, 1]$, independently for different e. When $|e| = 2$ for every hyperedge, this reduces to the bond model; thus hyperedge percolation strictly contains bond percolation as a special case.

Observable vertices. Throughout the paper we work with a chosen subset $V_{\text{obs}} \subseteq V$ called the *observable vertices*. We care only about how these vertices are connected to each other. Vertices in $V \setminus V_{\text{obs}}$, which we call *auxiliary*, may lie on paths but never serve as endpoints in the events we consider. In the bond and hyperedge models, we simply take $V_{\text{obs}} = V$. In the site model we also set $p_v = 1$ for every $v \in V_{\text{obs}}$, so each observable vertex is always open.

Definition 2.1.1 (Connectivity)**.** Let $\rho \in \{\mu, \sigma, \eta\}$ be either percolation on $G = (V, E)$ and let $u, v \in V_{\text{obs}}$. We write

$$u \underset{\rho}{\leftrightarrow} v$$

if there exists a path $P = (w_0 = u, \ldots, w_k = v)$ in G such that

- when $\rho = \mu$ (bond case), every edge w_i, w_{i+1} of P is open;
- when $\rho = \sigma$ (site case), every vertex w_i of P is open, including the endpoints u and v;
- when $\rho = \eta$ (hyperedge case), for each i there exists an open hyperedge $e_i \in E$ with $w_i, w_{i+1} \subseteq e_i$.

We denote the probability of this event by

$$\rho(uv) \ := \ \rho(u \underset{\rho}{\leftrightarrow} v).$$

One can ask many questions about the probabilistic properties of clusters connected via open vertices and edges. There are well known inequalities comparing critical probabilities of site and bond percolation on the same infinite graph [GS98].

To motivate our problem, recall Exercise 3.4 in [G18] (see also Exercise 6 in [D18b]):

> "Show that bond percolation on a graph G may be reformulated in terms of site percolation on a graph derived suitably from G."

Here is a formal definition.

Definition 2.1.2 (Exact simulation)**.** Let ρ be a percolation measure on a graph $G = (V, E)$ with observable set V_{obs}, and let ρ' be a percolation measure on a (possibly different) graph $G' = (V', E')$ with observable set V'_{obs}. We say that ρ *simulates* ρ' if there exists a map $f : V_{\text{obs}} \to V'_{\text{obs}}$ such that, for every Boolean combination of the events $\{ v_i \underset{\rho}{\leftrightarrow} v_j \}$ with $v_i, v_j \in V_{\text{obs}}$,

$$\rho(\mathcal{E}) \ = \ \rho'(\mathcal{E} \circ f),$$

where $\mathcal{E} \circ f$ is obtained from $\mathcal{E}$ by replacing each v_i with $f(v_i)$ and each connectivity symbol $\underset{\rho}{\leftrightarrow}$ with $\underset{\rho'}{\leftrightarrow}$.

Remark 2.1.3. By this definition, the simulation preserves events such as "At least n out of m vertices $v_1, \ldots, v_m$ are in the same cluster", but is not guaranteed to preserve the probability of "There is a path from a to b avoiding vertex c".

The following theorem [F61, FE61] solves Exercise 3.4 in [G18] by constructing a site percolation that simulates any given bond percolation.

Theorem 2.1.4. *For every graph G equipped with a bond percolation μ there exists a graph G' together with a site percolation σ that simulates μ.*

Proof. Make a copy G' of G and insert a new auxiliary vertex w_e in the middle of each edge $e = \{u, v\}$. We declare all original vertices observable and open with probability 1, while each auxiliary vertex w_e is declared open independently with probability p_e. Two original vertices are connected in σ iff every w_e on the corresponding path is open, i.e. exactly when every edge e on that path is open in μ. Thus σ simulates μ in the sense of Definition 2.1.2. □

Similarly, it is natural to ask whether site percolation can be simulated by bond percolation. Fisher [F61] noted that the other direction cannot be true since the argument proving Theorem 2.1.4 is only invertible for line graphs. We make his argument precise in Theorem 2.2.4 proved in Section 2.2. However, the question becomes more interesting if we consider approximate simulations.

Definition 2.1.5 (Approximate simulation)**.** Let $\{\rho_i\}$ be a sequence of percolation measures on graphs $G_i = (V_i, E_i)$, each with observable set $V_{i,\text{obs}} \subseteq V_i$. Let ρ be a percolation measure on a (possibly different) graph $G = (V, E)$ with observable set $V_{\text{obs}} \subseteq V$.

We say that $\{\rho_i\}$ *approximately simulates* ρ if there exist vertex maps

$$f_i : V_{\text{obs}} \longrightarrow V_{i,\text{obs}} \qquad (i = 1, 2, \ldots)$$

such that for every Boolean combination $\mathcal{E}$ of connectivity events $\{ v_j \underset{\rho}{\leftrightarrow} v_k\}$ with $v_j, v_k \in V_{\text{obs}}$,

$$\rho_i\big(\mathcal{E} \circ f_i\big) \;\longrightarrow\; \rho\big(\mathcal{E}\big) \qquad \text{as } i \to \infty,$$

where $\mathcal{E} \circ f_i$ is obtained from $\mathcal{E}$ by replacing each v_j with $f_i(v_j)$ and each instance of $\underset{\rho}{\leftrightarrow}$ with $\underset{\rho_i}{\leftrightarrow}$.

We denote by $K_{1,n}$ the star graph with one internal vertex and n leaves. Throughout this section we work with the site percolation model on $K_{1,n}$ ($n \geq 3$) whose *observable* set is the collection of leaves. Every observable leaf is deterministically open ($p_v = 1$), while the unique internal (auxiliary) vertex is open with probability $p \in (0,1)$. Equivalently, this model can be viewed as a *full hyperedge percolation* on the same leaf set: opening the internal vertex corresponds to opening the single hyperedge {all leaves} with probability p; when that hyperedge is closed ($1 - p$), no two leaves are connected, whereas when it is open all leaves belong to one common cluster.

The main results of the paper are the following theorems.

Theorem 2.1.6. *Site percolation on $K_{1,4}$ cannot be approximately simulated by bond percolation.*

Theorem 2.1.6 is deduced from Theorem 2.3.2.

Theorem 2.1.7. *Site percolation on $K_{1,3}$ cannot be approximately simulated by bond percolation.*

Theorem 2.1.7 follows from Theorem 2.4.6. Its proof relies on new inequalities for connectivity events. We introduce auxiliary events defined by decision trees to establish (2.7), and use computer search to discover further inequalities, including (2.11) and (2.12) in Section 2.5.

2.2 Preliminary remarks

In what follows we mostly work with a hyperedge viewpoint on the site percolation on $K_{1,n}$ introduced above, since for the full hyperedge percolation one can conveniently assume $V_{\text{obs}} = V$. The next result justifies this flexibility.

Theorem 2.2.1 (Site–hyperedge equivalence)**.** *Every Bernoulli hyperedge percolation measure can be simulated by a Bernoulli site percolation measure, and vice versa, in the sense of Definition 2.1.2.*

Proof. **(i) Hyperedge simulated by site percolation.** Let $H = (V, E)$ be a hypergraph equipped with Bernoulli hyperedge percolation η. Construct a graph $G' = (V', E')$ and a site percolation measure σ as follows:

- *Vertices:* $V' := V \cup \{w_e : e \in E\}$, adding auxiliary vertex w_e for each hyperedge e.
- *Edges:* for every $e \in E$ and $v \in e$ include $\{v, w_e\}$ in E'.
- *Site probabilities:* $\sigma(v \text{ open}) = 1$ for $v \in V$ and $\sigma(w_e \text{ open}) = p_e$.

Take observable set $V_{\text{obs}} \subseteq V$ and set $V'_{\text{obs}} := V_{\text{obs}}$ together with the identity map $f : V_{\text{obs}} \to V'_{\text{obs}}$, a σ-open path $u - w_{e_1} - \cdots - w_{e_m} - v$ exists if and only if the hyperedges $e_1, \ldots, e_m$ are η-open. Hence σ simulates η.

(ii) Site simulated by hyperedge percolation. Let $G = (V, E)$ carry Bernoulli site percolation σ with $\sigma(v \text{ open}) = p_v$. Build a hypergraph $H' = (V', E')$ and percolation η' as follows:

$$V' := V \cup \{w_e : e \in E\}, \qquad E' := \{\, e_v : v \in V\},$$

$$e_v := \{v\} \cup \{w_e : v \in e\}, \qquad \eta'(e_v \text{ open}) = p_v.$$

Take observable set $V_{\text{obs}} \subseteq V$ and set $V'_{\text{obs}} := V_{\text{obs}}$; let $g : V_{\text{obs}} \to V'_{\text{obs}}$ be the identity inclusion.

If $u, v \in V_{\text{obs}}$ are σ-connected, there is a path $u = x_0, x_1, \ldots, x_k = v$ in G with all x_i open. Each step $x_i x_{i+1}$ uses the auxiliary vertex $w_{x_i x_{i+1}}$, so the open hyperedges e_{x_i} give an η'-path from $g(u)$ to $g(v)$.

Conversely, any η'-path between $g(u)$ and $g(v)$ alternates e_{x_i} and $w_{x_i x_{i+1}}$; hence $u = x_0, x_1, \ldots, x_k = v$ is a σ-open path in G. Thus connectivity events coincide, and η' simulates σ. □

Note that the hypergraph percolation in the sense of [WZ11] is more general than our full hypergraph percolation and is capable of modeling more phenomena.

Now we know that simulating site percolation is equivalent to simulating full hyperedge percolation. It is easy to see that bond percolation cannot simulate *exactly* even a hyperedge of size 3 with probability $0 < p < 1$, thus proving Fisher's remark.

Fix a Bernoulli bond percolation μ on a graph $G = (V, E)$ and write $\mu(\,\cdot\,)$ for probabilities taken with respect to this measure.

Definition 2.2.2. For pairwise disjoint, non-empty vertex sets $A_1, \ldots, A_k \subseteq V$ we write

$$A_1|A_2|\ldots|A_k$$

for the event that

- all vertices inside each A_i lie in the *same* open cluster, and
- the clusters corresponding to different A_i's are *distinct* (no vertex of A_i is connected to a vertex of A_j for $i \neq j$).

Remark 2.2.3. Throughout the paper we will use the following shortcuts:

$$\mu(abc) := \mu(\,\{a, b, c\}\,), \qquad \mu(ab|c) := \mu(\{a, b\}|\{c\}), \qquad \mu(a|b|c) := \mu(\{a\}|\{b\}|\{c\}), \;\ldots$$

Throughout, a vertical bar "|" indicates that the vertices on either side of the bar must be in different clusters in the configuration and vertices between consecutive bars are in the same connected cluster.

First, we show that the exact simulation for 3-hyperedges is impossible.

Theorem 2.2.4. *Let $G = (V, E)$ be any graph, let μ be a Bernoulli bond percolation on G, and let $a, b, c \in V$. Then either*

$$\mu(abc) + \mu(a|b|c) < 1 \qquad \text{or} \qquad \max\{\mu(abc),\, \mu(a|b|c)\} = 1.$$

Proof. Delete every edge whose parameter is 0 and contract every edge whose parameter is 1; all remaining edges now have parameters strictly between 0 and 1.

Assume for a contradiction that

$$0 < \mu(abc),\ \mu(a|b|c) < 1 \quad \text{and} \quad \mu(abc) + \mu(a|b|c) = 1. \tag{2.1}$$

Because $\mu(a|b|c) > 0$, each of the mixed-cluster events

$$ab|c, \quad ac|b, \quad a|bc$$

must have probability 0, otherwise the left-hand side of (2.1) would be strictly smaller than 1.

Let us show that $\mu(ab|c) = 0$ forces *every* a–b path in G to pass through c. Indeed, if a path γ joining a to b while avoiding c existed in G, the configuration in which the edges of γ are declared open and all other edges are declared closed would realise the event $ab|c$ with positive probability (all edge parameters lie in $(0, 1)$), contradicting $\mu(ab|c) = 0$. Repeating the same argument for the other two zero-probability events we deduce

- every a–b path visits c,
- every a–c path visits b, and
- every b–c path visits a.

These three properties cannot be satisfied simultaneously. Indeed, the first property forces any a–b path to begin with an a–c sub-path; by the second property that sub-path must pass through b before it reaches c. Hence *any* a–b path would have to contain vertex b twice, which is impossible for a (simple) path in a graph. Thus these properties are impossible, contradicting (2.1).

Consequently our assumption is false and the theorem follows. □

The theorem implies that bond percolation cannot simulate a non-trivial hyperedge percolation on the graph containing exactly three vertices and a single hyperedge connecting these. Indeed, in such a hyperedge percolation, the probability that all three vertices are connected equals p (the probability that the hyperedge is open), while the probability that they are all disconnected equals $1 - p$. Theorem 2.4 shows that no bond percolation can achieve both these probabilities simultaneously.

Although Theorem 2.2.4 prohibits exact simulation of hyperedge percolation by bond percolation, we consider whether it is possible to have an arbitrarily good approximation.

Question 2.2.5. *For given k, $p \notin \{0,1\}$ and $\varepsilon > 0$, does there exist a graph $G = (V, E)$, containing vertices $x_1, \ldots, x_k$ and a bond percolation μ on it with $\mu(x_1x_2\ldots x_k) > p-\varepsilon$ and $\mu(x_1|x_2|\ldots|x_k) > 1-p-\varepsilon$?*

In Section 2.3 we show that approximate simulation is impossible for $k \geq 4$ using a lemma due to Hutchcroft [H21], thus proving Theorem 2.1.6. Finally, we develop a new technique using decision trees to resolve Question 2.2.5 for $k = 3$ (thereby proving Theorem 2.1.7) in Section 2.4.

2.3 Simulating k-hyperedge for $k \geq 4$

In [H21], the following theorem is proved using the vdBK inequality [BK85], where K_u is the cluster containing vertex u, and for each finite subset $\Lambda \subseteq V$

$$|K_{max}(\Lambda)| = \max\{|K_v \cap \Lambda| : v \in V\}$$

is the maximal number of vertices from Λ belonging to the same cluster.

Theorem 2.3.1 ([H21, Theorem 2.3])**.** *Let $G = (V, E)$ be a countable graph and let $\Lambda \subseteq V$ be finite and non-empty. Then for any Bernoulli bond percolation μ on G one has*

$$\mu(|K_{max}(\Lambda)| \geq 3^k \lambda) \leq \mu(|K_{max}(\Lambda)| \geq \lambda)^{3^{k-1}+1} \tag{2.2}$$

and

$$\mu(|K_u \cap \Lambda| \geq 3^k \lambda) \leq \mu(|K_{max}(\Lambda)| \geq \lambda)^{3^{k-1}} \mu(|K_u \cap \Lambda| \geq \lambda) \tag{2.3}$$

for every $\lambda \geq 1$ (not necessarily integer), integer $k \geq 0$ and $u \in V$.

This allows us to prove that one cannot even approximately simulate the 4-hyperedge.

Theorem 2.3.2. *For any $\delta > 0$ there exists an $\varepsilon > 0$ such that, for every graph $G = (V, E)$, every Bernoulli bond percolation μ on G, and every choice of four vertices $a, b, c, d \in V$, at least one of the following holds:*

(i) $\mu(abcd) \geq 1 - \delta$;

(ii) $\mu(a|b|c|d) \geq 1 - \delta$;

(iii) $\mu(abcd) + \mu(a|b|c|d) < 1 - \varepsilon$.

Proof. Let $\delta > 0$ and set $\varepsilon = \delta^2/2$. Suppose for some graph G, bond percolation μ, and vertices a, b, c, d we have

$$\mu(abcd) \leq 1 - \delta \quad \text{and} \quad \mu(a|b|c|d) \leq 1 - \delta.$$

Write $t = \mu(a|b|c|d) \in [0, 1 - \delta]$. Then (2.2) with $\lambda = \frac{4}{3}$ and $\Lambda = \{a, b, c, d\}$ gives

$$\mu(abcd) \leq \mu(ab \cup ac \cup ad \cup bc \cup bd \cup cd)^2. \tag{2.4}$$

Since $ab \cup ac \cup ad \cup bc \cup bd \cup cd$ is the complement of $a|b|c|d$, its probability is $1 - t$, so

$$\mu(abcd) \leq (1 - t)^2.$$

Hence

$$\mu(abcd) + \mu(a|b|c|d) = \mu(abcd) + t \le \min\{\, 1 - \delta + t,\ t + (1-t)^2 \}.$$

If $t \le \delta/2$ then $1 - \delta + t \le 1 - \delta/2 \le 1 - \varepsilon$, while if $t \ge \delta/2$ then

$$t + (1-t)^2 = 1 - t(1-t) \le 1 - \tfrac{\delta}{2}\,\delta = 1 - \varepsilon.$$

In either case $\mu(abcd) + \mu(a|b|c|d) \le 1 - \varepsilon$, which is alternative (iii). □

Remark 2.3.3. Another proof of (2.4) can be obtained by applying the vdBK inequality directly. For two events A, B, their disjoint occurrence $A \square B$ is defined as the event consisting of configurations x whose memberships in A and in B can be verified on disjoint subsets of edges. In this case, vdBK inequality asserts that

$$\mu(A \,\square\, B) \le \mu(A)\mu(B).$$

Let E be the event $ab \cup ac \cup ad \cup bc \cup bd \cup cd$. Then $E \square E$ represents the event where there exist two edge-disjoint paths between the vertices a, b, c, and d. Hutchcroft's argument in this context reduces to demonstrating that $abcd \subseteq E \square E$, which is established in the following lemma.

Lemma 2.3.4. *Let G be a connected graph, and let a_1, a_2, a_3, a_4 be any four vertices in G. Then, it is always possible to partition these vertices into two disjoint pairs, $\{a_{i_1}, a_{i_2}\}$ and $\{a_{j_1}, a_{j_2}\}$, such that there exist edge-disjoint paths γ_1 and γ_2 in G, where γ_1 connects a_{i_1} to a_{i_2} and γ_2 connects a_{j_1} to a_{j_2}.*

Proof. Partition the vertices a_1, a_2, a_3, a_4 into two disjoint pairs, $\{a_{i_1}, a_{i_2}\}$ and $\{a_{j_1}, a_{j_2}\}$, such that the sum of edge distances, $d(a_{i_1}, a_{i_2}) + d(a_{j_1}, a_{j_2})$, is minimized across all possible partitions. Without loss of generality, assume that the vertices in the first pair are a_1 and a_2, and the vertices in the second pair are a_3 and a_4.

Let γ_1 and γ_2 be the shortest paths connecting a_1 to a_2 and a_3 to a_4, respectively. Suppose the edge uv belongs to both paths. Without loss of generality, we assume that the order of vertices in γ_1 is $a_1 \to u \to v \to a_2$; otherwise, we can swap the labels of a_1 and a_2. Similarly, we assume that the order of vertices in γ_2 is $a_3 \to u \to v \to a_4$.

Then, we calculate the following:

$$\begin{aligned} d(a_1, a_3) + d(a_2, a_4) &\le \big(d(a_1, u) + d(u, a_3)\big) + \big(d(a_2, v) + d(v, a_4)\big) \\ &= \big(d(a_1, u) + d(v, a_2)\big) + \big(d(a_3, u) + d(v, a_4)\big) \\ &= \big(d(a_1, a_2) - 1\big) + \big(d(a_3, a_4) - 1\big). \end{aligned}$$

This inequality contradicts the selection of the pairs, as the pairs $\{a_1, a_2\}$ and $\{a_3, a_4\}$ were chosen to minimize $d(a_1, a_2) + d(a_3, a_4)$. Therefore, the paths γ_1 and γ_2 must be edge-disjoint. □

Proof of Theorem 2.1.6. Fix an arbitrary $p \in (0, 1)$ and let σ_p be the site percolation on the star $K_{1,4}$ in which the centre is open with probability p while the four leaves a, b, c, d are always open. Then

$$\sigma_p(abcd) = p, \qquad \sigma_p(a|b|c|d) = 1 - p, \qquad \sigma_p(abcd) + \sigma_p(a|b|c|d) = 1.$$

Choose $\delta := \frac{1}{2} \min\{p, 1-p\} > 0$ and let $\varepsilon > 0$ be the constant provided by Theorem 2.3.2. Assume, for contradiction, that σ_p can be approximately simulated by some bond-percolation measure μ. Make the simulation so accurate that

$$|\mu(abcd) - p| < \delta, \qquad |\mu(a|b|c|d) - (1 - p)| < \delta, \qquad |\mu(abcd) + \mu(a|b|c|d) - 1| < \varepsilon/2.$$

Because each of $\mu(abcd)$ and $\mu(a|b|c|d)$ lies in $(\delta, 1 - \delta)$, conditions (1) and (2) of Theorem 2.3.2 fail. The third bound yields $\mu(abcd) + \mu(a|b|c|d) > 1 - \varepsilon$, so condition (3) fails as well. This contradiction shows that no bond percolation can approximately simulate σ_p. □

2.4 Simulating 3-hyperedge: human proof

Now we see that it is impossible to even approximately simulate site percolation on $K_{1,4}$ with bond percolation, as promised in Theorem 2.1.6. To prove Theorem 2.1.7, we need the following lemma.

Definition 2.4.1. For two configurations $C_1, C_2 \in \Omega = 2^{[E]}$ and a set $S \subseteq E$ we denote by $C_1 {\to_S} C_2$ the configuration which coincides with C_1 on S and C_2 on its complement $\overline{S}$.

Lemma 2.4.2. *Consider two independent Bernoulli bond percolations C_1 and C_2 having the same distribution μ on the same graph G. Let a decision tree T select each edge and reveal it in both C_1 and C_2. Furthermore, allow on each step, before revealing, to decide if this edge will go to the set S (thus dependent on C_1 and C_2) or to its complement $\overline{S}$. Then $C_1 {\to_S} C_2$ is independent of $C_1 {\to_{\bar{S}}} C_2$ and both of them are distributed as μ.*

Example 2.4.3. If the graph is a path of length 2 from a to b, then the tree T in Figure 2.1 builds a set S of all edges with one end in the component of a in C_1.

Proof of Lemma 2.4.2. For every pair of configurations C_3, C_4 and given decision tree T there exist unique C_1 and C_2 such that $C_1 {\to_{S(C_1,C_2)}} C_2 = C_3$ and $C_1 {\to_{\bar{S}(C_1,C_2)}} C_2 = C_4$. Indeed, the path in T leading to $C_1 {\to_{S(C_1,C_2)}} C_2 = C_3$ and $C_1 {\to_{\bar{S}(C_1,C_2)}} C_2 = C_4$ is determined uniquely at each step, and the probability of this path is equal to $\mu(C_3)\mu(C_4)$, which is equal to $\mu(C_1)\mu(C_2)$ since the probability in Bernoulli percolation is a product of probabilities for individual edges. $\square$

Example 2.4.4. One can take T querying all the edges from the vertices already known to connect to the vertex a in C_1. It will assign all these edges to S and then discover the remaining edges, assigning them to $\bar{S}$. Then S will be the set of all open and closed edges with at least one edge in the component of a.

Note that this set S depends only on C_1. Given this, the configuration $C_1 {\to_S} C_2$ can be interpreted as follows. We take the configuration C_1 and resample all the edges not connected to the cluster of a. Lemma 2.4.2 claims that the resulting configuration has a distribution μ. Moreover, if instead we resampled the edges connected to the cluster of a, it would also result in measure μ.

Remark 2.4.5. Notice that the Markov chain method from [BHK06] is based on the fact that resampling edges in $\overline{S}$ from Example 2.4.4 preserves the measure restriction $\mu|_{a|b}$. In our notation, it means that for $A = a|b$ and any B, one has

$$\mu(C_1 \in A \text{ and } C_1 \underset{S}{\to} C_2 \in B) = \mu(C_1 \underset{S}{\to} C_2 \in A \cap B) = \mu(A \cap B) \tag{2.5}$$

Theorem 2.4.6. *For any $\delta > 0$ there exists an $\varepsilon > 0$ such that, for every graph $G = (V, E)$, every Bernoulli bond percolation μ on G, and every choice of three vertices $a, b, c \in V$, at least one of the following holds:*

(i) $\mu(abc) \geq 1 - \delta$;

(ii) $\mu\big(a|b|c\big) \geq 1 - \delta$;

(iii) $\mu(abc) + \mu\big(a|b|c\big) < 1 - \varepsilon$.

Remark 2.4.7. It is worth noting that Theorem 2.4.6 directly implies Theorem 2.3.2.

Proof. We will need multiple sets S_i for our purpose. So, we define sets S_1, S_2 and S_3, which are somewhat complex (See Figure 2.2).

To build S_1, we query all edges connected to c and put them in S. Then we query all not queried edges connected to a (this is vacuous if a was connected to c) and put them in $\bar{S}$. Then we query all not queried edges connected to b and put them in S. Finally, we put the rest of the edges in $\bar{S}$. We denote by K_x the set of vertices connected to x via edges open in C_1. Then,

$$S_1 = \begin{cases} E \cap \big(K_c \times V \cup K_b \times \overline{K_a}\big) \text{ if } C_1 \in a|b|c; \\ E \cap \big(K_c \times V\big) \text{ if } C_1 \text{ is in } abc,\, a|bc \text{ or } ab|c; \\ E \cap \big((K_b \cup K_c) \times V\big) \text{ if } C_1 \in ac|b. \end{cases}$$

The only case we will actually use is $a|b|c$. S_2 is defined analogously with b and c interchanged.

$$S_2 = \begin{cases} E \cap \left(K_b \times V \cup K_c \times \overline{K_a}\right) \text{ if } C_1 \in a|b|c; \\ E \cap \left(K_b \times V\right) \text{ if } C_1 \text{ is in } abc,\ a|bc \text{ or } ac|b; \\ E \cap \left((K_b \cup K_c) \times V\right) \text{ if } C_1 \in ab|c. \end{cases}$$

Finally, for S_3 we put all edges connected to a in $\bar{S}$, all not queried edges connected to b or c to S and the rest of the edges to $\bar{S}$.

$$S_3 = \begin{cases} E \cap \left((K_b \cup K_c) \times \overline{K_a}\right) \text{ if } C_1 \in a|b|c \text{ or } a|bc; \\ \varnothing \text{ if } C_1 \in abc; \\ \text{Something else otherwise.} \end{cases}$$

The key observation is the following:

Proposition 2.4.8. *For any two configurations C_1 and C_2 such that*

$$C_1 \in a|b|c \text{ and } C_1 \underset{S_3}{\rightarrow} C_2 \in ab \cup ac,$$

one has either $C_1 \rightarrow_{S_1} C_2 \in ab$ or $C_1 \rightarrow_{S_2} C_2 \in ac$.

Proof. Consider a path $\mathcal{P}$ from a to b or c in $C_1 \rightarrow_{S_3} C_2$. Along this path, consider the first edge uv incident to a vertex v in $K_b \cup K_c$. Denote by e the set of all the vertices of G which do not belong to K_a, K_b or K_c. The segment of $\mathcal{P}$ before uv is contained in the complement of $K_b \cup K_c$, which is $K_a \cup e$, and thus it is contained in all of the sets $\bar{S_1}$, $\bar{S_2}$, and $\bar{S_3}$ (see Figure 2.2: the sets K_a and e are shown in red, along with all the edges between them, in all the sets S_i).

The vertex u belongs to $K_a \cup e$. We now show that u must be in K_a. Indeed, suppose it is not; then the edge uv connects e with $K_b \cup K_c$. In $C_1 \rightarrow_{S_3} C_2$, this edge comes from C_1 (on Figure 2.2, all such edges are blue). However, in C_1 the edges between e and $K_b \cup K_c$ are closed, which would imply uv is closed in $C_1 \rightarrow_{S_3} C_2$, a contradiction since uv lies on the open path $\mathcal{P}$.

