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By

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ABSTRACT

Shock propagation in solids which exhibit a nonlinear stress strain curve when subjected to instantaneous loadings is studied. General three-dimensional shock waves are first examined and then specialized to the one-dimensional case. The method of solution involves two steps, the generation of an infinite set of coupled nonlinear ordinary differential equations to replace the original partial differential equations and the solution of this system. Explicit solutions are exhibited for a restricted class of materials using a series solution for the system and also a perturbation technique. The two methods are compared showing that a low order perturbation solution can give good results for the decay of the wave front. A way of extending the method to a general type of material is shown and the effect of the boundary conditions on the shock decay is calculated. A steady state solution is also obtained.

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I. INTRODUCTION

Wave propagation in solids has been studied for over one hundred years. Until the 1940's the work had been limited almost exclusively to linear problems. During the Second World War, however, high velocity impact of projectiles became a subject of great interest. Since the stresses involved in such cases are of such a magnitude to throw the material into the nonlinear range, new theories were needed. In response to this the first solutions in dynamic plasticity were obtained. Assuming that the material response could be characterized by a nonlinear stress strain law independent of time, Karman (1950), Rakmatulin (1945), and G. I. Taylor (1946) independently obtained results that were simple, elegant and yet strikingly useful. For the past twenty years the field has developed rapidly and is today an area of great activity.

Most of the work to date has dealt with nonlinear effects produced by nonlinearities introduced in the material characterization. This is understandable since in most problems, especially waves in metals, the geometric nonlinearities are of secondary influence.

Depending upon the type of constitutive law used, a variety of wave propagation phenomena can be produced. In linear materials it has long been known that only two types of waves are possible in an unbounded, isotropic medium namely equivoluminal waves and dilatational waves. In nonlinear materials, however, coupled waves are also possible. Such coupled waves contain components of both dilatation and rotation.

In the solution of boundary value problems, transform techniques have been used with great success in the linear theories, but have been of almost no use in the solution of nonlinear wave propagation problems. Even in the linear case the inversion process becomes quite formidable in many cases.

Probably the method most associated with nonlinear problems is the method of characteristics. It has also been a powerful tool for linear problems where the geometry or the material was of a complex nature. In the linear case the characteristics are straight lines and are known a priori. Integration along these characteristics then becomes a simple matter when a computer is used. Even for certain nonlinear materials the characteristics become straight lines. This is true for rate-independent nonlinear materials in the special case of simple waves (see Courant and Friedrichs (1948)), and in general for semilinear materials, i.e., those which are nonlinear in the time-dependent response but whose instantaneous response is linear (Lubliner and Secor (1967)). In the semilinear class are included viscoplastic materials, for which Malvern (1950) and others (see Simmons et al., (1961)) have successfully used the method of characteristics to study wave-propagation problems. When materials are instantaneously nonlinear and have a viscous response the method is faced with certain difficulties. In this case the characteristics are curved and are not known beforehand. Usually a trial and error procedure is used to determine the characteristic curves. Even with these difficulties the method is still used with some success.

Another method used in nonlinear problems is power series expansions about the wave front. This method in essence transforms a set of partial differential equations into an infinite set of ordinary differential equations. These equations are linear in the linear problem but become nonlinear for nonlinear materials. In most cases these equations can be solved sequentially.

Wave propagation is in essence the movement of discontinuities through a material. The discontinuities fall into two general types. First, acceleration waves where the dependent variables (stress, strain, and velocity) are continuous at the wave front but their derivations may be discontinuous. Such discontinuities, as is well known, travel along characteristics. Most of the work in wave propagation in materials with memory has dealt with this type of wave (Varley (1965), Coleman and Gurtin (1965), Lubliner (1967)). This type of wave carries discontinuities in stress, strain, and velocity (for first order discontinuities). Such waves can be propagated only in certain types of materials, namely those whose stress-strain curves are concave upward. This is because the material must be such that small disturbances do not dissipate the wave immediately after impact. If the boundary conditions are discontinuous of the first order a linear material will propagate such a wave. If the boundary conditions are continuous the solution will likewise be continuous for the linear case. One striking feature of waves in nonlinear materials of the hardening type is that continuous boundary conditions may give rise to discontinuous solutions. This happens when characteristics bunch up making the wave ever steeper until a shock wave is produced. Since the characteristic velocity increases with the slope of the stress-strain curve, the requirement of upward concavity for this occurrence is evident. Another important observation about shock waves is that they do not propagate along characteristics in the nonlinear material. Only in materials with instantaneous linearity does the shock wave propagate along characteristic lines. In this case since the characteristics are running parallel to the shock, the decay of the shock depends only on the boundary conditions. In the fully nonlinear case however characteristics are constantly intersecting the shock front and influencing the decay. Consequently this significantly complicates the problem since the solution for the shock decay depends upon the total solution.

Many of the techniques now available for the solution of shock wave problems in solids originated in the field of aerodynamics. At times the connection between compressible fluid flow and waves in solids is quite striking. Indeed, one-dimensional motion of gas in a tube is mathematically equivalent to the one-dimensional wave propagation problem in a nonlinear elastic solid. Consequently some of the methods developed to a high degree in gas dynamics such as the method of characteristics and the hodograph transformation can be used directly for solids. Use of the hodograph transformation however cannot be used to advantage if the material is viscoelastic.

The purpose of the present work is to investigate the shock wave problem in materials that exhibit nonlinear instantaneous response. In most cases the material considered will be viscoelastic. Initially the multidimensional case is looked at, then attention is focused on the one-dimensional problem. A method of solution is exhibited for the one-dimensional case and explicit solutions obtained. The behavior of a wave after very long times when transient effects can be neglected is also examined.

II. MULTI-DIMENSIONAL SHOCK WAVES

Consider an isotropic solid through which a surface of discontinuity is being propagated. It will be assumed that the stresses, velocities and strains are continuous everywhere except on the surface of discontinuity. The displacements will be required to be continuous everywhere and sufficiently small so that the linear strain-displacement relation can be used. If $u_i(X_j,t)$ denotes the displacement at any time t in terms of the coordinates X_j then the strains can be defined by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \tag{2.1}$$

The equation of motion is

$$\frac{\partial \sigma_{ij}}{\partial X_j} = \rho \frac{\partial v_i}{\partial t} \tag{2.2}$$

where σ_{ij} is the symmetric stress tensor, v_i are the particle velocities and ρ is the density of the material. Repeated subscripts will denote summation in the usual way. The relation between stress and strain will be defined in terms of two equations. For instantaneous deformations (encountered on the discontinuity) the relation will be

$$\sigma_{ij} = F_{ij}(\varepsilon_{Kl}) \tag{2.3}$$

Since the material is isotropic this equation can be written

$$\sigma_{ij} = A_o\delta_{ij} + A_1\varepsilon_{ij} + A_2\varepsilon_{iK}\varepsilon_{Kj} \tag{2.4}$$

where the A_i 's are scalar functions of the three invariants of the strain tensor.

For deformations that are not instantaneous it will be assumed that the material behaves according to a constitutive law of the form

$$\frac{\partial \sigma_{ij}}{\partial t} = f_{ijKl} \frac{\partial \varepsilon_{Kl}}{\partial t} + g_{ij} \tag{2.5}$$

where f_{ijKl} and g_{ij} are tensor functions of stress and strain. Obviously equations (2.3) and (2.5) must be compatible. For instantaneous deformations where the stress and strain rates are theoretically infinite, the g_{ij} term in equation (2.5) drops out. Thus substituting (2.3) into (2.5) gives

$$\frac{\partial F_{ij}}{\partial \varepsilon_{Kl}} \frac{\partial \varepsilon_{Kl}}{\partial t} = f_{ijKl} \frac{\partial \varepsilon_{Kl}}{\partial t}$$

or

$$\frac{\partial F_{ij}}{\partial \varepsilon_{Kl}} (\cdot) = f_{ijKl} (F_{mn}(\cdot),\cdot) \tag{2.6}$$

Thus if f_{ijKl} is known (2.6) can be integrated to calculate F_{ij} . The reverse however is not true. Given F_{ij} it is not possible in general to calculate f_{ijKl} since the second argument in f_{ijKl} contains information about the long time behavior of which F_{ij} is ignorant. Only if f_{ijKl} is a function of, say, stress only will it be possible to calculate it from a given F_{ij} .

Let the surface of discontinuity be defined by the equation

$$t - \phi(X) = 0$$

where X denotes a point (X_1,X_2,X_3) . Assume before the wave arrives the material is at rest and all dependent variables are zero.

The dependent variables can now be expanded in a Taylor series in time for the wave at position X . The expansions for the dependent variables are

$$\sigma_{ij}(X,t) = H(t - \phi(X)) \sum_{n=0}^{\infty} \frac{\sigma_{ij}^{(n)}(X)}{n!} [t - \phi(X)]^n \tag{2.14a}$$

$$\varepsilon_{ij}(X,t) = H(t - \phi(X)) \sum_{n=0}^{\infty} \frac{\varepsilon_{ij}^{(n)}(X)}{n!} [t - (X)]^n \tag{2.14b}$$

$$v_i(X,t) = H(t - \phi(X)) \sum_{n=0}^{\infty} \frac{v_i^{(n)}(X)}{n!} [t - \phi(X)]^n \tag{2.14c}$$

Substitution of (2.15a), (2.14c) into (2.3) yields

$$\begin{aligned} - \frac{\partial\phi}{\partial X_j} \delta(t - \phi) \sigma_{ij}^{(o)} + H(t - \phi) \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial\sigma_{ij}^{(n)}}{\partial X_j} - \sigma_{ij}^{(n+1)} \frac{\partial\phi}{\partial X_j} \cdot \\ \cdot [t - \phi]^n = \rho\{\delta(t - \phi) v_i^{(o)}\} \\ + \rho H(t - \phi) \sum_{n=0}^{\infty} \frac{1}{n!} v_i^{(n+1)} [t - \phi]^n \end{aligned} \tag{2.15}$$

Continuity of displacements demand

$$\frac{\partial P_{ij}}{\partial t} = \frac{\partial v_i}{\partial X_j} \tag{2.16}$$

where

$$P_{ij} = \frac{\partial u_i}{\partial X_j} \tag{2.17}$$

Substitution of (1.4c) and an expansion for P_{ij} into (1.6) results in

$$\begin{aligned} P_{ij}^{(o)} \delta(t - \phi) + H(t - \phi) \sum_{n=0}^{\infty} \frac{1}{n!} P_{ij}^{(n+1)} [t - \phi]^n \\ = - \frac{\partial\phi}{\partial X_j} v_i^{(o)} \delta(t - \phi) + H(t - \phi) \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial v_i^{(n)}}{\partial X_j} - v_i^{(n+1)} \frac{\partial\phi}{\partial X_j} [t - \phi]^n \end{aligned} \tag{2.18}$$

Equating terms of order $\delta(t - \phi)$ in (2.15) and (2.18) gives

$$- \frac{\partial \phi}{\partial X_j} \sigma_{ij}^{(0)} = \rho v_i^{(0)} \tag{2.19}$$

$$- \frac{\partial \phi}{\partial X_j} v_i^{(0)} = P_{ij}^{(0)}$$

The local wave speed C is defined by the relation

$$C \frac{\partial \phi}{\partial X_j} = n_j \tag{2.20}$$

where n_j are the components of the unit normal vector to the wave front. Define a new vector v_i by

$$\omega_i = - \frac{v_i}{C} \tag{2.21}$$

Using (2.20), (2.21), and the constitutive equation (2.30) of the wave surface with equation (2.19) yields

$$C^2 \rho \omega_i^{(0)} = n_j F_{ij}\left(\frac{1}{2} n_K \omega_i^{(0)} + \frac{1}{2} n_i \omega_K^{(0)}\right) \tag{2.22}$$

For the linear case as is well known two distinct wave speeds are possible: one associated with rotational waves and one associated with dilatational waves. No such simplification is present in the case considered here.

Assume A_o can be put into the form

$$A_o = B_o I \tag{2.23}$$

where I is the first invariant of the strain matrix and B_o is a function of the three invariants of the strain tensor and is regular for $I = 0$. This is a necessary assumption for the stress-strain relation to be smooth at the origin. In terms of equations (2.4) and (2.23), equation (2.22) can be written

$$C^2 \rho \, \omega_i^{(0)} = n_i n_K \omega_K^{(0)} B_o + \frac{A_1}{2} (n_j n_i \omega_j^{(0)} + \omega_i^{(0)})$$

$$+ \frac{A_2}{4} n_j (n_i \omega_K^{(0)} + n_K \omega_i^{(0)})(n_K \omega_j^{(0)} + n_j \omega_K^{(0)}) \tag{2.24}$$

Multiplying through by ω_i and then dividing by $\omega_i \omega_i$ gives

$$C^2 \rho = \frac{I^2}{|\omega^{(0)}|^2} B_o + \frac{A_1 I^2}{2|\omega^{(0)}|^2} + \frac{A_1}{2}$$

$$+ \frac{A_2}{2|\omega^{(0)}|^2} \left(n_j n_i \omega_K^{(0)} + n_j n_K \omega_i^{(0)} \quad n_K \omega_j^{(0)} + n_j \omega_K^{(0)} \right) \omega_i \tag{2.25}$$

Consider the special case where $A_2 = 0$. For transverse waves

$$n_K \omega_K^{(0)} = 0$$

and since

$$u_{K,K}^{(0)} = n_K \omega_K^{(0)} \tag{2.26}$$

these waves are equivoluminal immediately behind the shock front. Equation (2.24) demands for not all $\omega_i = 0$

$$C = \sqrt{\frac{A_1}{2\rho}} \tag{2.27}$$

For waves where $n_i \omega_i^{(0)} \neq 0$ multiply (2.24) by n_i and sum on i . This gives for the wave speed

$$C = \sqrt{\frac{B_o + A_1}{\rho}} \tag{2.28}$$

In the linear case A_1 and B_o become constants and are related to the Lame's coefficients λ and μ by

$$A_1 = 2\mu \; ; \; B_o = \lambda \; .$$

and the wave speed has a constant value. In the nonlinear case however A_1 and B_o depend on the local strain field and hence the wave will travel with variable velocity depending upon how the strain varies with time at the wave front.

For the material assumed

$$C^2 \rho \, \omega_i^{(o)} = n_i n_K \omega_K^{(o)} B_o + \frac{A_1}{2} \left(n_j n_i \omega_j^{(o)} + \omega_i^{(o)} \right) \tag{2.29}$$

Substitution of (2.28) in (2.29) yields

$$\left(B_o + \frac{A_1}{2} \right) \left(\omega_j^{(o)} - n_j \theta \right) = 0$$

where $\theta = n_K \omega_K^{(o)}$. Hence

$$n_j = \frac{\omega_j^{(o)}}{\theta} \tag{2.30}$$

From (2.17) and (2.19)

$$P_{ij}^{(o)} = \frac{\partial u_i^{(o)}}{\partial X_j} = - n_j \frac{v_i^{(o)}}{C} \tag{2.31}$$

Using (2.30) and (2.21) gives

$$\frac{\partial u_i^{(o)}}{\partial X_j} = n_j \, \omega_i^{(o)} = \omega_j^{(o)} \frac{\omega_i^{(o)}}{\theta} \tag{2.32}$$

and therefore

$$\frac{\partial u_i^{(o)}}{\partial X_j} - \frac{\partial u_j^{(o)}}{\partial X_i} = 0 \qquad (2.33)$$

Thus waves traveling with speed given by (2.28) will be irrotational at the wave front. This is a dilatational wave.

Now substitute (2.27) in (2.29). This gives

$$\frac{A_1}{2} \omega_i^{(o)} = n_i n_K \omega_K^{(o)} B_o + \frac{A_1}{2} (n_j n_i \omega_j^{(o)} + \omega_i^{(o)})$$

or

$$0 = \left(B_o + \frac{A_1}{2}\right) \theta n_i \qquad (2.34)$$

This implies $\theta = 0$ and hence is an equivoluminal wave at the shock front.

