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UNIVERSITY OF CALIFORNIA,
IRVINE

Applications of Symmetry Beyond the Standard Model: Invariants and Dualities

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Physics

by

Eric Bryan

Dissertation Committee:
Professor Arvind Rajaraman, Chair
Professor Yuri Shirman
Professor Shirley Li

2026

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DEDICATION

To my wife, Isabella, for her love, patience, and support throughout this journey.

To my parents, for their unwavering love, encouragement, and support throughout my life.

To my advisor, Arvind Rajaraman, for his mentorship and guidance.

Because I know
Today has been
The most perfect day
I've ever seen

— Thom Yorke[1]

All glory goes to my Lord and Savior Jesus Christ.

[1]Radiohead, "Videotape," from *In Rainbows* (2007).

Contents

LIST OF TABLES **v**

ACKNOWLEDGMENTS **vi**

VITA **vii**

ABSTRACT OF THE DISSERTATION **viii**

1 Introduction **1**

1.1 Symmetries and The Standard Model 1

1.2 Open Problems in Physics 3

1.2.1 Strong Coupling 3

1.2.2 CP Violation & Matter-Antimatter Asymmetry 4

1.3 Beyond the Standard Model 5

1.3.1 Multi-Higgs Doublet Models 5

1.3.2 Supersymmetry 6

1.4 Moduli Spaces and Invariants 9

1.4.1 Supersymmetric Gauge Theories and Moduli Spaces 9

1.4.2 Invariant Theory 10

1.5 Overview of the Dissertation 10

Bibliography 11

2 The Hilbert Series and the Flavor Invariants of the 3HDM **15**

2.1 Introduction . . . 15

2.2 Overview of the 3HDM . . . 20

2.3 Hilbert Series . . . 22

2.4 3HDM Hilbert Series Calculation . . . 25

2.4.1 Issues with the Direct Computation . . . 25

2.4.2 Calculation of Hilbert Series . . . 30

2.5 Results for the 3HDM Hilbert Series . . . 33

2.6 Calculation of Invariants up to $\mathcal{O}(\lambda^3)$ . . . 35

2.6.1 The Background Field Approach . . . 37

2.6.2 The $U(1)_1 \times U(1)_2$ Invariants . . . 38

2.6.3 3HDM Matching . . . 39

2.7 Results . . . 41

2.8 CP Properties of the Invariants . . . 46

2.8.1 CP Building Blocks . . . 47

2.8.2 CP on $U(1)_1 \times U(1)_2$ Invariants . . . 49

2.9 Conclusion . . . 53

2.10 Appendix . . . 55

Bibliography . . . 61

3 On the Calculation of Invariant Tensors in Gauge Theories **69**

3.1 Introduction . . . 69

3.2 SU(2) with fields in the dimension 4 representation . . . 72

3.2.1 Overview . . . 72

3.2.2 Counting Invariant Candidates . . . 72

3.2.3 Construction of Invariants . . . 74

3.3 SO(3) with fields in the dimension-5 representation . . . 80

3.3.1 Overview . . . 80

3.3.2 Counting Invariant Candidates 81
3.3.3 Construction of Invariants 83
3.4 SU(3) with sextets 90
3.4.1 Overview 90
3.4.2 Counting Candidate Invariants 91
3.4.3 Construction of Invariants 93
3.5 Summary and Conclusion 95
Bibliography 96

4 The Potency of Nilpotence **100**
4.1 Introduction 100
4.2 Review of A_k and D_{k+2} Dualities 103
4.2.1 A_k Dualities 103
4.2.2 D_{k+2} Dualities 105
4.3 New Tests of Duality Conjectures 107
4.3.1 Existing Tests of the Dualities 107
4.3.2 Dynamics of Nilpotent Flat Directions 108
4.3.3 Duality Tests for A_k Theory 110
4.3.4 Duality Tests for D_{k+2} Theory 113
4.4 Discussion 116
Bibliography 116

LIST OF TABLES

2.1 $U(1)_1 \times U(1)_2$ charges of the field components 38

2.2 3HDM Invariant Operators 42

2.3 CP-even and CP-odd $U(1)_1 \times U(1)_2$ invariants 51

2.4 3HDM CP-even and CP-odd invariants 53

3.1 Invariant tensors for SU(2) with matter in the dimension-four representation 74

3.2 Invariant tensors for SO(3) with matter in the dimension-five representation 82

3.3 Invariant tensors for SU(3) with matter in the sextet representation 92

4.1 Matter content and global charges of the A_k electric theory. 103

4.2 Matter content and global charges of the A_k magnetic theory. 105

4.3 Matter content and global charges of the D_{k+2} electric theory. 105

4.4 Matter content and global charges of the D_{k+2} magnetic theory. 106

4.5 Matter content and global charges of the A_k IR electric theory. 111

4.6 Matter content and global charges of the A_k Dual of the Electric Theory on Higgs Branch. 112

4.7 Matter content and global charges of the D_{k+2} IR electric theory. 114

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The text of Chapter 3 of this dissertation is a reprint of the material as it appears in Yahya Almumin, Jason Baretz, Eric Bryan, and Arvind Rajaraman, "On the Calculation of Invariant Tensors in Gauge Theories," Acta Physica Polonica B, 54, 5-A2 (2023), DOI: 10.5506/APhysPolB.54.5-A2. This material is reproduced under the terms of the Creative Commons Attribution 4.0 International License (CC BY 4.0). The coauthors of this publication are Yahya Almumin, Jason Baretz, and Arvind Rajaraman. Arvind Rajaraman directed and supervised the research forming the basis of this chapter.

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ABSTRACT OF THE DISSERTATION

Applications of Symmetry Beyond the Standard Model:
Invariants and Dualities

by

Eric Bryan

Doctor of Philosophy in Physics

University of California, Irvine, 2026

Professor Arvind Rajaraman, Chair

Symmetries play a fundamental role in quantum field theory, constraining the structure of physical interactions and providing powerful tools for understanding strongly coupled systems. This dissertation investigates two aspects of symmetry in theoretical particle physics: the construction and classification of invariant quantities in gauge theories and the study of infrared dualities in supersymmetric gauge theories.

The first part of this dissertation examines the three-Higgs-doublet model (3HDM), an extension of the Standard Model containing three scalar doublets. The scalar potential possesses a basis-change freedom under transformations of the Higgs fields, motivating the construction of basis-independent quantities that characterize its physical properties. We calculate the multigraded Hilbert series of the 3HDM scalar potential, which encodes the number of independent flavor invariants at each degree. We construct explicit flavor invariants using the background-field method, which simplifies the construction by breaking the flavor symmetry to a residual subgroup. We also determine the transformation properties of these invariants under CP, identifying CP-even and CP-odd invariants.

We further investigate the application of the background-field method to the construction of invariant tensors for more general groups and representations. By studying the residual

symmetries of generic background fields, we systematically identify independent invariant structures and construct corresponding invariant tensor contractions. This provides a general framework for constructing and classifying invariants in gauge theories.

The final part of this dissertation investigates proposed infrared dualities in four-dimensional supersymmetric gauge theories with adjoint matter. In particular, we study theories associated with the A_k and D_{k+2} superpotentials by examining their moduli spaces and the behavior of their proposed dual descriptions along mesonic flat directions. We find that the electric and magnetic descriptions of the A_k theories remain consistent under these deformations, providing an additional nontrivial test of the proposed duality. In contrast, analogous analyses of the D_{k+2} theories reveal inconsistencies between the electric and magnetic descriptions, challenging the proposed duality in these cases.

Together, these results demonstrate how symmetry-based methods can be used both to classify physical observables and to test nonperturbative relationships between quantum field theories.

Chapter 1

Introduction

1.1 Symmetries and The Standard Model

Modern high-energy particle physics is described by the Standard Model Lagrangian, which encodes the behavior and interactions of the known elementary particles, their intrinsic properties, and the symmetries of the universe. The structure of the Standard Model is deeply connected to the principle of symmetry, the idea that certain properties remain unchanged under particular transformations, which provides a common framework for understanding both the properties of elementary particles and the fundamental forces through which they interact [1–3].

The Standard Model possesses a rich variety of symmetries. Continuous spacetime symmetries, such as rotations and Lorentz transformations, describe the invariance of the laws of physics under changes of reference frame. Discrete symmetries, such as charge conjugation (C), parity (P), and time reversal (T), together with the CPT theorem, further constrain the structure of quantum field theories. The fundamental interactions of the Standard Model are governed by continuous local gauge symmetries, whose transformations may vary from point to point in spacetime. Requiring the theory to remain invariant under these local transformations necessitates the introduction of gauge fields, which mediate the fundamen-

tal interactions [1–3].

Mathematically, symmetries in physics are described using group theory. For example, the gauge symmetries of the Standard Model are described by the Lie group

$$SU(3)_C \times SU(2)_L \times U(1)_Y \tag{1.1}$$

where $SU(3)_C$ describes the strong interaction, while $SU(2)_L \times U(1)_Y$ describes the electroweak interaction. The corresponding gauge fields give rise to the gluons, the $W^{\pm}$ and Z bosons, and the photon [1–3].

Symmetries also determine the properties of elementary particles and the interactions between them. Particles are characterized by how they transform under the symmetries of the Standard Model, that is, by their representations under the corresponding symmetry groups. For the gauge symmetries, these representations determine their gauge charges and, consequently, which of the fundamental interactions they experience. For continuous spacetime symmetries, these representations determine the particles' intrinsic angular momentum, or spin. Particles with half-integer spin are called fermions, while particles with integer spin are called bosons. The fermions of the Standard Model consist of the quarks and leptons, while the bosonic sector contains the gauge bosons responsible for mediating the fundamental interactions, as well as the scalar Higgs boson.

The Higgs field plays a unique role in the Standard Model by allowing elementary particles to acquire mass while preserving the underlying gauge structure of the theory. Unlike the other fields of the Standard Model, the Higgs field acquires a nonzero value throughout the vacuum. This vacuum configuration spontaneously breaks the electroweak symmetry, distinguishing the electromagnetic interaction from the weak interaction. Through their interactions with the Higgs field, the $W^{\pm}$ and Z bosons acquire mass, while the photon remains massless. Finally, the quarks and leptons acquire their masses through Yukawa interactions, which couple the fermion fields to the Higgs field [1–3].

1.2 Open Problems in Physics

The Standard Model's predictions have been verified experimentally with remarkable precision. Nevertheless, many fundamental questions remain unresolved, including the behavior of gauge theories at strong coupling, the origin of sufficient CP violation to explain the observed matter–antimatter asymmetry, the nature of dark matter, the hierarchy problem, and the incorporation of gravity into a consistent quantum description of the fundamental interactions [1–4]. In this dissertation, we will focus particularly on questions related to strong coupling and CP violation.

1.2.1 Strong Coupling

Quantum Chromodynamics (QCD) is the quantum field theory describing the strong interaction and governs the interactions of quarks and gluons. QCD is a gauge theory based on the local gauge symmetry $SU(3)_C$, with quarks transforming as matter fields charged under this symmetry [1–3, 5].

In QCD, the beta function is negative, meaning that the interaction strength decreases at high energies and increases at low energies. This behavior is known as asymptotic freedom. Consequently, while perturbative methods provide an accurate description of QCD at sufficiently high energies, they eventually break down as the theory becomes strongly coupled in the infrared. At low energies, QCD exhibits several important nonperturbative phenomena, including confinement. Confinement refers to the fact that quarks and gluons, which carry color charge, are not observed as isolated particles, but instead appear within color-neutral bound states such as mesons and baryons [1–5].

Although the low-energy behavior of QCD can be studied through a variety of nonperturbative and effective approaches, obtaining an analytic understanding directly from the strongly coupled fundamental theory remains difficult. In particular, understanding confinement from first principles remains one of the central problems in quantum field theory [6].

1.2.2 CP Violation & Matter-Antimatter Asymmetry

Observations show that the universe is dominated by matter, with very little free antimatter. The bulk of the observed matter is baryonic, consisting primarily of protons and neutrons [4]. Explaining how a net baryon–antibaryon asymmetry could have developed from the early universe requires three conditions, known as the Sakharov conditions [7]:

- **Baryon Number Violation:** Baryon number counts the number of quarks minus anti-quarks. Without B-violation, any initial baryon–antibaryon asymmetry would remain zero forever.
- **C & CP Violation:** Charge conjugation (C) exchanges particles with antiparticles, while the combined CP transformation additionally reverses their spatial orientation. If C and CP were exact symmetries, processes producing baryons and antibaryons would occur at equal rates. Their violation therefore allows matter and antimatter to be produced differently.
- **Departure from Thermal Equilibrium:** The universe must transition fast enough that these asymmetry-producing interactions do not reverse, preventing the system from returning to balance.

The Standard Model contains CP violation through the mixing of different generations of quarks in the weak interaction, which is described by the Cabibbo–Kobayashi–Maskawa (CKM) matrix [8, 9]. This CP violation can be characterized by the Jarlskog invariant, a quantity that remains unchanged under arbitrary rephasings of the quark fields and would vanish if CP was conserved [10]. The Jarlskog invariant therefore provides a basis-independent characterization of physical CP violation in the quark sector. However, the level of CP violation in the SM is far too small to account for the observed matter–antimatter imbalance [11]. Therefore, this motivates the search for new sources of CP violation in extended models.

1.3 Beyond the Standard Model

The problems discussed above, together with other unresolved questions such as the nature of dark matter, the incorporation of gravity into a quantum description of the fundamental interactions, and the hierarchy problem, suggest that the Standard Model is not a complete description of nature. These open questions motivate the study of physics beyond the Standard Model, in which the particle content, symmetries, or interactions of the Standard Model are extended. In this dissertation, we focus on two important classes of such extensions: multi-Higgs-doublet models and supersymmetric gauge theories.

1.3.1 Multi-Higgs Doublet Models

N-Higgs-Doublet Models (NHDMs) extend the scalar sector of the Standard Model by introducing additional Higgs doublets. The enlarged scalar sector allows for new interactions and additional sources of CP violation that are absent in the Standard Model. These models therefore provide a natural framework for investigating possible extensions of the Higgs sector and new sources of CP violation [12, 13]. The first set of NHDMs, the two-Higgs-doublet models (2HDMs) have been extensively studied as minimal extensions of the Higgs sector [14]. However, it was found that the minimal 2HDM cannot simultaneously accommodate explicit CP violation and a viable dark matter candidate [15].

In this dissertation, we focus on the Three-Higgs-Doublet Model (3HDM), which extends the Standard Model to contain three scalar Higgs doublets. The enlarged scalar sector allows for a richer CP-violating structure and provides additional possibilities for physics beyond the Standard Model. As discussed previously, CP violation in the Standard Model can be characterized through invariant quantities such as the Jarlskog invariant. This motivates the search for analogous invariant quantities in the 3HDM that can characterize the CP properties of its enlarged scalar sector. The presence of three Higgs doublets similarly allows for the construction of many quantities that remain invariant under transformations among the

Higgs fields. Among these invariants, one can identify combinations with definite transformation properties under CP, including CP-odd invariants analogous to the Jarlskog invariant. Such quantities provide a basis-independent means of characterizing possible sources of CP violation in the extended Higgs sector [16–19].

1.3.2 Supersymmetry

Supersymmetry is an extension of spacetime symmetry that relates bosonic and fermionic degrees of freedom. Supersymmetric transformations convert bosons into fermions and fermions into bosons, organizing the fields of a theory into supermultiplets. If supersymmetry were realized in nature, the particles of the Standard Model would therefore be accompanied by superpartners whose spins differ by one-half. Supersymmetric theories are characterized by the number of independent supersymmetry generators, conventionally denoted by $\mathcal{N}$. Throughout this dissertation we restrict our attention to $\mathcal{N} = 1$ supersymmetry [20, 21].

Supersymmetry has been extensively studied as a possible extension of the Standard Model for several reasons. Historically, one of its most important motivations is the hierarchy problem. In the Standard Model, the Higgs mass receives large quantum corrections that tend to drive it toward much larger values than observed. In supersymmetric theories, contributions from bosonic and fermionic superpartners can cancel these large corrections, providing a mechanism for keeping the Higgs mass naturally small. Supersymmetry also improves gauge-coupling unification: in the Minimal Supersymmetric Standard Model, the strong, weak, and electromagnetic gauge couplings evolve toward a common value at high energies [20, 21].

Supersymmetry is also important as a theoretical tool for understanding quantum field theory. The additional symmetry places powerful constraints on the dynamics of supersymmetric theories, allowing certain quantities to be determined exactly even when perturbative methods fail. Non-renormalization theorems, holomorphy, and exact results for the renormalization-group evolution provide access to phenomena that are otherwise extremely

difficult to study analytically. Supersymmetric gauge theories therefore provide an important laboratory for investigating strongly coupled phenomena such as confinement, conformal dynamics, and dualities [20, 21].

Supersymmetric Dualities

In physics, a duality occurs when two seemingly different theoretical descriptions, potentially involving different variables, fields, interactions, or Lagrangians, describe the same underlying physics. Dualities are particularly powerful in quantum field theory when a strongly coupled theory admits an equivalent weakly coupled description, allowing otherwise intractable strong-coupling phenomena to be studied using perturbative methods. Dualities of various forms appear throughout theoretical physics, including in condensed matter physics, string theory, and supersymmetric gauge theories.

Supersymmetric Quantum Chromodynamics (SQCD) provides an important example of duality in quantum field theory. SQCD generalizes QCD by introducing supersymmetry and can be characterized by an $SU(N)$ gauge group with F flavors of quark and antiquark superfields. Like ordinary QCD, these theories can become strongly coupled in the infrared, where perturbative methods are no longer reliable. However, the additional constraints provided by supersymmetry allow considerably more information about their low-energy behavior to be determined [20].

For certain values of N and F, Seiberg proposed that the infrared physics of this theory can be equivalently described by a different supersymmetric gauge theory. The original theory, conventionally called the electric theory, has gauge group $SU(N)$ and F flavors, while its magnetic dual has gauge group $SU(F-N)$, F flavors of magnetic quarks, and additional gauge-singlet meson fields. Despite their different gauge groups and field content, the two theories are conjectured to describe the same physics in the infrared. This equivalence is known as Seiberg duality [20, 22].

Since the two theories are conjectured to describe the same infrared physics, quantities

that characterize their low-energy behavior must agree between the two descriptions. A number of nontrivial consistency checks provide evidence for this equivalence. The electric and magnetic theories possess the same global symmetries and matching 't Hooft anomalies associated with these symmetries. Gauge-invariant operators, quantities that remain unchanged under the local gauge symmetries, can also be mapped between the two descriptions, with electric mesons and baryons corresponding to appropriate operators in the magnetic theory [20, 22].

The importance of Seiberg duality is particularly apparent in the study of strongly coupled physics. In certain regimes, one description of the theory may be strongly coupled while its dual description is weakly coupled, allowing perturbative methods in one theory to provide information about the nonperturbative behavior of the other. Furthermore, for certain values of N and F, the electric and magnetic theories flow to the same interacting fixed point in the infrared, where they provide equivalent descriptions of a superconformal field theory. Supersymmetry places powerful constraints on these theories, making it possible to obtain exact results even in regimes where ordinary perturbative methods fail. Seiberg duality therefore provides an important theoretical laboratory for studying strongly coupled quantum field theories in a controlled setting [20, 22].

While Seiberg duality is well understood for supersymmetric gauge theories with relatively simple matter content, extending these ideas to theories containing additional matter fields leads to a much richer structure. Additional fields can produce more complicated vacuum structures, as well as new possible superpotentials and infrared behavior. Of particular interest to this dissertation are theories containing matter fields transforming in the adjoint representation of the gauge group, for which a number of generalizations of Seiberg duality have been proposed. In this dissertation, we investigate two classes of such proposed dualities, known as the A_k and D_{k+2} dualities [23–25]. The increased complexity of these theories motivates the development of additional consistency checks to determine whether the proposed dualities remain valid.

1.4 Moduli Spaces and Invariants

Both the 3HDM and the conjectured supersymmetric dualities discussed above can be studied through their symmetries and invariant structures. In the 3HDM, Jarlskog-like invariants provide a way to characterize possible new sources of CP violation. In supersymmetric dualities, on the other hand, the agreement of invariant structures between the electric and magnetic descriptions provides an important consistency check of the proposed duality. These applications motivate the introduction of the mathematical tools used to construct and analyze invariant structures in quantum field theories.

1.4.1 Supersymmetric Gauge Theories and Moduli Spaces

When supersymmetry is combined with gauge theories, the fields of the theory are organized into supersymmetric multiplets. The gauge bosons are accompanied by fermionic superpartners known as gauginos, while matter fields are organized into chiral supermultiplets containing both fermionic and scalar degrees of freedom. Supersymmetry places strong constraints on the interactions among these fields and, in particular, on the form of the scalar potential [20].

Of particular importance are field configurations that minimize the scalar potential while preserving supersymmetry. These configurations are constrained by conditions conventionally known as F-flatness and D-flatness. Because many different field configurations can satisfy these conditions, a supersymmetric theory may possess a continuous family of physically distinct vacuum states. The space formed by these possible vacuum states is referred to as the moduli space of the theory [20].

The moduli space can be characterized in terms of the gauge-invariant operators of the theory, whose expectation values distinguish physically inequivalent vacuum configurations. In particular, there is a close correspondence between D-flat field configurations modulo gauge transformations and gauge-invariant operators [26, 27]. This connection provides

a geometric way of studying the invariant structure of a gauge theory and will play an important role throughout this dissertation.

1.4.2 Invariant Theory

Invariant theory studies quantities that remain unchanged under the action of a group. For many of the symmetry groups and representations encountered in particle physics, the set of invariant operators is finitely generated. This means that although infinitely many invariant operators may be constructed, they can all be expressed in terms of a finite set of fundamental invariants, known as generators [28].

Knowing these generators is beneficial because they provide a finite set of quantities that can be used to characterize the invariant structure of a theory. Rather than considering the infinitely many invariant operators that can be constructed, one can instead focus on a finite generating set and the relations among its elements.

1.5 Overview of the Dissertation

The central theme of this dissertation is the study of invariant structures arising in quantum field theory. Although the physical applications range from Higgs-sector phenomenology to supersymmetric gauge theories, they are united by a common mathematical framework based on invariant theory and moduli spaces. The following chapters develop and apply these techniques in several distinct settings. This dissertation is organized as follows.

In Chapter 2, we introduce the mathematical machinery necessary to analyze the invariant structure of the Three-Higgs-Doublet Model (3HDM). In particular, we construct a basis of invariant operators of the scalar potential using techniques from invariant theory, including Hilbert series and the correspondence between gauge-invariant operators and D-flat field configurations. We further investigate the CP properties of these invariants and classify the independent CP-even and CP-odd combinations through the orders considered.

Chapter 3 extends these ideas to the more general problem of constructing invariant tensors in gauge theories. We develop systematic techniques for determining invariant tensors associated with various gauge groups and representations, illustrating the broad applicability of the methods introduced in the previous chapters.

Finally, in Chapter 4, we consider a different application of moduli spaces and gauge-invariant operators in the study of supersymmetric gauge theories. We use the geometry of the moduli space of D-flat field configurations to investigate proposed supersymmetric dualities involving adjoint matter. In particular, we study the A_k and D_{k+2} dualities by moving along specific flat directions in the moduli space and comparing the resulting low-energy theories on the electric and magnetic sides. This provides a new consistency check of the proposed dualities beyond the conventional matching of symmetries, anomalies, and gauge-invariant operators. We find that the A_k dualities pass these consistency checks, while inconsistencies arise for the proposed D_{k+2} dualities in the cases considered.

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Chapter 2

The Hilbert Series and the Flavor Invariants of the 3HDM

This chapter is based on joint work with Arvind Rajaraman and was published in *Physical Review D* [1]. In this work, we apply techniques from invariant theory to systematically construct and classify the flavor invariants of the Three-Higgs-Doublet Model (3HDM). We calculate the multigraded Hilbert series of the 3HDM for the first time, utilizing computational tools together with techniques from complex analysis. Furthermore, we employ background-field methods to explicitly construct a basis of invariant operators in the 3HDM through cubic order in the quartic coupling. The presentation in this chapter follows the published work, with additional discussion and details included where useful.

2.1 Introduction

The N-Higgs-doublet-models (NHDM) are natural extensions of the Standard Model that introduce additional Higgs doublets [2–6]. These are interesting because they can supply new sources for CP violation, possible dark matter candidates, and novel new patterns of electroweak symmetry breaking.

The simplest realization of this framework is the two-Higgs-doublet model (2HDM), which

has been studied extensively as a minimal extension of the Standard Model scalar sector. In the 2HDM, the enlarged Higgs potential allows for both explicit and spontaneous CP violation [7–15]. The 2HDM has also been explored as a framework for generating neutrino masses, either through extended Yukawa structures or via radiative mechanisms that naturally suppress the neutrino mass scale [16–21]. Depending on the imposed symmetries, the model can also accommodate stable scalar particles that serve as viable dark matter candidates [21–32]. Moreover, extended Higgs sectors exhibit nontrivial symmetry-breaking patterns, including electroweak and soft symmetry breaking, that lead to interesting low-energy symmetry structures [33–35]. The Minimal Supersymmetric Standard Model (MSSM) also requires a 2HDM stucture [36], where a second Higgs doublet is required to ensure anomaly cancellation and to provide masses separately to up- and down-type fermions.

However, it was found that the minimal 2HDM cannot simultaneously accommodate explicit CP violation and a viable dark matter candidate [37]. This motivates consideration of the next simplest multi-Higgs-doublet extension. In this work, we consider the three-Higgs-doublet model (3HDM), which constitutes the next nontrivial extension beyond the extensively studied two-Higgs-doublet model. The introduction of a third scalar doublet substantially enlarges the scalar potential and leads to a richer symmetry structure. Consequently, the 3HDM admits a wide array of discrete and continuous global symmetries, together with intricate patterns of spontaneous symmetry breaking that are inaccessible in simpler Higgs sectors. These features give rise to new sources of CP violation, viable dark matter candidates, and novel possibilities for organizing flavor symmetries of quarks and leptons.

Studies of the 3HDM [38–42] have classified the symmetry groups realizable in the scalar potential of the 3HDM. Such symmetries play an important role in determining the phenomenology of multi-Higgs models, including the structure of the scalar spectrum, CP properties, and the pattern of symmetry breaking. It has also been shown that symmetries acting on the three Higgs doublets may provide a natural framework for describing the observed

flavor structure of the Standard Model, and may offer a possible connection to the existence of three generations of quarks and leptons [41, 43].

Previous authors [44–47] have investigated CP violation in a variety of 3HDM scenarios, including models with imposed discrete symmetries. By analyzing the symmetries of the scalar potential and the vacuum structures that arise in these models, they identified regions of parameter space exhibiting either explicit or spontaneous CP violation. There has also been work extending symmetry-based analyses of the 3HDM to investigate viable scalar dark matter candidates [48–54]. These studies demonstrate how residual symmetries—such as S_3 or unbroken $U(1)$ symmetries—can stabilize dark matter within the 3HDM.

Many of the above methods rely on classifying symmetries of the 3HDM and the parameters which are invariant under such symmetries. Therefore a central challenge in the study of extended Higgs sectors is the identification of physical parameters that are independent of basis choices and field redefinitions. Indeed, all observable quantities—such as CP-violating observables analogous to the Jarlskog invariant, as well as invariant combinations appearing in renormalization group equations [55–57]—must ultimately be expressible in terms of operators that are invariant under the full set of local and global symmetries of the theory.

In the case of the 2HDM, this problem has been addressed systematically. A minimal generating set of invariant operators was identified in [58–63]. These approaches made use of bilinears and birdtrack techniques to construct the invariants explicitly. Such methods enabled researchers to address important questions about the 2HDM, for example the identification of CP-odd combinations of physical observables. However, a similar systematic study has not yet been carried out for the 3HDM, primarily due to the much larger parameter space and the increased complexity of the global symmetry and its irreducible representation content.

In this work, we aim to perform a similar analysis for the 3HDM. We present three main results. First, we compute the complete closed-form multigraded Hilbert series of the 3HDM, which encodes both the number of independent invariant operators and their representation

content. As will be described in the following sections, the Hilbert series provides a compact and powerful way to organize invariant operators. Closely related techniques appear in a variety of other physical contexts [64]. For example, in $\mathcal{N} = 1$ supersymmetric QCD, Hilbert series have been used to characterize and match the chiral rings of theories related by Seiberg duality [65–67]. Furthermore, these techniques have also been employed in model building to systematically construct Lagrangians consistent with imposed symmetries [68].