Therefore, $u \in K_a$ and $v \in K_b \cup K_c$. Depending on whether $v \in K_b$ or $v \in K_c$, the edge uv belongs to $\bar{S}_1$ or $\bar{S}_2$. Since all internal edges in K_b and K_c belong to S_1 and S_2, it follows that $C_1 \rightarrow_{S_1} C_2 \in ab$ or $C_1 \rightarrow_{S_2} C_2 \in ac$. □

The consequence of the Proposition 2.4.8 is the following inequality:

$$\mu\Big(C_1 \in a|b|c \text{ and } C_1 \underset{S_3}{\rightarrow} C_2 \in (ab \cup ac)\Big)$$
$$\leq \mu(C_1 \in a|b|c \text{ and } C_1 \underset{S_1}{\rightarrow} C_2 \in ab) + \mu(C_1 \in a|b|c \text{ and } C_1 \underset{S_2}{\rightarrow} C_2 \in ac). \quad (2.6)$$

Let's proceed to estimate the probabilities of these events. For $C_1 \in a|b|c$, we have $C_1 \rightarrow_{S_1} C_2 \in a|c$, so

$$\mu(C_1 \in a|b|c \text{ and } C_1 \underset{S_1}{\rightarrow} C_2 \in ab) \leq \mu(C_1 \underset{S_1}{\rightarrow} C_2 \in ab|c) = \mu(ab|c).$$

Similarly,

$$\mu(C_1 \in a|b|c \text{ and } C_1 \underset{S_2}{\rightarrow} C_2 \in ac) \leq \mu(ac|b).$$

Finally, we estimate $\mu\left(C_1 \in a|b|c \text{ and } C_1 \rightarrow_{S_3} C_2 \in (ab \cup ac)\right)$. If C_1 belongs to $a|b \cap a|c$, then $\bar{S}_3$ contains a cut from a to b and c, so $C_1 \rightarrow_{\bar{S}_3} C_2$ also belongs to $a|b \cap a|c$.

$$\mu\Big(C_1 \in a|b|c \text{ and } C_1 \underset{S_3}{\rightarrow} C_2 \in (ab \cup ac)\Big)$$
$$\geq \mu\Big(C_1 \in (a|b \cap a|c) \text{ and } C_1 \underset{S_3}{\rightarrow} C_2 \in (ab \cup ac)\Big) - \mu(a|bc)$$
$$= \mu\Big(C_1 \underset{\bar{S}_3}{\rightarrow} C_2 \in (a|b \cap a|c) \text{ and } C_1 \underset{S_3}{\rightarrow} C_2 \in (ab \cup ac)\Big) - \mu(a|bc)$$
$$= \mu(a|b \cap a|c)\mu(ab \cup ac) - \mu(a|bc).$$

Substituting our bounds into (2.6), we conclude

$$\mu(a|b \cap a|c)\mu(ab \cup ac) \leq \mu(ab|c) + \mu(ac|b) + \mu(a|bc). \quad (2.7)$$

To conclude the proof of Theorem 2.4.6, assume that alternatives (i) and (ii) both fail, and verify that alternative (iii) holds. Let $\delta > 0$ and put

$$\varepsilon \;=\; \frac{\delta^2}{4}.$$

Suppose for some graph G, Bernoulli percolation μ, and vertices a, b, c we have

$$\mu(abc) \leq 1 - \delta \quad \text{and} \quad \mu(a|b|c) \leq 1 - \delta.$$

Since trivially $\mu(ab \cup ac) \geq \mu(abc)$ and $\mu(a|b \cap a|c) \geq \mu(a|b|c)$, the displayed inequality gives

$$\mu(abc)\,\mu(a|b|c) \;\leq\; \mu(ab|c) + \mu(ac|b) + \mu(a|bc) \;=\; 1 - \mu(abc) - \mu(a|b|c).$$

Hence

$$\mu(abc) + \mu(a|b|c) \;\leq\; 1 - \mu(abc)\,\mu(a|b|c).$$

If $\mu(abc) \leq \delta/2$ (or similarly $\mu(a|b|c) \leq \delta/2$), then

$$\mu(abc) + \mu(a|b|c) \leq \frac{\delta}{2} + (1 - \delta) = 1 - \frac{\delta}{2} < 1 - \frac{\delta^2}{4} = 1 - \varepsilon.$$

Otherwise $\mu(abc), \mu(a|b|c) \geq \delta/2$ and so

$$\mu(abc) + \mu(a|b|c) \leq 1 - \frac{\delta}{2} \cdot \frac{\delta}{2} = 1 - \frac{\delta^2}{4} = 1 - \varepsilon.$$

In either case $\mu(abc) + \mu(a|b|c) \leq 1 - \varepsilon$, which is precisely alternative (iii). This completes the proof. □

From the equation (2.7), one can conclude that if $\mu(abc)$ and $\mu(a|b|c)$ are simultaneously greater than p, then $p(1-p) \leq 1 - 2p$ and so $p \leq \frac{3-\sqrt{5}}{2} \approx 0.382$. If we denote the maximal possible value of $\min\big(\mu(abc), \mu(a|b|c)\big)$ for any bond percolation by α_3, we get an estimate $\alpha_3 < 0.382$, which we improve in the next section. The lower bound $\alpha_3 > 0.29065$ is given in Appendix 2.7.

2.5 Simulating 3-hyperedge: computer-assisted proof

Consider a bond percolation on a graph G with specified vertices a, b, and c. We examine a set of decision trees T_k for $k = 1, \ldots, n$. These trees generate the sets S_k and $\overline{S}_k$. Each tree maps every pair of configurations C_1 and C_2 on G into a product space J^2, with $J = \{a|b|c, a|bc, ac|b, ab|c, abc\}$. The first coordinate maps the partition of vertices a, b, c in the graph $C_1 {\rightarrow_{S_k}} C_2$ into connected clusters, and the second coordinate corresponds to the partition of vertices in the graph $C_1 {\rightarrow_{\overline{S}_k}} C_2$.[1]

The bond percolation on G induces a joint probability distribution ρ on $(J^2)^n$. However, it is important to note that ρ may be restricted in several ways. Not every combination of partitions $(p_k, \bar{p}_k)$ for $k = 1, \ldots, n$ corresponds to a pair of actual configurations C_1 and C_2 that satisfy conditions $C_1 {\rightarrow_{S_k}} C_2 = p_k$ and $C_1 {\rightarrow_{\overline{S}_k}} C_2 = \bar{p}_k$. Denote by $F \subseteq (J^2)^n$ the set of all feasible combinations of these partitions. Thus, the support of ρ is contained within F.

Let μ be a probability distribution on J induced by bond percolation on G. Lemma 2.4.2 implies that all the marginal projections of ρ onto each $(J^2)_k$ are identical and equal to $\mu \times \mu$. Given these restrictions, we can formulate a necessary condition for the implementability of the distribution μ on J, expressed through the feasibility of a linear programming problem.

Proposition 2.5.1. *For any feasible distribution μ on J, there exists a distribution $\rho(p) \geq 0$ defined on the set $p \in F$, satisfying the following condition:*

$$\sum_{\substack{p \in F: \\ p_k = q, \overline{p}_k = \overline{q}}} \rho(p) = \mu(q) \cdot \mu(\bar{q}) \quad \textit{for all } k = 1, \ldots, n, \textit{ and for each pair } (q, \overline{q}) \in J^2. \tag{2.8}$$

The above proposition does not offer a necessary condition that can be expressed solely in terms of μ without the introduction of additional variables. To achieve a formulation that depends only on μ, we consider the dual linear programming problem to the one considered in Proposition 2.5.1.

[1] Take into account that there is a trivial tree T that generates the set of all the edges $S = E$; for this tree, $C_1 {\rightarrow_S} C_2 = C_1$ and $C_1 {\rightarrow_{\overline{S}}} C_2 = C_2$.

Definition 2.5.2. *Feasible potentials* are the collection of functions $\varphi_k \colon J^2 \to \mathbb{R}$ satisfying the inequalities

$$\sum_{k=1}^{n} \varphi_k(p_k, \overline{p}_k) \geq 0 \quad \text{for all feasible } p \in F. \tag{2.9}$$

Each function φ_k can be interpreted as a variable dual to the marginal projection constraint (2.8).

Next, we utilize the principle that the feasibility of the primal linear program is equivalent to the boundedness of the dual linear program.

Theorem 2.5.3. *Let φ_k be feasible potentials. Then, any feasible distribution μ on J satisfies the inequality*

$$\sum_{k=1}^{n} \sum_{(p,\overline{p}) \in J^2} \varphi_k(p, \overline{p}) \mu(p) \mu(\overline{p}) \geq 0. \tag{2.10}$$

Proof. Let μ be a feasible distribution on J. By Proposition 2.5.1, we can find a joint law ρ supported on the set F of feasible combinations of partitions with all marginal projections equal to $\mu \times \mu$. The latter condition implies that

$$\sum_{k=1}^{n} \sum_{(p,\overline{p}) \in J^2} \varphi_k(p, \overline{p}) \mu(p) \mu(\overline{p}) = \sum_{k=1}^{n} \sum_{p \in F} \varphi_k(p_k, \overline{p}_k) \rho(p).$$

By the definition of feasible potentials, the inequality

$$\sum_{k=1}^{n} \varphi_k(p_k, \overline{p}_k) \geq 0$$

holds for all $p \in F$; thus, the right-hand side of the equation above is non-negative. □

This theorem allows to prove the inequality

$$\mu(a|b \cap a|c)\mu(ab \cup ac) \leq \mu(ab|c) + \mu(ac|b) + \mu(a|bc) - \mu(ab|c)^2 - \mu(ac|b)^2, \tag{2.11}$$

which is obviously better than inequality (2.7) and leads to an estimate $\alpha_3 \leq 0.369$. Moreover, surprisingly, this theorem also proves the inequality

$$\mu(abc)\mu(a|b|c) \geq \mu(ab|c)\mu(ac|b) + \mu(ab|c)\mu(a|bc) + \mu(ac|b)\mu(a|bc), \tag{2.12}$$

which was first conjectured in an unpublished work by Erik Aas and proved in [G24a]. It is stronger than what the Harris–Kleitman inequality can tell about these events.

To prove the inequality (2.11), we formulate it in terms of the feasible distribution μ.

Proposition 2.5.4. *Any feasible distribution μ on J satisfies the inequality*

$$\mu(ab \cup ac) \cdot \mu(a|b \cap a|c) + \mu(ac|b)^2 + \mu(ab|c)^2 \leq \mu(a|bc) + \mu(ac|b) + \mu(ab|c).$$

Proof. Consider the decision trees constructing the sets S_1, S_2, S_3, and their complements as introduced in the proof of Theorem 2.4.6. In addition, we include a trivial decision tree that constructs the sets $S_0 = E$ and $\overline{S}_0 = \emptyset$, so that $C_1 \!\rightarrow_{S_0} C_2 = C_1$ and $C_1 \!\rightarrow_{\overline{S}_0} C_2 = C_2$. Define functions $\varphi_k \colon J^2 \to \mathbb{R}$, for $k = 0, \ldots, 3$, as follows:

$$\begin{aligned}
&\varphi_0(p, \overline{p}) = \mathbf{1}[p = a|bc], \\
&\varphi_1(p, \overline{p}) = \mathbf{1}[p = ab|c], \\
&\varphi_2(p, \overline{p}) = \mathbf{1}[p = ac|b], \\
&\varphi_3(p, \overline{p}) = -\mathbf{1}[p \in ab \cup ac \text{ and } \overline{p} \in a|b \cap a|c] - \mathbf{1}[p = \overline{p} = ac|b] - \mathbf{1}[p = \overline{p} = ab|c].
\end{aligned}$$

For each vertex $u \in G$, let K_u be the cluster in C_1 containing u. Denote by E_u the set of open edges within K_u in C_1. Additionally, denote by $\tilde{E}_u$ the set of all edges in G for which at least one endpoint is in K_u. By construction, the states of all the edges in $\tilde{E}_c$ are identical in the configurations C_1 and $C_1 \!\rightarrow_{S_1} C_2$; therefore, any event of the form "there is an open path between u and c", where u is any vertex of G, occurs simultaneously in the configurations C_1 and $C_1 \!\rightarrow_{S_1} C_2$. Similarly,

- C_1 and $C_1 \!\rightarrow_{S_2} C_2$ coincide on $\tilde{E}_b$; thus, whether an open path between u and b exists is consistent across both C_1 and $C_1 \!\rightarrow_{S_2} C_2$ for any vertex u.
- C_1 and $C_1 \!\rightarrow_{\overline{S}_3} C_2$ coincide on $\tilde{E}_a$; therefore, the existence of an open path between u and a is the same in both C_1 and $C_1 \!\rightarrow_{\overline{S}_3} C_2$ for any vertex u.

Finally, we recall an important observation from the proof of Theorem 2.4.6: if $C_1 \in a|b|c$ and $C_1 {\rightarrow_{S_3}} C_2 \in ab \cup ac$, then either $C_1 {\rightarrow_{S_1}} C_2 \in ab$ or $C_1 {\rightarrow_{S_2}} C_2 \in ac$.

Now, we are ready to prove that the functions φ_k are feasible potentials: the inequality

$$\sum_{k=0}^{3} \varphi_k(p_k, \overline{p}_k) \geq 0$$

holds for any feasible combination of partitions $(p_k, \overline{p}_k)$. Since only φ_3 can take negative values, it is sufficient to consider the following cases where φ_3 contributes negatively:

- $p_3 \in ab \cup ac$ and $\overline{p}_3 \in a|b \cap a|c$,
- $p_3 = \overline{p}_3 = ab|c$,
- $p_3 = \overline{p}_3 = ac|b$.

In each of these cases, it is sufficient to demonstrate that either $p_0 = a|bc$, or $p_1 = ab|c$, or $p_2 = ac|b$.

Consider the first case: $p_3 \in ab \cup ac$ and $\overline{p}_3 \in a|b \cap a|c$. Since C_1 and $C_1 {\rightarrow_{\overline{S}_3}} C_2$ coincide on $\tilde{E}_a$, and because the partition of $C_1 {\rightarrow_{\overline{S}_3}} C_2$ is $\overline{p}_3 \in a|b \cap a|c$, we conclude that $C_1 \in a|b \cap a|c$. This implies that either $C_1 \in a|bc$ or $C_1 \in a|b|c$.

In the case where $C_1 \in a|bc$, we have $p_0 = a|bc$. Alternatively, if $C_1 \in a|b|c$ and $C_1 {\rightarrow_{S_3}} C_2 \in ab \cup ac$, then by Proposition 2.4.8 from the proof of Theorem 2.4.6, either $C_1 {\rightarrow_{S_1}} C_2 \in ab$ or $C_1 {\rightarrow_{S_2}} C_2 \in ac$. This condition translates to either $p_1 \in ab$ or $p_2 \in ac$. Furthermore, since C_1 and $C_1 {\rightarrow_{S_1}} C_2$ coincide on $\tilde{E}_c$ and $C_1 \in c|a \cap c|b$, it follows that $C_1 {\rightarrow_{S_1}} C_2 \in c|a \cap c|b$ and hence $p_1 \in c|a \cap c|b$. Similarly, $p_2 = b|a \cap b|c$. Thus, $C_1 \in a|b|c$ and $C_1 {\rightarrow_{S_3}} C_2 \in ab \cup ac$ implies that either $p_1 = ab|c$ or $p_2 = ac|b$, covering all scenarios that ensure that the sum $\varphi_k(p_k, \overline{p}_k)$ is non-negative. This completes the analysis for the first case.

Consider the second case: $p_3 = \overline{p}_3 = ab|c$. Repeating the argument that the configurations C_1 and $C_1 {\rightarrow_{\overline{S}_3}} C_2$ coincide on $\tilde{E}_a$, we find that there is an open path between a and b

in C_1, and a and c are in different clusters. Thus, $C_1 \in ab|c$, implying that $p_0 = ab|c$ as well.

Next, we observe that if $C_1 \in ab|c$, then the set S_1 coincides with $\tilde{E}_c$. This specifically implies that $C_1 \!\to_{S_1}\! C_2 \in c|a \cap c|b$, and we need only to verify that $C_1 \!\to_{S_1}\! C_2 \in ab$.

Consider an open path γ between a and b in the configuration $C_1 \!\to_{S_3}\! C_2$. Note that no vertex from this path can belong to K_c; otherwise, since the component K_c is connected in $C_1 \!\to_{S_3}\! C_2$, it would imply that $C_1 \!\to_{S_3}\! C_2 \in abc$, which contradicts our assumption. Therefore, all the edges of γ must belong to $E \setminus \tilde{E}_c$. Given that $C_1 \in ab|c$, the set $\overline{S}_3$ contains all edges from $E \setminus \tilde{E}_c$; thus, the edge states of γ are derived from the configuration C_2, ensuring that γ is an open path in C_2. Additionally, since $\overline{S}_1 = E \setminus \tilde{E}_c$, all edge states of γ in $C_1 \!\to_{S_1}\! C_2$ are also from C_2, confirming that γ is an open path in $C_1 \!\to_{S_1}\! C_2$, which implies $C_1 \!\to_{S_1}\! C_2 \in ab$.

Therefore, $C_1 \!\to_{S_1}\! C_2 \in c|a \cap c|b \cap ab = ab|c$, translating to $p_1 = ab|c$. The third case, where $p_3 = \overline{p}_3 = ac|b$, is fully symmetric, leading to $p_2 = ac|b$.

Altogether, this demonstrates that the functions φ_k are feasible potentials. According to Theorem 2.5.3, any feasible distribution μ on J must satisfy the inequality:

$$\sum_{k=0}^{3} \sum_{(p,\overline{p}) \in J^2} \varphi_k(p,\overline{p})\mu(p)\mu(\overline{p}) = \mu(a|bc) + \mu(ac|b) + \mu(ab|c)$$
$$- \mu(ab \cup ac) \cdot \mu(a|b \cap a|c) - \mu(ac|b)^2 - \mu(ab|c)^2 \geq 0.$$

□

Remark 2.5.5. Notice that feasible potentials form a convex cone. An interesting computational task is to enumerate all its extreme rays. We have performed this numerically for the decision trees constructing the sets S_k for $k = 0, \ldots, 3$, and their complements, finding exactly three non-trivial rays that form this cone. The first two are responsible for the inequalities (2.11) and (2.12). The final one leads to the inequality:

$$\mu(ab \cup ac) \cdot \mu(a|b \cap a|c) + \mu(a|bc)^2 + \mu(ac|b)^2 + \mu(ab|c)^2$$
$$\leq \mu(a|bc) + \mu(ac|b) + \mu(ab|c) + \mu(abc) \cdot \mu(a|bc),$$

which is very similar to (2.11). The potentials leading to these inequalities can be found on GitHub.[2]

2.6 Further questions

Inequalitites (2.7) and (2.11) prove that if all three probabilities $\mu(ab|c)$, $\mu(ac|b)$ and $\mu(a|bc)$ are 0, then one of $\mu(abc)$ and $\mu(a|b|c)$ should be 0. In fact, the stronger statement holds:

Proposition 2.6.1. *If $\mu(ab|c) = 0$, then*

$$\mu(a|b|c)\mu(abc) = \mu(ac|b)\mu(a|bc) \text{ and } \mu(abc) = \mu(ac)\mu(bc).$$

Proof. As in the proof of Theorem 2.2.4, we first delete the edges having probability 0 and contract the edges having probability 1. Now all paths from a to b should pass through c, otherwise, there will be a nonzero probability of one such path being open and the rest of the edges closed. This means c splits the graph in two parts with a and b belonging to different parts. Thus, the events ac and bc are determined by different sets of edges and consequently are independent.

□

However, contrary to the inequalitites (2.7) and (2.11), this proof tells nothing when $\mu(ab|c) < \varepsilon$. So, we pose two conjectures increasing in strength:

Conjecture 2.6.2. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mu(ab|c) < \delta$ and $\mu(ac|b) < \delta$, then $\mu(abc)$ or $\mu(a|b|c)$ is less than ε.*

Conjecture 2.6.3. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mu(ab|c) < \delta$, then*

$$\mu(abc) - \mu(ac)\mu(bc) < \varepsilon.$$

[2] https://github.com/Kroneckera/bunkbed

Remark 2.6.4. On the contrary, if $\mu(abc) - \mu(ac)\mu(bc) < \varepsilon$, then, by inequality (2.12), one gets

$$\begin{aligned}\mu(ab|c)\Big(\mu(abc) + \mu(ac|b) + \mu(a|bc)\Big) \\ \leq \mu(abc)\Big(\mu(abc) + \mu(ab|c) + \mu(ac|b) + \mu(a|bc) + \mu(a|b|c)\Big) \\ - \Big(\mu(abc) + \mu(ac|b)\Big)\Big(\mu(abc) + \mu(ab|c)\Big)\end{aligned}$$

which simplifies to

$$\mu(ab|c) < \frac{\varepsilon}{\mu(ac \text{ or } bc)}.$$

And for the final question, finding the exact value for α_3 would also be interesting. The best boundaries are given in the Appendix 2.7.

2.7 Appendix: optimizing α_3

Let us recall that α_3 denotes the largest possible value of $\min\big(\mu(abc), \mu(a|b|c)\big)$ for the bond percolation. Let us restrict ourselves to the triangle graph with all three probabilities equal to p. Then $\mu(a|b|c) = (1-p)^3$ and $\mu(abc) = p^3 + 3p^2(1-p)$. These numbers coincide for $p \approx 0.3473$, and we get $\alpha_3 \geq \mu(abc) = \mu(a|b|c) \approx 0.278$, a root of the equation $x^3 - 24x^2 + 3x + 1 = 0$.

One can do better by utilizing the graph in Figure 2.3 where each red-blue edge has a probability of 0.32537 and both blue-blue edges have a probability of 0.19231. This way we get $\mu(abc) \approx \mu(a|b|c) \approx 0.29065$.

Our computer search using algorithms from Wagner [W21] wasn't able to beat this estimate (See the best $\min\big(\mu(abc), \mu(a|b|c)\big)$ achieved on each training epoch in Figure 2.4).

In fact, if $\mu(abc) = \mu(a|b|c)$, it seems this probability can only lie in a narrow range from 0.27 to 0.291. Indeed, in this case inequality (2.12) gives the lower bound of $2 - \sqrt{3} \approx 0.2679$.

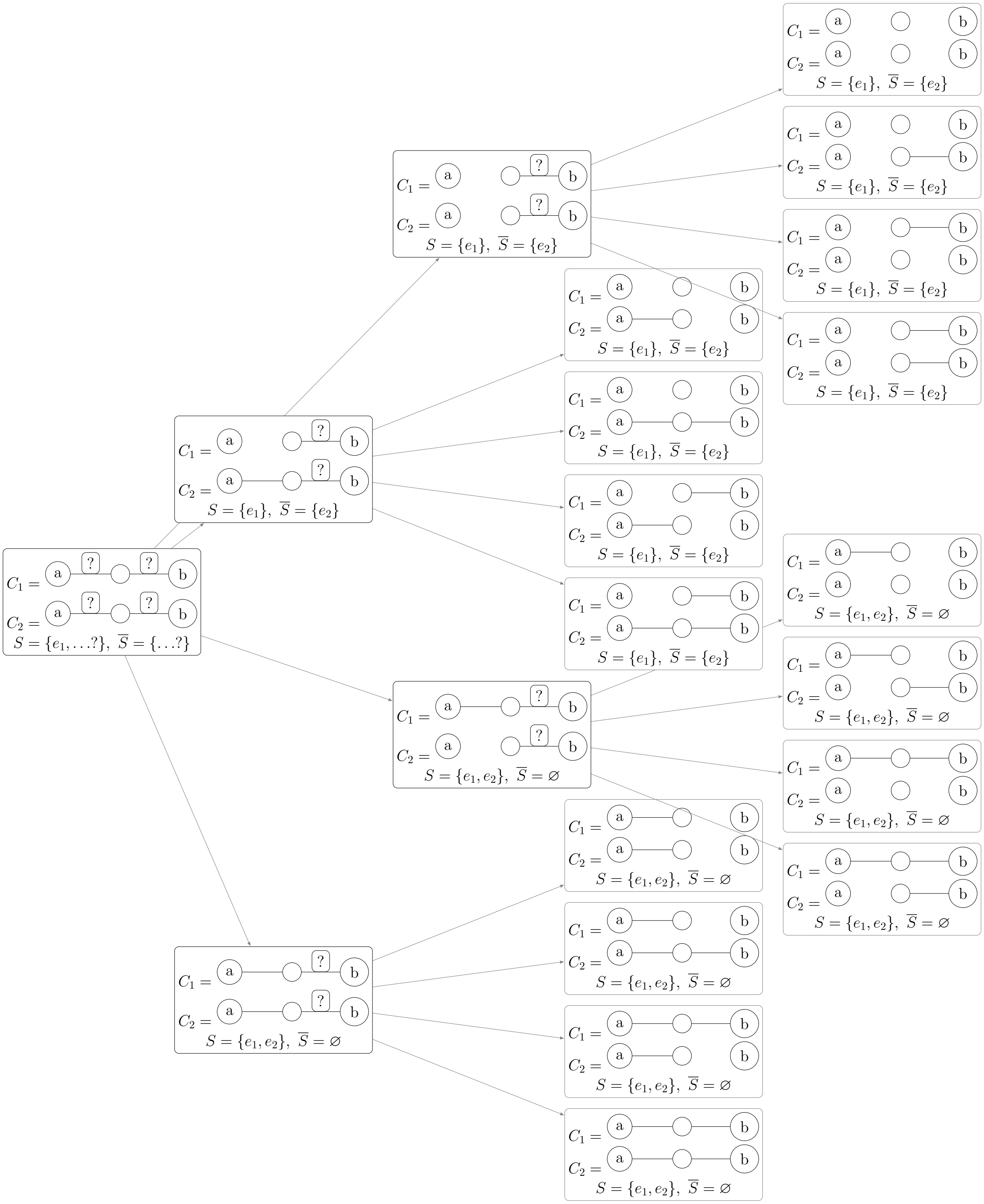

Figure 2.1: T corresponding to the Example 2.4.3

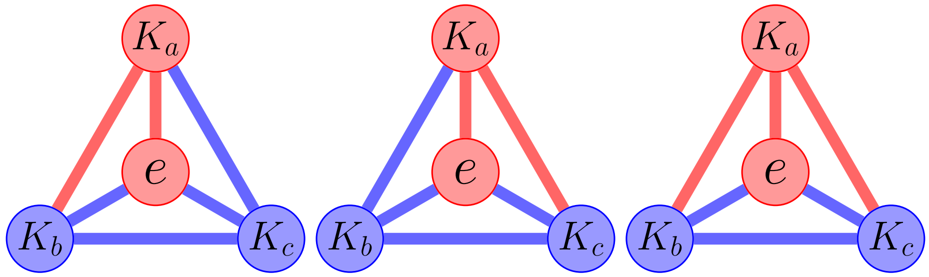

Figure 2.2: S_1, S_2 and S_3 for the case $C_1 \in a|b|c$. Regions surrounding a, b, c depict K_a, K_b and K_c. Respective sets are in blue and their complements are in red.

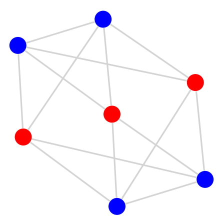

Figure 2.3: Graph for α_3.

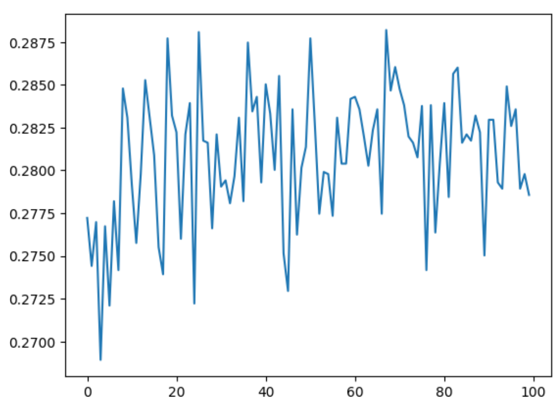

Figure 2.4: Best min $\big(\mu(abc), \mu(a|b|c)\big)$ achieved on each training epoch.

CHAPTER 3

Percolation inequalities and decision trees

3.1 Introduction

In percolation theory, key tools include the Harris–Kleitman (HK) and van den Berg–Kesten (vdBK) inequalities. These tools give lower and upper bounds on various connection probabilities for Bernoulli bond and site percolation on finite and infinite graphs.

Most percolation results hold for specific graphs such as lattices or Cayley graphs. HK and vdBK inequalities are rare exceptions that apply to general graphs. Other inequalities include those proved by van den Berg, Kahn, and Häggström in [BK01, BHK06] and the author in [G24a] and their corollaries. The recent work by Kozma and Nitzan [KN24] proposes a conjectured inequality for percolation on general graphs that would imply $\theta(p_c) = 0$ for bond percolation on $\mathbb{Z}^d$, which is an old conjecture. They prove a plethora of corollaries of the inequalities above, aimed to prove their conjecture. The celebrated bunkbed conjecture can also be seen as an inequality for connection probabilities in a general graph.