For linear materials and the partially linearized material considered here it is possible to talk of two distinct wave speeds and a single type of wave associated with each wave speed. In the most general material however this will not usually be the case. As is evident from (2.25) rotational and dilatational effects are coupled by the third term. We may expect however that in many instances either rotational or dilatational effects will still predominate and the coupled influence will be of secondary importance.

Let us now derive the equation that governs the decay of the amplitude of the shock wave. Equating coefficients of order $n = 0$ in (2.15) and (2.18) gives

$$\frac{\partial \sigma_{ij}^{(o)}}{\partial X_j} - \sigma_{ij}^{(1)} \frac{\partial \phi}{\partial X_j} = \rho\, v_i^{(1)}$$

and

$$\frac{\partial v_i^{(0)}}{\partial X_j} - v_i^{(1)} \frac{\partial \phi}{\partial X_j} = P_{ij}^{(1)} \tag{2.35}$$

The third necessary equation comes from the constitutive law

$$\sigma_{ij}^{(1)} = \frac{\partial F_{ij}^{o}}{\partial \varepsilon_{Kl}} \varepsilon_{Kl}^{(1)} + g_{ij}^{o}$$

where

$$\frac{\partial F_{ij}^{o}}{\partial \varepsilon_{Kl}} = \frac{\partial F_{ij}}{\partial \varepsilon_{Kl}} (\varepsilon_{pq}^{(0)})$$

$$g_{ij}^{o} = g_{ij}(F_{pq}(\varepsilon_{mn}^{(0)}), \varepsilon_{pq}^{(0)})$$

Let $(\ )_{,j}$ denote differentiation with respect to X_j . Combining equations (2.35) gives

$$- \frac{1}{2\rho}\left[(\phi,_K F_{iK}^{o}),_j + (\phi,_K F_{jK}^{o}),_i + F_{iK,K}^{o}\, \phi,_j + F_{jK,K}^{o}\, \phi,_i\right] \tag{2.36}$$

$$+ \frac{1}{\rho} g_{iK}^{o} \phi,_K \phi,_j = \varepsilon_{ij}^{(1)} - \frac{1}{2\rho}\left(\frac{\partial F_{iK}^{o}}{\partial \varepsilon_{pq}} \phi,_j + \frac{\partial F_{jK}^{o}}{\partial \varepsilon_{pq}} \phi,_i\right)\phi,_K\, \varepsilon_{pq}^{(1)}$$

Note that the shock decay involves $\varepsilon_{pq}^{(1)}$ which is not known beforehand and so only under special assumptions will (2.36) be directly integrable for $\varepsilon_{ij}^{(0)}$.

2.1 Plane Dilatational--Shear Wave

Consider the motion of a plane shock wave propagating into an infinite solid at rest in the x_1 direction. Therefore

$$n_2 = n_3 = 0 \; ; \; n_1 = 1 \tag{2.41}$$

Assume the geometry and boundary conditions are such that the only nonzero components of strain are ε_{11} , and ε_{12} . The wave consequently has dilatation as represented by the ε_{11} strain and shearing deformation as represented by the ε_{12} term. In this case

$$\varepsilon_{11}^{(0)} = \omega_1^{(0)} \; ; \; \varepsilon_{12}^{o} = \frac{\omega_2^{(0)}}{2} \tag{2.42}$$

Equations (2.24) produce two relevant equations. In what follows the superscripts will be dropped and it will be understood that all quantities are to be evaluated at the shock front. Equations (2.24) produce two relevant equations

$$c^2 \rho \, \omega_2 = A_1 \frac{\omega_2}{2} + \frac{A_2}{2} \omega_1 \omega_2 \tag{2.43}$$

and

$$c^2 \rho \, \omega_1 = (B_o + A_1)\omega_1 + A_2(\omega_1^2 + \frac{\omega_2^2}{4}) \tag{2.44}$$

Consider first the possibility of producing a pure dilatational wave. This means

$$\omega_2 = 0 \; ; \; \omega_1 \neq 0 \tag{2.45}$$

and (2.43) is satisfied identically. The wave speed is calculated from (2.44).

$$c^2 \rho = (B_o + A_1) \Big|_{\omega_2=0} + A_2 \Big|_{\omega_2=0} \omega_1 \qquad (2.46)$$

As expected the wave speed depends on the magnitude of the dilatation.

Consider now the possibility of propagating a pure shear wave. This is where

$$\omega_2 \neq 0 \ ; \ \omega_1 = 0 \qquad (2.47)$$

(2.43) gives the wave speed as

$$c^2 \rho = \frac{A_1}{2} \Big|_{\omega_1=0}$$

(2.44) however would demand

$$A_2 \Big|_{\omega_1=0} \frac{\omega_2^2}{4} = 0 \qquad (2.48)$$

This cannot be satisfied for arbitrary nonzero ω_2. Hence a pure shearing shock wave cannot in general be produced. Shearing motions will necessarily produce dilatational effects simultaneously. Thus two types of waves can be propagated. A dilatational shock with wave speed given by (2.46) and a coupled dilatational-shear wave whose speed is given by

$$c^2 \rho = \frac{A_1}{2} + \frac{A_2}{2} \omega_1 \qquad (2.49)$$

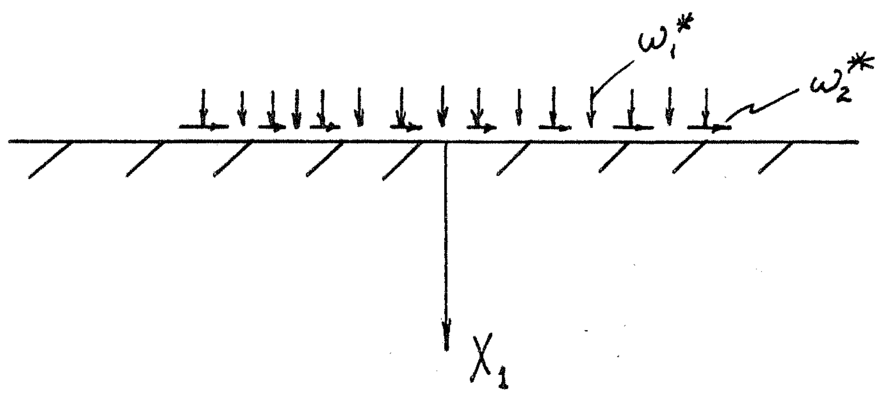
This coupled wave however demands a relation between ω_1 and ω_2. This is found by combining (2.43) and (2.44) which results in

$$\frac{A_2}{2} \omega_1^2 + (B_o + \frac{A_1}{2}) \omega_1 + \frac{A_2}{4} \omega_2^2 = 0 \qquad (2.50)$$

Let the solution of this equation for ω_2 be given by

$$\omega_2 = R(\omega_1) \tag{2.51}$$

Let us now see what happens when the surface of an infinite half space is impacted by instantaneous straining in the X_1 and X_2 direction. We will consider only what happens immediately after impact.



To make things more transparent let us take a more specific material. Let B_o and A_1 be positive constants and

$$A_2 = KI = K\,\omega_1 \ .$$

K is a positive constant. For fixed ω_2, the constitutive law for σ_{11} is seen to be of the hardening type and involves only first and third powers of ω_1, thus making the material behave the same in tension as in compression. For fixed ω_1 the law for σ_{12} is linear. In such a material we would expect shocks to be possible.

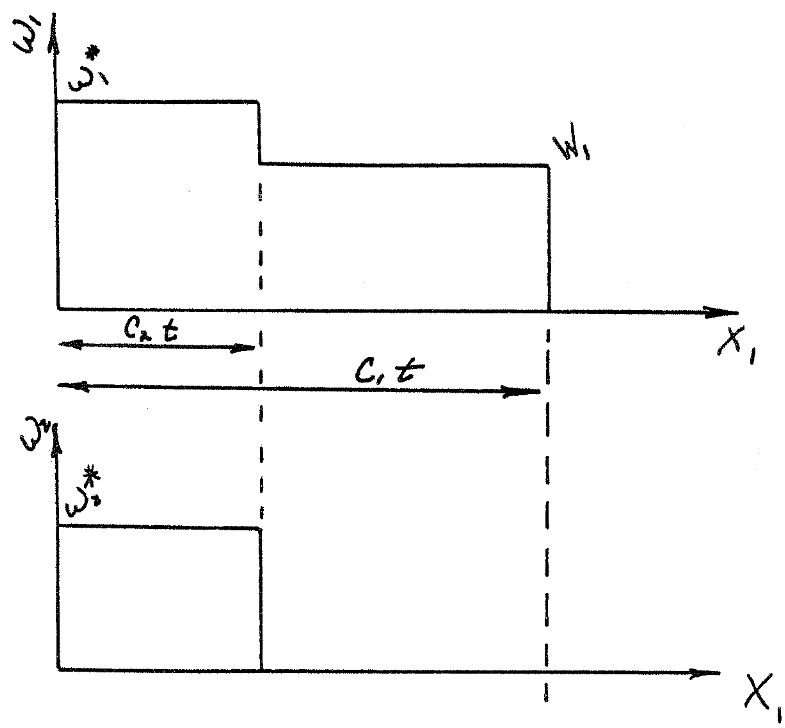
In such a material a pressure shock will move out followed by a coupled dilatational-shear shock wave. This must be the case because a coupled leading wave would require by equation (2.50)

$$(B_o + \frac{A_1}{2}) + \frac{K}{2}\omega_1^2 + \frac{K}{4}\omega_2^2 = 0 \qquad (2.52a)$$

But

$$B_o \ , A_1 \ , K > 0$$

Hence (2.52a) cannot be satisfied by real values of ω_1 and ω_2. This situation of a pressure wave followed by a shear wave is what of course happens in the linear case. The situation here is pictured below.



The pressure wave velocity as given by (2.46) is

$$C_1 = \sqrt{\frac{1}{\rho}((B_o + A_1) + K(W_1)^2)} \qquad (2.52b)$$

The necessary compatibility relation that must be satisfied at the second shock front is analogous to (2.50) and yields an equation for the determination of W_1.

$$W_1^3 + \alpha_1 W_1 + \alpha_2 = 0 \qquad (2.53)$$

where

$$\alpha_1 = \frac{1}{K}\left(B_o + \frac{A_1}{2} - \frac{K}{2}(\omega_1^*)^2\right)$$

$$\alpha_2 = -\frac{1}{K}\left(\omega_1^*(\omega_2^*)^2\frac{K}{4} + \left(B_o + \frac{A_1}{2}\right)\omega_1^* + \frac{K}{2}(\omega_1^*)^3\right)$$

The speed of the second wave is given by

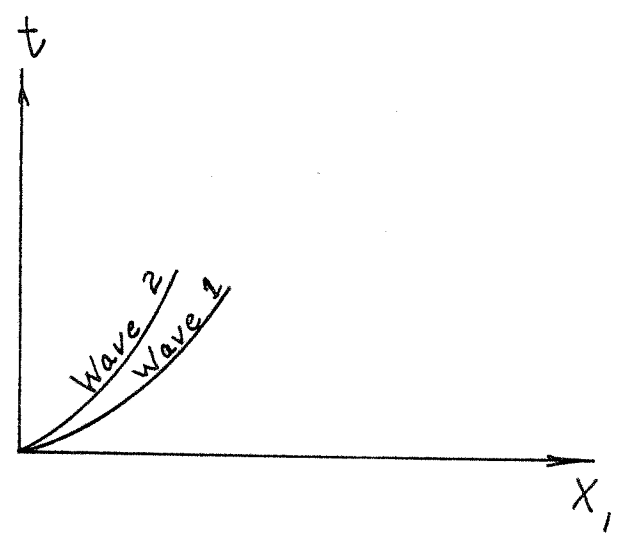
$$C_2 = \sqrt{\frac{1}{2\rho}\left(A_1 + K(\omega_1^*)^2\right)} \qquad (2.54)$$

It can be shown that

$$\frac{(\alpha_2)^2}{4} + \frac{(\alpha_1)^3}{27} > 0$$

for all loadings and values of the positive material constants. This condition insures that equation (5.4) has only one real root.

The preceding analysis would hold for all times if the material is non-linear elastic and the strains at the surface are kept constant after impact. If the material has viscosity or if the boundary conditions change in time the wave profiles will change and consequently the shock speeds will vary as indicated in the following diagram.



III. ONE-DIMENSIONAL SHOCK WAVES

Because of the difficulties involved in obtaining numerical solution to nonlinear wave propagation problems, almost all solutions have been confined to one-dimensional cases. Such problems arise when the dependent variables can be considered to depend on a single space variable. Examples of such cases are plane dilatational waves, torsional waves in a thin tube, and longitudinal waves in a thin bar where radial inertia effects can be neglected. In what follows the problem will be thought of as waves in a thin bar, however, everything naturally will hold for other one-dimensional cases with suitable modification of the constants involved.

As indicated earlier the solution by the method of characteristics is quite involved for the fully nonlinear case. In response to this an alternative method of solution will be exhibited from which explicit solutions are obtained. Lubliner & Secor (1966) derived the shock decay equation for this problem but gave no solution. Their actual solutions were confined to semi-linear materials (Lubliner and Secor (1967)).

Some of the development of Section I will be repeated here for the one-dimensional wave in order to make this section self-contained.

3.2 Problem Statement

Consider the problem of impact on the end of a semi-infinite bar composed of a nonlinear viscoelastic material. Let

X = distance from the end of the bar before deformation

t = time

ε = strain

σ = stress referred to original cross section of bar

V = velocity

ρ = density in reference state

The equation of continuity is

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial V}{\partial X} \tag{3.1}$$

The equation of motion is

$$\frac{\partial \sigma}{\partial X} = \rho \frac{\partial V}{\partial t} \tag{3.2}$$

The constitutive law relating stress and strain will be taken to be

$$\frac{\partial \varepsilon}{\partial t} = f(\sigma,\varepsilon) \frac{\partial \sigma}{\partial t} + g(\sigma,\varepsilon) \tag{3.3}$$

Using a constitutive law of this form has two advantages. Firstly this equation is capable of describing closely the behavior of a wide variety of materials and can be shown to be the most general constitutive law of the first order for a viscoelastic material of the differential type (see Lubliner (1964)). Secondly, it is much easier to handle than the law based on a multiple integral representation.

Returning to the problem of impact, different types of waves can be set up in the bar after impact. Of crucial importance in this regard is the form of $f(\sigma,\varepsilon)$. This function determines the instantaneous response of the material to loads. For rapid deformations equation (3.3) reduces to

$$\frac{d\varepsilon}{d\sigma} = f(\sigma,\varepsilon) \tag{3.4}$$

Let the integral of this equation passing through $(0,0)$ be given by $\varepsilon = F(\sigma)$.

The system (3.1), (3.2), (3.3) is a system of first order, quasilinear, hyperbolic partial differential equations. The characteristics are given by

$$dX = 0 \quad \text{and} \quad \frac{dX}{dt} = \pm\left(\frac{1}{\rho\ f(\sigma,\varepsilon)}\right)^{\frac{1}{2}} \tag{3.5}$$

Since these latter characteristics are dependent on stress and strain, they are curved and computation of wave profiles becomes much more difficult than in the linear case.

The velocity of a shock wave, as will be verified later, is

$$V_s = \left(\frac{\sigma}{\rho\ F(\sigma)}\right)^{\frac{1}{2}} \tag{3.6}$$

From the theory of hyperbolic equations, small disturbances are propagated along characteristics. Hence for a shock wave to exist the constitutive law must be such that the velocity of a shock is greater than the characteristic velocities at lower stress levels. This condition will exist if the instantaneous stress strain curve is concave upward in some region. This will happen if

$$\frac{d^2F}{d\sigma^2} < 0$$

Since we are interested in shock waves it is these kinds of materials that will be concentrated on.

