Title
Virtual invariants of Quot schemes of surfaces

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Publication Date
2020

UNIVERSITY OF CALIFORNIA SAN DIEGO

Virtual invariants of Quot schemes of surfaces

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy

in

Mathematics

by

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2020

The Dissertation of Woonam Lim is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California San Diego

2020

DEDICATION

To my parents Hansung and Hyosook, and my sister Hyerin

EPIGRAPH

Wir müssen wissen.
Wir werden wissen.

David Hilbert

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ACKNOWLEDGEMENTS

There are many people who I would like to express my sincere thanks. First of all, I would like to thank my advisor Dragos Oprea for his guidance and support in the past years. I am deeply indebted to him ranging from his inspirational course on algebraic geometry that solely made up my mind to purse algebraic geometry to all the discussions we had that made this thesis possible. The lessons that I have learned from him will stay with me for the rest of my career.

I am very grateful to my parents for their constant support and unconditional trust in me throughout my life. It is my pleasure to dedicate the present work to them. I am also grateful to my girlfriend Mingjie Chen for her love and for sharing every moments with me. Thanks to her presence, my graduate school life had been always wonderful.

There are also many friends who have taken important part in my life at San Diego. I would particularly like to thank Justin Lacini, Iacopo Brivio, Samir Canning, Shubham Sinha, David Stapleton, Michail Savvas, Bochao Kong, Patrick Girardet for numerous seminars and discussions we had together. I would also like to thank Nandagopal Ramachandran, Sindhana P.S., Thomas Grubb, Jun Bo Lau, Jiaxi Nie, Yuwen Li, Suhan Zhong, Baiming Qiao, Zongze Liu, Wei Yin, Shanyang Fang, Xiudi Tang, Ray Kim, John Lee for being good friends of mine.

Chapters 2-6 are, in part, being prepared for submission for publication. The dissertation author was the primary investigator and the author of the material below.

- Woonam Lim "Virtual χ_{-y}-genera of Quot schemes on surfaces "

Chapters 2, 4, 7 and 8 are, in part, being prepared for submission for publication. The dissertation author was the collaborator and the coauthor for the material below.

- Noah Arbesfeld, Drew Johnson, Woonam Lim, Dragos Oprea, Rahul Pandharipande "The virtual K-theory of Quot schemes of surfaces"

I would like to thank all the collaborators – Noah Arbesfeld, Drew Johnson, Dragos Oprea, Rahul Pandharipande – for many helpful conversations and permitting to include the material in the paper to the thesis.

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ABSTRACT OF THE DISSERTATION

Virtual invariants of Quot schemes of surfaces

by

Woonam Lim

Doctor of Philosophy in Mathematics

University of California San Diego, 2020

Professor Dragos Oprea, Chair

Quot schemes are fundamental objects in the moduli theory of algebraic geometry. Quot schemes of surfaces admit natural perfect obstruction theories if we consider 1-dimensional quotients of trivial vector bundles. We study various virtual invariants of such Quot schemes using the structure of Seiberg-Witten invariants and Hilbert schemes of points.

The main result expresses the virtual Quot scheme invariants universally in terms of Seiberg-Witten invariants and certain cohomological data of a surface. When a curve class is of Seiberg-Witten length N, we prove the multiplicative structural formula for the generating series of equivariant virtual Quot scheme invariants in terms of the universal series and Seiberg-Witten invariants. Furthermore, the universal series are completely determined up to the change of variables. As an application, we prove the rationality of the homological and K-theoretic descendent series for any curve classes of Seiberg-Witten length N. Explicit formulas are available in several cases for specializations to the generating series of virtual Euler characteristics and virtual χ_{-y}-genera.

For $K3$ surfaces, the usual virtual Quot scheme invariants vanish. We thus define and study the reduced invariants of Quot schemes. Rather surprisingly, we show that the reduced χ_{-y}-genera of Quot schemes and Pair spaces are equal when $N = 1$. This implies that the reduced χ_{-y}-genera of Quot schemes are given by the Kawai-Yoshioka formula.

We study the virtual Segre and Verlinde series of Quot schemes as a variation of the non-virtual Segre and Verlinde series of Hilbert schemes of points. Analogously to the conjectural algebraicity of the Segre and Verlinde series of Hilbert schemes of points, we prove that the virtual Segre and Verlinde series of Quot schemes are algebraic for any curve classes of Seiberg-Witten length N. Furthermore, we prove the virtual Segre and Verlinde correspondence relating the universal series. This, in particular, matches the virtual Segre and Verlinde series for punctual Quot schemes up to a sign.

Chapter 1

Preliminaries

All schemes in this paper are finite type over $\mathbb{C}$. By a surface, we always mean smooth projective surfaces.

1.1 Introduction

Let S be a smooth projective surface. There are various sheaf-theoretic moduli spaces on S whose invariants have been studied. The simplest and most studied moduli space in this direction is the Hilbert scheme of points, denoted by $S^{[n]}$. The famous formula of Göttsche [G] computes their topological Euler characteristics:

$$\sum_{n=0}^{\infty} q^n e(S^{[n]}) = \left(\prod_{n=1}^{\infty} \frac{1}{1-q^n} \right)^{e(S)}. \tag{1.1}$$

This formula illustrates the common phenomenon in enumerative geometry; the structure of invariants of moduli spaces are best studied by putting them into a generating series. In the above case, it is the modularity that comes as a structure of the invariants. For virtual invariants of Quot schemes that we study in this paper, rationality will often appear as a structure.

There exist two different interpretations of Hilbert scheme of points. First, we can consider $S^{[n]}$ as the moduli of ideal sheaves I_Z. This point of view can be generalized to moduli space of higher rank Gieseker semistable sheaves of fixed determinant. Second, we can consider

$S^{[n]}$ as parametrizing zero dimensional quotients of the structure sheaf

$$0 \to I_Z \to \mathscr{O}_S \to \mathscr{O}_Z \to 0.$$

This point of view brings Grothendieck's Quot scheme as another natural direction of generalization.

Unlike the Hilbert scheme of points, none of the two generalizations above (moduli of sheaves, Quot schemes) are smooth in general. However, moduli spaces often have a notion of expected dimension coming from a natural obstruction theory. Under certain technical conditions, obstruction theory defines a virtual structure on moduli spaces with which one can define the ensuing virtual invariants. Specifically, one can define various virtual homological (or K-theoretic) invariants of moduli spaces using the virtual fundamental class (or virtual structure sheaf). These virtual invariants are extensively studied for moduli of sheaves under the name of (K-theoretic) Donaldson invariants, (K-theoretic) Vafa-Witten invariants, virtual Segre/Verlinde numbers. See Section 1.5 for a brief summary of existing results on virtual invariants of moduli of sheaves. In this paper, we study analogous theories for Quot schemes.

We end the introduction by briefly explaining the main technical tools used in this paper. Hilbert scheme of points $S^{[n]}$ and Hilbert scheme of curves Hilb_S^β are two basic building blocks, at least philosophically, of various sheaf-theoretic moduli spaces on S. Fundamental tool to study Hilbert scheme of points was developed in [EGL] where the authors use the recursive structure to obtain universality and multiplicativity. On the other hand, [DKO] linked Hilbert scheme of curves to the Seiberg-Witten invariants. Putting the philosophical comments in practice by his seminar work, Mochizuki [Mo] derives a formula for descendent invariants of moduli of sheaves on a surface with $p_g > 0$ in terms of integrals over the product of Hilbert schemes of points and Seiberg-Witten invariants. On the Quot scheme side, the dissertation author [L] derives an analogous formula for virtual invariants of Quot schemes via virtual localization of [GP]. We then apply this structural formula repeatedly to study various virtual invariants Quot schemes.

1.2 Virtual homological and K-theoretic invariants

Let M be a projective scheme with a cotangent complex $\mathbb{L}_M$. A 2-term perfect obstruction theory of M, in the sense of Behrend and Fantechi, is a 2-term perfect complex

$$E^\bullet = \left[E^{-1} \to E^0\right]$$

together with a morphism in derived category

$$\phi : E^\bullet \to \mathbb{L}_M$$

such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is a surjection. Denote the dual complex of $E^\bullet$ by

$$E_\bullet = [E_0 \to E_1].$$

Given M with a 2-term perfect obstruction theory ϕ, we define the virtual tangent bundle and virtual cotangent bundle in the K-theory $K^0(M)$ as

$$T_M^{\mathrm{vir}} := E_0 - E_1, \quad \Omega_M^{\mathrm{vir}} := E^0 - E^{-1}.$$

We also define the virtual dimension as

$$\mathrm{vd}(M) := \mathrm{rk}(T_M^{\mathrm{vir}}) = \mathrm{rk}(E_0) - \mathrm{rk}(E_1).$$

Note that these definitions depend on the choice of an obstruction theory ϕ which we omit from the notation.

Associated to a scheme with a 2-term perfect obstruction theory is the virtual fundamental class [BF, LT]

$$[M]^{\mathrm{vir}} \in A_{\mathrm{vd}(M)}(M),$$

and the virtual structure sheaf [BF, CFK, Lee]

$$\mathcal{O}_M^{\mathrm{vir}} \in K_0(M).$$

These virtual structures replace the role of the fundamental class and the structure sheaf. We define virtual homological invariants of M by integrating some cohomology class $\tau \in H^*(M)$ over the virtual fundamental class

$$\int_{[M]^{\mathrm{vir}}} \tau.$$

Similarly, we define virtual K-theoretic invariants of M by taking holomorphic Euler characteristic of some $\alpha \in K^0(M)$ tensored with the virtual structure sheaf

$$\chi^{\mathrm{vir}}(M,\alpha) := \chi(M, \alpha \otimes \mathcal{O}_M^{\mathrm{vir}}).$$

Virtual Hirzebruch-Riemann-Roch theorem of [FG] relates virtual K-theoretic invariants to virtual homological invariants:

$$\chi^{\mathrm{vir}}(M,\alpha) = \int_{[M]^{\mathrm{vir}}} \mathrm{ch}(\alpha)\,\mathrm{td}(T_M^{\mathrm{vir}}).$$

Now we review the virtual invariants of particular interest; the virtual Euler characteristic and the virtual χ_{-y}-genus. See [FG] for the detailed treatment. We define the virtual Euler characteristic of M as

$$e^{\mathrm{vir}}(M) := \int_{[M]^{\mathrm{vir}}} c_{\mathrm{vd}}(T_M^{\mathrm{vir}}) \in \mathbb{Z}.$$

The above definition is motivated from the Poincaré-Hopf theorem. Indeed, when the obstruction sheaf $h^1(E_\bullet)$ vanishes, M is smooth and $e^{\mathrm{vir}}(M)$ is equal to the topological Euler characteristic $e(M)$.

For any vector bundle E, we denote

$$\Lambda_{-y}E := \sum_{n=0}^{\infty} [\Lambda^n E]\,(-y)^n, \quad S_y E := \sum_{n=0}^{\infty} [S^n E]\, y^n$$

in the power series ring $K^0(M)[\![y]\!]$ over the K-theory. Note that this definition extends multiplicatively to virtual bundles, i.e. K-theory classes of M. Using this K-theoretic operation, we define the virtual χ_{-y}-genus as

$$\chi_{-y}^{\mathrm{vir}}(M) := \chi^{\mathrm{vir}}(M, \Lambda_{-y}\Omega_M^{\mathrm{vir}}) \in \mathbb{Z}[y].$$

It was shown in [BF] that $\chi_{-y}^{\mathrm{vir}}(M)$ is a polynomial in y of degree less than or equal to $\mathrm{vd}(M)$ that specializes to $e^{\mathrm{vir}}(M)$ at $y = 1$. In this sense, the virtual χ_{-y}-genus is a natural K-theoretic refinement of the virtual Euler characteristic. One can also define the shifted virtual χ_{-y}-genus as

$$\overline{\chi}_{-y}^{\mathrm{vir}}(M) := y^{-\frac{\mathrm{vd}(M)}{2}} \cdot \chi_{-y}^{\mathrm{vir}}(M) \in \mathbb{Z}[y^{\frac{1}{2}}, y^{-\frac{1}{2}}].$$

The virtual Serre duality of [FG] implies that the shifted virtual χ_{-y}-genus is symmetric under $y \leftrightarrow y^{-1}$.

1.3 Deformation theory of Quot schemes

Let X be a smooth projective variety. Fix a coherent sheaf E on X and a vector $v \in H^*(X, \mathbb{Q})$ encoding the topological data of quotients. Denote by $\mathsf{Quot}_X(E, v)$ the Quot scheme parametrizing quotients

$$0 \to S \to E \to Q \to 0 \quad \text{such that} \quad \mathrm{ch}(Q) = v.$$

Deformation theory of the quotients can be summarized as

$$\mathrm{Tan} = \mathrm{Hom}(S,Q), \quad \mathrm{Obs} = \mathrm{Ext}^1(S,Q).$$

If the higher obstruction spaces $\mathrm{Ext}^{>1}(S,Q)$ vanish for all quotients parametrized by the Quot scheme, then $\mathsf{Quot}_X(E,v)$ carries a 2-term perfect obstruction theory with a virtual tangent bundle given by

$$T^{\mathrm{vir}}_{\mathsf{Quot}_X(E,v)} = \mathrm{R}\mathscr{H}om_\pi(\mathscr{S},\mathscr{Q}).$$

Here $\mathscr{S}$ and $\mathscr{Q}$ are the universal subsheaf and quotient on $\mathsf{Quot}_X(E,v)\times X$, and π is the projection map onto $\mathsf{Quot}_X(E,v)$.

Vanishing of the higher Ext groups as above are relatively rare. Whenever it happens, however, it led to many interesting results. Here we list a few cases in low dimensions.

1. X curve, E arbitrary and v arbitrary,
2. X surface, E torsion free and v Chern character of torsion sheaf,
3. X surface, E satisfying $\mathrm{Hom}(E,E\otimes K_X)=0$ and v arbitrary.

In the first case, vanishing of $\mathrm{Ext}^{>1}(S,Q)$ follows from the dimension argument which was observed by [CFK] and [MO1]. Virtual intersection theory of Quot schemes over curves had important applications to the study of moduli of stable bundles of curves by Marian and Oprea, including the proof of strange duality [MO2] and recalculation of the Verlinde formula [MO3].

The second and third cases were observed in [MOP1] and [OP], respectively. In general, we have

$$\begin{aligned}
\mathrm{Ext}^2(S,Q)^\vee &= \mathrm{Hom}(Q,S\otimes K_X)\\
&\hookrightarrow \mathrm{Hom}(Q,E\otimes K_X)\\
&\hookrightarrow \mathrm{Hom}(E,E\otimes K_X)
\end{aligned}$$

by Serre duality for surfaces. Desired vanishing for the second case follows from the equality in the first line because Q is torsion and $S \otimes K_X$ is torsion free. Third case follows directly from the injectivity. In [MOP1], the authors further observed that Quot scheme of $K3$ surfaces admit reduced obstruction theory which they use to prove Lehn's conjecture for $K3$ surfaces. See Chapter 6 for reduced obstruction theory and reduced invariants of Quot schemes on $K3$ surfaces.

In [OP], Oprea and Pandharipande initiated the systematic study of virtual invariants of Quot schemes over surfaces in the second case. In particular, virtual Euler characteristics of Quot schemes on minimal surfaces of general type with $p_g > 0$ are studied in connections to the Seiberg-Witten invariants. Furthermore, reduced Euler characteristics of certain Quot schemes of $K3$ surfaces are shown to be governed by the Kawai-Yoshioka formula.

In this paper, we restrict our perspective to the second case with E being trivial vector bundle of arbitrary rank. More precisely, fix a smooth projective surface S with integers N, n and a curve class $\beta \in H^2(S, \mathbb{Z})$. Then we study Quot scheme $\mathsf{Quot}_S(\mathbb{C}^N, \beta, n)$ parametrizing torsion quotients

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_S \to Q \to 0 \quad \text{such that} \quad c_1(Q) = \beta, \quad \chi(Q) = n.$$

By Hirzebruch-Riemann-Roch calculation, we have

$$\mathrm{vd}(\mathsf{Quot}_S(\mathbb{C}^N, \beta, n)) = Nn + \beta^2.$$

In the case of Hilbert scheme of points

$$\mathsf{Quot}_S(\mathbb{C}^1, \beta = 0, n) \simeq S^{[n]},$$

there is a difference between the virtual dimension n and the actual dimension $2n$. This can be explained by the following obstruction bundle of rank n

$$\begin{aligned}
\mathscr{E}xt^1_\pi(I_{\mathscr{Z}}, \mathscr{O}_{\mathscr{Z}}) &= \mathscr{E}xt^2_\pi(\mathscr{O}_{\mathscr{Z}}, \mathscr{O}_{\mathscr{Z}}) \\
&= \Big(\mathscr{H}om_\pi(\mathscr{O}_{\mathscr{Z}}, \mathscr{O}_{\mathscr{Z}} \otimes K_S)\Big)^\vee \\
&= \Big(\mathscr{H}om_\pi(\mathscr{O}_S, \mathscr{O}_{\mathscr{Z}} \otimes K_S)\Big)^\vee \\
&= \Big(K_S^{[n]}\Big)^\vee
\end{aligned} \tag{1.2}$$

where $\mathscr{Z}$ denotes the universal subscheme in $S^{[n]} \times S$. This excess obstruction bundle will play the crucial role in the computations.

1.4 Seiberg-Witten invariants

Seiberg-Witten invariants has played a central role in various enumerative geometry of surfaces. For instance, see the following references for the relationship of Seiberg-Witten invariants with each topic; Donaldson invariants and Witten's conjecture [GNY, Mo], Gromov-Witten invariants [Ta1, Ta2], Stable pair invariants [K] and Vafa-Witten invariants [GK1, GK3, La1, La2, TT1, TT2]. Not surprisingly after all these developments, Quot scheme invariants can also be expressed by Seiberg-Witten invariants and integrals over the product of Hilbert schemes of points [L].

We review Seiberg-Witten invariants of surfaces. See [DKO] for the foundation of algebro-geometric Seiberg-Witten invariants. There are two ingredients to define Seiberg-Witten invariants; Hilbert scheme of curves and Abel-Jacobi map. We begin with the Hilbert scheme of curves and its virtual fundamental class.

Let S be a smooth projective surface and $\beta \in H^2(S, \mathbb{Z})$ be an effective curve class. Consider the moduli space Hilb^β_S parametrizing curves (or more precisely, effective divisors)

$$0 \to \mathscr{O}(-D) \to \mathscr{O}_S \to \mathscr{O}_D \to 0 \quad \text{such that} \quad c_1(\mathscr{O}(D)) = \beta.$$

As a fine moduli space, there is the universal curve

$$\mathscr{D}_\beta \hookrightarrow \mathsf{Hilb}_S^\beta \times S$$

with the associated ideal exact sequence

$$0 \to \mathscr{O}(-\mathscr{D}_\beta) \to \mathscr{O}_{\mathsf{Hilb}_S^\beta \times S} \to \mathscr{O}_{\mathscr{D}_\beta} \to 0.$$

Since Hilb_S^β parametrizes torsion quotients from the torsion free sheaf $\mathscr{O}_S$, it possesses a 2-term perfect obstruction theory with the virtual tangent bundle

$$T^{\mathrm{vir}}_{\mathsf{Hilb}_S^\beta} = \mathrm{R}\mathscr{H}om_\pi(\mathscr{O}(-\mathscr{D}_\beta), \mathscr{O}_{\mathscr{D}_\beta}) = \mathrm{R}\pi_*(\mathscr{O}_{\mathscr{D}_\beta}(\mathscr{D}_\beta)).$$

This induces a virtual fundamental class

$$[\mathsf{Hilb}_S^\beta]^{\mathrm{vir}} \in A_{\mathrm{vd}_\beta}(\mathsf{Hilb}_S^\beta), \quad \mathrm{vd}_\beta := \frac{\beta(\beta - K_S)}{2}.$$

Another ingredient for the definition is the Abel Jacobi map from the Hilbert scheme of curves to the Picard group

$$\mathsf{AJ}^\beta : \mathsf{Hilb}_S^\beta \to \mathsf{Pic}_S^\beta, \quad D \mapsto \mathscr{O}_S(D).$$

Denote the Poincaré line bundle over $\mathsf{Pic}_S^\beta \times S$ by $\mathscr{L}_\beta$ which we normalize at $p \in S$. By See-Saw principle, we have

$$\mathscr{O}(\mathscr{D}_\beta) = \mathscr{L}_\beta \otimes \mathscr{O}(1) \tag{1.3}$$

where $\mathscr{O}(1)$ denotes the line bundle over Hilb_S^β defined as

$$\mathscr{O}(1) := \mathscr{O}(\mathscr{D}_\beta)\Big|_{\mathsf{Hilb}_S^\beta \times \{p\}}.$$

We denote its first Chern class by

$$h := c_1(\mathscr{O}(1)) \in H^2(\mathsf{Hilb}_S^\beta, \mathbb{Z}).$$

Notation of $\mathscr{O}(1)$ comes from the fact that if β is positive enough, then the Abel-Jacobi map is naturally a projectivization with a relative ample line bundle $\mathscr{O}(1)$.

Putting these ingredients together, we define the higher Seiberg-Witten invariants as

$$\mathrm{SW}^k(\beta) := \mathsf{AJ}_*^\beta\left(h^k \cap \left[\mathsf{Hilb}_S^\beta\right]^{\mathrm{vir}}\right), \quad k \geq 0$$

which lives in

$$H_{2(\mathrm{vd}_\beta - k)}(\mathsf{Pic}_S^\beta, \mathbb{Z}) \simeq \Lambda^{2(\mathrm{vd}_\beta - k)} H^1(S, \mathbb{Z}).$$

For convenience, we also define the total Seiberg-Witten invariant

$$\mathrm{SW}(\beta) := \sum_{k \geq 0} \mathrm{SW}^k(\beta) \in \Lambda^* H^1(S, \mathbb{Z}).$$

Note that when $\mathrm{vd}_\beta = 0$, the only non-trivial Seiberg-Witten invariant is $\mathrm{SW}^0(\beta)$ for degree reasons. In this case, the total Seiberg-Witten invariant is simply a degree of a zero dimensional virtual fundamental class

$$\mathrm{SW}(\beta) = \mathrm{SW}^0(\beta) = \deg\left(\left[\mathsf{Hilb}_S^\beta\right]^{\mathrm{vir}}\right) \in \mathbb{Z}.$$

Behavior of the Seiberg-Witten invariants of a surface depends heavily on whether the underlying surface has a non-zero holomorphic 2-form or not. Most importantly, total Seiberg-Witten invariants of surfaces with $p_g > 0$ vanish unless $\mathrm{vd}_\beta = 0$. In other words, the only non-trivial Seiberg-Witten invariants of surfaces with $p_g > 0$ is a degree of the zero dimensional virtual fundamental class of Hilbert scheme of curves. In contrast, surfaces with $p_g = 0$ always have infinitely many curve classes β of non-zero virtual dimension whose total Seiberg-Witten invariant is non trivial. This property of the Seiberg-Witten invariants is responsible for the difficulty in studying virtual invariants of Quot schemes of surfaces with $p_g = 0$.

1.5 Virtual invariants of moduli of sheaves

Recall from the introduction that $S^{[n]}$ admits two different generalizations: moduli of sheaves and Quot schemes. In this section, we review some of the results on the moduli of sheaves in literatures about virtual χ_{-y}-genera [GK1, GK3] and virtual Segre and Verlinde numbers [GK4, GKW]. See also [GK2] for the detailed survey of a beautiful set of conjectures and results in this direction.

Let (S,H) be a smooth projective surface with a polarization H. For each $\rho \in \mathbb{Z}^+$, $c_1 \in H^2(S,\mathbb{Z})$, $c_2 \in H^4(S,\mathbb{Z}) \simeq \mathbb{Z}$, let

$$M = M_S^H(\rho, c_1, c_2)$$

be a moduli space parametrizing rank ρ Gieseker H-semistable sheaves E with $c_1(E) = c_1$ and $c_2(E) = c_2$. Through out this section, we make the following assumptions:

$$\begin{aligned} &1.\ S \text{ satisfies } b_1(S) = 0 \text{ and } p_g > 0, \\ &2.\ M \text{ does not contain strictly Gieseker } H-\text{semistable sheaves}, \\ &3.\ \text{There exists a universal sheaf } \mathscr{E} \text{ on } M \times S. \end{aligned} \tag{1.4}$$

Under these assumptions, the moduli space M admits a perfect obstruction theory with

$$T_M^{\mathrm{vir}} = \mathrm{R}\pi_* \mathrm{R}\mathscr{H}om(\mathscr{E},\mathscr{E})_0[1]$$

where π and q denotes the projection maps

$$
\begin{tikzcd}
 & M \times S \arrow[dl, "\pi"'] \arrow[dr, "q"] & \\
M & & S
\end{tikzcd}
$$

and the subscript 0 denotes the trace-free part. This in turn gives a virtual fundamental class

$$[M]^{\mathrm{vir}} \in A_{\mathrm{vd}}(M), \quad \mathrm{vd} = 2\rho c_2 - (\rho - 1)c_1^2 - (\rho^2 - 1)\chi(\mathscr{O}_S).$$

We first review the conjecture of [GK3] describing shifted virtual χ_{-y}-genera of moduli of rank 2 sheaves. To do so, we introduce the following functions:

$$
\begin{aligned}
\phi(x,y) &:= \prod_{n=1}^{\infty} \frac{1}{(1-x^{2n})^{10}(1-x^{2n}y)(1-x^{2n}y^{-1})}, \\
\theta_3(x,y) &:= \prod_{n=1}^{\infty} (1-x^{2n})(1+x^{2n-1}y)(1+x^{2n-1}y^{-1}), \\
\overline{\eta}(x) &:= \prod_{n=1}^{\infty} \frac{1}{1-x^n}.
\end{aligned}
$$

Conjecture ([GK3])**.** *Under the assumptions* (1.4)*, the shifted* χ_{-y}*-genus of* $M = M_S^H(2, c_1, c_2)$ *is given by the coefficient of* $x^{\mathrm{vd}(M)}$ *of*

$$4\left(\frac{1}{2}\phi(x,y)\right)^{\chi(\mathscr{O}_S)} \left(\frac{2\overline{\eta}(x^4)^2}{\theta_3(x,y^{\frac{1}{2}})}\right)^{K_S^2} \sum_{a \in H^2(S,\mathbb{Z})} \mathrm{SW}(a)(-1)^{a.c_1} \left(\frac{\theta_3(x,y^{\frac{1}{2}})}{\theta_3(-x,y^{\frac{1}{2}})}\right)^{a.K_S}.$$

Based on the above conjecture, the authors also conjecture the formula for the generating series

$$Z_{S,H,2,c_1}^{\mathrm{inst}}(q) := q^{-\frac{1}{4}\chi(\mathscr{O}_S)+\frac{1}{12}K_S^2} \sum_{c_2 \in \mathbb{Z}} \overline{\chi}_{-y}^{\mathrm{vir}}(M_S^H(2,c_1,c_2))q^{\frac{\mathrm{vd}}{4}}$$

which is related to the S-duality conjecture of [VW]. See also [GK1] for the analogous study on moduli of rank 3 sheaves.

We now review the virtual Segre numbers studied in [GK4]. Let $\alpha \in K^0(S)$ be a K-theory class of rank s. Over the moduli space $M = M_S^H(\rho, c_1, c_2)$, we define the Chern character of the tautological class

$$\mathrm{ch}(\alpha_M) := \mathrm{ch}(-\mathrm{R}\pi_*(\mathscr{E} \otimes \det(\mathscr{E})^{-\frac{1}{\rho}} \otimes q^*\alpha)) \in A^*(M)_{\mathbb{Q}}.$$

When the root $\det(\mathscr{E})^{-\frac{1}{\rho}}$ does not exist, we define the right hand side by Grothendieck-Riemann-Roch formula. By the universal relation between Chern character and Chern class, this in turn defines $c(\alpha_M)$. The authors conjecture the following structural formula for the virtual Segre numbers $\int_{[M]^{\mathrm{vir}}} c(\alpha_M)$ together with the explicit formulas for certain universal series.

Conjecture ([GK4])**.** *There exists*

$$\mathsf{V}_s\ ,\ \mathsf{W}_s\ ,\ \mathsf{X}_s\ ,\ \mathsf{Y}_s\ ,\ \mathsf{Z}_s\ ,\ \mathsf{Y}_{i,s}\ ,\ \mathsf{Z}_{ij,s} \in \mathbb{C}[\![z^{\frac{1}{2}}]\!] \quad \textit{for} \quad 1 \le i \le j \le \rho - 1$$

with the following properties.[1] *For any $M = M_S^H(\rho, c_1, c_2)$ under the assumptions* (1.4) *and $\alpha \in K^0(S)$ of rank s, the virtual Segre number $\int_{[M]^{\mathrm{vir}}} c(\alpha_M)$ is given by the coefficient of $z^{\frac{1}{2}\mathrm{vd}(M)}$ of*

$$\rho^{2-\chi(\mathscr{O}_S)+K_S^2}\mathsf{V}_s^{c_2(\alpha)}\mathsf{W}_s^{c_1(\alpha)^2}\mathsf{X}_s^{\chi(\mathscr{O}_S)}\mathsf{Y}_s^{c_1(\alpha).K_S}\mathsf{Z}_s^{K_S^2}\sum_{(a_1,\ldots,a_{\rho-1})}\prod_{i=1}^{\rho-1}\varepsilon_\rho^{ia_i.c_1}\mathrm{SW}(a_i)\mathsf{Y}_{i,s}^{c_1(\alpha).a_i}\prod_{1\le i\le j\le \rho-1}\mathsf{Z}_{ij,s}^{a_i.a_j}$$

where the sum is over all $(a_1,\ldots,a_{\rho-1}) \in H^2(S,\mathbb{Z})^{\rho-1}$ and $\varepsilon_\rho := e^{\frac{2\pi i}{\rho}}$. Moreover

$$\begin{aligned}
\mathsf{V}_s(z) &= \left(1+\left(1-\frac{s}{\rho}\right)t\right)^{1-s}\left(1+\left(2-\frac{s}{\rho}\right)t\right)^{s}\left(1+\left(1-\frac{s}{\rho}\right)t\right)^{\rho-1},\\
\mathsf{W}_s(z) &= \left(1+\left(1-\frac{s}{\rho}\right)t\right)^{\frac{1}{2}s-1}\left(1+\left(2-\frac{s}{\rho}\right)t\right)^{\frac{1}{2}(1-s)}\left(1+\left(1-\frac{s}{\rho}\right)t\right)^{\frac{1}{2}-\frac{1}{2}\rho},\\
\mathsf{X}_s(z) &= \left(1+\left(1-\frac{s}{\rho}\right)t\right)^{\frac{1}{2}s^2-s}\left(1+\left(2-\frac{s}{\rho}\right)t\right)^{-\frac{1}{2}s^2+\frac{1}{2}}\left(1+\left(1-\frac{s}{\rho}\right)\left(2-\frac{s}{\rho}\right)t\right)^{-\frac{1}{2}}\\
&\qquad\cdot\left(1+\left(1-\frac{s}{\rho}\right)t\right)^{-\frac{(\rho-1)^2}{2\rho}s},
\end{aligned}$$

[1] We suppress the dependence of these universal series on ρ.

where
$$z = t\left(1+\left(1-\frac{s}{\rho}\right)t\right)^{1-\frac{s}{\rho}}.$$
Furthermore, $\mathsf{Y}_s, \mathsf{Z}_s, \mathsf{Y}_{i,s}, \mathsf{Z}_{ij,s}$ *are algebraic functions for all* s, i, j*.*

We also review the virtual Verlinde numbers studied in [GK4]. Recall the determinant line bundle construction on $M = M_S^H(\rho, c_1, c_2)$ [HL, Chapter 8] from the orthogonal part of the K-theory classes
$$\lambda : K_{(\rho,c_1,c_2)} \to \mathsf{Pic}(M).$$
Fix $r \in \mathbb{Z}$ and $L \in \mathsf{Pic}(S)_{\mathbb{Q}}$ such that
$$\mathscr{L} := L \otimes c_1^{-\frac{r}{\rho}} \in \mathsf{Pic}(S), \quad \rho \ \text{ divides } \ \mathscr{L}.c_1 + r\left(\frac{c_1.(c_1-K_S)}{2} - c_2\right).$$
We define a line bundle $\mu(L) \otimes E^{\otimes r}$ on M as
$$\mu(L) \otimes E^{\otimes r} := \lambda(v)$$
where $v \in K_{(\rho,c_1,c_2)}$ is a K-theory class satisfying $\mathrm{rk}(v) = r$, $\det(v) = \mathscr{L}$. Note that $c_2(v) \in \mathbb{Z}$ is determined by the orthogonality condition with respect to (ρ, c_1, c_2). The authors conjecture the following structural formula for the virtual Verlinde numbers $\chi^{\mathrm{vir}}(M, \mu(L) \otimes E^{\otimes r})$ together with the explicit formulas for certain universal series.