However, Hilbert series are in general difficult to compute for large representations composed of many irreducible components, such as those appearing in the 3HDM. As will be shown below, the task of computing the full Hilbert series of the 3HDM is technically challenging due to the appearance of higher-order poles, branch-cut ambiguities arising in the contour integrals, and computational difficulties associated with summing large polynomials with large coefficients. Indeed, previous studies of the 3HDM Hilbert series were restricted either to power-series expansions of the integrand or to the computation of the ungraded Hilbert series [59], which made use of Ω calculus and Machon's partition analysis [69]. The ungraded Hilbert series is obtained by identifying all grading variables of the multigraded Hilbert series, thereby counting invariant operators only according to their total degree. While this provides the total number of invariants at a given order, it does not retain information about the representation content from which those invariants are constructed. One of the main results of this work is the computation of the closed form multigraded Hilbert series of the 3HDM.

In addition to counting invariant operators using the Hilbert series, our second result resides in constructing the basis of invariant operators explicitly. To this end, we employ a method based on background field techniques. These methods were originally developed and proven in the context of supersymmetric theories [70, 71], but have since found broader application. The basic idea is to treat some parameters (such as the masses) as background values that partially break the global symmetry group. The set of invariants under this reduced symmetry can be classified more easily. One then combines these invariants with

the background fields to reconstruct the invariants of the full symmetry group. As we show, this can be done efficiently by generating candidate invariants of the full theory and successively matching them to the invariants in the theory with reduced symmetry. This approach allows one to systematically build a basis of invariant operators. Variants of this method have been successfully applied in a number of related contexts [72–74].

As will be discussed in the following section, the 3HDM contains three adjoint representations and one **27** representation of a global flavor symmetry group $SU(3)$. While in principle one can compute all independent invariants of the theory, their sheer number makes it impractical to present the full set explicitly. Instead, we restrict our attention to operators up to cubic order in the quartic couplings. For the special case of invariants containing no **27** fields, however, we determine the generating set to arbitrary order in the quartic couplings. We expect this truncation to be sufficient for phenomenological applications.

Our third and final result is the application of these invariants in a phenomenological context. In particular, we use our basis of invariants up to cubic order in the couplings to begin the construction of CP-even and CP-odd basis invariants. We explicitly identify CP-even and CP-odd combinations for invariants involving up to four fields and discuss how the construction may be systematically extended to higher orders.

Our paper is organized as follows. In Section 2.2, we provide an overview of the 3HDM Lagrangian and identify the building blocks required to construct invariant operators. In Section 2.3, we introduce the Hilbert series and present the explicit formula used to compute the Hilbert series for the 3HDM. In Section 2.4, we describe the steps involved in performing this calculation, emphasizing the technical obstacles that previously prevented its completion and explaining how each of these challenges can be overcome. In Section 2.5, we present the results of the Hilbert series calculation.

We then turn to the explicit construction of the independent invariant operators of the 3HDM. In Section 2.6, we describe our approach and present the calculation of the lowest-order invariants of the theory. Section 2.7 presents the resulting basis of invariant operators

through cubic order in the couplings. In Section 2.8, we apply these results in a phenomenological context by constructing CP-even and CP-odd invariants up to quartic order in the field content. Finally, we close with a discussion of our results.

2.2 Overview of the 3HDM

The most general renormalizable scalar potential of the NHDM contains quadratic and quartic terms involving the N new Higgs fields. It may be written as

$$V_H = (\mu^2)_i{}^j(\Phi^\dagger)^i\Phi_j + \lambda_{ij}{}^{kl}\big((\Phi^\dagger)^i\Phi_k\big)\big((\Phi^\dagger)^j\Phi_l\big) \tag{2.1}$$

where the Higgs fields are denoted by Φ_i, with $i = 1, 2, \ldots, N$. Throughout, the $SU(2)_L$ gauge indices are suppressed and we assume $SU(2)_L$ scalar doublets. The introduction of these new terms to the Standard Model Lagrangian induces a global $SU(N)$ symmetry under which the Higgs fields Φ_i transform as fundamentals. We have adopted the convention that fundamental indices are written with lower indices (subscripts), while anti-fundamental indices are written with upper indices (superscripts). Since the center of $SU(N)$ acts trivially on the gauge-invariant bilinears $(\Phi^\dagger)^i\Phi_j$, the faithful symmetry group is $PSU(N) = SU(N)/Z_N$. Nevertheless, we will continue to use the standard $SU(N)$ representation labels when discussing the representation content of the theory. One may also note that although the Higgs fields transform in the fundamental representation of $SU(N)$, the scalar potential depends only on the gauge-invariant bilinears, which transform in the representation $\bar{\mathbf{N}}\otimes\mathbf{N} = \mathbf{1}\oplus\mathbf{Adj}$. Consequently, questions concerning basis transformations, symmetries, and basis-invariant observables are naturally formulated in terms of the singlet and adjoint representations rather than the Higgs fields themselves.

The NHDM contains $(N^4 + N^2 + 2)/2$ physical parameters [75]. Subsequently the 3HDM contains a large number of free parameters and due to the large gauge symmetry, this makes the identification of physical observables particularly nontrivial. All observables must ulti-

mately be expressible in terms of the bilinear parameters $(\mu^2)_i{}^j$ and quartic couplings $\lambda_{ij}{}^{kl}$, subject to invariance under the global $SU(3)$ flavor symmetry. It is therefore natural to begin by classifying these parameters according to their transformation properties under $SU(3)$.

Due to the Hermiticity of the scalar potential and the symmetry of the quartic interaction under interchange of the two gauge-invariant bilinears, the couplings μ and λ satisfy the relations

$$(\mu^2)_i{}^j = ((\mu^2)_j{}^i)^*, \qquad \lambda_{ij}{}^{kl} = \lambda_{ji}{}^{lk} = (\lambda_{kl}{}^{ij})^* \,. \tag{2.2}$$

Under the $SU(3)$ action, the mass parameters $(\mu^2)_i{}^j$ transform in the representation

$$\bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \,. \tag{2.3}$$

The singlet corresponds to the trace component, $(\mu^2)_i{}^i$, while the traceless part transforms in the adjoint representation.

Since the quartic potential is written in terms of gauge-invariant bilinears $(\Phi^\dagger)^i\Phi_j$, the $SU(2)_L$ gauge indices have already been contracted. The remaining representation-theoretic analysis therefore concerns only the flavor indices transforming under $SU(3)$. In order for λ to satisfy the required symmetry properties, its irreducible components must have their fundamental and anti-fundamental indices either both symmetrized or both antisymmetrized. If they are simultaneously symmetrized, the representation is

$$\lambda_{(ij)}^{(kl)} \to \mathbf{6} \otimes \bar{\mathbf{6}} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{27} \,. \tag{2.4}$$

If they are simultaneously antisymmetrized, the representation is

$$\lambda_{[ij]}^{[kl]} \to \mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8} \,. \tag{2.5}$$

Combining the representation content of the bilinear parameters μ^2 and the quartic couplings λ, the scalar potential is parameterized by

$$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{27}\,. \tag{2.6}$$

We are therefore interested in constructing invariant structures built from the **8** and **27** representations of $SU(3)$. We will explicitly construct these representations in Section 2.6 but this is not needed for the calculation of the Hilbert series. Therefore, we will begin first by constructing the Hilbert series for this representation content.

2.3 Hilbert Series

In this section, we describe the basic formulae for the Hilbert series. Our discussion here will be brief and oriented toward physical applications; one can see [59, 61, 64, 65, 67] for more detailed treatments of the Hilbert series.

The Hilbert series is defined as the power series

$$H(t) = \sum_{k=0}^{\infty} d_k\, t^k\,, \tag{2.7}$$

where d_k is the number of independent invariant operators at order k and t is a fugacity that grades the operators according to their degree.

Although the Hilbert series is defined as an infinite power series, it can in fact be expressed as the power series expansion about $t = 0$ of a rational function $N(t)/D(t)$. In this convention, the Hilbert series is not unique: multiplying both the numerator and denominator by an arbitrary polynomial in t leaves the power series expansion unchanged.

To compute the Hilbert series explicitly, one may use a set of standard results from invariant theory. We state these formulae here without proof. For a finite group G, the

Hilbert series is given by the Molien formula

$$H(t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{det(1 - t\rho(g))} \tag{2.8}$$

where $\rho(g)$ denotes the representation of the group element g acting on the fields. This expression admits a natural generalization to continuous groups, where the discrete sum is replaced by an integral over the group,

$$H(t) = \int_G d\mu_G \frac{1}{det(1 - t\rho(g))} \tag{2.9}$$

where $d\mu_G$ is the Haar measure on G. To compute the integrand one makes use of the Plethystic exponential which is defined as

$$\mathrm{PE}[z_j, t, \mathbf{r}] = \exp\left(\sum_{k \geq 1} \frac{t^k \chi_{\mathbf{r}}(z_j^k)}{k}\right) \tag{2.10}$$

where $\chi_{\mathbf{r}}(z_j^k)$ is the character of the representation $\mathbf{r}$. Explicitly, for a matrix representation, ρ, one has

$$\frac{1}{\det(1 - t\rho(g))} = \exp\left(\sum_{k \geq 1} \frac{t^k \,\mathrm{Tr}\left(\rho(g)^k\right)}{k}\right) . \tag{2.11}$$

For Lie groups, the Haar measure has the form

$$\int_G d\mu_G = \frac{1}{(2\pi i)^m} \int_{|z_1|=1} \cdots \int_{|z_m|=1} \frac{dz_1}{z_1} \cdots \frac{dz_m}{z_m} \prod_{\alpha^+} \left(1 - \prod_{i=1}^{m} z_i^{\alpha^+}\right) \tag{2.12}$$

where z_i are the coordinates of the maximal torus of the gauge group G, and m and α^+ are the rank and positive roots of G, respectively, which exist in the $\bar{\mathbf{N}} \otimes \mathbf{N}$ representation of G [67].

In the 3HDM, as discussed above, the relevant field content decomposes into four non-

trivial $SU(3)$ representations: three adjoint representations **8** and one **27**. Consequently, the Hilbert series takes the form of a generating function in four variables, one for each irreducible component. We denote these variables by s, t, u, and q, where s, t, and u correspond to the three **8** representations, while q corresponds to the **27**. The grading variables are treated as formal fugacities and, for the contour-integral representation, are taken inside the unit disk.

We are interested in polynomial invariants of the 3HDM couplings under $SU(3)$ basis transformations. The corresponding Hilbert series, which counts independent invariant polynomials, may be written as [59]

$$H(s,t,u,q) = \frac{1}{(2\pi i)^2}\oint_{|z_1|=1}\frac{dz_1}{z_1}\oint_{|z_2|=1}\frac{dz_2}{z_2}(1-z_1z_2)\left(1-\frac{z_1^2}{z_2}\right)\left(1-\frac{z_2^2}{z_1}\right)\times$$
$$\mathrm{PE}[z_1,z_2,s,\mathbf{8}]\,\mathrm{PE}[z_1,z_2,t,\mathbf{8}]\,\mathrm{PE}[z_1,z_2,u,\mathbf{8}]\,\mathrm{PE}[z_1,z_2,q,\mathbf{27}]\,. \tag{2.13}$$

The functions $\mathrm{PE}[z_1,z_2,s,\mathbf{8}]$ and $\mathrm{PE}[z_1,z_2,q,\mathbf{27}]$ are calculated to be [59]

$$\mathrm{PE}[z_1,z_2,s,\mathbf{8}] = \left[(1-s)^2\left(1-s\frac{z_1^2}{z_2}\right)\left(1-s\frac{z_2}{z_1^2}\right)\left(1-s\frac{1}{z_1z_2}\right)\times\right.$$
$$\left.(1-sz_1z_2)\left(1-s\frac{z_1}{z_2^2}\right)\left(1-s\frac{z_2^2}{z_1}\right)\right]^{-1}, \tag{2.14}$$

and

$$\mathrm{PE}[z_1,z_2,q,\mathbf{27}] = \left[(1-q)^3\left(1-q\frac{1}{z_2^3}\right)\left(1-qz_2^3\right)\left(1-q\frac{1}{z_1z_2}\right)^2(1-qz_1z_2)^2\left(1-q\frac{z_2^2}{z_1}\right)^2\times\right.$$
$$\left(1-q\frac{z_1}{z_2^2}\right)^2\left(1-q\frac{z_2}{z_1^2}\right)^2\left(1-q\frac{z_1^2}{z_2}\right)^2\left(1-q\frac{1}{z_1^2z_2^2}\right)\left(1-qz_1^2z_2^2\right)\times$$
$$\left(1-q\frac{z_2^4}{z_1^2}\right)\left(1-q\frac{z_1^2}{z_2^4}\right)\left(1-q\frac{1}{z_1^3}\right)\left(1-qz_1^3\right)\left(1-q\frac{z_2^3}{z_1^3}\right)\times$$
$$\left.\left(1-q\frac{z_1^3}{z_2^3}\right)\left(1-q\frac{z_2^2}{z_1^4}\right)\left(1-q\frac{z_1^4}{z_2^2}\right)\right]^{-1}. \tag{2.15}$$

The ungraded Hilbert series is recovered by identifying all grading variables, $s = t = u = q$. This specialization counts invariant operators according only to their total degree and therefore retains less information than the multigraded Hilbert series. Nevertheless, the ungraded Hilbert series remains a useful quantity, as it determines the total number of independent invariants at each order. The corresponding integral was evaluated in Ref. [59]. In addition, Ref. [59] obtained a power-series expansion of the Molien–Weyl integrand about $z_1 = z_2 = 0$, allowing the Hilbert series to be determined through low orders in the couplings. These results provide valuable consistency checks on the multigraded Hilbert series computed in the present work.

2.4 3HDM Hilbert Series Calculation

The evaluation of this contour integral for the Hilbert series however presents several difficulties. Previous studies of the 3HDM Hilbert series have been limited either to local expansions of the integrand or to the computation of the ungraded Hilbert series [59].

In this section, we describe these difficulties and the sequence of standard computational techniques used to overcome them. The methods employed are based on established tools from Hilbert-series computations, complex analysis, and invariant theory rather than a new algorithmic framework. Readers interested only in the final form of the Hilbert series may skip directly to Section 2.5.

2.4.1 Issues with the Direct Computation

A direct evaluation of the integral faces several serious technical obstacles. First, the integrand typically contains higher-order poles arising from quadratic factors in the denominator, such as $(1 - qz_1z_2)^2$. These lead to poles whose residues require the evaluation of derivatives and rapidly become algebraically cumbersome when many such factors are present. In practice, this already renders a naive residue computation inefficient and unstable in symbolic

computation software.

A second complication arises from the structure of terms with quadratic or quartic poles in either z_1 or z_2. Depending on the order of integration, performing one contour integral can introduce branch points in the remaining complex variable, thereby generating branch cuts that obstruct a straightforward application of the residue theorem. This complicates the analytic structure of the integrand and obscures the interpretation of the contour prescription.

Finally, even if one were to formally classify all poles of the integrand, their number typically grows into the thousands. Summing the corresponding residues is computationally prohibitive in practice, either in `Mathematica` and in numerical or symbolic implementations in `Python` [59].

Higher Order Poles

We first address the higher order poles. These factors are

$$\left[\left(1-q\frac{z_2}{z_1^2}\right)^2\left(1-q\frac{z_1^2}{z_2}\right)^2\left(1-q\frac{1}{z_1z_2}\right)^2(1-qz_1z_2)^2\left(1-q\frac{z_2^2}{z_1}\right)^2\left(1-q\frac{z_1}{z_2^2}\right)^2\right]^{-1}. \quad (2.16)$$

These higher-order poles can be handled by rewriting the expression in terms of derivatives. For example, we can write

$$\left[\left(1-q\frac{z_2}{z_1^2}\right)^2\left(1-q\frac{z_1^2}{z_2}\right)^2\right]^{-1} = \frac{\partial^2}{\partial q_2\partial q_1}\left[\left(1-q_1\frac{z_2}{z_1^2}\right)\left(1-q_2\frac{z_1^2}{z_2}\right)\right]^{-1}\Bigg|_{q_1\to q,q_2\to q} \quad (2.17)$$

Since we have six higher order poles, we introduce six variables $q_1, q_2, \ldots, q_6$. These are treated as independent during the differentiation, a standard technique in complex analysis. After performing the contour integrals, we take the limit in which all q_i are set equal to q.

The Hilbert Series can be written as

$$H(s,t,u,q) = \frac{\partial^6}{\partial q_1 \partial q_2 \dots \partial q_6} \oint dz_1 dz_2 \, \frac{N(z_1,z_2)}{D(z_1,z_2,s,\dots,q) \times D_{cubic}(z_1,z_2,q)} \Bigg|_{q_1 \to q, \dots, q_6 \to q} \tag{2.18}$$

where

$$N(z_1,z_2) = \frac{1}{(1-s)^2(1-t)^2(1-u)^2(1-q)^3} \frac{1}{(2\pi i)^2} \times \frac{(1-z_1 z_2)(1-\frac{z_1^2}{z_2})(1-\frac{z_2^2}{z_1})}{z_1 z_2}\,, \tag{2.19}$$

and the denominator of the integrand has now only simple poles

$$\begin{aligned} D(z_1,z_2,s,\dots,q) = &\prod_i^6 (1-\alpha_i^- \frac{z_2}{z_1^2}) \prod_i^6 (1-\alpha_i^+ \frac{z_1^2}{z_2}) \prod_i^6 (1-\beta_i^- \frac{1}{z_1 z_2}) \prod_i^6 (1-\beta_i^+ z_1 z_2) \times \\ &\prod_i^6 (1-\gamma_i^- \frac{z_2^2}{z_1}) \prod_i^6 (1-\gamma_i^+ \frac{z_1}{z_2^2})\,, \end{aligned} \tag{2.20}$$

$$D_{cubic}(z_1,z_2,q) = \left(1-q\frac{1}{z_2^3}\right)\left(1-qz_2^3\right)\left(1-q\frac{1}{z_1^3}\right)\left(1-qz_1^3\right)\left(1-q\frac{z_2^3}{z_1^3}\right)\left(1-q\frac{z_1^3}{z_2^3}\right). \tag{2.21}$$

Here, we have defined the sets of variables

$$\alpha^- = \{s,t,u,\sqrt{q},-\sqrt{q},q_1\} \tag{2.22}$$

$$\alpha^+ = \{s,t,u,\sqrt{q},-\sqrt{q},q_2\} \tag{2.23}$$

$$\beta^- = \{s,t,u,\sqrt{q},-\sqrt{q},q_3\} \tag{2.24}$$

$$\beta^+ = \{s,t,u,\sqrt{q},-\sqrt{q},q_4\} \tag{2.25}$$

$$\gamma^- = \{s,t,u,\sqrt{q},-\sqrt{q},q_5\} \tag{2.26}$$

$$\gamma^+ = \{s,t,u,\sqrt{q},-\sqrt{q},q_6\}\,. \tag{2.27}$$

Since we are assuming all variables s,t,u,q have magnitude < 1, variables in the α^- array are the ones which contribute poles during the z_1 integration while variables in the α^+

array do not contribute poles.

The Partial Fraction Expansion

Despite the first simplification, the integrand still has too many poles to be evaluated by symbolic codes. We therefore continue the simplification.

We decompose the following functions into partial fraction form

$$A = \left[\prod_i^6 (1-\alpha_i^- \frac{z_2}{z_1^2}) \prod_i^6 (1-\alpha_i^+ \frac{z_1^2}{z_2})\right]^{-1} = \sum_{i=1}^6 A_i^- \frac{z_2}{z_1 - \alpha_i^- z_2^2} + \sum_{i=1}^6 A_i^+ \frac{z_2}{\alpha_i^+ z_1 - z_2^2} \tag{2.28}$$

$$B = \left[\prod_i^6 (1-\beta_i^- \frac{1}{z_1 z_2}) \prod_i^6 (1-\beta_i^+ z_1 z_2)\right]^{-1} = \sum_{i=1}^6 B_i^- \frac{1}{z_1 z_2 - \beta_i^-} + \sum_{i=1}^6 B_i^+ \frac{1}{\beta_i^+ z_1 z_2 - 1} \tag{2.29}$$

$$C = \left[\prod_i^6 (1-\gamma_i^- \frac{z_2^2}{z_1}) \prod_i^6 (1-\gamma_i^+ \frac{z_1}{z_2^2})\right]^{-1} = \sum_{i=1}^6 C_i^- \frac{z_2^2}{z_1^2 - \gamma_i^- z_2} + \sum_{i=1}^6 C_i^+ \frac{z_2^2}{\gamma_i^+ z_1^2 - z_2} \tag{2.30}$$

where

$$A_i^- = \frac{(\alpha_i^-)^6}{\prod_j (1-\alpha_i^- \alpha_j^+) \prod_{j\neq i} (\alpha_i^- - \alpha_j^-)} \tag{2.31}$$

$$A_i^+ = \frac{(\alpha_i^+)^5}{\prod_j (1-\alpha_i^+ \alpha_j^-) \prod_{j\neq i} (\alpha_i^+ - \alpha_j^+)} \tag{2.32}$$

and B_i^-, B_i^+, C_i^-, and C_i^+ can similarly be computed.

This transformation reduces the Hilbert series integrand from an expression that is prohibitively difficult to evaluate to one that is considerably more tractable. Naively, the computation requires the evaluation and summation of $6 \times 6 \times 6 \times 8 = 1728$ complex contour integrals. However, this number can be reduced by temporarily treating the variables $\alpha_i^\pm$, $\beta_i^\pm$, and $\gamma_i^\pm$ as independent dummy variables, which we denote by a, b, and c, respectively.

We then need to calculate eight distinct integrals

$$I_1(a,b,c) = \oint_{|z_1|=1} dz_1 \oint_{|z_2|=1} dz_2 \frac{(z_2)^3}{(z_1^2 - az_2)(z_1 z_2 - b)(z_1 - cz_2^2)} \frac{N(z_1,z_2)}{D_{cubic}(z_1,z_2)} \tag{2.33}$$

$$I_2(a,b,c) = \oint_{|z_1|=1} dz_1 \oint_{|z_2|=1} dz_2 \frac{(z_2)^3}{(az_1^2 - z_2)(z_1 z_2 - b)(z_1 - cz_2^2)} \frac{N(z_1,z_2)}{D_{cubic}(z_1,z_2)} \tag{2.34}$$

$$I_3(a,b,c) = \oint_{|z_1|=1} dz_1 \oint_{|z_2|=1} dz_2 \frac{(z_2)^3}{(z_1^2 - az_2)(bz_1 z_2 - 1)(z_1 - cz_2^2)} \frac{N(z_1,z_2)}{D_{cubic}(z_1,z_2)} \tag{2.35}$$

$$\vdots$$

$$I_8(a,b,c) = \oint_{|z_1|=1} dz_1 \oint_{|z_2|=1} dz_2 \frac{(z_2)^3}{(az_1^2 - z_2)(bz_1 z_2 - 1)(cz_1 - z_2^2)} \frac{N(z_1,z_2)}{D_{cubic}(z_1,z_2)} \,. \tag{2.36}$$

Calculation of Residues

Each of these integrands has nine distinct factors and can now be calculated within the computational powers of symbolic programs. However, there is still a technical issue related to the occurrence of poles with square roots of the second variable.

To see how this problem arises, consider the factors in the denominator of the integrand in I_1

$$[(z_1^2 - az_2)(z_1 - cz_2^2)]^{-1} \,. \tag{2.37}$$

If the z_1 contour integral were performed first, these poles would occur at $z_1 = \pm\sqrt{az_2}$, and would lead to an integral depending not just on z_2 but on its square root. This would introduce an apparent branch cut in the subsequent z_2 integration.

To avoid this issue, we first perform a partial fraction decomposition of the integrand with respect to the variable z_1, treating z_2 as a fixed parameter. This isolates the factor $(z_1^2 - az_2)$ into a separate term. For the term containing this factor, we then evaluate the z_2 contour integral before carrying out the z_1 integration. With this approach, the integration over z_2 and the subsequent z_1 integral can be evaluated using standard residue techniques

without introducing any branch-cut ambiguities.

A second issue is the appearance of poles involving ratios of the parameters a, b, c, and q. For example, after performing the first contour integration over z_1, we may obtain new terms in the integrand of the form, for example, $(b^2 - az_2^3)$ with poles proportional to expressions such as b^2/a. Ambiguities arise for such poles involving ratios of variables, such as s/q or q/s, and even expressions like $q/q = 1$. Since no relative ordering between the variables is assumed, it is not a priori clear whether such poles should be enclosed by the integration contour.

Fortunately, we find, both analytically and computationally, that all poles involving ratios of variables cancel in the full expression for the integrand. As a result, the final Hilbert series is independent of any arbitrary ordering among s, t, u, and q.

We can therefore perform the contour integrals over z_1, z_2 for each I_i using `Mathematica`.

2.4.2 Calculation of Hilbert Series

Once the integrals $I_1, \ldots, I_8$ have been evaluated, the Hilbert series can be written as

$$H(s,t,u,q) = \sum_{i=1}^{6}\sum_{j=1}^{6}\sum_{k=1}^{6} I_{ijk}\,, \tag{2.38}$$

where the quantities I_{ijk} are

$$I_{ijk} = \frac{\partial^6}{\partial q_1\,\partial q_2 \cdots \partial q_6}\Big(A_i^- B_j^- C_k^-\, I_1(\alpha_i^-, \beta_j^-, \gamma_k^-) + \cdots + A_i^+ B_j^+ C_k^+\, I_8(\alpha_i^+, \beta_j^+, \gamma_k^+)\Big)\Big|_{q_1=\cdots=q_6=q}\,. \tag{2.39}$$

For fixed values of i, j, and k, the computation of each individual I_{ijk} is straightforward. Now, however, each I_{ijk} is of the form $I_{ijk} = N_{ijk}/D_{ijk}$, where both the numerator and denominator are lengthy expressions. For example, the denominator of $I_{6,6,6}$ contains on the order of one hundred factors, while its expanded numerator contains roughly 10^4 terms, with

coefficients as large as $\mathcal{O}(10^{10})$ and monomials of degree up to $\mathcal{O}(s^7t^7u^7q^{59})$. This complexity makes summation over i, j, k computationally difficult.

To address this issue, we first bring the sum of terms to a common denominator

$$\sum_{i,j,k} I_{ijk} = \frac{N_{\text{LCM}}}{D_{\text{LCM}}}\,. \tag{2.40}$$

To compute D_{LCM} we used the `PolynomialLCM` function in `Mathematica`. Computing N_{LCM}, however, poses a significantly greater challenge. Indeed, one may write

$$N_{LCM} = \sum_{ijk} N_{ijk} \times \frac{D_{LCM}}{D_{ijk}} \tag{2.41}$$

$$= \sum_{ijk} N_{ijk} \times D'_{ijk} \tag{2.42}$$

where D'_{ijk} is obtained by canceling all factors of D_{ijk} from D_{LCM}, which is always possible since D_{LCM} is the least common multiple of all D_{ijk}.

Fully expanding each product $N_{ijk} \times D'_{ijk}$ is computationally demanding. The expressions D'_{ijk} typically contain hundreds of factors, the maximum powers of s, t, u, and q appearing in $N_{ijk} \times D'_{ijk}$ are of order 90, 90, 90, and 300, respectively, and the integer coefficients in these expansions can reach values as large as $\mathcal{O}(10^{27})$. Furthermore, at this stage N_{LCM} and D_{LCM} are not polynomials in q, but rather in $\sqrt{q}$, due to the dependence introduced by the arrays defined in Eqs. (2.22)–(2.27).

We found that the most effective way to address the computational complexity is to stop using symbolic algebra packages such as `Mathematica` or `Python`'s `SymPy`, and instead work directly with coefficient lists of `Python` integers. Since the coefficients can exceed $2^{63} - 1 \simeq 9 \times 10^{18}$, standard `NumPy Int64` types are insufficient, and one must instead use arbitrary-precision `Python` integers.

Defining $Q \equiv \sqrt{q}$, we write

$$N_{ijk} = \sum_{i_1,i_2,i_3,i_4} A_{i_1,i_2,i_3,i_4} s^{i_1} t^{i_2} u^{i_3} Q^{i_4} \tag{2.43}$$

where the coefficient tensor A_{i_1,i_2,i_3,i_4} can be computed directly in `Mathematica` using the `CoefficientList` function.

Now, for instance, to multiply N_{ijk} by the first factor appearing in D'_{ijk}—such as $(-1+q)$—we compute

$$N_{ijk}(-1+q) = -\left(\sum_{i_1,i_2,i_3,i_4} A_{i_1,i_2,i_3,i_4} s^{i_1} t^{i_2} u^{i_3} Q^{i_4}\right) + \left(\sum_{i_1,i_2,i_3,i_4} A_{i_1,i_2,i_3,i_4} s^{i_1} t^{i_2} u^{i_3} Q^{i_4+2}\right).$$

This multiplication then reduces to summing two four-dimensional arrays. This procedure generalizes to all factors appearing in D'_{ijk}.

Since each product $N_{ijk} D'_{ijk} \equiv N'_{ijk}$ is a four-dimensional array with dimensions as large as $(90, 90, 90, 600)$—corresponding to the maximal exponents of s, t, u, and $\sqrt{q}$—and with integer coefficients reaching values of order 10^{27}, this procedure consumes a substantial amount of RAM. Nevertheless, it is still computationally less demanding than manipulating the corresponding polynomials directly in `Mathematica` or `Python`'s `SymPy`.