The OSSS inequality is an inequality originating from the analysis of Boolean functions [OSSS05]. It was first applied to percolation models in [DRT17] and was the key component in the proofs of several results about critical exponents [H20, DRT19]. This allowed discussions about an "OSSS method" [K20]. The method uses the concept of a (random) decision tree, that reveals the edges in an order dependent on the already revealed edges.

In [GZ24], Aleksandr Zimin and the author have built several decision trees querying the edges in different order. We used them to build multiple percolation configurations.

Their independence properties turn out to be enough to prove several new inequalities for connection probabilities for bond percolation in general graphs, including the proof that it is impossible for three vertices a, b, c to be in the same cluster with probability $0 < p < 1$ and in three different clusters with probability $1 - p - \varepsilon$ for small enough ε. In this paper, we explore the dependencies between the percolation configurations obtained by the same tree and prove the decision tree generalizations of the HK and vdBK inequalities, as well as the inequality from [GP24a] and the correct form of the inequality from [R04]. This allows us to prove new inequalities for connection probabilities in graph percolation.

The structure of the paper is as follows. Section 3.2 introduces notation and the definitions for our method. Section 3.3 illustrates the method and proves the version of the HK inequality for decision trees. Section 3.4, analogously, proves the version of the vdBK inequality. Section 3.5 utilizes the Cauchy–Schwarz inequality and finishes the groundwork for proving the inequalities we are interested in.

Put together, these results allow us to show in Section 3.6 the following inequality (see the full version in Theorem 3.6.2). In what follows, let $G = (V, E)$ be a locally finite connected simple graph and $\mathbf{P}$ is the probability in a Bernoulli bond percolation model where each edge $e \in E$ is assigned a probability p_e of being open.

Theorem 3.1.1 (see Theorem 3.6.2)**.** *Let a, b, c be distinct vertices of graph G. Then*

$$\mathbf{P}(abc)^2 \leq 8\mathbf{P}(ab)\mathbf{P}(ac)\mathbf{P}(bc), \tag{3.1}$$

where $\mathbf{P}(abc)$ is the probability that a, b and c are in the same percolation cluster.

The proof of this inequality combines together the decision tree versions of HK and vdBK inequalities as well as ideas from Section 3.5. This inequality can be seen as the $\sqrt{8}$ bound on the Delfino–Viti constant for every graph [DV11]. Moreover, when the graph G is planar and a, b and c belong to the same face, we bring the constant 8 in (3.1) down to 2.

Next, in Section 3.7, we prove the following technical asymmetric inequality on connection probabilities.

Lemma 3.1.2. *For vertices a, b and c in G one has*

$$\mathbf{P}(a|b|c) + \mathbf{P}(a|b \cup a|c)^2 \geq \frac{\mathbf{P}(a|b|c)^2}{\mathbf{P}(a|b \cup b|c)} + \frac{\mathbf{P}(a|b|c)^2}{\mathbf{P}(a|c \cup b|c)}, \tag{3.2}$$

where $\mathbf{P}(a|b|c)$ is the probability that a, b and c are in three different clusters and $v|u$ denotes the event that vertices u and v are in two different clusters.

This lemma allows us to resolve the following conjecture:

Theorem 3.1.3 (formerly [GZ24, Conj. 6.2])**.** *For $\varepsilon > 0$, there exists $\delta > 0$, such that*

$$\Big[\mathbf{P}(ab|c) < \delta \text{ and } \mathbf{P}(ac|b) < \delta\Big] \implies \Big[\mathbf{P}(abc) < \varepsilon \text{ or } \mathbf{P}(a|b|c) < \varepsilon\Big],$$

where $\mathbf{P}(a|b|c)$ is the probability that a, b and c are in three different clusters, $\mathbf{P}(abc)$ is the probability that a, b and c are in the same cluster, $\mathbf{P}(ab|c)$ is the probability that a and b are in the same cluster different from the cluster of c and $\mathbf{P}(ac|b)$ is the probability that a and c are in the same cluster different from the cluster of b.

In fact, we believe the stronger Conjecture 3.10.1 ([GZ24, Conj. 6.3]). It describes the relation between four other connection events dependent on vertices a, b and c when $\mathbf{P}(ab|c) < \delta$. Substituting $\mathbf{P}(ac|b) < \delta$ into it recovers the Theorem 3.1.3.

In Section 3.8, we use decision trees to prove the main technical result (Main Lemma 3.8.1), that generalizes the proofs of the decision tree versions of the HK and vdBK inequalities. This general form makes it easier to prove various generalizations of the vdBK inequality. Additional implications include the positive mutual dependence for colored percolation (Theorem 3.8.6), proved in [GP24a] as well as an inequality from [R04].

In Section 3.9 we turn to inequalities concerning connection probabilities for just two points. We study the events of the form $ab^{\square n}$, which stands for the existence of n disjoint open paths between a and b. It is easy to see from the vdBK inequality, that for every n and m we have

$$\mathbf{P}(ab^{\square n+m}) \leq \mathbf{P}(ab^{\square n})\mathbf{P}(ab^{\square m}).$$

In other words, $f(n) = \mathbf{P}(ab^{\square n})$ is submultiplicative.

Conjecture 3.1.4. *The function $f(n)$ is log-concave. Moreover, $\log(f(n))/n$ is decreasing.*

We provide a partial result in the direction of Conjecture 3.1.4.

Theorem 3.1.5 (cf. Theorem 3.9.1)**.** *Let G be planar. Suppose a and b belong to the same face. Then*

$$\mathbf{P}(ab^{\square 3})^2 \leq \mathbf{P}(ab^{\square 2})^3. \tag{3.3}$$

We also believe a stronger statement, see Conjecture 3.10.2.

3.2 Definitions and notation

Throughout this paper, $G = (V, E)$ is a locally finite connected simple graph. We also assume that $a, b, c, d \in V$ are distinct vertices of G. A percolation configuration $C = (C(e) : e \in E)$ on G is a function from E to $\{0, 1\}$. If $\omega_e = 1$, the edge e is said to be open, otherwise e is said to be closed. We deal with a bond percolation measure μ on the probability space $\Omega = \{0, 1\}^E$ of all percolation configurations. We assume that μ is a product measure, where each edge e has its own probability p_e of being open. This model is called the Bernoulli bond percolation. We have $\mathbf{P}$ refer to the probability of an event with respect to μ. We also use the following notation from [GZ24].

Definition 3.2.1. We denote by "$v_{11}v_{12}\ldots v_{1i_1}|v_{21}\ldots v_{2i_2}|\ldots|v_{n1}\ldots v_{ni_n}$" the event that the vertices $v_{11}, \ldots, v_{1i_1} \in V$ belong to the same cluster, vertices $v_{21}, \ldots, v_{2i_2}$ belong to the same cluster, ..., vertices $v_{n1}, \ldots, v_{ni_n}$ belong to the same cluster, and, moreover, these clusters are all different. By $\mathbf{P}(v_{11}v_{12}\ldots v_{1i_1}|v_{21}\ldots v_{2i_2}|\ldots|v_{n1}\ldots v_{ni_n})$ we denote the probability of this event in the underlying bond percolation. In particular, $\mathbf{P}(abc)$ denotes the probability that vertices $a, b, c \in V$ lie in the same cluster, and $\mathbf{P}(a|b|c)$ is the probability that a, b and c belong to 3 different clusters.

Definition 3.2.2. We call the event $A \subseteq 2^\Omega$ *closed upward* if for every percolation configuration $C_1 \in A$ and every other configuration C_2 such that $C_1 \leq C_2$ coordinatewise, one has $C_2 \in A$. For example, events *ab* and *abc* are closed upward.

Definition 3.2.3. For two percolation configurations $C_1, C_2 \in \Omega$ and a set $S \subseteq E$ we denote by $C_1 \to_S C_2$ the configuration that coincides with C_1 on S and C_2 on its complement $\overline{S}$.

$$C_1 \underset{S}{\to} C_2(e) = \begin{cases} C_1(e), & \text{if } e \in S, \\ C_2(e), & \text{otherwise.} \end{cases} \tag{3.4}$$

The OSSS inequality introduced the concept of decision trees from computer science to percolation. A decision tree is an algorithm using a tree-like flowchart. Each node of the tree tests an edge of G whether it is open or closed and uses this information to move to the next node. The tree *decides* an event A if for all the configurations leading to the same leaf node L, event A is either simultaneously true or simultaneously false. Since we are working with probabilistic configurations, it can be beneficial to think that initially the states of edges are closed from us and an edge is *revealed* when it is queried by the tree.

Until Section 3.8, we deal with the decision trees that accept two configurations C_1, C_2 and build a set $S \subset E$ based on them. Each node can make a decision based only on the edges revealed so far.

Definition 3.2.4. Let $G = (V, E)$ be finite. Let T be a decision tree, where each node selects an edge, decides whether this edge goes to the set S or $\overline{S}$ and reveals it in both C_1 and C_2. In this case, we say that the set $S = S(C_1, C_2)$ is *built by T*.

Formally, a tree T on a finite graph G is an oriented network, containing nodes of two types – decision nodes and leaf nodes. Each decision node N contains an edge e that it queries, a decision $D \in \{$"S", "$\overline{S}$"$\}$ and 4 links to descendants indexed by $\{00, 01, 10, 11\}$. All nodes should be accessible via links from the initial node N_0 and the nodes on every path from N_0 should query pairwise distinct edges. The set $S(C_1, C_2)$ is then built using Algorithm 1.

Example 3.2.5. Assume that T first reveals the edges adjacent to some specific vertex a. Then T reveals the edges connected to the vertices connected to a via the revealed open edges and so on, until all edges with one end in the cluster of a are revealed. This is the breadth-first search (BFS) algorithm, as opposed to the depth-first search (DFS) Algorithm 2. Assume that T puts all the revealed edges in S. Then the set S built by T is the set of edges with at least one end in the cluster of a.

For a more detailed and visual example, see [GZ24, Figure 1].

3.3 HK inequality for decision trees

The key lemma used by Zimin and the author is the following independence result:

Lemma 3.3.1 ([GZ24, Lemma 4.2])**.** *Let G be finite. Let $S(C_1, C_2)$ be built by some decision tree. Then $C_1 \to_S C_2$ is independent of $C_2 \to_S C_1 = C_1 \to_{\bar{S}} C_2$ and both are distributed as μ.*

This lemma alone is enough to justify some inequalities of the new type. Moreover, it turns out that many classic correlation inequalities can be transferred to work with the events of type $C_1 \to_S C_2$. First, we prove a positive correlation result. When S is the set of all edges E, this result gives the usual HK inequality.

The HK inequality was independently discovered by Harris [H60] in the context of percolation and Kleitman [Kle66] in the context of set families. It ensures that every two closed upward events have a nonnegative correlation. The HK inequality was later generalized to the broader class of measures by Fortuin, Kasteleyn and Ginibre in [FKG71], so it is often also called the FKG inequality.

Theorem 3.3.2 (Decision tree HK inequality)**.** *Let G be finite. Let $S(C_1, C_2)$ be built by some decision tree. Assume A and B are some events in Ω closed upward. Then*

$$\mathbf{P}(C_1 \in A, C_1 \underset{S}{\to} C_2 \in B) \geq \mathbf{P}(C_1 \in A)\mathbf{P}(C_1 \underset{S}{\to} C_2 \in B) = \mu(A)\mu(B).$$

Proof. We use induction on the number of nodes in T with $D(N) =$ "S". In case when T always sends edges to $\overline{S}$, the inequality becomes an equality. Otherwise, consider all nodes of T with $D(N) =$ "S" and choose out of them a node N with edge $e(N)$ lying on the lowest level. Then, all descendants of N send their edges to $\overline{S}$. Consider the tree T' building a set S' that coincides with T in all nodes except for N, with the distinction that $D(N') =$ "$\bar{S}'$". Now, let Ω' be the probability space for all edges except for e. For each configuration C in Ω' there are two ways to extend it to a configuration on Ω, namely C^+ where the edge e is open and C^- where e is closed.

Now assume that the restriction of $C_1 \times C_2$ to $\Omega' \times \Omega'$ is fixed. We will get the induction step inequality

$$\mathbf{P}(C_1 \in A, C_1 \underset{S}{\to} C_2 \in B) \geq \mathbf{P}(C_1 \in A, C_1 \underset{S'}{\to} C_2 \in B) \tag{3.5}$$

by summing over all restrictions. Since all edges, that were not queried until node N, are sent to $\overline{S}$, the configuration $C_1 \to_S C_2$ is defined up to edge e. We will call the possible configurations C_3^+ and C_3^-.

$$C_3^+ = C_1^+ \underset{S}{\to} C_2^- = C_1^+ \underset{S}{\to} C_2^+ = C_1^- \underset{S'}{\to} C_2^+ = C_1^+ \underset{S'}{\to} C_2^+,$$

and

$$C_3^- = C_1^- \underset{S}{\to} C_2^- = C_1^- \underset{S}{\to} C_2^+ = C_1^- \underset{S'}{\to} C_2^- = C_1^+ \underset{S'}{\to} C_2^-.$$

Moreover, since B is closed up, we have three possibilities: both C_3^+ and C_3^- belong to B, none of them belong to B or only C_3^+ does. In the first case,

$$\mathbf{P}(C_1 \in A, C_1 \underset{S}{\to} C_2 \in B) = \mathbf{P}(C_1 \in A) = \mathbf{P}(C_1 \in A, C_1 \underset{S'}{\to} C_2 \in B).$$

In the second case,

$$\mathbf{P}(C_1 \in A, C_1 \underset{S}{\to} C_2 \in B) = 0 = \mathbf{P}(C_1 \in A, C_1 \underset{S'}{\to} C_2 \in B).$$

Finally, the third case is split into 3 subcases as well. If both C_1^+ and C_1^- belong to A or do not belong to A, the induction step is still trivial. The only nontrivial subcase is when C_1^+

belongs to A, but C_1^- does not. In this case, $C_1 \in A, C_1 \rightarrow_S C_2 \in B$ means that e is open in C_1. At the same time, $C_1 \in A, C_1 \rightarrow_{S'} C_2 \in B$ means that e is open in both C_1 and C_2. So, inequality (3.5) holds in this case. Finally, summing over all the restrictions on $\Omega' \times \Omega'$ we prove the inequality (3.5) and complete the induction. □

3.4 Decision tree vdBK inequality

The counterpart to the HK inequality is the vdBK inequality, which can be thought of as a sort of negative correlation inequality. For decision trees, these inequalities can beautifully work together, providing simple lower and upper bounds on the probabilities of events dependent on S.

Definition 3.4.1. For a space $\Omega = \prod_{i=1}^n \Omega_i$, a *witness* of an event A in a configuration C is a subset I of $[n]$, such that for any configuration C' that has the same coordinate as C for all Ω_i for $i \in I$ one has $C' \in A$.

One defines the *disjoint occurrence of A and B* denoted by $A \,\square\, B$ as

$$A \,\square\, B := \{C \in \Omega, \text{ s.t. there exist } I, J \subset [n] \\ \text{s.t. } I \text{ is a witness of } A \text{ in } C,\ J \text{ is a witness of } B \text{ in } C \text{ and } I \cap J = \varnothing\}.$$

The natural generalization to the decision trees involves the set S.

Definition 3.4.2. For the decision trees setup, the disjoint occurrence $A \,\square_S\, B$ is given by

$$A \,\square_S\, B := \{C_1, C_2 \in \Omega, \text{ s.t. there exist } I, J \subset [n] \\ \text{s.t. } I \text{ is a witness of } A \text{ in } C_1,\ J \text{ is a witness of } B \text{ in } C_1 \underset{S}{\rightarrow} C_2 \text{ and } I \cap J \subseteq \overline{S}\}.$$

For $S = E$, this definition turns into the usual disjoint occurrence of A and B in C_1. For $S = \varnothing$, the event $A \,\square_S\, B$ coincides with $A \times B$.

Theorem 3.4.3 (Decision tree vdBK inequality)**.** *Let G be finite. Let the decision tree T build a set $S(C_1, C_2)$ and A and B be two closed upward events. Then $P(A \square_S B) \leq P(A)P(B)$.*

Proof. As in the proof of Theorem 3.3.2, we induct on the number of nodes in T sending their edge to S. Again, N is such a node lying on the lowest level, e is an edge N sends to S, T' coincides with T in all nodes except for e and Ω' is the probability space for all edges except for e.

Again, we assume that the restriction of $C_1 \times C_2$ to $\Omega' \times \Omega'$ is fixed and prove the inequality

$$\mathbf{P}\big((C_1, C_2) \in A \square_S B\big) \leq \mathbf{P}\big((C_1, C_2) \in A \square_{S'} B\big) \tag{3.6}$$

for each restriction.

Assume that $(C_1, C_2) \in A \square_S B$, but $(C_1, C_2) \notin A \square_{S'} B$. Since A and B are closed upward, that means that e is open in C_1 and the set J used in the witness for $(C_1, C_2) \in A \square_S B$ contains e. Moreover, one can see that C_1 has e closed and C_2 has e open. Then notice that the configuration (C_1^-, C_2^+) has the same probability as (C_1, C_2), has the same restriction to $\Omega' \times \Omega'$, but it would, in contrast, lie in $A \square_{S'} B$, but not $A \square_S B$. Indeed, if (I, J) was the witness for $(C_1, C_2) \in A \square_S B$, then one can assume I does not contain e since A is closed upward. So, (I, J) would witness $(C_1^-, C_2^+) \in A \square_{S'} B$. Also, assume (I', J') witnesses $(C_1^-, C_2^+) \in A \square_{S'} B$. Then again by upward closeness we assume that I' does not contain e, but J' does and so the pair $(I' \cup \{e\}, J' \setminus \{e\})$ is the witness for $(C_1, C_2) \in A \square_S B$. Thus, for each restriction, the inequality (3.6) holds, and so the induction step is complete. □

3.5 Approach via Cauchy–Schwarz inequality

By Definition 3.2.4, the decision tree can have some leaf nodes such that not all edges are queried on the path leading to them. According to Algorithm 1, such edges are not assigned to S and therefore are assigned to $\overline{S}$. If we replace some of the leaf nodes with the subtrees, we will get a new tree. We say that the new tree is a continuation of the old tree.

Definition 3.5.1. We say that the decision tree T_2 *continues* decision tree T_1, when T_1 is a subset of nodes of T_2, where with each node $N \in T_2$, T_1 includes all its ancestors. So all nodes of T_1 put their edges in S or $\overline{S}$ the same way as their counterparts in T_2. In particular, if T_1 builds the set S_1 and T_2 builds the set S_2, then $S_1(C_1, C_2) \subseteq S_2(C_1, C_2)$. We also say that the decision tree T *decides* an event $A \subseteq \Omega$, when for every leaf L of T, the set of edges revealed on the path from the root to L witnesses either the event $C_1 \in A$ or the event $C_1 \notin A$.

By this definition, T_2 is able to decide finer events than T_1.

Theorem 3.5.2. *Let G be finite. Let T_1 and T_2 be decision trees for events $C_1 \in A$ and $C_1 \in B$ respectively, such that T_2 continues T_1 and $B \subset A$ is an intersection of A with an increasing or decreasing event in Ω. In addition, assume that all nodes of T_1 send the edges to S. Then*

$$\mathbf{P}(C_1 \in B, C_1 \underset{S_2}{\rightarrow} C_2 \in B) \geq \frac{\mathbf{P}(B)^2}{\mathbf{P}(A)}. \tag{3.7}$$

Proof. Let $\delta(N)$ be the probability that T_1 visits node N. We cal; it the *influence* of N. It is easy to see that the sum of the influences of the leaves of T_1 is equal to 1. Then we can write the probabilities of A and B as a sum over the leaves of T_1. Denote the set of leaves of T_1 where T_1 concludes A by X. Then,

$$\mathbf{P}(A) = \sum_{N \in X} \delta(N). \tag{3.8}$$

Since B is a subset of A, we can break the probability of B by which node of X it came through in T_2.

$$\mathbf{P}(B) = \sum_{N \in X} \delta(N)\mathbf{P}(B \mid T_2 \text{ goes through } N). \tag{3.9}$$

Finally, since T_1 only sent the vertices to S, for each $N \in X$ we can consider the subtree T_N of T_2 after the node N and apply Lemma 3.3.1 there to conclude that the conditional distributions of C_1 and $C_1 \to C_2$ coincide. Since B is an intersection of A with a monotone event, B is monotone in T_N. By Theorem 3.3.2 applied to T_N and the events $C_1 \in B$ and $C_1 \to_{S_2} C_2 \in B$ we get the representation

$$\mathbf{P}(C_1 \in B, C_1 \underset{S_2}{\to} C_2 \in B) \geq \sum_{N \in X} \delta(N) \mathbf{P}(B \mid T_2 \text{ goes through } N)^2. \tag{3.10}$$

Let us enumerate the nodes in X and consider the vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ indexed by X:

$$\overrightarrow{v} = \{\sqrt{\delta(N)}\}_{N \in X}, \ \overrightarrow{w} = \{\sqrt{\delta(N)} \mathbf{P}(B \mid T_2 \text{ goes through } N)\}_{N \in X}.$$

Finally, applying the Cauchy–Schwarz inequality to these vectors and using equations (3.8), (3.9) and (3.10), we get (3.7). □

Corollary 3.5.3. *Assume that some tree T first queries the edges from the component of a in C_1 and puts them in S. Then, regardless of what it does further,*

$$\mathbf{P}(C_1 \in a|b|c, C_1 \underset{S}{\to} C_2 \in a|b|c) \leq \frac{\mathbf{P}(a|b|c)^2}{\mathbf{P}(a|b \cup a|c)} \tag{3.11}$$

and

$$\mathbf{P}(C_1 \in a|bc, C_1 \underset{S}{\to} C_2 \in a|bc) \leq \frac{\mathbf{P}(a|bc)^2}{\mathbf{P}(a|b \cup a|c)}. \tag{3.12}$$

Proof. Indeed, let T_1 be the subtree of T cut at the moment where T queries all the edges from the component of a. By Theorem 3.5.2 applied to the trees T_1 and T and the decreasing events $A = a|b \cup a|c$ and $B = a|b|c$, we get Equation (3.11).

To obtain equation (3.12), consider the same trees and $A = a|b \cup a|c$ and $B = A \cap bc = a|bc$. □

3.6 Delfino–Viti constant for general graphs is less than $2\sqrt{2}$

If graph G is planar, we can say more about bond percolation on it. First, it allows for some graph simplifications like the star–triangle transformation, the effect of which on bond percolation is explained in [W81]. In the context of the bunkbed conjecture, the star–triangle transformations were also used by Linusson in [Lin11] (See also [L19]).

What is more, the assumption of planarity allows the decision trees to use the right-hand and left-hand rules for solving mazes: put your right (left) hand on the wall and keep it there until you find an exit. In our setup, it means the following: query the edges in the order of the depth-first search (DFS) and in each vertex choose the node visiting order starting from where you came, right to left (left to right). When the initial node is on the outer face, we choose the visiting order right to left (left to right) starting from the outer face. The DFS with the right order together with the theorems from the previous sections gives the following results.

Theorem 3.6.1. *Let G be a finite planar graph and a, b, c lie on the outer face. Then*

$$\mathbf{P}(abc)^2 \le 2\mathbf{P}(ab)\mathbf{P}(bc)\mathbf{P}(ac).$$

For nonplanar graphs, there are two ways to prove a weaker inequality.

Theorem 3.6.2 (cf. Theorem 3.1.1)**.** *For Bernoulli bond percolation on a graph G with vertices a, b, c one has*

$$\mathbf{P}(abc)^2 \le 8\mathbf{P}(ab)\mathbf{P}(ac)\mathbf{P}(bc) \tag{3.13}$$

and

$$\mathbf{P}(abc)^2 \le 2\mathbf{P}(ab \cup ac)^2\mathbf{P}(bc). \tag{3.14}$$

3.6.1 Proof of Theorem 3.6.1

Assume that a, b and c lie on the outer face in this clockwise order. Let us build a decision tree T using the DFS decision tree from Algorithm 2 as

$$T = DFS\_Decision\_tree(G, a, visited_e = \varnothing, S = \varnothing, \text{right-hand rule}, decision),$$

where *decision* returns "S" if c is not yet visited and "$\overline{S}$" otherwise.

So our tree would first use a right-hand rule to build a route from a to c in C_1 and add it to S. With probability $\mathbf{P}(a|c)$, the tree queries the whole component of a in C_1, since it does not contain c. If this is the case, we stop T. Suppose that this is not what happens. Then we effectively stop T anyway after reaching c, adding the rest of the edges to $\overline{S}$. The vertices visited by T until this moment then will be the vertices to the right of the rightmost path by open edges from a to c. Denote this path by P. All edges of P belong to S as well as all edges to the right of it.

Let T' be the continuation of T that reveals the remaining edges and puts them in $\overline{S}$. Then T' can decide $C_1 \in abc$. We are interested in $\mathbf{P}(C_1 \in abc, C_1 \rightarrow_S C_2 \in abc)$. Applying Theorem 3.5.2 for $T_1 = T$, $T_2 = T'$ and the increasing events $A = ac$ and $B = abc$, we get

$$\mathbf{P}(C_1 \in abc, C_1 \underset{S}{\rightarrow} C_2 \in abc) = \frac{\mathbf{P}(abc)^2}{\mathbf{P}(ac)}. \tag{3.15}$$

On the other hand, assume $C_1 \in abc$ and $C_1 \rightarrow_S C_2 \in abc$ occurred. Then there is a path from b to a in C_1. The first time this path intersects with P, it must do it from the left side of P and so all the edges before this point belong to $\overline{S}$. Denote this initial fragment of the path by P_1. Similarly, we consider a path from b to a in $C_1 \rightarrow_S C_2$ and its initial fragment before meeting P and denote it by P_2. So there are paths P_1 and P_2, consisting of edges from $\overline{S}$ such that they both connect b to P and the edges of P_1 are open in C_1 and the edges of P_2 are open in C_2.

Let v_1 be the vertex in P that is connected to b through P_1 and v_2 be the vertex on P connected to b via P_2. If on the path P the vertex v_1 lies closer to a than v_2, denote the segments of path P by $a \rightsquigarrow_P v_1$, $v_1 \rightsquigarrow_P v_2$, $v_2 \rightsquigarrow_P c$. Then the paths $a \rightsquigarrow_P v_1 \cup P_1$ and $v_2 \rightsquigarrow_P c \cup P_2$ are witnesses of the events ab and bc respectively and their intersection belongs to $\overline{S}$. It shows that $(C_1, C_1 \rightarrow_S C_2) \in (ab \,\square_S\, bc)$. On the contrary, if v_2 is closer to c than v_1, the paths $v_1 \rightsquigarrow_P c \cup P_1$ and $v_2 \rightsquigarrow_P a \cup P_2$ are the witnesses that prove $(C_1, C_1 \rightarrow_S C_2) \in (bc \,\square_S\, ac)$. Altogether, we get the estimate

$$\mathbf{P}(C_1 \in abc, C_1 \underset{S}{\rightarrow} C_2 \in abc) \leq \mathbf{P}(ab \,\square_S\, bc \cup bc \,\square_S\, ab). \tag{3.16}$$

By Theorem 3.4.3, the right side is bounded from above by $2\mathbf{P}(ab)\mathbf{P}(bc)$. Combining with (3.15), we get $\frac{\mathbf{P}(abc)^2}{\mathbf{P}(ac)} \leq 2\mathbf{P}(ab)\mathbf{P}(bc)$, which is equivalent to the theorem statement. □

Remark 3.6.3. By Theorem 3.8.3 one is able to improve over the vdBK estimate of the right side in (3.16) and show $\frac{\mathbf{P}(abc)^2}{\mathbf{P}(ac)} \leq 2\mathbf{P}(ab)\mathbf{P}(bc) - \mathbf{P}(abc)^2$.

3.6.2 Proof of Theorem 3.6.2

Assume G is finite. We first prove the inequality (3.14). Let tree T perform a DFS starting with the vertex a and put the edges it meets in S. With probability $\mathbf{P}(a|b \cup a|c)$, the tree queries the whole component of a in C_1 since it does not contain b or c. After reaching b or c, the tree T stops (and so puts the rest of the edges in $\overline{S}$).