3.2 Conversion to a System of Ordinary Differential Equations

The constitutive law (3.3) can be cast in another form. Consider the system of equations for fixed X

$$\varepsilon^{(1)} = M(\sigma) \tag{3.7a}$$

$$\frac{d\varepsilon^{(2)}}{dt} = G(\sigma,\varepsilon^{(2)}) \tag{3.7b}$$

$$\varepsilon = P(\varepsilon^{(1)},\varepsilon^{(2)}) \tag{3.7c}$$

Assume G is a regular function of σ and $\varepsilon^{(2)}$; then for a discontinuous stress input $\varepsilon^{(2)}$ will remain continuous since integration of (3.7b) gives

$$\varepsilon^{(2)}(t_1) - \varepsilon^{(2)}(t_o) = \int_{t_o}^{t_1} g(\sigma,\varepsilon^{(2)})dt$$

Then as $t_1 \to t_o$

$$\varepsilon^{(2)}(t_1) \to \varepsilon^{(2)}(t_o)$$

since

$$\lim_{t_1 \to t_o} \int_{t_o}^{t_1} G(\sigma,\varepsilon^{(2)}) = 0$$

The functions M,G,P will be assumed to be regular for all domains of use and are such that

$$M(0) = G(0,0) = P(0,0) = 0$$

Since $\varepsilon^{(2)}$ vanishes for rapid loadings the instantaneous response from an at rest condition is given by

$$\varepsilon = P(M(\sigma),0)$$

The function P represents the coupling between the instantaneous strain $\varepsilon^{(1)}$ and the viscous strain $\varepsilon^{(2)}$.

Consider the special case where the coupling is linear and P is linear. Then equation (3.7c) is

$$\varepsilon = \varepsilon^{(1)} + \varepsilon^{(2)}$$

Elimination of $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ in equations (3.7) gives

$$\frac{d\varepsilon}{dt} = M'(\sigma)\frac{d\sigma}{dt} + G(\sigma,\varepsilon - M(\sigma))$$

A prime denotes differentiation with respect to the argument. This uncoupled version of equation (3.3) will be used in what follows since the algebra is somewhat simplified but the essence of the problem is retained. Extension to the more general material is straightforward and presents no difficulty.

Let the boundary condition for system (3.1), (3.2), (3.3) be given by

$$\sigma(0,t) = H(t)P(t)$$

$$H(t) = \text{heaviside step function}$$

$$P(0) \neq 0$$

P is assumed to be continuous for $t \geq 0$. Let the path of the shock wave be defined by

$$t - f(X) = 0$$

$f(X)$ is kept as an unknown in the problem and is part of the complete solution. Consider a Taylor series expansion about the shock front for the dependent variables σ, ε, V .

The expansion for stress is

$$\sigma(X,t) = H(t - f(X)) \sum_{n=0}^{\infty} \frac{\sigma_n(X)}{n!} (t - f(X))^n$$

Let $\tau = t - f(X)$

$$V(X,t) = H(\tau) \sum_{n=0}^{\infty} \frac{V_n \tau^n}{n!}$$

$$\varepsilon^{(k)}(X,t) = H(\tau) \sum_{n=0}^{\infty} \frac{\varepsilon_n^{(k)} \tau^n}{n!} \qquad k = 1,2 \tag{3.8}$$

The partial derivatives of σ in X and t are given by

$$\sigma_X = - f' \delta(\tau) \sum_{n=0}^{\infty} \frac{\sigma_n}{n!} \tau^n + H(\tau) \sum_{n=0}^{\infty} \frac{\sigma_n' - f'\sigma_{n+1}}{n!} \tau^n \tag{3.9}$$

$$\sigma_t = \delta(\tau) \sum_{n=0}^{\infty} \frac{\sigma_n}{n!} \tau^n + H(\tau) \sum_{n=0}^{\infty} \frac{\sigma_{n+1}}{n!} \tau^n$$

It is now convenient to use these expansions to eliminate the partial strains in equations (3.7) and obtain direct relations between ε_n and σ_n .

$$\sum_{n=0}^{\infty} \varepsilon_n^{(1)} \frac{\tau^n}{n!} = M\left(\sum_{n=0}^{\infty} \frac{\sigma_n \tau^n}{n!} \right) \tag{3.10}$$

Let $L(\tau) = M(\sigma(\tau))$. Then

$$\varepsilon_n^{(1)} = \frac{d^n L(0)}{d\tau^n} \tag{3.11}$$

This gives for the first three terms

$$\varepsilon_0^{(1)} = M(\sigma_o)$$

$$\varepsilon_1^{(1)} = M'(\sigma_o)\sigma_1 \tag{3.12}$$

$$\varepsilon_2^{(1)} = M''(\sigma_o)(\sigma_1)^2 + M'(\sigma_o)\sigma_2$$

Let $N(\tau) = G(\sigma(\tau), \varepsilon(\tau))$. Then

$$\varepsilon_{n+1}^{(2)} = \frac{d^n N(0)}{d\tau^n} \tag{3.13}$$

This gives for the first three terms

$$\varepsilon_0^{(2)} = 0$$

$$\varepsilon_1^{(2)} = G(\sigma_o, \varepsilon_o) \tag{3.14}$$

$$\varepsilon_2^{(2)} = \frac{\partial G}{\partial \sigma}(\sigma_o, \varepsilon_o)\sigma_1 + \frac{\partial G}{\partial \varepsilon}(\sigma_o, \varepsilon_o)\varepsilon_1$$

The sets of equations (3.12) and (3.14) can be solved sequentially for the ε_n in terms of the σ_n . Elimination for the first three terms gives

$$\varepsilon_o = M(\sigma_o)$$

$$\varepsilon_1 = M'(\sigma_o)\sigma_1 + G(\sigma_o, M(\sigma_o)) \tag{3.15}$$

$$\varepsilon_2 = M''(\sigma_o)(\sigma_1)^2 + M'(\sigma_o)\sigma_2 + \frac{\partial G}{\partial \sigma}(\sigma_o, M(\sigma_o))\sigma_1 +$$

$$+ \frac{\partial G}{\partial \varepsilon}(\sigma_o, M(\sigma_o))[G(\sigma_o, M(\sigma_o)) + M'(\sigma_o)\sigma_1]$$

These relations actually define approximations to the constitutive law that are useful for very short times. Given a stress input the strain can be determined to any degree of approximation. Thus for a second order approximation with a stress input defined by

$$\sigma(t) = H(t)[\sigma_o + \sigma_1 t + \ldots]$$

the strain response will be

$$\varepsilon(t) \approx H(t)[M(\sigma_o) + (M'\sigma_1 + G)t +$$

$$+ (M''\sigma_1^2 + M'\sigma_2 + \frac{\partial G}{\partial \sigma}\sigma_1 + \frac{\partial G}{\partial \varepsilon}\{G + M^2\sigma_1\})t^2]$$

Equations (3.1) and (3.2) become in power series form

$$\delta(\tau) \sum_{n=0}^{\infty} \frac{\varepsilon_n}{n!}\tau^n + H(\tau) \sum_{n=0}^{\infty} \frac{\varepsilon_{n+1}}{n!}\tau^n =$$

$$= -f'\delta(\tau) \sum_{n=0}^{\infty} \frac{V_n}{n!}\tau^n + H(\tau) \sum_{n=0}^{\infty} \frac{V_n' - V_{n+1}f'}{n!}\tau^n \tag{3.16}$$

and

$$\rho\,\delta(\tau) \sum_{n=0}^{\infty} \frac{V_n}{n!}\tau^n + H(\tau) \sum_{n=0}^{\infty} \frac{V_{n+1}}{n!}\tau^n =$$

$$= -f'\delta(\tau) \sum_{n=0}^{\infty} \frac{\sigma_n}{n!}\tau^n + H(\tau) \sum_{n=0}^{\infty} \frac{\sigma_n' - \sigma_{n+1}f'}{n!}\tau^n \tag{3.17}$$

Comparing terms of the order $\delta(\tau)$ at $\tau = 0$ gives the relations

$$\varepsilon_o = -f'\,V_o \tag{3.18}$$

$$\rho\,V_o = -f'\,\sigma_o \tag{3.19}$$

These are the familiar shock relations usually derived by other methods. Combining equations (3.18) and noting that $\varepsilon_o = M(\sigma_o)$ gives for the speed of the shock wave C ,

$$\frac{1}{C} = \frac{df}{dX} = \rho\left(\frac{M(\sigma_o)}{\sigma_o}\right)^{\frac{1}{2}} \tag{3.20}$$

Equating coefficients of powers of τ gives

$$\varepsilon_{n+1} = \frac{dV_n}{dX} - \left(\rho \frac{M_o}{\sigma_o}\right)^{\frac{1}{2}} V_{n+1}$$

$$\rho V_{n+1} = \frac{d\sigma_n}{dX} - \left(\rho \frac{M_o}{\sigma_o}\right)^{\frac{1}{2}} \sigma_{n+1} \tag{3.21}$$

$$\varepsilon_{n+1} = \frac{d^n N}{d\tau^n}(0) + \frac{d^{n+1}L}{d\tau^{n+1}}(0)$$

$$M_o = M(\sigma_o)$$

Solving equations (3.21) for the σ_n 's defines an infinite number of coupled nonlinear differential equations. The first equation is

$$\frac{d\sigma_0}{dX} = -\frac{G(\sigma_o, M_o)}{\ell(\sigma_o)} - \frac{\left[M_o' - \frac{M_o}{\sigma_o}\right]}{\ell(\sigma_o)} \sigma_1 \tag{3.22}$$

where

$$\ell(\sigma_o) = \frac{2}{\rho}\left(\rho \frac{M_o}{\sigma_o}\right)^{\frac{1}{2}} + \frac{1}{2}\left(M_o' - \frac{M_o}{\sigma_o}\right)\left(\rho \frac{M_o}{\sigma_o}\right)^{-\frac{1}{2}}$$

This is the shock decay equation and describes how the shock amplitude varies along the bar. Note that unlike the linear or semilinear material the shock decay equation cannot be solved directly since there is a dependence on σ_1 . The complete set of equations has the form

$$-\frac{d\sigma_o}{dX} \ell(\sigma_o) - G(\sigma_o, M_o) = \sigma_1 [M_o' - \frac{M_o}{\sigma_o}]$$

$$\frac{d^2\sigma_o}{dX^2} - 2\left(\frac{M_o}{\sigma_o}\rho\right)^{\frac{1}{2}} \frac{d\sigma_1}{dX} - \sigma_1 J\left(\sigma_o, \sigma_1, \frac{d\sigma_o}{dX}\right) = \sigma_2\left(M_o' - \frac{M_o}{\sigma_o}\right)\rho + B_1(\sigma_o)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \tag{3.23}$$

$$\frac{d^2\sigma_{n-1}}{dX^2} - 2\left(\frac{M_o}{\sigma_o}\rho\right)^{\frac{1}{2}} \frac{d\sigma_n}{dX} - \sigma_n J\left(\sigma_o, \sigma_1, \frac{d\sigma_o}{dX}\right) = \sigma_{n+1}\left(M_o' - \frac{M_o}{\sigma_o}\right)\rho + B_n(\sigma_o, \ldots, \sigma_{n-1})$$

The function J is given by

$$J\left(\sigma_o, \sigma_1, \frac{d\sigma_o}{dX}\right) = \frac{\rho}{2\sigma_o}\left(\frac{M_o}{o}\rho\right)^{-\frac{1}{2}}\left(M_o' - \frac{M_o}{\sigma_o}\right)\frac{d\sigma_o}{dX}$$

$$- \rho M'' \sigma_1 - \rho \frac{\partial G}{\partial \sigma}(\sigma_o, M_o) - \rho \frac{\partial G}{\partial \varepsilon^{(2)}}(\sigma_o, M_o)$$

One advantage of dealing with system (3.23) instead of equations (3.1), (3.2), and (3.3) is that the shock relations are contained directly in the governing differential equations and are not separate entities. Equations (3.23) are coupled only in the right hand term. Thus the equations become uncoupled only if

$$M(\sigma_o) = M'(\sigma_o)\sigma_o \tag{3.24}$$

This can be satisfied only if

1) $\sigma_o = 0$

or

2) $M(\sigma_o) = K\sigma_o$

where K is a constant.

Recall one of our restrictions on M was

$$M(0) = 0 .$$

Hence 1) identically satisfies ((3.24). If this condition is met the shock amplitude is zero and the dependent variables become continuous at the wave front. However, discontinuities can be present in the derivatives of the dependent variables. These waves are commonly called acceleration waves. The wave speed in this case becomes

$$C = \lim_{\sigma \to 0} \left(\frac{\sigma_o}{\rho M(\sigma_o)} \right)^{\frac{1}{2}}$$

If $\sigma_o \neq 0$ then for uncoupling to occur, case 2) must be satisfied. The complete constitutive law for this type of material is

$$\frac{d\varepsilon}{dt} = K \frac{d\sigma}{dt} + g(\sigma,\varepsilon)$$

Materials of this type have acquired the designation of semi-linear materials in analogy to semi-linear equations in mathematics. In general we may define a material to be semilinear which obeys a linear stress strain law for instantaneous loadings. A material of the differential type described by

$$\frac{d^n\varepsilon}{dt^n} = N_1 \frac{d^n\sigma}{dt^n} + N_2$$

$$N_i = N_i\left(\frac{d^{n-1}\varepsilon}{dt^{n-1}} , \ldots, \varepsilon_j, \frac{d^{n-1}\sigma}{dt^{n-1}} , \ldots, \sigma\right) \qquad i = 1,2$$

will usually be semilinear if N_1 is a constant.

Equation (3.22) deserves closer scrutiny. The condition that a shock decrease in amplitude is

$$\frac{1}{\sigma_o}\frac{d\sigma_o}{dX} < 0 \ .$$

From (3.22) this means

$$-\frac{G_o}{\sigma_o \ell(\sigma_o)} - \frac{\left(M_o' - \frac{M_o}{\sigma_o}\right)}{\sigma_o \ell(\sigma_o)}\sigma_1 < 0 \qquad (3.25)$$

where $G_o = G(\sigma_o, M_o)$. Now

$$f'\ell(\sigma_o) = \frac{3}{2}\frac{M_o}{\sigma_o} + \frac{M_o'}{2} \ .$$

For the shock speed to be real and positive requires

$$f' > 0$$

$$\frac{M_o}{\sigma_o} > 0$$

by equation (3.20). For the overall system to be hyperbolic requires

$$M_o' > 0 \ .$$

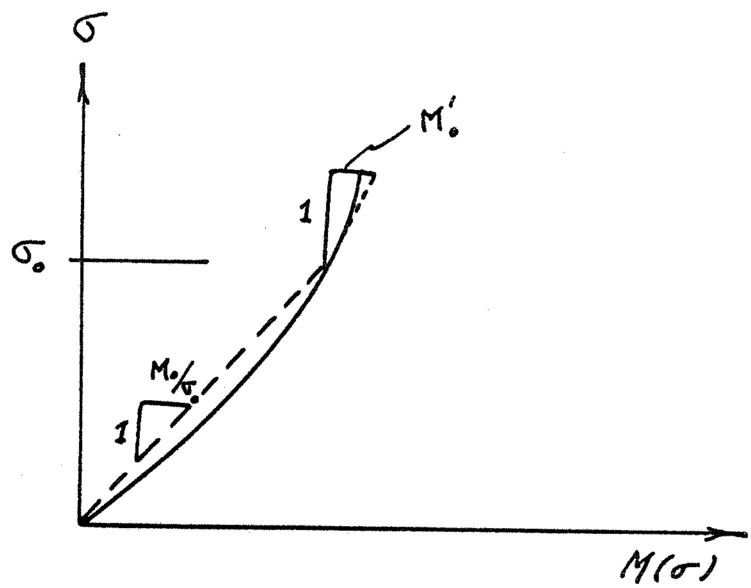
This then implies

$$\ell(\sigma_o) > 0$$

M is assumed to be of the type

$$\frac{d^2M}{d\sigma^2} < 0$$

for shock wave propagation. This material can be illustrated by the following diagram.