At this stage, the Hilbert series has been written in the form

$$H(s,t,u,q) = \frac{N_{LCM}}{D_{LCM}}, \tag{2.44}$$

where D_{LCM} is the least common multiple of the denominators appearing in the intermediate expressions. However, N_{LCM} and D_{LCM} may still contain common polynomial factors.

Dividing out all common factors using standard polynomial algorithms in `Mathematica` or `Python` would require writing N_{LCM} explicitly as a polynomial, which is prohibitively memory intensive. Instead, it is far more practical to retain N_{LCM} in its coefficient-array representation and carry out the division directly at the level of arrays polynomial synthetic

division formulas.

For example, consider the factor, $(s-t)$. We first verify whether this divides N_{LCM} by checking whether $N_{LCM}=0$ when $s=t$. This can be done straightforwardly at the level of the coefficient array for N_{LCM}. If N_{LCM} is divisible by $(s-t)$, we write

$$N_{LCM}=\sum a_i(t,u,Q)s^i \qquad \frac{N_{LCM}}{(s-t)}=\sum b_i(t,u,Q)s^i \tag{2.45}$$

One derives

$$b_{n-1}=a_n \qquad b_i=a_{i+1}+tb_{i+1} \tag{2.46}$$

where each a_i and b_i is a three dimensional coefficient array in t, u, and q. The division can then be done directly by manipulating the coefficient arrays.

Using these methods, we can find an explicit expression for the Hilbert series after all common factors have been divided through.

2.5 Results for the 3HDM Hilbert Series

In this section, we present the results of our Hilbert series calculation for the 3HDM. The computation of the multigraded Hilbert series was carried out using several independent Mathematica and Python scripts, including the evaluation of the Molien–Weyl integral, partial-fraction decomposition, and the subsequent synthetic division used to obtain the minimal form of the Hilbert series. As these calculations were performed in multiple stages, an exact total runtime is difficult to determine. However, the cumulative runtime was approximately 100 CPU-hours.

The calculation required access to a high-memory computing cluster with 124 GB of RAM. In particular, several intermediate expressions exceeded the memory available on standard cloud-computing instances with 51 GB of RAM, preventing the full calculation

from being completed in that environment.

The Hilbert series can be written as

$$H(s,t,u,q) = \frac{N}{D} \tag{2.47}$$

where the denominator is

$$\begin{aligned} D = {} & (1-q)(1+q)^2(1-q^3)^5(1-q^4)^4(1-q^5)^4(1-q^6)^2(1-q^7)^2(1-q^9) \\ & (1-qs)(1-q^2s)^3(1-q^3s)^2(1-s^2)(1-qs^2)^4(1-q^2s^2)^2(1-q^3s^2)^2 \\ & (1-s^3)(1-qs^3)^2(1-q^2s^3)(1-q^3s^3)(1-qs^4)(1-q^3s^4) \\ & (1-qt)(1-q^2t)^3(1-q^3t)^2(1-st)^2(1-qst)^2(1-s^2t)(1-qs^2t) \\ & (1-t^2)(1-qt^2)^4(1-q^2t^2)^2(1-q^3t^2)^2(1-st^2)(1-qst^2)(1-qs^2t^2) \\ & (1-t^3)(1-qt^3)^2(1-q^2t^3)(1-q^3t^3)(1-qt^4)(1-q^3t^4) \\ & (1-qu)(1-q^2u)^3(1-q^3u)^2(1-su)^2(1-qsu)^2(1-s^2u)(1-qs^2u) \\ & (1-tu)^2(1-qtu)^2(1-stu)(1-t^2u)(1-qt^2u) \\ & (1-u^2)(1-qu^2)^4(1-q^2u^2)^2(1-q^3u^2)^2(1-su^2)(1-qsu^2)(1-qs^2u^2) \\ & (1-tu^2)(1-qtu^2)(1-qt^2u^2)(1-u^3)(1-qu^3)^2(1-q^2u^3)(1-q^3u^3) \\ & (1-qu^4)(1-q^3u^4)\,. \end{aligned} \tag{2.48}$$

The fully expanded numerator of the multigraded Hilbert series contains 31,689,364 terms and is therefore far too large to display explicitly. The complete numerator will therefore be made available in an online repository [76]. Here, we present the first few terms of the

numerator. Up to $\mathcal{O}\big((stuq)^6\big)$, the numerator is

$$\begin{aligned}
N = 1 &+ s^3tu + s^2t^2u + s^2t^2 + s^2tu^2 + 3s^2tu + s^2u^2 + st^3u + st^2u^2 + 3st^2u + stu^3 \\
&+ 3stu^2 + stu - st - su + t^2u^2 - tu + q - qu - 3qu^2 - qu^3 - qt - 2qtu \\
&+ 3qtu^2 + 7qtu^3 - 3qt^2 + 3qt^2u + 8qt^2u^2 - qt^3 + 7qt^3u - qs - 2qsu + 3qsu^2 \\
&+ 7qsu^3 - 2qst + 10qstu + 20qstu^2 + 3qst^2 + 20qst^2u + 7qst^3 - 3qs^2 + 3qs^2u \\
&+ 8qs^2u^2 + 3qs^2t + 20qs^2tu + 8qs^2t^2 - qs^3 + 7qs^3u + 7qs^3t - 3q^2u - 2q^2u^2 \\
&+ 4q^2u^3 - 3q^2t + 4q^2tu + 20q^2tu^2 - 2q^2t^2 + 20q^2t^2u + 4q^2t^3 - 3q^2s + 4q^2su \\
&+ 20q^2su^2 + 4q^2st + 40q^2stu + 20q^2st^2 - 2q^2s^2 + 20q^2s^2u + 20q^2s^2t \\
&+ 4q^2s^3 - 3q^3u + 9q^3u^2 - 2q^3t + 26q^3tu + 9q^3t^2 - 2q^3s + 26q^3su \\
&+ 26q^3st + 9q^3s^2 - 3q^3 - 4q^4 + 10q^4u + 10q^4t + 10q^4s - q^5 + \mathcal{O}\big((stuq)^6\big) \,. \qquad (2.49)
\end{aligned}$$

The Hilbert series can be expanded as a power series, and we find that this expansion exactly reproduces the results of [59]. In addition, ungraded Hilbert series may be obtained by setting appropriate subsets of the grading variables equal to one another or to zero. Performing this procedure provides a further consistency check, and we verify that the resulting ungraded Hilbert series also agrees with the results of [59].

2.6 Calculation of Invariants up to $\mathcal{O}(\lambda^3)$

We now return to the 3HDM (discussed above in Section 2.2) and attempt to construct the invariants of this theory. As described there, physical observables must be polynomials in the parameters $(\mu^2)_i{}^j$ and $\lambda_{ij}{}^{kl}$ that are invariant under the $SU(3)$ group of field redefinitions. Consequently, the problem of identifying physical observables reduces to determining all polynomial combinations of these parameters that remain invariant under this symmetry.

Recall, under the $SU(3)$ action, the mass parameters $(\mu^2)_i{}^j$ transform in the representa-

tion

$$\bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8}\,, \tag{2.50}$$

Explicitly, these are the $SU(3)$ singlet

$$det(\mu^2) = \mu_{11}^2 \mu_{22}^2 \mu_{33}^2 \tag{2.51}$$

as well as an $SU(3)$ adjoint

$$(V^1)_i{}^j = (\mu^2)_i{}^j - \frac{1}{3}\sum_k (\mu^2)_k{}^k \delta_i{}^j\,. \tag{2.52}$$

$\lambda_{ij}{}^{kl}$ contains two adjoints; these are

$$(V^{(2)})_i{}^k = \sum_{l,j} \left(\lambda_{ij}{}^{kl}\delta_l{}^j + \lambda_{ij}{}^{lk}\delta_l{}^j\right) - \frac{1}{3}\sum_{n,m}(\lambda_{nm}{}^{nm} + \lambda_{nm}{}^{mn})\delta_i{}^k \tag{2.53}$$

$$(V^{(3)})_m{}^n = \sum_{i,j,k,l} \lambda_{ij}{}^{kl}\epsilon_{klm}\epsilon^{ijn} - \frac{1}{3}\sum_{i,j,k,l,p} \lambda_{ij}{}^{kl}\epsilon_{klp}\epsilon^{ijp}\delta_m{}^n \tag{2.54}$$

and the remaining components are organized into a traceless $W_{ij}{}^{kl}$ in the **27** of $SU(3)$,

$$\begin{aligned} W_{ij}{}^{kl} = \lambda_{ij}{}^{kl} + \lambda_{ji}{}^{kl} &+ \frac{1}{20}\sum_{m,n}(\lambda_{mn}{}^{mn} + \lambda_{mn}{}^{nm})(\delta_i{}^k\delta_j{}^l + \delta_j{}^k\delta_i{}^l) \\ &- \frac{1}{5}\sum_n \big((\lambda_{in}{}^{nk} + \lambda_{in}{}^{kn})\delta_j{}^l + (\lambda_{jn}{}^{nk} + \lambda_{jn}{}^{kn})\delta_i{}^l \\ &\qquad + (\lambda_{in}{}^{nl} + \lambda_{in}{}^{ln})\delta_j{}^k + (\lambda_{jn}{}^{nl} + \lambda_{jn}{}^{ln})\delta_i{}^k\big)\,. \end{aligned} \tag{2.55}$$

We now look for combinations of these parameters which are invariant under the symmetry. The number of independent invariants grows rapidly with the degree of the operator. We therefore restrict our explicit classification to invariants containing up to three quartic couplings, $\mathcal{O}(\lambda^3)$. This already yields a substantial set of independent invariants and is suf-

ficient to illustrate the structure of the invariant ring. (For the special case of invariants containing no **27** fields, we extend the analysis to arbitrary order in the quartic coupling λ.) A complete classification of the full invariant basis is left for future work.

Unlike the quartic couplings, however, we do not impose any restriction on the number of mass insertions. Consequently, the invariants may contain arbitrary powers of the mass parameters. In our case, the masses are in a $\mathbf{1} \otimes \mathbf{8}$ of $SU(3)$. The $\mathbf{1}$ is an invariant by itself, and can be factored out of any invariants in which it occurs; however, the $\mathbf{8}$ cannot be treated so easily. Invariants may a priori contain several factors of the $\mathbf{8}$.

To find these invariants, we use a method motivated by the background field approaches in field theory [70, 71].

2.6.1 The Background Field Approach

We move to the mass basis, where the mass terms are diagonal

$$\mu^2 = \begin{pmatrix} \mu_{11}^2 & 0 & 0 \\ 0 & \mu_{22}^2 & 0 \\ 0 & 0 & \mu_{33}^2 \end{pmatrix}. \tag{2.56}$$

We assume a generic point in parameter space with non-degenerate masses. This may be viewed as choosing a representative on a generic $SU(3)$ orbit, or equivalently as turning on background values for the singlet and adjoint components of the mass matrix.

For a generic diagonal mass matrix, the subgroup of $SU(3)$ which leaves the background invariant is the maximal torus $U(1)_1 \times U(1)_2$, generated by

$$t^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad t^8 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.57}$$

We denote the abelian symmetries generated by t^3 and t^8 as $U(1)_1$ and $U(1)_2$, respectively.

The invariants of the theory, which were originally polynomials in $(\mu^2)_i{}^j$ and $\lambda_{ij}{}^{kl}$ invariant under $SU(3)$, now just become polynomials in $\lambda_{ij}{}^{kl}$ which are invariant under the residual $U(1)_1 \times U(1)_2$ symmetry. We shall refer to these below as the $U(1)_1 \times U(1)_2$ invariants.

2.6.2 The $U(1)_1 \times U(1)_2$ Invariants

To find these invariants, we first find the charges of the components of $\lambda_{ij}{}^{kl}$. These are summarized in the table below. The first column lists all distinct $U(1)_1 \times U(1)_2$ charges, while the second column indicates the corresponding field components that carry each charge.

In each case, we have eliminated components which are constrained. For example, since the adjoint is traceless, we must have $(V^I)_1{}^1 + (V^I)_2{}^2 + (V^I)_3{}^3 = 0$, we can eliminate $(V^I)_3{}^3$ in favor of the other fields. Similarly, we can eliminate components of W with both a fundamental 3 index as well as an antifundamental 3 index.

$U(1)_1 \times U(1)_2$ Charge	Field(s)
$Q_{4,0}$	$W_{22}{}^{33}$
$Q_{3,1}$	$W_{22}{}^{13}$
$Q_{3,-1}$	$W_{12}{}^{33}$
$Q_{2,2}$	$W_{22}{}^{11}$
$Q_{2,0}$	$W_{12}{}^{13}$, $W_{22}{}^{23}$, $(V^2)_2{}^3$, $(V^3)_2{}^3$
$Q_{2,-2}$	$W_{11}{}^{33}$
$Q_{1,1}$	$W_{12}{}^{11}$, $W_{22}{}^{12}$, $(V^2)_2{}^1$, $(V^3)_2{}^1$
$Q_{1,-1}$	$W_{11}{}^{13}$, $W_{12}{}^{23}$, $(V^2)_1{}^3$, $(V^3)_1{}^3$
$Q_{0,2}$	$W_{23}{}^{11}$
$Q_{0,0}$	$W_{11}{}^{11}$, $W_{12}{}^{12}$, $W_{22}{}^{22}$, $(V^2)_1{}^1$, $(V^3)_1{}^1$,$(V^2)_2{}^2$,$(V^3)_2{}^2$
$Q_{0,-2}$	$W_{11}{}^{23}$
$Q_{-1,1}$	$W_{13}{}^{11}$, $W_{23}{}^{12}$, $(V^2)_3{}^1$, $(V^3)_3{}^1$
$Q_{-1,-1}$	$W_{11}{}^{12}$, $W_{12}{}^{22}$, $(V^2)_1{}^2$, $(V^3)_1{}^2$
$Q_{-2,2}$	$W_{33}{}^{11}$
$Q_{-2,0}$	$W_{13}{}^{12}$, $W_{23}{}^{22}$, $(V^2)_3{}^2$, $(V^3)_3{}^2$
$Q_{-2,-2}$	$W_{11}{}^{22}$
$Q_{-3,1}$	$W_{33}{}^{12}$
$Q_{-3,-1}$	$W_{13}{}^{22}$
$Q_{-4,0}$	$W_{33}{}^{22}$

It is now straightforward to find combinations of these parameters which are invariant under the residual symmetry. These $U(1)_1 \times U(1)_2$-invariant polynomials are listed in the Appendix.

2.6.3 3HDM Matching

The $U(1)_1 \times U(1)_2$-invariants are in fact a complete set of invariants of the 3HDM. However, they are written in a form that is basis dependent, since we have treated the mass terms as a background field breaking the symmetry. We will now find the invariants with full $SU(3)$ symmetry, which is done by combining the $U(1)_1 \times U(1)_2$ invariant polynomials listed in the Appendix with powers of the masses.

As a first illustrative example, consider the components $(V^2)_1{}^1$ and $(V^2)_2{}^2$, which are completely invariant under $U(1)_1 \times U(1)_2$. These objects correspond to $\mathcal{O}(\lambda)$ invariants of the theory. They must correspond to $SU(3)$-invariant contractions involving a single V^2 multiplying an arbitrary number of V^1 insertions (recall that we have denoted the adjoint in the mass terms as V^1).

A natural guess for such invariants are

$$I_1^{1,2} = \mathrm{Tr}(V^1 V^2) \tag{2.58}$$

$$I_2^{1,1,2} = \mathrm{Tr}(V^1 V^1 V^2) \tag{2.59}$$

In fact, these two structures are sufficient. Explicitly, if one now treats the mass terms as being the background value, Eqn. 2.56, then one finds

$$(V^2)_1{}^1 = \frac{2\mu_{11}^2 - \mu_{22}^2 - \mu_{33}^2}{6(\mu_{11}^2 - \mu_{22}^2)(\mu_{11}^2 - \mu_{33}^2)} I_1^{2,1} + \frac{2}{3(\mu_{11}^2 - \mu_{22}^2)(\mu_{11}^2 - \mu_{33}^2)} I_2^{2,1,1} \tag{2.60}$$

$$(V^2)_2{}^2 = \frac{\mu_{11}^2 - 2\mu_{22}^2 + \mu_{33}^2}{6(\mu_{11}^2 - \mu_{22}^2)(\mu_{11}^2 - \mu_{33}^2)} I_1^{2,1} - \frac{2}{3(\mu_{11}^2 - \mu_{22}^2)(\mu_{11}^2 - \mu_{33}^2)} I_2^{2,1,1} \,. \tag{2.61}$$

Hence the two invariants $(V^2)_1{}^1$ and $(V^2)_2{}^2$ arise from the two invariants $I^{1,2}, I^{1,1,2}$. It

will turn out to be convenient to keep the flavor structure more general. That is, we will take the invariants to be

$$I^{I,J} = \mathrm{Tr}(V^I V^J) \tag{2.62}$$

$$I^{I,J,K} = \mathrm{Tr}(V^I V^J V^K) \tag{2.63}$$

where $I, J, K = 1, 2, 3$.

We can extend this procedure to invariants involving the **27** representation. As an example, consider the components ${W_{11}}^{11}$, ${W_{12}}^{12}$, and ${W_{22}}^{22}$, which are completely invariant under the residual $U(1)_1 \times U(1)_2$ symmetry. They must correspond to $SU(3)$-invariant contractions involving a single W multiplying an arbitrary number of V^1 insertions

We can write several candidate $SU(3)$ invariants involving a single W and up to four factors of V^1:

$$I_1 = {W_{kl}}^{ij}{(V^1)_i}^k{(V^1)_j}^l \tag{2.64}$$

$$I_2 = {W_{kl}}^{ij}{(V^1V^1)_i}^k{(V^1)_j}^l \tag{2.65}$$

$$I_3 = {W_{kl}}^{ij}{(V^1V^1)_i}^k{(V^1V^1)_j}^l \tag{2.66}$$

$$I_4 = {W_{kl}}^{ij}{(V^1)_i}^k{(V^1V^1V^1)_j}^l \tag{2.67}$$

Since there are only three independent $U(1)_1 \times U(1)_2$ invariants in this sector, one of the above structures must be linearly dependent on the others.

These invariants can be evaluated straightforwardly once V^1 is assigned its background value. For example, one finds

$$I_1 = (\mu_{11}^2 - \mu_{33}^2)^2 {W_{11}}^{11} + 2(\mu_{11}^2 - \mu_{33}^2)(\mu_{22}^2 - \mu_{33}^2){W_{12}}^{12} + (\mu_{22}^2 - \mu_{33}^2)^2 {W_{22}}^{22} \tag{2.68}$$

Analogous expressions may be obtained for I_2, I_3, and I_4. One then finds that I_1, I_2, and I_3

are linearly independent, while

$$I_4 = \mathrm{Tr}(V^1 V^1) I_1 \tag{2.69}$$

and is therefore not an independent invariant.

This procedure can be automated efficiently using `Mathematica`. For each set of proposed basis invariants, we use `NullSpace` to determine the dimension of the space they generate, and `SubspaceBasis` to extract a linearly independent set spanning that space. Repeating this process allows one to systematically construct a basis of independent $SU(3)$ invariants that reproduces all $U(1)_1 \times U(1)_2$ invariants. The results are listed in Section 2.7.

2.7 Results

By applying the method described in Section 2.6.3, we find all $SU(3)$-invariants that are required to reproduce the list of invariants in the Appendix. This is therefore a full set of invariants for the 3HDM up to cubic order in the couplings. In the special case of invariants containing no **27** fields, the results are obtained to arbitrary order in the quartic coupling λ. It is organized by the number of W followed by the number of V.

	3HDM Invariants
Zero **27**, Two **8**	$\mathrm{Tr}(V^I V^J)$
Zero **27**, Three **8**	$\mathrm{Tr}(V^I V^J V^K)$ $\mathrm{Tr}([V^I V^J V^K])$
Zero **27**, Four **8**	$\mathrm{Tr}([V^I V^J][V^K V^L])$ $\mathrm{Tr}([V^I V^J V^K]V^L)$
Zero **27**, Five **8**	$\mathrm{Tr}([V^I V^J V^K]V^L V^M)$ $\mathrm{Tr}([V^I V^J V^K][V^L V^M])$
Zero **27**, Six **8**	$\mathrm{Tr}([V^I V^J][V^K V^L][V^M V^N])$
One **27**, Two **8**	$W_{ij}{}^{kl}(V^I)_k{}^i(V^J)_l{}^j$
One **27**, Three **8**	$W_{ij}{}^{kl}(V^I V^J)_k{}^i(V^K)_l{}^j$
One **27**, Four **8**	$W_{ij}{}^{kl}(V^I V^J V^K)_k{}^i(V^L)_l{}^j$ $W_{ij}{}^{kl}(V^I V^J)_k{}^i(V^K V^L)_l{}^j$
One **27**, Five **8**	$W_{ij}{}^{kl}(V^I V^J V^K V^L)_k{}^i(V^M)_l{}^j$ $W_{ij}{}^{kl}(V^I V^J V^K)_k{}^i(V^L V^M)_l{}^j$
One **27**, Six **8**	$W_{ij}{}^{kl}(V^I V^J V^K V^L)_k{}^i(V^M V^N)_l{}^j$ $W_{ij}{}^{kl}(V^I V^J V^K V^L V^M)_k{}^i(V^N)_l{}^j$
One **27**, Seven **8**	$W_{ij}{}^{kl}(V^I V^J V^K V^L V^M)_k{}^i(V^N V^O)_l{}^j$
Two **27**, Zero **8**	$W_{ij}{}^{kl}W_{kl}{}^{ij}$
Two **27**, One **8**	$W_{ij}{}^{ka}W_{bk}{}^{ij}(V^I)_a{}^b$
Two **27**, Two **8**	$W_{ij}{}^{ka}W_{bk}{}^{ij}(V^I V^J)_a{}^b$ $W_{ib}{}^{ak}W_{kd}{}^{ci}(V^I)_a{}^b(V^J)_c{}^d$
Two **27**, Three **8**	$W_{ib}{}^{ak}W_{kd}{}^{ci}(V^I V^J)_a{}^b(V^K)_c{}^d$ $W_{fb}{}^{ak}W_{kd}{}^{ce}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f$ $W_{df}{}^{ak}W_{kb}{}^{ce}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f$ $W_{ij}{}^{ka}W_{bk}{}^{ij}(V^I V^J V^K)_a{}^b$

	3HDM Invariants
Two **27**, Four **8**	$W_{fb}{}^{ak}W_{kd}{}^{ce}(V^IV^J)_a{}^b(V^K)_c{}^d(V^L)_e{}^f$ $W_{fb}{}^{ak}W_{kd}{}^{ce}(V^I)_a{}^b(V^JV^K)_c{}^d(V^L)_e{}^f$ $W_{fh}{}^{ac}W_{bd}{}^{eg}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h$ $W_{fb}{}^{ag}W_{hd}{}^{ce}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h$ $W_{ij}{}^{ka}W_{bk}{}^{ij}(V^IV^JV^KV^L)_a{}^b$ $W_{ib}{}^{ak}W_{kd}{}^{ci}(V^IV^JV^K)_a{}^b(V^L)_c{}^d$ $W_{fb}{}^{ak}W_{kd}{}^{ce}(V^I)_a{}^b(V^J)_c{}^d(V^KV^L)_e{}^f$ $W_{db}{}^{ek}W_{kf}{}^{ac}(V^I)_a{}^b(V^JV^K)_c{}^d(V^L)_e{}^f$ $W_{db}{}^{ek}W_{kf}{}^{ac}(V^I)_a{}^b(V^J)_c{}^d(V^KV^L)_e{}^f$
Two **27**, Five **8**	$W_{fb}{}^{ak}W_{kd}{}^{ce}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^M)_e{}^f$ $W_{fb}{}^{ak}W_{kd}{}^{ce}(V^IV^J)_a{}^b(V^K)_c{}^d(V^LV^M)_e{}^f$ $W_{fb}{}^{ak}W_{kd}{}^{ce}(V^I)_a{}^b(V^JV^K)_c{}^d(V^LV^M)_e{}^f$ $W_{df}{}^{ak}W_{kb}{}^{ec}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^M)_e{}^f$ $W_{ib}{}^{ak}W_{kd}{}^{ci}(V^IV^JV^K)_a{}^b(V^LV^M)_c{}^d$ $W_{fb}{}^{ak}W_{kd}{}^{ce}(V^IV^JV^K)_a{}^b(V^L)_c{}^d(V^M)_e{}^f$ $W_{fb}{}^{ak}W_{kd}{}^{ce}(V^I)_a{}^b(V^JV^KV^L)_c{}^d(V^M)_e{}^f$
Two **27**, Six **8**	$W_{fb}{}^{ak}W_{kd}{}^{ce}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^MV^N)_e{}^f$ $W_{fh}{}^{ac}W_{bd}{}^{eg}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^M)_e{}^f(V^N)_g{}^h$ $W_{fb}{}^{ag}W_{hd}{}^{ce}(V^IV^J)_a{}^b(V^K)_c{}^d(V^LV^M)_e{}^f(V^N)_g{}^h$ $W_{fb}{}^{ak}W_{kd}{}^{ce}(V^IV^JV^K)_a{}^b(V^L)_c{}^d(V^MV^N)_e{}^f$ $W_{fb}{}^{ak}W_{kd}{}^{ce}(V^IV^JV^K)_a{}^b(V^LV^M)_c{}^d(V^N)_e{}^f$ $W_{fb}{}^{ak}W_{kd}{}^{ce}(V^IV^J)_a{}^b(V^KV^LV^M)_c{}^d(V^N)_e{}^f$ $W_{fb}{}^{ak}W_{kd}{}^{ce}(V^I)_a{}^b(V^JV^KV^L)_c{}^d(V^MV^N)_e{}^f$ $W_{bd}{}^{ek}W_{kf}{}^{ac}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^MV^N)_e{}^f$ $W_{fb}{}^{ag}W_{hd}{}^{ce}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^M)_e{}^f(V^N)_g{}^h$
Two **27**, Seven **8**	$W_{fb}{}^{ak}W_{kd}{}^{ce}(V^IV^JV^K)_a{}^b(V^LV^M)_c{}^d(V^NV^O)_e{}^f$ $W_{fb}{}^{ak}W_{kd}{}^{ce}(V^IV^J)_a{}^b(V^KV^LV^M)_c{}^d(V^NV^O)_e{}^f$ $W_{fb}{}^{ag}W_{hd}{}^{ce}(V^IV^JV^K)_a{}^b(V^LV^M)_c{}^d(V^N)_e{}^f(V^O)_g{}^h$ $W_{fb}{}^{ag}W_{hd}{}^{ce}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^MV^N)_e{}^f(V^O)_g{}^h$
Two **27**, Eight **8**	$W_{fh}{}^{ac}W_{bd}{}^{eg}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^MV^N)_e{}^f(V^OV^P)_g{}^h$

	3HDM Invariants
Three **27**, Zero **8**	$W_{ij}{}^{kl}W_{kl}{}^{ab}W_{ab}{}^{ij}$ $W_{ij}{}^{kl}W_{ka}{}^{ib}W_{lb}{}^{ja}$
Three **27**, One **8**	$W_{ib}{}^{ak}W_{kj}{}^{lm}W_{ml}{}^{ij}(V^I)_a{}^b$ $W_{bi}{}^{kl}W_{jm}{}^{ai}W_{kl}{}^{jm}(V^I)_a{}^b$
Three **27**, Two **8**	$W_{bd}{}^{kl}W_{ij}{}^{ac}W_{kl}{}^{ij}(V^I)_a{}^b(V^J)_c{}^d$ $W_{ib}{}^{ak}W_{jd}{}^{cl}W_{kl}{}^{ij}(V^I)_a{}^b(V^J)_c{}^d$ $W_{ib}{}^{ck}W_{jd}{}^{al}W_{kl}{}^{ij}(V^I)_a{}^b(V^J)_c{}^d$ $W_{ib}{}^{kl}W_{jd}{}^{ac}W_{kl}{}^{ij}(V^I)_a{}^b(V^J)_c{}^d$ $W_{ij}{}^{ka}W_{bd}{}^{cl}W_{kl}{}^{ij}(V^I)_a{}^b(V^J)_c{}^d$ $W_{bd}{}^{kl}W_{ki}{}^{aj}W_{lj}{}^{ci}(V^I)_a{}^b(V^J)_c{}^d$ $W_{ib}{}^{ak}W_{kd}{}^{lj}W_{lj}{}^{ci}(V^I)_a{}^b(V^J)_c{}^d$ $W_{ib}{}^{ck}W_{kd}{}^{lj}W_{lj}{}^{ai}(V^I)_a{}^b(V^J)_c{}^d$
Three **27**, Three **8**	$W_{bd}{}^{kl}W_{ij}{}^{ac}W_{kl}{}^{ij}(V^IV^J)_a{}^b(V^K)_c{}^d$ $W_{ib}{}^{kl}W_{jd}{}^{ac}W_{kl}{}^{ij}(V^I)_a{}^b(V^JV^K)_c{}^d$ $W_{bf}{}^{ac}W_{id}{}^{kl}W_{kl}{}^{ei}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f$ $W_{bd}{}^{le}W_{ij}{}^{ac}W_{fl}{}^{ij}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f$ $W_{bd}{}^{la}W_{ij}{}^{ec}W_{fl}{}^{ij}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f$ $W_{bd}{}^{al}W_{if}{}^{ck}W_{kl}{}^{ei}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f$ $W_{bd}{}^{al}W_{if}{}^{ek}W_{kl}{}^{ci}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f$ $W_{bd}{}^{el}W_{if}{}^{ak}W_{kl}{}^{ci}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f$ $W_{ib}{}^{ac}W_{df}{}^{kl}W_{kl}{}^{ei}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f$ $W_{if}{}^{ac}W_{db}{}^{kl}W_{kl}{}^{ei}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f$ $W_{ib}{}^{ac}W_{jf}{}^{ek}W_{dk}{}^{ij}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f$ $W_{if}{}^{ac}W_{jb}{}^{ek}W_{dk}{}^{ij}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f$ $W_{bd}{}^{kl}W_{fk}{}^{ei}W_{li}{}^{ac}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f$ $W_{bd}{}^{kl}W_{fk}{}^{ai}W_{li}{}^{ec}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f$