Conjecture ([GK4])**.** *There exists*
$$\mathsf{G}_r\,,\ \mathsf{F}_r\,,\ \mathsf{A}_r\,,\ \mathsf{B}_r\,,\ \mathsf{A}_{i,r}\,,\ \mathsf{B}_{ij,r} \in \mathbb{C}[\![w^{\frac{1}{2}}]\!] \quad \textit{for} \quad 1 \le i \le j \le \rho - 1$$
with the following properties.[2] *For any* $M = M_S^H(\rho, c_1, c_2)$ *under the assumptions* (1.4) *and* $L \in \mathsf{Pic}(S)$*, the virtual Verlinde number* $\chi^{\mathrm{vir}}(M, \mu(L) \otimes E^{\otimes r})$ *is given by the coefficient of* $w^{\frac{1}{2}\mathrm{vd}(M)}$

[2] We suppress the dependence of these universal series on ρ.

of

$$\rho^{2-\chi(\mathscr{O}_S)+K_S^2}\,\mathsf{G}_r^{\chi(L)}\,F_r^{\frac{1}{2}\chi(\mathscr{O}_S)}\,\mathsf{A}_r^{L.K_S}\,\mathsf{B}_r^{K_S^2}\sum_{(a_1,\dots,a_{\rho-1})}\prod_{i=1}^{\rho-1}\varepsilon_\rho^{ia_i.c_1}\mathrm{SW}(a_i)\mathsf{A}_{i,r}^{a_i.L}\prod_{1\le i\le j\le\rho-1}\mathsf{B}_{ij,r}^{a_i.a_j}$$

where the sum is over all $(a_1,\dots,a_{\rho-1})\in H^2(S,\mathbb{Z})^{\rho-1}$ *and* $\varepsilon_\rho := e^{\frac{2\pi i}{\rho}}$. *Moreover*

$$\mathsf{G}_r(w) = 1+v,$$
$$\mathsf{F}_r(w) = (1+v)^{\frac{r^2}{\rho^2}}\left(1+\frac{r^2}{\rho^2}v\right)^{-1},$$

where

$$w = v(1+v)^{\frac{r^2}{\rho^2}-1}.$$

Furthermore, $\mathsf{A}_r, \mathsf{B}_r, \mathsf{A}_{i,r}, \mathsf{B}_{ij,r}$ *are algebraic functions for all* r, i, j.

Remark. There exists several evidence for the above conjectures. First of all, conjectures about virtual χ_{-y}-genera and virtual Verlinde numbers are true for $K3$ surfaces. This is because we may reduce the conjectures significantly using the fact that M is deformation equivalent to the Hilbert scheme of points. For Hilbert schemes of points on $K3$ surfaces, their χ_{-y}-genera [GSo] and Verlinde numbers [EGL] were previously known and these are compatible with the conjectures. Furthermore, Göttsche and Kool check all three conjectures for some numerics of ρ, s, r for various surfaces up to certain virtual dimensions, depending on the examples. Main technics of their computation include Mochizuki's formula [Mo], multiplicative universality argument for Hilbert schemes of points [EGL] and toric computations via localization. Surprising feature in their approach is that actual computations are done for toric surfaces although toric surfaces are not subject to the above conjectures as they are of $p_g = 0$.

Lastly, we review the surprising conjectural relationship between virtual Segre theory and virtual Verlinde theory. This extends the conjectural (non-virtual) Segre/Verlinde correspondence for Hilbert schemes of points by [J, MOP2]. Before we state the conjecture, we remark that a direct calculation shows that the universal series with explicit conjectural formulas in virtual Segre and Verlinde numbers are related as follows:

$$\mathsf{F}_r(w) = \mathsf{V}_s(z)^{\frac{s}{\rho}(\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})^2}\mathsf{W}_s(z)^{-\frac{4s}{\rho}}\mathsf{X}_s(z)^2,$$
$$\mathsf{G}_r(w) = \mathsf{V}_s(z)\mathsf{W}_s(z)^2,$$

where $s = \rho - r$ and

$$w = v(1+v)^{\frac{r^2}{\rho^2}-1}, \quad z = t\left(1+\left(1-\frac{s}{\rho}\right)t\right)^{1-\frac{s}{\rho}}, \quad v = t\left(1+\frac{r}{\rho}t\right)^{-1}. \tag{1.5}$$

The following conjecture expresses the relationship between the universal series in virtual Segre and Verlinde theory whose explicit formulas are not yet conjectured.

Conjecture ([GK4]). *For any $\rho > 0$, $r \in \mathbb{Z}$, $s = \rho - r$, and under the formal variable change* (1.5)*, we have*

$$\mathsf{A}_r(w^{\frac{1}{2}}) = \mathsf{W}_s(z)\mathsf{Y}_s(z^{\frac{1}{2}}), \qquad \mathsf{A}_{i,r}(w^{\frac{1}{2}}) = \mathsf{Y}_{i,s}(z^{\frac{1}{2}}),$$
$$\mathsf{B}_r(w^{\frac{1}{2}}) = \mathsf{Z}_s(z^{\frac{1}{2}}), \qquad \mathsf{B}_{ij,r}(w^{\frac{1}{2}}) = \mathsf{Z}_{ij,s}(z^{\frac{1}{2}}),$$

for all $1 \le i \le j \le \rho - 1$.

Chapter 2

Summary of results

In this chapter, we summarize the main results in this paper whose proofs and details can be found in the subsequent chapters.

This paper considers various virtual invariants of Quot schemes; virtual Euler characteristics, virtual χ_{-y}-genera, homological and K-theoretic descendent invariants and virtual Segre and Verlinde numbers. Here we introduce a general form of virtual invariants of Quot schemes that can specialize to all the cases of interest.

We first define the tautological classes of Quot schemes for descendent insertions. Let S be a smooth projective surface. For $\alpha \in K^0(S)$, we define the tautological class

$$\alpha^{[n]} := \mathrm{R}\pi_*(\mathscr{Q} \otimes q^*\alpha) \in K^0(\mathsf{Quot}_S(\mathbb{C}^N, \beta, n)).$$

Here $\mathscr{Q}$ is the universal quotient, and π, q are two projections.

Let $f(z) \in F[\![z]\!]$ be an invertible formal power series with coefficients in a field F. For our purposes, F will be a rational function field in several variables over $\mathbb{Q}$. For a vector bundle V over a scheme M, we define the following multiplicative characteristic of V

$$f(V) := \prod_i f(x_i) \in \Big(H^*(M, F)\Big)^\times$$

where x_i are Chern roots of V. This definition extends multiplicatively to K-theory

$$f : K^0(M) \to \Big(H^*(M,F)\Big)^{\times}$$

which we denote by the same letter f by abuse of notation.

Fix K-theory classes and invertible power series

$$\alpha_1,\dots,\alpha_\ell \in K^0(S), \quad f_1,\dots,f_\ell, g \in F[\![z]\!].$$

Consider the associated generating series of virtual Quot scheme invariants

$$Z^{\underline{f},g}_{S,N,\beta}(q\,|\,\underline{\alpha}) := \sum_{n\in\mathbb{Z}} q^n \int_{\left[\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)\right]^{\mathrm{vir}}} f_1(\alpha_1^{[n]})\cdots f_\ell(\alpha_\ell^{[n]})\cdot g\Big(T^{\mathrm{vir}}_{\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)}\Big). \tag{2.1}$$

The underline notation will be used to denote vector of integers, curve classes, K-theory classes, power series, etc. As virtual dimension $Nn+\beta^2$ becomes negative for sufficiently negative n, this generating series is a formal Laurent series in the variable q

$$Z^{\underline{f},g}_{S,N,\beta}(q\,|\,\underline{\alpha}) \in F(\!(q)\!).$$

The above general expression specializes to the generating series of each case of interest as follows:

1. virtual Euler characteristics: $\ell = 0$ and $g(z) = 1+z$,

2. virtual χ_{-y}-genera: $\ell = 0$ and $g(z) = \frac{z(1-ye^{-z})}{1-e^{-z}}$,

3. (master) homological descendent invariants: $f_s(z) = 1+x_s z$ and $g(z) = 1+z$,

4. (master) K-theoretic descendent invariants: $f_s(z) = 1+x_s e^{\varepsilon_s z}$ and $g(z) = \frac{z(1-ye^{-z})}{1-e^{-z}}$,

5. virtual Segre numbers: $f(z) = \frac{1}{1+z}$ and $g(z) = 1$,

6. virtual Verlinde numbers: $f(z) = e^z$ and $g(z) = \frac{z}{1-e^{-z}}$.

2.1 Universality of Quot scheme invariants

In this section, we explain the universal nature of the virtual invariants of Quot schemes appearing in the generating series (2.1). Roughly speaking, the universality here refers to certain independence of these invariants on the underlying surface. Consider an integral

$$\int_{\left[\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)\right]^{\mathrm{vir}}} f_1(\alpha_1^{[n]}) \cdots f_\ell(\alpha_\ell^{[n]}) \cdot g\left(T^{\mathrm{vir}}_{\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)}\right). \tag{2.2}$$

By virtual localization of [GP], computation of this integral can be recovered from the residual contributions of the each fixed loci

$$F[\underline{m},\underline{\beta}] = \left(S^{[m_1]} \times \cdots \times S^{[m_N]}\right) \times \left(\mathsf{Hilb}_S^{\beta_1} \times \cdots \times \mathsf{Hilb}_S^{\beta_N}\right)$$

with respect to a $\mathbb{C}^*$-action. See Section 3.3 for the definition of the residual contributions.

To state the first theorem about universality of the residual invariants, we introduce a few notations. Consider cohomology classes $\gamma \in H^{2*}(S)$ given by monomials in

$$c_1(\alpha_s)\ ,\ c_2(\alpha)\ ,\ \beta_i\ ,\ c_1(T_S)\ ,\ c_2(T_S).$$

For such γ's and $t_1, \ldots, t_N \geq 0$ satisfying $\deg(\gamma) + \sum t_i = 4$, we define

$$[\gamma]_{t_1,\ldots,t_N} \in \Lambda^{t_1} H^1(S)_1^\vee \otimes \cdots \otimes \Lambda^{t_N} H^1(S)_N^\vee \subseteq H^*(\mathsf{Pic}_S^{\underline{\beta}})$$

in Section 3.6. Since (higher) Seiberg-Witten invariants are defined in the homology of the Picard groups, it makes sense to integrate a polynomial in variables of the form $[\gamma]_{t_1,\ldots,t_N}$ across the product of Seiberg-Witten invariants as in the theorem below. Also, see (3.9) for the definition of $r_{s,i}, r_i, r_{ij}$ in the statement below.

Theorem A. *There exists a collection of the universal power series*

$$\mathsf{D}_{\underline{k},\underline{m}} \in F(r_{s,i}, r_i, r_{ij})((w_1,\ldots,w_N))[\![*]\!]$$

in the variables of the form $[\gamma]_{t_1,\ldots,t_N}$ *as defined above with the following properties.*[1] *The residual contribution of* $F[\underline{m},\underline{\beta}]$ *to the integral* (2.2) *is given by*

$$\prod_{s,i} f_s(w_i)^{\chi(\alpha_s)-r_{s,i}} \cdot \prod_i g(0)^{r_i-r_{ii}} \cdot \prod_{i\neq j}\left(\frac{g(w_j-w_i)}{w_j-w_i}\right)^{r_i-r_{ij}} \cdot \sum_{k_1,\ldots,k_N\geq 0}\int_{\mathrm{SW}^{\underline{k}}(\underline{\beta})} \mathsf{D}_{\underline{k},\underline{m}}.$$

Furthermore, $\mathsf{D}_{\underline{k},\underline{m}}$ *only depends on* $N,\underline{f},g,\underline{k},\underline{m}$ *and ranks of* α_s.

Remark. The crux of the matter of Theorem A is that $\mathsf{D}_{\underline{k},\underline{m}}$ is independent on the surface S and the decomposition $\underline{\beta}=(\beta_1,\ldots,\beta_N)$ of the curve class. The only non-trivial inputs to the formula are higher Seiberg-Witten invariants and various constructions using cohomology ring $H^*(S)$ and Chern class inputs from α_s,β_i and T_S.

2.2 Multiplicative structural formula

Even though Theorem A is helpful for understanding Quot scheme invariants conceptually, it is not suited for the computations. When Seiberg-Witten theoretic aspect of the curve class is simplified, we obtain the multiplicative structural formula for the generating series of equivariant Quot scheme invariants. The structural formula can be effectively used for various types of Quot scheme invariants.

Let β be a curve class on S. We say that β is of Seiberg-Witten length N if for every decomposition

$$\beta=\beta_1+\cdots+\beta_N \quad \text{with} \quad \mathrm{SW}(\beta_i)\neq 0 \ \text{ for all } \ i=1,\ldots,N,$$

[1] Double bracket $[\![*]\!]$ is a place for the variables $[\gamma]_{t_1,\ldots,t_N}$ for various γ and $t_1,\ldots,t_N$.

it satisfies

$$\mathrm{vd}_{\beta_i} = 0 \ \text{ for all } \ i = 1, \ldots, N.$$

There are various examples of Seiberg-Witten classes of length N. Most importantly, any curve class on a surface with $p_g > 0$ is of Seiberg-Witten length N for all $N \geq 1$. See Section 4.1 for more examples. When curve class is of Seiberg-Witten length N, we have a multiplicative structural formula for the generating series of equivariant Quot scheme invariants.

Theorem B. *There exist universal series*

$$\mathsf{A}\ ,\ \mathsf{B}_s\ ,\ \mathsf{U}_i\ ,\ \mathsf{V}_{i,s}\ ,\ \mathsf{W}_{i,j}\ \in\ \Big(F((w_1, \ldots, w_N))[\![q]\!]\Big)^{\times}$$

that only depend on $N, \underline{f}, g, \underline{r}$ with the following properties. Let β be a curve class of Seiberg-Witten length N and let $\alpha_1, \ldots, \alpha_\ell$ be K-theory classes with $\mathrm{rk}(\alpha_s) = r_s$. *Then the generating series of equivariant Quot scheme invariants is given by*

$$Z^{f,g}_{S,N,\beta}(q\,|\,\underline{\alpha}) = q^{-\beta.K_S} \cdot \mathsf{A}^{K_S^2} \cdot \prod_s \mathsf{B}_s^{K_S.c_1(\alpha_s)} \cdot \sum_{\substack{\beta=\beta_1+\cdots+\beta_N \\ \mathrm{vd}_{\beta_i}=0}} \mathrm{SW}(\beta_1)\cdots\mathrm{SW}(\beta_N) \cdot \prod_i \mathsf{U}_i^{\beta_i.K_S} \cdot \prod_{i,s} \mathsf{V}_{i,s}^{\beta_i.c_1(\alpha_s)} \cdot \prod_{i<j} \mathsf{W}_{i,j}^{\beta_i.\beta_j}.$$

In addition to the structural formula, we in fact determine all the universal series up to the change of variables

$$q = \prod_s \frac{1}{f_s(-H_i)^{r_s}} \cdot \prod_{k=1}^{N} \frac{-H_i - w_k}{g(-H_i - w_k)} \quad \text{with} \quad H_i(q=0) = -w_i. \tag{2.3}$$

Theorem C. *The universal series in Theorem B are explicitly given by formulas in Section 4.4 with respect to the change of variables* (2.3).

To give some flavor of the shape of the formulas, we record one of them here:

$$\mathsf{A} = g(0)^N \cdot \prod_s \prod_{i=1}^N \frac{f_s(-H_i)^{r_s}}{f_s(w_i)^{r_s}} \cdot \prod_{i,k\in[N]} \frac{H_i + w_k}{g(H_i+w_k)} \cdot \prod_{\substack{i_1\neq i_2 \\ i_1,i_2\in[N]}} \frac{g(H_{i_1}-H_{i_2})}{H_{i_1}-H_{i_2}}$$
$$\cdot \prod_{i=1}^N \left(\sum_s r_s \cdot \frac{f_s'}{f_s}(-H_i) + \sum_{k=1}^N \left(\frac{g'}{g}(-H_i - w_k) + \frac{1}{H_i+w_k} \right) \right).$$

It is worth mentioning the idea behind these computations. The key ingredient is a cosection of the obstruction bundle corresponding to the non-zero holomorphic 2-form $\theta \in H^0(S, K_S)$, if exists. At each point on the Quot scheme, the cosection is defined as a composition

$$\sigma : \mathrm{Ext}^1(S,Q) \xrightarrow{\delta} \mathrm{Ext}^2(Q,Q) \xrightarrow{\mathrm{trace}} H^2(\mathscr{O}_S) \xrightarrow{\int_S \theta \cup -} \mathbb{C}.$$

One can check by Serre duality that σ is zero if and only if the quotient sheaf Q is scheme theoretically supported on the canonical curve $C_\theta := \mathrm{Zero}(\theta)$. Therefore the zero locus of the cosection σ is, at least set theoretically, identified with a Quot scheme over the canonical curve C_θ. By the cosection localized class construction of [KL], computation can be expected to be localized on the Quot scheme over a curve C_θ.

Applying the above lines of idea is promising, but we follow a slightly different and more effective path. We work with the convenient geometry, namely a blow up of a $K3$ surface at a point, which determines all the universal series. In this case, the unique canonical curve is the exceptional divisor $E \simeq \mathbb{P}^1$. Using the equivariant localization and the Euler class of the obstruction bundle, we express the Quot scheme invariants on the surface by certain integrations over the products of Hilbert schemes of points on E. The generating series of such invariants is then of the form to which we can naturally apply the Lagrange-Bürmann formula. The change of variables (2.3) appears naturally in this procedure. See Chapter 4 for details.

2.3 Virtual Euler characteristics and χ_{-y}-genera

The study of invariants of various moduli spaces has its long history. Amongst others, Euler characteristics of moduli of sheaves on surfaces were of particular interest due to S-duality conjecture of Vafa and Witten [VW]. Based on their conjecture, certain modular property is expected for the generating series of Euler characteristics of moduli of shaves on surfaces. The first instance of this was Göttsche's formula (1.1). Subsequently, higher rank analogues of this formula also appeared in several cases when the moduli spaces are smooth of expected dimension.

Recently, Vafa-Witten theory in mathematics has undergone rapid development. Tanaka and Thomas [TT1, TT2] laid a foundation of the theory by giving a mathematical definition of Vafa-Witten invariants as residual virtual Euler characteristics of moduli of semistable Higgs sheaves. K-theoretic refinement of these invariants can also be defined using the twisted virtual structure sheaf [T]. The Vafa-Witten invariants consist of two different contributions: instanton branches and monopole branches. It is proven that the instanton branch contributions to the K-theoretic Vafa-Witten invariants are the (signed) shifted virtual χ_{-y}-genera of moduli of sheaves which is explained in Section 1.5. On the other hand, vertical components of the monopole branch contributions are studied by Laarakker [La1, La2] using the theory of nested Hilbert schemes.

Motivated in part from the Vafa-Witten theory, Oprea and Pandharipande [OP] initiated the study of virtual Euler characteristics of Quot schemes. As a result of the computations in various examples, the authors conjecture rationality, opposed to modularity in the Vafa-Witten theory, of the generating series of virtual Euler characteristics of Quot schemes. The dissertation author [L] proves that the generating series of virtual χ_{-y}-genera, a K-theoretic refinement of the virtual Euler characteristic, of Quot schemes are rational functions for arbitrary surfaces with $p_g > 0$.

Denote the generating series of virtual χ_{-y}-genera of Quot schemes as

$$Z^{\chi^{\mathrm{vir}}_{-y}}_{S,N,\beta}(q) := \sum_{n\in\mathbb{Z}} q^n \cdot \chi^{\mathrm{vir}}_{-y}(\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)) \in \mathbb{Q}(y)((q)).$$

As noted in Section 1.2, this specializes to the virtual Eulercharacteristic series at $y=1$. By the virtual Hirzebruch-Riemann-Roch theorem of [FG], this corresponds to the general series $Z^{f,g}_{S,N,\beta}(q\,|\,\underline{\alpha})$ with

$$\ell = 0, \quad g(z) := \frac{z(1-ye^{-z})}{1-e^{-z}}.$$

Applying Theorem B and Theorem C to this specialization, we obtain the explicit answers for several cases with simple Seiberg-Witten theories.

To state the theorem, we introduce some functions. For each partition $I\sqcup J$ of a set $[N]:=\{1,\dots,N\}$, we define a rational function

$$\mathsf{M}_{I\sqcup J} := (-N)^{N-|I|}\cdot\prod_{j\in J}\left(\frac{-z_j}{1-(1+y)z_j}\right)^N\cdot\prod_{j\in J}\frac{(1-z_j)(1-yz_j)}{z_j}\cdot\prod_{\substack{j_1\neq j_2\\ j_1,j_2\in J}}\frac{1-(1+y)z_{j_1}+yz_{j_1}z_{j_2}}{z_{j_2}-z_{j_1}}$$

where $z_1,\dots,z_N$ are distinct roots of $q=z^N$. By using these rational functions, we define new rational functions

$$\overline{\mathsf{A}} := \mathsf{M}_{\emptyset\sqcup[N]}, \quad \overline{\mathsf{G}}_{N,\ell,g} := \sum_{\substack{I\sqcup J=[N]\\ \text{s.t. } |I|=\ell}}\left(\mathsf{M}_{I\sqcup J}\right)^{g-1},$$

for each $0\le\ell\le N$ and $g\ge 0$. It is clear from the definition that these are symmetric in $z_1,\dots,z_N$. Therefore we can regard them as a rational function in the q variable

$$\overline{\mathsf{A}},\ \overline{\mathsf{G}}_{N,\ell,g}\in\mathbb{Q}(y)(q)$$

by symmetric reduction algorithm. Bar notations over $\overline{\mathsf{A}}$, $\overline{\mathsf{G}}_{N,\ell,g}$ are to distinguish them from their equivariant analogues in Chapter 5.

Theorem D. *In each case below, the generating series of virtual χ_{-y}-genera of Quot schemes are given by explicit rational functions.*

1. *S arbitrary, $\beta = 0$:*
$$Z^{\chi^{\mathrm{vir}}_{-y}}_{S,N,0}(q) = \overline{\mathsf{A}}^{K_S^2},$$

2. *S relatively minimal elliptic surface, β rational multiple of the fiber :*
$$Z^{\chi^{\mathrm{vir}}_{-y}}_{S,N,\beta}(q) = \sum_{\substack{\beta=\beta_1+\cdots+\beta_N \\ \mathrm{vd}_{\beta_i}=0}} \mathrm{SW}(\beta_1)\cdots\mathrm{SW}(\beta_N)\,,$$

3. *S minimal surface of general type with $p_g > 0$, $\beta = \ell K_S$ for some $0 \le \ell \le N$:*
$$Z^{\chi^{\mathrm{vir}}_{-y}}_{S,N,\ell K_S}(q) = q^{-\ell K_S^2} \cdot \mathrm{SW}(K_S)^{\ell} \cdot \overline{\mathsf{G}}_{N,\ell,g}\,,$$
where $g = K_S^2 + 1$ is the arithmetic genus of the canonical curve of S.

Remark. All Seiberg-Witten invariants in Theorem D are known. For minimal surface S of general type with $p_g > 0$, it was conjectured in [DKO] and proven in [CK] that $\mathrm{SW}(K_S) = (-1)^{\chi(\mathscr{O}_S)}$. See Section 4.1 for the relatively minimal elliptic surface cases.

The theorem below can be used to extend the formulas in Theorem D to non minimal surfaces via blow up formula.

Theorem E. *Let $\widetilde{S} \to S$ be a blow up of a point with the exceptional divisor E. Assume that a curve class β is of Seiberg-Witten length N. For each $0 \le \ell \le N$, we have*
$$Z^{\chi^{\mathrm{vir}}_{-y}}_{\widetilde{S},N,\widetilde{\beta}+\ell E}(q) = q^{\ell} \cdot \overline{\mathsf{G}}_{N,\ell,0} \cdot Z^{\chi^{\mathrm{vir}}_{-y}}_{S,N,\beta}(q).$$

Theorem D and Theorem E are sufficient to conclude that the generating series of virtual χ_{-y}-genera of Quot schemes are rational in the q variable for arbitrary surfaces with $p_g > 0$. However, we explain more general rationality statement in Section 2.5.

2.4 Reduced invariants of $K3$ surfaces

Surfaces with trivial canonical bundle often have zero enumerative invariants due to the trivial factor in the obstruction spaces. In such cases, one can attempt to define non-trivial invariants by reducing the obstruction theory. For $K3$ surfaces, reduced classes and the ensuing invariants have been studied in Gromov-Witten theory, and stable pair theory. See for instance [BL, KT1, KT2, MPT]. Since Yau and Zaslow [YZ] initiated the curve counting theory on $K3$ surfaces, there has been a significant development in enumerative geometry of $K3$ surfaces culminating in the proof of Katz-Klemm-Vafa conjecture [PT2].

Keeping with a theme of the paper, we study the reduced invariants of Quot schemes on $K3$ surfaces. For a $K3$ surface S and a non-zero effective curve class β, we consider a Quot scheme

$$\mathsf{Quot}_S(\mathbb{C}^1,\beta,n) \simeq S_\beta^{[m,0]}$$

which admits a reduced obstruction theory. Through the above isomorphism, their reduced virtual structures are also identified. On the other hand, one can also consider a Pair space

$$\mathsf{Pair}_S(\mathbb{C}^1,\beta,n) \simeq S_\beta^{[0,m]}$$

that also admits a reduced obstruction theory identified through the isomorphism.

Under certain positivity and irreducibility condition on β, surprising identity

$$e^{\mathrm{red}}(\mathsf{Quot}_S(\mathbb{C}^1,\beta,n)) = e(\mathsf{Pair}_S(\mathbb{C}^1,\beta,n))$$

between reduced invariants of Quot schemes and Pair spaces was found by [OP]. The proof of [OP] can be modified to yield the same identity for the K-theoretic refinements [L]. In this paper, we obtain the following theorem without any assumption on the curve class.

Theorem F. *Let S be a $K3$ surface and β be a non-zero effective curve class. Then we have*

$$\chi^{\mathrm{red}}_{-y}(\mathsf{Quot}_S(\mathbb{C}^1,\beta,n)) = \chi^{\mathrm{red}}_{-y}(\mathsf{Pair}_S(\mathbb{C}^1,\beta,n)).$$

Furthermore, these invariants depend upon β only through the intersection number $\beta^2 = 2g-2$.

Remark. In this paper, we in fact obtain the generalized version of Theorem F on the level of reduced algebraic cobordism classes. See Remark 18.

By Theorem F, the reduced χ_{-y}-genus of $\mathsf{Quot}_S(\mathbb{C}^1,\beta,n)$ can be computed by choosing any $K3$ surface S' with a curve class β' whose arithmetic genus is equal to that of β. We can always find such a pair with irreducible β'. Under the irreducibility assumption, the Pair space is smooth as the reduced obstruction space vanishes. In such cases, Hodge numbers of Pair spaces $\mathsf{Pair}_{S'}(\mathbb{C}^1,\beta',n)$ are computed by Kawai and Yoshioka [KY]. See (6.7) for the Kawai-Yoshioka formula specialized to χ_{-y}-genus. These observations lead us to the following theorem.

Theorem G. *Let S be a $K3$ surface and β be a non-zero effective curve class of arithmetic genus g. Then the generating series of reduced shifted χ_{-y}-genera of Quot schemes*

$$Z^{\overline{\chi}^{\mathrm{red}}_{-y}}_{S,1,\beta}(t) := \sum_{n\in\mathbb{Z}} t^n \cdot \overline{\chi}^{\mathrm{red}}_{-y}\big(\mathsf{Quot}_S(\mathbb{C}^1,\beta,n)\big)$$

is given by the coefficient of q^g of

$$\frac{1}{\left(t+\frac{1}{t}-\sqrt{y}-\frac{1}{\sqrt{y}}\right)} \cdot \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{18}(1-q^ny)(1-q^n/y)(1-q^n\sqrt{y}/t)(1-q^nt/\sqrt{y})(1-q^nt\sqrt{y})(1-q^n/t\sqrt{y})}.$$

In particular, the generating series is rational in the t variable.

2.5 Rationality of descendent series

Study of the generating series of virtual Euler characteristics (or virtual χ_{-y}-genera) of Quot schemes can be extended to the homological (or K-theoretic) descendent series. Given a collection of K-theory classes and integers

$$\alpha_1, \ldots, \alpha_\ell \in K^0(S), \quad k_1, \ldots, k_\ell \geq 0,$$

we define a homological descendent series

$$Z^{\mathsf{H}}_{S,N,\beta}(q \,|\, \underline{\alpha}, \underline{k}) := \sum_{n \in \mathbb{Z}} q^n \int_{\left[\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)\right]^{\mathrm{vir}}} c_{k_1}(\alpha_1^{[n]}) \cdots c_{k_\ell}(\alpha_\ell^{[n]}) \cdot c\left(T^{\mathrm{vir}}_{\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)}\right),$$

and a K-theoretic descendent series

$$Z^{\mathsf{K}}_{S,N,\beta}(q \,|\, \underline{\alpha}, \underline{k}) := \sum_{n \in \mathbb{Z}} q^n \cdot \chi^{\mathrm{vir}}\left(\mathsf{Quot}_S(\mathbb{C}^N, \beta, n), \Lambda^{k_1}(\alpha_1^{[n]})^{\varepsilon_1} \otimes \cdots \otimes \Lambda^{k_\ell}(\alpha_\ell^{[n]})^{\varepsilon_\ell} \otimes \Lambda_{-y}\Omega^{\mathrm{vir}}_{\mathsf{Quot}}\right).$$

Here $\varepsilon_1, \ldots, \varepsilon_\ell$ are ± 1 that we omit from the notation. When there are no tautological insertions, i.e. $\ell = 0$ cases, these series specialize to the series of virtual Euler characteristics and virtual χ_{-y}-genera, respectively.

The following rationality conjecture was one of the driving forces in recent developments of virtual geometry of Quot schemes.

Conjecture. *The homological descendent series and the K-theoretic descendent series are the Laurent expansion of rational functions in q.*

The conjecture is due to several papers: [OP] for the generating of virtual Euler characteristics, [JOP] for the homological descendent series and [AJLOP] for the K-theoretic descendent series. When curve class is of Seiberg-Witten length N, the affirmative answer for the conjecture is given in [AJLOP]. In loc. cit., proof was explicitly written for the K-theoretic descendent series without $\Lambda_{-y}\Omega^{\mathrm{vir}}_{\mathsf{Quot}}$ factor and it was noted that the same method applies to general homological and K-theoretic descendent series. In this paper, we spell out the necessary details for the claimed proof in [AJLOP] for both homological and K-theoretic descendent series.

Theorem H. *Let S be a surface and β be a curve class of Seiberg-Witten length N. Then the homological and K-theoretic descendent series are given by rational functions in the q variable*:

$$Z^{\mathsf{H}}_{S,N,\beta}(q\,|\,\underline{\alpha},\underline{k}) \in \mathbb{Q}(q), \quad Z^{\mathsf{K}}_{S,N,\beta}(q\,|\,\underline{\alpha},\underline{k}) \in \mathbb{Q}(y)(q).$$

Besides curve classes of Seiberg-Witten length N, little is known for the rationality conjecture. When $N = 1$, evidences for the rationality conjecture include the followings: for a simply connected surface S and any curve class β, we have

$$\begin{aligned} Z^{e^{\mathrm{vir}}}_{S,1,\beta}(q) &:= \sum_{n\in\mathbb{Z}} q^n \cdot e^{\mathrm{vir}}(\mathsf{Quot}_S(\mathbb{C}^1,\beta,n)) \in \mathbb{Q}(q) \quad \text{[JOP]}, \\ Z^{\mathsf{K}}_{S,1,\beta}(q\,|\,\alpha,k=1)\Big|_{y=0} &:= \sum_{n\in\mathbb{Z}} q^n \cdot \chi^{\mathrm{vir}}(\mathsf{Quot}_S(\mathbb{C}^1,\beta,n),\alpha^{[n]}) \in \mathbb{Q}(q) \quad \text{[AJLOP]}. \end{aligned}$$

Remark. We remark that the rationality phenomenon is not always guaranteed for virtual geometry of Quot schemes. For example, it was proven in [AJLOP] that the generating series of the cobordism classes are not given by a rational function, in general. In a similar direction, one can check with the same techniques as in this paper that the rationality does not occur for the following modified descendent series:

$$\begin{aligned} \widetilde{Z}^{\mathsf{H}}_{S,N,\beta}(q\,|\,\underline{\alpha},\underline{k}) &:= \sum_{n\in\mathbb{Z}} q^n \int_{\left[\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)\right]^{\mathrm{vir}}} c_{k_1}(\alpha_1^{[n]})\cdots c_{k_\ell}(\alpha_\ell^{[n]})\cdot s\Big(T^{\mathrm{vir}}_{\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)}\Big), \\ \widetilde{Z}^{\mathsf{K}}_{S,N,\beta}(q\,|\,\underline{\alpha},\underline{k}) &:= \sum_{n\in\mathbb{Z}} q^n \cdot \chi^{\mathrm{vir}}\Big(\mathsf{Quot}_S(\mathbb{C}^N,\beta,n),\Lambda^{k_1}(\alpha_1^{[n]})^{\varepsilon_1}\otimes\cdots\otimes\Lambda^{k_\ell}(\alpha_\ell^{[n]})^{\varepsilon_\ell}\otimes\Lambda_{-y}T^{\mathrm{vir}}_{\mathsf{Quot}}\Big). \end{aligned}$$

The only differences from the previous definitions are that we replaced the total Chern class by the total Segre class and replaced the virtual cotangent bundle by the virtual tangent bundle. For these modifications, the answers are expected to be only algebraic functions. These non rational examples make the above rationality theorem more mysterious.

2.6 Virtual Segre/Verlinde series

Consider a smooth projective surface S and a K-theory class $\alpha \in K^0(S)$ of rank r. Using the usual, i.e. non-virtual, geometry of Hilbert scheme of points, we define the (non-virtual) Segre series as

$$\mathsf{S}_{\alpha}^{\mathsf{Hilb}}(q) := \sum_{n=0}^{\infty} q^n \int_{S^{[n]}} s(\alpha^{[n]}).$$

Major driving force in the study of the Segre series was Lehn's conjecture computing the Segre series with some exotic change of variable when $r = 1$. This is now proven by work of several papers: K-trivial surfaces by [MOP1] and the general surfaces by [MOP1, MOP2, V]. See also [MOP3] for developments in higher rank Segre series where the authors obtain the formulas for all K-trivial surfaces in arbitrary rank r and for the general surfaces in relatively small rank $r \in \{\pm 1, \pm 2\}$. They further conjecture the algebraicity of the Segre series for arbitrary ranks.

With the same set of data, we can also define the (non-virtual) Verlinde series as

$$\mathsf{V}_{\alpha}^{\mathsf{Hilb}}(q) := \sum_{n=0}^{\infty} q^n \cdot \chi(S^{[n]}, \det \alpha^{[n]}).$$

This series was first studied in [EGL] where they determine the formulas for all K-trivial surfaces in arbitrary rank r and for the general surfaces in small ranks $r \in \{0, \pm 1\}$. Since the Picard group of $S^{[n]}$ consists only of the determinant line bundles as above, the study of the Verlinde series is self-justified. Apart from this, the Verlinde numbers can also be used for testing numerical strange duality for surfaces.