Thus

$$M_o' - \frac{M_o}{\sigma_o} < 0$$

For positive σ_o (3.25) can now be written

$$\sigma_1 < \frac{G_o}{\left(\frac{M_o}{\sigma_o} - M_o'\right)} \tag{3.26}$$

If (3.26) is not satisfied the shock front will actually increase in amplitude. Note that as

$$\frac{M_o}{\sigma_o} \to M'$$

the restriction on σ_1 becomes weaker. Indeed for a semilinear material there is no restriction on the magnitude of σ_1 .

3.3 Solution by Series

Many numerical techniques for solving nonlinear differential equations break down when applied to an infinite set of coupled equations. Some may be used in a modified form.

The first mode of attack will be a series solution. The σ_i 's will be expanded in a power series about $x = 0$. Thus

$$\sigma_i(X) = \sum_{n=0}^{\infty} a_{in} X^n \qquad (3.27)$$

The a_{io} are determined from the boundary conditions. If

$$\sigma(0,t) = P(t)H(t)$$

is the boundary condition then expanding $P(t)$ in powers of t gives

$$P(t) = P_o + \frac{P_i t}{1!} + \ldots \frac{P_n t^n}{n!} + \ldots$$

Since

$$\sigma(0,t) = \sigma_o(0) + \sum_{n=1}^{\infty} \sigma_n(0) \frac{t^n}{n!}$$

$$a_{io} = \sigma_i(0) = P_i \ .$$

If the infinite matrix a_{ij} is known the σ_i 's are known and the problem is solved. Our problem now is to generate the matrix a_{ij} .

The i^{th} equation of the set (3.23) has the form

$$\frac{d^2\sigma_{i-1}}{dX^2} + \lambda_1 \frac{d\sigma_i}{dX} + \lambda_2\sigma_{i+1} = \lambda_3 \qquad (3.28)$$

$$\lambda_k = \lambda_k(\sigma_o, \frac{d\sigma_o}{dX}, \sigma_1, \sigma_2, \ldots, \sigma_i) \qquad k=1,2,3$$

It is extremely important that this equation contains $\frac{d^2\sigma_{n-1}}{dX^2}$, $\frac{d\sigma_n}{dX}$, and σ_{n+1} linearly. If this were not the case we would be faced with solving a set of nonlinear algebraic equations instead of a linear set.

Consider what information is needed to calculate a typical element of a_{ij}. Supstituting (3.27) in (3.28) and equating coefficients of order n gives an equation of the form

$$B^{(n)}_{i,i-1}a_{i-1,n+1} + B^{(n)}_{i,i}a_{in} + B^{(n)}_{i,i+1}a_{i+1,n-1} = C^{(n)}_i \tag{3.28}$$

where

$$C^{(n)}_i, B^{(n)}_{i,i-1}, B^{(n)}_{i,i}, B^{(n)}_{i,i+1}$$

are functions of a_{KJ} where

$$K,J < i + n$$

It now becomes clear how a_{ij} can be generated. This can be accomplished by solving sets of algebraic aquations in the following manner.

$$B^{(1)}a_{o1} = C^{(1)}$$

$$\begin{bmatrix} B^{(2)}_{11} & B^{(2)}_{12} \\ B^{(2)}_{21} & B^{(2)}_{22} \end{bmatrix} \begin{Bmatrix} a_{o2} \\ a_{11} \end{Bmatrix} = \begin{Bmatrix} C^{(2)}_1 \\ C^{(2)}_2 \end{Bmatrix}$$

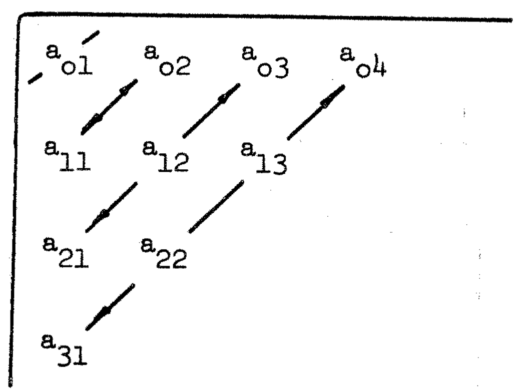
$$\begin{bmatrix} B^{(3)}_{11} & B^{(3)}_{12} & 0 \\ B^{(3)}_{21} & B^{(3)}_{22} & B^{(3)}_{23} \\ 0 & B^{(3)}_{32} & B^{(3)}_{33} \end{bmatrix} \begin{Bmatrix} a_{o3} \\ a_{12} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} C^{(3)}_1 \\ C^{(3)}_2 \\ C^{(3)}_3 \end{Bmatrix} \tag{3.29}$$

$$\begin{bmatrix} B_{11}^{(n)} & B_{12}^{(n)} & 0 & 0 & \cdots & \\ B_{21}^{(n)} & B_{22}^{(n)} & B_{23}^{(n)} & 0 & & \\ 0 & B_{32}^{(n)} & B_{33}^{(n)} & B_{34}^{(n)} & & \\ \vdots & & & \ddots & & B_{n-1,n}^{(n)} \\ & \cdots & & 0 & B_{n,n-1}^{(n)} & B_{nn}^{(n)} \end{bmatrix} \begin{Bmatrix} a_{on} \\ a_{1,n-1} \\ \vdots \\ a_{n-1,1} \end{Bmatrix} = \begin{Bmatrix} C_1^{(n)} \\ \vdots \\ \vdots \\ C_n^{(n)} \end{Bmatrix} \tag{3.29}$$

This system can now be solved sequentially since the coefficients of each matrix equation depend only on the values of the a_{ij} 's previously calculated. That is to say

$$B_{ij}^{(n)} = B_{ij}^{(n)}(a_{Kj}) \;;\; C_i^{(n)} = C_i^{(n)}(a_{Kj})$$

where $K + j \nless n$. Thus the infinite matrix a_{ij} is generated along diagonals as indicated.



a_{o1} is given by

$$a_{o1} = -\frac{G(P_o, M(P_o))}{\ell(P_o)} - \frac{\left(M'(P_o) - \frac{M(P_o)}{P_o}\right)}{\ell(P_o)} P_1 \tag{3.29a}$$

Equation (3.29a) shows clearly the two factors that influence the decay of a shock wave. The first term in (3.29a) represents the damping effect of the viscous properties of the material. The second term represents the effect of the boundary conditions as transmitted by the nonlinear character of the instantaneous response of the material. Clearly for a linear or semilinear material the second term drops out.

It is apparent that because of the order in which the a's are solved, the solution for σ_o will approach the true value fastest. This is fortunate since the solution for σ_o gives the amplitude and position of the shock wave that in many cases is the most important information desired.

For numerical computations it is fortunate that the B matrix is a banded one with each row containing at most three elements positioned about the diagonal. When solving for the a's this fact becomes important since the B matrix can now easily be diagonalized even when B is a very large matrix.

3.4 Convergence Proof

The standard proofs of convergence that are used for finite systems of differential equations break down when applied to an infinite system of equations as are being dealt with here. Infinite systems such as this arise also in celestial mechanics and consideration of the motion of infinitely many stars. The following proof is based on a similar proof by Moulton (1930).

Define new variables by

$$T_{2n-1} = V_n - V_n(0)$$

$$T_{2n} = \sigma_n - \sigma_n(0)$$

$$e_n = \varepsilon_n - \varepsilon_n(0)$$

Then equations (21) can be written

$$\frac{dT_{2n-1}}{dX} = e_{n+1} + f' T_{2n+1} + \varepsilon_{n+1}(0) + f' V_{n+1}(0)$$

$$\frac{dT_{2n}}{dX} = \rho\, T_{2n+1} + f' T_{2n+2} + f' \sigma_{n+1}(0) + \rho V_{n+1}(0) \tag{3.30}$$

It will be assumed that σ_o is bounded. Hence by (3.20) f' is bounded. It will further be assumed that there exist finite positive constants C_i such that the series

$$C = C_o T_o + C_1 T_1 + ,\ldots, C_n T_n + ,\ldots \tag{3.31}$$

converges for

$$r_i \geq |T_i|$$

Assume also that there exists an A such that

$$Cr_{2n-1}A \geq |\ _{n+1}| + |f'||T_{2n+1}| + |\varepsilon_{n+1}(0)| + |f'||V_{n+1}(0)|$$
$$Cr_{2n}A \geq \rho|T_{2n+1}| + |f'||T_{2n+2}| + |f'||\sigma_{n+1}(0)| + \rho|V_{n+1}(0)| \quad (3.32)$$

Consider now the system

$$\frac{dD_i}{dX} = A\ r_i\ P\ ;\ P = C_oD_o + C_1D_1\ r\ ,\ \ldots$$

P will converge if $|D_i| \leq r_i$ by equation (31). By symmetry

$$\frac{dD_1}{dX}\frac{1}{r_1} = \frac{dD_2}{dX}\frac{1}{r_2} = ,\ldots, = \frac{dD}{dX}$$

where $D = D_i/r_i$. Therefore (33) becomes

$$\frac{dD}{dX} = AP$$

But

$$P = [C_or_oD + C_1r_1D + ,\ldots,]$$

Hence

$$\frac{dD}{dX} = AD[C_or_o + C_1r_1 + ,\ldots,]$$

But by hypothesis

$$C = C_or_o + C_1r_1 + ,\ldots,$$

converges. Hence

$$\frac{dD}{dX} = ACD\ .$$

The solution of this is

$$D = e^{ACX} - 1 .$$

The expansion of this is

$$D(X) = \frac{(AC)X}{1!} + \frac{(AC)^2X^2}{2!} + ,\ldots,$$

which converges for $|X| < \infty$. This means that the series for $D_i(X)$ converges. Since the right members of (3.33) dominate the right members of (3.30) by assumption (3.32), then by the comparison test the expansions for T_n must also converge.

3.5 Effect of Boundary Conditions on the Decay of the Shock Front

A few observations can now be made concerning whether or not a shock will decay. Restating the shock decay equation,

$$\frac{d\sigma_o}{dX}\left(\frac{3}{2}\left(\rho\,\frac{M}{\sigma_o}\right)^{\frac{1}{2}} + \left(\rho\,\frac{M}{\sigma_o}\right)^{\frac{1}{2}}\frac{M'}{2}\,\rho\right) = \sigma_1\left(\frac{M}{\sigma_o} - M'\right)\rho - \rho\,G(\sigma_o, M(\sigma_o))$$

If the material is semilinear a shock wave must decay. In a semilinear material

$$M = M'\sigma_o$$

This equation then reduces to

$$\frac{d\sigma_o}{dX}\left(\frac{3}{2}\,(\rho\mu)^{\frac{1}{2}} + (\rho\mu)^{-\frac{1}{2}}\,\frac{\mu}{2}\,\rho\right) = -\,\rho\,G(\sigma_o, \mu\sigma_o)$$

where μ is a positive constant. Since $\frac{G}{\sigma_o}$ is positive on thermodynamic grounds, $\frac{1}{\sigma_o}\frac{d\sigma_o}{dX}$ is always negative and the wave must decay no matter what the boundary conditions.

If the material is linear elastic $\frac{d\sigma_o}{dX}$ is obviously zero and hence the wave travels with constant amplitude and profile no matter what the boundary conditions.

Thus it is only in the nonlinear elastic case and the nonlinear visco-elastic case that the question of decay of a shock is tied up with the boundary conditions. We may ask ourselves under what boundary conditions will the wave front travel with constant amplitude. This condition means $\frac{d\sigma_o}{dX} \equiv 0$.

In the nonlinear elastic case

$$\frac{d\sigma_o}{dX} \equiv 0 \rightarrow \sigma_1 \equiv 0 \rightarrow \sigma_2 \equiv 0 \rightarrow ,\ldots, \sigma_n \equiv 0$$

Hence only if the stress impact on the rod is in the form of a heavyside step function will this condition be satisfied.

For the nonlinear viscoelastic case the necessary input is no longer a step input of stress but is a specific function. For the case where $M(\sigma) = a + b\sigma$ the recursive relation

$$\sigma_{n+1} = \frac{\sigma_1}{a} \eta \, \sigma_n$$

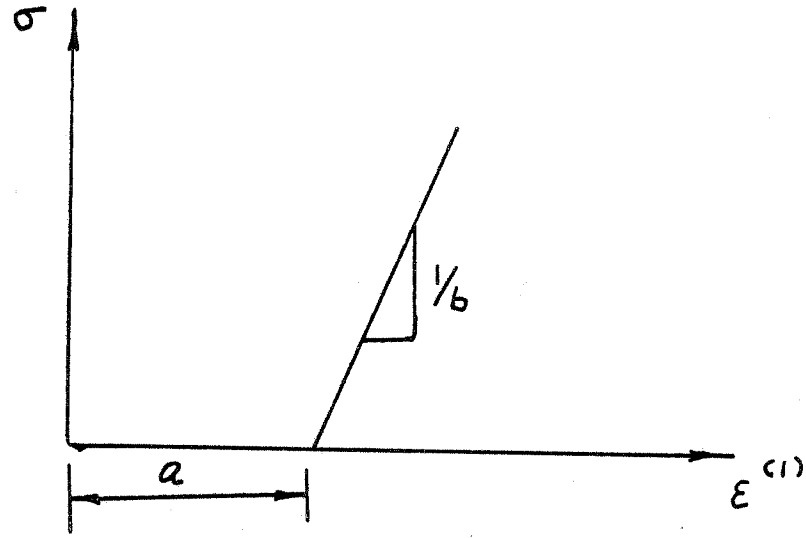
must be satisfied. This results in a boundary condition of the form

$$\sigma(0,t) = AH(t)e^{\frac{\eta}{a} At}$$

where A is a constant and $H(t)$ is the heavyside step function. It can be seen from this relation that as the material becomes more linear and a gets smaller, more and more energy is required to maintain a steady shock since $e^{\frac{\eta}{a} At}$ gets very large. If η is zero this of course indicates $AH(t)$ is the necessary initial condition.

3.6 A Restricted Class of Materials

Consider now a more restricted class of materials where the instantaneous response is shown by the following figure.



Thus

$$\varepsilon^{(1)} = M(\sigma) = a + b\sigma \tag{3.34}$$

For what follows it will be assumed that σ is always greater than zero. An analogous procedure can be used for compressional waves. b is taken to be greater than zero since if $b = 0$ the equations will be parabolic and if $b < 0$ the equations will be elliptic. If $a > 0$ impacting such a material will produce a shock that has variable velocity and hence follows a curved path in the X,t plane. If $a = 0$ the material is semilinear and the shock moves with constant velocity. For what follows a will be taken to be greater than zero since we are interested in the nonlinear shock problem. For simplicity take

$$g(\sigma,\varepsilon) = \eta\sigma$$

since the nonlinearity we are interested in is the instantaneous nonlinearity.