	3HDM Invariants
Three **27**, Four **8**	$W_{ib}{}^{ak}W_{jd}{}^{cl}W_{kl}{}^{ij}(V^1V^1)_a{}^b(V^1V^1)_c{}^d$
	$W_{ib}{}^{kl}W_{jd}{}^{ac}W_{kl}{}^{ij}(V^IV^J)_a{}^b(V^KV^L)_c{}^d$
	$W_{fb}{}^{ac}W_{id}{}^{kl}W_{kl}{}^{ei}(V^I)_a{}^b(V^JV^K)_c{}^d(V^L)_e{}^f$
	$W_{nd}{}^{le}W_{ij}{}^{ac}W_{fl}{}^{ij}(V^IV^J)_a{}^b(V^K)_c{}^d(V^L)_e{}^f$
	$W_{bd}{}^{le}W_{ij}{}^{ac}W_{fl}{}^{ij}(V^I)_a{}^b(V^J)_c{}^d(V^KV^L)_e{}^f$
	$W_{bd}{}^{la}W_{ij}{}^{ec}W_{fl}{}^{ij}(V^IV^J)_a{}^b(V^K)_c{}^d(V^L)_e{}^f$
	$W_{bf}{}^{ac}W_{dh}{}^{kl}W_{kl}{}^{eg}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h$
	$W_{hf}{}^{ac}W_{db}{}^{kl}W_{kl}{}^{eg}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h$
	$W_{bd}{}^{ae}W_{if}{}^{ck}W_{kh}{}^{gi}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h$
	$W_{bd}{}^{ge}W_{if}{}^{ak}W_{kh}{}^{ci}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h$
	$W_{bd}{}^{ai}W_{fj}{}^{eg}W_{hi}{}^{cj}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h$
	$W_{bd}{}^{ai}W_{fj}{}^{cg}W_{hi}{}^{ej}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h$
	$W_{bd}{}^{ei}W_{fj}{}^{ac}W_{hi}{}^{gj}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h$
	$W_{bd}{}^{ei}W_{fj}{}^{ag}W_{hi}{}^{cj}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h$
	$W_{bd}{}^{ai}W_{fh}{}^{cj}W_{ij}{}^{eg}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h$
	$W_{bd}{}^{ai}W_{fh}{}^{ej}W_{ij}{}^{cg}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h$
	$W_{bd}{}^{ei}W_{fh}{}^{aj}W_{ij}{}^{cg}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h$
	$W_{ib}{}^{ac}W_{jf}{}^{eg}W_{dh}{}^{ij}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h$
Three **27**, Five **8**	$W_{bd}{}^{le}W_{ij}{}^{ac}W_{fl}{}^{ij}(V^I)_a{}^b(V^JV^K)_c{}^d(V^LV^M)_e{}^f$
	$W_{fb}{}^{ac}W_{id}{}^{kl}W_{kl}{}^{ei}(V^IV^J)_a{}^b(V^K)_c{}^d(V^LV^M)_e{}^f$
	$W_{bf}{}^{ac}W_{dh}{}^{kl}W_{kl}{}^{eg}(V^I)_a{}^b(V^JV^K)_c{}^d(V^L)_e{}^f(V^M)_g{}^h$
	$W_{hf}{}^{ac}W_{db}{}^{kl}W_{kl}{}^{eg}(V^I)_a{}^b(V^JV^K)_c{}^d(V^L)_e{}^f(V^M)_g{}^h$
	$W_{bd}{}^{ae}W_{if}{}^{ck}W_{kl}{}^{gi}(V^I)_a{}^b(V^JV^K)_c{}^d(V^L)_e{}^f(V^M)_g{}^h$
	$W_{bd}{}^{ce}W_{if}{}^{gk}W_{kh}{}^{ai}(V^I)_a{}^b(V^JV^K)_c{}^d(V^L)_e{}^f(V^M)_g{}^h$
	$W_{bd}{}^{ge}W_{if}{}^{ak}W_{kh}{}^{ci}(V^I)_a{}^b(V^JV^K)_c{}^d(V^L)_e{}^f(V^M)_g{}^h$
	$W_{bd}{}^{ai}W_{fj}{}^{ce}W_{hi}{}^{gj}(V^I)_a{}^b(V^JV^K)_c{}^d(V^L)_e{}^f(V^M)_g{}^h$
	$W_{bd}{}^{ai}W_{fj}{}^{eg}W_{hi}{}^{cj}(V^I)_a{}^b(V^JV^K)_c{}^d(V^L)_e{}^f(V^M)_g{}^h$
	$W_{bd}{}^{ai}W_{fh}{}^{cj}W_{ij}{}^{eg}(V^I)_a{}^b(V^JV^K)_c{}^d(V^L)_e{}^f(V^M)_g{}^h$
	$W_{bd}{}^{ai}W_{fh}{}^{ej}W_{ij}{}^{cg}(V^I)_a{}^b(V^JV^K)_c{}^d(V^L)_e{}^f(V^M)_g{}^h$
	$W_{bd}{}^{ae}W_{fh}{}^{ck}W_{jk}{}^{gi}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h(V^M)_i{}^j$
	$W_{bd}{}^{ae}W_{fh}{}^{ik}W_{jk}{}^{cg}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h(V^M)_i{}^j$
	$W_{bd}{}^{eg}W_{fh}{}^{ak}W_{jk}{}^{ci}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h(V^M)_i{}^j$
	$W_{bd}{}^{eg}W_{fh}{}^{ik}W_{jk}{}^{ac}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h(V^M)_i{}^j$
	$W_{bd}{}^{ai}W_{fh}{}^{ck}W_{jk}{}^{eg}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h(V^M)_i{}^j$

	3HDM Invariants
Three **27**, Six **8**	$W_{bd}{}^{le}W_{ij}{}^{ac}W_{fl}{}^{ij}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^MV^L)_e{}^f$
	$W_{bd}{}^{al}W_{if}{}^{ck}W_{kl}{}^{ei}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^MV^L)_e{}^f$
	$W_{ib}{}^{ac}W_{df}{}^{kl}W_{kl}{}^{ei}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^MV^L)_e{}^f$
	$W_{ib}{}^{ac}W_{jf}{}^{ek}W_{dk}{}^{ij}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^MV^L)_e{}^f$
	$W_{bf}{}^{ac}W_{dh}{}^{kl}W_{kl}{}^{eg}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^M)_e{}^f(V^N)_g{}^h$
	$W_{hf}{}^{ac}W_{db}{}^{kl}W_{kl}{}^{eg}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^M)_e{}^f(V^N)_g{}^h$
	$W_{bd}{}^{ae}W_{if}{}^{ck}W_{kh}{}^{gi}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^M)_e{}^f(V^N)_g{}^h$
	$W_{bd}{}^{ae}W_{fh}{}^{ck}W_{jk}{}^{gi}(V^IV^J)_a{}^b(V^K)_c{}^d(V^L)_e{}^f(V^M)_g{}^h(V^N)_i{}^j$
	$W_{bd}{}^{ae}W_{fh}{}^{gk}W_{jk}{}^{ci}(V^IV^J)_a{}^b(V^K)_c{}^d(V^L)_e{}^f(V^M)_g{}^h(V^N)_i{}^j$
	$W_{bd}{}^{eg}W_{fh}{}^{mk}W_{jk}{}^{ci}(V^IV^J)_a{}^b(V^K)_c{}^d(V^L)_e{}^f(V^M)_g{}^h(V^N)_i{}^j$
	$W_{bd}{}^{eg}W_{fh}{}^{ik}W_{jk}{}^{ac}(V^IV^J)_a{}^b(V^K)_c{}^d(V^L)_e{}^f(V^M)_g{}^h(V^N)_i{}^j$
	$W_{bd}{}^{ai}W_{fh}{}^{ck}W_{jk}{}^{eg}(V^IV^J)_a{}^b(V^K)_c{}^d(V^L)_e{}^f(V^M)_g{}^h(V^N)_i{}^j$
	$W_{bd}{}^{ei}W_{fh}{}^{ak}W_{jk}{}^{cg}(V^IV^J)_a{}^b(V^K)_c{}^d(V^L)_e{}^f(V^M)_g{}^h(V^N)_i{}^j$
	$W_{bd}{}^{ae}W_{fh}{}^{gk}W_{jk}{}^{ci}(V^I)_a{}^b(V^J)_c{}^d(V^KV^L)_e{}^f(V^M)_g{}^h(V^N)_i{}^j$
	$W_{bd}{}^{ik}W_{fh}{}^{ac}W_{jl}{}^{eg}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h(V^M)_i{}^j(V^N)_k{}^l$
	$W_{db}{}^{ie}W_{fh}{}^{ak}W_{jl}{}^{cg}(V^I)_a{}^b(V^J)_c{}^d(V^K)_e{}^f(V^L)_g{}^h(V^M)_i{}^j(V^N)_k{}^l$
Three **27**, Seven **8**	$W_{hf}{}^{ac}W_{db}{}^{kl}W_{kl}{}^{eg}(V^I)_a{}^b(V^JV^K)_c{}^d(V^LV^M)_e{}^f(V^NV^O)_g{}^h$
	$W_{bd}{}^{ce}W_{if}{}^{gk}W_{kh}{}^{ai}(V^I)_a{}^b(V^JV^K)_c{}^d(V^LV^M)_e{}^f(V^NV^O)_g{}^h$
	$W_{bd}{}^{ai}W_{fj}{}^{ce}W_{hi}{}^{gj}(V^I)_a{}^b(V^JV^K)_c{}^d(V^LV^M)_e{}^f(V^NV^O)_g{}^h$
	$W_{bd}{}^{ai}W_{fh}{}^{cj}W_{ij}{}^{eg}(V^I)_a{}^b(V^JV^K)_c{}^d(V^LV^M)_e{}^f(V^NV^O)_g{}^h$
	$W_{bd}{}^{ie}W_{fh}{}^{ak}W_{jl}{}^{cg}(V^I)_a{}^b(V^JV^K)_c{}^d(V^L)_e{}^f(V^M)_g{}^h(V^N)_i{}^j(V^O)_k{}^l$
	$W_{bd}{}^{ie}W_{fh}{}^{ak}W_{jl}{}^{cg}(V^I)_a{}^b(V^J)_c{}^d(V^KV^L)_e{}^f(V^M)_g{}^h(V^N)_i{}^j(V^O)_k{}^l$
Three **27**, Eight **8**	$W_{bd}{}^{ai}W_{fh}{}^{cj}W_{ij}{}^{eg}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^MV^N)_e{}^f(V^OV^P)_g{}^h$
	$W_{db}{}^{ie}W_{fh}{}^{ak}W_{jl}{}^{cg}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^M)_e{}^f(V^N)_g{}^h(V^O)_i{}^j(V^P)_k{}^l$
Three **27**, Nine **8**	$W_{bd}{}^{ai}W_{fh}{}^{ck}W_{jk}{}^{eg}(V^IV^J)_a{}^b(V^KV^L)_c{}^d(V^MV^N)_e{}^f(V^OV^P)_g{}^h(V^Q)_i{}^j$

2.8 CP Properties of the Invariants

As a first application of our results, we investigate the CP properties of the invariants appearing in the 3HDM. In particular, we classify the invariants into CP-even and CP-odd combinations. This classification is purely algebraic: it refers to the transformation of the invariant polynomials under the CP transformation, rather than by itself implying the presence of physical CP violation. We first carry out this analysis in the $U(1)_1 \times U(1)_2$

basis obtained after gauge fixing, where the CP transformation properties are particularly transparent. We then use these results to determine the corresponding CP-even and CP-odd invariants of the full $SU(3)$ theory. We carry out this analysis for the invariants identified up to four fields. A complete classification of CP-even and CP-odd invariants is left for future work, together with a determination of the full invariant ring.

2.8.1 CP Building Blocks

Following [61], we begin with the most general CP transformation acting on the Higgs fields. Suppressing the $SU(2)_L$ indices, the transformation takes the form

$$\Phi_i \xrightarrow{\text{CP}} (\Phi^*)^j [U^T]_j{}^i \tag{2.70}$$

$$(\Phi^*)^i \xrightarrow{\text{CP}} [U^*]_i{}^j \Phi_j \tag{2.71}$$

where U is a unitary matrix acting in flavor space.

To determine the transformation properties of the mass coupling, consider

$$(\Phi^*)^j (\mu^2)_j{}^i \Phi_i \xrightarrow{\text{CP}} [U^*]_j{}^k \Phi_k \, (\mu^2)_j{}^i (\Phi^*)^l [U^T]_l{}^i \tag{2.72}$$

$$= (\Phi^*)^l [U^T]_l{}^i (\mu^2)_j{}^i [U^*]_j{}^k (\Phi)_k \tag{2.73}$$

$$= (\Phi^*)^j ([U^T](\mu^2)^T [U^*])_j{}^i \Phi_i \, . \tag{2.74}$$

Hence,

$$(\mu^2) \xrightarrow{\text{CP}} (U^T (\mu^2)^T U^*) \, . \tag{2.75}$$

Similarly, for the quartic coupling, one obtains

$$\lambda_{ij}^{kl} \xrightarrow{\text{CP}} [U^T]_i^n [U^T]_j^{n'} (\lambda^T)_{nn'}^{mm'} [U^*]_m^k [U^*]_{m'}^l \, . \tag{2.76}$$

Restricting ourselves to CP transformations of order two, one may always choose a basis in flavor space in which

$$U \propto \mathbf{1}. \tag{2.77}$$

Working in such a basis, the transformation properties of the coupling tensors simplify to

$$(\mu^2)_i{}^j \xrightarrow{\text{CP}} (\mu^2)_j{}^i = ((\mu^2)^*)_i{}^j \tag{2.78}$$

$$\lambda_{ij}^{kl} \xrightarrow{\text{CP}} \lambda_{kl}^{ij} = (\lambda^*)_{ij}^{kl} \tag{2.79}$$

where the final equalities follow from Hermiticity.

Using the definitions of the adjoint fields and the **27** in terms of μ^2 and λ, it is straightforward to show that CP acts on the irreducible $SU(3)$ building blocks by exchanging fundamental and antifundamental flavor indices. The resulting transformation properties are

$$(V^I)_i{}^j \xrightarrow{\text{CP}} (V^I)_j{}^i \tag{2.80}$$

$$W_{ij}{}^{kl} \xrightarrow{\text{CP}} W_{kl}{}^{ij}\,. \tag{2.81}$$

Consequently, determining the CP properties of an invariant reduces to studying how it transforms under the exchange of upper and lower flavor indices.

As a simple example, consider the invariant

$$W_{ij}{}^{kl} V_k{}^i V_l{}^j. \tag{2.82}$$

Since all fundamental and antifundamental indices are contracted pairwise, this object is invariant under the exchange of upper and lower indices and is therefore CP even.

More generally, however, tensor structures need not possess this symmetry. For example,

$$W_{ij}{}^{ka} W_{bd}{}^{cl} W_{kl}{}^{ij} (V^I)_a{}^b (V^J)_c{}^d \tag{2.83}$$

is not manifestly invariant under the exchange of fundamental and antifundamental indices. Such structures are therefore not necessarily CP even by themselves and may instead be combined to form either CP-even or CP-odd invariants.

2.8.2 CP on $U(1)_1 \times U(1)_2$ Invariants

It is often more convenient to determine the CP properties of invariants using the $U(1)_1 \times U(1)_2$ building blocks introduced in Section 2.6.2. Before doing so, we must verify that the gauge fixing used to obtain these building blocks does not itself break CP. Since

$$\langle V^1 \rangle \xrightarrow{\text{CP}} \langle V^1 \rangle^T = \langle V^1 \rangle \tag{2.84}$$

the chosen vacuum configuration is CP invariant, as $\langle V^1 \rangle$ is diagonal.

Since the gauge fixing preserves CP, the CP properties of the $U(1)_1 \times U(1)_2$ invariants may be used to determine the CP properties of the corresponding 3HDM invariants. If a $U(1)_1 \times U(1)_2$ field component carries charge Q_{q_1,q_2}, CP acts by exchanging all fundamental and antifundamental indices. Consequently,

$$Q_{q_1,q_2} \xrightarrow{\text{CP}} Q_{-q_1,-q_2}\,. \tag{2.85}$$

We first consider invariants quadratic in the field content. Such invariants must be of the form

$$Q_{q_1,q_2} Q_{-q_1,-q_2}, \tag{2.86}$$

in order to be neutral under $U(1)_1 \times U(1)_2$. In some cases the field components associated with the charges Q_{q_1,q_2} and $Q_{-q_1,-q_2}$ are already related by CP. An example is

$$(V^2)_2{}^1(V^2)_1{}^2, \tag{2.87}$$

which is manifestly CP even. In general, however, the two field components need not be CP conjugates of one another. Examples include

$$W_{12}{}^{11}W_{12}{}^{22}, \qquad W_{12}{}^{11}(V^2)_1{}^2. \tag{2.88}$$

In such cases, CP-even and CP-odd invariants are obtained by taking the symmetric and antisymmetric combinations, respectively.

We therefore define the quadratic CP-even and CP-odd invariants

$$\mathcal{I}_+^{I,J} = Q^{(I}_{q_1,q_2} Q^{J)}_{-q_1,-q_2} \tag{2.89}$$

$$\mathcal{I}_-^{I,J} = Q^{[I}_{q_1,q_2} Q^{J]}_{-q_1,-q_2} \tag{2.90}$$

where $Q^I_{q_1,q_2}$ denotes any field component carrying charge (q_1, q_2). We list out all of these invariants below.

$U(1)_1 \times U(1)_2$ Invariant	$\mathcal{I}_+^{I,J}$	$\mathcal{I}_-^{I,J}$
$Q_{-1,-1}Q_{1,1}$	$(V^{(I)}{}_1{}^2(V^{J)})_2{}^1$ $W_{12}{}^{11}(V^I)_1{}^2 + W_{11}{}^{12}(V^I)_2{}^1$ $W_{22}{}^{12}(V^I)_1{}^2 + W_{12}{}^{22}(V^I)_2{}^1$ $W_{11}{}^{12}W_{22}{}^{12} + W_{12}{}^{22}W_{12}{}^{11}$ $W_{11}{}^{12}W_{12}{}^{11}$ $W_{12}{}^{22}W_{22}{}^{12}$	$(V^{[I}{}_1{}^2(V^{J]})_2{}^1$ $W_{12}{}^{11}(V^I)_1{}^2 - W_{11}{}^{12}(V^I)_2{}^1$ $W_{22}{}^{12}(V^I)_1{}^2 - W_{12}{}^{22}(V^I)_2{}^1$ $W_{11}{}^{12}W_{22}{}^{12} - W_{12}{}^{22}W_{12}{}^{11}$ – –
$Q_{1,-1}Q_{-1,1}$	$(V^{(I}{}_1{}^3(V^{J)})_3{}^1$ $W_{13}{}^{11}(V^I)_1{}^3 + W_{11}{}^{13}(V^I)_3{}^1$ $W_{23}{}^{12}(V^I)_1{}^3 + W_{12}{}^{23}(V^I)_3{}^1$ $W_{11}{}^{13}W_{23}{}^{12} + W_{12}{}^{23}W_{13}{}^{11}$ $W_{11}{}^{13}W_{13}{}^{11}$ $W_{12}{}^{23}W_{23}{}^{12}$	$(V^{[I}{}_1{}^3(V^{J]})_3{}^1$ $W_{13}{}^{11}(V^I)_1{}^3 - W_{11}{}^{13}(V^I)_3{}^1$ $W_{23}{}^{12}(V^I)_1{}^3 - W_{12}{}^{23}(V^I)_3{}^1$ $W_{11}{}^{13}W_{23}{}^{12} - W_{12}{}^{23}W_{13}{}^{11}$ – –
$Q_{2,0}Q_{-2,0}$	$(V^{(I}{}_2{}^3(V^{J)})_3{}^2$ $W_{13}{}^{12}(V^I)_2{}^3 + W_{12}{}^{13}(V^I)_3{}^2$ $W_{23}{}^{22}(V^I)_2{}^3 + W_{22}{}^{23}(V^I)_3{}^2$ $W_{12}{}^{13}W_{23}{}^{22} + W_{22}{}^{23}W_{13}{}^{12}$ $W_{12}{}^{13}W_{13}{}^{12}$ $W_{22}{}^{23}W_{23}{}^{22}$	$(V^{[I}{}_2{}^3(V^{J]})_3{}^2$ $W_{13}{}^{12}(V^I)_2{}^3 - W_{12}{}^{13}(V^I)_3{}^2$ $W_{23}{}^{22}(V^I)_2{}^3 - W_{22}{}^{23}(V^I)_3{}^2$ $W_{12}{}^{13}W_{23}{}^{22} - W_{22}{}^{23}W_{13}{}^{12}$ – –
$Q_{-3,-1}Q_{3,1}$	$W_{13}{}^{22}W_{22}{}^{13}$	–
$Q_{-2,-2}Q_{2,2}$	$W_{11}{}^{22}W_{22}{}^{11}$	–
$Q_{0,-2}Q_{0,2}$	$W_{11}{}^{23}W_{23}{}^{11}$	–
$Q_{2,-2}Q_{-2,2}$	$W_{11}{}^{33}W_{33}{}^{11}$	–
$Q_{3,-1}Q_{-3,1}$	$W_{12}{}^{33}W_{33}{}^{12}$	–
$Q_{4,0}Q_{-4,0}$	$W_{22}{}^{33}W_{33}{}^{22}$	–

The case of $U(1)_1 \times U(1)_2$ invariants involving three or more fields is slightly different. Consider an invariant constructed from field components with charges

$$Q_{q_1^1,q_2^1} \cdots Q_{q_1^n,q_2^n}\,, \tag{2.91}$$

where

$$\sum_{i=1}^{n}(q_1^i, q_2^i) = (0,0)\,. \tag{2.92}$$

For an independent invariant, no charge Q_{q_1,q_2} can appear simultaneously with its conjugate charge $Q_{-q_1,-q_2}$. Indeed, if both charges were present, the invariant would factorize as

$$Q_{q_1,q_2}Q_{-q_1,-q_2} \tag{2.93}$$

times a lower-order $U(1)_1 \times U(1)_2$ invariant and would therefore not be independent.

Consequently, under CP the invariant is mapped to a distinct invariant obtained by reversing the sign of every charge,

$$Q_{q_1^1,q_2^1} \cdots Q_{q_1^n,q_2^n} \xrightarrow{\text{CP}} Q_{-q_1^1,-q_2^1} \cdots Q_{-q_1^n,-q_2^n}. \tag{2.94}$$

CP-even and CP-odd combinations are therefore obtained by taking the symmetric and antisymmetric linear combinations

$$\mathcal{I}_+ = Q_{q_1^1,q_2^1} \ldots Q_{q_1^n,q_2^n} + Q_{-q_1^1,-q_2^1} \cdot \ldots Q_{-q_1^n,-q_2^n} \tag{2.95}$$

$$\mathcal{I}_- = Q_{q_1^1,q_2^1} \ldots Q_{q_1^n,q_2^n} - Q_{-q_1^1,-q_2^1} \ldots Q_{-q_1^n,-q_2^n}\,. \tag{2.96}$$

One can build all CP-even and CP-odd $U(1)_1 \times U(1)_2$ invariants by utilizing the tables listed in the Appendix.

Having classified the $U(1)_1 \times U(1)_2$ invariants, the CP properties of the corresponding 3HDM invariants may then be determined either by matching them to the $U(1)_1 \times U(1)_2$ basis, as described in Section 2.6.3, or directly from the transformation properties

$$(V^I)_i{}^j \xrightarrow{CP} (V^I)_j{}^i, \tag{2.97}$$

$$W_{ij}{}^{kl} \xrightarrow{CP} W_{kl}{}^{ij} \tag{2.98}$$

under CP.

We carried out this procedure for all independent invariants we discovered involving up to four fields. In the special case of invariants involving no **27** fields, i.e. only **8** fields, we applied the analysis to all the invariants. The resulting CP-even and CP-odd invariants are summarized below. For invariants involving four or more fields, CP-even and CP-odd combinations may similarly be constructed by symmetrizing and antisymmetrizing invariants related by CP. A complete classification of all CP-even and CP-odd 3HDM invariants is left for future work.

Field content	CP-even invariants $\mathcal{I}_+$	CP-odd invariants $\mathcal{I}_-$
Zero **27**, Two **8**	$\mathrm{Tr}(V^I V^J)$	–
Zero **27**, Three **8**	$\mathrm{Tr}(V^{(I}V^J V^{K)})$	$\mathrm{Tr}(V^{[I}V^J V^{K]})$
Zero **27**, Four **8**	$\mathrm{Tr}(V^{[I}V^{J]})\mathrm{Tr}(V^{[K}V^{L]})$	$\mathrm{Tr}(V^{[I}V^J V^K]V^{L)})$
Zero **27**, Five **8**	$\mathrm{Tr}([V^I V^J V^K][V^L V^M])$	$\mathrm{Tr}([V^I V^J V^K]V^{(L}V^{M)})$
Zero **27**, Six **8**	–	$\mathrm{Tr}([V^I V^J][V^K V^L][V^M V^N])$
One **27**, Two **8**	$W_{ij}{}^{kl}(V^I)_k{}^i(V^J)_l{}^j$	–
One **27**, Three **8**	$W_{ij}{}^{kl}(V^{(I}V^{J)})_k{}^i(V^K)_l{}^j$	$W_{ij}{}^{kl}(V^{[I}V^{J]})_k{}^i(V^K)_l{}^j$
Two **27**, Zero **8**	$W_{ij}{}^{kl}W_{kl}{}^{ij}$	–
Two **27**, One **8**	$W_{ij}{}^{ka}W_{bk}{}^{ij}(V^I)_a{}^b$	–
Two **27**, Two **8**	$W_{ij}{}^{ka}W_{bk}{}^{ij}(V^{(I}V^{J)})_a{}^b$ $W_{ib}{}^{ak}W_{kd}{}^{ci}(V^I)_a{}^b(V^J)_c{}^d$	$W_{ij}{}^{ka}W_{bk}{}^{ij}(V^{[I}V^{J]})_a{}^b$ –
Three **27**, Zero **8**	$W_{ij}{}^{kl}W_{kl}{}^{ab}W_{ab}{}^{ij}$ $W_{ij}{}^{kl}W_{ka}{}^{ib}W_{lb}{}^{ja}$	– –
Three **27**, One **8**	$W_{ib}{}^{ak}W_{kj}{}^{lm}W_{ml}{}^{ij}(V^I)_a{}^b$ $W_{bi}{}^{kl}W_{jm}{}^{ai}W_{kl}{}^{jm}(V^I)_a{}^b$	– –

2.9 Conclusion

In this work, we have studied aspects of the invariant structure of the 3HDM.

In the first place, we have computed the complete Hilbert series of the 3HDM. Owing to the size of the 3HDM Hilbert series, standard computational techniques were insufficient, requiring the development of alternative methods to render the calculation tractable. We

expect that these techniques will be useful more generally for the study of large Hilbert series arising in other models with extended scalar sectors or nontrivial flavor symmetries.

Secondly, we have presented an approach to a systematic classification of invariant operators built from the parameters of the 3HDM. Using these methods, we explicitly constructed a basis of independent invariant operators up to $\mathcal{O}(\lambda^3)$ in the quartic coupling. However, in the special case of invariants containing no **27** fields, the results are obtained to arbitrary order in the quartic coupling λ. While the complete classification of invariants at arbitrary order is straightforward, the operators identified here are expected to already capture the most phenomenologically relevant structures. In particular, they provide the necessary building blocks for the applications in 3HDM phenomenology and model building at low order in λ.