As much of a surprise, interesting conjectural connection between the Segre and Verlinde series was suggested by Johnson [J], motivated from strange duality. Based on his work, conjectural relations, called Segre/Verlinde correspondence, between the universal series in the Segre series and the Verlinde series are formulated in [MOP3].

As explained in the introduction, most of the theory developed for Hilbert scheme of points can be generalized into two directions: virtual geometry of moduli of sheaves and Quot schemes. See Section 1.5 for the review of the virtual Segre and Verlinde theory for moduli of sheaves. Here we summarize instead the virtual Segre and Verlinde theory for Quot schemes studied in [AJLOP].

Using the virtual geometry of Quot schemes, we define the virtual Segre and Verlinde series as

$$\mathsf{S}_{S,N,\beta}(q\,|\,\alpha) := \sum_{n\in\mathbb{Z}} q^n \int_{\left[\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)\right]^{\mathrm{vir}}} s(\alpha^{[n]}),$$
$$\mathsf{V}_{S,N,\beta}(q\,|\,\alpha) := \sum_{n\in\mathbb{Z}} q^n \cdot \chi^{\mathrm{vir}}(\mathsf{Quot}_S(\mathbb{C}^N,\beta,n), \det\alpha^{[n]}).$$

Note that the virtual Segre and Verlinde numbers has already been appeared as a special case of the homological and K-theoretic descendent invariants, respectively. More precisely, we have

$$\int_{\left[\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)\right]^{\mathrm{vir}}} s(\alpha^{[n]}) = \int_{\left[\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)\right]^{\mathrm{vir}}} c_{\mathrm{vd}}(-\alpha^{[n]}) \cdot c(T^{\mathrm{vir}}_{\mathsf{Quot}})$$

where vd is the virtual dimension of the Quot scheme. Similarly, we have

$$\chi^{\mathrm{vir}}(\mathsf{Quot}_S(\mathbb{C}^N,\beta,n), \det\alpha^{[n]}) = \chi^{\mathrm{vir}}(\mathsf{Quot}_S(\mathbb{C}^N,\beta,n), \Lambda^{\mathrm{rk}(\alpha^{[n]})}\alpha^{[n]} \otimes \Lambda_{-y}\Omega^{\mathrm{vir}}_{\mathsf{Quot}})$$

if $y=0$ and $\alpha^{[n]}$ is a vector bundle. However, the difference with descendent series is that the degree of the descendent insertions, namely vd and $\mathrm{rk}(\alpha^{[n]})$, in virtual Segre and Verlinde series varies as n changes. This is responsible to the failure of virtual Segre and Verlinde series being rational functions, in general. Nevertheless, we show that the virtual Segre and Verlinde series are given by algebraic functions in the q variable for Seiberg-Witten classes of length N. This is analogous to the conjectural algebraicity of the Segre and Verlinde series for Hilbert scheme of points [MOP3] and moduli of sheaves [GK4].

Theorem I. *Let S be a surface and β be a curve class of Seiberg-Witten length N. For any $\alpha \in K^0(S)$, the virtual Segre series and the virtual Verlinde series*

$$\mathsf{S}_{S,N,\beta}(q\,|\,\alpha),\ \mathsf{V}_{S,N,\beta}(q\,|\,\alpha) \in \mathbb{Q}((q))$$

are given by algebraic functions in the q variable.

Base on Theorem I, it is natural to state the following conjecture.

Conjecture. *For any surface S, curve class β and K-theory class α, the virtual Segre and Verlinde series are given by algebraic functions in the q variable.*

Apart from the algebraicity, we also have a virtual Segre/Verlinde correspondence describing the relations between the universal series in the Segre side and the Verlinde side. Suppose from now that β is of Seiberg-Witten length N. By application of Theorem B, we have a multiplicative structural formula for the virtual Segre series

$$\mathsf{S}_{S,N,\beta}(q,\underline{w}\,|\,\alpha) = q^{-\beta.K_S}\cdot \mathsf{A}^{K_S^2}\cdot \mathsf{B}^{K_S.c_1(\alpha)}\cdot \sum_{\substack{\beta=\beta_1+\cdots+\beta_N \\ \mathrm{vd}_{\beta_i}=0}} \mathrm{SW}(\underline{\beta})\cdot \prod_i \mathsf{U}_i^{\beta_i.K_S}\cdot \prod_i \mathsf{V}_i^{\beta_i.c_1(\alpha)}\cdot \prod_{i<j}\mathsf{W}_{i,j}^{\beta_i.\beta_j},$$

and for the virtual Verlinde series

$$\mathsf{V}_{S,N,\beta}(\widetilde{q},\underline{\widetilde{w}}\,|\,\alpha) = \widetilde{q}^{\,-\beta.K_S}\cdot \widetilde{\mathsf{A}}^{K_S^2}\cdot \widetilde{\mathsf{B}}^{K_S.c_1(\alpha)}\cdot \sum_{\substack{\beta=\beta_1+\cdots+\beta_N \\ \mathrm{vd}_{\beta_i}=0}} \mathrm{SW}(\underline{\beta})\cdot \prod_i \widetilde{\mathsf{U}}_i^{\beta_i.K_S}\cdot \prod_i \widetilde{\mathsf{V}}_i^{\beta_i.c_1(\alpha)}\cdot \prod_{i<j}\widetilde{\mathsf{W}}_{i,j}^{\beta_i.\beta_j}.$$

We used the tilde notations to distinguish the Segre side and the Verlinde side. For comparison, we set up the relations between the equivariant parameters and the counting parameters:

$$e^{-\widetilde{w}_k} = 1+w_k, \quad (-1)^N q = \prod_{k=1}^{N}(1+w_k)\cdot \widetilde{q}.$$

Define power series

$$H_1, \dots, H_N \in \mathbb{Q}(w_1, \dots, w_N)[\![q]\!]$$

by the equation with the initial data

$$(-1)^N q = (1-H_i)^r \cdot \prod_{k=1}^{N} (H_i + w_k) \quad \text{with} \quad H_i(q=0) = -w_i. \tag{2.4}$$

Then explicit formulas of the universal series[2] according to Theorem C satisfy

$$\mathsf{A}\ ,\ \widetilde{\mathsf{A}}\ ,\ \mathsf{B}\ ,\ \widetilde{\mathsf{B}}\ ,\ \mathsf{U}_i\ ,\ \widetilde{\mathsf{U}}_i\ ,\ \mathsf{V}_i\ ,\ \widetilde{\mathsf{V}}_i\ ,\ \mathsf{W}_{i,j}\ ,\ \widetilde{\mathsf{W}}_{i,j} \in\ \mathbb{Q}(w_1, \dots, w_N)(H_1, \dots, H_N).$$

These universal series are subject to the relations below.

Theorem J. *With resepect to the change of variables* (2.4)*, we have the following relations between the universal series in the Segre side and the Verlinde side*:

$$\begin{aligned}
\mathsf{A} &= \widetilde{\mathsf{A}}, \\
\mathsf{B} &= \widetilde{\mathsf{B}}, \\
\mathsf{V}_i &= \widetilde{\mathsf{V}}_i, \\
\mathsf{W}_{i,j} &= \widetilde{\mathsf{W}}_{i,j} \cdot \mathsf{V}_i \cdot \mathsf{V}_j, \\
\mathsf{U}_i &= (-1)^{N-1} \cdot \prod_{k=1}^{N} (1+w_k) \cdot \mathsf{B} \cdot \mathsf{V}_i \cdot \widetilde{\mathsf{U}}_i.
\end{aligned}$$

Theorem J, in particular, yields rather simple virtual Segre/Verlinde correspondence for punctual Quot schemes.

Corollary. *Let S be a surface and $\alpha \in K^0(S)$ be a K-theory class. Then the virtual Segre series*

[2] We suppress the dependence of these universal series on N and r.

and the virtual Verlinde series matches (*up to sign*) *for the punctual Quot schemes*:

$$\mathsf{S}_{S,N,\beta=0}((-1)^N q\,|\,\alpha) = \mathsf{V}_{S,N,\beta=0}(q\,|\,\alpha).$$

This chapter is, in part, being prepared for submission for publication. The dissertation author was the primary investigator and the author of the first material and the collaborator and the coauthor for the second material.

- Woonam Lim "Virtual χ_{-y}-genera of Quot schemes on surfaces "
- Noah Arbesfeld, Drew Johnson, Woonam Lim, Dragos Oprea, Rahul Pandharipande "The virtual K-theory of Quot schemes of surfaces"

Chapter 3

Universality of Quot scheme invariants

3.1 Nested Hilbert schemes

Recently, there has been a rapid progress in understanding of the nested Hilbert schemes of points and curves mainly due to their applications to Vafa-Witten theory [TT1, TT2, La1, La2] and reduced DT theory of local surfaces [GSY1]. It was noted in [L] that nested Hilbert schemes also have applications to Quot scheme theory.

We review the notion of 2-step nested Hilbert scheme of points and curves. Define the nested Hilbert scheme $S_\beta^{[m_1,m_2]}$ as a moduli space parametrizing zero dimensional subschemes (Z_1,Z_2) and an effective divisor D such that

$$\mathrm{length}(Z_i)=m_i, \quad c_1(\mathscr{O}(D))=\beta, \quad I_{Z_1}(-D)\subseteq I_{Z_2}.$$

The virtual fundamental class of nested Hilbert schemes

$$\left[S_\beta^{[m_1,m_2]}\right]^{\mathrm{vir}} \in A_{m_1+m_2+\mathrm{vd}_\beta}(S_\beta^{[m_1,m_2]})$$

are constructed by Gholampour, Sheshmani and Yau [GSY2]. Even though explicit description of virtual fundamental classes of moduli spaces are usually not accessible, Gholampour and Thomas [GT1, GT2] managed to give explicit formulas for the virtual fundamental class of nested Hilbert schemes. The authors consider a natural embedding into the (virtually) smooth space

$$f_{\mathsf{nest}}: S_\beta^{[m_1,m_2]} \hookrightarrow S^{[m_1]} \times S^{[m_2]} \times \mathsf{Hilb}_S^\beta \tag{3.1}$$

together with a Carlsson-Okounkov K-theory class

$$\mathsf{CO}_\beta^{[m_1,m_2]} := \mathrm{R}\pi_*\mathscr{O}(\mathscr{D}_\beta) - \mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}_1}, I_{\mathscr{Z}_2}(\mathscr{D}_\beta))$$

on the ambient space $S^{[m_1]} \times S^{[m_2]} \times \mathsf{Hilb}_S^\beta$. The main theorem of [GT2] is the following:

Theorem 1 ([GT2]). *Pushforward of the virtual fundamental class gives*

$$f_{\mathsf{nest}*}\left[S_\beta^{[m_1,m_2]}\right]^{\mathrm{vir}} = c_{m_1+m_2}\big(\mathsf{CO}_\beta^{[m_1,m_2]}\big) \cap \Big(\big[S^{[m_1]}\big] \times \big[S^{[m_2]}\big] \times \big[\mathsf{Hilb}_S^\beta\big]^{\mathrm{vir}}\Big).$$

Theory of nested Hilbert schemes specializes to several existing theories by letting some of the parameters m_1, m_2, β to be 0. In each case, identification also matches the virtual fundamental classes as proven in [GSY2, Proposition 3.1]:

$$\left[S_{\beta=0}^{[m,m]}\right]^{\mathrm{vir}} = \left[S^{[m]}\right], \quad \left[S_\beta^{[0,0]}\right]^{\mathrm{vir}} = \left[\mathsf{Hilb}_S^\beta\right]^{\mathrm{vir}}, \quad \left[S_\beta^{[0,m]}\right]^{\mathrm{vir}} = \left[\mathsf{Pair}_S(\mathbb{C}^1, \beta, n)\right]^{\mathrm{vir}}.$$

Here $\mathsf{Pair}_S(\mathbb{C}^1, \beta, n)$ is the moduli space of stable pairs. As we see in the next section, Quot scheme theory is related to the one of the missing specialization $S_\beta^{[m,0]}$ as observed in [L].

3.2 Virtual fundamental class of Quot schemes with $N = 1$

In this section, we apply the well-developed theory of nested Hilbert schemes to Quot schemes with $N = 1$. As a main result, we express the virtual class of Quot schemes rather explicitly. This result will be used in the next section for Quot schemes with general N by means of virtual localization.

We first identify Quot schemes with $N = 1$ and certain nested Hilbert schemes. This is the content of the following lemma of Fogarty [F] that allows us to separate points and curves in the Quot schemes with $N = 1$.

Lemma 2 ([F])**.** *Let S be a smooth projective surface and β be an effective curve class. Then*

$$\mathsf{Quot}_S(\mathbb{C}^1, \beta, n) \simeq S^{[m]} \times \mathrm{Hilb}_S^{\beta},$$

where

$$m = n + \frac{\beta(\beta + K_S)}{2}.$$

We remark that the isomorphism in the above lemma relates

$$\Big[\, 0 \to I_Z(-D) \to \mathscr{O}_S \to Q \to 0 \,\Big] \longleftrightarrow (Z, D). \tag{3.2}$$

The formula relating $n = \chi(Q)$ to $m = \mathrm{length}(Z)$ then follows from the Riemann-Roch calculation. According to the definition of nested Hilbert schemes, the lemma identify the Quot scheme with $N = 1$ and a nested Hilbert scheme

$$\mathsf{Quot}_S(\mathbb{C}^1, \beta, n) = S_\beta^{[m,0]}.$$

The proposition below further matches the virtual fundamental classes.

Proposition 3 ([L])**.** *Through the identification $\mathsf{Quot}_S(\mathbb{C}^1, \beta, n) = S_\beta^{[m,0]}$, the virtual fundamental class of the Quot scheme on the left and the virtual fundamental class of the nested Hilbert scheme on the right agree*:

$$\Big[\mathsf{Quot}_S(\mathbb{C}^1, \beta, n)\Big]^{\mathrm{vir}} = \Big[S_\beta^{[m,0]}\Big]^{\mathrm{vir}}.$$

Proof. By work of Siebert [S, Theorem 4.6], the virtual fundamental class of a projective scheme with a 2-term perfect obstruction theory depends only on the virtual tangent bundle. Therefore it suffices to equate the virtual tangent bundle of $\mathsf{Quot}_S(\mathbb{C}^1, \beta, n)$ and $S_\beta^{[m,0]}$ through the identification.

By family version of the correspondence (3.2), the universal quotient over the product $\mathsf{Quot}_S(\mathbb{C}^1,\beta,n)\times S$ is given by

$$0\to I_{\mathscr{Z}}(-\mathscr{D}_\beta)\to\mathscr{O}_{\mathsf{Quot}_S(\mathbb{C}^1,\beta,n)\times S}\to\mathscr{Q}\to 0.$$

where $\mathscr{Z}$ and $\mathscr{D}_\beta$ denote the universal subscheme and the universal divisor, respectively. We compute the virtual tangent bundle of $\mathsf{Quot}_S(\mathbb{C}^1,\beta,n)$ as follows:

$$\begin{aligned}
T^{\mathrm{vir}}_{\mathsf{Quot}} &= \mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}}(-\mathscr{D}_\beta),\mathscr{Q})\\
&= \mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}}(-\mathscr{D}_\beta),\mathscr{O}-I_{\mathscr{Z}}(-\mathscr{D}_\beta))\\
&= -\mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}},I_{\mathscr{Z}})+\mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}},\mathscr{O}(\mathscr{D}_\beta))\\
&= \mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}},I_{\mathscr{Z}})_0[1]-\mathrm{R}\mathscr{H}om_\pi(\mathscr{O},\mathscr{O})+\mathrm{R}\mathscr{H}om_\pi(\mathscr{O}-\mathscr{O}_{\mathscr{Z}},\mathscr{O}+\mathscr{O}_{\mathscr{D}_\beta}(\mathscr{D}_\beta))\\
&= T_{S^{[m]}}+\mathrm{R}\pi_*(\mathscr{O}_{\mathscr{D}_\beta}(\mathscr{D}_\beta))-\mathrm{R}\mathscr{H}om_\pi(\mathscr{O}_{\mathscr{Z}},\mathscr{O}(\mathscr{D}_\beta))\\
&= T_{S^{[m]}}+T^{\mathrm{vir}}_{\mathsf{Hilb}^\beta_S}-\mathrm{R}\mathscr{H}om_\pi(\mathscr{O}_{\mathscr{Z}},\mathscr{O}(\mathscr{D}_\beta)).
\end{aligned}$$

The subscript 0 denotes the trace free part and π denotes the projection map from S to a point.

By [GSY2, Proposition 2.2], the virtual tangent bundle of $S_\beta^{[m,0]}$ as a nested Hilbert scheme is

$$T^{\mathrm{vir}}_{S_\beta^{[m,0]}}=\mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}},\mathscr{O}(\mathscr{D}_\beta))-\Big[\mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}},I_{\mathscr{Z}})+\mathrm{R}\mathscr{H}om_\pi(\mathscr{O},\mathscr{O})\Big]_0$$

where the bracket with subscript 0 denotes again the trace free part. This removes one copy of $\mathrm{R}\mathscr{H}om_\pi(\mathscr{O},\mathscr{O})$, hence

$$T^{\mathrm{vir}}_{S_\beta^{[m,0]}}=-\mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}},I_{\mathscr{Z}})+\mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}},\mathscr{O}(\mathscr{D}_\beta)).$$

This agrees with the third line of the computation of $T^{\mathrm{vir}}_{\mathsf{Quot}}$. $\square$

Consider the morphism (3.1) in the case of Quot scheme with the identification

$$f_{\mathsf{Quot}} : \mathsf{Quot}_S(\mathbb{C}^1, \beta, n) = S_\beta^{[m,0]} \xrightarrow{\sim} S^{[m]} \times \mathsf{Hilb}_S^\beta.$$

Then Theorem 1 and Proposition 3 yields the following Corollary.

Corollary 4. *Pushforward of the virtual fundamental class gives*

$$f_{\mathsf{Quot}*}\Big[\mathsf{Quot}_S(\mathbb{C}^1, \beta, n)\Big]^{\mathrm{vir}} = e\big(\mathsf{CO}_\beta^{[m,0]}\big) \cap \Big(\big[S^{[m]}\big] \times \big[\mathsf{Hilb}_S^\beta\big]^{\mathrm{vir}}\Big)$$

where

$$\mathsf{CO}_\beta^{[m,0]} = \mathrm{R}\mathscr{H}om_\pi(\mathscr{O}_\mathscr{Z}, \mathscr{O}(\mathscr{D}_\beta))$$

is a vector bundle of rank m.

Proof. We can easily check that

$$\begin{aligned}
\mathsf{CO}_\beta^{[m,0]} &= \mathrm{R}\pi_*\mathscr{O}(\mathscr{D}_\beta) - \mathrm{R}\mathscr{H}om_\pi(I_\mathscr{Z}, \mathscr{O}(\mathscr{D}_\beta)) \\
&= \mathrm{R}\pi_*\mathscr{O}(\mathscr{D}_\beta) - \mathrm{R}\mathscr{H}om_\pi(\mathscr{O} - \mathscr{O}_\mathscr{Z}, \mathscr{O}(\mathscr{D}_\beta)) \\
&= \mathrm{R}\mathscr{H}om_\pi(\mathscr{O}_\mathscr{Z}, \mathscr{O}(\mathscr{D}_\beta)).
\end{aligned}$$

On the other hand,

$$\mathrm{R}\mathscr{H}om_\pi(\mathscr{O}_\mathscr{Z}, \mathscr{O}(\mathscr{D}_\beta)) = \mathrm{R}\mathscr{H}om_\pi(\mathscr{O}(\mathscr{D}_\beta), \mathscr{O}_\mathscr{Z}(K_S))^\vee = \Big(\pi_*\mathscr{O}_\mathscr{Z}(K_S - \mathscr{D}_\beta)\Big)^\vee$$

is a vector bundle of rank m by vanishing of the higher pushforwards. The rest follows directly from Theorem 1 and Proposition 3. $\square$

3.3 Virtual localization formula

In this section, we analyze the necessary ingredients in the virtual localization formula for the integral

$$\int_{\left[\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)\right]^{\mathrm{vir}}} f_1(\alpha_1^{[n]})\cdots f_\ell(\alpha_\ell^{[n]})\cdot g\left(T^{\mathrm{vir}}_{\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)}\right) \tag{3.3}$$

in the generating series (2.1). Then virtual localization formula together with Corollary 4 expresses this as certain integrals over the product of Hilbert schemes of points and Hilbert schemes of curves.

We begin by reviewing the virtual localization formula of Graber and Pandharipande [GP]. Let M be a projective scheme with a $\mathbb{C}^*$-action. Suppose that M is equipped with a $\mathbb{C}^*$-equivariant 2-term perfect obstruction theory

$$\phi : E^\bullet = \left[E^{-1}\to E^0\right]\to \mathbb{L}_M.$$

Let $\{M_\alpha\}$ be the set of connected components of the fixed loci $M^{\mathbb{C}^*}$. Then restriction of $E^\bullet$ to each fixed loci has a decomposition into the fixed and moving parts

$$E^\bullet\Big|_{M_\alpha} = \left(E^\bullet\Big|_{M_\alpha}\right)^{\mathrm{fix}} \oplus \left(E^\bullet\Big|_{M_\alpha}\right)^{\mathrm{mov}}.$$

Each M_α is equipped with an induced 2-term perfect obstruction theory

$$\phi_\alpha : \left(E^\bullet\Big|_{M_\alpha}\right)^{\mathrm{fix}} \to \left(\mathbb{L}_M\Big|_{M_\alpha}\right)^{\mathrm{fix}} \simeq \mathbb{L}_{M_\alpha}$$

which in turn defines the virtual fundamental class $[M_\alpha]^{\mathrm{vir}}$. Define the virtual normal bundle of

M_α as the moving part of the restriction of $E_\bullet$

$$N_\alpha^{\mathrm{vir}} := \left(E_\bullet\Big|_{M_\alpha}\right)^{\mathrm{mov}}.$$

The following theorem localizes the virtual fundamental class $[M]^{\mathrm{vir}}$ to the fixed loci $i : M^{\mathbb{C}^*} \hookrightarrow M$ in the localized equivariant Chow group

$$A_*^{\mathbb{C}^*}(M) \otimes \mathbb{Q}[t, t^{-1}].$$

Here t is the first Chern class of the $\mathbb{C}^*$-representation of weight 1.

Theorem 5 (Virtual localization formula, [GP]). *In the localized equivariant Chow group as above, we have*

$$[M]^{\mathrm{vir}} = i_* \sum_\alpha \frac{[M_\alpha]^{\mathrm{vir}}}{e(N_\alpha^{\mathrm{vir}})}$$

where $e(\cdot)$ denote the equivariant Euler class.

Consider a $\mathbb{C}^*$-action on $\mathbb{C}^N$ with distinct weights

$$w_1, \ldots, w_N.$$

This induces a $\mathbb{C}^*$-action on $\mathsf{Quot}_S(\mathbb{C}^N, \beta, n)$ by acting in the middle factor of

$$0 \to S \to \mathbb{C}^N \otimes \mathscr{O}_S \to Q \to 0.$$

By weight decomposition, $\mathbb{C}^*$-fixed quotients are of the form

$$0 \to \bigoplus_{i=1}^N S_i[w_i] \to \bigoplus_{i=1}^N \mathscr{O}_S[w_i] \to \bigoplus_{i=1}^N Q_i[w_i] \to 0 \tag{3.4}$$

where the brackets denote $\mathbb{C}^*$-weights. Therefore we have a natural splitting of the fixed loci into products of Quot schemes with $N=1$:

$$\left(\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)\right)^{\mathbb{C}^*}=\bigsqcup_{\substack{\beta=\beta_1+\cdots+\beta_N\\ n=n_1+\cdots+n_N}}\mathsf{Quot}_S(\mathbb{C}^1,\beta_1,n_1)\times\cdots\times\mathsf{Quot}_S(\mathbb{C}^1,\beta_N,n_N).$$

For each decomposition $\beta=\beta_1+\cdots+\beta_N$ and $n=n_1+\cdots+n_N$, define a vector

$$\underline{m}:=(m_1,\ldots,m_N)\quad\text{such that}\quad m_i=n_i+\frac{\beta_i(\beta_i+K_S)}{2}$$

encoding the length of the zero dimensional parts of each $\mathsf{Quot}_S(\mathbb{C}^1,\beta_i,n_i)$. Denote each components of fixed loci as

$$\begin{aligned}F=F[\underline{m},\underline{\beta}]&:=\mathsf{Quot}_S(\mathbb{C}^1,\beta_1,n_1)\times\cdots\times\mathsf{Quot}_S(\mathbb{C}^1,\beta_N,n_N)\\ &\simeq\left(S^{[m_1]}\times\cdots\times S^{[m_N]}\right)\times\left(\mathsf{Hilb}_S^{\beta_1}\times\cdots\times\mathsf{Hilb}_S^{\beta_N}\right).\end{aligned}$$

Now we analyze the induced virtual fundamental class $[F]^{\mathrm{vir}}$ and the virtual normal bundle N_F^{vir}. Recall that a 2-term perfect obstruction theory of $\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)$ is given by

$$\phi:\mathrm{R}\mathscr{H}om_\pi(\mathscr{S},\mathscr{Q})^\vee\to\mathbb{L}_{\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)}$$

where $\mathscr{S}$ and $\mathscr{Q}$ denotes the universal subsheaf and quotient over $\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)\times S$. By applying the family version of weight decomposition (3.4), we can show that restriction of the universal exact sequence to $F\times S$ is given by

$$0\to\bigoplus_{i=1}^N\mathscr{S}_i[w_i]\to\bigoplus_{i=1}^N\mathscr{O}_{F\times S}[w_i]\to\bigoplus_{i=1}^N\mathscr{Q}_i[w_i]\to 0,$$

where

$$0\to\mathscr{S}_i\to\mathscr{O}_{F\times S}\to\mathscr{Q}_i\to 0$$

is the pullpuback of the universal exact sequence over $\mathsf{Quot}_S(\mathbb{C}^1, \beta_i, n_i) \times S$. Since we have

$$\mathrm{R}\mathscr{H}om_\pi(\mathscr{S}, \mathscr{Q})^\vee \Big|_{F\times S} = \bigoplus_{i,j} \mathrm{R}\mathscr{H}om_\pi(\mathscr{S}_i, \mathscr{Q}_j)^\vee [w_i - w_j],$$

the fixed and moving part of the restriction is given by

$$\begin{aligned}
\left(\mathrm{R}\mathscr{H}om_\pi(\mathscr{S}, \mathscr{Q})^\vee \Big|_{F\times S}\right)^{\mathrm{fix}} &= \bigoplus_{i} \mathrm{R}\mathscr{H}om_\pi(\mathscr{S}_i, \mathscr{Q}_j)^\vee \\
\left(\mathrm{R}\mathscr{H}om_\pi(\mathscr{S}, \mathscr{Q})^\vee \Big|_{F\times S}\right)^{\mathrm{mov}} &= \bigoplus_{i\neq j} \mathrm{R}\mathscr{H}om_\pi(\mathscr{S}_i, \mathscr{Q}_j)^\vee [w_i - w_j].
\end{aligned}$$

Therefore the induced 2-term perfect obstruction theory of F is simply the direct sum of the usual obstruction theories of $N = 1$ Quot scheme factors

$$\phi_F : \bigoplus_i \mathrm{R}\mathscr{H}om_\pi(\mathscr{S}_i, \mathscr{Q}_i)^\vee \to \bigoplus_i \mathbb{L}_{\mathsf{Quot}_S(\mathbb{C}^1, \beta_i, n_i)}.$$

In turn, we obtain the splitting of the virtual fundamental class

$$[F]^{\mathrm{vir}} = \left[\mathsf{Quot}_S(\mathbb{C}^1, \beta_1, n_1)\right]^{\mathrm{vir}} \times \cdots \times \left[\mathsf{Quot}_S(\mathbb{C}^1, \beta_N, n_N)\right]^{\mathrm{vir}}.$$

In addition, the virtual tangent bundle and virtual normal bundle are

$$\begin{aligned}
T_F^{\mathrm{vir}} &= \bigoplus_i \mathrm{R}\mathscr{H}om_\pi(\mathscr{S}_i, \mathscr{Q}_i) \\
N_F^{\mathrm{vir}} &= \bigoplus_{i\neq j} \mathrm{R}\mathscr{H}om_\pi(\mathscr{S}_i, \mathscr{Q}_j)[w_j - w_i]
\end{aligned}$$

As the integral (3.3) includes the descendent insertions, we also record the computation of the tautological class $\alpha^{[n]}$ restricted to the fixed locus F:

$$\begin{aligned}
\alpha^{[n]}\Big|_F &= \mathrm{R}\pi_*(\mathscr{Q}\otimes q^*\alpha)\Big|_F \\
&= \bigoplus_i \mathrm{R}\pi_*(\mathscr{Q}_i\otimes q^*\alpha)[w_i] \\
&= \bigoplus_i \alpha_{\beta_i}^{[m_i]}[w_i].
\end{aligned}$$

Here $\alpha_{\beta_i}^{[m_i]}$ denotes the tautological class over the following Quot scheme with $N=1$:

$$f_{\mathsf{Quot}} : \mathsf{Quot}_S(\mathbb{C}^1,\beta_i,n_i) \xrightarrow{\sim} S^{[m_i]}\times \mathsf{Hilb}_S^{\beta_i}.$$

3.4 Appearance of Seiberg-Witten invariants

In this section, we study the residual contribution of each fixed locus to the integral (3.3). By virtual localization formula, the residual contribution of $F=F[\underline{m},\underline{\beta}]$ is

$$\int_{[F]^{\mathrm{vir}}} f_1(\alpha_1^{[n]}|_F)\cdots f_\ell(\alpha_\ell^{[n]}|_F)\cdot\frac{g(T_F^{\mathrm{vir}}+N_F^{\mathrm{vir}})}{e(N_F^{\mathrm{vir}})} \tag{3.5}$$

where we used the decomposition

$$T^{\mathrm{vir}}_{\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)}\Big|_F = T_F^{\mathrm{vir}}+N_F^{\mathrm{vir}}.$$

By Corollary 4 and the splitting of the virtual fundamental class $[F]^{\mathrm{vir}}$, we have

$$[F]^{\mathrm{vir}} = e(\mathrm{Obs}_1)\cdots e(\mathrm{Obs}_N)\cap\Big(\big[S^{[m_1]}\big]\times\cdots\times\big[S^{[m_N]}\big]\times\big[\mathsf{Hilb}_S^{\beta_1}\big]^{\mathrm{vir}}\times\cdots\times\big[\mathsf{Hilb}_S^{\beta_N}\big]^{\mathrm{vir}}\Big)$$

where

$$\mathrm{Obs}_i := \mathrm{R}\mathscr{H}om_\pi(\mathscr{O}_{\mathscr{Z}_i},\mathscr{O}(\mathscr{D}_{\beta_i})).$$

Therefore the integral (3.5) can be rewritten as

$$\int_{\left[S^{[\underline{m}]}\right]\times\left[\mathsf{Hilb}_S^{\underline{\beta}}\right]^{\mathrm{vir}}} e(\underline{\mathrm{Obs}})\cdot \underline{f}(\underline{\alpha}^{[n]}\big|_F)\cdot \frac{g(T_F^{\mathrm{vir}}+N_F^{\mathrm{vir}})}{e(N_F^{\mathrm{vir}})}. \tag{3.6}$$

where underline notations stand for the obvious product terms.

Recall that the Seiberg-Witten invariants are defined by the virtual fundamental class of Hilbert scheme of curves and pushforwards along the Abel-Jacobi maps. It is then natural to consider the product of the Abel-Jacobi maps:

$$\underline{\mathsf{AJ}} : S^{[\underline{m}]}\times \mathsf{Hilb}_S^{\underline{\beta}} \to S^{[\underline{m}]}\times \mathsf{Pic}_S^{\underline{\beta}}.$$

Let τ be a cohomology class of $S^{[\underline{m}]}\times \mathsf{Pic}_S^{\underline{\beta}}$. By the projection formula and definition of Seiberg-Witten invariants, we have

$$\int_{\left[S^{[\underline{m}]}\right]\times\left[\mathsf{Hilb}_S^{\underline{\beta}}\right]^{\mathrm{vir}}} h_1^{k_1}\cdots h_N^{k_N}\cdot \underline{\mathsf{AJ}}^*\tau = \int_{\left[S^{[\underline{m}]}\right]\times \mathrm{SW}^{\underline{k}}(\underline{\beta})}\tau \tag{3.7}$$

where

$$\mathrm{SW}^{\underline{k}}(\underline{\beta}) := \mathrm{SW}^{k_1}(\beta_1)\times\cdots\times \mathrm{SW}^{k_N}(\beta_N)\in H_*(\mathsf{Pic}_S^{\underline{\beta}},\mathbb{Z}).$$

To apply this to the integral (3.6), we need to write its integrand as a linear combination of terms of the form

$$h_1^{k_1}\cdots h_N^{k_N}\cdot \underline{\mathsf{AJ}}^*\tau. \tag{3.8}$$

In this regard, we begin by applying the identity (1.3) to each K-theory class in the integrand.

Recall the identity (1.3) between line bundles

$$\mathscr{O}(\mathscr{D}_{\beta_i}) = \mathscr{L}_{\beta_i}\otimes \mathscr{O}(h_i).$$

Here we used the notation $\mathscr{O}(h_i)$ instead of $\mathscr{O}(1)$ to distinguish the different factors of $\mathsf{Hilb}_S^{\beta_i}$.