Backtracking the DFS order leaves us with a path P from a to either b or c. Note that T queries all the edges of the path P and puts them to S. Let Q be the set of vertices visited by the DFS that are not in P. The set S witnesses that all vertices from Q are connected to a. Also, since vertices from Q do not belong to the final path P, it means that the DFS queried all edges from Q before backtracking and so all of these edges, including those closed in C_1, belong to S.

Consider the tree T' continuing T that reveals the remaining edges putting them in $\bar{S}$ and so is able to decide the event abc. Now, by Theorem 3.5.2 applied to the trees T and T' and the events $ab \cup ac$ and abc,

$$\mathbf{P}(C_1 \in abc, C_1 \underset{S}{\to} C_2 \in abc) \geq \frac{\mathbf{P}(abc)^2}{\mathbf{P}(ab \cup ac)}.$$

On the other hand, as in the previous proof, $\mathbf{P}(C_1 \in abc, C_1 \to_S C_2 \in abc)$ is bounded from above by $2\mathbf{P}(ab \cup ac)P(bc)$, by Theorem 3.4.3. It proves the inequality in the form (3.14).

Now we prove (3.13). Denote by R_b the event that T connects a to b and by R_c the event that T connects a to c. It is easy to see that $\mathbf{P}(R_b) + \mathbf{P}(R_c) = \mathbf{P}(ab \cup ac)$ and further

$\mathbf{P}(R_b) \le \mathbf{P}(ab)$ and $\mathbf{P}(R_c) \le \mathbf{P}(ac)$. Also, by Theorem 3.5.2 we have

$$\mathbf{P}(C_1 \in R_b \cap abc, C_1 \underset{S}{\to} C_2 \in R_b \cap abc) \ge \frac{\mathbf{P}(R_b \cap abc)^2}{\mathbf{P}(R_b)}$$

and

$$\mathbf{P}(C_1 \in R_c \cap abc, C_1 \underset{S}{\to} C_2 \in R_c \cap abc) \ge \frac{\mathbf{P}(R_c \cap abc)^2}{\mathbf{P}(R_c)}.$$

On the other hand, we can estimate the LHS of the two inequalities, as in the previous proof, using Theorem 3.4.3:

$$\mathbf{P}(C_1 \in R_b \cap abc, C_1 \underset{S}{\to} C_2 \in R_b \cap abc) \le 2\mathbf{P}(ac)\mathbf{P}(bc)$$

and

$$\mathbf{P}(C_1 \in R_c \cap abc, C_1 \underset{S}{\to} C_2 \in R_c \cap abc) \le 2\mathbf{P}(ab)\mathbf{P}(bc).$$

Combining these four inequalities, we get

$$\begin{aligned}\mathbf{P}(abc) = \; & \mathbf{P}(R_b \cap abc) + \mathbf{P}(R_c \cap abc) \\ & \le \sqrt{2\mathbf{P}(ac)\mathbf{P}(bc)\mathbf{P}(R_b)} + \sqrt{2\mathbf{P}(ab)\mathbf{P}(bc)\mathbf{P}(R_c)} \le 2\sqrt{2\mathbf{P}(ab)\mathbf{P}(ac)\mathbf{P}(bc)},\end{aligned}$$

which implies (3.13). This completes the proof for finite G. For infinite G, both (3.14) and (3.13) follows by passing to the limit. □

Remark 3.6.4. By the specific randomized ordering choice of vertices in DFS (called PDFS in [GP24a]), one can ensure that

$$\mathbf{P}(R_b \cap abc) = \mathbf{P}(R_c \cap abc) = \frac{\mathbf{P}(abc)}{2}.$$

Not only does it the proof more straightforward, it also gives a slightly tighter bound

$$\mathbf{P}(abc) \le \sqrt{2\mathbf{P}(ac)\mathbf{P}(bc)\left(\mathbf{P}(ab|c) + \frac{\mathbf{P}(abc)}{2}\right)} + \sqrt{2\mathbf{P}(ab)\mathbf{P}(bc)\left(\mathbf{P}(ac|b) + \frac{\mathbf{P}(abc)}{2}\right)}.$$

3.6.3 Implications

An interesting case emerges when one applies this to the critical mode of percolation on $\mathbb{Z}^2$. From the box-crossing (RSW) inequalities (see [Gri99, Section 11.7]) and the HK inequality, one can show that $\frac{\mathbf{P}(abc)}{\sqrt{\mathbf{P}(ab)\mathbf{P}(ac)\mathbf{P}(bc)}}$ is bounded. Our method allows for better estimates on this bound. According to the conjectured integral formula from [DV11][1], for the critical regime on $\mathbb{Z}^2$ this quantity converges to approximately 1.022 as a, b and c tend from each other. This number is consistent with our upper bound of $2\sqrt{2}$.

Moreover, as per the earlier result in [SZK09, BI12], for the bond percolation on the upper half-plane, if a, b and c lie on the real line, the scaling limit of $\frac{\mathbf{P}(abc)}{\sqrt{\mathbf{P}(ab)\mathbf{P}(ac)\mathbf{P}(bc)}}$ converges to

$$\frac{2^{\frac{7}{2}}\pi^{\frac{5}{2}}}{3^{\frac{3}{4}}\Gamma(\frac{1}{3})^{\frac{9}{2}}} \approx 1.02992\ldots$$

In this case, our Theorem 3.6.1 is applicable and gives a consistent upper bound of $\sqrt{2}$.

For the supercritical mode, denote by θ the density of the infinite cluster. Then equation (3.13) tends to $\theta^6 \leq 8\theta^6$ as a, b and c tend away from each other.

In the non-integrable cases, our inequality still leads to an inequality on the three-point exponent. Denote by $D(a,b,c)$ the maximum distance between a, b and c:

$$D(a,b,c) = \max(D(a,b), D(a,c), D(b,c)).$$

Corollary 3.6.5. *Let G be a vertex-transitive infinite graph and C and α be the constant such that $\mathbf{P}(ab) < CD(a,b)^\alpha$. Then*

$$\mathbf{P}(abc) < (2C)^{\frac{3}{2}} D(a,b,c)^{\frac{3}{2}\alpha}.$$

Note that for G being a two-dimensional lattice in the critical mode, [LSW02], assuming conformal invariance, establishes that $\mathbf{P}(ab) = D(a,b)^{-2\eta+o(1)}$, where $\eta = \frac{5}{48}$ is the one-arm

[1] The proof of this formula for critical site percolation on triangular lattice was recently announced by Morris Ang, Gefei Cai, Xin Sun and Baojun Wu (personal communication, 10 Aug 2024)

exponent. The $\mathbf{P}(abc)$, in turn, grows as $D(a,b,c)^{-3\eta+o(1)}$, which coincides with our bound. Note that the conformal invariance is only known for site percolation on the triangular lattice as per the celebrated result of Smirnov[S01].

3.7 Proof of Theorem 3.1.3

3.7.1 Proof of Lemma 3.1.2

We slightly modify the proof of Theorem 4.2 in [GZ24] and use the better bound from Theorem 3.5.2. Assume G is finite. The following lemma was a keystone in the proof:

Lemma 3.7.1. *Let the decision function $\mathcal{S}$ always output "S" and the decision function $\bar{\mathcal{S}}$ always output "$\overline{S}$". Let S_1, S_2 and S_3 be given by the decision trees from Figure 3.1.*

Then if $C_1 \in a|b|c$ and $C_1 \to_{S_3} C_2 \in ab \cup ac$, one has $C_1 \to_{S_1} C_2 \in ab$ or $C_1 \to_{S_2} C_2 \in ac$.

We use this lemma, and as in [GZ24], we bound the probability of the first event from above by

$$\mathbf{P}(C_1 \in a|b|c \text{ and } C_1 \underset{S_3}{\to} C_2 \in ab \cup ac)$$
$$\leq \mathbf{P}(ab \cup ac)\mathbf{P}(a|b \cap a|c) - \mathbf{P}(a|bc) = \mathbf{P}(a|b|c) - \mathbf{P}(a|b \cap a|c)^2.$$

Now, using Theorem 3.5.2 we can get a better lower bound for the probabilities of the two latter events. Indeed,

$$\mathbf{P}(C_1 \in a|b|c \text{ and } C_1 \underset{S_1}{\to} C_2 \in ab)$$
$$\leq \mathbf{P}(a|b|c) - \mathbf{P}(C_1 \in a|b|c \text{ and } C_1 \underset{S_1}{\to} C_2 \in a|b|c) = \mathbf{P}(a|b|c) - \frac{\mathbf{P}(a|b|c)^2}{\mathbf{P}(a|c \cup b|c)}$$

and

Algorithm 3 S_1 Decision Tree

procedure DFS_DECISION_TREE(G, a, b, c)
  $visited_e \leftarrow$ array of size $|E(G)|$ initialized with $false$, $S \leftarrow \varnothing$
  $DFS\_Decision\_tree(G, c, visited_e, S, id, \mathcal{S})$
  $DFS\_Decision\_tree(G, a, visited_e, S, id, \bar{\mathcal{S}})$
  $DFS\_Decision\_tree(G, b, visited_e, S, id, \mathcal{S})$
  return S
end procedure

Algorithm 4 S_2 Decision Tree

procedure DFS_DECISION_TREE(G, a, b, c)
  $visited_e \leftarrow$ array of size $|E(G)|$ initialized with $false$, $S \leftarrow \varnothing$
  $DFS\_Decision\_tree(G, b, visited_e, S, id, \mathcal{S})$
  $DFS\_Decision\_tree(G, a, visited_e, S, id, \bar{\mathcal{S}})$
  $DFS\_Decision\_tree(G, c, visited_e, S, id, \mathcal{S})$
  return S
end procedure

Algorithm 5 S_3 Decision Tree

procedure DFS_DECISION_TREE(G, a, b, c)
  $visited_e \leftarrow$ array of size $|E(G)|$ initialized with $false$, $S \leftarrow \varnothing$
  $DFS\_Decision\_tree(G, a, visited_e, S, id, \bar{\mathcal{S}})$
  $DFS\_Decision\_tree(G, b, visited_e, S, id, \mathcal{S})$
  $DFS\_Decision\_tree(G, c, visited_e, S, id, \mathcal{S})$
  return S
end procedure

Figure 3.1: Trees building S_1, S_2, S_3

$$\mathbf{P}(C_1 \in a|b|c \text{ and } C_1 \underset{S_2}{\to} C_2 \in ac)$$
$$\leq \mathbf{P}(a|b|c) - \mathbf{P}(C_1 \in a|b|c \text{ and } C_1 \underset{S_2}{\to} C_2 \in a|b|c) = \mathbf{P}(a|b|c) - \frac{\mathbf{P}(a|b|c)^2}{\mathbf{P}(a|b \cup b|c)}.$$

Combining these bounds, we get the needed equation (3.2).

For infinite G, the theorem follows by passing to the limit. □

Now we are equipped to prove Theorem 3.1.3.

3.7.2 Proof of Theorem 3.1.3

By interchanging vertices a and b in (3.2), we get

$$\mathbf{P}(a|b|c) + \mathbf{P}(a|b \cup b|c)^2 \geq \frac{\mathbf{P}(a|b|c)^2}{\mathbf{P}(a|b \cup a|c)} + \frac{\mathbf{P}(a|b|c)^2}{\mathbf{P}(a|c \cup b|c)}. \tag{3.17}$$

We rewrite it as

$$\mathbf{P}(a|b|c) + (\mathbf{P}(a|b|c) + \mathbf{P}(ac|b))^2 \geq \frac{\mathbf{P}(a|b|c)^2}{\mathbf{P}(a|b|c) + \mathbf{P}(a|bc)} + \frac{\mathbf{P}(a|b|c)^2}{\mathbf{P}(a|b|c) + \mathbf{P}(ab|c)}.$$

Assume the contrary: let ε be the counterexample to the conjecture. We will show that small enough δ contradicts this inequality. Since $\mathbf{P}(ab|c) < \delta$ and $\mathbf{P}(ac|b) < \delta$, we get

$$\mathbf{P}(a|b|c) + (\mathbf{P}(a|b|c) + \delta)^2 \geq \frac{\mathbf{P}(a|b|c)^2}{\mathbf{P}(a|b|c) + \delta} + \frac{\mathbf{P}(a|b|c)^2}{1 - \mathbf{P}(abc)}.$$

Let us move the summands:

$$\mathbf{P}(a|b|c) - \frac{\mathbf{P}(a|b|c)^2}{\mathbf{P}(a|b|c) + \delta} \geq \frac{\mathbf{P}(a|b|c)^2}{1 - \mathbf{P}(abc)} - (\mathbf{P}(a|b|c) + \delta)^2.$$

Converting to the common denominator:

$$\delta \geq \frac{\delta \mathbf{P}(a|b|c)}{\mathbf{P}(a|b|c)+\delta} \geq \frac{-2\delta \mathbf{P}(a|b|c) - \delta^2 + \mathbf{P}(abc)(\mathbf{P}(a|b|c)+\delta)^2}{1-\mathbf{P}(abc)}.$$

Finally, we multiply both parts by $1-\mathbf{P}(abc)$ and estimate assuming $\mathbf{P}(abc) \geq \varepsilon$ and $\mathbf{P}(a|b|c) \geq \varepsilon$:

$$4\delta \geq \delta - \delta\mathbf{P}(abc) + 2\delta\mathbf{P}(a|b|c) + \delta^2 \geq \mathbf{P}(abc)(\mathbf{P}(a|b|c)+\delta)^2 \geq \mathbf{P}(abc)\mathbf{P}(a|b|c)^2 \geq \varepsilon^3.$$

Now we see that $\delta < \frac{\varepsilon^3}{4}$ contradicts Lemma 3.1.2, thus proving the conjecture. □

3.7.3 Remarks on the result

We hope that our tools can attack the notorious Conjecture 3.10.1. For now, we can only say using Theorem 3.6.2 that

$$\mathbf{P}(ab|c) < \delta \implies \mathbf{P}(abc) - 8\mathbf{P}(ac)\mathbf{P}(bc) < \varepsilon.$$

However, we improved the bounds on $\min\big(\mathbf{P}(abc), \mathbf{P}(a|b|c)\big)$. In [GZ24] this quantity was called α_3 and the upper bound on it was 0.369. Without loss of generality,

$$\mathbf{P}(a|b \cup a|c) = \min(\mathbf{P}(a|b \cup a|c), \mathbf{P}(a|b \cup b|c), \mathbf{P}(a|c \cup b|c)).$$

Then from the inequality (3.2), we get the upper bound t on α_3. Indeed, the optimum is achieved when $\mathbf{P}(a|b \cup a|c)$ and $\mathbf{P}(a|b|c)$ are as large as possible and $\mathbf{P}(a|b \cup b|c), \mathbf{P}(a|c \cup b|c)$ are as small as possible. It leads us to an equation

$$t + \left(t + \frac{1-2t}{3}\right)^2 \geq 2\frac{t^2}{t+\frac{1-2t}{3}}.$$

This can be simplified to

$$\frac{t^3 - 42t^2 + 12t + 1}{t+1} \geq 0,$$

and the only root of the numerator on the $(0, 1)$ interval is ≈ 0.356. This is better than the previous upper bound on α_3, but is still quite apart from the best lower bound of 0.29065.

3.8 General form of decision tree inequalities

Here we consider a generalized setup, where the decision tree generates a configuration as it goes rather than reveals what was hidden. Assume that at each decision node N of the decision tree T, it chooses an edge $e(N)$ of the graph and makes a decision $D(N) \in \{1, 2\}$, and then generates a random element of one of two probability spaces $\Omega_1(e)$ or $\Omega_2(e)$ according to the measures $\mu_1(e)$ or $\mu_2(e)$ respectively. If N is a decision node, it also has links to $|\Omega_i|$ vertices, where $i = D(N)$. Assume that each path from N_0 to a leaf node contains all edges once. So, every edge is assigned an element of $\Omega_1(e)$ or $\Omega_2(e)$, and the tree T builds a random configuration $C \in \Omega = \prod_{e \in E} \big(\Omega_1(e) \cup \Omega_2(e)\big)$. We use $\mathbf{P}$ to refer to the induced distribution.

Assume that event $A \subseteq \Omega$ is such that for every edge e and every configuration $C \in \Omega$ the probability of A is bigger if e is resampled from $\mu_2(e)$ rather than if it is resampled from $\mu_1(e)$. It turns out to be the common setup for the HK, vdBK and other inequalities.

Main Lemma 3.8.1. *Let T be a decision tree in the setup above building configuration C. Let A be an event in Ω. Let e be an arbitrary edge and $C \in \Omega$ an arbitrary configuration. Denote by $X_1 = X_1(C, e)$ the subset of such $x \in \Omega_1$ and by $X_2 = X_2(C, e)$ the subset of such $x \in \Omega_2$ that $C \rightarrow_{E \setminus \{e\}} x \in A$. Assume that for every e and C one has*

$$\mu_1(X_1) \le \mu_2(X_2). \tag{3.18}$$

Then

$$\mathbf{P}(C_1 \in A) \le \mathbf{P}(C \in A) \le \mathbf{P}(C_2 \in A). \tag{3.19}$$

Proof. The proof generalizes the proofs of Theorems 3.3.2 and 3.4.3. We only prove the first inequality of (3.19), since the second inequality is proved in the same manner. We induct on the number of nodes of T with $D(N) = 2$. If no such nodes exist, the inequality turns into equality. Otherwise, consider the lowest node N with $D(N) = 2$. Let tree T' coincide with T up to the vertex N, but the node N' has $D(N') = 1$ and so its children should now be indexed by $\Omega_1(e(N))$. We achieve it by copying an arbitrary child of N with its subtree under all children of N', so that after getting to N' the edges that were not yet assigned would be assigned a random element of the corresponding Ω_1.

Now we see that each path in T not passing through N exists in T' as well and has the same probability there. For paths in T passing through N and the paths in T' passing through N' the configuration C is the same except for the edge $e(N) = e(N')$. This happens with probability $\delta(N)$ and, conditional on passing through N, the probability of A is $\mu_1(X_1(C, e))$ for T and $\mu_2(X_2(C, e))$ for T'. So,

$$\mathbf{P}(C(T) \in A) - \mathbf{P}(C(T') \in A) = \delta(N)\Big(\mu_1\big(X_1(C, e)\big) - \mu_2\big(X_2(C, e)\big)\Big) \geq 0,$$

by condition (3.18). Now, from the induction hypothesis we obtain $\mathbf{P}(C(T_1) \in A) \leq \mathbf{P}(C(T') \in A) \leq \mathbf{P}(C(T) \in A)$. The second inequality in (3.19) is proved analogously. $\square$

Theorems 3.3.2 and 3.4.3 follow from the Main Lemma 3.8.1 with the correct choice of μ_1 and μ_2. In Theorem 3.3.2 $\Omega_1(e) = \Omega_2(e) = \{00, 01, 10, 11\}$. If p is the probability of e being open in the original percolation, then μ_1 assigns probabilities of $(1-p)^2, p(1-p), p(1-p)$ and p^2 to the elements respectively and μ_2 assigns probabilities of $1-p, 0, 0$ and p.

In Theorem 3.4.3, unary nodes generate a random element of $\Omega_1(e) = \{0, 1\}$ and binary nodes generate a random element of $\Omega_2(e) = \{00, 01, 10, 11\}$. We consider the event $A \,\square_S\, B$ on Ω, where S is the set of edges generated by unary nodes. It is easy to check that condition (3.18) holds. Thus, Main Lemma 3.8.1 proves Theorem 3.4.3. In fact, the extra generality helps to spot further generalizations.

Theorem 3.8.2. *Assume each edge $e \in E$ is assigned some $p(e) \in [0, 1]$. Let $\Omega_1(e)$ be the set $0, 1, 2$ and μ_1 assign the probabilities $(1-p)^2$, $2p(1-p)$ and p^2 to these outcomes, respectively. Let Ω_2 be the set $\{00, 01, 10, 11\}$ and μ_2 assign the probabilities $(1-p)^2$, $p(1-p)$, $p(1-p)$ and p^2. Let A and B be two increasing events on $\{0, 1\}^E$. Denote by $A \bowtie B$ the following* *event on* $\prod_{e\in E}(\Omega_1(e)\cup\Omega_2(e))$*:*

$$A \bowtie B = \left\{ C \in \prod_{e\in E}(\Omega_1(e)\cup\Omega_2(e)) \ s.t.\ \exists w_1, w_2 \subseteq E \ s.t.\ \mathrm{Ind}[w_1]\in A, \mathrm{Ind}[w_2]\in B, \right.$$
$$\left. \text{and } C(e) \in \begin{cases} \{1,2,10,11\}, & \text{if } e \in w_1, \\ \{1,2,01,11\}, & \text{if } e\in w_2, \\ \{2,11\}, & \text{if } e \in w_1\cap w_2. \end{cases} \right\} \tag{3.20}$$

Let C_1*,* C *and* C_2 *be the configurations built by decision trees as above. Then,*

$$\mathbf{P}(C_1 \in A\bowtie B) \le \mathbf{P}(C\in A\bowtie B) \le \mathbf{P}(C_2 \in A\bowtie B).$$

Informally, $A\bowtie B$ is the same as $A\,\square\, B$, but there is an extra probability p^2 for a unary node to produce a double edge that can be used in both witnesses.

Proof. We need to prove the condition (3.18). Consider a configuration C on $\prod_{e\in E}(\Omega_1(e)\cup\Omega_2(e))$, defined up to some edge e.

Note that Ω_1 has a natural linear ordering and Ω_2 has a natural partial ordering. These orders agree with the definition of $\bowtie$ in the sense that if $x<y$ and $C{\to_{E\setminus\{e\}}}\, x \in A\bowtie B$, then $C{\to_{E\setminus\{e\}}}\, y\in A\bowtie B$. Let X_1 be the subset of such $x\in\Omega_1$ and X_2 be the subset of such $x\in\Omega_2$ that $C{\to_{E\setminus\{e\}}}\, x\in A\bowtie B$. Then both X_1 and X_2 are closed upward. The theorem statement is then equivalent to $\mu_1(X_1)\le\mu_2(X_2)$.

So now there are four possibilities for X_1. It can be either $\varnothing$, $\{2\}$, $\{1,2\}$, or $\{0,1,2\}$. We analyze these cases separately.

(i) $X_1=\varnothing$: then, obviously, $0=\mu(X_1)\le\mathbf{P}(X_2)$.

(ii) $X_1=\{2\}$: then $11\in X_2$. Indeed, consider the witnesses w_1, w_2 for $C{\to_{E\setminus\{e\}}}\, 2\in A\bowtie B$. Same w_1, w_2 would witness $C{\to_{E\setminus\{e\}}}\, 2\in A\bowtie B$, because the definition (3.20) does not distinguish between 2 and 11. So, the probability of X_2 is at least $\mu_2(11)=p^2=\mu_1(2)$.

(iii) $X_1 = \{1, 2\}$: then $01 \in X_2$ or $10 \in X_2$. Indeed, let w_1, w_2 be the witnesses for $C \rightarrow_{E\setminus\{e\}} 1 \in A \bowtie B$. At most one of them should contain e and they can not both not contain it, because otherwise they would be witnesses for $C \rightarrow_{E\setminus\{e\}} 0 \in A \bowtie B$ as well. Without loss of generality, $e \in w_1$. Then w_1 and w_2 are a witness for $C \rightarrow_{E\setminus\{e\}} 10 \in A \bowtie B$ and $10 \in X_2$ as well as 11. So, $\mu_2(X_2) \geq \mu_2(10) + \mu_2(11) = p = \mu_1(X_1)$. Note that this case contributes to the inequality if $X_2 = \{10, 01, 11\}$, since in other cases we can actually prove $\mu_1(X_1) = \mu_2(X_2)$.

(iv) $X_1 = \{0, 1, 2\}$: then $00 \in X_2$. Indeed, let w_1, w_2 be the witnesses for $C \rightarrow_{E\setminus\{e\}} 0 \in A \bowtie B$. Both of them avoid e, so they witness $C \rightarrow_{E\setminus\{e\}} 00 \in A \bowtie B$ as well. So $X_2 = \Omega_2$ and $\mu_1(X_1) = 1 = \mu_2(X_2)$.

□

Along with the vdBK inequality, the paper [BK85] shows the following stronger result:

Theorem 3.8.3 ([BK85, eq. (3.6)]). *Let A_1, …, A_n and B_1, …, B_n be increasing events on $\{0,1\}^E$.*

Then

$$\mathbf{P}(A_1 \square B_1 \cup \cdots \cup A_n \square B_n) \leq \mathbf{P}(A_1 \times B_1 \cup \cdots \cup A_n \times B_n),$$

where the second event is a subset of $\{00, 01, 10, 11\}^E$ with the probability measure as in Theorem 3.8.2.

We note that the proof of Theorem 3.8.2 also proves the similar statement:

Theorem 3.8.4 (cf. Main Lemma 3.8.1). *In the conditions of Theorem 3.8.2, for every* *increasing event* $A_1, \ldots, A_n$ *and* $B_1, \ldots, B_n$ *on* $\{0,1\}^E$ *one has*

$$\begin{aligned}\mathbf{P}(C_1 \in A_1 \bowtie B_1 \cup \cdots \cup A_n \bowtie B_n) \\ \le \mathbf{P}(C \in A_1 \bowtie B_1 \cup \cdots \cup A_n \bowtie B_n) \\ \le \mathbf{P}(C_2 \in A_1 \bowtie B_1 \cup \cdots \cup A_n \bowtie B_n).\end{aligned}$$

Now we continue applications of our method with the decision tree version of the main theorem from [GP24a]. In notation from this paper, $f : E \to \{a, b, c, d\}$ is a uniform random coloring of the edges of G, where each edge is colored uniformly and independently. Denote by E_s, $s \in \{a, b, c, d\}$, a subset of edges of the corresponding color. Similarly, for every two distinct colors $s, t \in \{a, b, c, d\}$, let $E_{st} := E_s \cup E_t$. One can think of E_{st} as either a $\frac{1}{2}$-percolation or a uniformly random subset of edges of G, so that $G_{st} = (V, E_{st})$ is a uniform random subgraph of G.

Theorem 3.8.5 ([GP24a, first part of Theorem 1])**.** *Let* $\mathcal{U}, \mathcal{V}, \mathcal{W}$ *be closed upward graph properties. Denote by* $\mathcal{U}_{ab}$*,* $\mathcal{V}_{ac}$ *and* $\mathcal{W}_{bc}$ *the corresponding properties of* G_{ab}*,* G_{ac} *and* G_{bc}*, respectively. Then the events* $\mathcal{U}_{ab}$*,* $\mathcal{V}_{ac}$ *and* $\mathcal{W}_{bc}$ *are pairwise independent, but have negative mutual dependence:*

$$\mathbf{P}(\mathcal{U}_{ab} \cap \mathcal{V}_{ac} \cap \mathcal{W}_{bc}) \le \mathbf{P}(\mathcal{U}_{ab})\mathbf{P}(\mathcal{V}_{ac})\mathbf{P}(\mathcal{W}_{bc}), \tag{3.21}$$

where the probability is over uniform random colorings $f : E \to \{a, b, c, d\}$*.*

From Main Lemma 3.8.1 we get the decision tree version. Let Ω_1 be the set of triplets $\{000, 011, 101, 110\}$ and μ_1 be the measure assigning $\frac{1}{8}$ to each element of Ω_2. Similarly, let Ω_2 be the full set of triplets

$$\{000, 001, 010, 011, 100, 101, 110, 111\}$$

and μ_2 be the measure that assigns $\frac{1}{4}$ to each element of Ω_2. We say that an element C of $\prod_{e \in E}(\Omega_1(e) \cup \Omega_2(e))$ belongs to $\mathcal{U} \times \mathcal{V} \times \mathcal{W}$ if the configuration $E_1 \in \{0,1\}^E$ formed by the first digits on the edges belongs to $\mathcal{U}$, the configuration E_2 formed by the second digits on the edges belongs to $\mathcal{V}$ and the configuration E_3 formed by the third digits on the edges belongs to $\mathcal{W}$.

Theorem 3.8.6. *Let C_1, C and C_2 be the configurations built by decision trees as above. Then,*

$$\mathbf{P}(C_1 \in \mathcal{U} \times \mathcal{V}) = \mathbf{P}(C \in \mathcal{U} \times \mathcal{V}) = \mathbf{P}(C_2 \in \mathcal{U} \times \mathcal{V}). \tag{3.22}$$

and

$$\mathbf{P}(C_1 \in \mathcal{U} \times \mathcal{V} \times \mathcal{W}) \leq \mathbf{P}(C \in \mathcal{U} \times \mathcal{V} \times \mathcal{W}) \leq \mathbf{P}(C_2 \in \mathcal{U} \times \mathcal{V} \times \mathcal{W}). \tag{3.23}$$

Proof. The "pairwise independent" part of Theorem 3.8.5 is an easy part. It also generalizes to the decision tree setup as (3.22), since, without the third coordinate, the first two coordinates are uniformly distributed in both μ_1 and μ_2.