Finally, we applied our basis of invariant operators to begin the construction of CP-even and CP-odd invariants. We explicitly constructed CP-even and CP-odd combinations for invariants involving up to four fields and described how this procedure naturally extends to invariants of higher degree.

Several directions for future work naturally follow from our analysis. Firstly, one important extension would be the systematic construction of higher-order invariant operators beyond $\mathcal{O}(\lambda^3)$, which would further clarify the structure of the invariant ring of the 3HDM. We estimate there are roughly over 2000 more independent $U(1)_1 \times U(1)_2$ invariants to arbitrary order in λ.

An important remaining problem is the determination of the syzygies among the generators of the invariant ring. While the Hilbert series determines the counting of independent invariants and our explicit construction provides a basis through $\mathcal{O}(\lambda^3)$, the syzygies are required for a complete description of the invariant ring. Determining these relations would therefore be a necessary step toward a full characterization of the 3HDM parameter space in terms of basis-invariant quantities.

Another interesting direction would be to search for a more transparent or compact repre-

sentation of the invariant operators identified here, which in their current form are somewhat cumbersome. It would be useful to discover a more efficient method for representing these invariants.

The invariant operators identified in this work may also serve as useful inputs for phenomenological studies of CP violation, vacuum structure, and dark matter candidates in the 3HDM. These invariants provide a basis-independent way to characterize the scalar sector of the 3HDM and can be used to identify physically distinct regions of parameter space. In particular, they may be employed to construct CP-odd invariants to search for explicit or spontaneous CP violation, as well as to study the symmetry structure of the scalar potential in a basis-independent manner. We have initiated this program in the present work; however, it would be interesting to determine whether a general criterion exists for identifying a priori whether a given invariant operator is CP-even or CP-odd. Such tools are particularly valuable when exploring the large parameter space of multi-Higgs models. We hope to report on these in future work.

Finally, it would also be interesting to apply the techniques developed here to other multi-Higgs models, including general N-Higgs-doublet models with $N > 3$, where the complexity of the invariant structure grows rapidly.

2.10 Appendix

Here we list all $U(1)_1 \times U(1)_2$ invariants up to $\mathcal{O}(\lambda^3)$. The left column shows the charges of the fields, while the right column lists the corresponding invariant combinations of fields. Note that the λ^3 invariants consist of products of three fields. From the table of $U(1)_1 \times U(1)_2$ charges, there is a symmetry between Q_{q_1,q_2} and $Q_{-q_1,-q_2}$, which corresponds to replacing the fundamental indices of the fields with their anti-fundamental indices, and vice versa.

Charges	$U(1)_1 \times U(1)_2$ Invariants
$Q_{0,0}$	$(V^I)_1{}^1$
	$(V^I)_2{}^2$
	$W_{11}{}^{11}$
	$W_{12}{}^{12}$
	$W_{22}{}^{22}$
$Q_{-1,-1}Q_{1,1}$	$(V^I)_1{}^2(V^J)_2{}^1$
	$W_{12}{}^{11}(V^I)_1{}^2$
	$W_{22}{}^{12}(V^I)_1{}^2$
	$W_{11}{}^{12}(V^I)_2{}^1$
	$W_{12}{}^{22}(V^I)_2{}^1$
	$W_{11}{}^{12}W_{12}{}^{11}$
	$W_{11}{}^{12}W_{22}{}^{12}$
	$W_{12}{}^{22}W_{12}{}^{11}$
	$W_{12}{}^{22}W_{22}{}^{12}$
$Q_{1,-1}Q_{-1,1}$	$(V^I)_1{}^3(V^J)_3{}^1$
	$W_{13}{}^{11}(V^I)_1{}^3$
	$W_{23}{}^{12}(V^I)_1{}^3$
	$W_{11}{}^{13}(V^I)_3{}^1$
	$W_{12}{}^{23}(V^I)_3{}^1$
	$W_{11}{}^{13}W_{13}{}^{11}$
	$W_{11}{}^{13}W_{23}{}^{12}$
	$W_{12}{}^{23}W_{13}{}^{11}$
	$W_{12}{}^{23}W_{23}{}^{12}$
$Q_{2,0}Q_{-2,0}$	$(V^I)_2{}^3(V^J)_3{}^2$
	$W_{13}{}^{12}(V^I)_2{}^3$
	$W_{23}{}^{22}(V^I)_2{}^3$
	$W_{12}{}^{13}(V^I)_3{}^2$
	$W_{22}{}^{23}(V^I)_3{}^2$
	$W_{12}{}^{13}W_{13}{}^{12}$
	$W_{12}{}^{13}W_{23}{}^{22}$
	$W_{22}{}^{23}W_{13}{}^{12}$
	$W_{22}{}^{23}W_{23}{}^{22}$

Charges	$U(1)_1 \times U(1)_2$ Invariants
$Q_{-3,-1}Q_{3,1}$	$W_{13}{}^{22}W_{22}{}^{13}$
$Q_{-2,-2}Q_{2,2}$	$W_{11}{}^{22}W_{22}{}^{11}$
$Q_{0,-2}Q_{0,2}$	$W_{11}{}^{23}W_{23}{}^{11}$
$Q_{2,-2}Q_{-2,2}$	$W_{11}{}^{33}W_{33}{}^{11}$
$Q_{3,-1}Q_{-3,1}$	$W_{12}{}^{33}W_{33}{}^{12}$
$Q_{4,0}Q_{-4,0}$	$W_{22}{}^{33}W_{33}{}^{22}$
$Q_{1,-1}Q_{1,1}Q_{-2,0}, \quad Q_{-1,1}Q_{-1,-1}Q_{2,0}$	$(V^I)_1{}^3(V^J)_2{}^1(V^K)_3{}^2, \quad (V^I)_3{}^1(V^J)_1{}^2(V^K)_2{}^3$ $W_{13}{}^{12}(V^I)_1{}^3(V^J)_2{}^1 \quad W_{12}{}^{13}(V^I)_3{}^1(V^J)_1{}^2$ $W_{23}{}^{22}(V^I)_1{}^3(V^J)_2{}^1 \quad W_{22}{}^{23}(V^I)_3{}^1(V^J)_1{}^2$ $W_{12}{}^{11}(V^I)_1{}^3(V^J)_3{}^2 \quad W_{11}{}^{12}(V^I)_3{}^1(V^J)_2{}^3$ $W_{12}{}^{11}W_{13}{}^{12}(V^I)_1{}^3 \quad W_{11}{}^{12}W_{12}{}^{13}(V^I)_3{}^1$ $W_{12}{}^{11}W_{23}{}^{22}(V^I)_1{}^3 \quad W_{11}{}^{12}W_{22}{}^{23}(V^I)_3{}^1$ $W_{22}{}^{12}(V^I)_1{}^3(V^J)_3{}^2 \quad W_{12}{}^{22}(V^I)_3{}^1(V^J)_2{}^3$ $W_{22}{}^{12}W_{13}{}^{12}(V^I)_1{}^3 \quad W_{12}{}^{22}W_{12}{}^{13}(V^I)_3{}^1$ $W_{22}{}^{12}W_{23}{}^{22}(V^I)_1{}^3 \quad W_{12}{}^{22}W_{22}{}^{23}(V^I)_3{}^1$ $W_{11}{}^{13}(V^I)_2{}^1(V^J)_3{}^2 \quad W_{13}{}^{11}(V^I)_1{}^2(V^J)_2{}^3$ $W_{11}{}^{13}W_{13}{}^{12}(V^I)_2{}^1 \quad W_{13}{}^{11}W_{12}{}^{13}(V^I)_1{}^2$ $W_{11}{}^{13}W_{23}{}^{22}(V^I)_2{}^1 \quad W_{13}{}^{11}W_{22}{}^{23}(V^I)_1{}^2$ $W_{11}{}^{13}W_{12}{}^{11}(V^I)_3{}^2 \quad W_{13}{}^{11}W_{11}{}^{12}(V^I)_2{}^3$ $W_{12}{}^{11}W_{13}{}^{12}W_{11}{}^{13} \quad W_{11}{}^{12}W_{12}{}^{13}W_{13}{}^{11}$ $W_{12}{}^{11}W_{23}{}^{22}W_{11}{}^{13} \quad W_{11}{}^{12}W_{22}{}^{23}W_{13}{}^{11}$ $W_{11}{}^{13}W_{22}{}^{12}(V^I)_3{}^2 \quad W_{13}{}^{11}W_{12}{}^{22}(V^I)_2{}^3$ $W_{22}{}^{12}W_{13}{}^{12}W_{11}{}^{13} \quad W_{12}{}^{22}W_{12}{}^{13}W_{13}{}^{11}$ $W_{22}{}^{12}W_{23}{}^{22}W_{11}{}^{13} \quad W_{12}{}^{22}W_{22}{}^{23}W_{13}{}^{11}$ $W_{12}{}^{23}(V^I)_2{}^1(V^J)_3{}^2 \quad W_{23}{}^{12}(V^I)_1{}^2(V^J)_2{}^3$ $W_{12}{}^{23}W_{13}{}^{12}(V^I)_2{}^1 \quad W_{23}{}^{12}W_{12}{}^{13}(V^I)_1{}^2$ $W_{12}{}^{23}W_{23}{}^{22}(V^I)_2{}^1 \quad W_{23}{}^{12}W_{22}{}^{23}(V^I)_1{}^2$ $W_{12}{}^{23}W_{12}{}^{11}(V^I)_3{}^2 \quad W_{23}{}^{12}W_{11}{}^{12}(V^I)_2{}^3$ $W_{12}{}^{11}W_{13}{}^{12}W_{12}{}^{23} \quad W_{11}{}^{12}W_{12}{}^{13}W_{23}{}^{12}$ $W_{12}{}^{11}W_{23}{}^{22}W_{12}{}^{23} \quad W_{11}{}^{12}W_{22}{}^{23}W_{23}{}^{12}$ $W_{12}{}^{23}W_{22}{}^{12}(V^I)_3{}^2 \quad W_{23}{}^{12}W_{12}{}^{22}(V^I)_2{}^3$ $W_{22}{}^{12}W_{13}{}^{12}W_{12}{}^{23} \quad W_{12}{}^{22}W_{12}{}^{13}W_{23}{}^{12}$ $W_{22}{}^{12}W_{23}{}^{22}W_{12}{}^{23} \quad W_{12}{}^{22}W_{22}{}^{23}W_{23}{}^{12}$

Charges	$U(1)_1 \times U(1)_2$ Invariants	
$Q_{-1,-1}Q_{3,1}Q_{-2,0}, \quad Q_{1,1}Q_{-3,-1}Q_{2,0}$	${W_{22}}^{13}{(V^I)_1}^2{(V^J)_3}^2$	${W_{13}}^{22}{(V^I)_2}^1{(V^J)_2}^3$
	${W_{22}}^{13}{W_{13}}^{12}{(V^I)_1}^2$	${W_{13}}^{22}{W_{12}}^{13}{(V^I)_2}^1$
	${W_{22}}^{13}{W_{23}}^{22}{(V^I)_1}^2$	${W_{13}}^{22}{W_{22}}^{23}{(V^I)_2}^1$
	${W_{11}}^{12}{W_{22}}^{13}{(V^I)_3}^2$	${W_{12}}^{11}{W_{13}}^{22}{(V^I)_2}^3$
	${W_{22}}^{13}{W_{13}}^{12}{W_{11}}^{12}$	${W_{13}}^{22}{W_{12}}^{13}{W_{12}}^{11}$
	${W_{22}}^{13}{W_{23}}^{22}{W_{11}}^{12}$	${W_{13}}^{22}{W_{22}}^{23}{W_{12}}^{11}$
	${W_{12}}^{22}{W_{22}}^{13}{(V^I)_3}^2$	${W_{22}}^{12}{W_{13}}^{22}{(V^I)_2}^3$
	${W_{22}}^{13}{W_{13}}^{12}{W_{12}}^{22}$	${W_{13}}^{22}{W_{12}}^{13}{W_{22}}^{12}$
	${W_{22}}^{13}{W_{23}}^{22}{W_{12}}^{22}$	${W_{13}}^{22}{W_{22}}^{23}{W_{22}}^{12}$
$Q_{0,-2}Q_{1,1}Q_{-1,1}, \quad Q_{0,2}Q_{-1,-1}Q_{1,-1}$	${W_{11}}^{23}{(V^I)_2}^1{(V^J)_3}^1$	${W_{23}}^{11}{(V^I)_1}^2{(V^J)_1}^3$
	${W_{11}}^{23}{W_{13}}^{11}{(V^I)_2}^1$	${W_{23}}^{11}{W_{11}}^{13}{(V^I)_1}^2$
	${W_{11}}^{23}{W_{23}}^{12}{(V^I)_2}^1$	${W_{23}}^{11}{W_{12}}^{23}{(V^I)_1}^2$
	${W_{11}}^{23}{W_{12}}^{11}{(V^I)_3}^1$	${W_{23}}^{11}{W_{11}}^{12}{(V^I)_1}^3$
	${W_{12}}^{11}{W_{13}}^{11}{W_{11}}^{23}$	${W_{11}}^{12}{W_{11}}^{13}{W_{23}}^{11}$
	${W_{12}}^{11}{W_{23}}^{12}{W_{11}}^{23}$	${W_{11}}^{12}{W_{12}}^{23}{W_{23}}^{11}$
	${W_{11}}^{23}{W_{22}}^{12}{(V^I)_3}^1$	${W_{23}}^{11}{W_{12}}^{22}{(V^I)_1}^3$
	${W_{22}}^{12}{W_{13}}^{11}{W_{11}}^{23}$	${W_{12}}^{22}{W_{11}}^{13}{W_{23}}^{11}$
	${W_{22}}^{12}{W_{23}}^{12}{W_{11}}^{23}$	${W_{12}}^{22}{W_{12}}^{23}{W_{23}}^{11}$
$Q_{2,0}Q_{1,-1}Q_{-3,1}, \quad Q_{-2,0}Q_{-1,1}Q_{3,-1}$	${W_{33}}^{12}{(V^I)_2}^3{(V^J)_1}^3$	${W_{12}}^{33}{(V^I)_3}^2{(V^J)_3}^1$
	${W_{11}}^{13}{W_{33}}^{12}{(V^I)_2}^3$	${W_{13}}^{11}{W_{12}}^{33}{(V^I)_3}^2$
	${W_{12}}^{23}{W_{33}}^{12}{(V^I)_2}^3$	${W_{23}}^{12}{W_{12}}^{33}{(V^I)_3}^2$
	${W_{12}}^{13}{W_{33}}^{12}{(V^I)_1}^3$	${W_{13}}^{12}{W_{12}}^{33}{(V^I)_3}^1$
	${W_{11}}^{13}{W_{33}}^{12}{W_{12}}^{13}$	${W_{13}}^{11}{W_{12}}^{33}{W_{13}}^{12}$
	${W_{12}}^{23}{W_{33}}^{12}{W_{12}}^{13}$	${W_{23}}^{12}{W_{12}}^{33}{W_{13}}^{12}$
	${W_{22}}^{23}{W_{33}}^{12}{(V^I)_1}^3$	${W_{23}}^{22}{W_{12}}^{33}{(V^I)_3}^1$
	${W_{11}}^{13}{W_{33}}^{12}{W_{22}}^{23}$	${W_{13}}^{11}{W_{12}}^{33}{W_{23}}^{22}$
	${W_{12}}^{23}{W_{33}}^{12}{W_{22}}^{23}$	${W_{23}}^{12}{W_{12}}^{33}{W_{23}}^{22}$
$Q_{-4,0}Q_{2,0}Q_{2,0} \quad Q_{4,0}Q_{-2,0}Q_{-2,0}$	${W_{33}}^{22}{(V^I)_2}^3{(V^J)_2}^3$	${W_{22}}^{33}{(V^I)_3}^2{(V^J)_3}^2$
	${W_{33}}^{22}{W_{12}}^{13}{(V^I)_2}^3$	${W_{22}}^{33}{W_{13}}^{12}{(V^I)_3}^2$
	${W_{33}}^{22}{W_{22}}^{23}{(V^I)_2}^3$	${W_{22}}^{33}{W_{23}}^{22}{(V^I)_3}^2$
	${W_{12}}^{13}{W_{12}}^{13}{W_{33}}^{22}$	${W_{13}}^{12}{W_{13}}^{12}{W_{22}}^{33}$
	${W_{12}}^{13}{W_{22}}^{23}{W_{33}}^{22}$	${W_{13}}^{12}{W_{23}}^{22}{W_{22}}^{33}$
	${W_{22}}^{23}{W_{22}}^{23}{W_{33}}^{22}$	${W_{23}}^{22}{W_{23}}^{22}{W_{22}}^{33}$

Charges	$U(1)_1 \times U(1)_2$ Invariants	
$Q_{-2,-2}Q_{1,1}Q_{1,1} \quad Q_{2,2}Q_{-1,-1}Q_{-1,-1}$	${W_{11}}^{22}{(V^I)_2}^1{(V^J)_2}^1$	${W_{22}}^{11}{(V^I)_1}^2{(V^J)_1}^2$
	${W_{11}}^{22}{W_{12}}^{11}{(V^I)_2}^1$	${W_{22}}^{11}{W_{11}}^{12}{(V^I)_1}^2$
	${W_{11}}^{22}{W_{22}}^{12}{(V^I)_2}^1$	${W_{22}}^{11}{W_{12}}^{22}{(V^I)_1}^2$
	${W_{12}}^{11}{W_{12}}^{11}{W_{11}}^{22}$	${W_{11}}^{12}{W_{11}}^{12}{W_{22}}^{11}$
	${W_{12}}^{11}{W_{22}}^{12}{W_{11}}^{22}$	${W_{11}}^{12}{W_{12}}^{22}{W_{22}}^{11}$
	${W_{22}}^{12}{W_{22}}^{12}{W_{11}}^{22}$	${W_{12}}^{22}{W_{12}}^{22}{W_{22}}^{11}$
$Q_{-2,2}Q_{1,-1}Q_{1,-1} \quad Q_{2,-2}Q_{-1,1}Q_{-1,1}$	${W_{33}}^{11}{(V^I)_1}^3{(V^J)_1}^3$	${W_{11}}^{33}{(V^I)_3}^1{(V^J)_3}^1$
	${W_{33}}^{11}{W_{11}}^{13}{(V^I)_1}^3$	${W_{11}}^{33}{W_{13}}^{11}{(V^I)_3}^1$
	${W_{33}}^{11}{W_{12}}^{23}{(V^I)_1}^3$	${W_{11}}^{33}{W_{23}}^{12}{(V^I)_3}^1$
	${W_{11}}^{13}{W_{11}}^{13}{W_{33}}^{11}$	${W_{13}}^{11}{W_{13}}^{11}{W_{11}}^{33}$
	${W_{11}}^{13}{W_{12}}^{23}{W_{33}}^{11}$	${W_{13}}^{11}{W_{23}}^{12}{W_{11}}^{33}$
	${W_{12}}^{23}{W_{12}}^{23}{W_{33}}^{11}$	${W_{23}}^{12}{W_{23}}^{12}{W_{11}}^{33}$
$Q_{-2,-2}Q_{3,1}Q_{-1,1}, \quad Q_{2,2}Q_{-3,-1}Q_{1,-1}$	${W_{11}}^{22}{W_{22}}^{13}{(V^I)_3}^1$	${W_{22}}^{11}{W_{13}}^{22}{(V^I)_1}^3$
	${W_{22}}^{13}{W_{13}}^{11}{W_{11}}^{22}$	${W_{13}}^{22}{W_{11}}^{13}{W_{22}}^{11}$
	${W_{22}}^{13}{W_{23}}^{12}{W_{11}}^{22}$	${W_{13}}^{22}{W_{12}}^{23}{W_{22}}^{11}$
$Q_{-2,2}Q_{3,-1}Q_{-1,-1}, \quad Q_{2,-2}Q_{-3,1}Q_{1,1}$	${W_{33}}^{11}{W_{12}}^{33}{(V^I)_1}^2$	${W_{11}}^{33}{W_{33}}^{12}{(V^I)_2}^1$
	${W_{12}}^{33}{W_{11}}^{12}{W_{33}}^{11}$	${W_{33}}^{12}{W_{12}}^{11}{W_{11}}^{33}$
	${W_{12}}^{33}{W_{12}}^{22}{W_{33}}^{11}$	${W_{33}}^{12}{W_{22}}^{12}{W_{11}}^{33}$
$Q_{0,-2}Q_{2,2}Q_{-2,0}, \quad Q_{0,2}Q_{-2,-2}Q_{2,0}$	${W_{11}}^{23}{W_{22}}^{11}{(V^I)_3}^2$	${W_{23}}^{11}{W_{11}}^{22}{(V^I)_2}^3$
	${W_{22}}^{11}{W_{13}}^{12}{W_{11}}^{23}$	${W_{11}}^{22}{W_{12}}^{13}{W_{23}}^{11}$
	${W_{22}}^{11}{W_{23}}^{22}{W_{11}}^{23}$	${W_{11}}^{22}{W_{22}}^{23}{W_{23}}^{11}$
$Q_{1,-1}Q_{3,1}Q_{-4,0}, \quad Q_{-1,1}Q_{-3,-1}Q_{4,0}$	${W_{22}}^{13}{W_{33}}^{22}{(V^I)_1}^3$	${W_{13}}^{22}{W_{22}}^{33}{(V^I)_3}^1$
	${W_{22}}^{13}{W_{33}}^{22}{W_{11}}^{13}$	${W_{13}}^{22}{W_{22}}^{33}{W_{13}}^{11}$
	${W_{22}}^{13}{W_{33}}^{22}{W_{12}}^{23}$	${W_{13}}^{22}{W_{22}}^{33}{W_{23}}^{12}$
$Q_{2,-2}Q_{0,2}Q_{-2,0}, \quad Q_{-2,2}Q_{0,-2}Q_{2,0}$	${W_{11}}^{33}{W_{23}}^{11}{(V^I)_3}^2$	${W_{33}}^{11}{W_{11}}^{23}{(V^I)_2}^3$
	${W_{23}}^{11}{W_{13}}^{12}{W_{11}}^{33}$	${W_{11}}^{23}{W_{12}}^{13}{W_{33}}^{11}$
	${W_{23}}^{11}{W_{23}}^{22}{W_{11}}^{33}$	${W_{11}}^{23}{W_{22}}^{23}{W_{33}}^{11}$
$Q_{3,-1}Q_{1,1}Q_{-4,0}, \quad Q_{-3,1}Q_{-1,-1}Q_{4,0}$	${W_{12}}^{33}{W_{33}}^{22}{(V^I)_2}^1$	${W_{33}}^{12}{W_{22}}^{33}{(V^I)_1}^2$
	${W_{12}}^{11}{W_{33}}^{22}{W_{12}}^{33}$	${W_{11}}^{12}{W_{22}}^{33}{W_{33}}^{12}$
	${W_{22}}^{12}{W_{33}}^{22}{W_{12}}^{33}$	${W_{12}}^{22}{W_{22}}^{33}{W_{33}}^{12}$
$Q_{0,-2}Q_{3,1}Q_{-3,1}, \quad Q_{0,2}Q_{-3,-1}Q_{3,-1}$	${W_{22}}^{13}{W_{33}}^{12}{W_{11}}^{23}$	${W_{13}}^{22}{W_{12}}^{33}{W_{23}}^{11}$
$Q_{2,-2}Q_{2,2}Q_{-4,0}, \quad Q_{-2,2}Q_{-2,-2}Q_{4,0}$	${W_{22}}^{11}{W_{33}}^{22}{W_{11}}^{33}$	${W_{11}}^{22}{W_{22}}^{33}{W_{33}}^{11}$

We next list the remaining $U(1)_1 \times U(1)_2$ invariants to arbitrary order in λ. To conserve space, these invariants are presented only in terms of their field charges rather than the

corresponding field components. Furthermore, we list only one representative from each pair of invariants related by the transformation $Q_{q_1,q_2} \to Q_{-q_1,-q_2}$. The omitted invariants are obtained by applying this transformation to every factor appearing in the listed invariant.

$\mathcal{O}(\lambda^4)$ $U(1)_1 \times U(1)_2$ invariants

$(Q_{0,-2})^2 Q_{2,2} Q_{-2,2}$	$Q_{0,-2} Q_{2,2} Q_{-3,-1} Q_{1,1}$	$Q_{0,-2} Q_{2,2} Q_{1,-1} Q_{-3,1}$
$Q_{0,-2} (Q_{1,1})^2 Q_{-2,0}$	$Q_{0,-2} Q_{3,1} Q_{-1,1} Q_{-2,0}$	$Q_{2,-2} Q_{0,2} Q_{-3,-1} Q_{1,1}$
$Q_{2,-2} Q_{0,2} Q_{1,-1} Q_{-3,1}$	$Q_{2,-2} Q_{3,1} Q_{-3,1} Q_{-2,0}$	$(Q_{-1,-1})^2 Q_{3,1} Q_{-1,1}$
$(Q_{1,-1})^2 Q_{1,1} Q_{-3,1}$	$Q_{2,0} Q_{-2,-2} Q_{3,1} Q_{-3,1}$	$Q_{2,0} Q_{0,-2} (Q_{-1,1})^2$
$Q_{2,0} Q_{0,-2} Q_{1,1} Q_{-3,1}$	$Q_{-2,-2} Q_{-2,2} (Q_{2,0})^2$	$Q_{-3,-1} Q_{-1,1} (Q_{2,0})^2$
$Q_{-1,-1} Q_{-3,1} (Q_{2,0})^2$	$Q_{-4,0} Q_{2,0} Q_{0,-2} Q_{2,2}$	$Q_{-4,0} Q_{2,0} Q_{2,-2} Q_{0,2}$
$Q_{-4,0} Q_{2,0} Q_{-1,-1} Q_{3,1}$	$Q_{-4,0} Q_{2,0} Q_{1,-1} Q_{1,1}$	$Q_{-4,0} Q_{2,0} Q_{3,-1} Q_{-1,1}$
$Q_{-2,-2} (Q_{3,1})^2 Q_{-4,0}$	$Q_{0,-2} Q_{3,1} Q_{1,1} Q_{-4,0}$	$Q_{2,-2} (Q_{1,1})^2 Q_{-4,0}$
$Q_{1,-1} Q_{3,1} Q_{-1,-1} Q_{-3,1}$	$Q_{4,0} Q_{-2,-2} (Q_{-1,1})^2$	$Q_{4,0} Q_{0,-2} Q_{-1,1} Q_{-3,1}$
$Q_{4,0} Q_{2,-2} (Q_{-3,1})^2$	$Q_{3,-1} Q_{3,1} Q_{-4,0} Q_{-2,0}$	