The governing equations now reduce to

$$\frac{d\sigma_o}{dX}\left(\frac{2}{\rho} R(\sigma_o) - \frac{a}{2\sigma_o} R(\sigma_o)^{-1}\right) + \sigma_o \eta = a \frac{\sigma_1}{\sigma_o}$$

$$\frac{d\sigma_1}{dX} 2R(\sigma_o) + \sigma_1\left(\rho\eta - R^{-1}(\sigma_o) \frac{a\rho}{\sigma_o^2} \frac{d\sigma_o}{dX}\right) = \rho \frac{a\sigma_2}{\sigma_o} + \frac{d^2\sigma_o}{dX^2} \qquad (3.35)$$

$$\vdots$$

$$\frac{d\sigma_n}{dX} 2R(\sigma_o) + \sigma_n\left(\rho\eta - R^{-1}(\sigma_o) \frac{a\rho}{\sigma_o^2} \frac{d\sigma_o}{dX}\right) = \rho \frac{a\sigma_{n+1}}{\sigma_o} + \frac{d^2\sigma_{n-1}}{dX^2}$$

where

$$R(\sigma_o) = \rho^{\frac{1}{2}} \left(\frac{a}{\sigma_o} + b\right)^{\frac{1}{2}}$$

Before proceeding to the solution of the problem it is possible to get an idea of how the σ_n 's behave for large n . It was assumed that the initial expansion of stress, strain, and velocity in powers of τ is a convergent series. Hence if the series is to converge for $\tau > 0$, the $\frac{\sigma_n}{n!}$'s must approach zero in the limit. It is also assumed that σ_X and σ_{XX} are convergent for sufficiently large τ . Consequently we can use the expansions for these three quantities and the governing equations to see how the σ_n 's behave for large n . The convergence of $\sigma, \sigma_X, \sigma_{XX}$ implies then

$$\lim_{n\to\infty} S_n \to 0$$

$$\lim_{n\to\infty} S_n' - (n + 1)f' S_{n+1} \to 0 \qquad (3.36a)$$

$$\lim_{n\to\infty} S_n'' - (n + 1)f'' S_{n+1} - 2(n + 1)f' S_{n+1}' + (n + 1)(n + 2)(f')^2 S_{n+2} \to 0$$

Where $S_n = \frac{\sigma_n}{n!}$. Combining these equations with equations (3.35) gives for very large n

$$\frac{S_{n+1}}{S_n} = - \frac{\eta}{(n+1)b} \tag{3.36b}$$

This equation illustrates the alternating sign of the σ_n 's. Thus the S_n 's decay essentially as $\frac{1}{n}$. It is also significant that the constant which governs the nonlinearity seems to have little effect on the decay. The viscosity and the local slope of the stress strain curve b has direct influence however.

SERIES SOLUTION

For a material with nonlinearity of the type of equation (3.34) and with linear viscosity of the Maxwell type, numerical solutions have been obtained. Recall that the matrix $B_{ij}^{(K)}$ has the form

$$\begin{bmatrix} B_{11}^{(K)} & B_{12}^{(K)} & 0 & 0 & 0 \cdots & 0 & 0 \\ B_{21}^{(K)} & B_{22}^{(K)} & B_{23}^{(K)} & 0 & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & B_{32}^{(K)} & B_{33}^{(K)} & B_{34}^{(K)} & 0 & 0 & 0 \\ & & \ddots & \ddots & \ddots & & \\ 0 & 0 \cdots & 0 & 0 & B_{n,n-1}^{(K)} & B_{nn}^{(K)} & B_{n,n+1}^{(K)} \end{bmatrix}$$

For the case under consideration the matrices $B_{ij}^{(K)}$ and $C_i^{(K)}$ are determined after some manipulation.

$$B_{11}^{(K)} = 2\, f_o(K + 1) - (K + 1)\frac{a\, y_o}{2\, f_o}$$

$$B_{12}^{(K)} = -y_o\, a$$

$$C_1^{(K)} = (K + 1)\frac{a}{2f_o} \sum_{i=1}^{K} y_i A_{1,K-i+1} - \sum_{i=1}^{K} f_i A_{1,K-i+1}(2K - i + 2) - \eta\, A_{1K}\, \rho + a \sum_{i=1}^{K-1} A_{2i} y_{K-i} \tag{3.36c}$$

$$B_{n,n-1}^{(K)} = (K - n + 2)f_o$$

$$B_{nn}^{(K)} = -(K - n + 3)(K - n + 2)$$

$$B_{n,n+1}^{(K)} = A_n y_o$$

$$C_n^{(K)} = -2 \sum_{i=1}^{K-n+1} f_{K-n+2-i} A_{ni}(i)$$

$$- A_{n,K-n+1} - \sum_{i=1}^{K-n+1} A_{n,K-n+2-i} f_i(i)$$

$$- a \sum_{i=1}^{K-n} A_{n+1,i}\, y_{K-n+1-i}$$

where

$$A_n = a_{no}$$

$$y_o = \frac{1}{A_{1o}}$$

$$F_o = (\rho b)^{\frac{1}{2}}$$

$$y_{K+1} = - y_o \sum_{i=0}^{K} y_i A_{1,K+1-i}$$

$$f_{K+1} = \frac{1}{2f_o}\left(a\, y_{K+1} - \sum_{i=1}^{K} f_i f_{K+1-i}\right)$$

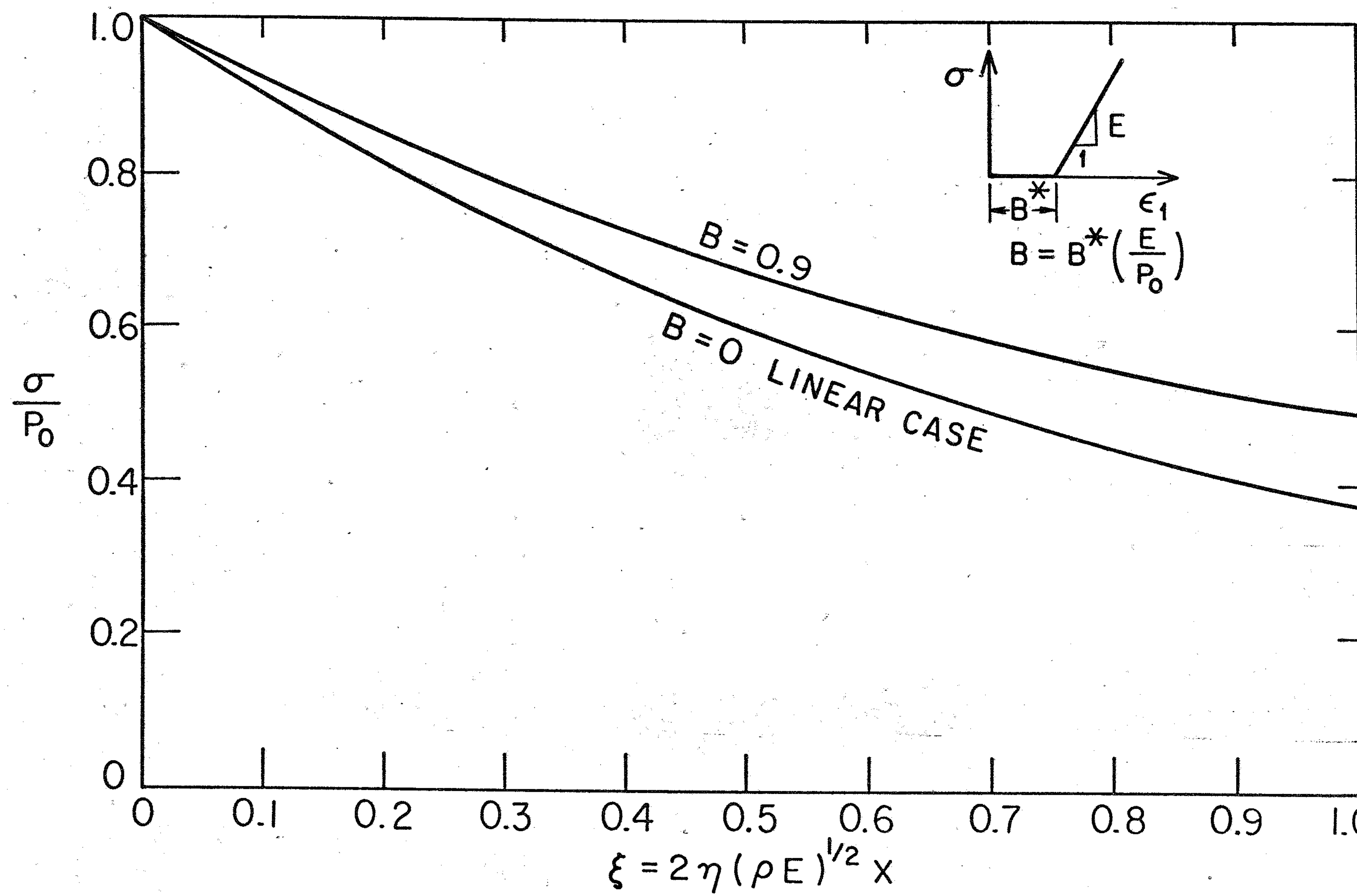


FIG. I SHOCK FRONT DECAY FOR STEP INPUT OF STRESS

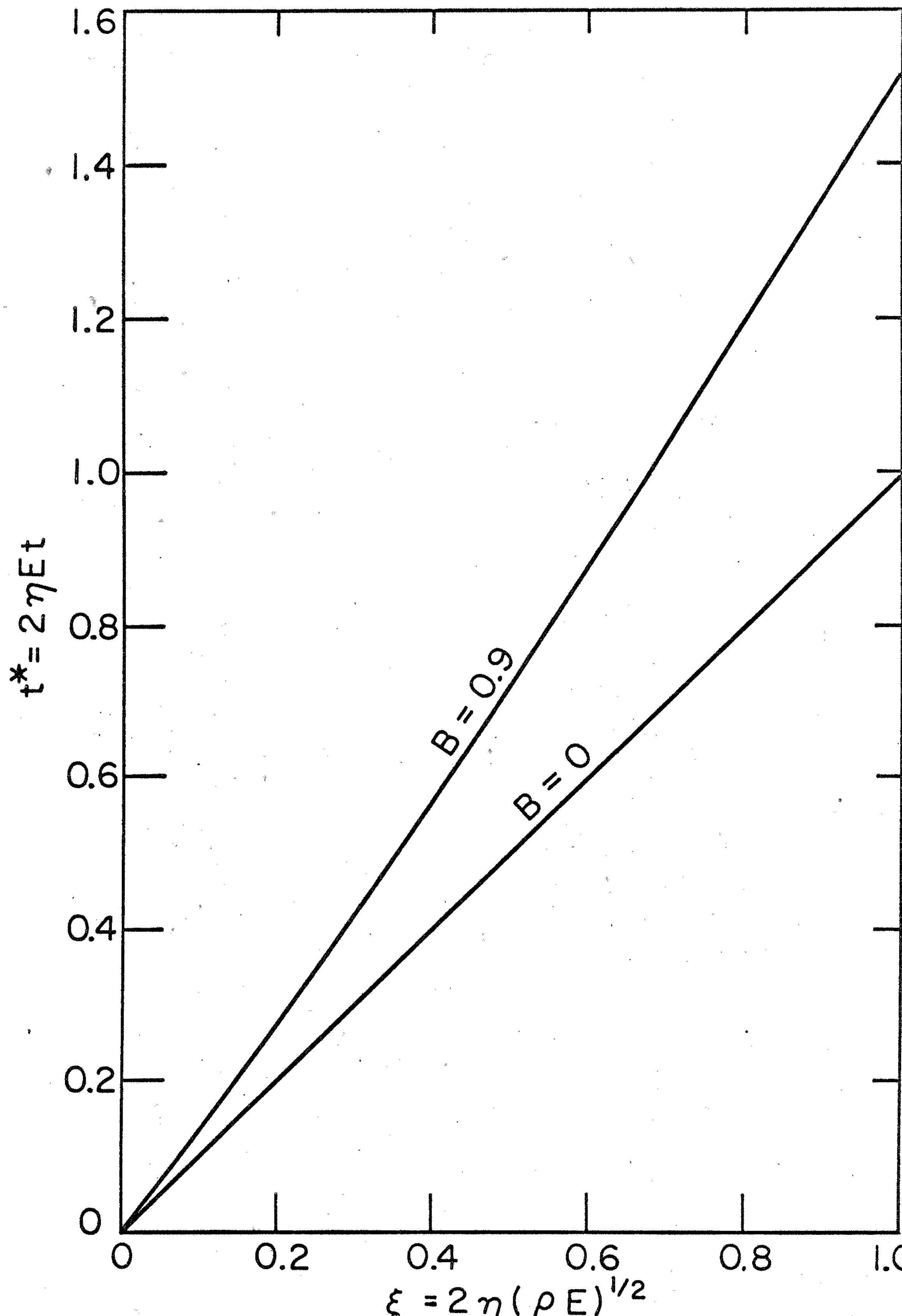


FIG. 2 SHOCK FRONT LOCATION FOR STEP INPUT OF STRESS

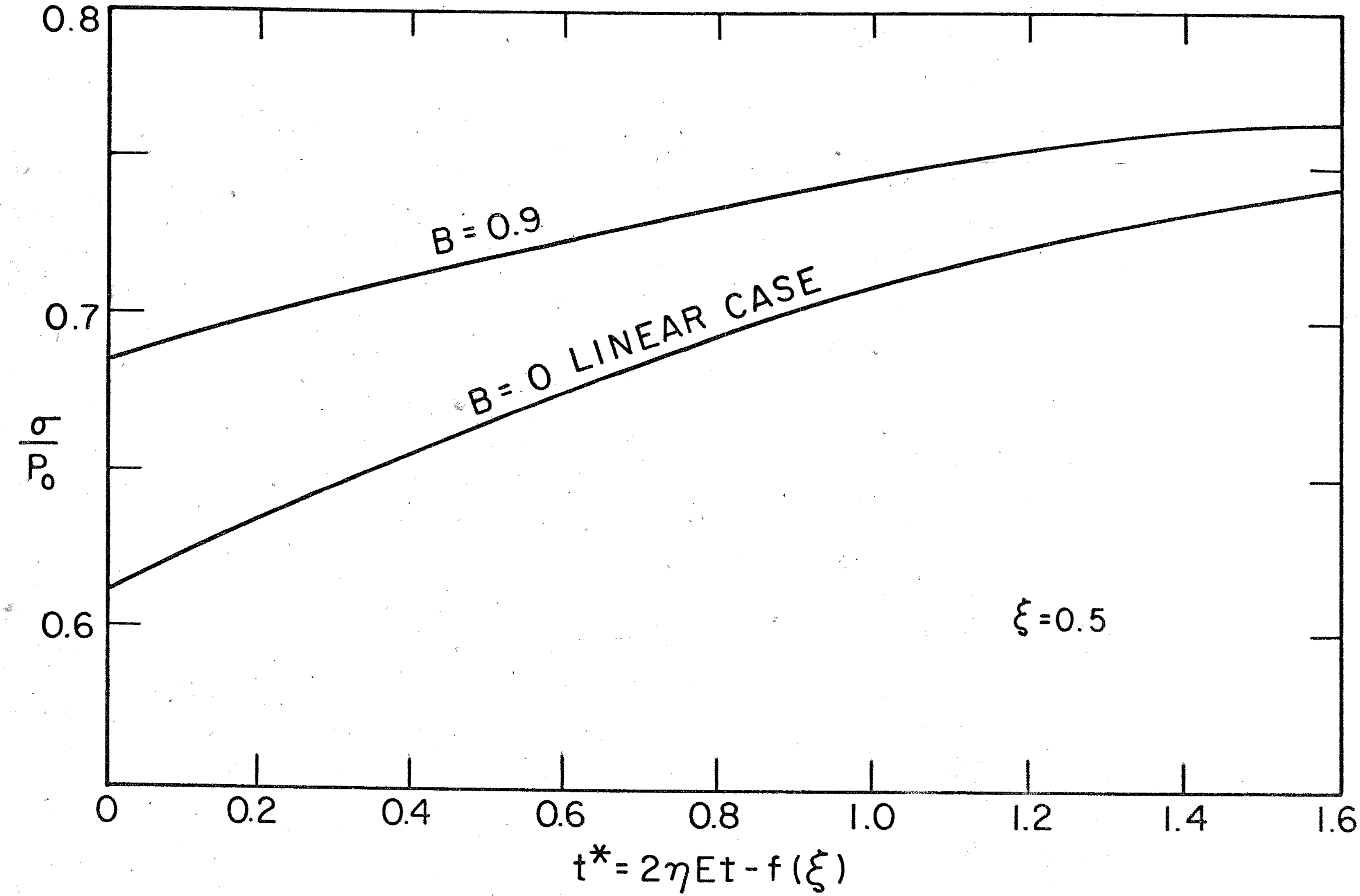


FIG. 3 WAVE PROFILE FOR STEP INPUT OF STRESS

3.7 A Method for More General Types of Materials

For materials with more complex constitutive equations than that of equation (3.34), the governing set of differential equations (eqs. 3.23) become very complex. If however, the shock amplitude at the front is all that is desired a simplified approach presents itself. The essence of the method is to assume that the material can be approximated by a piecewise continuous stress-strain law for rapid deformations. By proceeding in a stepwise fashion along X, general types of constitutive laws can be handled. Consider the X,t plane to be divided into equal segments along X.