If $C \in \prod_{e \in E} \Omega_1(e)$, then by the choice of Ω_1, we see that $E_3 = E_1 \oplus E_2$ is an edgewise exclusive or of independent E_1 and E_2, just like $E_{bc} = E_{ab} \oplus E_{ac}$. So, the probability on the left coincides with $\mathbf{P}(\mathcal{U}_{ab} \cap \mathcal{V}_{ac} \cap \mathcal{W}_{bc})$. The event on the right is an intersection of three independent events depending on E_1, E_2 and E_3, so its probability coincides with $\mathbf{P}(\mathcal{U}_{ab})\mathbf{P}(\mathcal{V}_{ac})\mathbf{P}(\mathcal{W}_{bc})$.

Let X_1 be the subset of such $x \in \Omega_1$ and X_2 be the subset of such $x \in \Omega_2$ that $C \rightarrow_{E \setminus \{e\}} x \in \mathcal{U} \times \mathcal{V} \times \mathcal{W}$. Note that $X_1 = X_2 \cap \Omega_2$, so we only need to analyze the possibilities for X_2. Since the condition for $x \in \mathcal{U} \times \mathcal{V} \times \mathcal{W}$ splits into 3 independent conditions for 3 coordinates, X_2 is a Cartesian product of 3 sets. Moreover, since $\mathcal{U}$, $\mathcal{V}$ and cW are increasing, X_2 is also increasing. This leaves us with a few options, up to the coordinate permutation.

(i) $X_2 = \varnothing$: then $X_1 = \varnothing$ and $\mu_1(X_1) = 0 = \mu_2(X_2)$.

(ii) $X_2 = \{111\}$: then $X_1 = \varnothing$ and $\mu_1(X_1) = 0 < \frac{1}{8} = \mu_2(X_2)$. Note that this is the only case where the inequality is strict.

(iii) $X_2 = \{111, 110\}$: then $X_1 = \{110\}$ and $\mu_1(X_1) = \frac{1}{4} = \mu_2(X_2)$.

(iv) $X_2 = \{111, 110, 101, 100\}$: then $X_1 = \{110, 101\}$ and $\mu_1(X_1) = \frac{1}{2} = \mu_2(X_2)$.

(v) $X_2 = \Omega_2$: then $X_1 = \Omega_1$ and $\mu_1(X_1) = 1 = \mu_2(X_2)$.

By Main Lemma 3.8.1, we are done. □

The last application is somewhat similar. Let

$$\Omega_1 = \Omega_2 = \{000, 001, 010, 011, 100, 101, 110, 111\}.$$

Let μ_1 be the mixture of the uniform distribution μ_{11} on $\{000, 111\}$ with the coefficient $\frac{2}{3}$ and the uniform distribution μ_{12} on Ω_1 with the coefficient $\frac{1}{3}$. Let μ_2 be the mixture of the uniform distribution μ_{21} on $\{000, 011, 100, 111\}$ with the coefficient $\frac{1}{3}$, the uniform distribution μ_{22} on $\{000, 010, 101, 111\}$ with the coefficient $\frac{1}{3}$, and the uniform distribution μ_{23} on $\{000, 001, 110, 111\}$ with the coefficient $\frac{1}{3}$.

Theorem 3.8.7. *Let C_1, C and C_2 be the configurations built by decision trees as above. Then,*

$$\mathbf{P}(C_1 \in \mathcal{U} \times \mathcal{V} \times \mathcal{W}) \geq \mathbf{P}(C \in \mathcal{U} \times \mathcal{V} \times \mathcal{W}) \geq \mathbf{P}(C_2 \in \mathcal{U} \times \mathcal{V} \times \mathcal{W}). \tag{3.24}$$

Proof. Let X be the subset of such $x \in \Omega_1 = \Omega_2$ that $C \to_{E\setminus\{e\}} x \in \mathcal{U} \times \mathcal{V} \times \mathcal{W}$. As in the previous proof, X can only be an increasing product of three events. By Main Lemma 3.8.1, we are left to check that $\mu_1(X) \geq \mu_2(X)$. So, without loss of generality, we have the following cases.

(i) $X = \varnothing$: then $\mu_1(X) = 0 = \mu_2(X)$.

(ii) $X = \{111\}$: then $\mu_1(X) = \frac{9}{24} > \frac{1}{4} = \mu_2(X)$. Note that this is one of the two cases where the inequality is strict.

(iii) $X = \{111, 110\}$: then $\mu_1(X_1) = \frac{10}{24} > \frac{4}{12} = \mu_2(X_2)$. Note that this is one of the two cases where the inequality is strict.

(iv) $X = \{111, 110, 101, 100\}$: then $\mu_1(X_1) = \frac{1}{2} = \mu_2(X_2)$.

(v) $X = \Omega_2$: then $\mu_1(X_1) = 1 = \mu_2(X_2)$.

$\square$

Remark 3.8.8. This final application of Main Lemma 3.8.1 stems from the work of Richards [R04]. His paper provides an incorrect proof for the inequality

$$2\mathbf{P}(\mathcal{U}\cap\mathcal{V}\cap\mathcal{W})+\mathbf{P}(\mathcal{U})\mathbf{P}(\mathcal{V})\mathbf{P}(\mathcal{W}) \geq \mathbf{P}(\mathcal{U})\mathbf{P}(\mathcal{V}\cap\mathcal{W})+\mathbf{P}(\mathcal{V})\mathbf{P}(\mathcal{U}\cap\mathcal{W})+\mathbf{P}(\mathcal{W})\mathbf{P}(\mathcal{V}\cap\mathcal{U}) \quad (3.25)$$

The proof mimics the proof of the HK inequality and utilizes induction. The induction step implicitly worked in the space of triples of configurations and effectively was equivalent to equation (3.24). Inequality (3.25) is still a conjecture. Sahi [S08] generalized this inequality to a series of conjectured inequalities. There are partial results in the direction of these conjectures [LS22].

3.9 Inequalities for disjoint paths between two vertices

3.9.1 Proof of Theorem 3.1.5

Finally, after studying the connectivity events for 3 vertices, we study the minimal case – inequalities concerning connections for just two points. Although it may seem that there is not enough variation – a and b can be either connected or disconnected, we study the events of the form $ab^{\square n} := ab \,\square\, ab \,\square \cdots \square\, ab$ (n times). Note that in general $\square$ is not associative, but this particular event means that there are n nonintersecting paths from a to b passing through open edges. Thus, this definition does not depend on the order of operations. When b is a ghost vertex, $\mathbf{P}(ab^{\square n})$ is related to the monochromatic arms exponents.

Proof of Theorem 3.1.5. Let G be finite. Without loss of generality, the face to which a and b both belong is an outer face. This allows us to run a right-hand rule walk on it and to talk about the "right" and "left" side of every path. Let T be a decision tree that runs a right-hand rule walk starting from a, until it runs into b, and put its edges in S. If the walk reaches b, then part of the edges in this walk form the path P_1 that is the rightmost path from a to b. It means that for every path P from a to b, all vertices of P_1 lie on P or to the right of it. In this case, run the second right-hand rule path from a, not taking the edges already considered. If this walk also reaches b, then part of the edges in the walk should form the path P_2 which is the second rightmost path from a to b. It means that for all paths P and Q that don't share edges and Q lies to the right of P, the path P_2 lies to the right of P.

Now T is a decision tree for the event $ab^{\square 2}$. If this event occurs, then we can continue T to the tree T' that runs the right-hand rule walk from a once again. Then T' is a decision tree for the event $ab^{\square 3}$. Now, from Theorem 3.5.2 we get

$$\mathbf{P}(C_1 \in ab^{\square 3}, C_1 \underset{S}{\rightarrow} C_2 \in ab^{\square 3}) \geq \frac{\mathbf{P}(ab^{\square 3})^2}{\mathbf{P}(ab^{\square 2})}. \tag{3.26}$$

Also, from Theorem 3.4.3, we get the other estimate. Indeed, if $C_1 \in ab^{\square 3}$ and $C_1 \rightarrow_S C_2 \in ab^{\square 3}$, then there are paths P_3 in $C_1|_{\overline{S}}$ that completes the triple of nonintersecting paths P_1, P_2 and P_3 in C_1 and P_3' in $C_2|_{\overline{S}}$ that completes the triple of nonintersecting paths P_1, P_2 and P_3' in $C_1 \rightarrow_S C_2$. So we have a pair of witnesses $(P_1 \cup P_3, P_2 \cup P_3')$ for the event $ab^{\square 2} \square_S ab^{\square 2}$. Now by Theorem 3.4.3 we get

$$\mathbf{P}(C_1 \in ab^{\square 3}, C_1 \underset{S}{\rightarrow} C_2 \in ab^{\square 3}) \leq \mathbf{P}(ab^{\square 2} \square_S ab^{\square 2}) \leq \mathbf{P}(ab^{\square 2})^2. \tag{3.27}$$

Combining equations (3.26) and (3.27), we get the needed (3.3).

For the infinite G, the result follows by standard limit arguments.

□

3.9.2 Generalization of Theorem 3.1.5

Theorem 3.9.1. *Let G be planar. Assume a and b belong to the same face, n is a natural number and $k, l, m \leq n$ are such that $k + l + m = 2n$. Then*

$$\mathbf{P}(ab^{\square n})^2 \leq \mathbf{P}(ab^{\square k})\mathbf{P}(ab^{\square l})\mathbf{P}(ab^{\square m}) \tag{3.28}$$

Proof. The proof is analogous to the previous one. Let T be a decision tree that runs k right-hand rule walks from a and puts the edges it meets in S. Then T is a decision tree for $ab^{\square k}$. By Theorem 3.5.2, we get

$$\mathbf{P}(C_1 \in ab^{\square n}, C_1 \underset{S}{\rightarrow} C_2 \in ab^{\square n}) \geq \frac{\mathbf{P}(ab^{\square n})^2}{\mathbf{P}(ab^{\square k})}. \tag{3.29}$$

On the other hand, if $C_1 \in ab^{\square n}$ and $C_1 \rightarrow_S C_2 \in ab^{\square n}$, then there are $n-k$ nonintersecting paths from a to b in $C_1|\overline{S}$ and other $n-k$ nonintersecting paths from a to b in $C_2|\overline{S}$. We add them to witnesses w_1, w_2 of $ab^{\square l} \square_S ab^{\square m}$. Now we split the k paths from S into $n-m$ and $n-l$ paths (we can do it since $n-m+n-l=k$) and add these paths to w_1 and w_2, respectively. Now this construction gives an estimate

$$\mathbf{P}(C_1 \in ab^{\square n}, C_1 \underset{S}{\rightarrow} C_2 \in ab^{\square n}) \leq \mathbf{P}(ab^{\square l} \square_S ab^{\square m}) \leq \mathbf{P}(ab^{\square l})\mathbf{P}(ab^{\square m}). \tag{3.30}$$

Combining equations (3.29) and (3.30), we get the needed (3.28). □

3.10 Open problems

Section 3.7 leaves some open questions. Despite Theorem 3.1.3, the more precise question remains open:

Conjecture 3.10.1. *For $\varepsilon > 0$, there exists $\delta > 0$, such that*

$$\mathbf{P}(ab|c) < \delta \implies \Big(\mathbf{P}(abc)\mathbf{P}(a|b|c) - \mathbf{P}(ac|b)\mathbf{P}(a|bc) < \varepsilon\Big).$$

Numerical simulations confirm this conjecture, which is as natural as could be.

We also propose a strengthening of the Conjecture 3.1.4 on the probabilities of $ab^{\Box k}$. Consider the example where G consists just of the vertices a and b connected via N edges (or disjoint paths, to keep G simple), each having a probability of $\frac{\lambda}{N}$. Then as $N \to \infty$, the distribution of the number of paths between a and b tends to the Poisson distribution with parameter λ, so we have $\mathbf{P}(ab^{\Box k}) \to \sum_{i=k}^{\infty} \frac{\lambda^i}{i! e^{\lambda}}$.

Conjecture 3.10.2. *For a given graph G we define the implied λ_k as the unique number such that*

$$\mathbf{P}(ab^{\Box k}) = \sum_{i=k}^{\infty} \frac{\lambda_k^i}{i! e^{\lambda_k}}.$$

We conjecture that $\{\lambda_k\}$ is a decreasing sequence.

Algorithm 1 Building Set S by Decision Tree T

```
procedure BuildSet(T, C_1, C_2)
    S ← ∅
    N ← N_0                          ▷ Start from the root node N_0 of the decision tree T
    while N is a decision node do
        e ← edge queried by N
        if decision of N is "S" then
            S ← S ∪ {e}
        end if
        if C_1(e) = 1 and C_2(e) = 1 then
            N ← N_11                 ▷ Both configurations have edge e open
        else if C_1(e) = 1 and C_2(e) = 0 then
            N ← N_10                 ▷ Configuration C_1 has edge e open and C_2 has it closed
        else if C_1(e) = 0 and C_2(e) = 1 then
            N ← N_01                 ▷ Configuration C_1 has edge e closed and C_2 has it open
        else
            N ← N_00                 ▷ Both configurations have edge e closed
        end if
    end while
    return S
end procedure
```

Algorithm 2 DFS Decision Tree

```
procedure DFS_Decision_tree(G, x, &visited_e, &S, order_decider, decision)    ▷
Graph G, source vertex x, reference to list of visited edges visited_e, reference to set S, function
order_decider deciding the order of neighbors, function decision outputting D ∈ {"S", "S̄"}
    visited_v ← array of size |V(G)| initialized with false
    stack ← empty stack
    Push(stack, x)
    while stack is not empty do
        v ← Pop(stack)
        if visited_v[v] is false then
            visited_v[v] ← true
            neighbors ← order_decider({neighbors of v}, v, stack, visited_e, visited_v)
            for each u ∈ neighbors do
                if visited_e[vu] is false then
                    Put vu in decision(vu, v, stack, visited_e, visited_v)    ▷ Decision node
                    if visited[u] is false then
                        Push(stack, u)
                    end if
                end if
            end for
        end if
    end while
end procedure
```

CHAPTER 4

The bunkbed conjecture is false

4.1 Introduction

The *bunkbed conjecture* (BBC) is a celebrated open problem in probability introduced by Kasteleyn in 1985, see [BK01, Remark 5]. The conjecture is both natural and intuitively obvious, but has defied repeated proof attempts; it is known only in a few special cases. In this paper we disprove the conjecture without resorting to computer experiments (cf. Section 4.7).

Let $G = (V, E)$ be a connected graph, possibly infinite and with multiple edges. In *Bernoulli bond percolation*, each edge is *deleted* independently at random with probability $1 - p$, and otherwise *retained* with probability $p \in [0, 1]$. Equivalently, this model gives a random subgraph of G weighted by the number of edges. For $p = \frac{1}{2}$ we obtain a uniform random subgraph of G. See [BR06, Gri99] for standard results and [D18a, Wer09] for recent overview of percolation.

Let $\mathbf{P}_p[u \leftrightarrow v]$ denote the probability that vertices $u, v \in V$ are connected. It is often of interest to compare these probabilities, as computing them exactly is #P-hard [PB83]. For example, the classical *Harris–Kleitman inequality*, a special case of the *FKG inequality*, implies that $\mathbf{P}_p[u \leftrightarrow v] \leq \mathbf{P}_p[u \leftrightarrow v \,|\, u \leftrightarrow w]$ for all $u, v, w \in V$, see e.g. [AS16, Ch. 6]. Harris used this to prove that the *critical probability* $p_c(G) := \inf\{p : \mathbf{P}_p(G) > 0\}$ satisfies $p_c(\mathbb{Z}^2) \geq \frac{1}{2}$ [H60], in the first step towards Kesten's remarkable exact value $p_c(\mathbb{Z}^2) = \frac{1}{2}$ [Kes80], where $\mathbf{P}_p(G)$ denotes the probability that there exists an infinite percolation cluster.

Considerations of percolation monotonicity on $\mathbb{Z}^2$ (see §4.8.4), led Kasteleyn to the following problem.

Fix a finite connected graph $G = (V, E)$ and a subset $T \subseteq V$. A *bunkbed graph* $\overline{G} = (\overline{V}, \overline{E})$ is a subgraph of the graph product $G \times K_2$ defined as follows. Take two copies of G, which we denote G and $G' = (V', E')$, and add all edges of the form (w, w'), where $w \in T$ and w' is a corresponding vertex in T'; we denote this set of edges by $\overline{T}$. The resulting bunkbed graph has $\overline{V} = V \cup V'$ and $\overline{E} = E \cup E' \cup \overline{T}$.

In the *bunkbed percolation*, the usual bond percolation is performed only on edges in G and G', while all edges in $\overline{T}$ are retained (i.e., not deleted). We use $\mathbf{P}_p^{\mathrm{bb}}[u \leftrightarrow v]$ to denote the connecting probability in this case. The vertices in T are called *transversal* and the edges in $\overline{T}$ are called *posts*, to indicate their special status. See e.g. [Lin11, RS16], for these and several other equivalent models of the bunkbed percolation. We refer also to [Gri23, §4.1], [Pak22, §5.5] and [Rud21] for recent overviews and connections to other areas.

Conjecture 4.1.1 (*bunkbed conjecture*)**.** Let $G = (V, E)$ be a connected graph, let $T \subseteq V$, and let $0 < p < 1$. Then, for all $u, v \in V$, we have:

$$\mathbf{P}_p^{bb}[u \leftrightarrow v] \geq \mathbf{P}_p^{bb}[u \leftrightarrow v'].$$

The bunkbed conjecture is known in a number of special cases, including wheels [Lea09], complete graphs [dB16, dB18, HL19], complete bipartite graphs [Ric22], and graphs symmetric w.r.t. the $u \leftrightarrow v$ automorphism [Ric22]. It is also known for one [Lin11, Lemma 2.4] and for two transversal vertices (Theorem 4.9.3, see also [Lohr18, §6.3]). Finally, the conjecture was recently proved in the $p \uparrow 1$ limit [HNK23, Hol24a].

Theorem 2. *There is a connected planar graph $G = (V, E)$ with $|V| = 7222$ vertices and $|E| = 14442$ edges, a subset $T \subset V$ with three transversal vertices, and vertices $u, v \in V$, such that*

$$\mathbf{P}_{\frac{1}{2}}^{\mathrm{bb}}[u \leftrightarrow v] < \mathbf{P}_{\frac{1}{2}}^{\mathrm{bb}}[u \leftrightarrow v'].$$

In particular, the bunkbed conjecture is false.

The result is surprising since analogous inequalities for simple random walks and for the Ising model on bunkbed graphs were proved by Häggström [Häg98, Häg03], cf. §4.8.5. Recall that three is the smallest number of transversal vertices we can have to disprove the conjecture. On the other hand, the total number of vertices is unlikely to be optimal, see Remark 4.4.2 and Section 4.7.

The proof of the theorem is based on an example of Hollom [Hol24b] refuting the 3-uniform hypergraph version of the BBC. Unfortunately, Hollom's example alone cannot disprove the conjecture since it is impossible to find a gadget graph simulating a single 3-hyperedge using bond percolation [GZ24, Thm 1.5].

We give a robust version of Hollom's construction using the approach in [G24a, GZ24]. The proof of Theorem 2 occupies most of the paper. It is self-contained modulo Hollom's result which is small enough to be checked by hand. In Section 4.6, we extend the theorem to the case when the set of transversal vertices is not fixed but chosen uniformly at random from V, see Theorem 3. We conclude with discussion of our computer experiments in Section 4.7, and final remarks in Section 4.8.

4.2 Notation

In percolation, deleted edges are called *closed* while retained edges are called *open*. Note that there are several different models of percolation and variations on the bunkbed conjecture (BBC), see §4.8.1.

A hypergraph is a collection of subsets of vertices; to simplify the notation we use the same letter to denote both. The hypergraph is called *uniform* if all hyperedges have the same size. A *path* in a hypergraph is a sequence $(v_0 \to v_1 \to \ldots \to v_\ell)$ of vertices, such that v_{i-1}, v_i lie in the same hyperedge, for all $1 \leq i \leq \ell$. We say that two vertices in a hypergraph are *connected* if there is there is a path between them. For further definitions and results on hypergraphs, see e.g. [Ber89, §1.2].

The notion of *hypergraph percolation* is is a natural extension of graph percolation, and goes back to the study of random hypergraphs, see e.g. [SPS85]. In recent years, the study of hypergraph percolation also emerged in probabilistic and statistical physics literature, see e.g. [WZ11] and [BD24], respectively.

4.3 Hypergraph percolation

4.3.1 Hollom's example

Let H be a finite connected hypergraph on the set V of vertices. We use $\mathbf{P}_p[u \leftrightarrow v]$ to denote probability of connectivity of vertices $u, v \in V$ in the *hypergraph percolation*, where each hyperedge e in H is retained with probability p, or deleted with probability $1-p$.

Let $T \subseteq V$ be the set of transversal vertices. Denote by $\overline{H}$ be the *bunkbed hypergraph* with levels $H \simeq H'$, and *vertical posts* which are the (usual) edges. Note that $\overline{H}$ has *horizontal* hyperedges and vertical posts.

In [Hol24b], Hollom considers the following natural hypergraph generalization of the *Alternative BBC*, see §4.8.1. In the *alternative bunkbed hypergraph percolation*, each hyperedge e in H is either deleted while the corresponding hyperedge e' in H' is retained with probability $\frac{1}{2}$, or vice versa: the hyperedge e is retained and e' is deleted.

Lemma 4.3.1 (Hollom [Hol24b, Claim 5.1]). *Let H be the hypergraph with six 3-edges as in the Figure 4.1, and let $T = \{u_2, u_7, u_9\}$ be the set of transversal vertices. In the alternative bunkbed hypergraph percolation, we have:*

$$\mathbf{P}^{\text{alt}}[u_1 \leftrightarrow u'_{10}] = \frac{13}{64} \quad \textit{and} \quad \mathbf{P}^{\text{alt}}[u_1 \leftrightarrow u_{10}] = \frac{12}{64}\,.$$

We give a robust version of Hollom's construction.

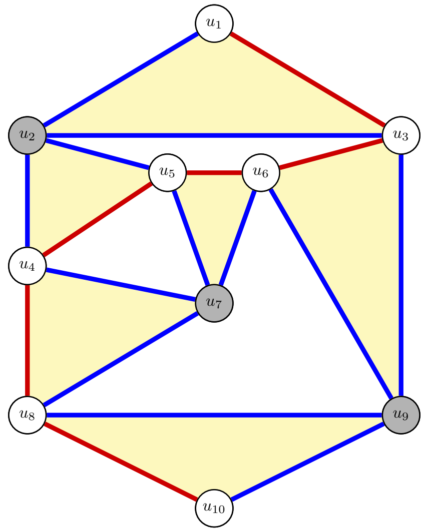

Figure 4.1: Hollom's 3-uniform hypergraph H.

4.3.2 Robust hyperedge lemma

Note that in Hollom's example, each hyperedge has exactly one transversal vertex. We explore this structure.

Consider the following *WZ hypergraph percolation model* introduced by Wierman and Ziff in [WZ11] (see also [GZ24]). We define this model only for the graph $\overline{H}$. Let $e = (a, b, c)$ be a hyperedge where a is a transversal vertex. We will fix the order of vertices in each hyperedge precisely in (4.5). In the model, hyperedge e is set to have

- probability p_{abc} to connect all three vertices,
- probability $p_{a|b|c}$ to not connect any of the vertices,
- probability $p_{a|bc}$ to connect two non-transversal vertices, and
- probability $p_{ab|c} = p_{ac|b}$ to connect a transversal to a nontransversal vertex,

and these events are independent on all hyperedges.

Finally, we assume that these five probabilities sum up to 1:

$$p_{abc} + p_{a|b|c} + p_{a|bc} + p_{ab|c} + p_{ac|b} = 1.$$

It is easy to see that the hypergraph percolation on $\overline{H}$ is a partial case of the WZ model, where $p_{abc} = p$ and $p_{a|b|c} = 1 - p$.

Definition 4.3.2 (Configurations in the WZ model)**.** A *configuration* in the WZ model is an assignment of one of the five states $\{abc,\ a|b|c,\ a|bc,\ ab|c,\ ac|b\}$ to each hyperedge in $H \cup H'$. Equivalently, we can represent it by a function

$$\psi : H \cup H' \to \Upsilon \quad \text{where} \quad \Upsilon = \{abc, ab|c, ac|b, a|bc, a|b|c\}.$$

The probability of a configuration ψ is given by

$$\mathbf{P}(\psi) = \prod_{e \in H \cup H'} p_{\psi(e)},$$

where p_α denotes the probability of the state $\alpha \in \Upsilon$.

We say that vertices u and v are *connected* (written $u \leftrightarrow v$) if they are connected by a path in the bunkbed hypergraph $\overline{H}$ in a way that every two vertices on a hyperedge are connected by the rules above. We use $\mathbf{P}^{\text{wz}}[\, u \leftrightarrow v\,]$ to denote these connection probabilities, omitting the superscript if the model is clear from context.

Lemma 4.3.3. *Let H be Hollom's hypergraph in the Figure 4.1, $\overline{H}$ be the bunkbed hypergraph built on it, and let $T = \{u_2, u_7, u_9\}$ be the set of transversal vertices. Consider the WZ hypergraph percolation as described above, where the connection probabilities satisfy*

$$400\, p_{a|bc} \,\le\, p_{abc}\, p_{a|b|c} \,-\, p_{ab|c} p_{ac|b}\,. \tag{4.1}$$

Then we have:

$$\mathbf{P}^{\text{wz}}(u_1 \leftrightarrow u_{10}) \,<\, \mathbf{P}^{\text{wz}}(u_1 \leftrightarrow u'_{10})\,. \tag{4.2}$$

It was noted in [G24a, Cor. 3.6], that the RHS in (4.1) is nonnegative if the hyperedge is simulated by a gadget in Bernoulli edge percolation:

$$p_{ab|c}\, p_{ac|b} \,=\, p^2_{ab|c} \,\le\, p_{abc}\, p_{a|b|c}\,. \tag{4.3}$$

This is a consequence of the Harris–Kleitman (HK) inequality. In fact, a slightly stronger inequality always holds (ibid.) Since the LHS in (4.1) is nonnegative, one can view this assumption as strengthening the HK inequality in this case (cf. §4.8.2). Also, it is easy to see that the hypergraph model satisfies the condition (4.1), since two of these terms become zero.

4.3.3 Refining the State Space

Let $\mathcal{C}$ be the set of configurations ψ that contain a path $u_1 \leftrightarrow u_{10}$, and let $\mathcal{C}'$ be the set of those containing a path $u_1 \leftrightarrow u'_{10}$. The probabilities of sets $\mathcal{C}$ and $\mathcal{C}'$ are given by

$$\mathbf{P}(\mathcal{C}) \;:=\; \sum_{\psi\in\mathcal{C}} \mathbf{P}(\psi) \quad \text{and} \quad \mathbf{P}(\mathcal{C}') \;:=\; \sum_{\psi\in\mathcal{C}'} \mathbf{P}(\psi).$$

Our goal is to prove $\mathbf{P}(\mathcal{C}) < \mathbf{P}(\mathcal{C}')$ by constructing a suitable map from $\mathcal{C}$ to $\mathcal{C}'$.