Also recall from the proof of Lemma 2 that the universal subsheaf $\mathscr{S}_i$ is given by

$$\mathscr{S}_i = I_{\mathscr{Z}_i}(-\mathscr{D}_{\beta_i}) = I_{\mathscr{Z}_i} \otimes \mathscr{L}_{\beta_i}^{-1} \otimes \mathscr{O}(-h_i).$$

Using these, we rewrite the K-theoretic classes

$$\alpha_{s,\beta_i}^{[m_i]}\ , \quad \mathrm{R}\mathscr{H}om_\pi(\mathscr{S}_i, \mathscr{Q}_j)\ , \quad \mathrm{Obs}_i$$

appearing as the ingredients of the integrand in (3.6). First, we have

$$\begin{aligned}
\alpha_{s,\beta_i}^{[m_i]} &= \mathrm{R}\pi_*(\mathscr{Q}_i \otimes q^*\alpha_s) \\
&= \mathrm{R}\pi_*\big((\mathscr{O} - I_{\mathscr{Z}_i} \otimes \mathscr{L}_{\beta_i}^{-1} \otimes \mathscr{O}(-h_i)) \otimes q^*\alpha_s\big) \\
&= \mathbb{C}^{\chi(\alpha_s)} - \mathrm{R}\pi_*(I_{\mathscr{Z}_i} \otimes \mathscr{L}_{\beta_i}^{-1} \otimes q^*\alpha_s) \otimes \mathscr{O}(-h_i).
\end{aligned}$$

With a similar procedure, we obtain

$$\begin{aligned}
\mathrm{R}\mathscr{H}om_\pi(\mathscr{S}_i, \mathscr{Q}_j) = \mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}_i} \otimes \mathscr{L}_{\beta_i}^{-1}, \mathscr{O}) \otimes \mathscr{O}(h_i) \\
- \mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}_i} \otimes \mathscr{L}_{\beta_i}^{-1}, I_{\mathscr{Z}_j} \otimes \mathscr{L}_{\beta_j}^{-1}) \otimes \mathscr{O}(h_i - h_j)
\end{aligned}$$

and

$$\mathrm{Obs}_i = \mathrm{R}\mathscr{H}om_\pi(\mathscr{O}_{\mathscr{Z}_i}, \mathscr{L}_{\beta_i}) \otimes \mathscr{O}(h_i).$$

For the ease of the notation, we introduce the following notations for various ranks

$$\begin{aligned} r_{s,i} &:= \text{rank } \mathrm{R}\pi_*(I_{\mathscr{Z}_i} \otimes \mathscr{L}_{\beta_i}^{-1} \otimes q^*\alpha_s), \\ r_i &:= \text{rank } \mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}_i} \otimes \mathscr{L}_{\beta_i}^{-1}, \mathscr{O}), \\ r_{ij} &:= \text{rank } \mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}_i} \otimes \mathscr{L}_{\beta_i}^{-1}, I_{\mathscr{Z}_j} \otimes \mathscr{L}_{\beta_j}^{-1}). \end{aligned}$$

These can be computed using Hirzebruch-Riemann-Roch formula

$$\begin{aligned} r_{s,i} &= \chi(\alpha_s(-D_i)) - \mathrm{rk}(\alpha_s) \cdot m_i, \\ r_i &= \chi(\mathscr{O}(D_i)) - m_i, \\ r_{ij} &= \chi(\mathscr{O}(D_i - D_j)) - m_i - m_j, \end{aligned} \tag{3.9}$$

where $D_i \in \mathsf{Hilb}_S^{\beta_i}$.

To analyze the structure of the integrand, we need lemmas about multiplicative characteristic of line bundle twist. See appendix for the statements and the proof. We first use Lemma 29 to rewrite the following part of the integrand:

$$\underline{f}(\underline{\alpha}^{[n]}|_F) = \prod_{s,i} \frac{f_s(w_i)^{\chi(\alpha_s) - r_{s,i}}}{\sum_{k \geq 0} (-h_i)^k \cdot f_s^{(k)}\Big(r_{s,i} \,|\, w_i \,|\, c_*\big(\mathrm{R}\pi_*(I_{\mathscr{Z}_i} \otimes \mathscr{L}_{\beta_i}^{-1} \otimes q^*\alpha_s)\big)\Big)}$$

and

$$g(T^{\mathrm{vir}} + N_F^{\mathrm{vir}}) = \prod_{i,j} g(w_j - w_i)^{r_i - r_{ij}} \cdot \frac{\sum_{k \geq 0} h_i^k \cdot g^{(k)}\Big(r_i \,|\, w_j - w_i \,|\, c_*\big(\mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}_i} \otimes \mathscr{L}_{\beta_i}^{-1}, \mathscr{O})\big)\Big)}{\sum_{k \geq 0} (h_i - h_j)^k \cdot g^{(k)}\Big(r_{ij} \,|\, w_j - w_i \,|\, c_*\big(\mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}_i} \otimes \mathscr{L}_{\beta_i}^{-1}, I_{\mathscr{Z}_j} \otimes \mathscr{L}_{\beta_j}^{-1})\big)\Big)}.$$

We use Lemma 30 to obtain

$$\frac{1}{e(N_F^{\mathrm{vir}})} = \prod_{i\neq j}(w_j - w_i)^{r_{ij}-r_i}$$
$$\cdot \frac{\sum_{k\geq 0}(h_i-h_j)^k \cdot e^{(k)}\Big(r_{ij}\,|\,w_j - w_i\,|\,c_*\big(\mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}_i}\otimes\mathscr{L}_{\beta_i}^{-1}, I_{\mathscr{Z}_j}\otimes\mathscr{L}_{\beta_j}^{-1}))\Big)}{\sum_{k\geq 0} h_i^k \cdot e^{(k)}\Big(r_i\,|\,w_j - w_i\,|\,c_*\big(\mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}_i}\otimes\mathscr{L}_{\beta_i}^{-1}, \mathscr{O}))\Big)}.$$

Lastly, we apply the standard formula for the Chern class of line bundle twist to obtain

$$\begin{aligned} e(\underline{\mathrm{Obs}}) &= \prod_i c_{m_i}\big(\mathrm{R}\mathscr{H}om_\pi(\mathscr{O}_{\mathscr{Z}_i}, \mathscr{L}_{\beta_i})\otimes\mathscr{O}(h_i)\big) \\ &= \prod_i \left(\sum_{k=0}^{m_i} h^k \cdot c_{m_i-k}\big(\mathrm{R}\mathscr{H}om_\pi(\mathscr{O}_{\mathscr{Z}_i}, \mathscr{L}_{\beta_i})\big)\right). \end{aligned}$$

As a result, we can express the integrand of (3.6) as

$$\prod_{s,i} f_s(w_i)^{\chi(\alpha_s)-r_{s,i}} \cdot \prod_i g(0)^{r_i - r_{ii}} \cdot \prod_{i\neq j}\left(\frac{g(w_j-w_i)}{w_j-w_i}\right)^{r_i-r_{ij}} \cdot \sum_{k_1,\ldots,k_N\geq 0} h_1^{k_1}\cdots h_N^{k_N}\mathsf{A}_{\underline{k},\underline{m}}$$

where $\mathsf{A}_{\underline{k},\underline{m}}$ is the universal power series

$$\mathsf{A}_{\underline{k},\underline{m}} \in F(r_{s,i}, r_i, r_{ij})((w_1,\ldots,w_N))[\![c_*]\!]$$

where c_* are Chern classes of

$$\mathrm{R}\pi_*(I_{\mathscr{Z}_i}\otimes\mathscr{L}_{\beta_i}^{-1}\otimes q^*\alpha_s), \mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}_i}\otimes\mathscr{L}_{\beta_i}^{-1}, \mathscr{O}), \mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}_i}\otimes\mathscr{L}_{\beta_i}^{-1}, I_{\mathscr{Z}_j}\otimes\mathscr{L}_{\beta_j}^{-1}), \mathrm{R}\pi_*(\mathscr{L}_{\beta_i}). \tag{3.10}$$

Here we used the relation

$$\mathrm{R}\mathscr{H}om_\pi(\mathscr{O}_{\mathscr{Z}_i}, \mathscr{L}_{\beta_i}) = \mathrm{R}\pi_*(\mathscr{L}_{\beta_i}) - \mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}_i}\otimes\mathscr{L}_{\beta_i}^{-1}, \mathscr{O})$$

to replace $\mathrm{R}\mathscr{H}om_\pi(\mathscr{O}_{\mathscr{Z}_i}, \mathscr{L}_{\beta_i})$ by $\mathrm{R}\pi_*(\mathscr{L}_{\beta_i})$. Power series $\mathsf{A}_{\underline{k},\underline{m}}$ depends only on $N, \underline{f}$, g, $\underline{k}$ and $\underline{m}$. Dependence on $\underline{m}$ only comes from the $e(\underline{\mathrm{Obs}})$ contribution. As K-theory classes in the list (3.10) are pullbacks from $S^{[\underline{m}]} \times \mathsf{Pic}^{\underline{\beta}}_S$, we can rewrite the integral (3.6) as

$$\prod_{s,i} f_s(w_i)^{\chi(\alpha_s)-r_{s,i}} \cdot \prod_i g(0)^{r_i - r_{ii}} \cdot \prod_{i \neq j} \left(\frac{g(w_j - w_i)}{w_j - w_i} \right)^{r_i - r_{ij}} \cdot \sum_{k_1,\dots,k_N \geq 0} \int_{\left[S^{[\underline{m}]}\right] \times \mathrm{SW}^{\underline{k}}(\underline{\beta})} \mathsf{A}_{\underline{k},\underline{m}} \tag{3.11}$$

using the observation (3.7). This expresses the residual virtual Quot scheme invariants of each fixed locus as universal integrals over the product of Hilbert scheme of points and Seiberg-Witten invariants.

3.5 Generalized recursive structure of Hilbert scheme of points

In their celebrated work, Ellingsrud, Göttsche and Lehn [EGL] used a recursive structure of Hilbert scheme of points to exhibit the universal nature of tautological integrals

$$\int_{S^{[m]}} \mathsf{P}(F_1^{[m]}, \dots, F_\ell^{[m]}, T_{S^{[m]}}).$$

Here P is a polynomial in the Chern classes of K-theoretic inputs. Their recursive argument can easily be generalized in the following sense. First, we can consider a relative version over a smooth projective base. Second, we can consider multiple factors of Hilbert scheme of points. This generalization will be used in the next section to study the integrals of the form

$$\int_{\left[S^{[\underline{m}]}\right] \times \mathrm{SW}^{\underline{k}}(\underline{\beta})} \mathsf{A}_{\underline{k},\underline{m}}.$$

Let S be a smooth projective surface and $B_1, \dots, B_N$ be smooth projective schemes. We consider $S_i \times B_i$ as a trivial family of surfaces over the base B_i where $S_1 = \cdots = S_N = S$.

Subscripts of S_i's are used to distinguish their corresponding base B_i's. Fix K-theory classes

$$\alpha_s \in K^0(S), \quad s = 1, \dots, \ell,$$

and

$$F_i \in K^0(S_i \times B_i), \quad i = 1, \dots, N.$$

Using these data, we define the following tautological classes:

$$\begin{aligned}
(F_i, \alpha_s)^{[m_i]} &:= \mathrm{R}\pi_*(I_{\mathscr{Z}_i} \otimes F_i^\vee \otimes \alpha_s) \in K^0(S_i^{[m_i]} \times B_i), \\
F_i^{[\![m_i]\!]} &:= \mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}_i} \otimes F_i^\vee, \mathscr{O}) \in K^0(S_i^{[m_i]} \times B_i), \\
(F_i, F_{i'})^{[\![m_i, m_{i'}]\!]} &:= \mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}_i} \otimes F_i^\vee, I_{\mathscr{Z}_{i'}} \otimes F_{i'}^\vee) \in K^0(S_i^{[m_i]} \times S_{i'}^{[m_{i'}]} \times B_i \times B_{i'}).
\end{aligned}$$

We further fix K-theory classes V_i of B_i. Consider a tautological integral

$$\int_{S_1^{[m_1]} \times \cdots \times S_N^{[m_N]} \times B_1 \times \cdots \times B_N} \mathsf{P} \tag{3.12}$$

where P is a power series in Chern classes of the K-theory inputs

$$(F_i, \alpha_s)^{[m_i]}, \quad F_i^{[\![m_i]\!]}, \quad (F_i, F_{i'})^{[\![m_i, m_{i'}]\!]}, \quad T_{S^{[m_i]}}, \quad V_i$$

with coefficients in some field.

As in [EGL], we use the following recursive structure

$$\begin{tikzcd}
& S^{[m,m-1]} \arrow[dl, "f"'] \arrow[dr, "g"] & \\
S^{[m]} & & S^{[m-1]} \times S.
\end{tikzcd}$$

Key observation is that f is a generically finite morphism of degree m. This allows us to transform integrals over the Hilbert scheme of m points to integrals over the Hilbert scheme of $m-1$ points with one copy of S. Applying this recursion $m = \sum m_i$ number of times, we can show that the integral (3.12) is equal to

$$\int_{S^m \times B_1 \times \cdots \times B_N} \mathsf{P}' \tag{3.13}$$

where P' is a power series in Chern classes of the K-theory inputs

$$\alpha_s^{(j)}, \quad F_i^{(j)}, \quad T_S^{(j)}, \quad \mathscr{O}_\Delta^{(j_1,j_2)}, \quad V_i$$

with coefficients in the same field. Superscript with (j) denotes the place of S factor in S^m where it is pulled back. Furthermore, P' only depends on P and ranks of F_i and α_s. By using Chern classes of the diagonals, we can further integrate over S^m factor to equate (3.13) with

$$\int_{B_1 \times \cdots \times B_N} \mathsf{P}'' \tag{3.14}$$

where P'' is a power series in the classes of the form

$$\pi_* \Big(\prod_s c_{I_s}(\alpha_s) \cdot \prod_i c_{J_i}(F_i) \cdot c_P(T_S) \Big) \cdot \prod_i c_{Q_i}(V_i). \tag{3.15}$$

Here $\pi : S \times \underline{B} \to \underline{B}$ is the projection and I_s, J_i, P and Q_i denote multisets defining monomials in Chern classes. Similarly, P'' only depends on P' and ranks of F_i and α_s.

The above procedure remains true even if we replace the fundamental classes $[B_i]$ with any homology classes $\sigma_i \in H^*(B_i, \mathbb{Z})$. In other words, we have

$$\int_{S_1^{[m_1]} \times \cdots \times S_N^{[m_N]} \times \sigma_1 \times \cdots \times \sigma_N} \mathsf{P} \;=\; \int_{\sigma_1 \times \cdots \times \sigma_N} \mathsf{P}''. \tag{3.16}$$

In the case of interest, we will consider

$$B_i = \mathsf{Pic}_S^{\beta_i}, \quad F_i = \mathscr{L}_{\beta_i}, \quad V_i = \mathrm{R}\pi_*(\mathscr{L}_{\beta_i}), \quad \sigma_i = \mathrm{SW}^{k_i}(\beta_i).$$

In this case, we can further analyze the classes (3.15) using the cohomological description of the

Poincaré line bundles $\mathscr{L}_{\beta_i}$. We do this in the next section.

3.6 Integration over the product of Hilbert schemes of points

In this section, we rewrite

$$\int_{\left[S^{[\underline{m}]}\right]\times\mathrm{SW}^{\underline{k}}(\underline{\beta})} \mathsf{A}_{\underline{k},\underline{m}} \tag{3.17}$$

by integrating over the product of Hilbert scheme of points $S^{[\underline{m}]}$. This integration process can be studied via recursion explained in Section 3.5. More precisely, there exist power series $\mathsf{B}_{\underline{k},\underline{m}}$ in classes of the form

$$\pi_*\Big(\prod_s c_{I_s}(\alpha_s)\cdot\prod_i c_1(\mathscr{L}_{\beta_i})^{t_i}\cdot c_P(T_S)\Big)\cdot\prod_i c_{Q_i}(\mathrm{R}\pi_*(\mathscr{L}_{\beta_i}))$$

such that

$$\int_{\left[S^{[\underline{m}]}\right]\times\mathrm{SW}^{\underline{k}}(\underline{\beta})} \mathsf{A}_{\underline{k},\underline{m}} = \int_{\mathrm{SW}^{\underline{k}}(\underline{\beta})} \mathsf{B}_{\underline{k},\underline{m}}.$$

By Grothendieck-Riemann-Roch, we have

$$\mathrm{ch}(\mathrm{R}\pi_*(\mathscr{L}_{\beta_i})) = \pi_*(\mathrm{ch}(\mathscr{L}_{\beta_i})\cdot\mathrm{td}(S)).$$

Therefore, we may replace $\mathsf{B}_{\underline{k},\underline{m}}$ by another power series $\mathsf{C}_{\underline{k},\underline{m}}$ in classes of the form

$$\pi_*\Big(\prod_s c_{I_s}(\alpha_s)\cdot\prod_i c_1(\mathscr{L}_{\beta_i})^{t_i}\cdot c_J(T_S)\Big) \tag{3.18}$$

such that

$$\mathsf{B}_{\underline{k},\underline{m}} = \mathsf{C}_{\underline{k},\underline{m}}.$$

Now we study classes of the form (3.18) in detail using the cohomological description of the Poincaré line bundles $\mathscr{L}_{\beta_i}$. Note that the first Chern class of the Poincaré line bundle decomposes

$$c_1(\mathscr{L}_{\beta_i}) = (\beta_i, \mathrm{id}_i, 0) \in H^2(S) \ \oplus \ \left(H^1(S) \otimes H^1(S)_i^\vee\right) \ \oplus \ \Lambda^2 H^1(S)_i^\vee$$

after identifications

$$H^1(\mathsf{Pic}_S^{\beta_i}) = H^1(S)_i^\vee, \quad H^2(\mathsf{Pic}_S^{\beta_i}) = \Lambda^2 H^1(S)_i^\vee.$$

Subscripts i expresses the position among N factors of Picard groups. This decomposition implies that the classes of the form (3.18) is a combination of

$$[\gamma]_{t_1,\ldots,t_N} := \pi_*\left(\gamma \cdot \mathrm{id}_1^{t_1} \cdots \mathrm{id}_N^{t_N}\right) \in \Lambda^{t_1} H^1(S)_1^\vee \otimes \cdots \otimes \Lambda^{t_N} H^1(S)_N^\vee$$

for each monomial γ in the Chern classes

$$c_1(\alpha_s)\ ,\ c_2(\alpha)\ ,\ \beta_i\ ,\ c_1(T_S)\ ,\ c_2(T_S).$$

For degree reasons, we only need to consider $[\gamma]_{t_1,\ldots,t_N}$ with

$$\deg(\gamma) + \sum t_i = 4.$$

This explains that there exist a power series $\mathsf{D}_{\underline{k},\underline{m}}$ in classes $[\gamma]_{t_1,\ldots,t_N}$ as above such that

$$\mathsf{C}_{\underline{k},\underline{m}} = \mathsf{D}_{\underline{k},\underline{m}}.$$

Putting all these together, we conclude that the residual contribution (3.5) is given by

$$\prod_{s,i} f_s(w_i)^{\chi(\alpha_s)-r_{s,i}} \cdot \prod_i g(0)^{r_i-r_{ii}} \cdot \prod_{i\neq j}\left(\frac{g(w_j-w_i)}{w_j-w_i}\right)^{r_i-r_{ij}} \cdot \sum_{k_1,\dots,k_N\geq 0} \int_{\mathrm{SW}^{\underline{k}}(\underline{\beta})} \mathsf{D}_{\underline{k},\underline{m}}. \tag{3.19}$$

Furthermore, $\mathsf{D}_{\underline{k},\underline{m}}$ only depends on $N, \underline{f}$, g, $\underline{k}$, $\underline{m}$ and ranks of α_s. Note that inputs of power series $\mathsf{D}_{\underline{k},\underline{m}}$ are $[\gamma]_{t_1,\dots,t_N}$ for various choices of γ which depend on α_s, β_i and T_S. One can also check by tracing back the argument that $\mathsf{D}_{\underline{k},\underline{m}}$ is polynomial in classes of the form $[\gamma]$ with $\deg(\gamma) = 4$, hence there is no issue of convergence. This proves Theorem A expressing virtual Quot scheme invariants universally in terms of Seiberg-Witten invariants and cohomological data of S.

This chapter is, in part, being prepared for submission for publication. The dissertation author was the primary investigator and the author of the material below.

- Woonam Lim "Virtual χ_{-y}-genera of Quot schemes on surfaces "

Chapter 4

Multiplicative structural formula

4.1 Seiberg-Witten classes of length N

In the previous chapter, we expressed virtual invariants of Quot schemes universally in terms of Seiberg-Witten invariants. Even though this provides a conceptual understanding and even an algorithm for the computation for small numerics, it is hard to deduce any explicit formula or structure of the generating series. In part, difficulty stems from the contributions of higher Seiberg-Witten invariants $\mathrm{SW}^k(\beta)$ for $k > 0$. This aspect of difficulty does not appear for Seiberg-Witten classes of length N which was defined in [L].

Definition 6. Let $\beta \in H^2(S,\mathbb{Z})$ a curve class.

1. We say that a decomposition $\beta = \beta_1 + \cdots + \beta_N$ is Seiberg-Witten non-trivial if $\mathrm{SW}(\beta_i) \neq 0$ for all $i = 1, \ldots, N$.

2. We say that β is of Seiberg-Witten length N if for any length N Seiberg-Witten non-trivial decomposition $\beta = \beta_1 + \cdots + \beta_N$, we have $\mathrm{vd}_{\beta_i} = 0$ for all $i = 1, \ldots, N$.

3. If β is of Seiberg-Witten length N for all $N \geq 1$, then we say that β is a Seiberg-Witten class of arbitrary length.

By the formula (3.19), the residual contribution of $F[\underline{m}, \underline{\beta}]$ is zero unless $\underline{\beta}$ is a Seiberg-Witten non-trivial decomposition. This implies that if β is of Seiberg-Witten length N, then we only need to consider decompositions by curve classes of virtual dimension 0 in order to compute virtual invariants of $\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)$. This observation will eventually yield multiplicativity of the generating series in the next section.

There are several examples of Seiberg-Witten classes of arbitrary length:

1. Zero curve class on arbitrary surfaces,
2. Curve classes in the fibers on relatively minimal elliptic surfaces,
3. Any curve classes on surfaces with $p_g > 0$.

The first example can be seen easily because zero curve class only admits trivial effective decompositions. Furthermore, we clearly have $\mathrm{SW}(0) = 1$ because $\mathsf{Hilb}_S^{\beta=0}$ is a reduced point representing the empty curve.

To explain the second example, we let $p : S \to C$ be a relatively minimal elliptic surface over a smooth curve and denote the fiber class of p by F. By the canonical bundle formula, K_S is a rational multiple of F. Let β be a curve class in the fibers, i.e. it satisfies $\beta.F = 0$. Consider the following decomposition of arbitrary lenth

$$\beta = \beta_1 + \cdots + \beta_N \quad \text{such that} \quad \mathrm{SW}(\beta_i) \neq 0 \quad \text{for all} \quad 1 \leq i \leq N.$$

Note that each β_i is effective class supported on the fibers. By Zariski's lemma, this implies $\beta_i^2 \leq 0$ with the equality if and only if β_i is a rational multiple of F. On the other hand, K_S being a rational multiple of F implies $\beta_i.K_S = 0$. These add up to

$$\mathrm{vd}_{\beta_i} = \frac{\beta_i(\beta_i - K_S)}{2} = \frac{\beta_i^2}{2} \leq 0$$

with equality if and only if β_i is a rational multiple of F. Since $\mathrm{SW}(\beta_i) \neq 0$, we must have non negative virtual dimension. Hence, we have $\mathrm{vd}_{\beta_i} = 0$ for all i. This argument proves not only the fact that β is of Seiberg-Witten class of length N, but also that it suffices to consider decompositions of β into rational multiples of F to compute Quot scheme invariants. It is worth noting that Seiberg-Witten invariants are known for rational multiples of F by [DKO, Proposition 5.8]. Let $F_1,\ldots,F_r$ be the multiple fibers of $p: S \to C$ with multiplicities $m_1,\ldots,m_r$. Then for each β_i, a rational multiple of F, we have

$$\mathrm{SW}(\beta_i) = \sum_{\substack{\beta_i = dF + \sum_j a_j F_j \\ \text{s.t. } 0 \le a_j < m_j}} (-1)^d \binom{2g-2+\chi(\mathscr{O}_S)}{d},$$

where g is the genus of the base curve C.

The third case is a huge collection of examples for Seiberg-Witten classes of arbitrary length. This is a direct consequence of [DKO, Proposition 4.20] saying that $\mathrm{SW}(\beta) \neq 0$ implies $\mathrm{vd}_\beta = 0$ for surfaces with $p_g > 0$. Much more is known for a minimal surface S of general type with $p_g > 0$. In this case, $\mathrm{SW}(\beta) \neq 0$ only for $\beta = 0$ or $\beta = K_S$. Furthermore, it was conjectured [DKO] and proven [CK] that

$$\mathrm{SW}(K_S) = (-1)^{\chi(\mathscr{O}_S)}.$$

More examples of curve classes of Seiberg-Witten length N can be produced via blow up construction as follows. Let $p: \widetilde{S} \to S$ be a blow up of a point with the exceptional divisor E. For each curve class $\beta \in H^2(S,\mathbb{Z})$, we denote by $\widetilde{\beta} \in H^2(\widetilde{S},\mathbb{Z})$ the pullback of β under the morphism p.

Lemma 7. *Assume that β is a curve class of Seiberg-Witten length N. Then $\widetilde{\beta} + \ell E$ is a curve class of Seiberg-Witten length N for all $\ell \in \mathbb{Z}$. Furthermore,*

$$\widetilde{\beta} + \ell E = \left(\widetilde{\beta_1} + \ell_1 E\right) + \cdots + \left(\widetilde{\beta_N} + \ell_N E\right)$$

is a Seiberg-Witten non-trivial decomposition if and only if

$$\beta = \beta_1 + \cdots + \beta_N$$

is a Seiberg-Witten non-trivial decomposition and

$$\ell_i = 0 \text{ or } 1 \quad \text{for all} \quad i = 1, \dots, N.$$

Proof. We use the blow up formula of Seiberg-Witten invariants [DKO, Theorem 4.12]. It says

$$\mathrm{SW}(\widetilde{\beta_i} + \ell_i E) = \tau_{\leq 2 \cdot \mathrm{vd}_{\widetilde{\beta}_i + \ell_i E}} \Big(\mathrm{SW}(\beta_i) \Big) \tag{4.1}$$

after the identification

$$H_* \Big(\mathsf{Pic}_{\widetilde{S}}^{\widetilde{\beta}_i + \ell_i E} \Big) = \Lambda^* H^1(\widetilde{S})^\vee = \Lambda^* H^1(S)^\vee = H_* \Big(\mathsf{Pic}_S^{\beta_i} \Big).$$

Here $\tau_{\leq n}$ denotes the truncation up to homological degree n. Simple calculation shows that

$$\mathrm{vd}_{\widetilde{\beta}_i + \ell_i E} = \frac{(\widetilde{\beta_i} + \ell_i E)(\widetilde{\beta_i} + \ell_i E - \widetilde{K_S} - E)}{2} = \mathrm{vd}_{\beta_i} - \frac{\ell_i(\ell_i - 1)}{2}. \tag{4.2}$$

Now assume that β is a curve class of Seiberg-Witten length N. Having a decomposition

$$\widetilde{\beta} + \ell E = \big(\widetilde{\beta_1} + \ell_1 E \big) + \cdots + \big(\widetilde{\beta_N} + \ell_N E \big)$$

is equivalent to

$$\beta = \beta_1 + \cdots + \beta_N \quad \text{and} \quad \ell = \ell_1 + \cdots + \ell_N.$$

First, we assume that $\mathrm{SW}(\widetilde{\beta_i} + \ell_i E) \neq 0$ for all i. By (4.1), we have $\mathrm{SW}(\beta_i) \neq 0$ for all i. Since β is a class of Seiberg-Witten length N, this implies that $\mathrm{vd}_{\beta_i} = 0$ for all i. According to (4.2),

$$\mathrm{vd}_{\widetilde{\beta}_i + \ell_i E} = - \frac{\ell_i(\ell_i - 1)}{2} \leq 0$$

with equality if and only if $\ell_i = 0$ or 1. Since $\mathrm{SW}(\widetilde{\beta_i} + \ell_i E) \neq 0$, the virtual dimension must be non negative, hence equality condition is satisfied. This proves the one direction of the statement. For the other direction, assume that

$$\mathrm{SW}(\beta_i) \neq 0 \quad \text{and} \quad \ell_i = 0 \text{ or } 1 \quad \text{for all} \quad i = 1, \ldots, N.$$

Since β is a class of Seiberg-Witten length N, we have $\mathrm{vd}_{\beta_i} = 0$ for all i. Again by the formula (4.2), we obtain $\mathrm{vd}_{\widetilde{\beta}_i + \ell_i E} = 0$. Therefore blow up formula (4.1) reads

$$\mathrm{SW}(\widetilde{\beta}_i + \ell_i E) = \tau_{\leq 0}\Big(\mathrm{SW}(\beta_i)\Big) = \mathrm{SW}(\beta_i) \neq 0,$$

as $\mathrm{SW}(\beta_i)$ is a pure degree zero term. This proves the other direction.

In particular, $\widetilde{\beta} + \ell E$ is of Seiberg-Witten length N because every Seiberg-Witten non-trivial decomposition of length N consists of virtual dimension 0 classes due to the formula (4.2). □

4.2 Multiplicative universal series

In this section, we define the shifted generating series of residual invariants for a fixed curve class decomposition $\underline{\beta} = (\beta_1, \ldots, \beta_N)$. When each curve class in the decomposition has virtual dimension zero, we prove that the shifted generating series admits multiplicative universality. This applies to the Quot scheme invariants for curve classes of Seiberg-Witten length N. The main result then expresses the generating series of equivariant Quot scheme invariants in terms of multiplicative universal series and degree 0 Seiberg-Witten invariants.

We begin by defining the shifted generating series of residual invariants corresponding to the decomposition $\underline{\beta} = (\beta_1, \ldots, \beta_N)$.

Definition 8. [1] For a given vector $\underline{\beta}$ of curve classes, we define the shifted generating series of

[1] We take different notational convention from [AJLOP, L].

residual invariants by

$$\widehat{Z}^{\underline{f},g}_{S,N,\underline{\beta}}(q,\underline{w}\,|\,\underline{\alpha}) := \sum_{m_1,\dots,m_N\geq 0} q^{\sum m_i} \int_{\left[S^{[\underline{m}]}\right]\times\left[\mathsf{Hilb}^{\underline{\beta}}_S\right]^{\mathrm{vir}}} e(\underline{\mathrm{Obs}})\cdot \underline{f}(\underline{\alpha}^{[\underline{m}]}_{\underline{\beta}}[\underline{w}])\cdot \frac{g(T^{\mathrm{vir}}_F+N^{\mathrm{vir}}_F)}{e(N^{\mathrm{vir}}_F)}. \quad (4.3)$$

In the above definition, we used the following simplified notation as before:

$$\underline{f}(\underline{\alpha}^{[\underline{m}]}_{\underline{\beta}}[\underline{w}]) := \prod_{s,i} f_s(\alpha^{[m_i]}_{s,\beta_i}[w_i]).$$

We call $\widehat{Z}$ as a "shifted" generating series because the exponent of q differs from the original holomorphic Euler characteristic of the quotient. By Lemma 2, they are related by

$$n = \sum_i \left(m_i - \frac{\beta_i(\beta_i+K_S)}{2}\right) = \sum_i m_i - \sum_i \mathrm{vd}_{\beta_i} - \beta.K_S$$

where $\beta = \sum \beta_i$. Therefore non-shifted generating series of residual invariants should be defined as

$$q^{-\beta.K_S-\sum \mathrm{vd}_{\beta_i}}\cdot \widehat{Z}^{\underline{f},g}_{S,N,\underline{\beta}}(q,\underline{w}\,|\,\underline{\alpha}).$$

If we add up these non-shifted residual generating series for all possible decompositions of β, then we recover the generating series of equivariant Quot scheme invariants

$$Z^{\underline{f},g}_{S,N,\beta}(q,\underline{w}\,|\,\underline{\alpha}) = \sum_{\beta=\beta_1+\cdots+\beta_N} q^{-\beta.K_S-\sum \mathrm{vd}_{\beta_i}}\cdot \widehat{Z}^{\underline{f},g}_{S,N,\underline{\beta}}(q,\underline{w}\,|\,\underline{\alpha}) \quad (4.4)$$

as a result of virtual localization.