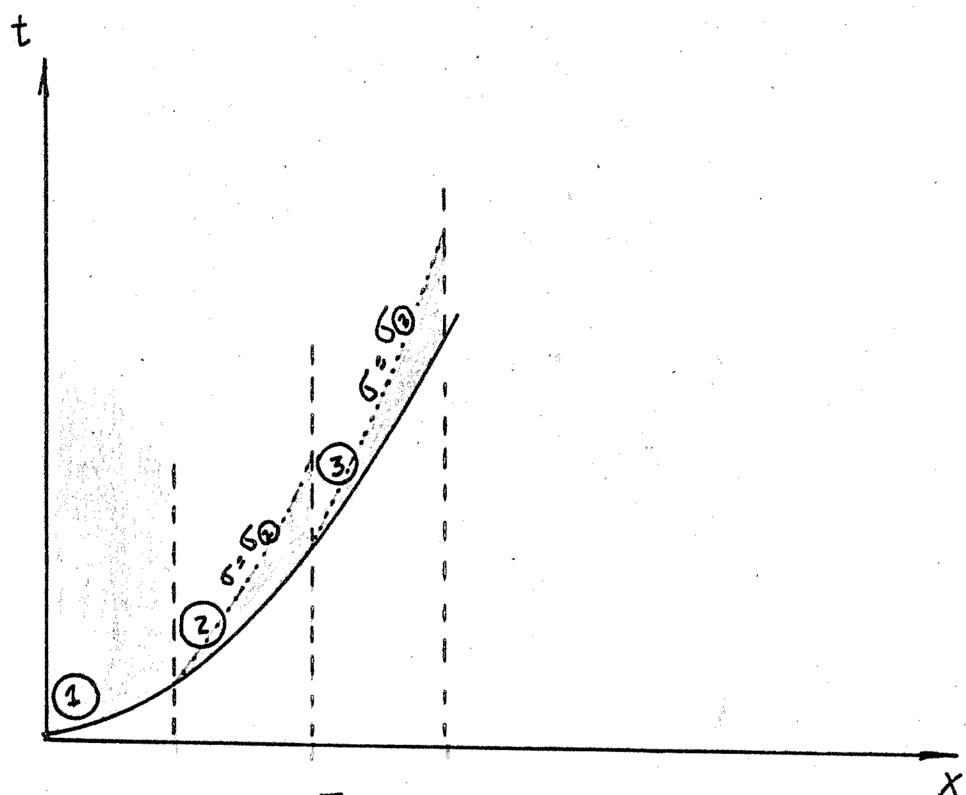


FIG. 4

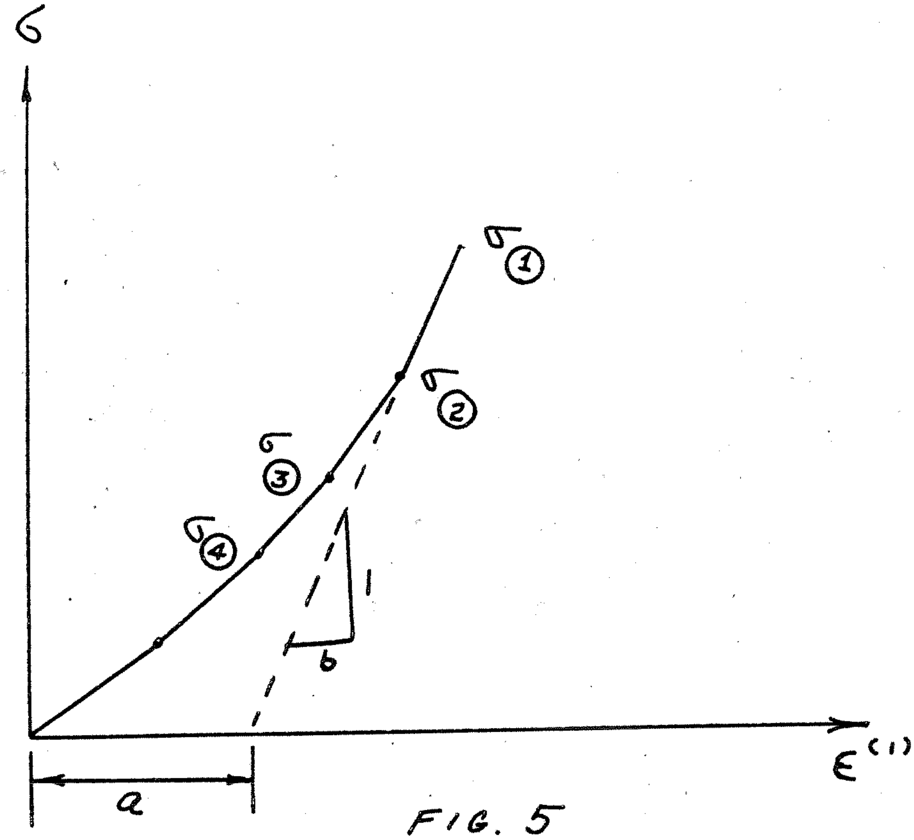


FIG. 5

In each segment the material is assumed to obey a law that is locally linear. This is the kind of material of the type of equation (3.34) which has already been solved. Thus, in segment ① - ② the stress strain law is $\varepsilon_1 = a + b$. The wave profile calculated at station ② then gives the boundary condition for the solution in segment ② - ③ and new values of a and b are determined from the stress-strain law. For a step input of stress the solutions obtained would have validity only within certain regions as indicated by the shaded portions of fig. 4. This general idea of progressing in steps along X can also be used merely to increase the accuracy of the general method.

The method outlined above was used to calculate the shock front for a nonlinear elastic material with a parabolic stress-strain law subjected to variable loadings. The results are indicated in fig. 7.

A tensile stress impact was considered that decays with time. As indicated earlier the shock wave produced will decrease in amplitude due to the nonlinearity of the material which introduces the effect of the boundary conditions. In a linear material there will be no such decrease in amplitude.

Since a parabolic stress strain law which is hardening in tension will be weakening in compression, a tensile impact will produce a shock but a compressive impact will generate an acceleration wave.

Figure 7 illustrates the important role nonlinear effects can play in wave propagation problems. Indeed it would seem that certain results ascribed to viscosity may in reality be due to a nonlinear mechanism.

The constitutive law was taken to be

$$\bar{\varepsilon} = 2\bar{\sigma} - .5(\bar{\sigma})^2$$

where

$$\bar{\varepsilon} = 2 \frac{E}{P_o} \varepsilon$$

$$\bar{\sigma} = \frac{\sigma}{P_o}$$

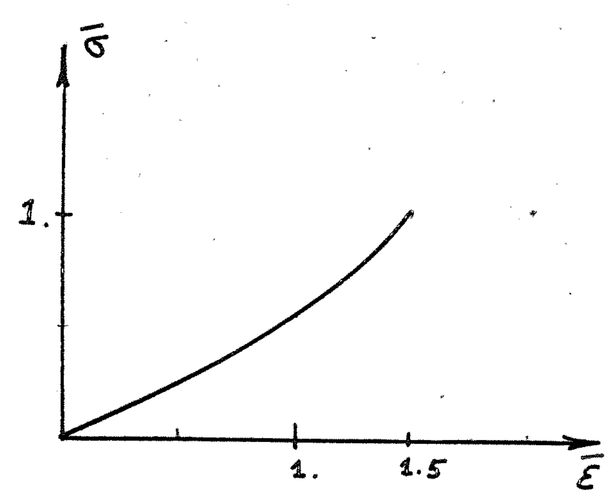


FIG. 6

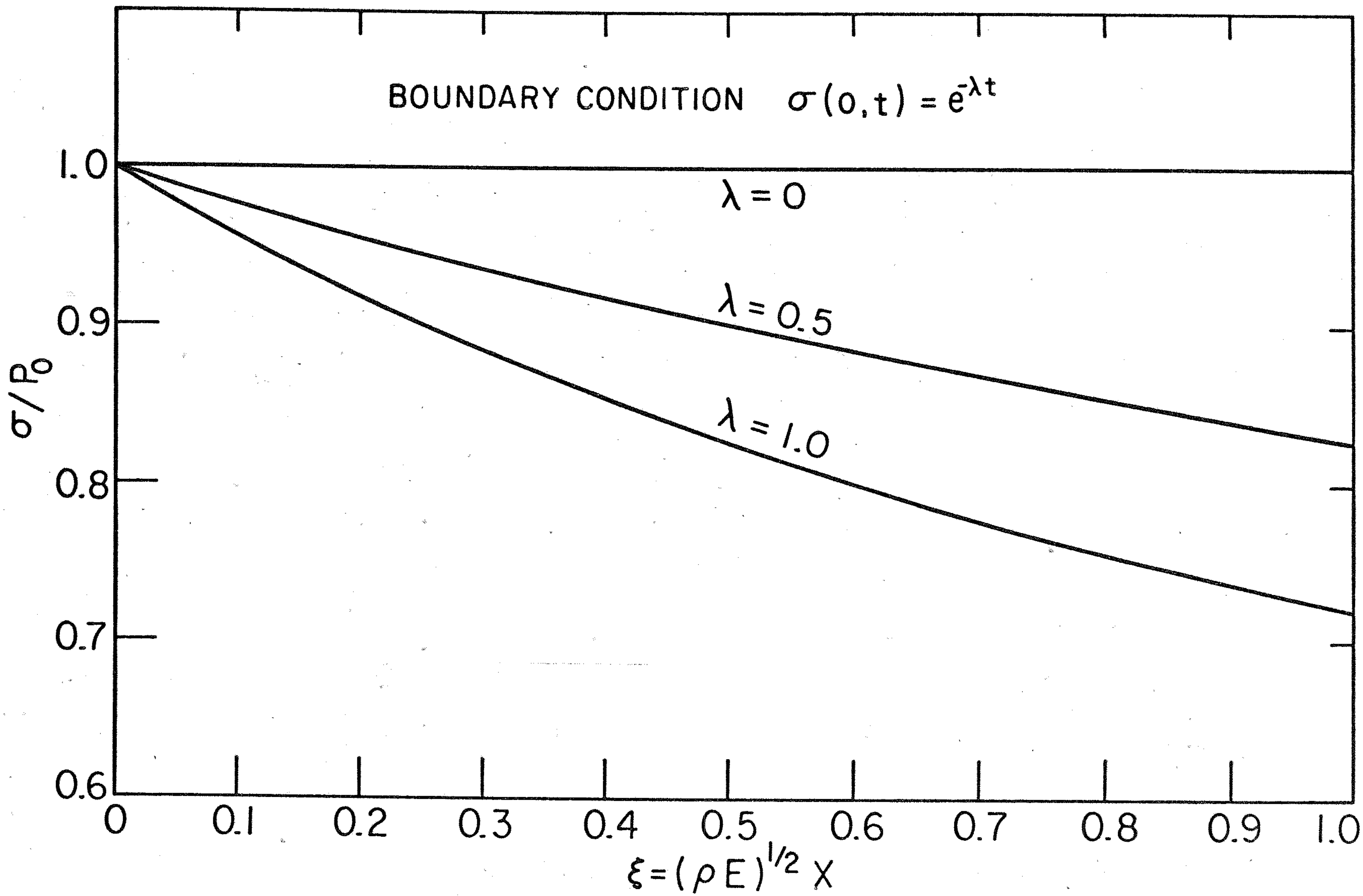


FIG. 7 SHOCK FRONT DECAY FOR NONLINEAR ELASTIC MATERIAL WITH DIFFERENT BOUNDARY CONDITIONS

3.8 Solution by Perturbation

Notice that the term in equations (3.23) that induces the coupling with the next term is involved only in the expression $M' - M/\sigma_o$. If this quantity is small, a perturbation method can be used. Let

$$M' - \frac{M}{\sigma_o} = aN(\sigma_o)$$

where a is a small number. The dependent variables are now expanded in powers of the parameter a .

$$\sigma_o = \sigma_o^{(0)} + a\,\sigma_o^{(1)} + a^2\sigma^{(2)} + \ldots$$
$$\sigma_n = \sigma_n^{(0)} + a\,\sigma_n^{(1)} + a^2\sigma_n^{(2)} + \ldots \qquad (3.37)$$

Substituting these equations into the system (3.23) and equating powers of a gives systems of equations that are now uncoupled in the forward direction and they can be solved sequentially. The shock decay equation for this first system of zero order is

$$\frac{d\sigma_o^{(0)}}{dX}\; 2(\rho M'(\sigma_o^{(0)}))^{\frac{1}{2}} = -\,\rho\; G(\sigma_o^{(0)},M(\sigma_o^{(0)})) \qquad (3.38)$$

which is solved easily at least numerically. The equation for $\sigma_1^{(0)}$ is

$$2(\rho M'(\sigma_o^{(0)}))^{\frac{1}{2}}\frac{d\sigma_1^{(0)}}{dX} + \rho\, M''(\sigma_o^{(0)})(\sigma_1^{(0)})^2 +$$

$$+\left[\rho\,\frac{\partial G}{\partial \sigma_o} + \rho\,\frac{\partial G}{\partial M}\,M'\right]_{\sigma_o = \sigma_o^{(0)}} \sigma_1^{(0)}$$

$$= \frac{d^2\sigma_o^{(0)}}{dX^2} + \left[\rho\,\frac{\partial G}{\partial M}\,G\right]_{\sigma_o = \sigma_o^{(0)}}$$

After the calculation of $\sigma_o^{(0)}$ this of course has the form

$$\frac{d\sigma_1^{(0)}}{dX} + b_1(X)(\sigma_1^{(0)})^2 + b_2(X)\sigma_1^{(0)} + b_3(X) = 0 \tag{3.40}$$

This is the famous Riccati equation whose solution is well known. The equation for $\sigma_o^{(1)}$ is of the form

$$\frac{d\sigma_o^{(1)}}{dX} = \sigma_1^{(0)} \; C_1(\sigma_o^{(0)}) + C_2(\sigma_o^{(0)},\sigma_o^{(1)}) \tag{3.41}$$

Since $\sigma_1^{(0)}$ and $\sigma_o^{(0)}$ have previously been calculated the equation has the form

$$\frac{d\sigma_o^{(1)}}{dX} = d_1(X) + d_2(X,\sigma_o^{(1)}) \tag{3.42}$$

which can readily be solved. Stopping at this point would give a first order approximation to the shock front, namely

$$\sigma_o(X) \approx \sigma_o^{(0)}(X) + a\sigma_o^{(1)} \tag{3.43}$$

For certain problems with very small nonlinearities this solution may be adequate. It is evident however that the process can be continued as far as necessary. It is also clear that if an n^{th} order approximation is desired to the stress at the shock front it is not necessary to even consider the equations in system (3.23) for $\sigma_i(X)$ where $i > n+1$. Thus the solution of $\sigma_o(X)$ Of a^2 order would involve only the first three equations of system (3.23).