Since each hyperedge in $\overline{H}$ has a symmetric counterpart, one can view a configuration not as a function $\psi : \overline{H} \to \Upsilon$, but as a function from H to Υ^2. For each hyperedge e in H, there are 25 possibilities for a pair $(\psi(e), \psi(e'))$. To handle certain configurations more precisely, we refine the pairs $(abc, a|b|c)$ and $(a|b|c, abc)$ by splitting each into two disjoint sub-events:

$$(abc, a|b|c) \mapsto (abc, a|b|c)_+ \cup (abc, a|b|c)_-, \quad (a|b|c, abc) \mapsto (a|b|c, abc)_+ \cup (a|b|c, abc)_-.$$

The probabilities (or *weights*) of these refined events are given by:

$$\mathbf{P}\big[(abc, a|b|c)_+\big] \;=\; p_{ab|c} \cdot p_{ac|b}\,, \quad \mathbf{P}\big[(abc, a|b|c)_-\big] \;=\; p_{abc} \cdot p_{a|b|c} - p_{ab|c} \cdot p_{ac|b}\,, \tag{4.4}$$

and similarly for $(a|b|c, abc)_+$ and $(a|b|c, abc)_-$. This refinement increases the total number of possible pairs in Υ^2 from 25 to 27, resulting in the *extended state space*

$$\Upsilon^2_{\text{ext}} \;:=\; \Upsilon^2 \smallsetminus \big\{(a|b|c, abc), (abc, a|b|c)\big\} \cup \big\{(abc, a|b|c)_+, (abc, a|b|c)_-, (abc, a|b|c)_+, (abc, a|b|c)_-\big\}.$$

In the original model, $\psi(e)$ and $\psi(e')$ were sampled independently from Υ. After the refinement, however, we consider a new framework where by configuration we mean a function

$\Psi : H \to \Upsilon^2_{\text{ext}}$. In the new model, the state of a pair (e, e') is sampled directly from Υ^2_{ext} independently with probabilities given by equation (4.4) for refined pairs and $\mathbf{P}\big[(\alpha, \beta)\big] = p_\alpha p_\beta$ for non-refined pairs. Then the states $\psi(e)$ and $\psi(e')$ are defined by $\Psi(e)$.

Building on the refinement of Υ^2 to Υ^2_{ext}, we now focus on a particularly important subset of states. Specifically, we consider a smaller set of interest:

$$\Lambda := \{abc, ab|c, ac|b, a|b|c\}.$$

The Cartesian product $\Lambda^2 := \Lambda \times \Lambda$ consists of all ordered pairs of states in Λ, giving $4 \times 4 = 16$ elements. To incorporate the refined structure introduced earlier, we replace the pairs $(abc, a|b|c)$ and $(a|b|c, abc)$ in Λ^2 with their "$_+$" counterparts.

Definition 4.3.4 (Refined Pair Set Λ^2_+)**.** The *refined set of pairs* Λ^2_+ is defined as:

$$\Lambda^2_+ := \big\{(\alpha, \beta) \in \Lambda^2 : (\alpha, \beta) \notin \{(abc, a|b|c), (a|b|c, abc)\}\big\} \cup \big\{(abc, a|b|c)_+, (a|b|c, abc)_+\big\} \subset \Upsilon^2_{\text{ext}}\,.$$

4.3.4 Involutions on Extended State Spaces

To proceed with the construction of a map from $\mathcal{C}$ to $\mathcal{C}'$, we define two weight-preserving involutions on the extended state space Υ^2_{ext}.

Definition 4.3.5 (Reflection Involution)**.** The *reflection involution* $\mathcal{R}$ is defined on the extended state space Υ^2_{ext}. For a pair $\Psi(e) \in \Upsilon^2_{\text{ext}}$, it swaps the states of e and e', and is formally given by:

$$\mathcal{R}\big((\alpha, \beta)\big) := (\beta, \alpha).$$

Additionally, for refined states, the reflection involution $\mathcal{R}$ is defined to preserve the sign.

The reflection involution $\mathcal{R}$ is weight-preserving because the weight of each configuration is symmetric under the swapping of $\psi(e)$ and $\psi(e')$. While $\mathcal{R}$ works by simply swapping states, making it straightforward to handle symmetry, the half-reflection involution η requires a more detailed approach. It is constructed to modify vertex connections as described in Proposition 4.3.7, while preserving weights.

Partition the set of pairs $\Lambda_+^2 = \Omega_0 \cup \Omega_1 \cup \Xi$ into the following three subsets:

$$\Omega_0 := \left\{ \begin{matrix} (abc, ac|b),\ (ac|b, abc),\ (a|b|c, ab|c),\ (ab|c, a|b|c), \\ (abc, abc),\ (a|b|c, a|b|c),\ (ab|c, ab|c),\ (ac|b, ac|b) \end{matrix} \right\},$$

$$\Omega_1 := \big\{(abc, ab|c), (ab|c, abc), (a|b|c, ac|b), (ac|b, a|b|c)\big\}, \text{ and}$$

$$\Xi := \big\{(abc, a|b|c)_+, (a|b|c, abc)_+, (ab|c, ac|b), (ac|b, ab|c)\big\}.$$

Definition 4.3.6 (Half-Reflection Involution)**.** The *half-reflection involution* η is defined on the set Λ_+^2 as follows:

- On Ω_0, the involution η is the identity map.
- On Ω_1, the half-reflection coincides with the reflection involution $\mathcal{R}$ defined earlier.
- On Ξ, the half-reflection involution η is given by:

$$\eta\big((abc, a|b|c)_+\big) := (ab|c, ac|b), \quad \eta\big((ab|c, ac|b)\big) := (abc, a|b|c)_+,$$

$$\eta\big((a|b|c, abc)_+\big) := (ac|b, ab|c), \quad \eta\big((ac|b, ab|c)\big) := (a|b|c, abc)_+.$$

This involution is weight-preserving because it satisfies the following conditions:

- ◦ On Ω_0, the involution is constant, making it trivially weight-preserving.
- ◦ On Ω_1, the involution coincides with the reflection involution $\mathcal{R}$, which has already been shown to preserve weights.
- ◦ On Ξ, the involution swaps pairs in such a way that the probabilities remain balanced. Specifically, since

$$\mathbf{P}\big((abc, a|b|c)_+\big) = \mathbf{P}\big((a|b|c, abc)_+\big) = p_{ab|c} \cdot p_{ac|b},$$

swapping $(abc, a|b|c)_+$ with $(ab|c, ac|b)$, and $(a|b|c, abc)_+$ with $(ac|b, ab|c)$, does not alter the total probability.

After defining the half-reflection involution η, we examine how it modifies connectivity in configurations. The following proposition describes the effect of η on states in Λ^2_+.

Proposition 4.3.7. *Let Ψ be a configuration and $e = (a, b, c) \in H$ such that $\Psi(e) \in \Lambda^2_+$. Let Ψ' be any configuration such that*

$$\Psi'(e) = \eta(\Psi(e)).$$

Then, the following properties hold:

i. If b and c are connected in Ψ within e, then b and c' are connected in Ψ'. Similarly, if b' and c' are connected in Ψ within e', then b' and c are connected in Ψ'.

ii. If a and b are connected in Ψ within e, then they remain connected in Ψ'. Similarly, if a' and b' are connected in Ψ within e', then they remain connected in Ψ'.

iii. If a and c are connected in Ψ within e, then a' and c' are connected in Ψ'. Similarly, if a' and c' are connected in Ψ within e', then a and c are connected in Ψ'.

Proof. We will prove only the first part of each statement. The second part follows directly from the symmetry of η. In particular, the relation

$$\eta\big(\mathcal{R}\big(\Psi(e)\big)\big) = \mathcal{R}\big(\eta\big(\Psi(e)\big)\big)$$

guarantees that the roles of e and e' are interchangeable under η.

For *(i)*, suppose b and c are connected in Ψ within e, which implies $\psi(e) = abc$. In Ψ', we claim there exists a path $b \to a \to a' \to c'$. This holds if:

- The first component of $\eta\big(\Psi(e)\big)$ belongs to $\{abc, ab|c\}$, and
- The second component of $\eta\big(\Psi(e)\big)$ belongs to $\{abc, ac|b\}$,

whenever $\psi(e) = abc$. These conditions follow directly from the definition of η and its action on Ω_0, Ω_1, and Ξ.

For *(ii)*, assume a and b are connected in Ψ within e, which implies $\psi(e) \in \{ab|c, abc\}$. In Ψ', we verify that a and b remain connected within e. This requires that the first component of $\eta\big(\Psi(e)\big)$ belongs to $\{ab|c, abc\}$, whenever $\psi(e) \in \{ab|c, abc\}$. Again, this follows from the definition of η.

For *(iii)*, suppose a and c are connected in Ψ within e, which implies $\psi(e) \in \{ac|b, abc\}$. In Ψ', we claim a' and c' are connected within e'. This is satisfied if the second component of $\eta\big(\Psi(e)\big)$ belongs to $\{ac|b, abc\}$, whenever $\psi(e) \in \{ac|b, abc\}$. The result follows directly from the definition of η.

$\square$

4.3.5 Proof of Lemma 4.3.3

We define the subset of configurations $\mathcal{X}$ as:

$$\mathcal{X} := \big\{\Psi : \Psi(e) \in \Lambda_+^2 \text{ for some } e \in H\big\}.$$

Our goal is to construct a weight-preserving involution $\phi : \mathcal{X} \to \mathcal{X}$, which satisfies:

$$\Psi \in \mathcal{C} \implies \phi(\Psi) \in \mathcal{C}', \quad \text{and} \quad \Psi \in \mathcal{C}' \implies \phi(\Psi) \in \mathcal{C}.$$

To define ϕ, we begin by introducing the *red path* ρ from u_1 to u_{10}, as shown in Figure 4.1. Observe that ρ traverses every hyperedge exactly once and avoids transversal vertices. Fix the order on the hyperedges of H according to their appearance along the path ρ:

$$(u_2, u_1, u_3),\ (u_9, u_3, u_6),\ (u_7, u_6, u_5),\ (u_2, u_5, u_4),\ (u_7, u_4, u_8),\ (u_9, u_8, u_{10}). \tag{4.5}$$

This notation also establishes a fixed ordering for the vertices within each hyperedge. Specifically, if a hyperedge $e = (a, b, c)$ corresponds to an entry (u_i, u_j, u_k) in the sequence above, then the vertices of e are assigned as $a = u_i$, $b = u_j$, and $c = u_k$, preserving the order within each tuple. In particular, the first vertex $a = u_i$ in each tuple is a transversal vertex.

The map $\phi : \mathcal{X} \to \mathcal{X}$ is defined as follows. Let $e = (a, b, c)$ be the first hyperedge along ρ such that $\Psi(e) \in \Lambda_+^2$. The configuration $\Psi' = \phi(\Psi)$ is constructed according to the following rule:

$$\Psi'(h) = \begin{cases} \Psi(h) & \text{if } h \text{ appears before } e \text{ along } \rho, \\ \eta\big(\Psi(h)\big) & \text{if } h = e, \\ \mathcal{R}\big(\Psi(h)\big) & \text{if } h \text{ appears after } e \text{ along } \rho. \end{cases}$$

Since both η and $\mathcal{R}$ are weight-preserving by their respective definitions, ϕ is also a weight-preserving involution.

Next, we establish that ϕ maps configurations in $\mathcal{C}$ to configurations in $\mathcal{C}'$, and vice versa.

Lemma 4.3.8. *Let $\Psi \in \mathcal{X} \cap \mathcal{C}$, and let Ψ' be the configuration obtained by applying the involution ϕ to Ψ. Then, $\Psi' \in \mathcal{C}'$. Conversely, if $\Psi \in \mathcal{X} \cap \mathcal{C}'$, then $\Psi' \in \mathcal{C}$.*

Proof. We have $V = \{u_1, \ldots, u_{10}\}$ and $T = \{u_2, u_7, u_9\}$. Let $L \subseteq (V \setminus T) \cup (V' \setminus T')$ denote the set of nontransversal vertices that lie along the path ρ between u_1 and b, inclusively, along with their counterparts from the other level. Similarly, let $R := (V \setminus T) \cup (V' \setminus T') \setminus L$. For any path γ in Ψ, we construct a corresponding path γ' in Ψ' as follows. For all vertices $u_i \in R$ on γ, replace u_i with its counterpart u_i' in γ', and vice versa: for all $u_i' \in R$ on γ, replace u_i' with u_i in γ'.

To confirm that γ' is a connected path in Ψ', consider two sequential vertices x_k and x_{k+1} in γ and their corresponding images y_k and y_{k+1} in γ'. We analyze the connectivity in the following cases:

- **Transversal edge:** If $x_k x_{k+1}$ is a transversal edge in γ, then it remains unchanged under ϕ. The corresponding edge $y_k y_{k+1}$ in γ' is also transversal, ensuring connectivity.

- **Hyperedge before e:** If x_k and x_{k+1} are connected in Ψ through a hyperedge h before e in ρ, then $y_k = x_k$, $y_{k+1} = x_{k+1}$, and $\Psi'(h) = \Psi(h)$. The connection is preserved in γ'.

- **Hyperedge after e:** If x_k and x_{k+1} are connected in Ψ through a hyperedge h after e in ρ, then $y_k = x'_k$, $y_{k+1} = x'_{k+1}$, where the prime indicates the symmetric component (not necessarily in H'). Since ϕ applies the reflection involution $\mathcal{R}$ to hyperedges after e, we have $\Psi'(h') = \Psi(h)$, ensuring that y_k and y_{k+1} remain connected in Ψ'.

- **Hyperedge e:** If x_k and x_{k+1} are connected in Ψ through the hyperedge e, Proposition 4.3.7 ensures that the connectivity is appropriately modified in Ψ'.

Thus, for every sequential pair of vertices x_k, x_{k+1} in γ, their images y_k, y_{k+1} in γ' are connected in Ψ'. Moreover, the mapping constructed to transform γ into γ' always maps u_1 to itself and u_{10} to u'_{10}. Consequently, if γ connects u_1 to u_{10}, then γ' connects u_1 to u'_{10}, and vice versa. This completes the proof. □

With ϕ established as a weight-preserving involution on $\mathcal{X}$ and Lemma 4.3.8 proving that ϕ maps $\mathcal{C} \cap \mathcal{X}$ to $\mathcal{C}' \cap \mathcal{X}$ and vice versa, it follows that:

$$\mathbf{P}(\mathcal{C} \cap \mathcal{X}) = \mathbf{P}(\mathcal{C}' \cap \mathcal{X}).$$

This equivalence allows us to focus on the complementary subset of configurations, $\mathcal{X}^c$, which consists of all Ψ such that for every hyperedge $e \in H$, the pair $\Psi(e)$ belongs to $\Upsilon^2_{\text{ext}} \setminus \Lambda^2_+$. Explicitly, the set $\Upsilon^2_{\text{ext}} \setminus \Lambda^2_+$ is described by the following pairs:

$$\begin{aligned} &(abc, a|b|c)_-, \quad (a|b|c, abc)_- \quad \text{with probability } p_{abc} \cdot p_{a|b|c} - p_{ac|b} \cdot p_{ab|c}, \\ &(a|bc, *), \quad (*, a|bc), \quad \text{and} \quad (a|bc, a|bc), \quad \text{where } * \in \Lambda. \end{aligned}$$

In this setting, the WZ hypergraph percolation model conditioned on $\mathcal{X}^c$ has the following probabilities for each remaining possible value of $\Psi(e)$:

$$\begin{cases} (abc, a|b|c)_-, \text{ with probability } \frac{1}{Z}\big(p_{abc}p_{a|b|c} - p_{ab|c}p_{ac|b}\big), \\ (a|b|c, abc)_-, \text{ with probability } \frac{1}{Z}\big(p_{abc}p_{a|b|c} - p_{ab|c}p_{ac|b}\big), \\ (a|bc, *), \text{ with probability } \frac{1}{Z}p_{a|bc} \cdot p_*, \quad \text{for } * \in \Lambda, \\ (*, a|bc), \text{ with probability } \frac{1}{Z}p_{a|bc} \cdot p_*, \quad \text{for } * \in \Lambda, \\ (a|bc, a|bc), \text{ with probability } \frac{1}{Z}p^2_{a|bc}, \end{cases}$$

where the normalizing constant is:

$$Z := 2\, p_{abc} \cdot p_{a|b|c} - 2\, p_{ab|c} \cdot p_{ac|b} + 2\, p_{a|bc} - p^2_{a|bc}.$$

We denote the corresponding conditional probabilities by $\mathbf{P}_{\mathcal{X}^c}$. This notation emphasizes the restriction to the subset $\mathcal{X}^c$, making the context of these probabilities explicit.

Denote by $\mathcal{A}$ the subset of events that for all $e \in H$, the value of $\Psi(e)$ belongs to $\{(abc, a|b|c)_-, (a|b|c, abc)_-\}$. Using the inequality $(1-x)^a \geq 1 - ax$ and the assumption (4.1) from the lemma, we compute $\mathbf{P}_{\mathcal{X}^c}(\mathcal{A})$ as follows:

$$\begin{aligned} \mathbf{P}_{\mathcal{X}^c}(\mathcal{A}) &= \left(1 - \frac{2p_{a|bc} - p^2_{a|bc}}{2(p_{abc}\, p_{a|b|c} - p_{ab|c}\, p_{ac|b}) + 2p_{a|bc} - p^2_{a|bc}}\right)^6 \\ &\geq \left(1 - \frac{p_{a|bc}}{(p_{abc}\, p_{a|b|c} - p_{ab|c}\, p_{ac|b}) + p_{a|bc}}\right)^6 \\ &\geq 1 - \frac{6p_{a|bc}}{(p_{abc}p_{a|b|c} - p_{ab|c}\, p_{ac|b}) + p_{a|bc}} >_{(4.1)} 1 - \frac{6p_{a|bc}}{401p_{a|bc}} > \frac{64}{65}. \end{aligned}$$

Conditioning on $\mathcal{A}$ effectively transforms the WZ model into the alternative bunkbed hypergraph percolation model. By Hollom's result (Lemma 4.3.1), we have:

$$\begin{aligned} \mathbf{P}_{\mathcal{X}^c}(u_1 \leftrightarrow u_{10} \mid \mathcal{A}) - \mathbf{P}_{\mathcal{X}^c}(u_1 \leftrightarrow u'_{10} \mid \mathcal{A}) = \mathbf{P}^{\text{alt}}(u_1 \leftrightarrow u_{10}) - \mathbf{P}^{\text{alt}}(u_1 \leftrightarrow u_{10}) \\ =_{\text{Lemma 4.3.1}} \tfrac{12}{64} - \tfrac{13}{64} = -\tfrac{1}{64}. \end{aligned}$$

Now, we combine these results:

$$\begin{aligned}\mathbf{P}_{\mathcal{X}^c}(u_1 \leftrightarrow u_{10}) - \mathbf{P}_{\mathcal{X}^c}(u_1 \leftrightarrow u'_{10}) \\ &\le \mathbf{P}_{\mathcal{X}^c}(\overline{\mathcal{A}}) + \mathbf{P}_{\mathcal{X}^c}(\mathcal{A})\Big(\mathbf{P}_{\mathcal{X}^c}(u_1 \leftrightarrow u_{10} \mid \mathcal{A}) - \mathbf{P}_{\mathcal{X}^c}(u_1 \leftrightarrow u'_{10} \mid \mathcal{A})\Big) \\ &< \tfrac{1}{65} - \tfrac{1}{64} \cdot \tfrac{64}{65} \;=\; 0.\end{aligned}$$

This completes the proof.

4.4 Disproof of the Bunkbed Conjecture

4.4.1 Hyperedge simulation

In this section, we construct a graph that simulates a hyperedge in the sense of WZ hypergraph percolation, adhering to the conditions of the Lemma 4.3.3. We prove the following technical result for the *weighted percolation*.

Lemma 4.4.1. *Let $n \ge 3$ and $0 < p < 1$. Consider a weighted graph G_n on $(n+1)$ vertices given in Figure 4.2. Denote $b := v_1$ and $c := v_n$. Then $p_{ab|c} = p_{ac|b}$ and*

$$p_{abc}\, p_{a|b|c} \,-\, p_{ab|c}\, p_{ac|b} \;>\; \big(n\, \tfrac{1-p}{1+p} \,-\, 1\big)\, p_{a|bc}\,, \tag{4.6}$$

where

$$p_{abc} \,:=\, \mathbf{P}_p[a \leftrightarrow b \leftrightarrow c], \quad p_{a|bc} \,:=\, \mathbf{P}_p[a \not\leftrightarrow b \leftrightarrow c], \quad p_{ab|c} \,:=\, \mathbf{P}_p[a \leftrightarrow b \not\leftrightarrow c],$$
$$p_{ac|b} \,:=\, \mathbf{P}_p[a \leftrightarrow c \not\leftrightarrow b] \quad \textit{and} \quad p_{a|b|c} \,:=\, \mathbf{P}_p[a \not\leftrightarrow b \not\leftrightarrow c \not\leftrightarrow a].$$

We prove the lemma in the next section, see Proposition 4.5.4.

4.4.2 Proof of Theorem 2

In notation of Lemma 4.3.3, let $p = \frac{1}{2}$ and let $n := 3 \cdot 401 + 1 = 1204$. The resulting graph G_n is planar, has 1205 vertices and 2407 edges.

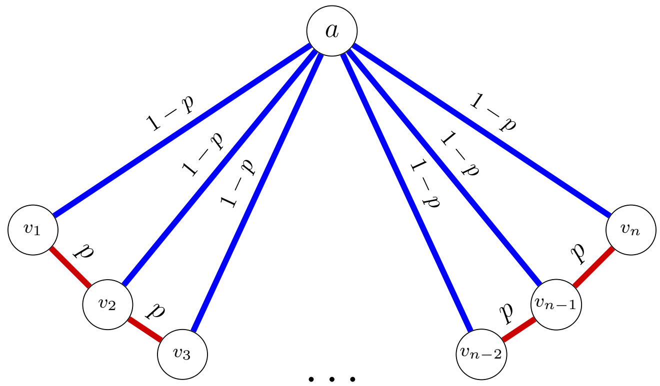

Figure 4.2: Graph G_n with $n+1$ vertices.

Take Hollom's hypergraph H from Figure 4.1 and substitute for each 3-hyperedge with a graph G_n from Lemma 4.4.1, placing it so a is a transversal vertex while $b = v_1$ and $c = v_n$ are the other two vertices. The resulting graph is still planar, has $10 + 6 \cdot 1202 = 7222$ vertices and $6 \cdot 2407 = 14442$ edges.

By Lemma 4.4.1, the $\frac{1}{2}$-percolation on G_n satisfies conditions of Lemma 4.3.3. Thus, by Lemma 4.3.3, we have:

$$\mathbf{P}(u_1 \leftrightarrow u_{10}) \ < \ \mathbf{P}(u_1 \leftrightarrow u'_{10}),$$

as desired. □

Remark 4.4.2. Due to the multiple conditionings and the gadget structure, the difference of probabilities given by the counterexample in Theorem 2 is less than 10^{-4331}, out of reach computationally. A computer-assisted computation shows that one can use G_n with $p = \frac{1}{2}$ and $n = 14$, giving a relatively small graph on 82 vertices. However, even in this case, the difference of the probabilities in the BBC is on the order 10^{-47}. This and other computations are collected on the author's website, see §4.8.2.

Since *Weighted BBC* is equivalent to BBC (see §4.8.1), once can instead take weighted graph G_n with $p = \frac{1}{2n}$ and $n = 402$. This graph is still too large to analyze experimentally. A computer-assisted computation shows that one can use G_n with $p = 0.0349$ and $n = 5$, giving a rather small graph on 28 vertices. However, even in this case, the difference of the probabilities in the Weighted BBC are on the order 10^{-78}.

4.5 Proof of Lemma 4.4.1

We prove the lemma as a consequence of elementary calculations.

Lemma 4.5.1. *We have:*

$$\mathbf{P}_p(a \leftrightarrow v_n) \;=\; \frac{1-p^{2n}}{1+p}\,.$$

Proof. Let $p_n := \mathbf{P}_p(a \leftrightarrow v_n)$ as in the lemma. We establish a recurrence relation for p_n. There are two cases:

(1) The edge (a, v_n) is open. This occurs with probability $1-p$. In this case, vertices a and v_n are directly connected.

(2) The edge (a, v_n) is closed. This occurs with probability p. In this case, vertex v_n can only connect to a through the edge (v_{n-1}, v_n), which is open with probability p. If this edge is closed, vertex v_n is isolated from a. If it is open, the probability that a and v_{n-1} are in the same connected component is p_{n-1}.

Combining these cases, we obtain the following recurrence relation:

$$p_n \;=\; (1-p) \,+\, p^2 p_{n-1}\,,$$

with the initial condition $p_0 = 0$. The result follows by induction. □

Lemma 4.5.2. *We have:*

$$\mathbf{P}_p(a \leftrightarrow v_1 \leftrightarrow v_n) \;=\; \frac{1-p^{2n}}{(1+p)^2} \,+\, \frac{n(1-p)p^{2n-1}}{1+p}\,.$$

Proof. Let $p_n := \mathbf{P}_p(a \leftrightarrow v_1 \leftrightarrow v_n)$ denote the probability as in the lemma. We calculate this probability by analyzing whether edges (a, v_1) and (a, v_n) are open or closed. There are four cases:

(1) Both edges (a, v_1) and (a, v_n) are open, each with probability $1-p$. Then a is directly connected to both v_1 and v_n. Thus, the probability is $(1-p)^2$.

(2) Edge (a, v_n) is closed. If the edge (a, v_n) is closed, vertex v_n is connected to the rest of the graph through the edge (v_{n-1}, v_n), which is open with probability p. This reduce the problem to G_{n-1}. Thus, the probability is $p^2 p_{n-1}$.

(3) The edge (a, v_1) is closed. Similarly, if the edge (a, v_1) is closed (with probability p). Thus, the probability is $p^2 p_{n-1}$.

(4) Both edges (a, v_1) and (a, v_n) are closed. If both edges (a, v_1) and (a, v_n) are closed (each with probability p), v_1 must connect to v_2 by the edge (v_1, v_2), and v_n must connect to v_{n-1} by the edge (v_{n-1}, v_n). The problem reduces to finding the probability that a, $\widehat{u}_1 = v_2$, and $\widehat{u}_{n-2} = v_{n-1}$ are in the same connected component in the graph $\widehat{G}_{n-2}$, the restriction of G_n to the vertices $a, v_2, \ldots, v_{n-1}$. Thus, the corresponding probability is $p^4 p_{n-2}$.

Using inclusion-exclusion of these four cases, we obtain the following recurrence relation:

$$p_n \,=\, (1-p)^2 \,+\, 2p^2 p_{n-1} \,-\, p^4 p_{n-2},$$

with initial conditions $p_0 = 0$ and $p_1 = 1 - p$. The result follows by induction. $\square$

Lemma 4.5.3. *We have:*

$$\mathbf{P}_p(a \not\leftrightarrow v_1 \leftrightarrow v_n) \,=\, p^{2n-1}.$$

Proof. If the vertices v_1 and v_n are in the same connected component that does not contain vertex a, they must be connected by the path $\gamma := (v_1 \to v_2 \to \ldots \to v_n)$. The probability that this path is open is p^{n-1}. In addition, any edge (a, v_k) must be closed for all $1 \le k \le n$, as otherwise vertex a is connected to the path γ. The probability that all these edges are closed is p^n. Thus, the probability in the lemma is p^{2n-1}. $\square$

We conclude with the following result which immediately implies Lemma 4.4.1.

Proposition 4.5.4. *In notation of Lemma 4.4.1, we have* $p_{a|bc} = p^{2n-1}$ *and*

$$p_{abc}\, p_{a|b|c} \, - \, p_{ac|b}\, p_{ab|c} \, \geq \, \left(n\, \tfrac{1-p}{1+p} \, - \, 1\right) p^{2n-1}.$$

Proof. The first part is given by Lemma 4.5.3. For the second part, using Lemmas 4.5.1, 4.5.2 and 4.5.3 and $p_{abc} \leq 1$, we have:

$$\begin{aligned}
p_{abc}\, p_{a|b|c} \, - \, p_{ac|b}\, p_{ab|c} \; &= \; p_{abc} \, - \, (p_{abc} + p_{ab|c})(p_{abc} + p_{ac|b}) \, - \, p_{abc}\, p_{a|bc} \\
&= \; \mathbf{P}_p(a \leftrightarrow v_1 \leftrightarrow v_n) \, - \, \mathbf{P}_p(a \leftrightarrow v_1) \cdot \mathbf{P}_p(a \leftrightarrow v_n) \, - \, p_{abc}\, p_{a|bc} \\
&\geq \; \left(\tfrac{1-p^{2n}}{(1+p)^2} \, + \, \tfrac{n\,(1-p)\,p^{2n-1}}{1+p}\right) \, - \, \left(\tfrac{1-p^{2n}}{1+p}\right)^2 - p^{2n-1} \\
&\geq \; \tfrac{p^{2n}(1-p^{2n})}{(1+p)^2} \, + \, \tfrac{n\,(1-p)\,p^{2n-1}}{1+p} \, - \, p^{2n-1} \\
&\geq \; \left(\tfrac{n\,(1-p)}{1+p} - 1\right) p^{2n-1},
\end{aligned}$$

as desired. □

4.6 Complete BBC

In notation of the Bunkbed Conjecture 4.1.1, one can ask if a version of the BBC holds for uniform $T \subseteq V$. This is equivalent to $\frac{1}{2}$-percolation on the product graph $G \times K_2$. To distinguish from BBC, we call this *Complete BBC*, see §4.8.1. It turns out that the proof of Theorem 2 extends to the proof of Complete BBC, but a counterexample is a little larger:

Theorem 3. *There is a connected graph* $G = (V, E)$ *with* $|V|, |E| < 10^6$*, and vertices* $u, v \in V$*, such that for the* $\frac{1}{2}$*-percolation on* $G \times K_2$ *we have:*

$$\mathbf{P}_{\frac{1}{2}}[u \leftrightarrow v] \, < \, \mathbf{P}_{\frac{1}{2}}[u \leftrightarrow v'].$$

In particular, the complete bunkbed conjecture is false.