Now we suppose that $\underline{\beta} = (\beta_1,\dots,\beta_N)$ consists of curve classes with $\mathrm{vd}_{\beta_i}=0$. In this case, the shifted series (4.3) can be greatly simplified because the virtual class

$$\left[\mathsf{Hilb}^{\beta_i}_S\right]^{\mathrm{vir}}$$

is of zero dimension with degree equal to $\mathrm{SW}(\beta_i)\in\mathbb{Z}$. For the integral calculations in this section, we may assume that

$$\left[\mathsf{Hilb}_S^{\beta_i}\right]^{\mathrm{vir}} = \mathrm{SW}(\beta_i)\cdot[\mathrm{pt}_i].$$

We choose $D_i \in \mathsf{Hilb}_S^{\beta_i}$ to represent a point class $[\mathrm{pt}_i]$. Define the embedding

$$i_{\underline{m}} : S^{[\underline{m}]} \hookrightarrow S^{[\underline{m}]} \times \mathsf{Hilb}_S^{\underline{\beta}},$$

corresponding to the point class representatives $D_1,\ldots,D_N$. Then the shifted residual generating series (4.3) decomposes into contributions from Seiberg-Witten invariants

$$\mathrm{SW}(\underline{\beta}) := \mathrm{SW}(\beta_1)\cdots\mathrm{SW}(\beta_N) \in \mathbb{Z}$$

and contributions from integrals over the products of Hilbert schemes of points

$$W_{S,N,\underline{\beta}}^{\underline{f},g}(q,\underline{w}\,|\,\underline{\alpha}) := \sum_{\underline{m}\geq 0} q^{\Sigma m_i} \int_{\left[S^{[\underline{m}]}\right]} i_{\underline{m}}^{*}\left(e(\underline{\mathrm{Obs}})\cdot \underline{f}(\underline{\alpha}_{\underline{\beta}}^{[\underline{m}]}[\underline{w}])\cdot \frac{g(T_F^{\mathrm{vir}}+N_F^{\mathrm{vir}})}{e(N_F^{\mathrm{vir}})}\right). \tag{4.5}$$

More precisely, we have

$$\widehat{Z}_{S,N,\underline{\beta}}^{\underline{f},g}(q,\underline{w}\,|\,\underline{\alpha}) = \mathrm{SW}(\underline{\beta})\cdot W_{S,N,\underline{\beta}}^{\underline{f},g}(q,\underline{w}\,|\,\underline{\alpha}). \tag{4.6}$$

We now study the series (4.5) to exhibit multiplicative universality following the proof of [EGL, Theorem 4.2]. We first record the pullbacks under $i_{\underline{m}}$ of K-theoretic ingredients of the integrand:

$$\begin{aligned} i_{\underline{m}}^{*}(\mathrm{Obs}_i) &= \mathrm{R}\mathscr{H}om_\pi(\mathscr{O}_{\mathscr{Z}_i}, L_i),\\ i_{\underline{m}}^{*}(\alpha_{s,\beta_i}^{[m_i]}) &= \mathbb{C}^{\chi(\alpha_s)} - \mathrm{R}\pi_*(I_{\mathscr{Z}_i}\otimes L_i^{-1}\otimes q^*\alpha_s),\\ i_{\underline{m}}^{*}(\mathrm{R}\mathscr{H}om_\pi(\mathscr{S}_i,\mathscr{Q}_j)) &= \mathrm{R}\mathscr{H}om_\pi(I_{\mathscr{Z}_i}\otimes L_i^{-1}, \mathscr{O} - I_{\mathscr{Z}_j}\otimes L_j^{-1}).\end{aligned}$$

Here $L_i = \mathscr{O}_S(D_i)$. In fact, the series (4.5) will be unchanged even if we replace L_i by any line bundle in $\mathsf{Pic}_S^{\beta_i}$. Moreover, one may define the series (4.5) even if β_i's are non-algebraic by means of Grothendieck-Riemann-Roch.

Let $\underline{r} = (r_1, \ldots, r_\ell)$ be a vector of ranks $r_s := \mathrm{rk}(\alpha_s)$ of tautological inputs. Define a set $\mathscr{K}_{N,\underline{r}}$ as a set of tuples

$$(S\,|\,\beta_1, \ldots, \beta_N\,|\,\alpha_1, \ldots, \alpha_\ell)$$

such that

$$S \text{ smooth projective surface}, \quad \beta_i \in H^2(S, \mathbb{Z}), \quad \alpha_s \in K^0(S) \;\text{ with }\; \mathrm{rk}(\alpha_s) = r_s.$$

The crucial point to the argument is that we allow disconnected surfaces in the above collection. Hence $\mathscr{K}_{N,\underline{r}}$ is equipped with a commutative monoid structure with respect to the disjoint union operation $\sqcup$. The series (4.5) then define a map

$$W^{\underline{f},g}_{*,N,*}(q, \underline{w}\,|\,*) : \mathscr{K}_{N,\underline{r}} \longrightarrow \Big(F((w_1, \ldots, w_N))[[q]]\Big)^{\times}$$

in an obvious manner. As we considered only multiplicative characteristics to define the series, this map is multiplicative with respect to disjoint union

$$W^{\underline{f},g}_{S\sqcup S',N,\underline{\beta}\sqcup\underline{\beta'}}(q\,|\,\underline{w}\,|\,\underline{\alpha}\sqcup\underline{\alpha'}) = W^{\underline{f},g}_{S,N,\underline{\beta}}(q\,|\,\underline{w}\,|\,\underline{\alpha}) \cdot W^{\underline{f},g}_{S',N,\underline{\beta'}}(q\,|\,\underline{w}\,|\,\underline{\alpha'}).$$

On the other hand, consider a map $c : \mathscr{K}_{N,\underline{r}} \longrightarrow \mathbb{Q}^d$ sending

$$(S\,|\,\beta_1, \ldots, \beta_N\,|\,\alpha_1, \ldots, \alpha_\ell)$$

to the collection of all possible Chern numbers

$$K_S^2\,,\; K_S.c_1(\alpha_s)\,,\; c_1(\alpha_s).c_1(\alpha_t)\,,\; c_2(\alpha_s)\,,\; \chi(\mathscr{O}_S)\,,\; \beta_i.K_S\,,\; \beta_i.c_1(\alpha_s)\,,\; \beta_i.\beta_j. \qquad (4.7)$$

Here d is the number of all possible choices of Chern numbers. Note that the map c is clearly additive with respect to disjoint union.

By argument in Chapter 3, each integral in the series (4.5) can be written as a universal polynomial in Chern numbers (4.7) with coefficient in $F((w_1,\dots,w_N))$. This defines a map h which makes the following diagram commutes:

$$\begin{tikzcd}
\mathscr{K}_{N,\underline{r}} \arrow[rr, "W^{\underline{f},g}_{*,N,*}(q|\underline{w}|*)"] \arrow[dr, "c"'] & & \Big(F((w_1,\dots,w_N))[[q]]\Big)^{\times} \\
& \mathbb{Q}^d \arrow[ur, "h"'] &
\end{tikzcd}$$

Note that the image of the Chern class map c is Zariski dense by toric surface examples. Therefore we conclude that h is a multiplicative function. Equivalently, there exist multiplicative universal series

$$\mathsf{A}\ ,\ \mathsf{B}_s\ ,\ \mathsf{C}_{s,t}\ ,\ \mathsf{D}_s\ ,\ \mathsf{E}\ ,\ \mathsf{U}_i\ ,\ \mathsf{V}_{i,s}\ ,\ \mathsf{W}_{i,j}\ \in \Big(F((w_1,\dots,w_N))[[q]]\Big)^{\times} \tag{4.8}$$

depending only on $N,\underline{f},g$ and $\underline{r}$ such that

$$W^{\underline{f},g}_{S,N,\underline{\beta}}(q,\underline{w}|\underline{\alpha}) = \mathsf{A}^{K_S^2}\cdot\prod_s \mathsf{B}_s^{K_S.c_1(\alpha_s)}\cdot\prod_{s\le t}\mathsf{C}_{s,t}^{c_1(\alpha_s).c_1(\alpha_t)}\cdot\prod_s \mathsf{D}_s^{c_2(\alpha_s)}\cdot\mathsf{E}^{\chi(\mathscr{O}_S)}\cdot$$
$$\prod_i \mathsf{U}_i^{\beta_i.K_S}\cdot\prod_{i,s}\mathsf{V}_{i,s}^{\beta_i.c_1(\alpha_s)}\cdot\prod_{i\le j}\mathsf{W}_{i,j}^{\beta_i.\beta_j}.$$

4.3 Vanishing of some universal series

In this section, we show that some of the universal series in (4.8) are trivial. Let $\beta_1=\cdots=\beta_N=K_S$. We may take $L_i=K_S$ to define the series (4.5), hence

$$i_{\underline{m}}^*(\mathrm{Obs}_i) = \mathrm{R}\mathscr{H}om_\pi(\mathscr{O}_{\mathscr{Z}_i},K_S) = \big(\mathscr{O}_S^{[m_i]}\big)^{\vee}.$$

Note that $\left(\mathscr{O}_S^{[m_i]}\right)^\vee$ has a trivial summand if $m_i > 0$. Therefore $i_{\underline{m}}^*(\underline{\mathrm{Obs}})$ vanishes unless $m_1 = \cdots = m_N = 0$. Since the integral is over a point in this case, we only need to consider equivariantly non trivial terms. By computation of ranks in (3.9), we have

$$\mathrm{rk}(\alpha_{s,K_S}^{[0]}) = \chi(\alpha_s) - \chi(\alpha_s \otimes K_S^{-1}) = -r_s \cdot K_S^2 + K_S.c_1(\alpha_s)$$

and

$$\mathrm{rk}\big(\mathrm{R}\mathscr{H}om_\pi(\mathscr{S}_i, \mathscr{Q}_j)\big) = \chi(K_S) - \chi(\mathscr{O}_S) = 0.$$

Therefore the series (4.5) becomes

$$\prod_{s,i} \left(\frac{1}{f_s(w_i)^{r_s}}\right)^{K_S^2} \cdot f_s(w_i)^{K_S.c_1(\alpha_s)}$$

which is equal to

$$\mathsf{A}^{K_S^2} \cdot \prod_s \mathsf{B}_s^{K_S.c_1(\alpha_s)} \cdot \prod_{s \le t} \mathsf{C}_{s,t}^{c_1(\alpha_s).c_1(\alpha_t)} \cdot \prod_s \mathsf{D}_s^{c_2(\alpha_s)} \cdot \mathsf{E}^{\chi(\mathscr{O}_S)} \cdot \prod_i \mathsf{U}_i^{K_S^2} \cdot \prod_{i,s} \mathsf{V}_{i,s}^{K_S.c_1(\alpha_s)} \cdot \prod_{i \le j} \mathsf{W}_{i,j}^{K_S^2}.$$

The independence of the Chern numbers implies

$$\mathsf{C}_{s,t} = \mathsf{D}_s = \mathsf{E} = 1.$$

Therefore we have

$$W_{S,N,\underline{\beta}}^{\underline{f},g}(q, \underline{w} \,|\, \underline{\alpha}) = \mathsf{A}^{K_S^2} \cdot \prod_s \mathsf{B}_s^{K_S.c_1(\alpha_s)} \cdot \prod_i \mathsf{U}_i^{\beta_i.K_S} \cdot \prod_{i,s} \mathsf{V}_{i,s}^{\beta_i.c_1(\alpha_s)} \cdot \prod_{i \le j} \mathsf{W}_{i,j}^{\beta_i.\beta_j}.$$

Note that this formula is for arbitrary choices of S, $\underline{\beta}$, $\underline{\alpha}$. In this paper, we will apply this formula only for curve classes of Seiberg-Witten length N. In such cases, we have extra relations

$$\beta_i.K_S = \beta_i^2 \quad \text{for all} \quad i = 1, \ldots, N,$$

coming from $\mathrm{vd}_{\beta_i} = 0$. Therefore we may rewrite the multiplicative formula in this case as

$$W^{f,g}_{S,N,\underline{\beta}}(q,\underline{w}\,|\,\underline{\alpha}) = \mathsf{A}^{K_S^2} \cdot \prod_s \mathsf{B}_s^{K_S.c_1(\alpha_s)} \cdot \prod_i \mathsf{U}_i^{\beta_i.K_S} \cdot \prod_{i,s} \mathsf{V}_{i,s}^{\beta_i.c_1(\alpha_s)} \cdot \prod_{i<j} \mathsf{W}_{i,j}^{\beta_i.\beta_j} \tag{4.9}$$

by replacing U_i with $\mathsf{U}_i \cdot \mathsf{W}_{i,i}$. The only difference from the previous formula is that we are missing $\mathsf{W}_{i,i}$'s. We will exclusively use this version of the formula in the paper.

By combining (4.4), (4.6), (4.9), we obtain the multiplicative structural formula for the generating series of equivariant Quot scheme invariants

$$\begin{aligned} Z^{f,g}_{S,N,\beta}(q,\underline{w}\,|\,\underline{\alpha}) = q^{-\beta.K_S} \cdot \mathsf{A}^{K_S^2} \cdot \prod_s \mathsf{B}_s^{K_S.c_1(\alpha_s)} & \\ \cdot \sum_{\substack{\beta=\beta_1+\cdots+\beta_N \\ \mathrm{vd}_{\beta_i}=0}} \mathsf{SW}(\underline{\beta}) \cdot \prod_i \mathsf{U}_i^{\beta_i.K_S} \cdot \prod_{i,s} \mathsf{V}_{i,s}^{\beta_i.c_1(\alpha_s)} \cdot \prod_{i<j} \mathsf{W}_{i,j}^{\beta_i.\beta_j} & \end{aligned} \tag{4.10}$$

whenever β is a curve class of Seiberg-Witten length N.

4.4 Calculation via blow up of $K3$ surfaces

In this section, we study the universal series appearing in (4.9) and find their closed formula up to certain change of variables. Crucial to the argument is the Euler class $i^*_{\underline{m}}(e(\underline{\mathrm{Obs}}))$ in the series (4.5) which allows us to localize the computations to curve geometry. This can be also thought of as cosection localization of [KL] as hinted by the excess obstruction bundle (1.2). We will use the convenient geometry, namely blow up of $K3$ surfaces, so that the curve involved in the calculation is simply $\mathbb{P}^1$. In this case, we can apply the Lagrange-Bürmann formula [Ge] where the aforementioned change of variables naturally arise.

Let S be a blow up of $K3$ surface with the exceptional divisor E. Let $I \sqcup J = [N]$ be a partition of the set $[N] = \{1,\ldots,N\}$ into two parts. In this section, we make a convention to denote the indices in I, J and $[N]$ by i, j and k, respectively. Define a vector of curve classes

$$\underline{E}_{I\sqcup J}=(\beta_1,\ldots,\beta_N) \quad \text{where} \quad \begin{cases}\beta_i=E, & \text{if } i\in I,\\ \beta_j=0, & \text{if } j\in J.\end{cases}$$

Note that every Hilbert scheme of curves for $\beta_1,\ldots,\beta_N$ is a reduced point

$$\mathsf{Hilb}_S^{\beta_i}=\{E\}, \quad \mathsf{Hilb}_S^{\beta_j}=\{0\}.$$

Hence the morphism $i_{\underline{m}}$ corresponding to the choices $D_i=E$ and $D_j=0$ is an isomorphism

$$i_{\underline{m}}: S^{[\underline{m}]} \xrightarrow{\sim} S^{[\underline{m}]}\times \mathsf{Hilb}_S^{\underline{E}_{I\sqcup J}}.$$

Since $K_S=E$ and $E^2=-1$, the equation (4.9) reads

$$W^{f,g}_{S,N,\underline{E}_{I\sqcup J}}(q,\underline{w}\,|\,\underline{\alpha})=\mathsf{A}^{-1}\cdot\prod_s \mathsf{B}_s^{E.c_1(\alpha_s)}\cdot\prod_{i\in I}\mathsf{U}_i^{-1}\cdot\prod_{i\in I,s}\mathsf{V}_{i,s}^{E.c_1(\alpha_s)}\cdot\prod_{\substack{i_1<i_2\\ i_1,i_2\in I}}\mathsf{W}_{i_1,i_2}^{-1}. \tag{4.11}$$

We now compute $W^{f,g}_{S,N,\underline{E}_{I\sqcup J}}(q,\underline{w}\,|\,\underline{\alpha})$ using the formula (4.5) for arbitrary $d_s=E.c_1(\alpha_s)$ and partitions $I\sqcup J=[N]$. Comparison of the outcome to the multiplicative formula (4.11) will determine all the universal series. For the computation, we begin by analyzing $i^*_{\underline{m}}(e(\underline{\mathrm{Obs}}))$. Note that if $i\in I$, we have

$$i^*_{\underline{m}}(\mathrm{Obs}_i)=\mathrm{R}\mathscr{H}om_\pi(\mathscr{O}_{\mathscr{Z}_i},E)\;=\big(\mathscr{O}_S^{[m_i]}\big)^\vee.$$

This follows from the relative Serre duality and $K_S=E$. Therefore we conclude that $i^*_{\underline{m}}(e(\mathrm{Obs}_i))$ vanishes if $m_i>0$ due to a trivial factor. Hence, we assume that $m_i=0$ for all $i\in I$. On the other hand, if $j\in J$, we have

$$i^*_{\underline{m}}(\mathrm{Obs}_j)=\mathrm{R}\mathscr{H}om_\pi(\mathscr{O}_{\mathscr{Z}_i},\mathscr{O}_S)=\big(E^{[m_j]}\big)^\vee$$

for the similar reason. Note that we have a tautological section of $E^{[m_j]}$ that vanishes exactly over $E^{[m_j]}$. This implies that

$$i_{\underline{m}}^*(e(\mathrm{Obs}_j)) = (-1)^{m_j}\left[E^{[m_j]}\right] = (-1)^{m_j}\left[\mathbb{P}^{m_j}\right].$$

Denote by

$$\iota_{\underline{m}} : \prod_{j\in J}\mathbb{P}^{m_j} \hookrightarrow \prod_{j\in J} S^{[m_j]}$$

the product of natural embeddings of the Hilbert schemes on E into the Hilbert schemes on S. Then the above discussion about the Euler class sums up to

$$i_{\underline{m}}^*(e(\underline{\mathrm{Obs}})) = (-1)^{\Sigma m_j}\cdot \iota_{\underline{m}*}\left[\prod_{j\in J}\mathbb{P}^{m_j}\right]$$

Therefore we obtain the formula below which localizes the calculation to the Hilbert scheme geometries of the exceptional curve E:

$$W^{\underline{f},g}_{S,N,\underline{E}_{I\sqcup J}}(q,\underline{w}\,|\,\underline{\alpha}) = \sum_{\forall j\in J,\ m_j\geq 0}(-q)^{\Sigma m_j}\int_{\prod_j \mathbb{P}^{m_j}}\iota_{\underline{m}}^*\left(\underline{f}(\underline{\alpha}_{\underline{\beta}}^{[\underline{m}]}[\underline{w}])\cdot\frac{g(T_F^{\mathrm{vir}}+N_F^{\mathrm{vir}})}{e(N_F^{\mathrm{vir}})}\right). \tag{4.12}$$

We now write down the integrand explicitly after pulling back under $\iota_{\underline{m}}$. This is possible because the Hilbert scheme of points on $\mathbb{P}^1$ is simply a projective space over which we understand the tautological classes explicitly. Recall that $\alpha_s\big|_E$ is a K-theory class on $E=\mathbb{P}^1$ of rank r_s and degree d_s. From the calculations of [L, OP], Sections 3 and 5 respectively, we have

1. $\iota_{\underline{m}}^*(\alpha_{s,E}^{[0]}[w_i]) = \mathbb{C}^{r_s+d_s}[w_i]$,
2. $\iota_{\underline{m}}^*(\alpha_{s,0}^{[m_j]}[w_j]) = \mathscr{O}(-h_j)^{\oplus r_s m_j}[w_j] + \big(\mathscr{O}-\mathscr{O}(-h_j)\big)^{\oplus(r_s+d_s)}[w_j]$,
3. $\iota_{\underline{m}}^*(T_F^{\mathrm{vir}}) = \sum_j \Big(\mathscr{O}(-h_j)^{\oplus m_j}+\mathscr{O}(h_j)-\mathscr{O}\Big)$,
4. $\iota_{\underline{m}}^*(N^{\mathrm{vir}}) = \sum_{j_1\neq j_2}\Big(\mathscr{O}(-h_{j_2})^{\oplus m_{j_2}}+\mathscr{O}(h_{j_1})-\mathscr{O}(h_{j_1}-h_{j_2})\Big)[w_{j_2}-w_{j_1}]$

$$+\sum_{i,j}\Big(\mathscr{O}(-h_j)^{\oplus m_j}[w_j-w_i]+\mathscr{O}(h_j)[w_i-w_j]\Big).$$

Here the h_j's denote the hyperplane classes of each projective spaces.

Having the integrands understood, now we perform the integral over the projective spaces. Note that the integration over the product of projective spaces can be thought of as extracting the suitable coefficient of h_j. We denote this operation by square brackets. After the careful analysis, we obtain that

$$W^{f,g}_{S,N,\underline{E}_{I\sqcup J}}(q,\underline{w}\,|\,\underline{\alpha})=\sum_{m_j\geq 0}(-q)^{\sum m_j}\left[\prod_{j\in J}h^{m_j}\right]\ \prod_{j\in J}\Phi_j(h_j)^{m_j}\cdot\Psi_{I\sqcup J}(\{h_j\}_{j\in J})$$

where

$$\begin{aligned}\Phi_j(h_j)&=(-h_j)\cdot\prod_s f_s(-h_j+w_j)^{r_s}\cdot\prod_k\frac{g(-h_j+w_j-w_k)}{-h_j+w_j-w_k},\\ \Psi_{I\sqcup J}&=\frac{1}{g(0)^{|J|}}\cdot\prod_j h_j\cdot\prod_{s,k}f_s(w_k)^{r_s+d_s}\cdot\prod_{s,j}\frac{1}{f_s(-h_j+w_j)^{r_s+d_s}}\cdot\prod_{j,k}\frac{g(h_j-w_j+w_k)}{h_j-w_j+w_k}\\ &\quad\cdot\prod_{j_1\neq j_2}\frac{h_{j_1}-w_{j_1}-h_{j_2}+w_{j_2}}{g(h_{j_1}-w_{j_1}-h_{j_2}+w_{j_2})}.\end{aligned}\tag{4.13}$$

Recall the convention that we use i,j,k for indices in $I,J,[N]$, respectively.

Even though the above expressions seem wild, we are able to apply the multivariable Lagrange-Bürmann formula [G] which says

$$W^{f,g}_{S,N,\underline{E}_{I\sqcup J}}(q,\underline{w}\,|\,\underline{\alpha})=\frac{\Psi_{I\sqcup J}}{K_{I\sqcup J}}(\{h_j\}_{j\in J})$$

up to change of variables

$$q=\frac{-h_j}{\Phi_j(h_j)}\quad\text{with}\quad h_j(q=0)=0,$$

where

$$K_{I\sqcup J} = \prod_{j\in J}\left(1 - h_j \frac{d}{dh_j}\log\Phi_j(h_j)\right).$$

Plugging in the functions (4.13), we obtain

$$q = \prod_s \frac{1}{f_s(-h_j+w_j)^{r_s}} \cdot \prod_{k=1}^{N} \frac{-h_j+w_j-w_k}{g(-h_j+w_j-w_k)} \quad \text{with} \quad h_j(q=0)=0, \tag{4.14}$$

and

$$K_{I\sqcup J} = \prod_j h_j \cdot \prod_j \left(\sum_s r_s \cdot \frac{f_s'}{f_s}(-h_j+w_j) + \sum_{k=1}^{N}\left(\frac{g'}{g}(-h_j+w_j-w_k) + \frac{1}{h_j-w_j+w_k}\right)\right).$$

Note that the change of variable formulas (4.14) are different for each h_j's. To work with a single formula, we make the additional change of variables

$$H_j = h_j - w_j. \tag{4.15}$$

According to this additional change of variables, we obtain

$$W^{f,g}_{S,N,\underline{E}_{I\sqcup J}}(q,\underline{w}\,|\,\underline{\alpha}) = \frac{\frac{1}{g(0)^{|J|}} \cdot \prod_{s,k} f_s(w_k)^{r_s+d_s} \cdot \prod_{s,j} \frac{1}{f_s(-H_j)^{r_s+d_s}} \cdot \prod_{j,k} \frac{g(H_j+w_k)}{H_j+w_k} \cdot \prod_{j_1\neq j_2} \frac{H_{j_1}-H_{j_2}}{g(H_{j_1}-H_{j_2})}}{\prod_j \left(\sum_s r_s \cdot \frac{f_s'}{f_s}(-H_j) + \sum_{k=1}^{N}\left(\frac{g'}{g}(-H_j-w_k) + \frac{1}{H_j+w_k}\right)\right)}, \tag{4.16}$$

where

$$q = \prod_s \frac{1}{f_s(-H_j)^{r_s}} \cdot \prod_{k=1}^{N} \frac{-H_j-w_k}{g(-H_j-w_k)} \quad \text{with} \quad H_j(q=0) = -w_j. \tag{4.17}$$

As claimed before, we now have a single change of variable formula together with distinct initial conditions. This observation will be crucial to the computations in the subsequent chapters.

The formula (4.16) determines all the universal series by comparing with

$$W^{\underline{f},g}_{S,N,\underline{E}_{I\sqcup J}}(q,\underline{w}\,|\,\underline{\alpha}) = \mathsf{A}^{-1}\cdot\prod_s \mathsf{B}_s^{d_s}\cdot\prod_{i\in I}\mathsf{U}_i^{-1}\cdot\prod_{i\in I,s}\mathsf{V}_{i,s}^{d_s}\cdot\prod_{\substack{i_1<i_2\\ i_1,i_2\in I}}\mathsf{W}_{i_1,i_2}^{-1}$$

for various choices of partitions $I\sqcup J$. In the formula of universal series below, we do not follow the previous convention to use i,j,k for indices in $I,J,[N]$ because partition $I\sqcup J$ is not involved in the universal series. First, we find

$$\mathsf{B}_s = \prod_{k=1}^{N}\frac{f_s(w_k)}{f_s(-H_k)},\quad \mathsf{V}_{i,s} = f_s(-H_i),$$

by considering the terms with exponent d_s and letting $I=\emptyset$ or $I=\{i\}$. Second, we find

$$\mathsf{A} = g(0)^N\cdot\prod_s\prod_{i=1}^{N}\frac{f_s(-H_i)^{r_s}}{f_s(w_i)^{r_s}}\cdot\prod_{i,k\in[N]}\frac{H_i+w_k}{g(H_i+w_k)}\cdot\prod_{\substack{i_1\neq i_2\\ i_1,i_2\in[N]}}\frac{g(H_{i_1}-H_{i_2})}{H_{i_1}-H_{i_2}}$$
$$\cdot\prod_{i=1}^{N}\left(\sum_s r_s\cdot\frac{f_s'}{f_s}(-H_i)+\sum_{k=1}^{N}\left(\frac{g'}{g}(-H_i-w_k)+\frac{1}{H_i+w_k}\right)\right)$$

and

$$\mathsf{U}_i = \frac{1}{g(0)}\cdot\prod_s\frac{1}{f_s(-H_i)^{r_s}}\cdot\prod_{k=1}^{N}\frac{g(H_i+w_k)}{H_i+w_k}\cdot\prod_{\substack{i'\neq i\\ i'\in[N]}}\frac{H_{i'}-H_i}{g(H_{i'}-H_i)}\cdot\frac{H_i-H_{i'}}{g(H_i-H_{i'})}$$
$$\cdot\left(\sum_s r_s\cdot\frac{f_s'}{f_s}(-H_i)+\sum_{k=1}^{N}\left(\frac{g'}{g}(-H_i-w_k)+\frac{1}{H_i+w_k}\right)\right)^{-1}$$

by comparing the terms with exponent (-1) and letting $I=\emptyset$ or $I=\{i\}$. Finally,

$$\mathsf{W}_{i_1,i_2} = \frac{g(H_{i_1}-H_{i_2})}{H_{i_1}-H_{i_2}}\cdot\frac{g(H_{i_2}-H_{i_1})}{H_{i_2}-H_{i_1}}$$

by considering $I=\{i_1,i_2\}$ for $i_1<i_2$.

This chapter is, in part, being prepared for submission for publication. The dissertation author was the primary investigator and the author of the first material and the collaborator and the coauthor for the second material.

- Woonam Lim “Virtual χ_{-y}-genera of Quot schemes on surfaces ”
- Noah Arbesfeld, Drew Johnson, Woonam Lim, Dragos Oprea, Rahul Pandharipande “The virtual K-theory of Quot schemes of surfaces”

Chapter 5

Virtual Euler characteristics and χ_{-y}-genera

In this chapter, we study the generating series of the virtual Euler characteristics and virtual χ_{-y}-genera of Quot schemes, denoted by

$$Z^{e^{\mathrm{vir}}}_{S,N,\beta}(q) \in \mathbb{Q}((q)), \quad Z^{\chi^{\mathrm{vir}}_{-y}}_{S,N,\beta}(q) \in \mathbb{Q}(y)((q)),$$

respectively. Both of the series are conjecturally given by rational functions in the q variable. Since virtual χ_{-y}-genus recovers virtual Euler characteristic at $y = 1$, we only record the virtual χ_{-y}-genus from now.

When β is a curve class of Seiberg-Witten length N, we can use the structural formula (4.10) and the computation of the universal series in the previous chapter to study $Z^{\chi^{\mathrm{vir}}_{-y}}_{S,N,\beta}(q)$. If Seiberg-Witten theory of decompositions of β is well-understood, the structural formula yields explicit formulas. We do this for the following three cases:

1. S arbitrary and $\beta = 0$,
2. S relatively minimal elliptic surface and β rational multiple of the fiber,
3. S minimal surface of general type with $p_g > 0$ and β multiple of K_S.

Furthermore, we prove blow up formulas of the generating series when the curve class is of Seiberg-Witten length N. Answers for the above three cases and the blow up formulas are enough to conclude that $Z^{\chi^{\mathrm{vir}}_{-y}}_{S,N,\beta}(q)$ is a rational function for arbitrary surfaces with $p_g > 0$. This will be generalized to K-theoretic descendent series for any Seiberg-Witten classes of length N in the next chapter.

Note that $Z^{\chi^{\mathrm{vir}}_{-y}}_{S,N,\beta}(q)$ is obtained by specializing the general series $Z^{f,g}_{S,N,\beta}(q\,|\,\underline{\alpha})$ to the case of

$$\ell = 0, \quad g(z) = \frac{z(1-ye^{-z})}{1-e^{-z}}$$

by the virtual Hirzebruch-Riemann-Roch theorem of [FG]. Therefore the structural formula (4.10) reads

$$Z^{\chi^{\mathrm{vir}}_{-y}}_{S,N,\beta}(q) = q^{-\beta.K_S} \cdot \mathsf{A}^{K_S^2} \sum_{\substack{\beta=\beta_1+\cdots+\beta_N \\ \mathrm{vd}_{\beta_i}=0}} \mathsf{SW}(\underline{\beta}) \cdot \prod_i \mathsf{U}_i^{\beta_i.K_S} \cdot \prod_{i<j} \mathsf{W}_{i,j}^{\beta_i.\beta_j} \tag{5.1}$$

where

$$\begin{aligned}
\mathsf{A} &= (1-y)^N \cdot \prod_{i,k\in[N]} \frac{1-e^{-H_i-w_k}}{1-ye^{-H_i-w_k}} \cdot \prod_{\substack{i_1\neq i_2 \\ i_1,i_2\in[N]}} \frac{1-ye^{-H_{i_1}+H_{i_2}}}{1-e^{-H_{i_1}+H_{i_2}}} \\
&\qquad\qquad\qquad\qquad \cdot \prod_{i=1}^N \left(\sum_{k=1}^N \Big(-\frac{(1-y)e^{H_i+w_k}}{(1-ye^{H_i+w_k})(1-e^{H_i+w_k})} \Big) \right), \\
\mathsf{U}_i &= \frac{1}{(1-y)} \cdot \prod_{k=1}^N \frac{1-ye^{-H_i-w_k}}{1-e^{-H_i-w_k}} \cdot \prod_{\substack{i'\neq i \\ i'\in[N]}} \left(\frac{1-e^{-H_{i'}+H_i}}{1-ye^{-H_{i'}+H_i}} \cdot \frac{1-e^{-H_i+H_{i'}}}{1-ye^{-H_i+H_{i'}}} \right) \\
&\qquad\qquad\qquad\qquad \cdot \left(\sum_{k=1}^N \Big(-\frac{(1-y)e^{H_i+w_k}}{(1-ye^{H_i+w_k})(1-e^{H_i+w_k})} \Big) \right)^{-1}, \\
\mathsf{W}_{i,j} &= \frac{1-ye^{-H_i+H_j}}{1-e^{-H_i+H_j}} \cdot \frac{1-ye^{-H_j+H_i}}{1-e^{-H_j+H_i}},
\end{aligned} \tag{5.2}$$

for change of variables

$$q = \prod_{k=1}^N \frac{1-e^{H_i+w_k}}{1-ye^{H_i+w_k}} \quad \text{with} \quad H_i(q=0) = -w_i. \tag{5.3}$$

Note that all the universal series in (5.2) and the change of variables (5.3) can be simplified by the exponential change of variables

$$e^{H_i} = x_i, \quad e^{w_k} = t_k. \tag{5.4}$$

We record the simplified version here:

$$\begin{aligned}
\mathsf{A} &= (1-y)^N \cdot \prod_{i,k\in[N]} \frac{1-x_i^{-1}t_k^{-1}}{1-yx_i^{-1}t_k^{-1}} \cdot \prod_{\substack{i_1\neq i_2 \\ i_1,i_2\in[N]}} \frac{1-yx_{i_1}^{-1}x_{i_2}}{1-x_{i_1}^{-1}x_{i_2}} \\
&\qquad\qquad \cdot \prod_{i=1}^N \left(\sum_{k=1}^N \Big(-\frac{(1-y)x_it_k}{(1-yx_it_k)(1-x_it_k)} \Big) \right), \\
\mathsf{U}_i &= \frac{1}{(1-y)} \cdot \prod_{k=1}^N \frac{1-yx_i^{-1}t_k^{-1}}{1-x_i^{-1}t_k^{-1}} \cdot \prod_{\substack{i'\neq i \\ i'\in[N]}} \left(\frac{1-x_{i'}^{-1}x_i}{1-yx_{i'}^{-1}x_i} \cdot \frac{1-x_i^{-1}x_{i'}}{1-yx_i^{-1}x_{i'}} \right) \\
&\qquad\qquad \cdot \left(\sum_{k=1}^N \Big(-\frac{(1-y)x_it_k}{(1-yx_it_k)(1-x_it_k)} \Big) \right)^{-1}, \\
\mathsf{W}_{i,j} &= \frac{1-yx_i^{-1}x_j}{1-x_i^{-1}x_j} \cdot \frac{1-yx_j^{-1}x_i}{1-x_j^{-1}x_i},
\end{aligned} \tag{5.5}$$

and

$$q = \prod_{k=1}^N \frac{1-x_it_k}{1-yx_it_k} \quad \text{with} \quad x_i(q=0) = t_k^{-1}. \tag{5.6}$$

Note that rationality starts to appear from the formulas. More precisely, we have

$$\mathsf{A}\ ,\ \mathsf{U}_i\ ,\ \mathsf{W}_{i,j}\ \in\ \mathbb{Q}(y)(t_1,\ldots,t_N)(x_1,\ldots,x_N).$$

We repeatedly use these formulas to study the following cases: punctual Quot schemes, Quot schemes on relatively minimal elliptic surfaces, Quot schemes on minimal general type surfaces with $p_g > 0$ and blow up formulas.