$\mathcal{O}(\lambda^5)$ $U(1)_1 \times U(1)_2$ invariants

$(Q_{0,-2})^2 Q_{2,2} (Q_{-1,1})^2$	$(Q_{0,-2})^2 Q_{2,2} Q_{1,1} Q_{-3,1}$	$(Q_{0,-2})^2 (Q_{1,1})^2 Q_{-2,2}$
$Q_{0,-2} (Q_{1,1})^3 Q_{-3,-1}$	$(Q_{0,-2})^2 Q_{3,1} Q_{-1,1} Q_{-2,2}$	$Q_{2,-2} Q_{3,1} Q_{-3,1} Q_{-2,-2} Q_{0,2}$
$Q_{-1,-1} Q_{3,1} Q_{0,-2} (Q_{-1,1})^2$	$Q_{1,-1} (Q_{1,1})^2 Q_{0,-2} Q_{-3,1}$	$Q_{3,-1} (Q_{-1,1})^3 Q_{0,-2}$
$Q_{-2,-2} Q_{-2,2} Q_{2,0} Q_{-1,-1} Q_{3,1}$	$Q_{-2,-2} Q_{-2,2} Q_{2,0} Q_{3,-1} Q_{-1,1}$	$Q_{-2,-2} (Q_{-1,1})^2 (Q_{2,0})^2$
$Q_{0,-2} Q_{-1,1} Q_{-3,1} (Q_{2,0})^2$	$Q_{2,-2} (Q_{-3,1})^2 (Q_{2,0})^2$	$Q_{-3,-1} Q_{-1,1} Q_{2,0} Q_{0,-2} Q_{2,2}$
$Q_{-3,-1} (Q_{-1,1})^2 Q_{2,0} Q_{3,-1}$	$Q_{-1,-1} Q_{-3,1} Q_{2,0} Q_{2,-2} Q_{0,2}$	$(Q_{-1,-1})^2 Q_{-3,1} Q_{2,0} Q_{3,1}$
$Q_{-4,0} (Q_{0,-2})^2 (Q_{2,2})^2$	$Q_{-4,0} (Q_{2,-2})^2 (Q_{0,2})^2$	$Q_{-4,0} Q_{-1,-1} Q_{3,1} Q_{2,-2} Q_{0,2}$
$Q_{-4,0} (Q_{-1,-1})^2 (Q_{3,1})^2$	$Q_{-4,0} Q_{1,-1} Q_{1,1} Q_{0,-2} Q_{2,2}$	$Q_{-4,0} Q_{1,-1} Q_{1,1} Q_{2,-2} Q_{0,2}$
$Q_{-4,0} (Q_{1,-1})^2 (Q_{1,1})^2$	$Q_{-4,0} Q_{3,-1} Q_{-1,1} Q_{0,-2} Q_{2,2}$	$Q_{-4,0} (Q_{3,-1})^2 (Q_{-1,1})^2$
$Q_{-4,0} Q_{2,0} Q_{0,-2} (Q_{1,1})^2$	$Q_{-4,0} Q_{2,0} Q_{0,-2} Q_{3,1} Q_{-1,1}$	$Q_{-4,0} Q_{2,0} Q_{2,-2} Q_{3,1} Q_{-3,1}$
$Q_{-4,0} Q_{0,2} (Q_{1,-1})^2 Q_{2,0}$	$Q_{-4,0} Q_{0,2} Q_{3,-1} Q_{-1,-1} Q_{2,0}$	$Q_{-4,0} Q_{2,2} Q_{3,-1} Q_{-3,-1} Q_{2,0}$
$Q_{-2,2} (Q_{-1,-1})^2 (Q_{2,0})^2$	$Q_{0,2} Q_{-1,-1} Q_{-3,-1} (Q_{2,0})^2$	$Q_{2,2} (Q_{-3,-1})^2 (Q_{2,0})^2$
$(Q_{-2,-2})^2 (Q_{3,1})^2 Q_{-2,2}$	$Q_{-2,-2} (Q_{3,1})^2 Q_{-1,-1} Q_{-3,1}$	$(Q_{2,-2})^2 Q_{2,2} (Q_{-3,1})^2$
$Q_{1,-1} Q_{3,1} Q_{2,-2} (Q_{-3,1})^2$	$Q_{-3,-1} Q_{-3,1} (Q_{2,0})^3$	$Q_{0,-2} (Q_{3,1})^2 Q_{-4,0} Q_{-2,0}$
$Q_{2,-2} Q_{3,1} Q_{1,1} Q_{-4,0} Q_{-2,0}$	$Q_{3,-1} Q_{3,1} Q_{-2,-2} Q_{-2,2} Q_{-2,0}$	$Q_{3,-1} Q_{3,1} Q_{-4,0} Q_{-2,-2} Q_{0,2}$
$Q_{3,-1} Q_{3,1} Q_{-4,0} Q_{0,-2} Q_{-2,2}$	$Q_{3,-1} Q_{3,1} Q_{-4,0} Q_{-1,-1} Q_{-1,1}$	$Q_{0,2} (Q_{3,-1})^2 Q_{-4,0} Q_{-2,0}$
$Q_{2,2} Q_{3,-1} Q_{1,-1} Q_{-4,0} Q_{-2,0}$	$Q_{-2,-2} (Q_{-3,1})^2 (Q_{4,0})^2$	$Q_{-2,2} (Q_{-3,-1})^2 (Q_{4,0})^2$

$\mathcal{O}(\lambda^6)$ $U(1)_1 \times U(1)_2$ invariants

$(Q_{0,-2})^2(Q_{1,1})^3Q_{-3,1}$	$(Q_{0,-2})^2Q_{3,1}(Q_{-1,1})^3$	$Q_{-2,-2}(Q_{-1,1})^3Q_{2,0}Q_{3,-1}$
$(Q_{0,-2})^2Q_{-1,1}Q_{-3,1}Q_{2,0}Q_{2,2}$	$(Q_{2,-2})^2(Q_{-3,1})^2Q_{2,0}Q_{0,2}$	$Q_{-3,-1}Q_{-1,1}(Q_{0,-2})^2(Q_{2,2})^2$
$Q_{-3,-1}(Q_{-1,1})^2Q_{3,-1}Q_{0,-2}Q_{2,2}$	$Q_{-3,-1}(Q_{-1,1})^2Q_{3,-1}Q_{0,-2}Q_{2,2}$	$Q_{-3,-1}(Q_{-1,1})^3(Q_{3,-1})^2$
$Q_{-1,-1}Q_{-3,1}(Q_{2,-2})^2(Q_{0,2})^2$	$(Q_{-1,-1})^2Q_{-3,1}Q_{3,1}Q_{2,-2}Q_{0,2}$	$(Q_{-1,-1})^2Q_{-3,1}Q_{3,1}Q_{2,-2}Q_{0,2}$
$(Q_{-1,-1})^3Q_{-3,1}(Q_{3,1})^2$	$Q_{-1,-1}(Q_{-3,1})^2Q_{2,0}Q_{2,-2}Q_{3,1}$	$Q_{-4,0}(Q_{0,-2})^2(Q_{1,1})^2Q_{2,2}$
$Q_{-4,0}(Q_{0,-2})^2Q_{3,1}Q_{-1,1}Q_{2,2}$	$Q_{-4,0}(Q_{2,-2})^2Q_{3,1}Q_{-3,1}Q_{0,2}$	$Q_{-4,0}Q_{1,-1}(Q_{1,1})^3Q_{0,-2}$
$Q_{-4,0}(Q_{0,2})^2(Q_{1,-1})^2Q_{2,-2}$	$Q_{-4,0}Q_{0,2}(Q_{1,-1})^3Q_{1,1}$	$Q_{-4,0}(Q_{0,2})^2Q_{3,-1}Q_{-1,-1}Q_{2,-2}$
$Q_{-4,0}(Q_{2,2})^2Q_{3,-1}Q_{-3,-1}Q_{0,-2}$	$Q_{-2,2}(Q_{-1,-1})^3Q_{2,0}Q_{3,1}$	$(Q_{0,2})^2Q_{-1,-1}Q_{-3,-1}Q_{2,0}Q_{2,-2}$
$(Q_{2,2})^2(Q_{-3,-1})^2Q_{2,0}Q_{0,-2}$	$Q_{2,2}(Q_{-3,-1})^2Q_{2,0}Q_{3,-1}Q_{-1,1}$	$Q_{-2,-2}(Q_{3,1})^2Q_{-2,2}(Q_{-1,-1})^2$
$Q_{-2,2}(Q_{3,-1})^2Q_{-2,-2}(Q_{-1,1})^2$	$Q_{-2,-2}Q_{-1,1}Q_{-3,1}(Q_{2,0})^3$	$Q_{0,-2}(Q_{-3,1})^2(Q_{2,0})^3$
$Q_{-3,-1}Q_{-3,1}(Q_{2,0})^2Q_{0,-2}Q_{2,2}$	$Q_{-3,-1}Q_{-3,1}(Q_{2,0})^2Q_{0,-2}Q_{2,2}$	$Q_{-3,-1}Q_{-3,1}(Q_{2,0})^2Q_{2,-2}Q_{0,2}$
$Q_{-3,-1}Q_{-3,1}(Q_{2,0})^2Q_{2,-2}Q_{0,2}$	$Q_{-2,2}Q_{-1,-1}Q_{-3,-1}(Q_{2,0})^3$	$Q_{0,2}(Q_{-3,-1})^2(Q_{2,0})^3$
$Q_{0,-2}(Q_{3,1})^2Q_{-2,-2}Q_{-2,2}Q_{-2,0}$	$(Q_{0,-2})^2(Q_{3,1})^2Q_{-4,0}Q_{-2,2}$	$Q_{0,-2}(Q_{3,1})^2Q_{-4,0}Q_{-1,-1}Q_{-1,1}$
$Q_{3,-1}Q_{3,1}(Q_{-2,-2})^2Q_{-2,2}Q_{0,2}$	$Q_{3,-1}Q_{3,1}Q_{-2,-2}(Q_{-2,2})^2Q_{0,-2}$	$Q_{3,-1}Q_{3,1}Q_{-4,0}Q_{0,-2}(Q_{-1,1})^2$
$Q_{3,-1}Q_{3,1}Q_{-4,0}Q_{0,2}(Q_{-1,-1})^2$	$Q_{0,2}(Q_{3,-1})^2Q_{-2,-2}Q_{-2,2}Q_{-2,0}$	$(Q_{0,2})^2(Q_{3,-1})^2Q_{-4,0}Q_{-2,-2}$
$Q_{0,2}(Q_{3,-1})^2Q_{-4,0}Q_{-1,-1}Q_{-1,1}$	$Q_{-2,-2}(Q_{-3,1})^2Q_{4,0}(Q_{2,0})^2$	$Q_{-2,2}(Q_{-3,-1})^2Q_{4,0}(Q_{2,0})^2$
$Q_{-2,-2}(Q_{-3,1})^2Q_{4,0}Q_{1,-1}Q_{3,1}$	$Q_{-2,2}(Q_{-3,-1})^2Q_{4,0}Q_{3,-1}Q_{1,1}$	$Q_{2,-2}(Q_{3,1})^2Q_{-2,-2}(Q_{-3,1})^2$

$\mathcal{O}(\lambda^7)$ $U(1)_1 \times U(1)_2$ invariants

$Q_{-2,-2}(Q_{-1,1})^4(Q_{3,-1})^2$	$(Q_{0,-2})^3Q_{-1,1}Q_{-3,1}(Q_{2,2})^2$	$(Q_{2,-2})^3(Q_{-3,1})^2(Q_{0,2})^2$
$(Q_{2,-2})^2(Q_{-3,1})^3Q_{2,0}Q_{3,1}$	$Q_{-4,0}(Q_{0,-2})^2(Q_{1,1})^4$	$Q_{-4,0}(Q_{0,2})^2(Q_{1,-1})^4$
$Q_{-2,2}(Q_{-1,-1})^4(Q_{3,1})^2$	$(Q_{0,2})^3Q_{-1,-1}Q_{-3,-1}(Q_{2,-2})^2$	$(Q_{2,2})^3(Q_{-3,-1})^2(Q_{0,-2})^2$
$(Q_{2,2})^2(Q_{-3,-1})^3Q_{3,-1}Q_{2,0}$	$(Q_{0,-2})^2(Q_{-3,1})^2(Q_{2,0})^2Q_{2,2}$	$(Q_{0,-2})^2(Q_{-3,1})^2(Q_{2,0})^2Q_{2,2}$
$(Q_{0,2})^2(Q_{-3,-1})^2(Q_{2,0})^2Q_{2,-2}$	$(Q_{0,2})^2(Q_{-3,-1})^2(Q_{2,0})^2Q_{2,-2}$	$(Q_{0,-2})^2(Q_{3,1})^2Q_{-2,-2}(Q_{-2,2})^2$
$(Q_{0,-2})^2(Q_{3,1})^2Q_{-4,0}(Q_{-1,1})^2$	$(Q_{0,2})^2(Q_{3,-1})^2(Q_{-2,-2})^2Q_{-2,2}$	$(Q_{0,2})^2(Q_{3,-1})^2Q_{-4,0}(Q_{-1,-1})^2$
$Q_{-2,-2}(Q_{-3,1})^2(Q_{2,0})^4$	$Q_{-2,2}(Q_{-3,-1})^2(Q_{2,0})^4$	$Q_{-2,-2}(Q_{-3,1})^2(Q_{1,-1})^2(Q_{3,1})^2$
$Q_{-2,-2}(Q_{-3,1})^2(Q_{1,-1})^2(Q_{3,1})^2$	$(Q_{-2,-2})^2(Q_{-3,1})^2Q_{4,0}(Q_{3,1})^2$	$Q_{-2,2}(Q_{-3,-1})^2(Q_{3,-1})^2(Q_{1,1})^2$
$Q_{-2,2}(Q_{-3,-1})^2(Q_{3,-1})^2(Q_{1,1})^2$	$(Q_{-2,2})^2(Q_{-3,-1})^2(Q_{3,-1})^2Q_{4,0}$	

$\mathcal{O}(\lambda^8)$ $U(1)_1 \times U(1)_2$ invariants

$(Q_{2,-2})^3(Q_{-3,1})^3Q_{3,1}Q_{0,2}$	$(Q_{2,2})^3(Q_{-3,-1})^3Q_{3,-1}Q_{0,-2}$	$Q_{-3,-1}Q_{-3,1}(Q_{0,-2})^3(Q_{2,2})^3$
$Q_{-3,-1}Q_{-3,1}(Q_{2,-2})^3(Q_{0,2})^3$		

$\mathcal{O}(\lambda^9)$ $U(1)_1 \times U(1)_2$ invariants

$(Q_{0,-2})^4(Q_{-3,1})^2(Q_{2,2})^3$	$(Q_{0,2})^4(Q_{-3,-1})^2(Q_{2,-2})^3$	$(Q_{-2,-2})^3(Q_{-3,1})^2(Q_{3,1})^4$
$(Q_{-2,2})^3(Q_{-3,-1})^2(Q_{3,-1})^4$		

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Chapter 3

On the Calculation of Invariant Tensors in Gauge Theories

This chapter is based on joint work with Arvind Rajaraman, Yahya Almumin, and Jason Baretz and was published in *Acta Physica Polonica B* [1]. In this work, we apply techniques from invariant theory to systematically construct bases of invariant tensors for $SU(2)$ with fields transforming in the four-dimensional representation, $SO(3)$ with fields transforming in the five-dimensional representation, and $SU(3)$ with fields transforming as sextets. The invariant bases are constructed using background-field techniques. My contribution to this work was the systematic analysis of the resulting invariants and their syzygies using LiE [2], a computer algebra program for computations involving reductive Lie groups and Lie algebras. The presentation in this chapter follows the published work, with additional discussion and details included where useful.

3.1 Introduction

A gauge theory is specified by the gauge group and the representation of the matter fields under the gauge group, but all observables, including the physical spectrum, are gauge invariant combinations of fields. The structure of these objects is found by contracting the

gauge-covariant fields with *invariant tensors* to form invariant objects. While there are an infinite number of invariants, there is expected to be a basis set of invariants such that all other invariants can be generated from these basis invariants. These motivate us to understand the basis set of invariant tensors for a general gauge theory.

Invariant tensors are known for the fundamental representations of the classical groups[1]. However, the tensors for many other representations are not classified. For many groups such as E_6, even the full set of tensors for the fundamental representation have not been found [3].

A brief example will suffice to show the kinds of difficulties that may occur. It is known that in the group $SO(3)$, the invariant tensors are $\delta^{ij}, \epsilon^{ijk}$. These tell us that in a theory where all fields V_i^I (we are using lower case letters for the gauge representation, and upper case to label the fields–a flavor index) are in the fundamental, the complete set of invariant polynomials are generated by $V_i^I V_j^J \delta^{ij}$ and $V_i^I V_j^J V_k^K \epsilon^{ijk}$. But in a theory with $SO(3)$ gauge symmetry and with fields V_{ij}^I in the symmetric tensor representation, one can produce an infinite set of invariants by contracting an arbitrarily long sequence $V_{ij}^{I_1} V_{jk}^{I_2} .. V_{li}^{I_M}$ (and there exist further invariants involving epsilon tensors). Only a small set of these are independent, but finding these is nontrivial.

In this paper, we present a new approach to finding a minimal set of invariants for more complicated gauge theories. Our approach will be to use the connection between invariant tensors and D-flat directions in field space, which was originally described for supersymmetric field theories (specifically in the context of dualities in these theories [4–7].) This theorem asserts that the independent gauge invariant polynomial invariants of the theory are in 1-1 correspondence with the orbits of constant field configurations where configurations differing by a *complex* gauge transformation are identified [20–26]. It is clear that at any point on configuration space, we can calculate the value of any gauge invariant combination of the fields. The theorem states that this can be reversed; a knowledge of the values of all the

[1]For some of the mathematical literature relevant to invariant theory see: [8]– [19].

independent gauge invariant polynomials is sufficient to reconstruct the orbits of constant field configurations quotiented by complex gauge transformations.

This theorem is often used in simple theories to characterize the field space in terms of the known operators. Here we will reverse the implication, and use the field space to find a complete set of gauge invariant objects in various theories.

The procedure for finding the tensors is then as follows. We take a set of fields in the relevant representation, and set each component to an arbitrary constant value. We then use a complex gauge transformation to set some field components to zero. If the gauge transformations are completely fixed by this procedure, then the nonzero components parametrize the orbits, and we must find a set of invariants such that each of these parameters can be written as a linear combination of invariants. Such a set of invariants would then be a basis set of invariants for the theory.

In practice, we find that the full gauge symmetry is not easy to fix with a single field. In each case, we find a remnant discrete symmetry, and occasionally, a larger continuous symmetry. One possibility is to use further fields to completely gauge fix the symmetry, but in each case we analyze below, we find that the residual symmetry is simple enough that we can find the complete set of combinations which are invariant under the residual symmetry. These combinations parametrize the gauge fixed space, and we must find a set of invariants such that each of these combinations can be written as a linear combination of invariants. Such a set of invariants would then be a basis set of invariants for the theory.

We now show the practicality of this approach by explicitly finding the basis set of invariants for three gauge theories –$SO(3)$ with symmetric tensor matter, $SU(2)$ with matter in the dimension 4 representation, and $SU(3)$ with matter in the sextet. To our knowledge, the last two are completely new analyses (the first case has been analyzed previously in [20]).

3.2 SU(2) with fields in the dimension 4 representation

3.2.1 Overview

We will take as our first example a theory with a gauge group $SU(2)$ and a field content where there are N fields in a representation of dimension 4; this is the simplest case for which the independent set of invariants has (to our knowledge) not been worked out.

The fields can be represented as three-index tensors V^I_{abc} where $a, b, c = 1, 2$ are acted on by the gauge symmetry, and $I = 1..N$ labels the different fields (we shall consistently use lower case indices for gauge indices and upper case indices to label the different fields, similar to a flavor index). The fields can also be represented as a column vector with four elements:

$$V^I = \begin{pmatrix} V^I_{111} \\ V^I_{112} \\ V^I_{122} \\ V^I_{222} \end{pmatrix} \tag{3.1}$$

The invariant tensor is ϵ^{ab}, but one can write an infinite set of invariants, and it is hard to find relations between them. We therefore find the gauge-fixed configuration space, and attempt to characterize this space by invariants.

3.2.2 Counting Invariant Candidates

We first perform a preliminary counting to establish some candidate choices for the invariants. The invariants can be classified by the symmetry of the fields (more specifically by the flavor symmetry). For each candidate flavor symmetry, we use the program LiE [2] to find the number of invariants with that symmetry. We then subtract any invariants that can be written as a product of smaller invariants.

For instance, if we consider two fields, they can be the symmetric combination $\square\!\square$ or

the antisymmetric combination $\begin{smallmatrix}\square\\\square\end{smallmatrix}$. From LiE, we find that there is one invariant with the symmetry $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, and none with $\square\square$. Proceeding further, we find from LiE that there is one invariant with four antisymmetrized fields. However, we also have the product of two of the invariants $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, which could have the same flavor symmetry. Subtracting this, we find there are actually no invariants with four antisymmetrized fields.

We proceed with this process. For each candidate flavor symmetry, in the first column (shown as a list of the number of boxes per row), we use the program LiE to find the number of invariants (shown in the second column) with that symmetry. We then subtract any invariants that can be written as a product of smaller invariants; these are shown in the following column of the table. The net number of independent invariants after subtraction is shown in the last column.

Tableau	# Invts	Subtract	Net
(2)	0	0	0
$(1,1)$	1	0	1
(3)	0	0	0
$(2,1)$	0	0	0
$(1,1,1)$	0	0	0
(4)	1	0	1
$(3,1)$	0	0	0
$(2,2)$	1	$(1,1)\otimes(1,1)$	0
$(2,1,1)$	0	0	0
$(1,1,1,1)$	1	$(1,1)\otimes(1,1)$	0
(5)	0	0	0
$(4,1)$	0	0	0
$(3,2)$	0	0	0
$(3,1,1)$	0	0	0
$(2,2,1)$	0	0	0
$(2,1,1,1)$	0	0	0
$(1,1,1,1,1)$	0	0	0
(6)	0	0	0
$(5,1)$	1	$(4)\otimes(1,1)$	0
$(4,2)$	0	0	0
$(3,3)$	2	$(1,1)\otimes(1,1)\otimes(1,1)$	1
$(4,1,1)$	1	$(4)\otimes(1,1)$	0
$(3,2,1)$	0	0	0
$(2,2,2)$	0	0	0
$(3,1,1,1)$	0	0	0
$(2,2,1,1)$	1	$(1,1)\otimes(1,1)\otimes(1,1)$	0
$(2,1,1,1,1)$	0	0	0
$(1,1,1,1,1,1)$	0	0	0

We have also verified that no new invariants of degree 8 exist. We, therefore, find that to six fields, we have three candidate invariants, of symmetry (1,1), (4), (3,3).

3.2.3 Construction of Invariants

Even though all invariants of eight fields canceled above, this cannot prove that we have achieved a complete count. To have confidence in our result, we apply the theorem from the introduction, by gauge fixing some fields

We begin by considering a single field V^1. By a complex gauge transformation, one can

set the second and third components to zero, and set the first component to 1. The field then has the form

$$V^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ d \end{pmatrix} \tag{3.2}$$

This breaks the continuous gauge symmetry but preserves a discrete symmetry, which can be understood as follows: if we interchange every gauge 1 index with a 2 index, this is equivalent to taking $\epsilon^{ab} \to -\epsilon^{ab}$. Then any invariant with $4n+2$ fields will pick up a minus sign, while any invariant with $4n$ fields is unchanged. This then indicates that the combined transformation of interchanging every 1 index with a 2 index, and multiplying every field by an overall factor of i should be a symmetry (this is in fact the gauge symmetry corresponding to a rotation by π around the x-axis).

Under this symmetry, we have

$$V^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ d \end{pmatrix} \to \begin{pmatrix} id \\ 0 \\ 0 \\ i \end{pmatrix} \tag{3.3}$$

We can further use a gauge transformation by e^{iL_3} (i.e. a gauge transform by the L_3 subgroup of $SU(2)$) to transform

$$V^I = \begin{pmatrix} V^I_{111} \\ V^I_{112} \\ V^I_{122} \\ V^I_{222} \end{pmatrix} \to \begin{pmatrix} e^{3i\alpha}V^I_{111} \\ e^{i\alpha}V^I_{112} \\ e^{-i\alpha}V^I_{122} \\ e^{-3i\alpha}V^I_{222} \end{pmatrix} \tag{3.4}$$

A suitable choice of complex α allows us to bring

$$V^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ d \end{pmatrix} \rightarrow \begin{pmatrix} id \\ 0 \\ 0 \\ i \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ -d \end{pmatrix} \tag{3.5}$$

We have therefore produced a configuration of the form (3.2) but with a change in sign of d. This implies that the sign of d can be changed by a gauge transformation. The gauge invariant combination is $(d)^2$.

As explained in the introduction, this parameter $(d)^2$ should be expressible as a polynomial in the invariants. We, therefore, look for an $SU(2)$ invariant tensor such that gauge fixing the field to be of the form (3.2) allows us to deduce the value of $(d)^2$.

There is no nonzero bilinear invariant involving V^1 alone. We can however find an invariant of degree 4 in V^1 as follows: we first construct a symmetric combination of two fields

$$W^{IJ}_{ab} = (V^I)_{acd}(V^J)_b{}^{cd} + (V^I)_{bcd}(V^J)_a{}^{cd} \tag{3.6}$$

(as always in $SU(2)$, indices are raised and lowered by the epsilon tensor). We can then construct the invariant

$$I_4^{IJKL} = W^{IJ}_{ab} W^{KLab} \tag{3.7}$$

We now evaluate

$$I_4^{1111} = -8(d)^2 \tag{3.8}$$

The knowledge of the invariant I_4^{1111} is therefore sufficient to deduce the value of d^2, and

therefore is hence sufficient to completely parametrize the gauge-inequivalent configurations of a single field. By the theorem cited in the introduction, I_4^{IJKL} is a basis set of invariants for a single field in the dimension 4 representation of $SU(2)$.

We now consider multiple fields. These can be brought by a complex gauge transformation to the form

$$V^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ d^1 \end{pmatrix} \qquad V^I = \begin{pmatrix} a^I \\ b^I \\ c^I \\ d^I \end{pmatrix} \quad \text{for I} > 1 \tag{3.9}$$

Exactly as above, this parametrization breaks the gauge symmetry but preserves a discrete Z_2 gauge symmetry. Under this symmetry, we have

$$V^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ d^1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ -d^1 \end{pmatrix} \tag{3.10}$$

This breaks the Z_2, but preserves a subgroup under which each 1 index gets a factor of ω, and each 2 index receives a factor of ω^2, where ω is a cube root of 1. This corresponds to a map $a^I \rightarrow a^I$, $d^I \rightarrow d^I$, $b^I \rightarrow \omega b^I$, $c^I \rightarrow \omega^2 c^I$. The invariants are then a^I, d^I, $b^I c^J$, $c^I c^J c^K$, $b^I b^J b^K$

Under the broken Z_2, these combinations are acted on as

$$d^1 \rightarrow -d^1 \qquad a^I(d^1) \leftrightarrow -d^I$$
$$b^I c^J \rightarrow -b^J c^I \qquad b^I b^J b^K d^1 \leftrightarrow c^I c^J c^K \,.$$

The gauge-invariant even combinations are

$$\begin{aligned} i_1^I &= a^I(d^1) - d^I \\ i_2^{IJ} &= b^I c^J - b^J c^I \\ i_3^{IJK} &= b^I b^J b^K d^1 + c^I c^J c^K \end{aligned} \tag{3.11}$$

A product of two combinations odd under the Z_2 is even under the Z_2. We can therefore generate a new set of gauge invariant even combinations:

$$\begin{aligned} i_4^I &= d^1(a^I d^1 + d^I) \\ i_5^{IJ} &= d^1(b^I c^J + b^J c^I) \\ i_6^{IJK} &= d^1(b^I b^J b^K d^1 - c^I c^J c^K) \end{aligned} \tag{3.12}$$

The combinations $i_{1\ldots6}$ are the complete set of invariant under the residual $U(1)$ group; by virtue of the theorem cited in the introduction, they are in 1-1 correspondence with the space of gauge-inequivalent configurations.

Now the set of $SU(2)$ invariants is also in 1-1 correspondence with the gauge-inequivalent configurations. This implies that the $SU(2)$ are in 1-1 with the $i_{1\ldots6}$ found above. In particular, we should be able to find the value of each of the $i_{1\ldots6}$ as functions of the complete set of $SU(2)$ invariants. We, therefore, look for $SU(2)$ invariants that can reproduce $i_{1\ldots6}$; that is, we look for $SU(2)$ invariant polynomials in the fields such that when these are evaluated on the gauge-fixed configuration (3.9), their values are sufficient to reconstruct combinations (3.11), (3.12).

The flavor symmetry is a guide. For instance, the combinations with one free index i.e. i_1^I, i_4^I should be reproduced from operators with one free index. One such operator is provided by the operator I_4^{IJKL} where three indices are replaced by 1. Another operator that we can

consider is the antisymmetric bilinear

$$I_2^{IJ} = V^I_{abc} V^{Jabc} \tag{3.13}$$

Indeed, we find that on the configuration (3.9)

$$I_2^{I1} = i_1^I \qquad I_4^{111I} = 4 i_4^I \tag{3.14}$$

Hence a knowledge of the invariants (3.7, 3.13) indeed allows us to reproduce the combinations with one free flavor index i_1^I, i_4^I.

The combinations with two free flavor indices are i_2^{IJ}, i_5^{IJ}. We find that on the configuration (3.9)

$$I_2^{IJ} = -3 i_2^{IJ} + \ldots \qquad I_4^{11IJ} = -4 i_5^{IJ} + \ldots \tag{3.15}$$

where the ellipses represent products of invariants with fewer indices.

We have two combinations with three flavor indices, i.e. i_3^{IJK}, i_6^{IJK}. One can be reproduced from I_4^{1IJK}, but we need another invariant. We, therefore, consider

$$I_6^{IJKLMN} = W^{IJ}_{ab} W^{KL}_{cd} V^{Mab}{}_e V^{Necd} \tag{3.16}$$

We find on the configuration (3.9)

$$I_4^{1IJK} = -8 i_3^{IJK} + \ldots \tag{3.17}$$

$$I_6^{111IJK} = -8 i_6^{IJK} + \ldots \tag{3.18}$$

where we have omitted terms which are composed of products of invariants of lower degree.

We find then that the invariants

$$I_2^{IJ}, I_4^{11IJ}, I_6^{IJKLMN} \tag{3.19}$$

are sufficient to reconstruct the gauge invariant parameter space of this theory. The theorem from the introduction then tells us that these are a complete set of independent polynomial invariants for the dimension-4 representation of $SU(2)$ (that is, none of the invariants in this set can be expressed as a polynomial in the others, but any other invariant is a polynomial in these invariants). This also agrees with the counting of the previous subsection.