Proof. Recall that Hollom's Model 4.3 in [Hol24b] is the hypergraph version of the Complete BBC. Hollom disproves it in [Hol24b, §5.1] by showing that his 3-hypergraph in Figure 4.1

is minimal in a sense that bunkbed probabilities $\mathbf{P}[u \leftrightarrow v]$ and $\mathbf{P}[u \leftrightarrow v']$ are equal for all subsets $\{u_2, u_7, u_9\} \subset T \subseteq \{u_1, \ldots, u_{10}\}$. He then makes $k = 102$ "clones" of vertices $\{u_2, u_7, u_9\}$ to make sure at least one is always in the percolation cluster with high probability.

We notice that our counterexample has a similar minimal structure because of the form of the gadget used in its construction. The only path ρ from u_1 to u_{10} avoiding transversal vertices still passes through all nontransversal vertices. From this point on, proceed as in the proof of Theorem 2. In notation of the proof of Lemma 4.3.3, we have that the only two ways we can have a nonzero probability gap is if one of the vertices $\{u_2, u_7, u_9\}$ is not in T or all the vertices along the red path ρ are not in T.

Now consider the difference of probabilities $\delta := \mathbf{P}(u_1 \leftrightarrow u_{10}) - \mathbf{P}(u_1 \leftrightarrow u'_{10})$ for the graph G and $T = \{u_2, u_7, u_9\}$. Then for the graph G where T is a random subset containing $\{u_2, u_7, u_9\}$ one has $\mathbf{P}(u_1 \leftrightarrow u_{10}) \; - \; \mathbf{P}(u_1 \leftrightarrow u'_{10}) = \delta \cdot 2^{-|G|+3}$.

For each of the vertices $t \in \{u_2, u_7, u_9\}$ replace it with the gadget – add k additional vertices $w_{t,i}$ for $i \in [k]$ and connect then to t. This gadget imitates a single vertex t having a probability of being transversal increased from $\frac{1}{2}$ to $1 - \frac{1}{2}\left(\frac{7}{8}\right)^k$. Let $\mathcal{A}$ be the event that all imitated vertices are transversal. Then $\mathbf{P}(\mathcal{A}) \geq 1 - \frac{3}{2}\left(\frac{7}{8}\right)^k$. We have:

$$\mathbf{P}_{\frac{1}{2}}[u \leftrightarrow v] \; - \; \mathbf{P}_{\frac{1}{2}}[u \leftrightarrow v'] \leq 1 - \mathbf{P}(\mathcal{A}) + \mathbf{P}(\mathcal{A}) \cdot \delta \cdot 2^{-|G|+3} \leq \frac{3}{2}\left(\frac{7}{8}\right)^k + \frac{1}{2}\delta \cdot 2^{-|G|+3}.$$

This is negative if $\delta \cdot 2^{-|G|+3} < -3\left(\frac{7}{8}\right)^k$. It is obvious such k exists. We use the computer estimate from Remark 4.4.2 that $\delta < -10^{-4332}$ to say that this is true for $k \geq 112182$. Therefore, for the graph G' obtained from G by adding $3k = 336546$ vertices and edges, we have

$$\mathbf{P}_{\frac{1}{2}}[u \leftrightarrow v] \; < \; \mathbf{P}_{\frac{1}{2}}[u \leftrightarrow v'],$$

as desired. □

4.7 Experimental testing

Versions of the bunkbed conjecture were repeatedly tested by various researchers, although failed attempts remain largely unreported, see e.g. [Rud21, §3.1]. In this section we describe our own attempt to refute the conjecture using a large scale computer computation.

4.7.1 Initial tests

We started with a series of brute force tests of the Polynomial BBC, see §4.8.1. We exhaustively tested all connected graphs with at most 8 vertices, and connected graph with at most 15 edges from the `House of Graphs` database, see [CDG23]. In each case, the Polynomial BBC held true. At this point we chose to develop a more refined approach.

4.7.2 The algorithm

Our starting point is the machine learning algorithm by Wagner [W21], which we adjusted and modified. Roughly, the algorithm inputs a neural network used in a randomized graph generating algorithm, various constraints and a function to optimize. It outputs new weights for the neural network with the function improved. In his remarkable paper, Wagner describes how he was able to disprove five open problems in graph theory, so we had high hopes that this approach might help to disprove the bunkbed conjecture.

To give a quick outline of our approach, we consider a graph $G = (V, E)$ on $n = |V|$ vertices, with the set of transversal vertices $T \subset V$, and fixed $u, v \notin W$. Flip a fair coin for each edge $e \in E$. Depending on the outcome, either retain e and delete e', or vice versa. Check whether $u \leftrightarrow v$ and $u \leftrightarrow v'$. Repeat this N times to estimate the corresponding probabilities p and p', respectively. Based on these probabilities, use Wagner's algorithm to obtain the next iteration. Repeat M times or until a potential counterexample with $p < p'$ is found.

4.7.3 Implementation and results

We first used Wagner's original code on a laptop computer, but when that proved too slow we made major changes. To speed up the performance and tweak the code, we implemented Wagner's algorithm in `Julia`.

We then ran the code on a shared `UCLA Hoffman2 Cluster`, which is a Linux compute cluster consisting of 800+ 64-bit nodes and over 26,000 cores, with an aggregate of over 174 TB of memory.[1] Each run of the algorithm required about 2 hours. After six runs with different parameters, the results were too similar to continue.

Specifically, we ran the algorithm on graphs with $n = 20$ and $n = 30$ vertices, and for 3, 4 and 5 transversal vertices. Although we started with relatively dense graphs, the algorithm converged to relatively sparse graphs with about 100 edges. We used $N = 4000$, pruning the Monte Carlo sampling when the desired probabilities were far apart.

We used $M = 2000$, after which the probabilities p, p' rapidly converged to $\frac{1}{2}$ and became nearly indistinguishable. More precisely, the *probability gap* $p - p'$ became smaller than 0.01 getting close to the Monte Carlo error, i.e. the point when we would need to increase N to avoid false positives. At all stages, we had $p > p'$ suggesting validity of the bunkbed conjecture. At the time of the experiments and prior to this work, we saw no evidence that an experimental approach could ever succeed.

4.7.4 Analysis

Having formally disproved the bunkbed conjecture, it is clear that our computational approach was misguided. For the uniform weights we tested, we could never have reached graphs of size anywhere close to that in Theorem 2, of course. Even when the number of vertices is optimized to 82 as suggested in Remark 4.4.2, the number of edges is still very

[1] The system overview is available here: `www.hoffman2.idre.ucla.edu/About/System-overview.html`

large while the probability gap in the theorem is on the order of 10^{-47}, thus undetectable in practice.[2]

In hindsight, to reach a small counterexample we should have used the weighted bunkbed percolation rather than the more efficient alternative model, with some edges having a very large weight and some very small weight. Of course, by Remark 4.4.2, the probability gap in the theorem is still prohibitively small, at least for the graphs we consider.

4.7.5 Conclusions

It seems, the Bunkbed Conjecture has some unique features making it very poorly suited for computer testing. In fact, one reason we stopped our computer experiments is that in our initial rush to testing we failed to contemplate societal implications of working with even moderately large graphs.

Suppose we *did* find a potential counterexample graph with only $m = 100$ edges and the probability gap was large enough to be statistically detectable. Since analyzing all of $2^m \approx 10^{30}$ subgraphs is not feasible, our Monte Carlo simulations could only confirm the desired inequality with high probability. While this probability could be amplified by repeated testing, one could never formally *disprove* the bunkbed conjecture this way, of course.

This raises somewhat uncomfortable questions whether the mathematical community is ready to live with an uncertainty over validity of formal claims that are only known with high probability. It is also unclear whether in this imaginary world the granting agencies would be willing to support costly computational projects to further increase such probabilities (cf. [GBCST16, Zei93]). Fortunately, our failed computational effort avoided this dystopian reality, and we were able to disprove the bunkbed conjecture by a formal argument.

---

[2] Johann Beurich showed that a 25 vertex modification of our construction can disprove the Weighted BBC with the probability gap on the other of $6.1 \cdot 10^{-95}$, thus again undetectable in practice (personal communication).

4.8 Final remarks

4.8.1 Variations on the BBC

Although the version of the Bunkbed Conjecture 4.1.1 given in [BK01] is considered the most definitive, over the years several closely related versions has been studied:

(0) *Counting BBC*, where one compares the number of subgraphs connecting vertices u, v and those that do not. This conjecture is a restatement of the BBC in the case $p = \frac{1}{2}$.

(1) *Weighted BBC*, where the edge probabilities $p_e = p_{e'}$ can depend on $e \in E$. This conjecture is equivalent to the BBC by [RS16], since edge probabilities can be approximated by series-parallel graphs.

(2) *Polynomial BBC*, where the edge probabilities above are viewed as variables. In this case the conjecture claims that the difference of polynomials corresponding to $\mathbf{P}[u \leftrightarrow v]$ and $\mathbf{P}[u \leftrightarrow v']$ is a polynomial with nonnegative coefficients. This conjecture is stronger than Weighted BBC as there are polynomials positive on $[0, 1]$ which have negative coefficients such as $(x - y)^2$. Although we did not find a counterexample on a graph with at most 8 vertices, it is likely that there is a sufficiently small counterexample in this case. Cf. [Ric22], where the difference is a sum of squares.

(3) *Computational BBC*, where one asks if the counting version of the probability gap is in $\#\mathsf{P}$, i.e. has a combinatorial interpretation, see [Pak22, Conj. 5.6]. Clearly, this conjecture implies BBC. We refer to [IP22] for a formal treatment of this problem for general polynomials.

(4) *Alternative BBC*, where fair coin flips determine whether the edge e is deleted and e' retains or vice versa. This conjecture implies BBC [Lin11, Prop. 2.6].

(5) *Complete BBC*, where one takes all $T = V$ and performs the weighted percolation on the full $\overline{G} := G \times K_2$, i.e. on all edges in $\overline{G}$ including the posts. The conjecture in this case

is weaker than the BBC, see e.g. [Lin11, Prop. 2.2].

In all but the last case, the corresponding conjecture is refuted by Theorem 2. In Complete BBC, the corresponding conjecture is refuted by Theorem 3 by a more involved counterexample (based on a more involved counterexample by Hollom).

4.8.2 Robustness lemma

Lemma 4.3.3 is a finite problem which can be reformulated as follows. By definition, probabilities on both sides of (4.2) are polynomials in 5 variables of degree at most 12, with at most 5^{12} nonzero coefficients. The lemma gives positivity of the difference of these two polynomials on a region of the unit cube $[0,1]^5$ cut out by the quadratic inequality (4.1).

Since our proof of Lemma 4.3.3 is somewhat cumbersome and uses a case-by-case analysis, we verified the lemma computationally. The results and the code are available on `GitHub`.[3] Of course, the advantage of our combinatorial proof is that it is elementary and amenable for generalizations.

4.8.3 Special cases

Our counterexample makes prior positive results somewhat more valuable. It would be interesting to find other families of graphs on which the Bunkbed Conjecture holds. We are especially interested in the corresponding problem for the Polynomial BBC. Note that we emphasized planarity in Theorem 2 since it was speculated in [Lin11] that planarity helps.

4.8.4 Percolation in $\mathbb{Z}^d$

For lattices, the connection probabilities $\mathbf{P}_p[u \leftrightarrow v]$ between vertices are known as the *two-point functions*. For percolation in higher dimensions, these were famously studied by Hara,

[3] *Generating Partitions of Graph Vertices into Connected Components*, description and code at GitHub.com/Kroneckera/bunkbed-counterexample

van der Hofstad and Slade [HHS03], and they are also of interest for other probabilistic models.

Curiously, it is not known whether connection probabilities are monotone as the distance $|u-v|$ increases. This claim would follow from the bunkbed conjecture. This suggests that investigating the BBC for grid-like graphs is still of interest even if the conjecture is false for general planar graphs. Note that the monotonicity is known in the $p \downarrow 0$ limit.

4.8.5 Random cluster model

It was shown in [Häg03, §3] that the analogue of the BBC holds for the random cluster model with parameter $q = 2$. Our Theorem 2 shows that one cannot take $q = 1$. It would be interesting to find the smallest $q > 1$ such that the BBC holds for all finite graphs. We note that monotonicity in q is unclear, so e.g. it is not known if BBC holds for all $q \geq 2$.[4]

4.9 Appendix: Bunkbed conjecture for two transversal vertices

In [BK01], the main result, proved by Ahlswede–Daykin (AD) inequality [AD78], is the following inequality:

Theorem 4.9.1. *For vertices a, b, c, d in G and an edge percolation on it, one has*

$$\mathbf{P}(abc|d)\mathbf{P}(a|d) \geq \mathbf{P}(ab|d)\mathbf{P}(ac|d). \tag{4.7}$$

The authors mention that their original motivation for the inequality (4.7) was the bunkbed conjecture. Indeed, they proved, for the bunkbed graph $\bar{G}$ with two transversal vertices x and y, the following inequality:

$$\mathbf{P}^{\mathrm{bb}}[u \leftrightarrow v \mid x \not\leftrightarrow y] \geq \mathbf{P}^{\mathrm{bb}}[u \leftrightarrow v' \mid x \not\leftrightarrow y],$$

[4] Alan Sokal suggested to us that BBC should hold for *integer* $q \geq 2$ and fail for *noninteger* $q > 2$ (personal communication, October 2, 2024).

where $\mathbf{P}^{\mathrm{bb}}$ is understood in the sense of Chapter 4. One can see that this inequality closely resembles the bunkbed conjecture. However, as we will demonstrate, it is their subsequent result, Theorem 4.9.2 from [BHK06], that directly leads to the bunkbed conjecture for two transversal vertices.

Theorem 4.9.2 ([BHK06, eq. (2)])**.** *For vertices a, b, c, d in G and an edge percolation on it, one has*

$$\mathbf{P}(ab|cd)\mathbf{P}(a|d) \leq \mathbf{P}(ab|d)\mathbf{P}(a|cd).$$

Theorem 4.9.3. *For the bunkbed graph $\bar{G}$ with transversal vertices u and v, based on graph G, we have:*

$$\mathbf{P}^{bb}[u \leftrightarrow v] \geq \mathbf{P}^{bb}[u \leftrightarrow v'].$$

Proof. Consider the graph G', which is graph G with vertices x and y contracted into a single vertex denoted by $[xy]$. The probabilities of vertices being connected in G' are closely related to the probabilities in $\bar{G}$ and G. We denote probabilities of connectedness in $\bar{G}$ by $\mathbf{P}^{\mathrm{bb}}$, and in G or G' by $\mathbf{P}$, with the relevant graph implied by whether the vertex $[xy]$ appears in the argument.

In particular, we have:

$$\mathbf{P}^{\mathrm{bb}}[u \leftrightarrow v] = \mathbf{P}(uv|[xy]) + \mathbf{P}(uv[xy]) - (\mathbf{P}(ux|vy) + \mathbf{P}(uy|vx))\mathbf{P}(x|y),$$

since for u and v to be connected, they must be connected in the lower component with vertices x and y contracted, excluding cases where u and v are connected to different vertices among x and y, and the upper component does not assist in connecting $x = x'$ and $y = y'$.

Similarly, for the connection between u and v', we have:

$$\mathbf{P}^{\mathrm{bb}}[u \leftrightarrow v'] = \mathbf{P}(u[xy])\mathbf{P}(v[xy]) - \mathbf{P}(ux|y)\mathbf{P}(vy|x) - \mathbf{P}(uy|x)\mathbf{P}(vx|y),$$

because for u and v' to be connected, they must be linked to x or y, and we subtract the cases where u and v' are connected to different vertices among x and y.

By subtracting $\mathbf{P}(uv|[xy])$ from $\mathbf{P}^{\mathrm{bb}}[u \leftrightarrow v]$ and noting that

$$\mathbf{P}(uv[xy]) \geq \mathbf{P}(u[xy])\mathbf{P}(v[xy])$$

due to the Harris–Kleitman inequality for G', we are left with:

$$\big(\mathbf{P}(ux|vy) + \mathbf{P}(uy|vx)\big)\mathbf{P}(x|y) \leq \mathbf{P}(ux|y)\mathbf{P}(vy|x) + \mathbf{P}(uy|x)\mathbf{P}(vx|y),$$

which corresponds to the sum of two inequalities from Theorem 4.9.2. □

CHAPTER 5

On Harris–Kleitman type inequalities

5.1 Introduction

This chapter presents a generalization of the Harris–Kleitman inequality and previous work by the authors [G24a, Corollary 3.6]. Our main contributions are:

- A new inequality for functions on the hypercube that generalizes the Harris–Kleitman inequality
- A framework for testing measure implementability using convex optimization
- New applications to Bernoulli percolation, including inequalities for connectivity events

We work in the setting of the hypercube $H_n = 2^{[n]}$ with a product measure μ. For $x, y \in H_n$ we say that $x \preceq y$ if x is coordinatewise less or equal to y. A subset of H_n is called closed upwards if with each x it contains all y such that $x \preceq y$. In particular, the indicator function of each closed upward subset is nondecreasing.

Our work extends the classical techniques for studying inequalities concerning the sizes of closed upward subsets. In our setting, sizes of sets can be replaced by the measures relative to some product measure μ on H_n. By the classical techniques (see a footnote in [K22] and its explanation in [G24a, Proposition 3.1]), one can go one step further and think about these inequalities as relative to any probability measure μ satisfying the FKG condition

$$\mu(x)\mu(y) \geq \mu(x \cup y)\mu(x \cap y).$$

The chapter is organized as follows. Section 5.2 presents our main theoretical results, including the generalization of the Harris–Kleitman inequality. Section 5.3 applies these results to Bernoulli percolation, deriving new inequalities for connectivity events. Section 5.4 introduces a computational framework for testing measure implementability using convex optimization. Finally, we discuss open problems and future directions in Section 5.5.

5.2 Main results

Our main result generalizes the Harris–Kleitman inequality by considering a broader class of functions on the hypercube. The key insight is that we can replace the requirement of nondecreasing functions with a more general condition on the function's behavior with respect to the partial order.

Theorem 5.2.1. *Let μ be a probability product measure on $2^{[n]}$. Let $g(x, y)$ be a function on $2^{[n]} \times 2^{[n]}$ such that for any $x \preceq y$, $z \preceq t$ one has*

$$g(x, z) + g(y, t) \leq g(x, t) + g(y, z). \tag{5.1}$$

Then

$$\mathbf{E}_{\mu\times\mu} g(x, y) \geq \mathbf{E}_{\mu} g(x, x). \tag{5.2}$$

Proof. Proof proceeds by the induction on n. For $n = 0$, equation (5.2) turns into equality. Consider the last coordinate and let μ' be the projection of μ onto the rest of the coordinates. It will also be a product measure. Moreover, the projection of μ to the last coordinate will assign probability p to 0 and $1 - p$ to 1. Let x' and y' be generated independently according to μ' and x^-, x^+, y^- and y^+ be defined as x and y supplied with the last coordinate equal to 0 and 1 respectively. From the induction hypothesis, we know that $\mathbf{E}_{\mu'\times\mu'} g(x^-, y^-) \geq \mathbf{E}_{\mu'\times\mu'} g(x^-, x^-)$ and $\mathbf{E}_{\mu'\times\mu'} g(x^+, y^+) \geq \mathbf{E}_{\mu'\times\mu'} g(x^+, x^+)$. Combining this with the condition

(5.1) applied to x^-, x^+, y^- and y^+ we get

$$\begin{aligned}
&\mathbf{E}_{\mu\times\mu} g(x,y) \\
&\quad= \mathbf{E}_{\mu'\times\mu'}\big(p^2 g(x^-,y^-) + p(1-p)g(x^-,y^+) + p(1-p)g(x^+,y^-) + (1-p)^2 g(x^+,y^+)\big) \\
&\qquad\qquad= p(1-p)\mathbf{E}\big(g(x^-,y^+) + g(x^+,y^-) - g(x^-,y^-) - g(x^+,y^+)\big) \\
&\qquad\qquad\qquad + p\mathbf{E}_{\mu'\times\mu'} g(x^-,y^-) + (1-p)\mathbf{E}_{\mu'\times\mu'} g(x^+,y^+) \\
&\qquad\qquad\qquad \geq 0 + p\mathbf{E}_{\mu'} g(x^-,x^-) + (1-p)\mathbf{E}_{\mu'} g(x^+,y^+) \\
&\qquad\qquad\qquad\qquad = \mathbf{E}_\mu g(x,x).
\end{aligned}$$

□

This theorem has several important consequences. First, it generalizes the original Harris-Kleitman theorem, as we show in the following corollary. Second, it provides a framework for studying more general inequalities on the hypercube, as we will see in the subsequent sections.

Corollary 5.2.2. *Let f_1 and f_2 be nondecreasing functions on H_n, then f_1 and f_2 correlate nonnegatively with respect to μ.*

Proof. Consider $g(x,y) = f_1(x)f_2(y)$. Then for $x \preceq y$, $z \preceq t$ one has

$$\begin{aligned}
g(x,z) + g(y,t) - g(x,t) - g(y,z) &= f_1(x)f_2(z) + f_1(y)f_2(t) - f_1(x)f_2(t) - f_1(y)f_2(z) \\
&= (f_1(y) - f_1(x))(f_2(t) - f_2(z)) \geq 0.
\end{aligned}$$

So, the condition of Theorem 5.2.1 holds and the conclusion is

$$\mathbf{E}_{\mu\times\mu} f_1(x)f_2(y) \leq \mathbf{E}_\mu f_1(x)\mathbf{E}_\mu f_2(x),$$

which shows that f_1 and f_2 have nonnegative correlation with respect to μ. □

We notice that it is convenient to consider g to be constant on subsets of H_n. This motivates the following partial case of Theorem 5.2.1.

Theorem 5.2.3. *Let $\mathcal{P}$ be a poset of size m and H_n be split into subsets S_p indexed by $p \in \mathcal{P}$ such that if $x \in S_a$ and $y \in S_b$ are such that $x \preceq y$, then $a \leq_{\mathcal{P}} b$. Let A be an $m \times m$ matrix satisfying the condition*

$$A_{a,c} + A_{b,d} \leq A_{a,d} + A_{b,c}, \tag{5.3}$$

whenever $a <_{\mathcal{P}} b$ and $c <_{\mathcal{P}} d$. Then for any probability product measure μ we have

$$\sum_{a,b \in \mathcal{P}} A_{a,b}\mu(S_a)\mu(S_b) \geq \sum_{a \in \mathcal{P}} A_{a,a}\mu(S_a) \tag{5.4}$$

Proof. We use Theorem 5.2.1 for the function g that is constant within the subsets S_p. For $x \in S_a$ and $y \in S_b$ we put $g(x, y) = A_{a,b}$ It is easy to check that condition (5.3) implies (5.1) for g and so

$$\sum_{a,b \in \mathcal{P}} A_{a,b}\mu(S_a)\mu(S_b) = \mathbf{E}_{\mu \times \mu} g(x, y) \geq \mathbf{E}_{\mu} g(x, x) = \sum_{a \in \mathcal{P}} A_{a,a}\mu(S_a).$$

□

An example poset to keep in mind here is the poset M_3 that is the poset of the smallest nondistributive lattice:

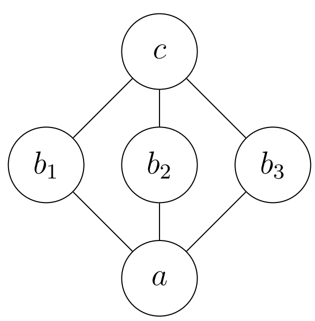

Figure 5.1: Poset M_3

This poset was used in [G24a]. It corresponds to a partition lattice of a set $\{1, 2, 3\}$. So for bond percolation one can think of inequalities on sizes of S_p as inequalities on Boolean combinations of the events "vertex i is connected to vertex j" (see Theorem 5.3.2).

Definition 5.2.4. We say that for the given poset $\mathcal{P}$ the vector of probabilities $\{m_p\}_{p\in\mathcal{P}}$ is *realizable* if there exists H_n that can be split into subsets S_p indexed by $p \in \mathcal{P}$ such that if $x \in S_a$ and $y \in S_b$ are such that $x \preceq y$, then $a \leq_{\mathcal{P}} b$ and there is a product measure μ on H_n such that $\mu(S_p) = m_p$.

So Theorem 5.2.3 can be seen as the set of restrictions on realizable vectors of probabilities for a given poset.

Example 5.2.5. For a diamond poset (see Figure 5.2), the vector of probabilities

$$\{m_a, m_{b_1}, m_{b_2}, m_c\}$$

is realizable if and only if $m_a m_c \geq m_{b_1} m_{b_2}$.

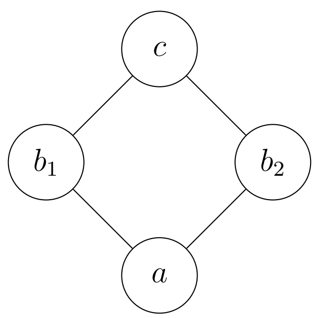

Figure 5.2: Poset M_2

Proof. Indeed, if the vector is realizable, then we apply Theorem 5.2.3 with

$$A = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 1 \\ b & 0 & 0 & -1 & 0 \\ c & 0 & -1 & 0 & 0 \\ d & 1 & 0 & 0 & 0 \end{array},$$

which gives the needed inequality. Conversely, if $m_a m_c \geq m_{b_1} m_{b_2}$, denote by p the excess of m_c over its minimal attainable value: $p = m_c - \frac{m_{b_1} m_{b_2}}{m_a}$ and $1 - p = \frac{(m_a+m_{b_1})(m_a+m_{b_2})}{m_a}$.

Then consider the following product measure on $\{0,1\}^3$: the first coordinate is 0 with probability $1-p = \frac{(m_a+m_{b_1})(m_a+m_{b_2})}{m_a}$ and 1 with probability p, the second coordinate is 0 with probability $\frac{m_a}{m_a+m_{b_1}}$ and 1 with probability $\frac{m_{b_1}}{m_a+m_{b_1}}$ and the third coordinate is 0 with probability $\frac{m_a}{m_a+m_{b_2}}$ and 1 with probability $\frac{m_{b_2}}{m_a+m_{b_2}}$. Then consider the sets

$$S_a = \{(0,0,0)\};$$
$$S_{b_1} = \{(0,1,0)\};$$
$$S_{b_2} = \{(0,0,1)\};$$
$$S_c = \{(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}.$$

One can see that the probabilities of these sets are equal to the corresponding m_p. □

For general posets, it is unclear what are the conditions on the vector $\{m_p\}$ to be realizable. One can merge some of the nodes of $\mathcal{P}$ to form a diamond poset and apply the Harris-Kleitman inequality to the merged parts. This operation gives restrictions which are a partial case of the restrictions given by Theorem 5.2.3.

In 2001, Richards [R04] stated a third-degree inequality for a cube poset $\mathcal{P}_{cube}$ (Figure 5.3). The proof was later found to contain significant problems.

Conjecture 5.2.6. *For any realizable vector $\{m_p\}$ corresponding to the poset $\mathcal{P}_{cube}$, define m_A, m_B and m_C as the sum of elements where the corresponding letter doesn't have an overline and m_{AB}, m_{AC} and m_{BC} as the sum of vectors where both letters don't have an overline. Then*

$$2m_{ABC} + m_A m_B m_C \geq m_A m_{BC} + m_B m_{AC} + m_C m_{AB}.$$

Sahi and Lieb were able to prove partial cases of this conjecture [S08, LS22].