3.9 Example of Perturbation Solution of Second Order

An explicit solution will now be exhibited for the type of material described by equation (3.34). It will further be assumed that the viscosity is of the linear Maxwell type. Since this is a solution of a^2 order for $\sigma_o(X)$, only the first three equations need be considered. They are

$$\frac{d\sigma_o}{dX} \rho^{\frac{1}{2}} 2\left(\frac{a}{\sigma_o} + b\right)^{\frac{1}{2}} - \frac{}{2\sigma_o} \rho\left(\frac{a}{\sigma_o} + b\right)^{-\frac{1}{2}} + \sigma_o \eta\rho = a \frac{\sigma_1}{\sigma_o} \rho$$

$$\frac{d\sigma}{dX}\left(2 \frac{a}{\sigma_o} + b\right)^{\frac{1}{2}} \rho^{\frac{1}{2}} + \sigma_1\left[\eta\rho - \left(\frac{a}{\sigma_o} + b\right)^{-\frac{1}{2}} \frac{a\sigma_o'}{(\sigma_o)^2} \rho^{\frac{1}{2}}\right] = \rho \frac{a\sigma_2}{\sigma_o} + \sigma_o'' \qquad (3.44)$$

$$\frac{d}{dX}^{2}\left(2 \frac{a}{\sigma_o} + b\right)^{\frac{1}{2}} \rho^{\frac{1}{2}} + \sigma_2\left[\eta\rho - \left(\frac{a}{\sigma_o} + b\right)^{-\frac{1}{2}} \frac{a\sigma_o'}{(\sigma_o)^2} \rho^{\frac{1}{2}}\right] = \rho \frac{a\sigma_3}{\sigma_o} + \sigma_1''$$

The magnitude of a determines the nonlinearity and is assumed to be small. The σ_i 's will be expanded in powers of a and inserted in equations (3.44). This leads to the sequence of linear differential equations

$$\frac{d\sigma_o^{(0)}}{dX} 2(b\rho)^{\frac{1}{2}} + \eta\sigma_o^{(0)} \rho = 0$$

$$\frac{d\sigma_1^{(0)}}{dX} 2(b\rho)^{\frac{1}{2}} + \sigma_1^{(0)}\eta\rho = \sigma_o^{(0)''} \qquad (3.45)$$

$$\frac{d\sigma_o^{(1)}}{dX} 2(b\rho)^{\frac{1}{2}} + \sigma_o^{(1)}\left[(\rho b)^{\frac{1}{2}} \frac{\sigma_o^{(0)'}}{\sigma_o^{(0)}} + 2\eta\rho\right] = -\frac{\sigma_o^{(0)'}}{\sigma_o^{(0)}} - \frac{\sigma_1^{(0)}}{\sigma_o^{(0)}} + \frac{\sigma_o^{(0)'}}{2\sigma_o^{(0)}(b\rho)^{\frac{1}{2}}}$$

$$\frac{d\sigma_2^{(0)}}{dX} \; 2(\rho b)^{\frac{1}{2}} + \sigma_2^{(0)} \; \eta\rho = \sigma_1^{(0)''}$$

$$\frac{d\sigma_1^{(1)}}{dX} \; 2(\rho b)^{\frac{1}{2}} + \eta \; \sigma_1^{(1)} \rho = \frac{\sigma_2^{(0)}}{\sigma_o^{(0)}} + \sigma_o^{(1)''} - \frac{\sigma_1^{(0)} \sigma_o^{(0)'}}{(\rho b)^{\frac{1}{2}} \sigma_o^{(0)}} - \frac{2\sigma_1^{(0)'}}{\sigma_o^{(0)}} \tag{3.45}$$

$$\frac{d\sigma_o^{(2)}}{dX} \; 2(\rho b)^{\frac{1}{2}} + \sigma_o^{(2)} \; 2\eta\sigma_o^{(0)}\rho = -\eta \; (\sigma_o^{(1)})^2 \rho + \sigma_1^{(1)}$$

$$- \sigma_o^{(0)'}\left(\sigma_o^{(2)} F_o + \sigma_o^{(1)} F_1 + \sigma_o^{(0)} F_2 - \frac{F_1}{2F_o^2}\right)$$

$$- \sigma_o^{(1)'}\left(\sigma_o^{(1)} F_o + \sigma_o^{(0)} F_1 + \frac{1}{2F_o}\right)$$

where

$$F_o = (\rho b)^{\frac{1}{2}}$$

$$F_1 = \frac{1}{\sigma_o^{(0)}}$$

$$F_2 = - \frac{1}{b(\sigma_o^{(0)})^2}\left(3\sigma_o^{(1)} + \frac{1}{\rho b}\right).$$

These equations are linear and simple enough to be solved exactly. For simplicity take $\rho = b = 1$. After some calculation the solutions are

$$\sigma_o^{(0)} = e^{-\eta X/2} \tag{3.46}$$

$$\sigma_1^{(0)} = \frac{X\eta^2}{8} \; e^{-\eta X/2}$$

$$\sigma_o^{(1)} = \frac{X\eta}{8}$$

$$\sigma_2^{(0)} = \frac{X}{16}\frac{X\eta}{8} - 1\ \eta^3\ e^{-\eta X/2}$$

$$\sigma_1^{(1)} = \frac{1}{2}\left[\frac{\eta^3}{32} X^2 e^{-\eta X/2} + \frac{X^2\eta^3}{64} + X\left(-\frac{3}{32}\eta^2\right)\right.$$

$$\left. + \eta\left(\frac{5\eta}{16} - \frac{1}{4}\right) - \frac{3}{16}\eta^2\ e^{-\eta X/2}\right]$$

$$\sigma_o^{(2)} = \frac{7}{(25)(16)} e^{\eta X/2}\left(\frac{5\eta X}{4} - 1\right) - \frac{e^{\eta X/2}}{10\eta}$$

$$- \frac{e^{-\eta X/2}}{8}\left(\frac{\eta X}{4} - 1\right) - \frac{e^{-\eta X/2}}{4}$$

$$+ e^{-3\eta X/4}\left[\frac{57}{25(16)} - \frac{1}{10\eta}\right]$$

The second order approximation is then given by

$$\sigma_o(X) \approx \sigma_o^{(0)} + a\sigma_o^{(1)} + a^2\sigma_o^{(2)}$$

In fig. 9 the first and second perturbation solutions for $a = .5$ are compared to the series solution. It is seen that the second perturbation gives a good approximation to the solution for this case.

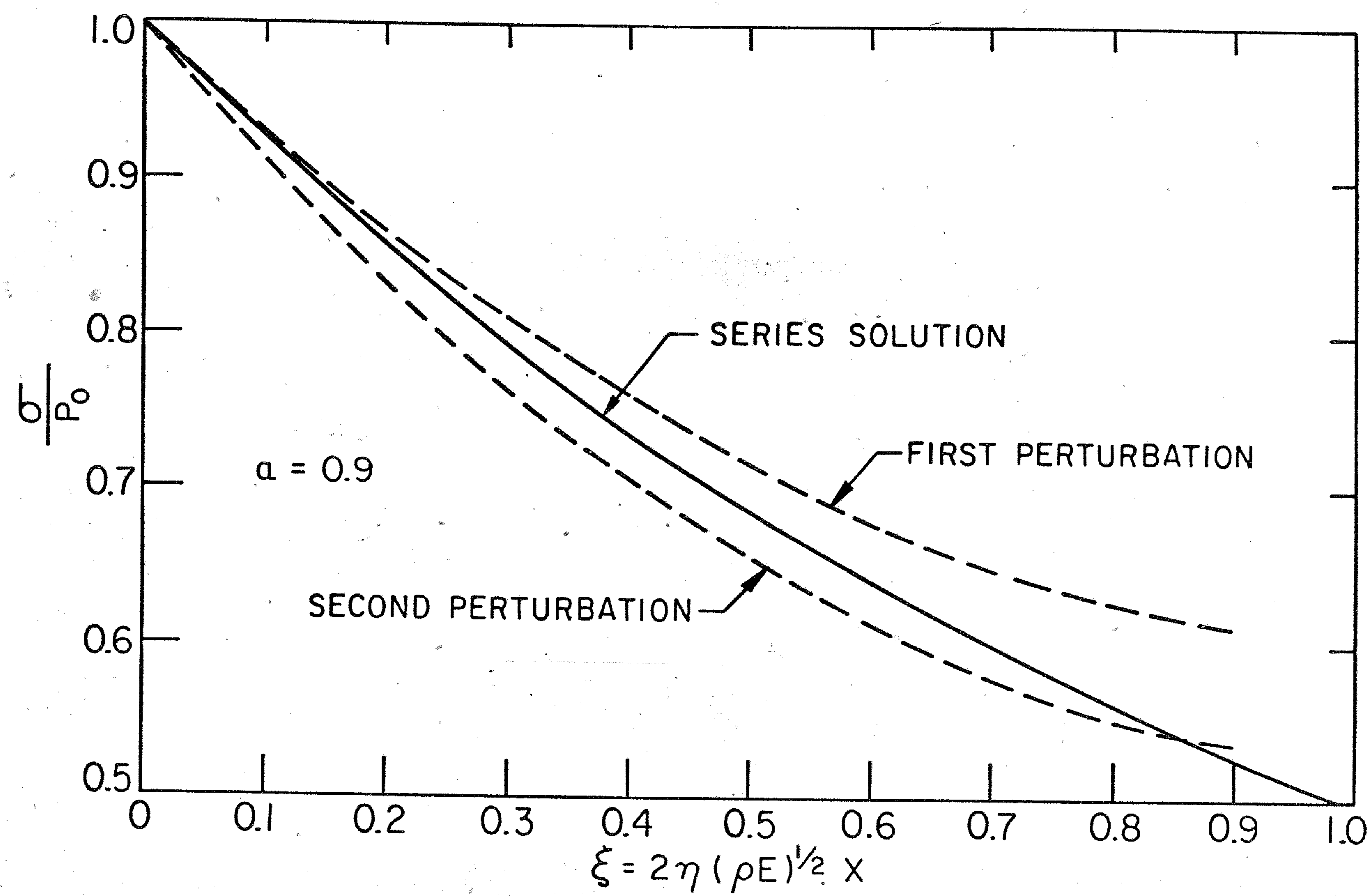
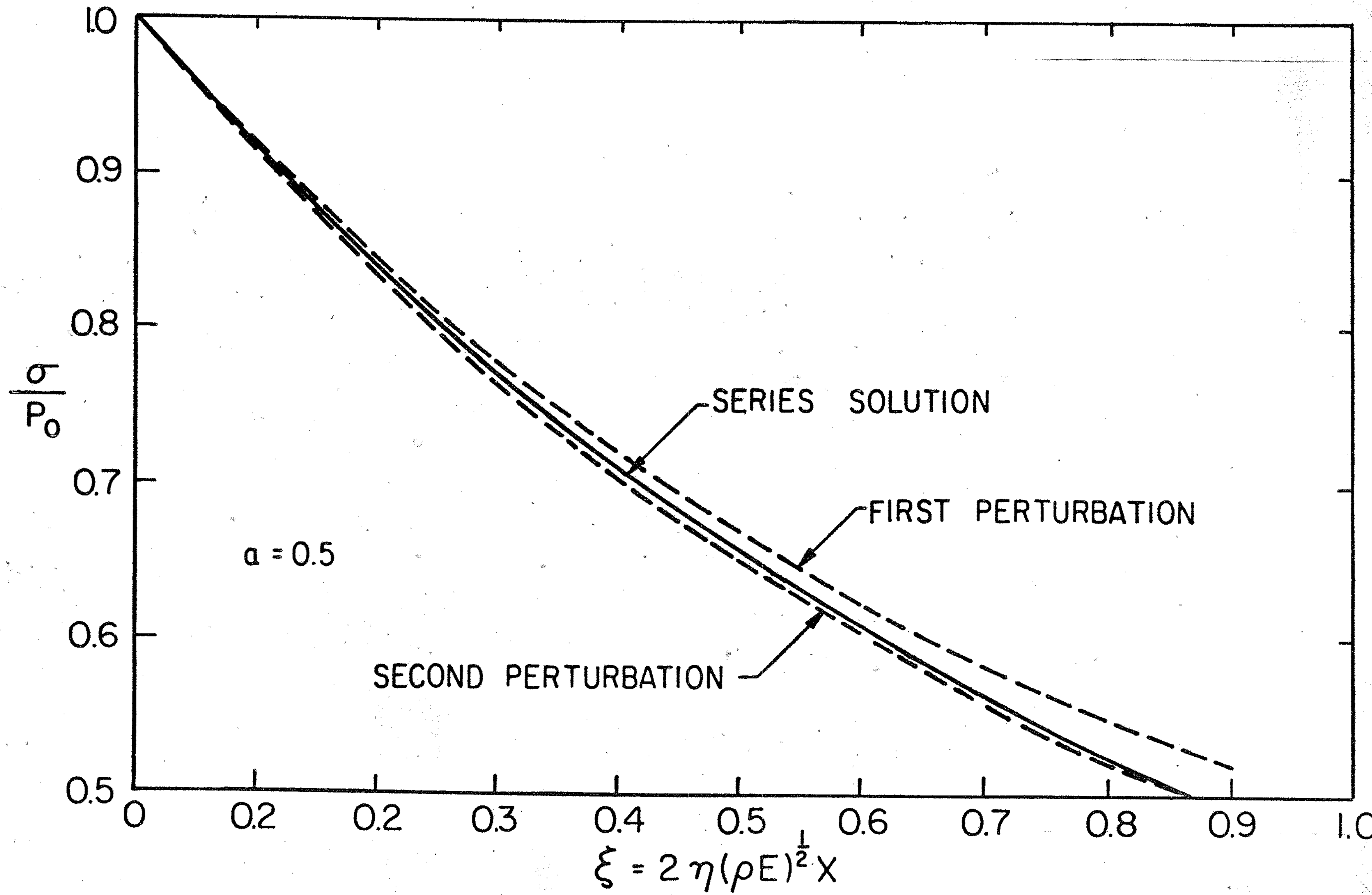

1.0
0.9
0.8
0.7
0.6
0.5
σ / P_o
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\xi = 2\eta(\rho E)^{1/2} X$
a = 0.9
SERIES SOLUTION
FIRST PERTURBATION
SECOND PERTURBATION

FIG. 8



1.0
0.9
0.8
0.7
0.6
0.5
σ/P₀
SERIES SOLUTION
FIRST PERTURBATION
SECOND PERTURBATION
a = 0.5
0
0.2
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\xi = 2\eta(\rho E)^{\frac{1}{2}} X$

FIG. 9

IV. STEADY STATE SOLUTIONS

At times one may be interested in the behavior of stress waves under steady state conditions. For very long times when transient effects can be neglected, the solution for such waves is greatly simplified. This is because the partial differential equations can now be replaced by an ordinary differential equation. Such steady state waves are of three types. The trivial solution where all dependent variables are constant, a continuous solution, and a solution that may be discontinuous at a certain point. Greenberg (1967) gives existence proofs for such waves in certain viscoelastic materials. It is the purpose here to look at a very simplified material for which an exact solution is obtainable.

Restating equations (3.1)

$$\varepsilon_t = f(\sigma,\varepsilon)\sigma_t + g(\sigma,\varepsilon) \qquad (4.1a)$$

$$\sigma_X = \rho V_t \qquad (4.1b)$$

$$\varepsilon_t = V_X \qquad (4.1c)$$

The boundary conditions will be taken to be

$$\lim_{X \to -\infty} \varepsilon(X) = \gamma \qquad (4.1d)$$

$$\lim_{X \to +\infty} \varepsilon(X) = 0$$

We seek a solution of (4.1) in which $\sigma = \sigma(\xi)$, $V = V(\xi)$, $\varepsilon = \varepsilon(\xi)$. Where

$$\xi = X - Ct \qquad (4.2)$$

and C is a constant. Substituting equation (4.2) in equations (4.1) gives the relations

$$- \varepsilon' C = - Cf(\sigma,\varepsilon)\sigma' + g(\sigma,\varepsilon) \tag{4.4a}$$

$$\sigma' = - \rho C V' \tag{4.4b}$$

$$V' = - C\varepsilon' \tag{4.4c}$$

Combining (4.4b) and (4.4c) gives

$$\sigma' = \rho\, C^2\, \varepsilon' \tag{4.5}$$

The integral of this is

$$\sigma = \rho\, c^2\, \varepsilon + K \tag{4.6}$$

where K is an arbitrary constant.

$$\varepsilon' = \frac{g(\rho C^2\varepsilon + K,\varepsilon)}{C\{C^2\rho f(\rho C^2\varepsilon + K,\varepsilon) - 1\}} \tag{4.7}$$

This is a first order nonlinear differential equation for ε containing the constant K which must be determined from the boundary conditions.