5.1 Punctual Quot schemes

The punctual Quot scheme $\mathsf{Quot}_S(\mathbb{C}^N, n)$ parametrizes zero dimensional quotients of a trivial vector bundle. We omitted $\beta = 0$ from the notation. Since $\beta = 0$ only admits a trivial decomposition, the structural formula for virtual χ_{-y}-genera (5.1) reads

$$Z^{\chi^{\mathrm{vir}}_{-y}}_{S,N,0}(q, \underline{w}) = \mathsf{A}^{K_S^2}.$$

Note that the universal series in (5.5)

$$\mathsf{A} \in \mathbb{Q}(y)(t_1, \ldots, t_N)(x_1, \ldots, x_N)$$

is symmetric in $x_1, \ldots, x_N$. On the other hand, $x_1, \ldots, x_N$ are N distinct roots of the equation

$$q = \prod_{k=1}^{N} \frac{1 - xt_k}{1 - yxt_k}$$

corresponding to distinct initial data

$$x_i(q = 0) = t_k^{-1}.$$

One can rewrite the above equation as a polynomial equation of degree N

$$q \prod_{k=1}^{N} (1 - yxt_k) - \prod_{k=1}^{N} (1 - xt_k) = 0.$$

Since $x_1, \ldots, x_N$ are roots of the polynomial equation, elementary symmetric polynomials

$$\sigma_i(x_1, \ldots, x_N), \quad i = 1, \ldots, N$$

can be expressed as a rational function in $t_1,\dots,t_N$ and q. Therefore we can express the universal series as a rational function in the q variable

$$\mathsf{A} \in \mathbb{Q}(y)(t_1,\dots,t_N)(q).$$

Furthermore, one can easily check that there are no poles at $t_1 = \cdots = t_N = 1$ or equivalently at $w_1 = \cdots = w_N = 0$.

To obtain non-equivariant answer, we need to find the limit

$$\mathsf{A}\Big|_{\underline{t}=1} \in \mathbb{Q}(y)(q).$$

In this limit, the equation becomes

$$q = \left(\frac{1-x}{1-yx}\right)^N.$$

We make a further substitutions

$$\frac{1-x}{1-yx} = z, \quad \frac{1-x_i}{1-yx_i} = z_i. \tag{5.7}$$

Using

$$x_i = \frac{1-z_i}{1-yz_i},$$

straightforward computations gives

$$\mathsf{A}\Big|_{\underline{t}=1} = (-N)^N \cdot \prod_{i=1}^{N} \left(\frac{-z_i}{1-(1+y)z_i}\right)^N \cdot \prod_{i=1}^{N} \frac{(1-z_i)(1-yz_i)}{z_i} \cdot \prod_{\substack{i\neq j \\ i,j\in[N]}} \frac{1-(1+y)z_i+yz_iz_j}{z_j-z_i}. \tag{5.8}$$

Here $z_1, \ldots, z_N$ are are roots of the equation

$$q = z^N$$

or equivalently

$$h(z) := z^N - q = 0.$$

We can factor the polynomial $h(z)$ using the roots

$$h(z) = \prod_{i=1}^{N} (z - z_i).$$

By plugging in certain values in the z variable and normalizing, we obtain

$$\prod_{i=1}^{N} (-z_i) = h(0) = -q$$

and

$$\prod_{i=1}^{N} (1 - uz_i) = u^N \cdot h(1/u) = 1 - u^N q.$$

Using the derivative formula and $z_i^N = q$, we further obtain

$$\prod_{\substack{i \neq j \\ i,j \in [N]}} (z_j - z_i) = \prod_{i=1}^{N} h'(z_i) = \prod_{i=1}^{N} N z_i^{N-1} = \prod_{i=1}^{N} \frac{Nq}{z_i} = N^N (-q)^{N-1}.$$

Substituting these identities to the equation (5.8) yields a simplified formula

$$\mathsf{A}\Big|_{t=1} = \frac{(1-q)(1-y^N q)}{\big(1-(1+y)^N q\big)^N} \cdot \prod_{\substack{i \neq j \\ i,j \in [N]}} (1-(1+y)z_i + y z_i z_j).$$

Functional equation of the term

$$\mathsf{P}_N(q,y) := \prod_{i\neq j}(1-(1+y)z_i + yz_iz_j)$$

can also be obtained. Straightforward computation shows

$$\begin{aligned}\mathsf{P}_N(q,y) &= \prod_{i\neq j}(yz_iz_j)\cdot\prod_{i\neq j}(1-(1+y^{-1})z_j^{-1}+y^{-1}z_i^{-1}z_j^{-1})\\ &= (y^Nq^2)^{N-1}\cdot\mathsf{P}_N(q^{-1},y^{-1})\end{aligned}$$

where we used the fact that $z_1^{-1},\dots,z_N^{-1}$ are roots of the equation

$$z^N = q^{-1}.$$

5.2 Relatively minimal elliptic surfaces

Let $p : S \to C$ be a relatively minimal elliptic surface with the fiber class F. Recall that the canonical bundle formula says that K_S is a rational multiple of F. Suppose that β is a curve class supported on fibers, i.e. $\beta.F = 0$. Since β is a Seiberg-Witten class of arbitrary length, the generating series of virtual χ_{-y}-genera follows the structural formula (5.1)

$$Z^{\chi_{-y}^{\mathrm{vir}}}_{S,N,\beta}(q,\underline{w}) = q^{-\beta.K_S}\cdot\mathsf{A}^{K_S^2}\sum_{\substack{\beta=\beta_1+\cdots+\beta_N\\ \mathrm{vd}_{\beta_i}=0}}\mathsf{SW}(\underline{\beta})\cdot\prod_i\mathsf{U}_i^{\beta_i.K_S}\cdot\prod_{i<j}\mathsf{W}_{i,j}^{\beta_i.\beta_j}.$$

On the other hand, we have seen in Section 4.1 that β_i in the effective decomposition

$$\beta = \beta_1+\cdots+\beta_N$$

satisfies $\mathrm{vd}_{\beta_i} = 0$ if and only if β_i is a rational multiple of F. Therefore all the exponents appearing in the above structural formula vanish

$$\beta.K_S = K_S^2 = \beta_i.K_S = \beta_i.\beta_j = 0.$$

Hence we conclude that

$$Z^{\chi_{-y}^{\mathrm{vir}}}_{S,N,\beta}(q) = \sum_{\substack{\beta=\beta_1+\cdots+\beta_N \\ \mathrm{vd}_{\beta_i}=0}} \mathrm{SW}(\beta_1)\cdots\mathrm{SW}(\beta_N).$$

In other words, we have

$$\chi_{-y}^{\mathrm{vir}}\big(\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)\big) = \begin{cases} \displaystyle\sum_{\substack{\beta=\beta_1+\cdots+\beta_N \\ \mathrm{vd}_{\beta_i}=0}} \mathrm{SW}(\beta_1)\cdots\mathrm{SW}(\beta_N)\,, & \text{if } n=0 \\ 0\,, & \text{otherwise.} \end{cases}$$

5.3 Minimal surfaces of general type with $p_g > 0$

Let S be a minimal surface of general type with $p_g > 0$. Recall that $\mathrm{SW}(\beta_i) \neq 0$ only if $\beta_i = 0$ or K_S. Therefore virtual χ_{-y}-genus of Quot scheme invariant vanish unless $\beta = \ell K_S$ for some $0 \leq \ell \leq N$. For the rest of the section, we suppose that

$$\beta = \ell K_S \quad \text{for some} \quad 0 \leq \ell \leq N.$$

Since $\mathrm{SW}(\underline{\beta}) = 0$ unless $\beta_i = 0$ or K_S for all i, it is enough to consider the decompositions

$$\underline{K}_{I\sqcup J} := (\beta_1,\ldots,\beta_N) \quad \text{where} \quad \begin{cases} \beta_i = K_S, & \text{if } i \in I \\ \beta_j = 0, & \text{if } j \in J. \end{cases}$$

for partitions $I \sqcup J = [N]$ such that $|I| = \ell$. Therefore the formula (5.1) reads

$$Z^{\chi^{\mathrm{vir}}_{-y}}_{S,N,\beta}(q,\underline{w}) = q^{-\ell K_S^2} \cdot \mathrm{SW}(K_S)^{\ell} \sum_{\substack{I \sqcup J = [N] \\ \text{s.t. } |I| = \ell}} \Big(\mathsf{A} \cdot \prod_{i \in I} \mathsf{U}_i \cdot \prod_{\substack{i_1 < i_2 \\ i_1, i_2 \in I}} \mathsf{W}_{i_1,i_2} \Big)^{K_S^2}.$$

In this section, we use i, j, k to denote indices in $I, J, [N]$, repsectively.

Let g be the arithmetic genus of a canonical curve given by $K_S^2 = g - 1$. For each $0 \leq \ell \leq N$ and $g \geq 0$, define

$$\mathsf{G}_{N,\ell,g} := \sum_{\substack{I \sqcup J = [N] \\ \text{s.t. } |I| = \ell}} \Big(\mathsf{A} \cdot \prod_{i \in I} \mathsf{U}_i \cdot \prod_{\substack{i_1 < i_2 \\ i_1, i_2 \in I}} \mathsf{W}_{i_1,i_2} \Big)^{g-1} \in \mathbb{Q}(y)(t_1, \dots, t_N)(x_1, \dots, x_N) \tag{5.9}$$

where A, U_i, W_{i_1,i_2} are as in the list (5.5). Note that the range of parameter g that is relevant to the case of minimal surface of general type with $p_g > 0$ is

$$g = K_S^2 + 1 \geq 2$$

because K_S is big and nef. We will see in the next section that $\mathsf{G}_{N,\ell,g}$ for $g = 0$ is related to blow up formulas.

Straightforward computation shows that

$$\Big(\mathsf{A} \cdot \prod_{i \in I} \mathsf{U}_i \cdot \prod_{\substack{i_1 < i_2 \\ i_1, i_2 \in I}} \mathsf{W}_{i_1,i_2} \Big)$$

is equal to

$$(1-y)^{N-\ell} \cdot \prod_{j,k} \frac{1 - x_j^{-1} t_k^{-1}}{1 - y x_j^{-1} t_k^{-1}} \cdot \prod_{j_1 \neq j_2} \frac{1 - y x_{j_1}^{-1} x_{j_2}}{1 - x_{j_1}^{-1} x_{j_2}} \cdot \prod_j \left(\sum_{k=1}^{N} \Big(- \frac{(1-y) x_j t_k}{(1 - y x_j t_k)(1 - x_j t_k)} \Big) \right).$$

After summing over these terms with the exponent $(g-1)$ for all partitions $I \sqcup J = [N]$ such that $|I| = \ell$, one can check that $\mathsf{G}_{N,\ell,g}$ is symmetric in $x_1, \dots, x_N$. By the same argument as in Section

5.1, we can rewrite $\mathsf{G}_{N,\ell,g}$ as a function in the q variable

$$\mathsf{G}_{N,\ell,g} \in \mathbb{Q}(y)(t_1,\dots,t_N)(q)$$

which has no pole at $t_1 = \cdots = t_N = 1$. Using the same extra change of variables as in (5.7), we find the non-equivariant answer

$$\mathsf{G}_{N,\ell,g}\Big|_{\underline{t}=1} = \sum_{\substack{I \sqcup J = [N] \\ \text{s.t. } |I|=\ell}} \left((-N)^{N-\ell} \cdot \prod_{j\in J} \left(\frac{-z_j}{1-(1+y)z_j} \right)^N \cdot \prod_{j\in J} \frac{(1-z_j)(1-yz_j)}{z_j} \cdot \right.$$
$$\left. \prod_{\substack{j_1 \neq j_2 \\ j_1, j_2 \in J}} \frac{1-(1+y)z_{j_1} + yz_{j_1}z_{j_2}}{z_{j_2}-z_{j_1}} \right)^{g-1} \tag{5.10}$$

where $z_1,\dots,z_N$ are roots of the equation $q = z^N$. Therefore, we conclude that the answer is given by a rational function in the q variable

$$Z^{\chi^{\mathrm{vir}}_{-y}}_{S,N,\ell K_S}(q) = q^{-\ell K_S^2} \cdot \mathrm{SW}(K_S)^{\ell} \cdot \mathsf{G}_{N,\ell,g}\Big|_{\underline{t}=1} \in \mathbb{Q}(y)(q)$$

where $g = K_S^2 + 1$ and $\mathrm{SW}(K_S) = (-1)^{\chi(\mathscr{O}_S)}$.

Rational functions $\mathsf{G}_{N,\ell,g}\Big|_{\underline{t}=1}$ can be explicitly calculated for choices of relatively small parameters N, ℓ, g using the symmetric reduction algorithm. We compute few examples here. Two extreme cases, when $\ell = 0$ or N, are given by

$$\mathsf{G}_{N,0,g}\Big|_{\underline{t}=1} = \left(\mathsf{A}\Big|_{\underline{t}=1} \right)^{g-1}, \quad \mathsf{G}_{N,N,g}\Big|_{\underline{t}=1} = 1.$$

When $\ell = N-1$, we have

$$\begin{aligned}
\mathsf{G}_{N,N-1,g}\Big|_{\underline{t}=1} &= \sum_{i=1}^{N} \left((-1)^{N+1} \cdot N \cdot q \cdot \frac{(1-z_i)(1-yz_i)}{z_i\big(1-(1+y)z_i\big)^N} \right)^{g-1} \\
&= (-1)^{(N+1)(g-1)} \cdot N^{g-1} \cdot q^{g-1} \cdot \sum_{n\in\mathbb{Z}} Nq^n [z^{Nn}] \left(\frac{(1-z)(1-yz)}{z\big(1-(1+y)z\big)^N} \right)^{g-1} \qquad (5.11)
\end{aligned}$$

where we used the roots of unity formula

$$\sum_{i=1}^{N} z_i^k = \begin{cases} Nq^n, & \text{if } k = Nn \\ 0, & \text{otherwise.} \end{cases}$$

The formula (5.11) for $g = 0$ case reads

$$\mathsf{G}_{N,N-1,0}\Big|_{\underline{t}=1} = (-1)^{N+1} \cdot \sum_{n=1}^{\infty} q^{n-1} [z^{Nn-1}] \frac{\big(1-(1+y)z\big)^N}{(1-z)(1-yz)}.$$

If we denote

$$\sum_{i=0}^{\infty} a_i z^i := \big(1-(1+y)z\big)^N, \qquad (5.12)$$

then

$$\begin{aligned}
[z^{Nn-1}] \frac{\big(1-(1+y)z\big)^N}{(1-z)(1-yz)} &= [z^{Nn-1}] \sum_{i=0}^{\infty} a_i z^i \cdot \sum_{k=0}^{\infty} \frac{1-y^{k+1}}{1-y} z^k \\
&= \sum_{i=0}^{\infty} a_i \cdot \frac{1-y^{Nn-i}}{1-y}.
\end{aligned}$$

Hence, we find the closed formula in this case

$$
\begin{aligned}
\mathsf{G}_{N,N-1,0}\Big|_{\underline{t}=1} &= \frac{(-1)^{N+1}}{(1-y)} \cdot \sum_{i=0}^{\infty} a_i \sum_{n=1}^{\infty} q^{n-1} \cdot (1-y^{Nn-i}) \\
&= \frac{(-1)^{N+1}}{(1-y)} \cdot \sum_{i=0}^{\infty} a_i \left(\frac{1}{1-q} - \frac{y^{N-i}}{1-y^N q} \right) \\
&= \frac{(-1)^{N+1}}{1-y} \left(\frac{(-y)^N}{1-q} - \frac{y^N (-y^{-1})^N}{1-y^N q} \right) \\
&= \frac{1}{1-y} \cdot \frac{(1-y^N) - (1-y^{2N})q}{(1-q)(1-y^N q)}.
\end{aligned}
$$

We used the equation (5.12) with the substitutions at $z = 1$ and $z = y^{-1}$ in the third equality. Lastly, we record few of the higher genus cases for $N = 2$ and $\ell = 1$ case. By symmetric reduction algorithm, we find

$$
\begin{aligned}
\mathsf{G}_{2,1,2}\Big|_{\underline{t}=1} &= -4q \cdot \frac{(1+y) - (1+y+y^2+y^3)q}{\big(1-(1+y)^2 q\big)^2}, \\
\mathsf{G}_{2,1,3}\Big|_{\underline{t}=1} &= 8q \cdot \frac{1-(1+y^2)q - (1-3y^2+y^4)q^2 + (1+y^2)(1-2y-2y^3+y^4)q^3 + y^2(1+y)^4 q^4}{\big(1-(1+y)^2 q\big)^4}.
\end{aligned}
$$

5.4 Blow up formula

Let S be a smooth projective surface and β be a curve class of Seiberg-Witten length N. Consider a blow up morphism $p : \widetilde{S} \to S$ with the exceptional divisor E. By Lemma 7, the curve class $\widetilde{\beta} + \ell E$ on $\widetilde{S}$ is also of Seiberg-Witten length N. Hence, the series of virtual χ_{-y}-genera of Quot schemes with this curve class follows the structural formula (5.1)

$$
Z^{\chi_{-y}^{\mathrm{vir}}}_{\widetilde{S},N,\widetilde{\beta}+\ell E}(q,\underline{w}) = q^{-(\widetilde{\beta}+\ell E).(\widetilde{K_S}+E)} \sum_{\substack{\widetilde{\beta}+\ell E = \gamma_1+\cdots+\gamma_N \\ \mathrm{vd}_{\gamma_i}=0}} \mathsf{SW}(\underline{\gamma}) \cdot \mathsf{A}^{(\widetilde{K_S}+E)^2} \cdot \prod_i \mathsf{U}_i^{\gamma_i.(\widetilde{K_S}+E)} \cdot \prod_{i<j} \mathsf{W}_{i,j}^{\gamma_i.\gamma_j}. \tag{5.13}
$$

Recall that Lemma 7 also implies that $\mathrm{SW}(\underline{\gamma}) \neq 0$ if and only if the decomposition of $\widetilde{\beta} + \ell E$ is of the form

$$\widetilde{\beta} + \ell E = (\widetilde{\beta_1} + \ell_1 E) + \cdots + (\widetilde{\beta_N} + \ell_N E)$$

such that

$$\mathrm{SW}(\beta_i) \neq 0,\ \ell_i = 0 \text{ or } 1$$

for all $i = 1, \ldots, N$. Such decompositions are in one-to-one correspondence to decompositions $\underline{\beta} = (\beta_1, \ldots, \beta_N)$ of β together with a choice of a partition $I \sqcup J = [N]$ such that $|I| = \ell$. Denote the corresponding decomposition of $\widetilde{\beta} + \ell E$ by

$$\underline{\widetilde{\beta}} + \underline{E}_{I \sqcup J} := (\widetilde{\beta_1} + \ell_1 E, \cdots, \widetilde{\beta_N} + \ell_N E) \quad \text{where} \quad \begin{cases} \ell_i = 1, & \text{if } i \in I \\ \ell_j = 0, & \text{if } j \in J. \end{cases}$$

Recall that blow up formula of Seiberg-Witten invariants (4.1) implies

$$\mathrm{SW}(\underline{\widetilde{\beta}} + \underline{E}_{I \sqcup J}) = \mathrm{SW}(\underline{\beta}).$$

Also recall that the intersection form of the blow up surface admits an orthonormal decomposition

$$H^2(\widetilde{S}, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus \mathbb{Z}[E] \quad \text{with} \quad E^2 = -1.$$

Applying these two observations to the formula (5.13) yields the decomposition of the generating series

$$Z^{\chi^{\mathrm{vir}}_{-y}}_{\widetilde{S},N,\widetilde{\beta}+\ell E}(q,\underline{w}) = \left(q^{-\beta.K_S}\cdot \sum_{\substack{\beta=\beta_1+\cdots+\beta_N \\ \text{s.t. } \mathrm{vd}_{\beta_i}=0}} \mathsf{SW}(\underline{\beta})\cdot \mathsf{A}^{K_S^2}\cdot \prod_{k=1}^{N}\mathsf{U}_k^{\beta_k.K_S}\cdot \prod_{\substack{k_1<k_2 \\ k_1,k_2\in[N]}} \mathsf{W}_{k_1,k_2}^{\beta_{k_1}.\beta_{k_2}}\right) \cdot \left(q^{\ell}\cdot \sum_{\substack{I\sqcup J=[N] \\ \text{s.t. } |I|=\ell}} \left(\mathsf{A}\cdot\prod_{i\in I}\mathsf{U}_i\cdot \prod_{\substack{i_1<i_2 \\ i_1,i_2\in I}} \mathsf{W}_{i_1,i_2}\right)^{-1}\right)$$

into contributions from the base surface S and the exceptional divisor. It is clear from the above decomposition that we have

$$Z^{\chi^{\mathrm{vir}}_{-y}}_{\widetilde{S},N,\widetilde{\beta}+\ell E}(q,\underline{w}) = Z^{\chi^{\mathrm{vir}}_{-y}}_{S,N,\beta}(q,\underline{w})\cdot\left(q^{\ell}\cdot \mathsf{G}_{N,\ell,0}\right)$$

where $\mathsf{G}_{N,\ell,0}$ is defined in (5.9). By taking limit at $w_1=\cdots=w_N=0$, we obtain the formula for the non-equivariant generating series

$$Z^{\chi^{\mathrm{vir}}_{-y}}_{\widetilde{S},N,\widetilde{\beta}+\ell E}(q) = Z^{\chi^{\mathrm{vir}}_{-y}}_{S,N,\beta}(q)\cdot\left(q^{\ell}\cdot \mathsf{G}_{N,\ell,0}\Big|_{\underline{t}=1}\right).$$

This chapter is, in part, being prepared for submission for publication. The dissertation author was the primary investigator and the author of the material below.

- Woonam Lim “Virtual χ_{-y}-genera of Quot schemes on surfaces ”

Chapter 6

Reduced invariants of $K3$ surfaces

In this chapter, we study the reduced classes and reduced invariants of certain Quot schemes on $K3$ surfaces. We first find several conditions under which Quot schemes on $K3$ surfaces admit reduced 2-term perfect obstruction theory. This mildly generalizes the result of [MOP1]. When $N = 1$, the reduced class of Quot schemes are identified with the reduced class of certain nested Hilbert schemes. Similar result exists for reduced classes in stable pair theory by works of [KT1, GSY2, GT2]. Using rather concrete descriptions of reduced classes in both Quot scheme theory and stable pair theory, we identify the reduced χ_{-y}-genera of Quot schemes and Pair spaces when $N = 1$. Therefore we conclude that reduced χ_{-y}-genera of Quot schemes of $K3$ surfaces are given by the Kawai-Yoshioka formula [KY]. In the last section, we use elliptic $K3$ surfaces to compute the reduced χ_{-y}-genera of Hilbert schemes of points without curve classes.

6.1 Reduced obstruction theory

In the proposition below, we give certain conditions under which Quot schemes on K-trivial surfaces admit reduced obstruction theory.

Proposition 9. *Let S be a surface with a trivial canonical line bundle. Then* $\mathsf{Quot}_S(E, v)$ *admits a reduced 2-term perfect obstruction theory in the following cases:*

1. *E simple sheaf and non-zero Chern character v distinct from* $\mathrm{ch}(E)$*,*

2. *E torsion free sheaf and v Chern character of non-zero torsion sheaf.*

Proof. Recall that $\mathrm{Quot}_S(E,v)$ parametrizes quotients

$$0 \to S \xrightarrow{f} E \xrightarrow{g} Q \to 0 \quad \text{such that} \quad \mathrm{ch}(Q) = v.$$

We have seen in Section 1.3 that Quot schemes of the second case admit a natural 2-term perfect obstruction theory by showing that higher obstruction spaces vanish. For the first case, recall that we have

$$\begin{aligned} \mathrm{Ext}^2(S,Q)^\vee &= \mathrm{Hom}(Q,S) \\ &\hookrightarrow \mathrm{Hom}(Q,E) \\ &\hookrightarrow \mathrm{Hom}(E,E). \end{aligned}$$

We claim that $\mathrm{Hom}(Q,E) = 0$. Suppose for contradiction that there is a non-zero morphism $\phi \in \mathrm{Hom}(Q,E)$. Since E is a simple sheaf, non-zero composition morphism

$$E \xrightarrow{g} Q \xrightarrow{\phi} E$$

is an isomorphism. In particular, g is injective. Since g is also a quotient, we conclude that g is an isomorphism. This contradicts the choice of v that is distinct from $\mathrm{ch}(E)$. This proves $\mathrm{Hom}(Q,E) = 0$, hence the higher obstruction space $\mathrm{Ext}^2(S,Q)$ vanishes.

Consider the composition of the morphisms

$$\sigma : \mathrm{Ext}^1(S,Q) \xrightarrow{\delta} \mathrm{Ext}^2(Q,Q) \xrightarrow{\text{trace}} H^2(\mathscr{O}_S).$$

Roughly speaking, δ sends obstruction classes for deforming the quotient to obstruction classes for deforming the sheaf Q and the trace map sends this further to obstruction classes for deforming the line bundle $\det(Q)$. Since Picard group is smooth, there is no obstructions to deform a line bundle. So obstruction classes for deforming the quotient lives in $\ker(\sigma)$.

Crucial to the construction of the reduced 2-term perfect obstruction theory is the surjectivity of the morphism σ. This guarantees that the difference between the dimension of tangent space $\mathrm{Hom}(S,Q)$ and the dimension of reduced obstruction space $\ker(\sigma)$ stays constant. To show that σ is surjective, we show that both δ and a trace map are surjective. By the long exact sequence of Ext groups, cokernel of δ lives in $\mathrm{Ext}^2(E,Q) = \mathrm{Hom}(Q,E)^\vee$. We have seen above that $\mathrm{Hom}(Q,E) = 0$ for the first case. In the second case, the same follows from the fact that Q is torsion and E is torsion free. So δ is surjective. Note that the trace map is Serre dual to the identity morphism

$$H^0(K_S) \xrightarrow{\mathrm{id}} \mathrm{Hom}(Q, Q \otimes K_S).$$

The fact that K_S is trivial and Q is non-zero implies injectivity of id, hence surjectivity of the trace. □

For the ease of the notation, we write the Quot scheme satisfying either of the conditions in Proposition 9 simply by Quot. Reduced 2-term perfect obstruction theory of Quot induces a reduced class

$$[\mathsf{Quot}]^{\mathrm{red}} \in A_*(\mathsf{Quot})$$

and a reduced structure sheaf

$$\mathscr{O}^{\mathrm{red}}_{\mathsf{Quot}} \in K_0(\mathsf{Quot}).$$

Furthermore, a reduced tangent bundle is given by

$$T^{\mathrm{red}}_{\mathsf{Quot}} = \mathrm{R}\mathscr{H}om_\pi(\mathscr{S}, \mathscr{Q}) + \mathscr{O}_{\mathsf{Quot}}$$

which have an extra copy of the trivial line bundle to the usual virtual tangent bundle. A reduced cotangent bundle is defined as a K-theoretic dual of the reduced tangent bundle

$$\Omega^{\mathrm{red}}_{\mathsf{Quot}} = \left(T^{\mathrm{red}}_{\mathsf{Quot}}\right)^{\vee}.$$

Using these definitions, the reduced χ_{-y}-genus is well-defined as

$$\chi^{\mathrm{red}}_{-y}(\mathsf{Quot}) := \chi\left(\mathsf{Quot}, (\Lambda_{-y}\Omega^{\mathrm{red}}_{\mathsf{Quot}}) \otimes \mathscr{O}^{\mathrm{red}}_{\mathsf{Quot}}\right).$$

By virtual Hirzebruch-Riemann-Roch theorem [BF], we also have

$$\chi^{\mathrm{red}}_{-y}(\mathsf{Quot}) = \int_{\left[\mathsf{Quot}\right]^{\mathrm{red}}} \mathrm{ch}\left(\Lambda_{-y}\Omega^{\mathrm{red}}_{\mathsf{Quot}}\right) \mathrm{td}\left(T^{\mathrm{red}}_{\mathsf{Quot}}\right).$$

6.2 Reduced class of Quot schemes with $N = 1$

In this subsection, we study the reduced class of Quot schemes on $K3$ surfaces with $N = 1$. Main output equates the reduced class of Quot schemes with that of the corresponding nested Hilbert schemes.

We begin by reviewing the reduced class of nested Hilbert schemes. Let S be a smooth projective surface. Recall that nested Hilbert schemes $S^{[m_1,m_2]}_{\beta}$ admit a natural perfect obstruction theory constructed by [GSY2]. Under the condition that

$$H^2(L) = 0 \quad \text{for all effective} \quad L \in \mathsf{Pic}^{\beta}_{S}, \tag{6.1}$$

it also possesses a reduced obstruction theory by [GSY2, GT2]. This generalizes the construction of a reduced obstruction theory in the case of Hilbert schemes of curves $\mathsf{Hilb}^{\beta}_{S}$ [DKO] and Pair spaces $\mathsf{Pair}_S(\mathbb{C}^1, \beta, n)$ [KT2] under the same condition (6.1). See Definition 13 for the definition of Pair spaces.

Now let S be a $K3$ surface and β be a non-zero effective curve class of Hodge type $(1,1)$. Recall that Lemma 2 says

$$\mathsf{Quot}_S(\mathbb{C}^1, \beta, n) = S_\beta^{[m,0]}$$

where $m = n + (g-1)$. Here g is the arithmetic genus of the curve class β given by $2g-2 = \beta^2$. Note that the above choice of Quot scheme satisfies the second condition of Proposition 9, hence admits a reduced obstruction theory. On the other hand, the condition (6.1) is also satisfied because we have

$$H^2(L_\beta) = H^0(L_\beta^{-1})^\vee = 0$$

as L_β is a non-trivial effective line bundle. Therefore $S_\beta^{[m,0]}$ above also admits reduced obstruction theory. The proposition below identifies the resulting reduced classes for the Quot scheme and the nested Hilbert scheme. This is analogous to Proposition 3.

Proposition 10. *Let S be a $K3$ surface and β be a non-zero effective curve class of Hodge type $(1,1)$. Through the identification $\mathsf{Quot}_S(\mathbb{C}^1, \beta, n) = S_\beta^{[m,0]}$, the reduced class of the Quot scheme on the left and the reduced class of the nested Hilbert scheme on the right agree:*

$$\left[\mathsf{Quot}_S(\mathbb{C}^1, \beta, n)\right]^{\mathrm{red}} = \left[S_\beta^{[m,0]}\right]^{\mathrm{red}}.$$

Proof. Recall that reduced classes only depend on the K-theory of reduced tangent bundles. Therefore we only need to equate reduced tangent bundles of the Quot scheme and the nested Hilbert scheme. Note that we have already identified the usual virtual tangent bundles of each side in the proof of Proposition 3. Since reduced tangent bundles are equal to the usual tangent bundles up to one additional copy of the trivial line bundle, this proves the statement. □

Consider the canonical isomorphism

$$f_{\mathsf{Quot}} : S_\beta^{[m,0]} \xrightarrow{\sim} S^{[m]} \times \mathsf{Hilb}_S^\beta.$$

Similarly to Corollary 4, we have the following explicit computation of the reduced class of Quot schemes.

Proposition 11 ([GT2]). *Let S and β be as in Proposition 10. Pushforward of the reduced class gives*

$$f_{\mathsf{Quot}*}\left[S_\beta^{[m,0]}\right]^{\mathrm{red}} = e\big(\mathsf{CO}_\beta^{[m,0]}\big) \cap \Big(\big[S^{[m]}\big] \times \big[\mathsf{Hilb}_S^\beta\big]^{\mathrm{red}}\Big)$$

where

$$\mathsf{CO}_\beta^{[m,0]} = \mathrm{R}\mathscr{H}om_\pi(\mathscr{O}_{\mathscr{X}}, \mathscr{O}(\mathscr{D}_\beta)).$$

Proof. This is a special case [GT2, Theorem 5]. □

To make the formula in the above proposition more explicit, we study the reduced class $\big[\mathsf{Hilb}_S^\beta\big]^{\mathrm{red}}$. Since S is a $K3$ surface, Pic_S^β consists of a reduced point $\{L_\beta\}$. Therefore the Hilbert scheme of curves is simply a projective space

$$\mathsf{Hilb}_S^\beta = \mathbb{P}(H^0(L_\beta)). \tag{6.2}$$

Since $H^2(L_\beta) = 0$, we have

$$h^0(L_\beta) - h^1(L_\beta) = 2 + \frac{\beta^2}{2} = g + 1$$

by Riemann-Roch formula. Therefore we have

$$\dim(\mathsf{Hilb}_S^\beta) \geq g$$

where

$$g = \frac{\beta^2}{2} + 1 = \mathrm{vd}_\beta + 1$$

is the reduced virtual dimension. When all higher cohomology groups of L_β vanishes, equivalently $H^1(L_\beta) = 0$, the actual dimension and the reduced virtual dimension are equal, hence

$$\left[\mathsf{Hilb}_S^\beta\right]^{\mathrm{red}} = \left[\mathsf{Hilb}_S^\beta\right].$$

As somewhat expected from the virtual fundamental class theory, the proposition below says that we can regard the reduced class $\left[\mathsf{Hilb}_S^\beta\right]^{\mathrm{red}}$ as if it is a fundamental class of a g dimensional projective space.