Definition 5.2.7. We say that for the given poset $\mathcal{P}$ the vector of probabilities $\{m^-_p\}_{p\in\mathcal{P}}$ can be *glued* with the vector $\{m^+_p\}_{p\in\mathcal{P}}$ if there exists some n, a probability product measure μ on H_n and two subdivisions of H_n into parts S^-_p and S^+_p such that

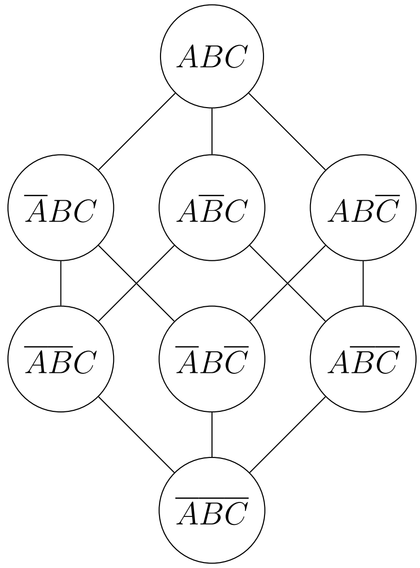

Figure 5.3: Cube poset

- S_a^- is contained in $\bigcup_{b<_{\mathcal{P}}a} S_b^+$;
- $\mu(S_p^-) = m_p^-$;
- $\mu(S_p^+) = m_p^+$.

Theorem 5.2.8. *If Let A be an $m \times m$ matrix satisfying the condition*

$$A_{a,c} + A_{b,d} \leq A_{a,d} + A_{b,c}, \tag{5.5}$$

whenever $a <_{\mathcal{P}} b$ and $c <_{\mathcal{P}} d$. Suppose in addition that whenever $a \leq_{\mathcal{P}} b$, $A_{a,b} \geq 0$. Then for any poset $\mathcal{P}$ and any vectors $\{m_p^-\}_{p\in\mathcal{P}}$ and $\{m_p^+\}_{p\in\mathcal{P}}$ that can be glued together, one has

$$\sum_{a,b\in\mathcal{P}} A_{a,b} m_a^- m_b^+ \geq 0$$

5.3 Applications to Bernoulli percolation

The Harris–Kleitman inequality was originally developed to study Bernoulli bond percolation, where it was used to show that the critical probability p_c for the square lattice is at least $\frac{1}{2}$. Our new inequalities provide a powerful tool for studying more complex connectivity events in percolation models.

We first introduce some notation and basic concepts. Consider a Bernoulli bond percolation $\mathbf{P}$ on a finite graph $G = (V, E)$ where each edge $e \in E$ has a probability p_e of being open. Let E_o be the resulting random set of open edges. We call the connected components of $G_o = (V, E_o)$ *clusters*.

The key insight is that connectivity events in percolation can be naturally represented as elements of a poset, where the partial order is given by containment of connected components. This allows us to apply our general results to derive new inequalities for percolation probabilities.

Definition 5.3.1. Consider a Bernoulli bond percolation on a finite graph $G = (V, E)$ where each edge $e \in E$ has a probability w_e of being open. Let E_o be the random set of open edges. We call the connected components of $G_o = (V, E_o)$ *clusters*. We denote by "$v_{11}v_{12}\ldots v_{1i_1}|v_{21}\ldots v_{2i_2}|\ldots|v_{n1}\ldots v_{ni_n}$" the event that the vertices $v_{11}, \ldots, v_{1i_1} \in V$ belong to the same cluster, vertices $v_{21}, \ldots, v_{2i_2}$ belong to the same cluster, …, vertices $v_{n1}, \ldots, v_{ni_n}$ belong to the same cluster, and, moreover, these clusters are all different. By $\mathbf{P}(v_{11}v_{12}\ldots v_{1i_1}|v_{21}\ldots v_{2i_2}|\ldots|v_{n1}\ldots v_{ni_n})$ we denote the probability of this event in the underlying bond percolation. In particular, $\mathbf{P}(abc)$ denotes the probability that vertices $a, b, c \in V$ lie in the same cluster, and $\mathbf{P}(a|b|c)$ is the probability that a, b and c belong to 3 different clusters.

The following theorem was proven in [G24a] (see also an alternative proof in [GZ24]). We put the proof in the context of our results.

Theorem 5.3.2. *Let $G = (V, E)$ be a finite graph and a, b, c are vertices in V. Let $\mathbf{P}$ be taken over Bernoulli percolation on G. Then*

$$\mathbf{P}(abc)\mathbf{P}(a|b|c) \geq \mathbf{P}(ab|c)\mathbf{P}(ac|b) + \mathbf{P}(ab|c)\mathbf{P}(a|bc) + \mathbf{P}(ac|b)\mathbf{P}(a|bc).$$

Proof. Since events ab, ac and bc are all increasing and if two of them happen, the third is forced, the events abc, $ab|c$, $ac|b$, $a|bc$ and $a|b|c$ form the poset on Figure 5.4.

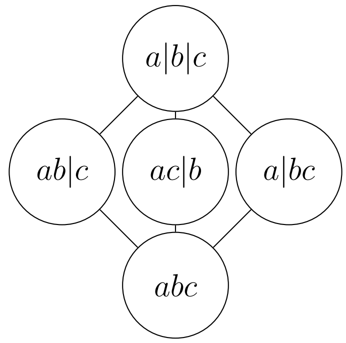

Figure 5.4: Partition lattice of $\{a, b, c\}$

Consider the following matrix A labeled by the elements of the poset

$$A = \begin{array}{c|ccccc} & abc & ab|c & ac|b & a|bc & a|b|c \\ \hline abc & 0 & 0 & 0 & 0 & 1 \\ ab|c & 0 & 0 & -1 & -1 & 0 \\ ac|b & 0 & -1 & 0 & -1 & 0 \\ a|bc & 0 & -1 & -1 & 0 & 0 \\ a|b|c & 1 & 0 & 0 & 0 & 0 \end{array} \tag{5.6}$$

It is easy to check the condition (5.3) for A, so by Theorem 5.2.3 we get

$$2\big(\mathbf{P}(abc)\mathbf{P}(a|b|c) - \mathbf{P}(ab|c)\mathbf{P}(ac|b) - \mathbf{P}(ab|c)\mathbf{P}(a|bc) - \mathbf{P}(ac|b)\mathbf{P}(a|bc)\big) \geq 0.$$

□

We can consider the larger partition lattices. The partition lattice on 4 element set $\{a, b, c, d\}$ is shown on Figure 5.5.

Our method allows one to find new inequalities for the probabilities of the connectivity events.

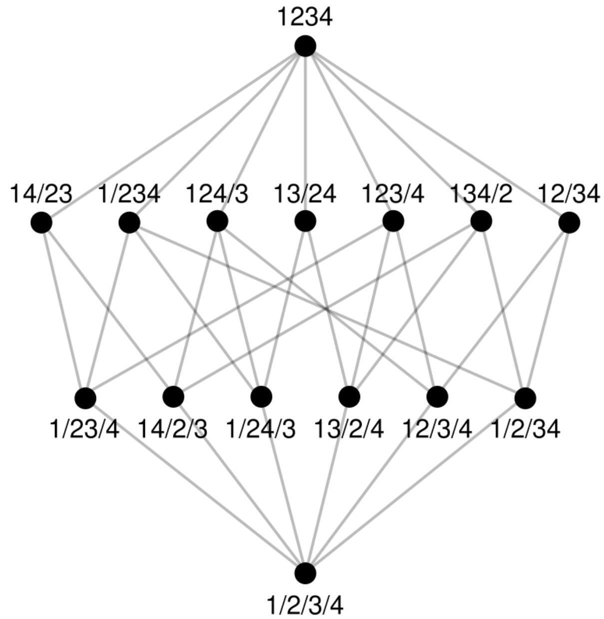

Figure 5.5: Partition lattice of $\{a, b, c, d\}$

Theorem 5.3.3. *Let $G = (V, E)$ be a finite graph and a, b, c, d are vertices in V. Let $\mathbf{P}$ be taken over Bernoulli percolation on G. Then*

$$\mathbf{P}(ab \cap cd) - \mathbf{P}(ab)\mathbf{P}(cd) \geq \mathbf{P}(ab \cup cd)(\mathbf{P}(ac|bd) + \mathbf{P}(ad|bc)) + \mathbf{P}(ac|bd)\mathbf{P}(ad|bc). \tag{5.7}$$

Proof. Consider the following matrix A:

It acts as a certificate for Theorem 5.2.3 to show the inequality (5.7) □

The work [BHK06] studies correlations for different connectivity events in the percolation measure μ conditioned on the event $a|b$. It shows that any two events with positive dependence on the component of a correlate nonnegatively and all such events correlate nonpositively with the events with positive dependence on the component of b.

In the heart of the proof [Hol24b] there is a sequence of product measures on hypercubes of increasing dimension that approximate the Bernoulli percolation conditioned on $a|b$. Additionally, it shows that the events positively dependent on the component of a are closed upwards in these hypercubes while events positively dependent on the component of b are closed downwards. In particular, if we only care about 4 vertices, it shows that the events in Figure 5.6 give rise to a realizable vector.

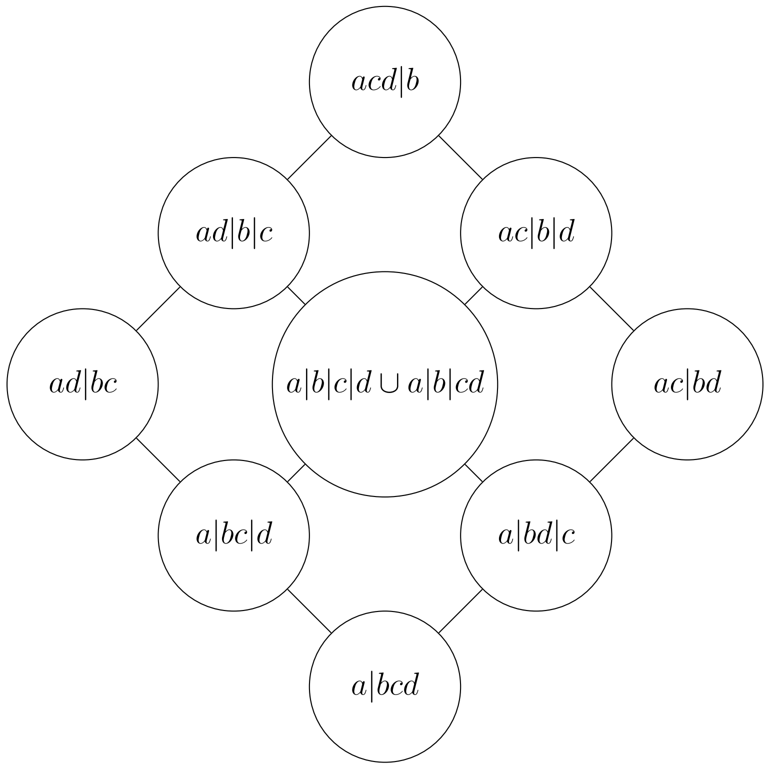

Figure 5.6: Poset of events conditioned on $a|b$

Papers [BHK06, BK01] found the inequalities that follow from this poset by applying the Harris-Kleitman inequality to the pair of the increasing events ac and ad and the pair of increasing events ac and $b|d$. However, there are more restrictions on the probability vectors realizable by the poset. In particular, one can notice a hidden M_3 in this poset.

Corollary 5.3.4. *Let $G = (V, E)$ be a finite graph and a, b, c, d are vertices in V. Let $\mathbf{P}$ be taken over Bernoulli percolation on G. Then*

$$\begin{aligned}(\mathbf{P}(acd|b) + \mathbf{P}(ad|b|c) + \mathbf{P}(ac|b|d))(\mathbf{P}(a|bc|d) + \mathbf{P}(a|bd|c) + \mathbf{P}(a|bcd))\\ \geq \mathbf{P}(ad|bc)\mathbf{P}(ac|bd) + (\mathbf{P}(ad|bc) + \mathbf{P}(ac|bd))(\mathbf{P}(a|b|c|d) + \mathbf{P}(a|b|cd)).\end{aligned} \quad (5.8)$$

Proof. Consider the following A:

$A =$

	$a\|bcd$	$a\|bc\|d$	$ad\|bc$	$a\|bd\|c$	$a\|b\|c\|d \cup a\|b\|cd$	$ad\|b\|c$	$ac\|bd$	$ac\|b\|d$	$acd\|b$
$a\|bcd$	0	0	0	0	0	1	0	1	1
$a\|bc\|d$	0	0	0	0	0	1	0	1	1
$ad\|bc$	0	0	0	0	-1	0	-1	0	0
$a\|bd\|c$	0	0	0	0	0	1	0	1	1
$a\|b\|c\|d \cup a\|b\|cd$	0	0	-1	0	0	0	-1	0	0
$ad\|b\|c$	1	1	0	1	0	0	0	0	0
$ac\|bd$	0	0	-1	0	-1	0	0	0	0
$ac\|b\|d$	1	1	0	1	0	0	0	0	0
$acd\|b$	1	1	0	1	0	0	0	0	0

One can see that it satisfies the condition (5.3). Indeed, it just comes from the matrix (5.6) for the poset M_3 one gets by merging together 3 top vertices of the poset as well as 3 bottom vertices. Writing the corresponding conclusion of Theorem 5.2.3 and multiplying it by $\mathbf{P}(a|b)^2$ to get rid of conditional probabilities we get the needed (5.8). □

The paper [BHK06] also defines a way to use two sets of vertices S, T instead of two vertices a, b. If we apply their method to $S = \{a, b, c\}$ and $T = \{d\}$, we will get the poset M_3 on Figure 5.7.

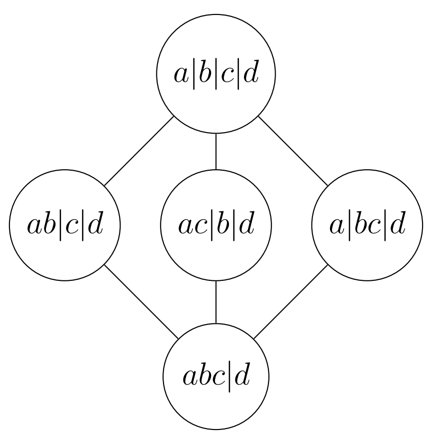

Figure 5.7: Poset of events conditioned on $a|d \cap b|d \cap c|d$

Writing the inequality corresponding to matrix (5.6) and multiplying it by $\mathbf{P}(a|d \cap b|d \cap c|d)^2$ we get the following inequality.

Corollary 5.3.5. *Let $G = (V, E)$ be a finite graph and a, b, c, d are vertices in V. Let $\mathbf{P}$ be taken over Bernoulli percolation on G. Then*

$$\mathbf{P}(abc|d)\mathbf{P}(a|b|c|d) \geq \mathbf{P}(ab|c|d)\mathbf{P}(ac|b|d) + \mathbf{P}(ab|c|d)\mathbf{P}(a|bc|d) + \mathbf{P}(ac|b|d)\mathbf{P}(a|bc|d).$$

5.4 Testing measure implementability

For a poset $\mathcal{P}$ consider a matrix $F_{\mathcal{P}}$ defined by the following rules

Definition 5.4.1. We say that a *covers* b (written as $a \lessdot b$) if $a <_{\mathcal{P}} b$ and there are no elements between a and b.

Assume poset $\mathcal{P}$ has m elements and p pairs of elements (a, b) in a covering relation. Then $F_{\mathcal{P}}$ is an $m \times p$ matrix where each row corresponds to an element of $\mathcal{P}$ and each column corresponds to a cover $a \lessdot b$ and

$$F_{e,ab} = \begin{cases} -1, & \text{if } e = a, \\ 1, & \text{if } e = b, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.4.2. *If vector $x = \{m_p\}_{p \in \mathcal{P}}$ is realizable, then there exists a nonnegative and nonnegatively determined $p \times p$ matrix M such that*

$$FMF^T = \text{diag}(x) - xx^T. \tag{5.9}$$

Proof. Assume x is realizable by a product measure μ on a hypercube H_n. For $n = 0$ one can take $M = 0$ and the statement would be true. For $n \geq 1$ we use induction. Split the cube into the upper and lower halves. Let p be the probability of the upper half and μ' be the probability measure on H_{n-1} obtained as the projection of μ to $n-1$ first coordinates. Then μ' coincides with μ conditioned on the upper or lower part of H_n. Let M^+ and M^- be the matrices and x^+ and x^- be the vectors corresponding to μ^+ and μ^-.

To use the fact that the upper and lower parts can be glued together, consider the subdivisions S^- and S^+ indexed by elements of $\mathcal{P}$. Let y be the vector indexed by the pairs $a \lessdot b$ from poset, such that $y_{ab} = \mu'(\{S_a^- \cap S_b^+\})$. By definition of F, $x^+ - x^- = Fy$. Note that all entries of y are nonnegative.

Now

$$\begin{aligned}
&\operatorname{diag}(x) - xx^t \\
&= p\operatorname{diag}(x^+) + (1-p)\operatorname{diag}(x^-) + p^2 x^+ x^{+T} + p(1-p)(x^+ x^{-T} + x^- x^{+T}) + (1-p)^2 x^- x^{-T} \\
&\qquad = p(FM^+F^T) + (1-p)(FM^-F^T) - p(1-p)(x^+ - x^-)(x^+ - x^-)^T \\
&\qquad\qquad = F(pM^+ + (1-p)M^- + yy^T)F^T,
\end{aligned}$$

and it is easy to see that $M = pM^+ + (1-p)M^- + yy^T$ is nonnegatively determined and has nonnegative entries. □

Testing if a particular vector x for a particular poset $\mathcal{P}$ satisfies condition (5.9) for some matrix M is a convex optimization problem – the restrictions of M being nonnegative and nonnegatively determinate are convex, so instead of checking all inequalities coming from various matrices A satisfying condition (5.3), one can run one instance of a convex optimization program.

Computationally, it is much more efficient. Even for a relatively small poset P_4 from Figure 5.5, possible matrices A form a 100-dimensional cone with more than 10 000 generators. So having a simple test helps.

Moreover, this test is as strong as checking all the generators. Let $\langle A, B\rangle = \operatorname{Tr}(A^T B)$ be the Frobenius product of two matrices – the component-wise inner product of two matrices as though they are vectors. Then (5.3) is equivalent to

$$\langle A, Fe_i e_j^T F^T \rangle \geq 0 \tag{5.10}$$

for all $0 \leq i, j \leq p$ and (5.4) is equivalent to

$$\langle A, \mathrm{diag}(x) - xx^t \rangle \tag{5.11}$$

for $x = \{m_p\}_{p\in\mathcal{P}}$.

Proposition 5.4.3. *If a vector x satisfies condition (5.9) for some nonnegative matrix M, then for every matrix A satisfying (5.10), it satisfies (5.11) as well.*

Proof. Note that any nonnegative matrix M is a sum of terms of the form $m_{ij}e_i e_j^T$. Then

$$\langle A, \mathrm{diag}(x) - xx^T \rangle = \langle A, FMF^T \rangle$$
$$= \langle A, F\left(\sum m_{ij}e_i e_j^T\right)F^T \rangle = \sum m_{ij}\langle A, Fe_i e_j^T F^T \rangle \geq 0$$

□

This shows that the realizability test of Theorem 5.4.2 is stronger than this of Theorem 5.2.3 since it also adds the restriction that M is nonnegative definite.

Definition 5.4.4. We say one vector $\{m_p^+\}_{p\in\mathcal{P}}$ *dominates* another vector $\{m_p^+\}_{p\in\mathcal{P}}$

5.5 Final remarks

We believe the following conjecture:

Conjecture 5.5.1. *If for some poset $\mathcal{P}$ for the neighborhood of vector x there exists matrix M satisfying (5.9), then x is realizable.*

This conjecture contradicts Sahi's conjecture as well as the following conjecture by Jeff Kahn:

Conjecture 5.5.2 (Jeff Kahn)**.** *Let $\{A_i\}$ be the closed upwards events on H_n with product measure μ such that $\mu(A_i) < \varepsilon$. Denote by A the set of points belonging to exactly one of A_i.*

Then $\mu(A) < f(\varepsilon)$ for some f such that $\lim_{\varepsilon \to 0} f(\varepsilon) = \frac{1}{e}$.

CHAPTER 6

The defect of the FKG inequality is not in #P

In [IP22], the following theorem is proved (see definitions in the original paper):

Theorem 6.0.1 (Proposition 2.5.1 in [IP22])**.** *Let*

$$S := \left\{ (\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1, \delta_0, \delta_1, h_1, h_2, h_3, h_4) \in \mathbb{N}^{12} \;\middle|\; \begin{array}{c} \alpha_0\beta_0 + h_1 = \gamma_0\delta_0, \alpha_0\beta_1 + h_2 = \gamma_0\delta_1, \\ \alpha_1\beta_0 + h_3 = \gamma_1\delta_0, \alpha_1\beta_1 + h_4 = \gamma_1\delta_1 \end{array} \right\}$$

and let

$$\varphi := (\gamma_0 + \gamma_1)(\delta_0 + \delta_1) - (\alpha_0 + \alpha_1)(\beta_0 + \beta_1).$$

Then, under the Univariate Binomial Basis Conjecture (see [IP22, 4.4.2]), we have

$$\varphi(\#\{\vec{P} \in S\}) \notin \#\mathbf{P}.$$

This theorem shows that, under some computer science assumptions, there is no combinatorial way to prove the Ahlswede–Daykin (AD) inequality [AD78], since it doesn't preserve the complexity class #P.

It turns out, one can prove that the weaker Fortuin–Kasteleyn–Ginibre (FKG) inequality [FKG71] also doesn't preserve #P. In fact, it doesn't preserve #P even in case when the lattice is $\{0,1\}^3$. Here is the usual formulation of the inequality:

Theorem 6.0.2 (FKG inequality)**.** *Let L be a finite lattice with operations $\wedge$ and $\vee$. Let $f, g : L \to \mathbb{R}$ be increasing functions. Moreover, let $\mu : L \to \mathbb{R}$ be a probability measure on L such that*

$$\mu(x \wedge y)\mu(x \vee y) \le \mu(x)\mu(y) \text{ for all } x, y \in L. \qquad (\diamond)$$

Then

$$\mathbf{E}[f(X)g(Y)] \geq \mathbf{E}[f(X)]\mathbf{E}[g(Y)].$$

Before proving the result, we need to define the set S analogously to Theorem 6.0.1.

Definition 6.0.3. Let $I = \{000, 001, 010, 011, 100, 101, 110, 111\}$ be the set of indices for the hypercube. The set S will include all vectors $(f_\alpha, g_\alpha, \mu_\alpha)$ for each $\alpha \in I$. Moreover, we will require that:

- f_α, g_α are increasing functions,
- μ_α satisfies the diamond condition ($\diamond$).

To encode the first condition in the set S, we will use the following functions f'_β, g'_β indexed by $E(H_3)$ – the edges of H_3, so each $\beta = (\alpha_1, \alpha_2), \alpha_1 \geqslant \alpha_2$:

$$f'_\beta = f_{\alpha_1} - f_{\alpha_2}, \tag{6.1}$$

$$g'_\beta = g_{\alpha_1} - g_{\alpha_2}. \tag{6.2}$$

To encode the second condition in the set S, we will use the following measure μ'_γ indexed by $D(H_3)$ – the set of valid diamonds in H_3, so each

$$\gamma = (\alpha_1, \alpha_2, \alpha_3, \alpha_4), \text{ such that } \alpha_1 \wedge \alpha_2 = \alpha_3, \alpha_1 \vee \alpha_2 = \alpha_4.$$

We define

$$\mu'_\gamma = \mu_{\alpha_1}\mu_{\alpha_2} - \mu_{\alpha_3}\mu_{\alpha_4}. \tag{6.3}$$

Finally, set S to be the set of all natural values of

- f_α, g_α and μ_α for $\alpha \in I$;
- f'_β and g'_β for $\beta \in E(H_3)$;
- μ'_γ for $\gamma \in D(H_3)$,

satisfying the conditions (6.1), (6.2) and (6.3). Naturally, we have $S \subseteq \mathbb{N}^{3\times 8+2\times 12+64}$, since I has 8 elements, $E(H_3)$ has 12 edges, and $D(H_3)$ has 64 diamonds.

Theorem 6.0.4. *Let I and S be defined as in Definition 6.0.3. Let*

$$\varphi := \sum_{\alpha\in I}(f_\alpha g_\alpha \mu_\alpha)\sum_{\alpha\in I}\mu_\alpha - \sum_{\alpha\in I}(f_\alpha \mu_\alpha)\sum_{\alpha\in I}(g_\alpha\mu_\alpha).$$

Then, under the Univariate Binomial Basis Conjecture (see [IP22, 4.4.2]), we have

$$\varphi(\#\{\vec{P}\in S\}) \notin \#\mathbf{P}.$$

Proof. The main ingredients of our proof will be the measure μ and the functions f, g depicted in Figures 6.1, 6.2 and 6.3.

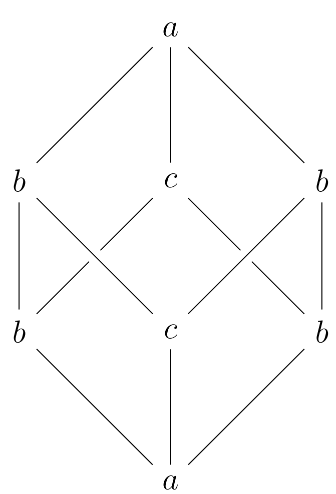

Figure 6.1: μ

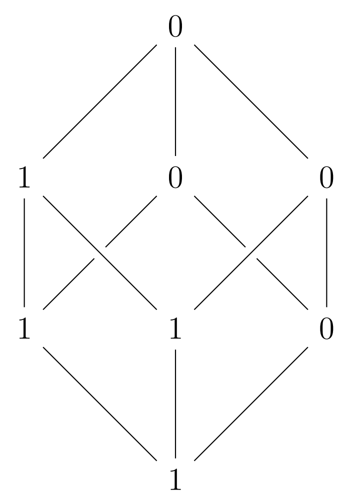

Figure 6.2: f

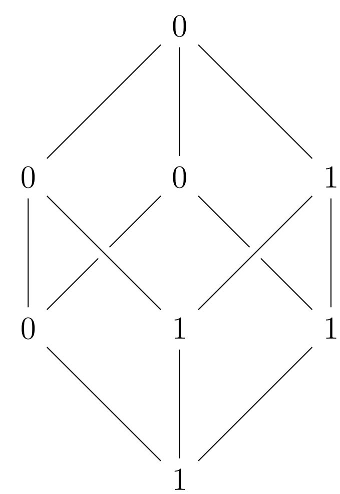

Figure 6.3: g

Let us treat a, b, c as polynomials in x. To satisfy the diamond conditions ($\diamond$) in a #P way, they should satisfy the inequalities $b^2 \leq_\# ac$, $bc \leq_\# ab$, $b^2 \leq_\# a^2$ and $c^2 \leq_\# a^2$, where $P(x) \leq_\# Q(x)$ means that in the decomposition of $Q-P$ in the binomial basis all coefficients are nonnegative (see [IP22, Theorem 4.3.2]). For these polynomials, the FKG inequality shows that $4b^2 \leq (a+c)^4$. However, we will see that $4b^2 \leq_\# (a+c)^4$ does not necessarily hold.

Indeed, consider $a = 30 + 91x + 200\binom{x}{2}$, $b = 110x$ and $c = 100x$. Then the conditions turn into

$$0 \leq_{\#} 60000\binom{x}{3} + 34000\binom{x}{2},$$

$$0 \leq_{\#} 66000\binom{x}{3} + 42020\binom{x}{2} + 2310x,$$

$$0 \leq_{\#} 240000 \times \binom{x}{4} + 58200 \times 6 \times \binom{x}{3} + 58581 \times 2 \times \binom{x}{2} + 1641x + 900,$$

$$0 \leq_{\#} 240000 \times \binom{x}{4} + 58200 \times 6 \times \binom{x}{3} + 60681 \times 2 \times \binom{x}{2} + 3741x + 900,$$

and the conclusion turns into

$$0 \leq 240000 \times \binom{x}{4} + 78200 \times 6 \times \binom{x}{3} + 90481 \times 2 \times \binom{x}{2} - 459x + 900,$$

which has a negative term in x, so $4b^2 \leq_{\#} (a+c)^4$ does not hold. $\square$

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