3.3 SO(3) with fields in the dimension-5 representation

3.3.1 Overview

We will take as our next example a theory with a gauge group $SO(3)$ and a field content where there are N fields are in a representation of dimension 5; such a field is a symmetric tensor V_{ij}^I of $SO(3)$, where we take $i, j = 1..3$, and $I = 1..N$ is a flavor index labeling the different fields. The field is a traceless symmetric tensor of $SO(3)$, and can therefore be written as

$$V^I = \begin{pmatrix} V_{11}^I & V_{12}^I & V_{13}^I \\ V_{12}^I & V_{22}^I & V_{23}^I \\ V_{13}^I & V_{23}^I & V_{33}^I \end{pmatrix} \tag{3.20}$$

with $V_{11}^I + V_{22}^I + V_{33}^I = 0$.

It will prove convenient to define a product

$$(A \cdot B)_{ij} = A_{ik} B^k{}_j \tag{3.21}$$

as well as a trace

$$Tr(A) = A_{ij}\delta^{ij} \tag{3.22}$$

We may then write a sequence of invariants

$$\begin{aligned} I_2^{IJ} &= Tr(V^I \cdot V^J) \\ I_3^{IJK} &= Tr(V^I \cdot V^J \cdot V^K) \\ I_4^{IJKL} &= Tr(V^I \cdot V^J \cdot V^K \cdot V^L) \\ I_5^{IJKLM} &= Tr(V^I \cdot V^J \cdot V^K \cdot V^L \cdot V^M) \end{aligned} \tag{3.23}$$

and so on.

To find the independent set of invariants, we now find the gauge-fixed configuration space, and attempt to characterize this space by invariants.

3.3.2 Counting Invariant Candidates

We follow the counting procedure described in the previous section. For each candidate tableau in the first column (shown as a list of the number of boxes per row), we use the program LiE to find the number of invariants (shown in the second column) with that symmetry. We then subtract any invariants that can be written as a product of smaller invariants; these are shown in the following column of the table. The net number of independent invariants after subtraction is shown in the last column.

Tableau	# Invts	Subtract	Net
(2)	1	0	1
$(1,1)$	0	0	0
(3)	1	0	1
$(2,1)$	0	0	0
$(1,1,1)$	0	0	0
(4)	1	$(2)\otimes(2)$	0
$(3,1)$	0	0	0
$(2,2)$	2	$(2)\otimes(2)$	1
$(2,1,1)$	0	0	0
$(1,1,1,1)$	0	0	0
(5)	1	$(2)\otimes(3)$	0
$(4,1)$	1	$(2)\otimes(3)$	0
$(3,2)$	1	$(2)\otimes(3)$	0
$(3,1,1)$	0	0	0
$(2,2,1)$	1	0	1
$(2,1,1,1)$	0	0	0
$(1,1,1,1,1)$	1	0	1
(6)	2	$(3)\otimes(3),\ (2)\otimes(2)\otimes(2)$	0
$(5,1)$	0	0	0
$(4,2)$	3	$(2,2)\otimes(2),\ (2)\otimes(2)\otimes(2),\ (3)\otimes(3)$	0
$(3,3)$	0	0	0
$(4,1,1)$	0	0	0
$(3,2,1)$	1	$(2,2)\otimes(2)$	0
$(2,2,2)$	2	$(2,2)\otimes(2),\ (2)\otimes(2)\otimes(2)$	0
$(3,1,1,1)$	1	0	1
$(2,2,1,1)$	0	0	0
$(2,1,1,1,1)$	0	0	0
$(1,1,1,1,1,1)$	0	0	0

We have also verified that no new invariants of degree 8 exist. We construct candidate

structures for each of the tableaux with a net nonzero number of invariants.

$$(2) \ : \ I_2^{IJ} = \mathrm{Tr}\left(V^I \cdot V^J\right) \tag{3.24}$$

$$(3) \ : \ I_3^{IJK} = \mathrm{Tr}\left(V^I \cdot V^J \cdot V^K\right) \tag{3.25}$$

$$(2,2) \ : \ \tilde{I}_4^{IJKL} = \mathrm{Tr}\left(V^{[I} \cdot V^{J]} \cdot V^{[K} \cdot V^{L]}\right) \tag{3.26}$$

$$(2,2,1) \ : \ I_5^{IJKLMN} = \mathrm{Tr}\left(V^{[I} \cdot V^J \cdot V^{K]} \cdot V^{[L} \cdot V^{M]}\right) \tag{3.27}$$

$$(1,1,1,1,1) \ : \ \tilde{I}_5^{IJKLM} = \mathrm{Tr}\left(V^{[I} \cdot V^J \cdot V^K \cdot V^L \cdot V^{M]}\right) \tag{3.28}$$

$$(3,1,1,1) \ : \ I_6^{IJKLMN} = \mathrm{Tr}\left(V^{[I} \cdot V^J \cdot V^K \cdot V^{L]} \cdot V^M \cdot V^N\right) \tag{3.29}$$

3.3.3 Construction of Invariants

Even though all invariants of eight fields canceled above, this cannon prove that we have achieved a complete count. To have confidence in our result, we apply the theorem from the introduction by gauge-fixing some fields.

We first consider the case where the matter content is a single field V^1. We can use complex gauge transformations to bring this to the form of a diagonal matrix

$$V^1 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \tag{3.30}$$

with $a + b + c = 0$.

This generically fixes the continuous gauge symmetry, but the ordering of the three diagonal elements can be changed by a gauge transformation. There is therefore a residual discrete $Z_2 \times Z_2$ gauge symmetry. In the special case when two eigenvalues coincide, the continuous symmetry is partially unbroken and there is a residual $U(1)$ symmetry.

The invariants are the symmetric combinations

$$i_1 = a + b + c = 0 \tag{3.31}$$

$$i_2 = a^2 + b^2 + c^2 \tag{3.32}$$

$$i_3 = a^3 + b^3 + c^3 \tag{3.33}$$

We look for $SO(3)$ invariants which can reproduce these combinations. We find

$$I_2^{11} = i_2 \qquad I_3^{111} = i_3 \tag{3.34}$$

The invariants I_2^{IJ}, I_3^{IJK} are hence sufficient to completely parametrize the gauge-inequivalent configurations of a single field, and are hence form a complete set of invariants for one field.

We now consider a generic configuration of multiple fields V^I. We gauge fix V^1 as before. The configuration is now

$$V^1 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix}$$

$$V^I = \begin{pmatrix} V_{11}^I & V_{12}^I & V_{13}^I \\ V_{21}^I & V_{22}^I & V_{23}^I \\ V_{31}^I & V_{32}^I & V_{33}^I \end{pmatrix} \qquad \text{for I} > 1 \tag{3.35}$$

We first consider the special case where two eigenvalues of V^1 coincide. Here we have

that

$$V^1 = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$
$$V^I = \begin{pmatrix} V^I_{11} & V^I_{12} & V^I_{13} \\ V^I_{21} & V^I_{22} & V^I_{23} \\ V^I_{31} & V^I_{32} & V^I_{33} \end{pmatrix} \quad \text{for I} > 1 \tag{3.36}$$

This particular configuration preserves a $U(1)$ subgroup of the $SU(2)$, so we could proceed by further fixing the gauge to find the completely gauge-fixed hypersurface in field space. But the gauge group is simple enough at this point that we can straightforwardly write down a set of independent polynomial combinations (invariant under the $U(1)$ symmetry) which parametrize the configuration space.

The fields in V^I can be organized into combinations with specific charges $2, 1, 0, -1, -2$ under the $U(1)$; these are

$$\begin{aligned} T^I_2 &= 2V^I_{12} + i(V^I_{11} - V^I_{22}) \\ T^I_1 &= V^I_{13} - iV^I_{23} \\ T^I_0 &= V^I_{11} + V^I_{22}, \end{aligned} \tag{3.37}$$

where the subscripts denote the respective charges. The negatively charged fields are the complex conjugates of the positive charges.

In addition to the $U(1)$, there is also a discrete Z_2 symmetry corresponding to charge conjugation

$$T^I_2 \leftrightarrow -T^I_{-2}, \quad T^I_1 \leftrightarrow -T^I_{-1}, \quad T^I_0 \leftrightarrow T^I_0 \tag{3.38}$$

The combinations of fields invariant under the $U(1)$ symmetry are

$$T_2^I T_{-2}^J,\ T_1^I T_{-1}^J,\ T_0^I,\ T_2^I T_{-1}^J T_{-1}^K,\ T_{-2}^I T_1^J T_1^K$$

Under the Z_2 action, these combinations are acted on as

$$\begin{aligned} T_0^I &\leftrightarrow T_0^I \\ T_2^I T_{-2}^J &\leftrightarrow T_2^J T_{-2}^I \\ T_1^I T_{-1}^J &\leftrightarrow T_1^J T_{-1}^I \\ T_2^I T_{-1}^J T_{-1}^K &\leftrightarrow -T_{-2}^I T_1^J T_1^K \end{aligned} \tag{3.39}$$

We make combinations which are even/odd under the Z_2; the gauge invariant even combinations are

$$\begin{aligned} i_1^I &\equiv T_0^I \\ i_2^{IJ} &\equiv T_2^I T_{-2}^J + T_2^J T_{-2}^I \\ i_3^{IJ} &\equiv T_1^I T_{-1}^J + T_1^J T_{-1}^I \\ i_4^{IJK} &\equiv T_2^I T_{-1}^J T_{-1}^K - T_{-2}^I T_1^J T_1^K \end{aligned} \tag{3.40}$$

A product of two combinations odd under the Z_2 is even under the Z_2. The only such combination which cannot be written in terms of the already obtained even combinations is

$$i_5^{IJKL} \equiv (T_2^I T_{-2}^J - T_2^J T_{-2}^I)(T_1^K T_{-1}^L - T_1^L T_{-1}^K) \tag{3.41}$$

The combinations $i_1^I, i_2^{IJ}, i_3^{IJ}, i_4^{IJK}, i_5^{IJKL}$ completely parametrize the gauge-inequivalent orbits of the configuration space.

We now promote these to $SO(3)$ invariants; that is, we look for $SO(3)$ invariants which

reduce to the combinations (3.40,3.41) on the gauge-fixed configuration space of equation (3.35). Once again, we use the flavor symmetry as a guide.

The combination with one free index i.e. i_1^I should be reproduced from operators with one free flavor index. One such operator is provided by the operator I_2^{IJ} where one of the fields is taken to be V^1. Indeed, we find

$$I_2^{1I} = 3ai_1^I \tag{3.42}$$

Hence a knowledge of the invariant I_2^{IJ} allows us to reproduce the combinations with one free flavor index i_1^I.

The combinations with two free flavor indices i.e. i_2^{IJ}, i_3^{IJ} should be reproduced from $SO(3)$ invariants with two free flavor indices IJ. Indeed, we find

$$I_2^{IJ} = \frac{1}{4}i_2^{IJ} + i_3^{IJ} + ... \tag{3.43}$$

$$I_3^{1IJ} = \frac{a}{4}(i_2^{IJ} - 2i_3^{IJ}) + ... \tag{3.44}$$

where the ellipses indicate terms with (already determined) lower degree combinations like $i_1^I i_1^J$. These two invariants therefore determine i_2^{IJ}, i_3^{IJ}.

For i_4^{IJK}, we have JK symmetrized. This will be reproduced by an $SO(3)$ invariant with three flavor indices IJK, where the JK indices are symmetrized. The flavor indices may be organized into various symmetries, and must map to $SO(3)$ invariants with the same structure. These may be guessed by

Tableau	Candidate $SO(3)$ invariant
$\begin{array}{\|c\|c\|c\|}\hline I & J & K \\ \hline \end{array}$	$\begin{array}{\|c\|c\|c\|}\hline I & J & K \\ \hline \end{array}$
$\begin{array}{\|c\|c\|}\hline J & K \\ \hline I \\ \cline{1-1} \end{array}$	$\begin{array}{\|c\|c\|}\hline J & K \\ \hline I & 1 \\ \hline \end{array}$

To verify that this guess is correct, we evaluate the invariants after gauge fixing. We find

$$I_3^{IJK} = -\frac{i}{4}(i_4^{IJK} + i_4^{JIK} + i_4^{KIJ}) + \dots \tag{3.45}$$

$$\tilde{I}_4^{IJK1} + \tilde{I}_4^{IKJ1} = -\left(\frac{3ai}{2}\right)(2i_4^{IJK} - i_4^{JIK} - i_4^{KIJ}) + \dots \tag{3.46}$$

where the ellipses indicate terms with (already determined) lower-degree combinations. Hence the invariants I_3^{IJK}, $\tilde{I}_4^{IJKL}$ indeed allow us to solve for i_4^{IJK}.

Finally, for i_5^{IJKL}, we have that IJ and KL are antisymmetrized. Once again, this invariant will be organized into various symmetries, and must map to $SO(3)$ invariants with the same structure.

These are guessed to be

Tableau	Candidate $SO(3)$ invariant
$\begin{array}{\|c\|c\|}\hline I & K \\ \hline J & L \\ \hline \end{array}$	$\begin{array}{\|c\|c\|}\hline I & K \\ \hline J & L \\ \hline \end{array}$
$\begin{array}{\|c\|c\|}\hline I & K \\ \hline J \\ \cline{1-1} L \\ \cline{1-1} \end{array}$	$\begin{array}{\|c\|c\|}\hline I & K \\ \hline J & 1 \\ \cline{1-2} L \\ \cline{1-1} \end{array}$
$\begin{array}{\|c\|}\hline I \\ \hline J \\ \hline K \\ \hline L \\ \hline \end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline I \\ \hline J \\ \hline K \\ \hline L \\ \hline \end{array}$

Evaluating these invariants on the gauge-fixed configuration, we find that indeed i_5^{IJKL} can be written as a combination of $\tilde{I}_4^{IJKL}$, $\tilde{I}_5^{IJKLM}$, I_6^{IJLK11}.

The parameters of the gauge-fixed configuration space with the enhanced $U(1)$ symmetry are therefore in 1-1 correspondence with the invariants (that is, every parameter of the gauge-fixed configuration with the enhanced $U(1)$ symmetry can be found as a polynomial in the

invariants)

$$I_2^{IJ}, I_3^{IJK}, \tilde{I}_4^{IJKL}, \tilde{I}_5^{IJKLM}, I_6^{IJLKM}$$

More generally, the eigenvalues of V^1 are all different, and we have

$$V^1 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix}$$
$$V^I = \begin{pmatrix} V_{11}^I & V_{12}^I & V_{13}^I \\ V_{21}^I & V_{22}^I & V_{23}^I \\ V_{31}^I & V_{32}^I & V_{33}^I \end{pmatrix} \quad \text{for I} > 1 \tag{3.47}$$

The parameters a, b are reproduced from I_2^{11}, I_3^{111}, following the analysis for a single field.

This parametrization of the configuration space preserves two Z_2 symmetries:

$$\begin{aligned} (Z_2)_A &: V_{12}^I \to -V_{12}^I, V_{13}^I \to -V_{13}^I \\ (Z_2)_B &: V_{12}^I \to -V_{12}^I, V_{23}^I \to -V_{23}^I \end{aligned} \tag{3.48}$$

To find the moduli space, we should find the combinations of V_{ij}^I which are gauge invariant under these discrete symmetries. Once again, the remaining symmetry is simple enough that we can just do this by inspection. We find that the polynomials invariant under the two discrete symmetries are generated by

$$V_{11}^I, \quad V_{22}^I, \quad V_{12}^I V_{12}^J, \quad V_{13}^I V_{13}^J, \quad V_{23}^I V_{23}^J, \quad V_{12}^I V_{13}^J V_{23}^K$$

We now promote these to $SO(3)$ invariants; that is, we look for $SO(3)$ invariants which reduce to these combinations on the configuration space of equation (3.47). We first check whether the invariants we have found are sufficient to do this.

We start with invariants with one flavor index I. We find

$$I_2^{1I} = aV_{11}^I + bV_{22}^I + (a+b)(V_{11}^I + V_{22}^I) \tag{3.49}$$

$$I_3^{11I} = a^2V_{11}^I + b^2V_{22}^I + (a+b)^2(-V_{11}^I - V_{22}^I) \tag{3.50}$$

which can be used to solve for V_{11}^I, V_{22}^I in terms of I_2^{1I}, I_2^{11I}. This inversion fails only if $(b+2a) = 0, (2b+a) = 0$ or $a = b$ i.e. when two eigenvalues in V^1 are equal, which we have already assumed to not be the case.

Similarly, it is easy to show that the invariants $I_2^{IJ}, I_3^{IJ1}, \tilde{I}_4^{I1J1}$ (symmetric in IJ) are sufficient to reconstruct $V_{12}^I V_{12}^J, V_{13}^I V_{13}^J, V_{23}^I V_{23}^J$.

The invariant $V_{12}^I V_{13}^J V_{23}^K$ can be organized in several symmetries, as in the analysis above. We find these are reproduced by I_3^{IJK}, $\tilde{I}_4^{IJK1}$, I_6^{I1J1K1}.

Our final result is then that every point on the gauge invariant configuration space can be reproduced by a knowledge of the invariants

$$I_2^{IJ},\ I_3^{IJK},\ \tilde{I}_4^{IJKL},\ I_5^{IKJLM},\ \tilde{I}_5^{IJKLM},\ I_6^{IJLKM} \tag{3.51}$$

Hence the theorem described in the introduction ensures that any gauge invariant polynomial in this theory can be generated by these invariants.

3.4 SU(3) with sextets

3.4.1 Overview

We now consider a theory with a $SU(3)$ symmetry and fields in the sextet representation V_{ij}^I.

We can form an infinite set of invariants by contracting these fields with the epsilon tensor ϵ^{ijk}. We now find an independent set of tensors by finding a set that can parametrize

the gauge fixed configuration space.

3.4.2 Counting Candidate Invariants

We follow the counting procedure described in the previous section. For each candidate tableau in the first column (shown as a list of the number of boxes per row) we use the program LiE to find the number of invariants (shown in the second column) with that symmetry. We then subtract any invariants that can be written as a product of smaller invariants; these are shown in the following column of the table. The net number of independent invariants after subtraction is shown in the last column.

Tableau	# Invts	Subtract	Net
(2)	0	0	0
(1, 1)	0	0	0
(3)	1	0	1
(2, 1)	0	0	0
(1, 1, 1)	0	0	0
(4)	0	0	0
(3, 1)	0	0	0
(2, 2)	0	0	0
(2, 1, 1)	0	0	0
(1, 1, 1, 1)	0	0	0
(5)	0	0	0
(4, 1)	0	0	0
(3, 2)	0	0	0
(3, 1, 1)	0	0	0
(2, 2, 1)	0	0	0
(2, 1, 1, 1)	0	0	0
(1, 1, 1, 1, 1)	0	0	0
(6)	1	$(3) \otimes (3)$	0
(5, 1)	0	0	0
(4, 2)	1	$(3) \otimes (3)$	0
(3, 3)	0	0	0
(4, 1, 1)	0	0	0
(3, 2, 1)	0	0	0
(2, 2, 2)	1	0	1
(3, 1, 1, 1)	0	0	0
(2, 2, 1, 1)	0	0	0
(2, 1, 1, 1, 1)	0	0	0
(1, 1, 1, 1, 1, 1)	1	0	1

We define the invariants

$$I_3^{IJK} \equiv \epsilon^{ikm}\epsilon^{jln}V_{ij}^I V_{kl}^J V_{mn}^K \tag{3.52}$$

$$I_6^{IJKLMN} \equiv \epsilon^{abc}V_{ad}^I V_{be}^J V_{cf}^K \epsilon^{eim}\epsilon^{fkn}\epsilon^{djl}V_{ij}^L V_{kl}^M V_{mn}^N \tag{3.53}$$

$$\tag{3.54}$$

3.4.3 Construction of Invariants

We begin by considering a single field. By a gauge transformation, we can bring it to the form

$$V^1 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \tag{3.55}$$

This form of the field preserves two $U(1)$ symmetries; the first is

$$\begin{aligned} V_{11} &\to e^{2i\alpha} V_{11} \qquad V_{22} \to e^{-2i\alpha} V_{22} \qquad V_{33} \to V_{33} \\ V_{13} &\to e^{i\alpha} V_{13} \qquad V_{23} \to e^{-i\alpha} V_{23} \qquad V_{12} \to V_{12} \end{aligned} \tag{3.56}$$

and the second is

$$\begin{aligned} V_{11} &\to e^{2i\alpha} V_{11} \qquad V_{33} \to e^{-2i\alpha} V_{33} \qquad V_{22} \to V_{22} \\ V_{12} &\to e^{i\alpha} V_{12} \qquad V_{23} \to e^{-i\alpha} V_{23} \qquad V_{13} \to V_{13} \end{aligned} \tag{3.57}$$

These symmetries alter the eigenvalues without changing the form; the first one takes $a \to e^{2i\alpha} a, b \to e^{-2i\alpha} b, c \to c$, and the second takes $a \to e^{2i\alpha} a, c \to e^{-2i\alpha} c, b \to b$. The only gauge invariant combination is the product abc.

We find

$$I_3^{111} = 6abc \tag{3.58}$$

This invariant therefore reproduces the gauge-fixed configuration space for a single field, and is therefore a complete set of invariants for a single sextet of $SU(3)$.

We now consider multiple fields, We can bring these to the form

$$V^1 = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$$
$$V^I = \begin{pmatrix} V^I_{11} & V^I_{12} & V^I_{13} \\ V^I_{12} & V^I_{22} & V^I_{23} \\ V^I_{13} & V^I_{23} & V^I_{33} \end{pmatrix} \quad \text{for I} > 1 \tag{3.59}$$

where we have used the $U(1)$ symmetries to further simplify the form of the first field.

The form of the configuration space still preserves an $SO(3)$ symmetry, and each sextet of $SU(3)$ decomposes to a 1+5 of $SO(3)$. Fortunately, we have analyzed this system already in the previous section, and so we can write down the invariants. The only new invariant is the singlet, which is the trace of the matrix. Combining with the previously derived $SO(3)$ invariants for a dimension 5 field, the $SO(3)$ invariant combinations for this theory are

$$\begin{aligned} i_1^I &\equiv Tr(V^I) \\ i_2^{IJ} &\equiv Tr((V^I \cdot V^J) \\ i_3^{IJK} &\equiv Tr(V^I \cdot V^J \cdot V^K) \\ \tilde{i}_4^{IJKL} &\equiv Tr(V^{[I} \cdot V^{J]} \cdot V^{[K} \cdot V^{L]}) \\ \tilde{i}_5^{IJKLM} &\equiv Tr(V^{[I} \cdot V^J \cdot V^K \cdot V^L \cdot V^{M]}) \\ i_6^{IJLKM} &\equiv Tr(V^{[I} \cdot V^J \cdot V^{K]} \cdot V^{[L} \cdot V^{M]}) \end{aligned} \tag{3.60}$$

We now find $SU(3)$ invariants which reproduce these combinations on the configuration space (3.59).

From the invariant that we have already defined, we obtain

$$\begin{aligned} I_3^{11I} &= 2a^2 i_1^I \\ I_3^{1IJ} &= -a i_2^{IJ} + .. \\ I_3^{IJK} &= 2 i_3^{IJK} + .. \end{aligned} \tag{3.61}$$

which reproduces all combinations with one, two or three free flavor indices.

For the remaining combinations, we need to consider invariants containing six fields contracted with 4 epsilon tensors. The structure of the combinations above suggests that we should look at combinations where there are two pairs of three fields, where the three fields are antisymmetrized. We find

$$I_6^{[IJ]1[KL]1} = -a^2 \tilde{i}_4^{IJKL} \tag{3.62}$$

$$I_6^{[IJKLM]1} = 6a i_5^{IJKLM} \tag{3.63}$$

$$I_6^{[IJK][LM]1} = 4a i_6^{IJKLM} \tag{3.64}$$

We thus find that every point on the gauge-fixed configuration space can be reproduced by a knowledge of the invariants

$$I_3^{IJK}, I_6^{IJKLMN} \tag{3.65}$$

Hence the theorem described in the introduction ensures that any gauge invariant polynomial in this theory can be generated by these invariants.

3.5 Summary and Conclusion

We have discussed a new method to efficiently find a set of independent invariant tensors in gauge theories. We have done this by using a theorem, familiar from supersymmetric field

theories, that relates D-flat directions to the invariants in a gauge theory. Specifically, this theorem asserts that the constant configurations, identified by complex gauge transformations, are in 1-1 correspondence with the gauge invariant operators in the theory.

We have shown that this provides a straightforward method to find the independent invariant tensors. We have explicitly applied these methods to three gauge theories– $SO(3)$ with fields in the symmetric tensor representation, $SU(2)$ with a dimension 4 representation, and $SU(3)$ with matter in the sextet– and in each case, we have found the set of independent polynomial invariants. This shows the practicality of the approach.

Our methods are general, and as far as we can see, can be applied to any group with any matter content. The immediate ones which would be interesting to analyze are exceptional groups with matter in the (anti)-fundamental representation. Knowing the invariant tensors would also help in looking for dual pairs in supersymmetric gauge theories with exceptional gauge groups.

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Chapter 4

The Potency of Nilpotence

This chapter is based on joint work with Arvind Rajaraman and Yuri Shirman and was published in *Physical Review D* [1]. In this work, we analyze the moduli spaces of proposed supersymmetric dualities involving adjoint superfields in $SU(N)$ gauge theories, focusing on the A_k and D_{k+2} dualities. We develop a new method for testing these dualities by studying their behavior along specific nilpotent flat directions in the moduli space. We find that the proposed A_k dualities pass these consistency checks, whereas the D_{k+2} dualities fail for the cases considered. The presentation in this chapter follows the published work, with additional discussion and details included where useful.

4.1 Introduction

One of the most striking features of $\mathcal{N} = 1$ supersymmetric gauge theories is Seiberg duality [2], which posits an infrared (IR) equivalence between seemingly different ultraviolet (UV) gauge theories. The initial examples of dualities in $SU(N)$ theories with matter in a fundamental representation were quickly expanded to more general theories, and even to chains of dualities.

The duality conjectures have been tested and extended in numerous settings, particularly for theories with simple superpotentials and gauge groups. Since at least one of the two dual

descriptions is typically strongly coupled in the IR, the duality is usually verified by indirect checks. Firstly, the global symmetries of the two theories must agree. Secondly, the 't Hooft anomalies must match for the two theories. These are very restrictive conditions, and the successful matching of the 't Hooft anomalies is taken as strong evidence for the duality.

Further tests of the dualities involve dynamics on the moduli space or the deformation of the theory by new superpotential terms. Often, these tests are complementary – turning on moduli VEVs on the electric side corresponds to superpotential deformations on the magnetic side and vice versa. Typically, turning on gauge invariant operators on the electric side of duality results in gauge symmetry breaking. On the magnetic side, this usually corresponds to superpotential deformations which may result in new Yukawa interactions, mass terms for some matter fields, and gauge symmetry breaking. For the duality to be valid, the two low energy theories must remain dual. This check is especially non-trivial along flat directions where one can show that the two low energy theories flow to the same IR fixed point.

A comprehensive study of dynamics on the moduli space has been done for some classes of models, for example, $SU(N)$ theories with matter in the fundamental representation [2]. However, due to the complexity of the moduli space, the analysis is far from complete in models with more general gauge groups, matter content, and the superpotentials.

In this paper we will focus on $\mathcal{N} = 1$ dualities in models with chiral superfields in both fundamental and adjoint representations and nontrivial superpotentials for the adjoints. While the exhaustive study of the dynamics on the moduli space of these theories is beyond the scope of this paper, we will extend the analysis to a new class of flat directions (we will refer to such flat directions as nilpotent) and show that new stringent tests of dualities can be performed along these directions. We expect such analysis to be useful in more general models with general matter representations and non-trivial superpotentials which allow the existence of nilpotent flat directions.

The first model of $\mathcal{N} = 1$ duality with adjoint superfields was proposed by Kutasov [3–5], where the electric description was based on an $SU(N)$ gauge group with F chiral flavors in

the fundamental representation, a single adjoint chiral superfield X and a superpotential[1]

$$W = \mathrm{Tr} X^{k+1} \,. \tag{4.1}$$

The F-flatness conditions coming from the superpotential truncate the chiral ring of the theory and are an essential condition for the existence of the duality map.

Kutasov's conjecture was extended in [6] to theories with F fundamental flavors, two adjoints X and Y, and a superpotential

$$W = \mathrm{Tr} X^{k+1} + \mathrm{Tr} X Y^2 \,. \tag{4.2}$$

It was also shown [6–9] that the superpotentials in these two classes of models correspond to A_k and D_{k+2} singularities of Arnold's singularity theory [10].