Proposition 12. *Let S and β be as in Proposition 10 and g be the arithmetic genus of the curve class β. Consider any linear embedding of g dimensional projective space*

$$\iota : \mathbb{P}^g \hookrightarrow \mathbb{P}(H^0(L_\beta)).$$

Then we have

$$\left[\mathsf{Hilb}_S^\beta\right]^{\mathrm{red}} = \iota_*\left[\mathbb{P}^g\right].$$

Proof. Through the identification (6.2), we have

$$\mathscr{O}(\mathscr{D}_\beta) = \mathscr{O}_{\mathbb{P}}(1) \boxtimes L_\beta \quad \text{over} \quad \mathbb{P}(H^0(L_\beta)) \times S. \tag{6.3}$$

We can calculate the reduced obstruction bundle as follows:

$$\begin{aligned}
\text{reduced obstruction} &= T_{\mathsf{Hilb}_S^\beta} - T^{\mathrm{red}}_{\mathsf{Hilb}_S^\beta} \\
&= \left(\mathscr{O}_{\mathbb{P}}(1)^{\oplus h^0(L_\beta)} - \mathscr{O}_{\mathbb{P}}\right) - \left(\mathrm{R}\mathscr{H}om_\pi(\mathscr{O}(-\mathscr{D}_\beta), \mathscr{O}_{\mathscr{D}_\beta}) + \mathscr{O}_{\mathbb{P}}\right) \\
&= \left(\mathscr{O}_{\mathbb{P}}(1)^{\oplus h^0(L_\beta)} - \mathscr{O}_{\mathbb{P}}\right) - \left(\mathscr{O}_{\mathbb{P}}(1)^{\oplus \chi(L_\beta)} - \mathscr{O}_{\mathbb{P}}^{\chi(\mathscr{O}_S)} + \mathscr{O}_{\mathbb{P}}\right) \\
&= \mathscr{O}_{\mathbb{P}}(1)^{h^1(L_\beta)}.
\end{aligned}$$

We used the Euler sequence and formula (6.3) for the second and the third equality, respectively. Since reduced class is the Euler class of reduced obstruction space, we conclude that the reduced class is

$$\left[\mathbb{P}(H^0(L_\beta))\right]^{\mathrm{red}} = e\big(\mathscr{O}_{\mathbb{P}}(1)^{h^1(L_\beta)}\big) \cap \left[\mathbb{P}(H^0(L_\beta))\right].$$

This proves the proposition. □

6.3 Reduced class of Pair spaces with $N = 1$

Let S be a smooth projective surface and β be a non-zero effective curve class.

Definition 13. Pair space $\mathsf{Pair}_S(\mathbb{C}^N, \beta, n)$ is a moduli space parametrizing morphisms

$$\mathbb{C}^N \otimes \mathscr{O}_S \xrightarrow{s} F \quad \text{with} \quad c_1(F) = \beta, \;\; \chi(Q) = n$$

such that F is a pure sheaf of dimension 1 and $\mathrm{coker}(s)$ is of zero dimension.

The main difference is that purity of sheaf F in Pair space is exchanged with surjectivity in Quot scheme. However, Pair spaces are in many sense similar to Quot schemes.

First of all, the Pair space is a fine moduli space with the universal stable pair

$$\mathbb{C}^N \otimes \mathscr{O}_{\mathsf{Pair}_S(\mathbb{C}^N,\beta,n)\times S} \xrightarrow{s} \mathscr{F}.$$

We denote the complex of universal morphism by

$$\mathbb{I}^\bullet := \left[\mathbb{C}^N \otimes \mathscr{O}_{\mathsf{Pair}_S(\mathbb{C}^N,\beta,n)\times S} \xrightarrow{s} \mathscr{F}\right]$$

placed at the zeroth and first places. By result of [Lin], the Pair space as above also admits a natural 2-term perfect obstruction space with a virtual tangent bundle

$$T^{\mathrm{vir}}_{\mathsf{Pair}_S(\mathbb{C}^N,\beta,n)} = \mathrm{R}\mathscr{H}om_\pi(\mathbb{I}^\bullet, \mathscr{F}).$$

Using the virtual structure of $\mathsf{Pair}_S(\mathbb{C}^N, \beta, n)$, one can develop the parallel story to Quot scheme theory. In this thesis, we focus on the special case of Pair spaces when $N = 1$.

When $N = 1$, the Pair space is naturally identified with a nested Hilbert scheme [PT1]:

$$\mathsf{Pair}_S(\mathbb{C}^1, \beta, n) = S_\beta^{[0,m]} \tag{6.4}$$

where

$$m = n + \frac{\beta(\beta + K_S)}{2}.$$

Furthermore, [GSY2, Proposition 3.1] below says that virtual fundamental class of the Pair space on the left hand side is identified with virtual fundamental class of the corresponding nested Hilbert scheme.

Proposition 14 ([GSY2])**.** *Through the identification* $\mathsf{Pair}_S(\mathbb{C}^1, \beta, n) = S_\beta^{[0,m]}$*, the virtual fundamental class of the Pair space on the left and the virtual fundamental class of the nested Hilbert scheme on the right agree:*

$$\left[\mathsf{Pair}_S(\mathbb{C}^1, \beta, n)\right]^{\mathrm{vir}} = \left[S_\beta^{[0,m]}\right]^{\mathrm{vir}}.$$

Proof. Proof in [GSY2] identifies the virtual tangent bundle of the Pair space with that of the nested Hilbert scheme. □

Under the condition (6.1), both $\mathsf{Pair}_S(\mathbb{C}^1, \beta, n)$ and $S_\beta^{[0,m]}$ admits a reduced obstruction theory. Analogously to the above proposition, we can identify the reduced classes.

Proposition 15. *Suppose that a surface S and a curve class β satisfies the condition* (6.1)*. Through the identification* $\mathsf{Pair}_S(\mathbb{C}^1, \beta, n) = S_\beta^{[0,m]}$*, the reduced class of the Pair space on the left and the reduced class of the nested Hilbert scheme on the right agree:*

$$\left[\mathsf{Pair}_S(\mathbb{C}^1, \beta, n)\right]^{\mathrm{red}} = \left[S_\beta^{[0,m]}\right]^{\mathrm{red}}.$$

Proof. The usual virtual tangent bundles of each agree as noted in the proof of Proposition 14. Since the reduced tangent bundles differ from the usual tangent bundles by trivial vector bundle of rank p_g, the reduced tangent bundles also agree. Therefore the ensuing reduced classes also agree. □

To state the formula for the reduced class, we consider the canonical embedding of the Pair space, or equivalently $S_\beta^{[0,m]}$, to the (virtually) smooth space

$$f_{\mathsf{Pair}} : S_\beta^{[0,m]} \hookrightarrow S^{[m]} \times \mathsf{Hilb}_S^\beta.$$

Proposition 16 ([GT2]). *Suppose that a surface S and a curve class β satisfies the condition* (6.1). *The reduced class of $S_\beta^{[0,m]}$ as a Pair space, or equivalently as a nested Hilbert scheme, satisfies*

$$f_{\mathsf{Pair}*}\left[S_\beta^{[0,m]}\right]^{\text{red}} = e\big(\mathsf{CO}_\beta^{[0,m]}\big) \cap \left(\left[S^{[m]}\right] \times \left[\mathsf{Hilb}_S^\beta\right]^{\text{red}}\right)$$

where

$$\mathsf{CO}_\beta^{[0,m]} = \mathrm{R}\pi_*(\mathscr{O}_{\mathscr{X}}(\mathscr{D}_\beta)).$$

Proof. This is a special case [GT2, Theorem 5]. □

6.4 Reduced invariants of Quot schemes and Pair spaces

In this section, we apply Proposition 11 and Proposition 16 to compare the reduced invariants of Quot schemes and Pair spaces with $N = 1$ on $K3$ surfaces. Let S be a $K3$ surface and β be a non-zero effective curve class of Hodge type $(1,1)$. Consider a Quot scheme and a Pair space

$$\mathsf{Quot}_S(\mathbb{C}^1, \beta, n), \quad \mathsf{Pair}_S(\mathbb{C}^1, \beta, n).$$

The following diagram summarizes the geometries involved in the comparison:

$$\begin{tikzcd}[column sep=small]
\mathsf{Quot}_S(\mathbb{C}^1,\beta,n) = S_\beta^{[m,0]} \arrow[dr, "f_{\mathsf{Quot}}", "\sim"'] & & \\
& S^{[m]} \times \mathsf{Hilb}_S^\beta & S^{[m]} \times \mathbb{P}^g. \arrow[l, hook', "\mathrm{id}_{S^{[m]}} \times \iota"'] \\
\mathsf{Pair}_S(\mathbb{C}^1,\beta,n) = S_\beta^{[0,m]} \arrow[ur, hook, "f_{\mathsf{Pair}}"'] & &
\end{tikzcd}$$

Rather surprisingly, the theorem below exactly matches the reduced χ_{-y}-genus of the Quot scheme and the Pair space above.

Theorem 17. *Let S be a $K3$ surface and β be a non-zero effective curve class of Hodge type $(1,1)$. Then we have*

$$\chi_{-y}^{\mathrm{red}}\big(\mathsf{Quot}_S(\mathbb{C}^1,\beta,n)\big) = \chi_{-y}^{\mathrm{red}}\big(\mathsf{Pair}_S(\mathbb{C}^1,\beta,n)\big).$$

Proof. Since reduced geometry of the Quot scheme and the Pair space in the statement are identified with the nested Hilbert schemes $S_\beta^{[m,0]}$ and $S_\beta^{[0,m]}$, respectively, we need to prove that

$$\chi_{-y}^{\mathrm{red}}\big(S_\beta^{[m,0]}\big) = \chi_{-y}^{\mathrm{red}}\big(S_\beta^{[0,m]}\big).$$

Calculation of a reduced χ_{-y}-genus involves two data, namely a reduced class and a reduced tangent bundle. In our case, we know the formula of pushforward of reduced classes under morphisms f_{Quot} and f_{Pair}:

$$f_{\mathsf{Quot}*}\Big(\big[S_\beta^{[m,0]}\big]^{\mathrm{red}}\Big) = e\big(\mathsf{CO}_\beta^{[m,0]}\big) \cap \Big(\big[S^{[m]}\big] \times \big[\mathsf{Hilb}_S^\beta\big]^{\mathrm{red}}\Big),$$
$$f_{\mathsf{Pair}*}\Big(\big[S_\beta^{[0,m]}\big]^{\mathrm{red}}\Big) = e\big(\mathsf{CO}_\beta^{[0,m]}\big) \cap \Big(\big[S^{[m]}\big] \times \big[\mathsf{Hilb}_S^\beta\big]^{\mathrm{red}}\Big).$$

Similarly, we can express the reduced tangent bundles naturally as pullbacks via f_{Quot} and f_{Pair}:

$$\begin{aligned} f^*_{\mathsf{Quot}}\left(T^{\mathrm{red}}_{\mathsf{Quot}}\right) &= T_{S^{[m]}} + T^{\mathrm{red}}_{\mathsf{Hilb}^\beta_S} - \mathsf{CO}^{[m,0]}_\beta, \\ f^*_{\mathsf{Pair}}\left(T^{\mathrm{red}}_{\mathsf{Pair}}\right) &= T_{S^{[m]}} + T^{\mathrm{red}}_{\mathsf{Hilb}^\beta_S} - \mathsf{CO}^{[0,m]}_\beta. \end{aligned}$$

Using the identification (6.3) and the definition of K-theoretic Carlsson-Okounkov classes, it is easy to check that

$$\mathsf{CO}^{[m,0]}_\beta = \left((L_\beta^{-1})^{[m]}\right)^\vee \boxtimes \mathscr{O}_{\mathbb{P}}(1), \quad \mathsf{CO}^{[0,m]}_\beta = L_\beta^{[m]} \boxtimes \mathscr{O}_{\mathbb{P}}(1).$$

where $\mathbb{P} = \mathbb{P}(H^0(L_\beta)) = \mathsf{Hilb}^\beta_S$. Serre duality is used for the computation of $\mathsf{CO}^{[m,0]}_\beta$ which is responsible for two dual signs. We show below that difference of extra duals in the Quot scheme side can be disregarded in our case because S is a $K3$ surface.

Using the above formulas for reduced class and reduced tangent bundle, we can express the reduced χ_{-y}-genus as an integral

$$\chi^{\mathrm{red}}_{-y}(S^{[m,0]}_\beta) = \int_{\left[S^{[m]}\right] \times \left[\mathsf{Hilb}^\beta_S\right]^{\mathrm{red}}} \frac{e}{g}(\mathsf{CO}^{[m,0]}_\beta) \cdot g(T_{S^{[m]}}) \cdot g(T^{\mathrm{red}}_{\mathsf{Hilb}^\beta_S})$$

with the similar formula for $S^{[0,m]}_\beta$. Here g is a power series

$$g(z) := \frac{z(1-ye^{-z})}{1-e^{-z}} \in \mathbb{Q}(y)[\![z]\!]$$

defining the multiplicative characteristic corresponding to the χ_{-y}-genus. We want to show that

$$\begin{aligned} &\int_{\left[S^{[m]}\right] \times \left[\mathsf{Hilb}^\beta_S\right]^{\mathrm{red}}} \frac{e}{g}(\mathsf{CO}^{[m,0]}_\beta) \cdot g(T_{S^{[m]}}) \cdot g(T^{\mathrm{red}}_{\mathsf{Hilb}^\beta_S}) \\ &\qquad = \int_{\left[S^{[m]}\right] \times \left[\mathsf{Hilb}^\beta_S\right]^{\mathrm{red}}} \frac{e}{g}(\mathsf{CO}^{[0,m]}_\beta) \cdot g(T_{S^{[m]}}) \cdot g(T^{\mathrm{red}}_{\mathsf{Hilb}^\beta_S}). \end{aligned}$$

In fact, we show this equality for arbitrary choice of power series

$$g \in F[\![z]\!], \quad g(0) \neq 0$$

for some field F. To do so, we define a new power series

$$\widetilde{g}(z) := \frac{z+x}{g(z+x)} \in F(x)[\![z]\!].$$

By definition of $\widetilde{g}$, we have

$$\frac{e}{g}(V \otimes L) = \widetilde{g}(V)\Big|_{x=c_1(L)}$$

for any K-theory class V and a line bundle L. Therefore we can rewrite the claimed equality as

$$\int_{\left[S^{[m]}\right] \times \left[\mathsf{Hilb}_S^\beta\right]^{\mathrm{red}}} \widetilde{g}\Big(\big((L_\beta^{-1})^{[m]}\big)^\vee\Big)\Big|_{x=h} \cdot g(T_{S^{[m]}}) \cdot g(T^{\mathrm{red}}_{\mathsf{Hilb}_S^\beta})$$
$$= \int_{\left[S^{[m]}\right] \times \left[\mathsf{Hilb}_S^\beta\right]^{\mathrm{red}}} \widetilde{g}\big(L_\beta^{[m]}\big)\Big|_{x=h} \cdot g(T_{S^{[m]}}) \cdot g(T^{\mathrm{red}}_{\mathsf{Hilb}_S^\beta})$$

where $h = c_1(\mathscr{O}_{\mathbb{P}}(1))$. Since cohomology class h and a K-theory class $T^{\mathrm{red}}_{\mathsf{Hilb}_S^\beta}$ are pullbacks from the projection map

$$S^{[m]} \times \mathsf{Hilb}_S^\beta \to \mathsf{Hilb}_S^\beta,$$

the above equality is implied by the following equality between two tautological integrals over a Hilbert scheme of points:

$$\int_{S^{[m]}} \widetilde{g}\Big(\big((L_\beta^{-1})^{[m]}\big)^\vee\Big) \cdot g(T_{S^{[m]}}) = \int_{S^{[m]}} \widetilde{g}\big(L_\beta^{[m]}\big) \cdot g(T_{S^{[m]}}) \in F(x). \tag{6.5}$$

Crucial to the argument is that S is holomorphic symplectic, hence so is $S^{[m]}$. In particular, we have $T_{S^{[m]}} \simeq T_{S^{[m]}}^\vee$. Together with the fact that $S^{[m]}$ is even dimensional, the left hand side of

(6.5) is equal to

$$\int_{S^{[m]}} \widetilde{g}\big((L_\beta^{-1})^{[m]}\big) \cdot g(T_{S^{[m]}}).$$

Therefore we must show

$$\int_{S^{[m]}} \widetilde{g}\big((L_\beta^{-1})^{[m]}\big) \cdot g(T_{S^{[m]}}) = \int_{S^{[m]}} \widetilde{g}\big(L_\beta^{[m]}\big) \cdot g(T_{S^{[m]}}). \tag{6.6}$$

By recursive argument of [EGL], there exist a polynomial

$$\mathsf{P}(u_1,u_2,u_3,u_4) \in F(x)[u_1,u_2,u_3,u_4]$$

which only depends on m and g such that

$$\int_{S^{[m]}} \widetilde{g}(M^{[m]}) \cdot g(T_{S^{[m]}}) = \mathsf{P}\Big(\, c_2(T_S)\,,\, c_1(T_S)^2\,,\, c_1(T_S).c_1(M)\ ,c_1(M)^2\,\Big)$$

for any choice of surface S and a line bundle M. By applying this to our case, the desired equality (6.6) is equivalent to

$$\mathsf{P}\Big(\, c_2(T_S)\,,\, c_1(T_S)^2\,,\, c_1(T_S).c_1(L_\beta^{-1})\ ,c_1(L_\beta^{-1})^2\,\Big)$$
$$= \mathsf{P}\Big(\, c_2(T_S)\,,\, c_1(T_S)^2\,,\, c_1(T_S).c_1(L_\beta)\ ,c_1(L_\beta)^2\,\Big).$$

This follows directly from the fact that $c_1(T_S) = 0$ for a $K3$ surface S. This completes the proof.

□

Remark 18. Since the proof in Theorem 17 works for an arbitrary choice of power series g, the same proof gives the equality between the reduced algebraic cobordism class

$$\big[\mathsf{Quot}_S(\mathbb{C}^1,\beta,n)\big]^{\mathrm{red}}_{\Omega_*} = \big[\mathsf{Pair}_S(\mathbb{C}^1,\beta,n)\big]^{\mathrm{red}}_{\Omega_*} \in \Omega_*(\mathrm{pt}),$$

under the same assumption. See [Sh] for the definition of algebraic cobordism class of a projective scheme with a perfect obstruction theory.

The proposition below allows us to compute the reduced χ_{-y}-genera of Quot schemes, or equivalently Pair spaces, by choosing a geometry (S,β) where the Pair space is smooth. In such case, reduced χ_{-y}-genus of $\mathsf{Pair}_S(\mathbb{C}^1,\beta,n)$ is equal to the non-virtual χ_{-y}-genus which is computed by work of Kawai and Yoshioka [KY].

Proposition 19. *Let S be a $K3$ surface and β be a non-zero effective curve class of Hodge type $(1,1)$. Then the reduced χ_{-y}-genera*

$$\chi_{-y}^{\mathrm{red}}\big(\mathsf{Quot}_S(\mathbb{C}^1,\beta,n)\big) = \chi_{-y}^{\mathrm{red}}\big(\mathsf{Pair}_S(\mathbb{C}^1,\beta,n)\big)$$

depend upon β only through the intersection number $\beta^2 = 2g-2$.

Proof. By the proof of Theorem 17, we have

$$\chi_{-y}^{\mathrm{red}}\big(S_\beta^{[0,m]}\big) = \int_{\left[S^{[m]}\right]\times\left[\mathsf{Hilb}_S^\beta\right]^{\mathrm{red}}} \frac{e}{g}\big(L_\beta^{[m]}\boxtimes\mathscr{O}_{\mathbb{P}}(1)\big)\cdot g(T_{S^{[m]}})\cdot g(T^{\mathrm{red}}_{\mathsf{Hilb}_S^\beta}).$$

where

$$g(z) = \frac{z(1-ye^{-z})}{1-e^{-z}}.$$

Recall that $\mathsf{Hilb}_S^\beta = \mathbb{P}(H^0(L_\beta))$ and $m = n+(g-1)$. Proposition 12 says that the reduced class of the Hilbert scheme of curve is given by

$$\left[\mathsf{Hilb}_S^\beta\right]^{\mathrm{red}} = \iota_*\left[\mathbb{P}^g\right]$$

for any linear embedding ι of $\mathbb{P}^g$ into the Hilbert scheme of curves. Note that the reduced tangent bundle of Hilb_S^β is

$$\begin{aligned} T^{\mathrm{red}}_{\mathsf{Hilb}_S^\beta} &= \mathrm{R}\mathscr{H}om_\pi(\mathscr{O}(-\mathscr{D}_\beta), \mathscr{O}_{\mathscr{D}_\beta}) + \mathscr{O}_{\mathbb{P}} \\ &= \mathscr{O}_{\mathbb{P}}(1)^{\chi(L_\beta)} - \mathscr{O}_{\mathbb{P}}^{\chi(\mathscr{O}_S)} + \mathscr{O}_{\mathbb{P}} \\ &= \mathscr{O}_{\mathbb{P}}(1)^{g+1} - \mathscr{O}_{\mathbb{P}}. \end{aligned}$$

Therefore the reduced tangent bundle pullbacks via ι to the usual tangent bundle of the g dimensional projective space

$$\iota^* T^{\mathrm{red}}_{\mathsf{Hilb}_S^\beta} = T_{\mathbb{P}^g}.$$

All these combined, we obtain

$$\chi^{\mathrm{red}}_{-y}(S_\beta^{[0,m]}) = \int_{[S^{[m]}]\times[\mathbb{P}^g]} \frac{e}{g}\big(L_\beta^{[m]} \boxtimes \mathscr{O}_{\mathbb{P}^g}(1)\big) \cdot g(T_{S^{[m]}}) \cdot g(T_{\mathbb{P}^g}).$$

Following the proof of Theorem 17, one can compute this integral by applying the recursion of [EGL] relative to $\mathbb{P}^g$. So we conclude that this integral depends only on m, genus g and the Chern numbers

$$c_2(T_S) = 24, \quad c_1(T_S)^2 = 0, \quad c_1(T_S).c_1(L_\beta) = 0, \quad c_1(L_\beta)^2 = 2g-2.$$

This completes the proof. $\square$

To state the Kawai-Yoshioka formula, we introduce a notation

$$\mathscr{C}_g^{[m]}$$

for the Pair space $\mathsf{Pair}_S(\mathbb{C}^1, \beta, n)$ corresponding to the irreducible curve class β of arithmetic genus g. Notation is partially supported from the above proposition because the invariant of $\mathscr{C}_g^{[m]}$ that we consider here depends only on m and g. Under the irreducibility condition on β, [KY] proved that $\mathscr{C}_g^{[m]}$ is smooth of dimension $m+g$ which is equal to the reduced virtual dimension as a Pair space. This implies that

$$\chi_{-y}(\mathscr{C}_g^{[m]}) = \chi_{-y}^{\text{red}}(\mathsf{Pair}_S(\mathbb{C}^1, \beta, n)).$$

For simplicity of the formula, we use the shifted χ_{-y}-genus. Recall that the shifted virtual χ_{-y}-genus is

$$\overline{\chi}_{-y}^{\text{vir}}(M) := y^{-\frac{\text{vd}}{2}} \cdot \chi_{-y}^{\text{vir}}(M).$$

Then the Kawai-Yoshioka formula reads

$$\begin{aligned}\sum_{g\geq 0}\sum_{m\geq 0} \overline{\chi}_{-y}(\mathscr{C}_g^{[m]}) t^{m+1-g} q^{g-1} = \frac{1}{(t+\frac{1}{t}-\sqrt{y}-\frac{1}{\sqrt{y}})\cdot q} \\ \cdot \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{18}(1-q^n y)(1-q^n/y)(1-q^n\sqrt{y}/t)(1-q^n t/\sqrt{y})(1-q^n t\sqrt{y})(1-q^n/t\sqrt{y})}.\end{aligned} \tag{6.7}$$

This is a specialization to the shifted χ_{-y}-genera of the original formula computing the Hodge polynomials. At $y=1$, the above formula is the Jacobi cusp form of weight 10 and index 1.

Define the generating series of shifted reduced χ_{-y}-genera of Quot schemes, or equivalently Pair spaces,

$$Z_{S,1,\beta}^{\overline{\chi}_{-y}^{\text{red}}}(t) := \sum_{n\in\mathbb{Z}} t^n \cdot \overline{\chi}_{-y}^{\text{red}}\big(\mathsf{Quot}_S(\mathbb{C}^1, \beta, n)\big) = \sum_{n\in\mathbb{Z}} t^n \cdot \overline{\chi}_{-y}^{\text{red}}\big(\mathsf{Pair}_S(\mathbb{C}^1, \beta, n)\big).$$

The above discussions sum up to the following corollary.

Corollary 20. *Let S be a $K3$ surface and β be a non-zero effective curve class of arithmetic genus g. Then the generating series*

$$Z_{S,1,\beta}^{\overline{\chi}_{-y}^{\text{red}}}(t)$$

of reduced invariants is given by the q^g coefficient of the expression

$$\frac{1}{\left(t+\frac{1}{t}-\sqrt{y}-\frac{1}{\sqrt{y}}\right)} \cdot \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{18}(1-q^n y)(1-q^n/y)(1-q^n\sqrt{y}/t)(1-q^n t/\sqrt{y})(1-q^n t\sqrt{y})(1-q^n/t\sqrt{y})}.$$

Remark 21. Note that Corollary 20 directly implies that the generating series of shifted reduced χ_{-y} genera is a rational function in t variable

$$Z^{\overline{\chi}^{\mathrm{red}}_{-y}}_{S,1,\beta}(t) \in \mathbb{Q}(\sqrt{y})(t)$$

which is symmetric under

$$t \longleftrightarrow t^{-1}.$$

We can also obtain the generating series of (non-shifted) reduced χ_{-y}-genera

$$\begin{aligned} Z^{\chi^{\mathrm{red}}_{-y}}_{S,1,\beta}(t) &:= \sum_{n\in\mathbb{Z}} t^n \cdot \chi^{\mathrm{red}}_{-y}(\mathsf{Quot}_S(\mathbb{C}^1,\beta,n)) \\ &= \frac{y^g}{\sqrt{y}} \cdot Z^{\overline{\chi}^{\mathrm{red}}_{-y}}_{S,1,\beta}(t\sqrt{y}) \end{aligned}$$

as the q^g coefficient of the expression

$$\frac{y^g}{\left(ty+\frac{1}{t}-y-1\right)} \cdot \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{18}(1-q^n y)(1-q^n/y)(1-q^n/t)(1-q^n t)(1-q^n ty)(1-q^n/ty)}.$$

From the formula, we clearly have the rationality of the non-shifted series

$$Z^{\chi^{\mathrm{red}}_{-y}}_{S,1,\beta}(t) \in \mathbb{Q}(y)(t).$$

Here we record a few computations of $Z^{\chi^{\mathrm{red}}_{-y}}_{S,1,\beta}(t)$ for low genus cases:

$$\begin{aligned}
&g=0: \frac{t}{(1-t)(1-yt)},\\
&g=1: \frac{(1+y)(1+yt^2)+(1+18y+y^2)t}{(1-t)(1-yt)},\\
&g=2: \frac{(1+y+y^2)(1+y^2t^4)+(1+20y+20y^2+y^3)(t+yt^3)+(1+20y+192y^2+20y^3+y^4)t^2}{t(1-t)(1-ty)}.
\end{aligned} \tag{6.8}$$

6.5 Reduced invariants of Hilbert schemes of points

In this section, we study the reduced χ_{-y}-genus of $\mathrm{Quot}_S(\mathbb{C}^1,n)=S^{[n]}$ for $K3$ surface S. Note that the reduced structure on $S^{[n]}$ is defined only for non-zero quotient, i.e. $n\geq 1$ cases. Define the generating series of reduced χ_{-y} genera of Hilbert schemes of points

$$Z^{\chi^{\mathrm{red}}_{-y}}_{S,1,\beta=0}(t):=\sum_{n\geq 1} t^n\cdot \chi^{\mathrm{red}}_{-y}(S^{[n]}).$$

Recall that the usual obstruction space of $S^{[n]}$ as a Quot scheme is given by

$$\big(K_S^{[n]}\big)^\vee \simeq \big(\mathscr{O}_S^{[n]}\big)^\vee.$$

The reduced obstruction bundle

$$\big(\mathscr{O}_S^{[n]}\big)^\vee - \mathscr{O}_{S^{[n]}}$$

is obtained by removing one copy of the trivial line bundle from the usual obstruction bundle. Therefore we can express the reduced χ_{-y}-genus of $S^{[n]}$ as an integral

$$\begin{aligned}
\chi^{\mathrm{red}}_{-y}(S^{[n]}) &= \int_{S^{[n]}} \frac{e}{g}\Big(\big(\mathscr{O}_S^{[n]}\big)^\vee - \mathscr{O}_{S^{[n]}}\Big)\cdot g(T_{S^{[n]}})\\
&= \int_{S^{[n]}} \frac{e}{g}\Big(\mathscr{O}_S^{[n]} - \mathscr{O}_{S^{[n]}}\Big)\cdot g(T_{S^{[n]}})
\end{aligned} \tag{6.9}$$

where we used the fact that $S^{[n]}$ is holomorphic symplectic variety for the second equality as in the proof of Theorem 17. The function g in the formula is again defined as

$$g(z)=\frac{z(1-ye^{-z})}{1-e^{-z}}.$$

By the recursive formula of [EGL], the integral value of (6.9) is identical to any choice of $K3$ surface S.

Since the answer is independent on the choice of $K3$ surface, we assume that S is an elliptic $K3$ surface with the fiber class f of arithmetic genus 1. Even though $S^{[n]}$ does not involve any curve classes, we show that the answer can be obtained from the previous section by using the fiber class. More precisely, we claim that

$$\chi_{-y}^{\mathrm{red}}(S^{[n]})=\chi_{-y}^{\mathrm{red}}(\mathsf{Pair}_S(\mathbb{C}^1,f,n)),\quad n\geq 1. \tag{6.10}$$

The proposition below follows easily from this claim.

Proposition 22. *The generating series of reduced χ_{-y}-genera of Hilbert schemes of points on a $K3$ surface S is given by*

$$Z_{S,1,\beta=0}^{\chi_{-y}^{\mathrm{red}}}(t)=\frac{(2+20y+2y^2)t}{(1-t)(1-yt)}.$$

Proof. We first assume the claim (6.10). The only difference of $Z_{S,1,\beta=0}^{\chi_{-y}^{\mathrm{red}}}(t)$ with the genus 1 computation in (6.8) is that we need to remove the t^0 coefficient. Therefore we obtain

$$\begin{aligned}Z_{S,1,\beta=0}^{\chi_{-y}^{\mathrm{red}}}(t)&=\frac{(1+y)(1+yt^2)+(1+18y+y^2)t}{(1-t)(1-yt)}-(1+y)\\&=\frac{(2+20y+2y^2)t}{(1-t)(1-yt)}.\end{aligned}$$

Now we prove the claim (6.10). Denote the Chern roots of $\mathscr{O}_S^{[n]}$ by $u_1,\ldots,u_n$ with $u_1=0$.

Using the splitting principle, we can rewrite the equality (6.9) as

$$\chi_{-y}^{\mathrm{red}}(S^{[n]}) = \int_{S^{[n]}} \prod_{i=2}^{n} \frac{u_i}{g(u_i)} \cdot g(T_{S^{[n]}}). \tag{6.11}$$

On the other hand, the proof of Proposition 19 for the genus 1 case reads

$$\chi_{-y}^{\mathrm{red}}\big(\mathsf{Pair}_S(\mathbb{C}^1,f,n)\big) = \int_{\left[S^{[n]}\right]\times\left[\mathbb{P}^1\right]} \frac{e}{g}\big(L_f^{[n]}\boxtimes \mathscr{O}_{\mathbb{P}^1}(1)\big)\cdot g(T_{S^{[n]}})\cdot g(T_{\mathbb{P}^1}).$$

Denote the Chern roots of $L_f^{[n]}$ by $v_1,\dots,v_n$ and let $h = c_1(\mathscr{O}_{\mathbb{P}^1}(1))$. Then the above integral can be rewritten as

$$\int_{\left[S^{[n]}\right]\times\left[\mathbb{P}^1\right]} \prod_{i=1}^{n} \frac{h+v_i}{g(h+v_i)} \cdot \frac{g(h)^2}{g(0)} \cdot g(T_{S^{[n]}}). \tag{6.12}$$

By integrating over $\mathbb{P}^1$, this is equal to

$$\int_{S^{[n]}} \left([h^1]\prod_{i=1}^{n} \frac{h+v_i}{g(h+v_i)} \cdot \frac{g(h)^2}{g(0)}\right) \cdot g(T_{S^{[n]}})$$

Note that

$$\begin{aligned}
[h^1]\prod_{i=1}^{n} \frac{h+v_i}{g(h+v_i)} \cdot \frac{g(h)^2}{g(0)} &= \left([h^1]\,\frac{g(h)^2}{g(0)}\right)\cdot \prod_{i=1}^{n}\frac{v_i}{g(v_i)} + g(0)\cdot\left([h^1]\prod_{i=1}^{n}\frac{h+v_i}{g(h+v_i)}\right)\\
&= 2g^{(1)}\cdot\prod_{i=1}^{n}\frac{v_i}{g(v_i)} + g^{(0)}\cdot\sum_{i=1}^{n}\left(\frac{g(v_i)-v_i g'(v_i)}{g(v_i)^2}\cdot\prod_{j\neq i}\frac{v_j}{g(v_j)}\right)
\end{aligned}$$

where we defined $g^{(k)}$ by

$$g(z) = g^{(0)} + g^{(1)}z + g^{(2)}z^2 + \cdots.$$

Therefore the reduced χ_{-y}-genus of $\mathsf{Pair}_S(\mathbb{C}^1,f,n)$ is

$$\int_{S^{[n]}} \left(2g^{(1)}\cdot\prod_{i=1}^{n}\frac{v_i}{g(v_i)} + g^{(0)}\cdot\sum_{i=1}^{n}\left(\frac{g(v_i)-v_i\cdot g'(v_i)}{g(v_i)^2}\cdot\prod_{j\neq i}\frac{v_j}{g(v_j)}\right)\right)\cdot g(T_{S^{[n]}}) \tag{6.13}$$

This is a tautological integral on $S^{[n]}$ with the insertion $L_f^{[n]}$. By [EGL], there exists a polynomial $\mathsf{P}(x_1, x_2, x_3, x_4)$ that only depends on n, g such that the integral (6.13) is equal to

$$\mathsf{P}\Big(c_2(T_S)\,,\,c_1(T_S)^2\,,\,c_1(T_S).c_1(L_f)\;,c_1(L_f)^2\Big).$$

Since $c_1(T_S) = 0$ and $c_1(L_f)^2 = 0$ from the fact that S is an elliptic $K3$ surface with a fiber class f, we may replace L_f by $\mathscr{O}_S$. By doing so, we conclude that the integral (6.13) is unchanged if we replace $v_1, \ldots, v_n$ with the Chern roots of $\mathscr{O}_S^{[n]}$. In other words, the reduced χ_{-y}-genus of $\mathsf{Pair}_S(\mathbb{C}^1, f, n)$ is

$$\int_{S^{[n]}} \left(2g^{(1)} \cdot \prod_{i=1}^{n} \frac{u_i}{g(u_i)} + g^{(0)} \cdot \sum_{i=1}^{n} \left(\frac{g(u_i) - u_i \cdot g'(u_i)}{g(u_i)^2} \cdot \prod_{j \neq i} \frac{u_j}{g(u_j)}\right)\right) \cdot g(T_{S^{[n]}}).$$

Using the fact that $u_1 = 0$, this further simplifies to

$$\int_{S^{[n]}} \left(g^{(0)} \cdot \frac{g(u_1) - u_1 \cdot g'(u_1)}{g(u_1)^2} \cdot \prod_{j \neq 1} \frac{u_j}{g(u_j)}\right) \cdot g(T_{S^{[n]}}) = \int_{S^{[n]}} \prod_{j=2}^{n} \frac{u_j}{g(u_j)} \cdot g(T_{S^{[n]}}).$$

This matches with the integral (6.11), hence completes the proof. $\square$

Remark 23. Since the proof of Proposition 22 works for arbitrary choice of the power series

$$g(z) \in \Big(F[\![z]\!]\Big)^{\times},$$

the same proof yields the equality between the reduced cobordism class

$$\big[S^{[n]}\big]^{\mathrm{red}}_{\Omega_*} = \big[\mathsf{Pair}_S(\mathbb{C}^1, f, n)\big]^{\mathrm{red}}_{\Omega_*} \in \Omega_*(\mathrm{pt}).$$

This chapter is, in part, being prepared for submission for publication. The dissertation author was the primary investigator and the author of the material below.