An additional requirement on the solution is that

$$\lim_{X \to +\infty} \varepsilon'(X) = 0$$

$$\lim_{X \to -\infty} \varepsilon'(X) = 0$$

These requirements plus conditions (4.1d) demand that

$$g(K,0) = 0 \;. \tag{4.9}$$

Thus K is determined from equation (4.9). Generally this implies $k = 0$. This condition $\varepsilon(-\infty) = \gamma$ gives a relation between γ and the wave speed.

$$g(\rho\ C^2\ \gamma,\gamma) = 0 \tag{4.10}$$

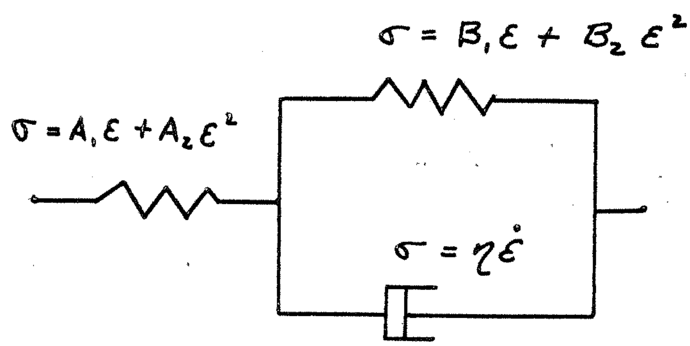
Thus if γ is prescribed, the wave speed is dependent on the stress strain relation at equilibrium.

Now for the problem posed, C , ρ , ε are greater than zero. On thermodynamic grounds $g(\sigma,\varepsilon)$ must be positive for positive arguments. On examining equation (4.7) it is seen that the solution for ε will be continuous and monotonically decreasing for increasing X if

$$C < \left(\frac{1}{\rho f(\rho C^2 \varepsilon,\varepsilon)}\right)^{\frac{1}{2}} \tag{4.12}$$

The right side of (4.12) is just the characteristic velocity. Hence the wave speed must be greater than the characteristic velocity for all values of ε .

Now let us look at a more specific material with a view to solving equation (4.7). The law chosen can be represented schematically as follows.



Thus there are two nonlinear springs but the viscosity behaves linearly. The model of fig. 1 represents the constitutive law

$$\varepsilon_t = (A_1 + 2A_2\sigma)\sigma_t + \frac{1}{\eta}\left(\varepsilon - (B_1 + A_1)\sigma - (A_2 + B_2)\sigma^2\right) \tag{4.13}$$

For this type of material equation (4.7) becomes

$$\frac{d\bar{\varepsilon}}{d\xi} = \frac{\bar{\varepsilon}\left(1 - \frac{B_1 + A_1}{A_1}\, V - \frac{B_2 + A_2}{A_1}\, V^2 \bar{\varepsilon}\right)}{C\eta\left(V - 1 + 2\, V^2\, \frac{A_2}{A_1}\, \bar{\varepsilon}\right)} \tag{4.14}$$

where

$$V = \rho\, C^2\, A_1$$

and

$$\bar{\varepsilon} = \varepsilon / A_1$$

Equation (4.14) can be integrated to give

$$(e/a - f/b)\log(a + b\bar{\varepsilon}) - \frac{e}{a}\log\bar{\varepsilon} = \frac{X}{\eta C} + \ell \tag{4.15}$$

where

$$\ell = \text{arbitrary constant}$$

$$a = 1 - V\, \frac{B_1 + A_1}{A_1}$$

$$b = -V^2\, \frac{A_2 + B_2}{A_1}$$

$$e = 1 - V$$

$$f = -2\, \frac{A_2}{A_1}\, V^2$$

The wave speed as function of prescribed strain is

$$C = \left(\frac{1}{\rho A_1}\right)^{\frac{1}{2}} \left\{ \frac{- \frac{B_1 + A_1}{A_1} + \sqrt{\left(\frac{B_1 + A_1}{A_1}\right)^2 + 4\gamma\, \frac{A_2 + B_2}{A_1}}}{-2\gamma\, \frac{A_2 + B_2}{A_1}} \right\}^{\frac{1}{2}}$$

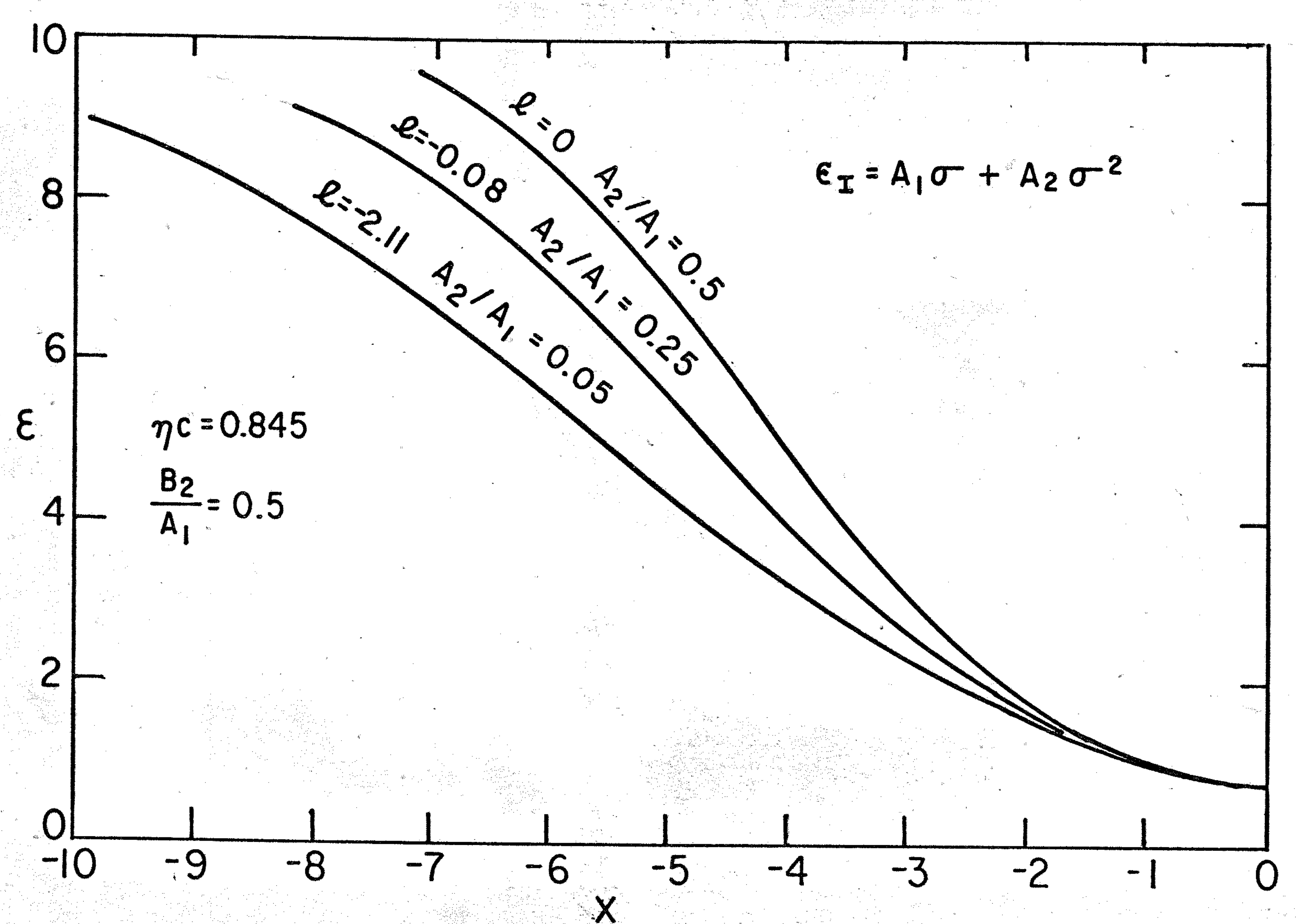


FIG. 10 VARIATION OF WAVE PROFILE WITH NON LINEAR INSTANTANEOUS RESPONSE

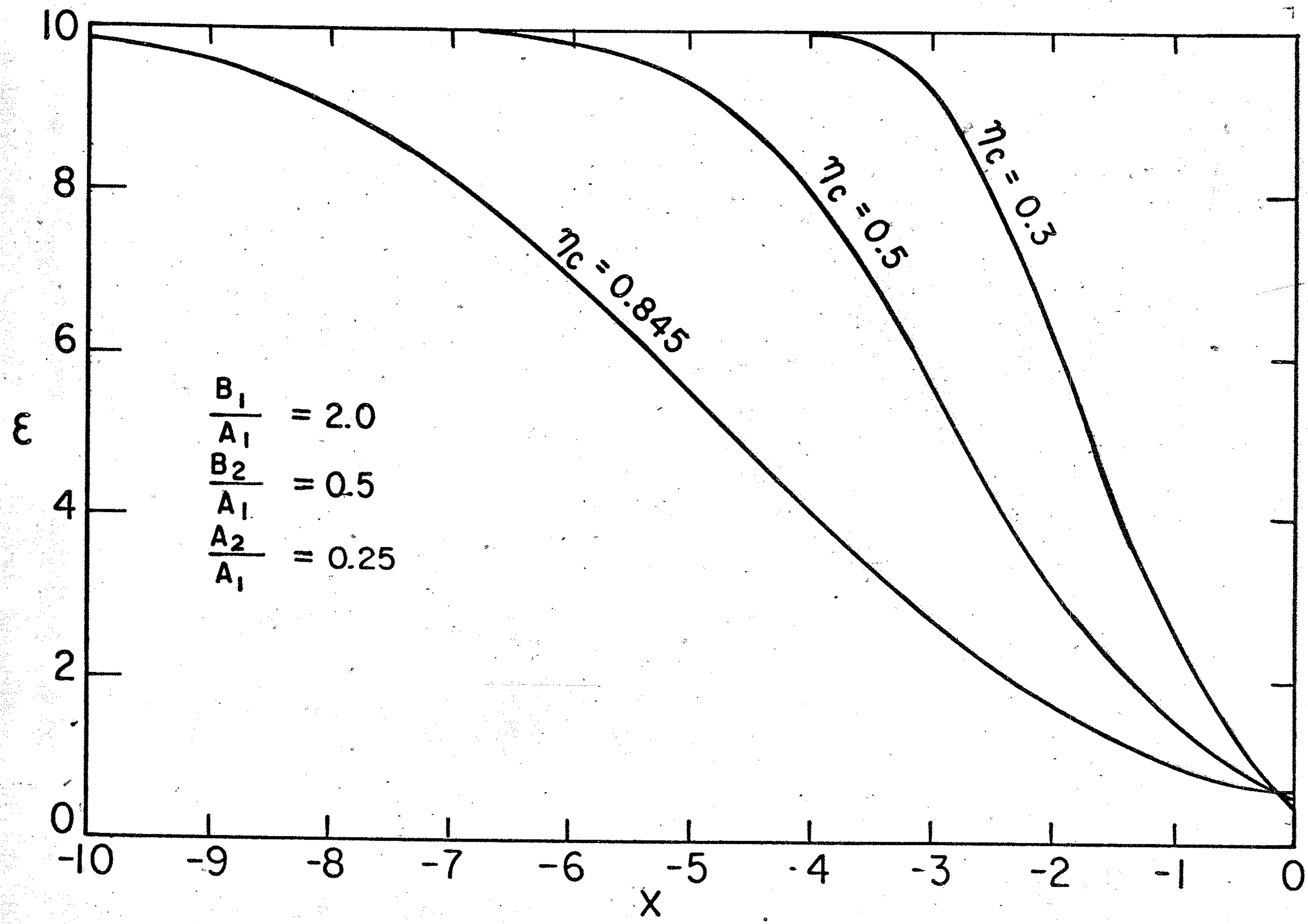


FIG. 11 VARIATION OF STEADY STATE WAVE WITH VISCOSITY

REFERENCES

1. Donnell, L. H., "Longitudinal Wave Transmission and Impact," Trans. ASME 52, p. 153, 1930.

2. Karman, T. V., and Durvez, P. E., "On the Propagation of Plastic Deformation in Solids," J. Appl Phys. 21, p. 987, 1950.

3. Taylor, G. I., "The Testing of Materials at High Rates of Loading," J. Instn. Civil Engrs. 26, p. 486, 1946.

4. Rahkmatulin, H. A., "On Plastic Waves of Loading," (in Russian), Prikl. Mat. Mekh. 9, p. 91, 1945.

5. Malvern, L. E., "The Propagation of Longitudinal Waves of Plastic Deformation in a Bar of Material Exhibiting a Strain-Rate Effect," J. Appl. Mech. 18, p. 203, 1951.

6. Kolsky, H., "Stress Waves in Solids," Dover, 1963.

7. Courant, R., and Hilbert, D., "Methods of Mathematical Physics, II, Interscience, 1962.

8. Courant, R., and Friedricks, "Supersonic Flow and Shock Waves," Interscience, 1948.

9. Coleman, B. D., Gurtin, M. E., and Herrera, I., "Waves in Materials with Memory, I. The Velocity of One-Dimensional Shock and Acceleration Waves," Arch. Rat. Mech. Anal., 19, p. 1, 1965.

10. Coleman, B. D., and Gurtin, M. E., "Waves in Materials with Memory, II. On the Growth and Decay of One-Dimensional Acceleration Waves," Arch. Rat. Mech. Anal., 19, p. 239, 1965.

11. Coleman, B. D., and Gurtin, M. E., "Waves in Materials with Memory, III. Thermodynamic Influences on the Growth and Decay of Acceleration Waves," Arch. Rat. Mech. Anal., 19, p. 266, 1965.

12. Coleman, B. D., and Gurtin, M. E., "Waves in Materials with Memory, IV. Thermodynamics and the Velocity of General Acceleration Waves," Arch. Rat. Mech. Anal., 19, p. 317, 1965.

13. Cristescu, N., "Dynamic Plasticity," North-Holland, 1967.

14. Moulton, F. R., "Differential Equations," Dover, 1930.

15. Morrison, J. A., "Waves in Rods of Voigt Material and Visco-Elastic Materials with Three-Parameter Models," Quart. Appl. Math., 14, p. 153, 1956.

16. Lubliner, J., "Short Time Approximations in Nonlinear Viscoelasticity, Int. J. Solids Structures, 3, p. 513, 1967.

17. Lubliner, J., "A Generalized Theory of Strain-Rate-Dependent Plastic Wave Propagation," J. Mech. Phys. Sol., 12, p. 59, 1964.

18. Bland, D. R., "On Shock Structure in a Solid," J. Inst. Maths. Applics. 1, p. 56, 1964.

19. Pipkin, A. C., "Shock Structure in a Viscoelastic Fluid," Quart. Appl. Math., 23, p. 297, 1966.

20. Glauz, R. D., and Lee, E. H., "Transient Wave Analysis in a Linear Time-Dependent Material," J. Appl. Phys., 25, p. 947, 1954.

21. Secor, G., and Lubliner, J., "The Propagation of Shock Waves in Semi-linear Viscoelastic Materials," Univ. of Calif. Tech. Report No. 66-12, 1966.

22. Secor, G., and Lubliner, J., "The Propagation of Shock Waves in Non-Linear Viscoelastic Materials," Univ. of Calif. Tech. Report No. 66-11, 1966.

23. Thomas, T. Y., "The Decay of Waves in Elastic Solids," J. Math. and Mech., 6, p. 759, 1957.

24. Craggs, J. W., "Progress in Solid Mechanics (ed. by I. N. Sneddon and R. Hill) Vol. II, North-Holland Publishing Co., p. 143, 1961.

25. Varley, E., "Acceleration Fronts in Viscoelastic Materials," Arch. Rat. Mech. Anal. 19, p. 215, 1965.

26. Greenberg, J. M., "The Existence of Steady Shock Waves in Nonlinear Materials with Memory," Arch. Rat. Mech. Anal., 24, p. 1, 1967.

27. Simmons, J. A., Hauser, S. and Dorn, J. E., "Mathematical Theories of Plastic Deformation Under Impulsive Loading," University of California Publications in Engineering, 5, p. 177, 1961.

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