However, already in the original paper [6] it was realized that the duality conjecture in D_{k+2} poses a puzzle. For any value of k, the theory and the dual have the same symmetries, and the global anomalies match across the theories. However, while the F-term equations lead to truncation of the chiral ring in models with odd k, the chiral ring is not truncated in models with even k. Since the truncation of the chiral ring is a necessary condition for the existence of duality, this puts in question the validity of the duality conjecture of [6], at least when k is even. While one may have more confidence in the duality for odd k, there is still a tension, which arises from the fact that one can flow between even and odd k theories either by deforming them with tree level superpotential terms or by simply moving along the moduli space.

One possible resolution of this puzzle could be that the chiral ring of even k model is truncated by non-perturbative dynamics. Further research led to better understanding of the dynamics of ADE models [7], however, no dynamical effects capable of truncating the chiral ring were found.

[1] We will ignore numerical coefficients throughout.

In this work we will take a different approach towards addressing the puzzle. In particular, we will carefully investigate a class of directions on the moduli space of the A_k and D_{k+2} models along which the VEVs of X is nilpotent[2], $X^k = 0$. In the case of A_k models our work will extend the results of [11] while in the case of D_{k+2} models our analysis will be completely new. Our results will further strengthen the evidence for duality in A_k models but the tests of duality will fail in D_{k+2} models even when k is odd.

The outline of the paper is as follows. In Section 4.2, we review the conjectured dualities involving adjoints. In Section 4.3 we discuss the tests of duality in A_k and D_{k+2} models. After reviewing the existing duality tests in Section 4.3.1 we discuss the dynamics along nilpotent flat directions in Section 4.3. We then test the duality of A_k models in Section 4.3.3 finding additional evidence in support of duality conjecture. Finally, in Section 4.3.4 we show that similar analysis of D_{k+2} models does not yield consistent results. We close with a discussion.

4.2 Review of A_k and D_{k+2} Dualities

4.2.1 A_k Dualities

The electric description of A_k models [3] is given in terms of an $SU(N)$ gauge group with an adjoint superfield X and F flavors in the fundamental representation. The matter content and symmetries of the model are given by Table 4.1. The $U(1)_R$ charges were chosen so that

Field	$SU(N)$	$SU(F)$	$SU(F)$	$U(1)_B$	$U(1)_R$
Q	$\square$	$\square$	1	1	$1-\frac{2}{k+1}\frac{N}{F}$
$\bar{Q}$	$\overline{\square}$	1	$\overline{\square}$	-1	$1-\frac{2}{k+1}\frac{N}{F}$
X	Adj	1	1	0	$\frac{2}{k+1}$

Table 4.1: Matter content and global charges of the A_k electric theory.

[2]When studying D_{k+2} models we will limit our attention to $Y = 0$ directions for simplicity.

the following superpotential is allowed by symmetries

$$W_{el} = \mathrm{Tr}(X^{k+1}) + \lambda \mathrm{Tr} X \,, \tag{4.3}$$

where we have included a Lagrange multiplier term $\lambda \mathrm{Tr} X$ in the superpotential to enforce tracelessness of the adjoint. The equation of motion ensures that X is nilpotent up to an identity operator

$$X^k = \lambda \mathbb{I} \,. \tag{4.4}$$

We will be interested in the moduli space directions consistent with $\lambda = 0$ and therefore we will drop the Lagrange multiplier terms in the following.

As was argued in [5], this equation truncates the chiral ring of the theory and deforms the quantum moduli space. The low energy physics can then be described in terms of "dressed" chiral meson and baryon operators that belong to the truncated chiral ring. In particular, the meson operators in the chiral ring are of the form

$$M_m = \bar{Q} X^{m-1} Q \,, \qquad m = 1, 2, ..., k \,. \tag{4.5}$$

The duality maps dressed mesons to the elementary mesons in the magnetic theory[3]. The magnetic theory is described in terms of an $SU(kF - N)$ gauge theory with matter content provided in table 4.2. and the superpotential

$$W_{mag} = \mathrm{Tr}(\tilde{X}^{k+1}) + \sum_{m=1}^{k} M_m \bar{q} \tilde{X}^{k-m} q \tag{4.6}$$

[3]The dressed baryons are mapped to dressed baryons of the magnetic description but we will not write them down explicitly since we will not turn them on in our discussion.

Field	$SU(kF-N)$	$SU(F)$	$SU(F)$	$U(1)_B$	$U(1)_R$
q	$\square$	$\overline{\square}$	1	$\frac{N}{kF-N}$	$1-\frac{2}{k+1}\frac{kF-N}{F}$
$\bar{q}$	$\overline{\square}$	1	$\square$	$-\frac{N}{kF-N}$	$1-\frac{2}{k+1}\frac{kF-N}{F}$
$\tilde{X}$	Adj	1	1	0	$\frac{2}{k+1}$
M_m	1	$\square$	$\overline{\square}$	0	$2-\frac{4}{k+1}\frac{N}{F}+\frac{2}{k+1}(m-1)$

Table 4.2: Matter content and global charges of the A_k magnetic theory.

4.2.2 D_{k+2} Dualities

The D_{k+2} model [6] can be obtained by extending the A_k model with a second adjoint superfield, Y, and a new superpotential term. The electric description is based on an $SU(N)$ gauge group with matter content given by Table 4.3 and the superpotential[4]

Field	$SU(N)$	$SU(F)$	$SU(F)$	$U(1)_B$	$U(1)_R$
Q	$\square$	$\square$	1	1	$1-\frac{N}{F(k+1)}$
$\bar{Q}$	$\overline{\square}$	1	$\overline{\square}$	-1	$1-\frac{N}{F(k+1)}$
X	Adj	1	1	0	$\frac{2}{k+1}$
Y	Adj	1	1	0	$\frac{k}{k+1}$

Table 4.3: Matter content and global charges of the D_{k+2} electric theory.

$$W = \mathrm{Tr}\left(X^{k+1}\right) + \mathrm{Tr}\left(XY^2\right) , \tag{4.7}$$

The equations of motion are given by

$$\begin{aligned} (k+1)X^k + Y^2 &= 0 \\ XY + YX &= 0 \end{aligned} \tag{4.8}$$

Clearly, in this model X is only nilpotent if Y is. On the other hand, we can see from

[4]As mentioned earlier, we have dropped Lagrange multiplier terms for brevity.

(4.8) that Y satisfies the equation

$$Y^3 = (-1)^k Y^3 \,, \tag{4.9}$$

implying that it is nilpotent for odd k while remaining unconstrained for even k. Thus the chiral ring is only truncated for odd values of k. It has been conjectured in the literature that for even k the chiral ring is truncated by non-perturbative dynamics, however no such dynamical effects have been found so far [7].

Assuming the truncation of the chiral ring, the mesonic directions of the quantum moduli space of the electric theory are parameterized by

$$M_{m,l} = \bar{Q} X^{m-1} Y^{l-1} Q \,, \qquad m = 1, 2 \ldots, k \,; \quad l = 1, 2, 3 \,. \tag{4.10}$$

As usual, the composite mesons map into the elementary mesons of the magnetic description. The dual theory has gauge group $SU(3kF - N)$ and matter content is given by Table 4.4. The superpotential of the magnetic dual is given by

Field	$SU(3kF-N)$	$SU(F)$	$SU(F)$	$U(1)_B$	$U(1)_R$
q	$\square$	$\overline{\square}$	1	$\frac{N}{3kF-N}$	$1 - \frac{3kF-N}{F(k+1)}$
$\bar{q}$	$\overline{\square}$	1	$\square$	$-\frac{N}{3kF-N}$	$1 - \frac{3kF-N}{F(k+1)}$
$\tilde{X}$	Adj	1	1	0	$\frac{2}{k+1}$
$\tilde{Y}$	Adj	1	1	0	$\frac{k}{k+1}$
$M_{m,l}$	1	$\square$	$\overline{\square}$	0	$2 - \frac{2}{k+1}\frac{N}{F} + \frac{2(m-1)+k(l-1)}{k+1}$

Table 4.4: Matter content and global charges of the D_{k+2} magnetic theory.

$$W = \mathrm{Tr}\tilde{X}^{k+1} + \mathrm{Tr}\tilde{X}\tilde{Y}^2 + \sum_{m}^{k}\sum_{l}^{3} M_{m,l}\bar{q}\tilde{X}^{k-m}\tilde{Y}^{3-l}q \,. \tag{4.11}$$

4.3 New Tests of Duality Conjectures

4.3.1 Existing Tests of the Dualities

The duality conjectures in A_k and D_{k+2} theories have been tested in a variety of ways. The global symmetries and anomalies of the electric and magnetic description agree. The t'Hooft anomalies of the two theories have been shown to match (notably, for the D_{k+2} theories, this agreement occurs for both odd and even k). Certain deformations of the superpotential, e.g. by mass terms, have also been considered, and the duality conjecture continues to hold.

A different set of tests considers the dynamics along certain directions on the moduli space. One such example is the analysis of [11]. These authors considered the dynamics of an A_2 model along the Higgs branch parameterized by the meson $M_1 \sim Q\bar{Q}$. Choosing the rank of the meson matrix to be $F_M < N-1$, one finds the electric theory gauge group is broken to $SU(N-F_M)$. The adjoint X decomposes into an adjoint X', F_M fundamental flavors Q', $\overline{Q}'$ and gauge singlets. Since F_M fundamental flavors are eaten by super-Higgs mechanism, the number of light fundamentals is still F. On the other hand, the global symmetry is broken to $SU(F-F_M)$ due to a superpotential term

$$\Delta W_{el} = \sum_{f}^{F_M} \overline{Q}'^{f} X' Q'_f \tag{4.12}$$

which arises from the X^3 superpotential of the UV theory.

A naive duality transformation suggests that on the Higgs branch the magnetic dual is described in terms of an $SU(2F-(N-F))$ theory, i.e. it the gauge group of this dual is larger than the $SU(2F-N)$ magnetic group in the UV. The resolution of this apparent puzzle arises from the observation that the superpotential deformation (4.12) leads to tadpoles in the superpotential of the magnetic theory, which in turn break the $SU(2N-(F-N))$ group down to the expected $SU(2F-N)$.

A more general analysis of ADE models was performed in [7, 8]. While the authors found

that the duality conjectures worked in most cases, the duality conjecture failed along some flat directions in even-k D_{k+2} models. The authors of [7] performed an extensive analysis of the dynamics but did not find non-perturbative effects that would lift the dangerous flat directions and restore the duality conjecture. Going beyond D_{k+2} models, it was shown in [7] that similar puzzles arise in E_7 superconformal field theories where different RG flow trajectories resulted in different number of higher dimensional representations in the IR.

4.3.2 Dynamics of Nilpotent Flat Directions

Despite a large body of work, the dynamics along the flat directions of ADE models has not been studied exhaustively. While a complete analysis of the moduli space is beyond the scope of our work, we will here consider a large new class of flat directions, the nilpotent flat directions along which $X \neq 0$ while $X^k = 0$. These flat directions are parameterized by mesons of the form $M_m \sim \overline{Q} X^{m-1} Q$. While some of these flat directions were considered in [11] in the case of A_k models, they have not been studied in D_{k+2} models.

When we move along a nilpotent direction, we can analyze the RG flow into the IR in two different ways. In the first approach we will start from the magnetic dual theory in the UV. The moduli space of this theory consists of the Higgs and Coulomb branches, parameterized by dressed mesons and baryons of the magnetic theory, as well as a mesonic branch parametrized by gauge singlet mesons M_m. Moving along along our chosen nilpotent flat direction on the electric side corresponds to turning on the VEVs of these mesons and moving onto *the meson branch of the magnetic theory*. The magnetic gauge group remains unbroken along the meson branch but the magnetic superpotential is modified.

In the second approach we start from the electric theory and move along the nilpotent flat direction, thus partially breaking the electric gauge group. Integrating out the heavy degrees of freedom we find an effective description in terms of an A_k or D_{k+2} theory with a smaller gauge group and a deformed superpotential. We will refer to this theory as the *electric theory on the Higgs branch*. At this point we can perform the duality transformation,

thus constructing a *magnetic dual of the Higgs branch theory*. Somewhat surprisingly, the gauge group of this magnetic dual is larger than the magnetic gauge group found in the first approach. This is not necessarily a failure of the duality. Indeed, we shall see that the superpotential deformation of the Higgs branch electric theory implies a spontaneous symmetry breaking in its magnetic dual, leading to a possible agreement with the analysis employed in our first approach. If the duality works the RG flow may be graphically represented as in Figure 4.1.

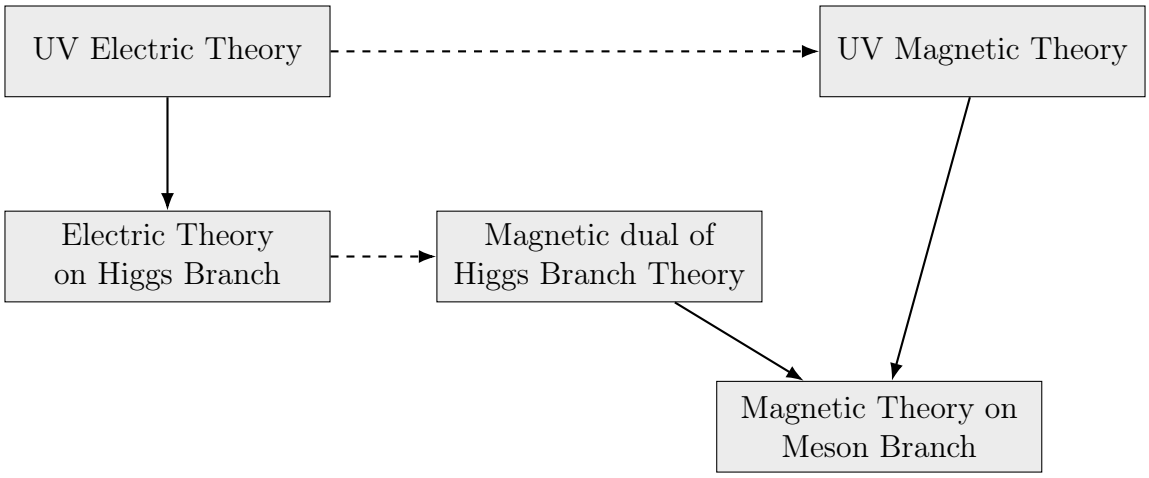


Figure 4.1: Flows between electric and magnetic theories under Higgsing and meson-branch deformations. The horizontal dashed lines indicate duality maps, and the vertical or diagonal lines represent flows from the UV theories to the IR theories.

Indeed, when studying the A_k models we will show that the theories follow the flows indicated by this diagram, further confirming the validity of the duality conjecture. On the other hand, we will argue that in D_{k+2} models the RG flow diagram can not close for any value of k, thus invalidating the duality conjecture in this class of models.

To be more specific, we note that the UV magnetic gauge group in the UV remains unbroken along the mesonic branch. On the other hand, we will find complementary pairs of nilpotent directions of the electric theory along which the effective description will be given in terms of D_{k+2} models with different gauge groups yet the same superpotential deformations. Since, in the magnetic description, the superpotential deformations appear as

symmetry breaking tadpoles, this precludes the possibility that the duals of the complementary nilpotent directions can flow to the same gauge group of the UV magnetic dual on the meson branch. It is interesting that the failure mode will be similar to the one found in [7] – this time different RG trajectories will lead to IR physics with the same superpotential but different magnetic gauge groups.

4.3.3 Duality Tests for A_k Theory

The Electric Theory on the Higgs branch

We consider a flat direction parameterized by a VEV of a single meson $M_n = \overline{Q}^F X^{n-1} Q_F$, $1 < n \leq k$. In terms of the elementary fields (and up to gauge and global symmetry transformations), this flat direction is given by

$$X_2^1 = X_3^2 = \cdots = X_n^{n-1} = Q_F^n = \overline{Q}_1^F = v \tag{4.13}$$

where the indices on the fundamental and anti fundamental are color indices. All other components of the fields are 0. It is easy to see that the adjoint is nilpotent along this direction, $X^n = 0$.

Along this direction the unbroken gauge group is $SU(N-n)$ and the adjoint X decomposes into an adjoint X', n fundamental flavors, and several gauge singlets. At the same time, n fundamental flavors are eaten by the super-Higgs mechanism. Thus the low energy $SU(N-n)$ theory still has F light flavors, one of which is still coupled in the superpotential..

The light degrees of freedom and their charges are given by Table 4.5 where we omitted gauge singlet fields with the R charge less than 2/3 since they saturate the unitarity bound and become free, decoupling from the interacting physics at the IR fixed point [2, 12–14]. The electric superpotential of the low energy theory takes the form

$$\mathrm{Tr}(X^{k+1}) = \mathrm{Tr}\left((X')^{k+1}\right) + \bar{x}\,(X')^{k-n}\,x + \ldots \tag{4.14}$$

Field	$SU(N-n)$	$SU(F-1)$	$SU(F-1)$	$U(1)_B$	$U(1)_R$
Q'	$\square$	$\square$	1	1	$\frac{F(k+1)+n-2N-1}{(F-1)(k+1)}$
$\bar{Q}'$	$\overline{\square}$	1	$\overline{\square}$	-1	$\frac{F(k+1)+n-2N-1}{(F-1)(k+1)}$
x	$\square$	1	1	1	$\frac{n+1}{k+1}$
$\bar{x}$	$\overline{\square}$	1	1	-1	$\frac{n+1}{k+1}$
X'	Adj	1	1	0	$\frac{2}{k+1}$

Table 4.5: Matter content and global charges of the A_k IR electric theory.

where x and $\overline{x}$ represent the light linear combination of the moduli left uneaten by the super-Higgs mechanism.

We note that one can interpret this effective theory as a deformation of an A_k model with $N' = N - n$ colors and F flavors by a superpotential term involving just one fundamental flavor. The deformation breaks the global symmetry to $SU(F-1)$.

The Magnetic Theory on the meson branch

The magnetic dual of the A_k theory along the nilpotent direction of the moduli space can be constructed in two ways. First we note that $(M_n)^F_F$ direction on the Higgs branch of the electric theory corresponds to a gauge singlet direction of the magnetic theory (we will refer to it as a mesonic direction). Below the scale of the meson VEV the physics is still described in terms of the $SU(kF-N)$ magnetic gauge group but now with the superpotential

$$W_{mag} = \mathrm{Tr}(\tilde{X}^{k+1}) + \sum_m M_m \bar{q} \tilde{X}^{k-m} q + \bar{q}^F \tilde{X}^{k-n} q_F \,, \tag{4.15}$$

although one should remember that the Yukawa coupling in the last term is given by a VEV of a massless field.

The Dual of the Electric Theory on Higgs Branch

We can now perform the duality transformation on the electric theory on the meson branch. Following [11] we find that this magnetic theory has an $SU(kF-N+n)$ gauge group, matter content given by Table 4.6. and the superpotential

Field	$SU(kF-N+n)$	$SU(F-1)$	$SU(F-1)$	$U(1)_B$	$U(1)_R$
q	$\square$	$\overline{\square}$	1	$\frac{N-n}{kF-N+n}$	$\frac{F(1-k)+2N-n}{(F-1)(k+1)}$
$\bar{q}$	$\overline{\square}$	1	$\square$	$-\frac{N-n}{kF-N+n}$	$\frac{F(1-k)+2N-n}{(F-1)(k+1)}$
q_F	$\square$	1	1	$\frac{N-n}{kF-N+n}$	$-\frac{n}{k+1}$
$\bar{q}^F$	$\overline{\square}$	1	1	$-\frac{N-n}{kF-N+n}$	$-\frac{n}{k+1}$
$\tilde{X}$	Adj	1	1	0	$\frac{2}{k+1}$
$(M'_m)_f^{f'}$	1	$\square$	$\overline{\square}$	0	$\frac{2(m(F-1)+Fk+n-2N-1)}{(F-1)(k+1)}$
$(M'_m)_F^f$	1	1	$\overline{\square}$	0	$\frac{F(2m+k+n+1)-2(m+N+1)}{(F-1)(k+1)}$
$(M'_m)_f^F$	1	$\square$	1	0	$\frac{F(2m+k+n+1)-2(m+N+1)}{(F-1)(k+1)}$
$(M'_m)_F^F$	1	1	1	0	$\frac{2(m+n+1)}{k+1}$
$(M'_{k-n})_F^F$	1	1	1	0	2

Table 4.6: Matter content and global charges of the A_k Dual of the Electric Theory on Higgs Branch.

$$W'_{mag} = \mathrm{Tr}(\tilde{X}^{k+1}) + \sum_m M'_m \bar{q}\tilde{X}^{k-m}q + (M'_{k-n})_F^F \,. \tag{4.16}$$

To fully analyze the vacuum structure of the deformed magnetic dual, one must observe a presence of a tadpole for the M'_{k-n} meson in the superpotential. This tadpole drives the VEVs of the composite $\bar{q}\tilde{X}^n q$ meson to non-zero value

$$0 \neq \bar{q}\tilde{X}^{k-(k-n)}q = \bar{q}\tilde{X}^n q \,. \tag{4.17}$$

At the minimum of the superpotential the unbroken gauge group is $SU(kF-N)$ in an agreement with the gauge group of the magnetic theory on the meson branch. Moreover,

integrating out the heavy fields, one arrives at the superpotential (4.15).

We therefore find that the duality conjectures work along these directions in the moduli space of the A_k theory.

4.3.4 Duality Tests for D_{k+2} Theory

We will now proceed in the same manner for the D_{k+2} theory. However, to analyze the D_{k+2} theory we will consider two related flat directions, parameterized by a pair of dressed mesons $M_{\tilde{n},1} = \overline{Q}^F X^{\tilde{n}-1} Q_F$ where the first meson has $\tilde{n} = n$ and the second meson has $\tilde{n} = k-n$. It is clear that the gauge symmetry breaking pattern along these flat directions can be inferred from the analysis of the A_k models. Thus we can build on the results obtained in section 4.3.3.

The Electric Theory on the Higgs branch

Along the $M_{\tilde{n},1}$ direction of the Higgs branch the $SU(N)$ gauge group is broken to $SU(N-\tilde{n})$ subgroup. Each of the adjoints, X and Y, decomposes into an adjoint (X' and Y' respectively), $\tilde{n}$ new fields in the fundamental representation ($x, \bar{x}$, and $y, \bar{y}$ respectively) and several gauge singlet fields. The low energy superpotential takes the form:

$$\begin{aligned} W = &\operatorname{Tr}\left((X')^{k+1}\right) + \operatorname{Tr}\left(X'(Y')^2\right) + \bar{x}\,(X')^{k-\tilde{n}}\,x + \bar{y}_1 Y' x + \bar{x} Y' y_{\tilde{n}} \\ &+ \sum_{f=1}^{\tilde{n}} \bar{y}_f X' y_f + \sum_{f=1}^{\tilde{n}-1} \bar{y}_{f+1} y_f + \dots\,, \end{aligned} \tag{4.18}$$

where x, $\bar{x}$, y_f, and $\bar{y}_f$ represent the $2\tilde{n}$ new flavors in the fundamental representation. It is easy to see that this superpotential gives mass to $\tilde{n}-1$ flavors while $\tilde{n}$ flavors are eaten by the super-Higgs mechanism. After integrating out the massive fields, the superpotential becomes

$$W = \operatorname{Tr}\left((X')^{k+1}\right) + \operatorname{Tr}\left(X'(Y')^2\right) + \bar{x} X'^{k-\tilde{n}} x + \bar{y} Y' x + \bar{x} Y' y + \bar{y}\,(X')^{\tilde{n}}\,y\,, \tag{4.19}$$

where $\bar{y} = \bar{y}_1$ and $y = y_n$ while the matter content of the theory is summarized in Table (4.7).

Field	$SU(N-n)$	$SU(F-1)$	$SU(F-1)$	$U(1)_B$	$U(1)_R$
Q'	$\square$	$\square$	1	1	$\frac{F(k+1)-N-1+n}{(F-1)(k+1)}$
$\bar{Q}'$	$\overline{\square}$	1	$\overline{\square}$	-1	$\frac{F(k+1)-N-1+n}{(F-1)(k+1)}$
x	$\square$	1	1	1	$\frac{n+1}{k+1}$
$\bar{x}$	$\overline{\square}$	1	1	-1	$\frac{n+1}{k+1}$
y	$\square$	1	1	1	$1-\frac{n}{k+1}$
$\bar{y}$	$\overline{\square}$	1	1	-1	$1-\frac{n}{k+1}$
X'	Adj	1	1	0	$\frac{2}{k+1}$
Y'	Adj	1	1	0	$\frac{k}{k+1}$

Table 4.7: Matter content and global charges of the D_{k+2} IR electric theory.

The Magnetic Theory on the meson branch

Starting in the UV the magnetic dual of the D_{k+2} model is given in terms of an $SU(3kF-N)$ gauge group discussed in Section 4.2.2. The nilpotent flat directions of the Higgs branch considered in the previous subsection correspond to the mesonic branch of the magnetic description. While the gauge group of the magnetic description remains unbroken, the meson VEV converts one of the superpotential terms into a tadpole for a dressed magnetic meson

$$M_{\tilde{n},1}\overline{q}\tilde{X}^{k-\tilde{n}}\tilde{Y}^2 q \longrightarrow \tilde{M}_{k-n,3}\,. \tag{4.20}$$

This tadpole drives the magnetic theory to a new IR fixed point.

The Dual of the Electric Theory on Higgs Branch

As in the case of the A_k models, the analysis is more involved if we construct a dual of the low energy $SU(N-\tilde{n})$ electric description. The magnetic theory has an $SU(N')$ gauge group

with $N' = 3k(F+1) - N + \tilde{n}$. The matter content consists two adjoint supermultiplets, $F+1$ flavors in the fundamental representation and a set of elementary mesons. However, the global non-Abelian symmetry under which the fundamentals and mesons transform is only $SU(F-1)_L \times SU(F-1)_R$. The superpotential of the dual description is given by a sum of the usual magnetic superpotential (4.11) of a D_{k+2} and the deformation superpotential

$$\Delta W'_{mag} \sim \left(M'_{k-\tilde{n},1}\right)^F_F + \left(M'_{1,2}\right)^F_{F+1} + \left(M'_{1,2}\right)^{F+1}_F + \left(M'_{\tilde{n},1}\right)^{F+1}_{F+1} . \tag{4.21}$$

The Failure of the Duality Map

Let us now compare the duals of the electric theories on the Higgs brach for the cases $\tilde{n} = n$ and $\tilde{n} = k - n$. It is easy to see that the two theories have the same unbroken global symmetry and the same superpotential (4.21). On the other hand, they have different gauge groups (specifically $SU(3k(F+1) - N + n)$ and $SU(3k(F+1) - N + k - n)$). In both cases, the number of colors in this dual description is greater than the number of colors in the UV magnetic theory $N' > 3kF - N$.

Just as in the case of A_k models, the gauge group of these magnetic theories has to be broken down to $SU(3kF - N)$ if duality is to work. Indeed, the tadpoles (4.21) induce VEVs for four dressed magnetic mesons:

$$\begin{aligned}
(M'_{k-n,1})^F_F &\to \bar{q}_F \tilde{X}^n \tilde{Y}^2 q_F \neq 0 \\
(M'_{1,2})^{F+1}_F &\to \bar{q}_F \tilde{X}^{k-1} \tilde{Y} q_{F+1} \neq 0 \\
(M'_{1,2})^F_{F+1} &\to \bar{q}_{F+1} \tilde{X}^{k-1} \tilde{Y} q_F \neq 0 \\
(M'_{n,1})^{F+1}_{F+1} &\to \bar{q}_{F+1} \tilde{X}^{k-n} \tilde{Y}^2 q_{F+1} \neq 0
\end{aligned} \tag{4.22}$$

partially breaking the magnetic $SU(N')$ group. However, our choice of $\tilde{n}$ ensures that these VEVs are invariant under the change $n \to k - n$. It is clear that the same VEVs can not break both $SU(3k(F+1) - N + n)$ and $SU(3k(F+1) - N + k - n)$ down to $SU(3kF - N)$.

In other words, the duality must fail for at least one of these nilpotent directions.

4.4 Discussion

Duality conjectures in N=1 supersymmetric theories have been tested in a variety of ways, including matching of symmetries and 't Hooft anomalies. We have analyzed here a new set of tests, which involve analysis of dynamics along nilpotent directions on the moduli space. The construction of the duality map along such flat directions leads to two potential magnetic duals which must be equivalent in the infrared.

We have applied these tests to $\mathcal{N} = 1$ duality with adjoint superfields and nontrivial superpotentials. We have found that models with A_k superpotentials theory indeed pass this test, lending further support to the duality conjecture. On the other hand, the models with D_{k+2} theories with two adjoints fail the test, suggesting that the conjecture is invalid. While previous authors had already pointed out difficulties on the duality conjecture for even k, we emphasize that our tests show that the conjecture fails for both k odd and k even.

There are several further directions to pursue. These tests can be applied to other theories with superpotentials, as for example, theories with symmetric and antisymmetric tensors [15, 16]. It would also be interesting to study a-maximization in these theories.

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