- Woonam Lim "Virtual χ_{-y}-genera of Quot schemes on surfaces "

Chapter 7

Rationality of descendent series

In Chapter 5, 6, we have studied virtual (reduced) Euler characteristics and χ_{-y}-genera of Quot schemes. Main outcome of the study is the rationality of the generating series of such invariants for Seiberg-Witten classes of length N. Note that all these invariants are integration across the virtual (reduced) classes of certain cohomological classes defined by virtual (reduced) tangent bundles.

In [JOP], the authors suggest more general framework for rationality of Quot scheme invariants. Specifically, they conjectured rationality of homological descendent series which they prove for punctual Quot schemes. It was commented in [AJLOP, Remark 18] that the same argument with the proof of [AJLOP, Theorem 1] gives a proof of rationality of homological descendent series for curve classes of Seiberg-Witten length N. We explain this in Section 7.1.

Refining the work of [JOP] using the K-theory, [AJLOP] studied K-theoretic descendent series. As a main result, they prove the rationality of K-theoretic descendent series for Seiberg-Witten classes of length N. In section 7.2, we study the generalization of K-theoretic descendent series by including $\Lambda_{-y}\Omega^{\mathrm{vir}}$ factor following the comment in [AJLOP, page 4].

7.1 Homological descendent series

Define a homological descendent series

$$Z^{\mathsf{H}}_{S,N,\beta}(q\,|\,\underline{\alpha},\underline{k}) := \sum_{n\in\mathbb{Z}} q^n \int_{\left[\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)\right]^{\mathrm{vir}}} c_{k_1}(\alpha_1^{[n]})\cdots c_{k_\ell}(\alpha_\ell^{[n]})\cdot c\Big(T^{\mathrm{vir}}_{\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)}\Big). \tag{7.1}$$

When there is no descendent insertions, i.e. $\ell = 0$, this specializes to the generating series of virtual Euler characteristics. Note that one can define the similar series using Chern character insertions instead of Chern classes. These parallel theories are related by the universal polynomial expressing Chern characters by Chern classes and vice versa. In particular, rationality question is equivalent for either convention. We take Chern class convention because it has the advantage of being multiplicative.

To work with multiplicative characteristics, we form the master homological descendent series using the Chern polynomials rather than Chern classes of fixed degrees:

$$Z^{\dagger,\mathsf{H}}_{S,N,\beta}(q,\underline{x}\,|\,\underline{\alpha}) := \sum_{n\in\mathbb{Z}} q^n \int_{\left[\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)\right]^{\mathrm{vir}}} c_{x_1}(\alpha_1^{[n]})\cdots c_{x_\ell}(\alpha_\ell^{[n]})\cdot c\Big(T^{\mathrm{vir}}_{\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)}\Big). \tag{7.2}$$

We recover (7.1) for each $\underline{k}$ by taking derivatives of the master series

$$Z^{\mathsf{H}}_{S,N,\beta}(q\,|\,\underline{\alpha},\underline{k}) = \frac{1}{k_1!\cdots k_\ell!}\cdot\left(\frac{\partial}{\partial x_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial x_\ell}\right)^{k_\ell} Z^{\dagger,\mathsf{H}}_{S,N,\beta}(q,\underline{x}\,|\,\underline{\alpha})\Big|_{\underline{x}=0}. \tag{7.3}$$

For the rest of the section, we suppose that β is of Seiberg-Witten length N. Note that the series (7.2) corresponds to the general series (2.1) with the choice of

$$F = \mathbb{Q}(x_1,\ldots,x_N), \quad f_s(z) = 1 + x_s z, \quad g(z) = 1+z.$$

Therefore the structural formula (4.10) reads

$$Z^{\dagger,\mathsf{H}}_{S,N,\underline{\beta}}(q,\underline{x},\underline{w}\,|\,\underline{\alpha}) = q^{-\beta.K_S}\cdot \mathsf{A}^{K_S^2}\cdot\prod_s \mathsf{B}_s^{K_S.c_1(\alpha_s)}$$
$$\cdot\sum_{\substack{\beta=\beta_1+\cdots+\beta_N\\ \mathrm{vd}_{\beta_i}=0}} \mathsf{SW}(\underline{\beta})\cdot\prod_i \mathsf{U}_i^{\beta_i.K_S}\cdot\prod_{i,s}\mathsf{V}_{i,s}^{\beta_i.c_1(\alpha_s)}\cdot\prod_{i<j}\mathsf{W}_{i,j}^{\beta_i.\beta_j} \quad (7.4)$$

where

$$\begin{aligned}
\mathsf{A} &= \prod_s\prod_{i=1}^N \frac{(1-x_sH_i)^{r_s}}{(1+x_sw_i)^{r_s}}\cdot\prod_{i,k\in[N]}\frac{H_i+w_k}{1+H_i+w_k}\cdot\prod_{\substack{i_1\neq i_2\\ i_1,i_2\in[N]}}\frac{1+H_{i_1}-H_{i_2}}{H_{i_1}-H_{i_2}}\\
&\quad\cdot\prod_{i=1}^N\left(\sum_s r_s\cdot\frac{x_s}{1-x_sH_i}+\sum_{k=1}^N\left(\frac{1}{1-H_i-w_k}+\frac{1}{H_i+w_k}\right)\right),\\
\mathsf{U}_i &= \prod_s\frac{1}{(1-x_sH_i)^{r_s}}\cdot\prod_{k=1}^N\frac{1+H_i+w_k}{H_i+w_k}\cdot\prod_{\substack{i'\neq i\\ i'\in[N]}}\frac{H_{i'}-H_i}{1+H_{i'}-H_i}\cdot\frac{H_i-H_{i'}}{1+H_i-H_{i'}}\\
&\quad\cdot\left(\sum_s r_s\cdot\frac{x_s}{1-x_sH_i}+\sum_{k=1}^N\left(\frac{1}{1-H_i-w_k}+\frac{1}{H_i+w_k}\right)\right)^{-1}, \qquad (7.5)\\
\mathsf{W}_{i,j} &= \frac{1+H_i-H_j}{H_i-H_j}\cdot\frac{1+H_j-H_i}{H_j-H_i},\\
\mathsf{B}_s &= \prod_{k=1}^N\frac{1+x_sw_k}{1-x_sH_k},\\
\mathsf{V}_{i,s} &= 1-x_sH_i,
\end{aligned}$$

for change of variables

$$q=\prod_s\frac{1}{(1-x_sH_i)^{r_s}}\cdot\prod_{k=1}^N\frac{-H_i-w_k}{1-H_i-w_k}\quad\text{with}\quad H_i(q=0)=-w_i. \qquad (7.6)$$

To show rationality of $Z^{\mathsf{H}}_{S,N,\beta}(q\,|\,\underline{\alpha},\underline{k})$, we analyze the above universal series and the change of variables. First, observe that all the universal series are rational functions

$$\mathsf{A}\ ,\ \mathsf{U}_i\ ,\ \mathsf{W}_{i,j}\ ,\ \mathsf{B}_s\ ,\ \mathsf{V}_{i,s}\ \in \mathbb{Q}(\underline{x},\underline{w})(H_1,\ldots,H_N).$$

As we have seen in various calculations in Chapter 5, it is desirable to have symmetry in variables. Even though individual universal series are not necessarily symmetric in $H_1,\ldots,H_N$, we can write $Z^{\dagger,\mathsf{H}}_{S,N,\underline{\beta}}(q,\underline{x}\,|\,\underline{\alpha})$ as a combination of symmetric ones. More precisely, it can be written by using the symmetrized summations permuting the curve classes

$$\frac{1}{|\mathsf{Stab}|}\sum_{\sigma\in S_N}\left(\mathsf{A}^{K_S^2}\cdot\prod_s \mathsf{B}_s^{K_S.c_1(\alpha_s)}\cdot\prod_i \mathsf{U}_i^{\beta_{\sigma(i)}.K_S}\cdot\prod_{i,s}\mathsf{V}_{i,s}^{\beta_{\sigma(i)}.c_1(\alpha_s)}\cdot\prod_{i<j}\mathsf{W}_{i,j}^{\beta_{\sigma(i)}.\beta_{\sigma(j)}}\right) \tag{7.7}$$

where $|\mathsf{Stab}|$ is the size of the stabilizer group fixing $(\beta_1,\ldots,\beta_N)$.

We work more generally by considering a function R that satisfies the following conditions:

$$\begin{aligned} &1.\ \mathsf{R}\in\mathbb{Q}(\underline{x},\underline{w})(H_1,\ldots,H_N),\\ &2.\ \text{no poles at the evaluation } \underline{x}=\underline{w}=0,\\ &3.\ \text{symmetric in variables } H_1,\ldots,H_N. \end{aligned} \tag{7.8}$$

It is clear from the formulas (7.5) that every universal series satisfies the first two conditions. The symmetrized summation (7.7) further satisfies the third condition. Based on the derivative formulation (7.3), the following lemma proves that $Z^{\mathsf{H}}_{S,N,\beta}(q\,|\,\underline{\alpha},\underline{k})$ is rational in the q variable if the curve class β is of Seiberg-Witten length N.

Lemma 24. *Let* R *be a function satisfying the conditions in* (7.8). *Let* $H_1,\ldots,H_N$ *be formal power series in the q variable with coefficients in* $\mathbb{Q}(\underline{x},\underline{w})$ *defined recursively from the equation* *and the initial condition in* (7.6)*. Then we have*

$$\frac{1}{k_1!\cdots k_\ell!}\cdot\left(\frac{\partial}{\partial x_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial x_\ell}\right)^{k_\ell}\mathsf{R}\Big|_{\underline{x}=\underline{w}=0}\in\mathbb{Q}(q)$$

for arbitrary $\underline{k}$*.*

Proof. Proof consists of two steps. First, we prove that the conditions (7.8) are preserved by taking arbitrary number of derivatives with respect to $x_1,\dots,x_\ell$. This reduces the proof to the case with $\underline{k}=0$. We do this in the second part of the proof.

For the first part of the proof, we apply the chain rule

$$\frac{\partial\mathsf{R}}{\partial x_s}=\partial_{x_s}\mathsf{R}+\sum_{i=1}^{N}\partial_{H_i}\mathsf{R}\cdot\frac{\partial H_i}{\partial x_s}.$$

In the above formula, partial derivative notations with subscripts, as in $\partial_{x_s}\mathsf{R}$ and $\partial_{H_i}\mathsf{R}$, denote partial derivatives while considering $\underline{x},\underline{w}$ and $\underline{H}$ as independent variables. On the other hand, fractional partial derivative notations, as in $\frac{\partial\mathsf{R}}{\partial x_s}$ and $\frac{\partial H_i}{\partial x_s}$, denote partial derivatives where $H_1,\dots,H_N$ are considered to be functions in the variables $\underline{x},\underline{w}$ and q. It is clear that $\partial_{x_s}\mathsf{R}$ satisfies all the conditions in (7.8). Note that $\partial_{H_i}\mathsf{R}$ is a rational function without poles at $\underline{x}=\underline{w}=0$ which transforms to $\partial_{H_{\sigma(i)}}\mathsf{R}$ under the permutation σ of $H_1,\dots,H_N$.

To analyze $\frac{\partial H_i}{\partial x_s}$, we introduce a function

$$\mathsf{P}(\underline{x},\underline{w},H):=\prod_s\frac{1}{(1-x_sH)^{r_s}}\cdot\prod_{k=1}^{N}\frac{-H-w_k}{1-H-w_k}\in\mathbb{Q}(\underline{x},\underline{w},H).\tag{7.9}$$

Then the equation (7.6) can be written as

$$q=\mathsf{P}(\underline{x},\underline{w},H_i).\tag{7.10}$$

Taking partial derivatives by x_s to the above equation yields

$$0 = \partial_{x_s} \mathsf{P}(\underline{x}, \underline{w}, H_i) + \partial_H \mathsf{P}(\underline{x}, \underline{w}, H_i) \cdot \frac{\partial H_i}{\partial x_s}$$

from the chain rule again. This gives the formula

$$\frac{\partial H_i}{\partial x_s} = -\frac{\partial_{x_s} \mathsf{P}}{\partial_H \mathsf{P}}(\underline{x}, \underline{w}, H_i),$$

expressing $\frac{\partial H_i}{\partial x_s}$ as an explicit rational function in variables $\underline{x}, \underline{w}$ and H_i. Since P has no poles at $\underline{x} = \underline{w} = 0$, so does $\frac{\partial H_i}{\partial x_s}$.

In conclusion, we have shown that

$$\frac{\partial \mathsf{R}}{\partial x_s} = \partial_{x_s} \mathsf{R} - \sum_{i=1}^{N} \partial_{H_i} \mathsf{R} \cdot \frac{\partial_{x_s} \mathsf{P}}{\partial_H \mathsf{P}}(\underline{x}, \underline{w}, H_i)$$

satisfies all the conditions in (7.8). We used the summation over $i = 1, \ldots, N$ and the transformation rule for the summands for the third condition. By repeating the argument, we show that conditions (7.8) are preserved under taking arbitrary number of derivatives with respect to $x_1, \ldots, x_\ell$.

For the second part of the proof, we show that

$$\mathsf{R}(\underline{x}, \underline{w}, H_1, \ldots, H_N)\Big|_{\underline{x}=\underline{w}=0} \in \mathbb{Q}(q).$$

We perform two specializations $\underline{x} = 0$ and $\underline{w} = 0$ one at a time. Recall that

$$H_1, \ldots, H_N \in \mathbb{Q}(\underline{x}, \underline{w})[\![q]\!]$$

are solutions to the equation

$$q = \mathsf{P}(\underline{x}, \underline{w}, H) \tag{7.11}$$

with distinct initial conditions at $q = 0$. We first show that the equation and the solutions behave well under $\underline{x} = 0$ specialization. More precisely, we show that coefficients of the H_i's have no poles at $\underline{x} = 0$. Furthermore, $\underline{x} = 0$ specializations of the solutions

$$H_1\Big|_{\underline{x}=0}, \ldots, H_N\Big|_{\underline{x}=0} \in \mathbb{Q}(\underline{w})[\![q]\!]$$

are solutions to the $\underline{x} = 0$ specialized equation

$$q = \mathsf{P}(\underline{0}, \underline{w}, H) = \prod_{k=1}^{N} \frac{-H - w_k}{1 - H - w_k}$$

with distinct initial conditions at $q = 0$ as before. This can be checked easily by induction on the degree of q for each H_i. By the initial condition in (7.6), the constant term of the q power series expansion of H_i is given by

$$H_i\Big|_{q=0} = -w_i.$$

This clearly has no poles at $\underline{x} = 0$ as a rational function in $\mathbb{Q}(\underline{x}, \underline{w})$. Suppose for the induction hypothesis that coefficients of H_i up to degree $p - 1$, for some $p \geq 1$, have no poles at $\underline{x} = 0$. Recall that the p'th degree part of H_i is recursively determined by taking the derivative $\left(\frac{\partial}{\partial q}\right)^p$ followed by the evaluation at $q = 0$ to the equation (7.10) which yields

$$\left(\frac{\partial}{\partial q}\right)^p q\Big|_{q=0} = \partial_H \mathsf{P}(\underline{x}, \underline{w}, -w_i) \cdot \frac{\partial^p H_i}{\partial q^p}\Big|_{q=0} + \text{(lower order terms)}.$$

Here the lower order term is a polynomial in

$$\partial_H^u \mathsf{P}(\underline{x}, \underline{w}, -w_i), \qquad \frac{\partial^v H_i}{\partial q^v}\Big|_{q=0}$$

for $1 \leq u \leq p$ and $1 \leq v < p$. By induction hypothesis, the lower order term is a rational function with no poles at $\underline{x} = 0$. Therefore we only need to check that $\partial_H \mathsf{P}(\underline{x}, \underline{w}, -w_i)$ is a rational function which is non-zero at $\underline{x} = 0$ so that we can divide. Explicit calculation using the definition (7.9) gives

$$\partial_H \mathsf{P}(\underline{x}, \underline{w}, -w_i) = \prod_s \frac{1}{(1 + x_s w_i)^{r_s}} \cdot \prod_{k \neq i} \frac{w_i - w_k}{1 + w_i - w_k} \cdot (-1) \in \mathbb{Q}(\underline{x}, \underline{w}).$$

This proves the first claim that H_i is a power series with coefficients in rational functions without poles at $\underline{x} = 0$. Furthermore, since $\underline{x}$ is independent on the variable q, the above procedure shows the second claim that $\underline{x}$ specializations of $H_1, \ldots, H_N$ are solutions of the $\underline{x} = 0$ specialized equation.

To finish the proof, we need to show that

$$\mathsf{S}(\underline{w}, z_1, \ldots, z_N)\Big|_{\underline{w}=0} \in \mathbb{Q}(q)$$

where we use the simplified notations

$$z_i := H_i\Big|_{\underline{x}=0}, \quad \mathsf{S}(\underline{w}, z_1, \ldots, z_N) := \mathsf{R}(\underline{0}, \underline{w}, z_1, \ldots, z_N).$$

From the previous argument, $z_1, \ldots, z_N$ are solutions of the equation

$$q = \prod_{k=1}^{N} \frac{-z - w_k}{1 - z - w_k}$$

with the initial condition

$$z_i(q = 0) = -w_i.$$

Note that the conditions (7.8) on R implies that $\mathsf{S} \in \mathbb{Q}(\underline{w})(z_1, \ldots, z_N)$ is symmetric in variables $z_1, \ldots, z_N$ without poles at $\underline{w} = 0$. Let e_i be the elementary symmetric polynomials in $z_1, \ldots, z_N$ with a sign $(-1)^i$. Since S is symmetric in $z_1, \ldots, z_N$, there exists $\mathsf{T} \in \mathbb{Q}(\underline{w})(e_1, \ldots, e_N)$ without poles at $\underline{w} = 0$ such that

$$\mathsf{S}(\underline{w}, z_1, \ldots, z_N) = \mathsf{T}(\underline{w}, e_1, \ldots, e_N).$$

By rewriting the equation as

$$\prod_{k=1}^{N}(z+w_k) - q\prod_{k=1}^{N}(z+w_k-1) = 0,$$

we obtain the equality

$$\begin{aligned}\prod_{k=1}^{N}(z+w_k) - q\prod_{k=1}^{N}(z+w_k-1) &= (1-q)\cdot\prod_{i=1}^{N}(z-z_i)\\ &= (1-q)\cdot\left(z^N + e_1 z^{N-1} + e_2 z^{N-2} + \cdots + e_N\right).\end{aligned}$$

We used the factorization with respect to the roots and definition of the signed elementary symmetric polynomials. From this formula, it is clear that

$$e_i \in \mathbb{Q}(\underline{w})(q)$$

without poles at $\underline{w} = 0$. Therefore we conclude that

$$\mathsf{T}(\underline{w}, e_1, \ldots, e_N)\Big|_{\underline{w}=0} \in \mathbb{Q}(q).$$

This completes the proof. □

7.2 K-theoretic descendent series

Define a K-theoretic descendent series

$$Z^{\mathsf{K}}_{S,N,\beta}(q\,|\,\underline{\alpha},\underline{k}) := \sum_{n\in\mathbb{Z}} q^n \cdot \chi^{\mathrm{vir}}\Big(\mathsf{Quot}_S(\mathbb{C}^N,\beta,n), \Lambda^{k_1}(\alpha_1^{[n]})^{\varepsilon_1}\otimes\cdots\otimes\Lambda^{k_\ell}(\alpha_\ell^{[n]})^{\varepsilon_\ell}\otimes\Lambda_{-y}\Omega^{\mathrm{vir}}_{\mathsf{Quot}}\Big) \tag{7.12}$$

where $\varepsilon_1, \ldots, \varepsilon_\ell$ are ± 1. For the ease of the notation, we omitted dependence on the vector of signs $\underline{\varepsilon}$ in the notation of generating series. This definition generalizes some of the virtual K-

theoretic series that have been studied before. When there is no descendent insertions, i.e. $\ell = 0$, this specializes to the generating series of virtual χ_{-y}-genera. On the other hand, when $y = 0$ and $\varepsilon_1 = \cdots = \varepsilon_\ell = 1$, this specializes to K-theoretic invariants studied in [AJLOP]. However, it was already noted in [AJLOP] that the same method of theirs works for this generalization.

Even though homological descendent series and K-theoretic descendent series are defined by different structures of Quot schemes, essentially the same method applies to both cases. Rationality statement for K-theoretic descendent series can be seen analogously to the homological case after the extra exponential change of variables. This is not unexpected because K-theory and homology are connected via Chern character map. The exponential change of variables were already seen in the study of virtual χ_{-y}-genera in (5.4).

To work with multiplicative characteristics, we form the master K-theoretic descendent series using the total wedge products with formal variable $x_1, \ldots, x_\ell$:

$$Z^{\dagger,\mathsf{K}}_{S,N,\beta}(q,\underline{x}\,|\,\underline{\alpha}) := \sum_{n\in\mathbb{Z}} q^n \cdot \chi^{\mathrm{vir}}\Big(\mathsf{Quot}_S(\mathbb{C}^N,\beta,n), \Lambda_{x_1}(\alpha_1^{[n]})^{\varepsilon_1} \otimes \cdots \otimes \Lambda_{x_\ell}(\alpha_\ell^{[n]})^{\varepsilon_\ell} \otimes \Lambda_{-y}\Omega^{\mathrm{vir}}_{\mathsf{Quot}}\Big). \tag{7.13}$$

We recover (7.12) for each $\underline{k}$ by taking derivatives of the master series

$$Z^{\mathsf{K}}_{S,N,\beta}(q\,|\,\underline{\alpha},\underline{k}) = \frac{1}{k_1!\cdots k_\ell!} \cdot \left(\frac{\partial}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial}{\partial x_\ell}\right)^{k_\ell} Z^{\dagger,\mathsf{K}}_{S,N,\beta}(q,\underline{x}\,|\,\underline{\alpha})\Big|_{\underline{x}=0}.$$

By virtual Hirzebruch-Riemann-Roch formula, we can express the master K-theoretic descendent series using the integrations over the virtual fundamental classes:

$$Z^{\dagger,\mathsf{K}}_{S,N,\beta}(q,\underline{x}\,|\,\underline{\alpha}) \;=\; \sum_{n\in\mathbb{Z}} q^n \int_{\left[\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)\right]^{\mathrm{vir}}} \prod_{s=1}^{\ell} \mathrm{ch}^{\varepsilon_\ell}\Big(\Lambda_{x_\ell}(\alpha_\ell^{[n]})\Big) \cdot \mathrm{ch}\Big(\Lambda_{-y}\Omega^{\mathrm{vir}}_{\mathsf{Quot}}\Big) \cdot \mathrm{td}(T^{\mathrm{vir}}_{\mathsf{Quot}}). \tag{7.14}$$

For the rest of the section, we suppose that β is of Seiberg-Witten length N. Note that the series (7.14) corresponds to the general series (2.1) with the choice of

$$F = \mathbb{Q}(x_1, \ldots, x_N, y), \quad f_s(z) = 1 + x_s e^{\varepsilon_s z}, \quad g(z) = \frac{z(1 - ye^{-z})}{1 - e^{-z}}.$$

Therefore the structural formula (4.10) reads

$$\begin{aligned} Z^{\dagger,\mathsf{K}}_{S,N,\underline{\beta}}(q, \underline{x}, \underline{w} \,|\, \underline{\alpha}) &= q^{-\beta.K_S} \cdot \mathsf{A}^{K_S^2} \cdot \prod_s \mathsf{B}_s^{K_S.c_1(\alpha_s)} \\ &\cdot \sum_{\substack{\beta = \beta_1 + \cdots + \beta_N \\ \mathrm{vd}_{\beta_i} = 0}} \mathsf{SW}(\underline{\beta}) \cdot \prod_i \mathsf{U}_i^{\beta_i.K_S} \cdot \prod_{i,s} \mathsf{V}_{i,s}^{\beta_i.c_1(\alpha_s)} \cdot \prod_{i<j} \mathsf{W}_{i,j}^{\beta_i.\beta_j} \end{aligned} \tag{7.15}$$

where

$$\begin{aligned} \mathsf{A} &= (1-y)^N \prod_s \prod_{i=1}^N \frac{(1 + x_s e^{-\varepsilon_s H_i})^{r_s}}{(1 + x_s e^{\varepsilon_s w_i})^{r_s}} \cdot \prod_{i,k \in [N]} \frac{1 - e^{-H_i - w_k}}{1 - ye^{-H_i - w_k}} \cdot \prod_{\substack{i_1 \neq i_2 \\ i_1, i_2 \in [N]}} \frac{1 - ye^{-H_{i_1} + H_{i_2}}}{1 - e^{-H_{i_1} + H_{i_2}}} \\ &\quad \cdot \prod_{i=1}^N \left(\sum_s r_s \cdot \frac{\varepsilon_s x_s e^{-\varepsilon_s H_i}}{1 + x_s e^{-\varepsilon_s H_i}} + \sum_{k=1}^N \Big(- \frac{e^{-H_i - w_k} + ye^{H_i + w_k}}{(1 - ye^{H_i + w_k})(1 - e^{-H_i - w_k})} \Big) \right), \\ \mathsf{U}_i &= \frac{1}{1-y} \prod_s \frac{1}{(1 + x_s e^{-\varepsilon_s H_i})^{r_s}} \cdot \prod_{k=1}^N \frac{1 - ye^{-H_i - w_k}}{1 - e^{-H_i - w_k}} \cdot \prod_{\substack{i' \neq i \\ i' \in [N]}} \frac{1 - e^{-H_{i'} + H_i}}{1 - ye^{-H_{i'} + H_i}} \cdot \frac{1 - e^{-H_i + H_{i'}}}{1 - ye^{-H_i + H_{i'}}} \\ &\quad \cdot \left(\sum_s r_s \cdot \frac{\varepsilon_s x_s e^{-\varepsilon_s H_i}}{1 + x_s e^{-\varepsilon_s H_i}} + \sum_{k=1}^N \Big(- \frac{e^{-H_i - w_k} + ye^{H_i + w_k}}{(1 - ye^{H_i + w_k})(1 - e^{-H_i - w_k})} \Big) \right)^{-1}, \\ \mathsf{W}_{i,j} &= \frac{1 - ye^{-H_i + H_j}}{1 - e^{-H_i + H_j}} \cdot \frac{1 - ye^{-H_j + H_i}}{1 - e^{-H_j + H_i}}, \\ \mathsf{B}_s &= \prod_{k=1}^N \frac{1 + x_s e^{\varepsilon_s w_k}}{1 + x_s e^{-\varepsilon_s H_k}}, \\ \mathsf{V}_{i,s} &= 1 + x_s e^{-\varepsilon_s H_i}, \end{aligned}$$

for change of variables

$$q = \prod_s \frac{1}{(1 + x_s e^{-\varepsilon_s H_i})^{r_s}} \cdot \prod_{k=1}^N \frac{1 - e^{H_i + w_k}}{1 - ye^{H_i + w_k}} \quad \text{with} \quad H_i(q=0) = -w_i.$$

Now we use the following exponential change of variables as in (5.4)

$$e^{H_i} = z_i, \quad e^{w_k} = t_k.$$

We record the universal series with respect to this extra change of variables

$$\begin{aligned}
\mathsf{A} &= (1-y)^N \prod_s \prod_{i=1}^N \frac{(1+x_s z_i^{-\varepsilon_s})^{r_s}}{(1+x_s t_i^{\varepsilon_s})^{r_s}} \cdot \prod_{i,k\in[N]} \frac{1-z_i^{-1}t_k^{-1}}{1-yz_i^{-1}t_k^{-1}} \cdot \prod_{\substack{i_1\neq i_2 \\ i_1,i_2\in[N]}} \frac{1-yz_{i_1}^{-1}z_{i_2}}{1-z_{i_1}^{-1}z_{i_2}} \\
&\quad \cdot \prod_{i=1}^N \left(\sum_s r_s \cdot \frac{\varepsilon_s x_s z_i^{-\varepsilon_s}}{1+x_s z_i^{-\varepsilon_s}} + \sum_{k=1}^N \Big(-\frac{z_i^{-1}t_k^{-1} + yz_i t_k}{(1-yz_i^{-1}t_k^{-1})(1-z_i^{-1}t_k^{-1})} \Big) \right), \\
\mathsf{U}_i &= \frac{1}{1-y} \prod_s \frac{1}{(1+x_s z_i^{-\varepsilon_s})^{r_s}} \cdot \prod_{k=1}^N \frac{1-yz_i^{-1}t_k^{-1}}{1-z_i^{-1}t_k^{-1}} \cdot \prod_{\substack{i'\neq i \\ i'\in[N]}} \frac{1-z_{i'}^{-1}z_i}{1-yz_{i'}^{-1}z_i} \cdot \frac{1-z_i^{-1}z_{i'}}{1-yz_i^{-1}z_{i'}} \\
&\quad \cdot \left(\sum_s r_s \cdot \frac{\varepsilon_s x_s z_i^{-\varepsilon_s}}{1+x_s z_i^{-\varepsilon_s}} + \sum_{k=1}^N \Big(-\frac{z_i^{-1}t_k^{-1} + yz_i t_k}{(1-yz_i^{-1}t_k^{-1})(1-z_i^{-1}t_k^{-1})} \Big) \right)^{-1}, \\
\mathsf{W}_{i,j} &= \frac{1-yz_i^{-1}z_j}{1-z_i^{-1}z_j} \cdot \frac{1-yz_j^{-1}z_i}{1-z_j^{-1}z_i}, \\
\mathsf{B}_s &= \prod_{k=1}^N \frac{1+x_s t_k^{\varepsilon_s}}{1+x_s z_k^{-\varepsilon_s}}, \\
\mathsf{V}_{i,s} &= 1+x_s z_i^{-\varepsilon_s},
\end{aligned} \tag{7.16}$$

and

$$q = \prod_s \frac{1}{(1+x_s z_i^{-\varepsilon_s})^{r_s}} \cdot \prod_{k=1}^N \frac{1-z_i t_k}{1-yz_i t_k} \quad \text{with} \quad z_i(q=0) = t_i^{-1}. \tag{7.17}$$

It is clear from the formula that

$$\mathsf{A}\ ,\ \mathsf{U}_i\ ,\ \mathsf{W}_{i,j}\ ,\ \mathsf{B}_s\ ,\ \mathsf{V}_{i,s}\ \in \mathbb{Q}(\underline{x},\underline{t})(y)(z_1,\ldots,z_N).$$

By the symmetrization method as in the previous section, we can write

$$Z^{\dagger,\mathsf{K}}_{S,N,\underline{\beta}}(q,\underline{x}\,|\,\underline{\alpha})$$

as a summation of functions R satisfying the following conditions:

$$\begin{aligned}&1.\ \mathsf{R}\in\mathbb{Q}(\underline{x},\underline{t})(y)(z_1,\ldots,z_N),\\&2.\ \text{no poles at the evaluation } \underline{x}=0,\ \underline{t}=1,\\&3.\ \text{symmetric in variables } z_1,\ldots,z_N.\end{aligned}\tag{7.18}$$

Then the rationality statement

$$Z^{\mathsf{H}}_{S,N,\beta}(q\,|\,\underline{\alpha},\underline{k})\in\mathbb{Q}(y)(q)$$

for curve classes β of Seiberg-Witten length N follows from the lemma below.

Lemma 25. *Let* R *be a function satisfying the conditions in* (7.18). *Let* $z_1,\ldots,z_N$ *be formal power series in the* q *variable with coefficients in* $\mathbb{Q}(\underline{x},\underline{t})(y)$ *defined recursively from the equation and the initial condition in* (7.17). *Then we have*

$$\frac{1}{k_1!\cdots k_\ell!}\cdot\left(\frac{\partial}{\partial x_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial x_\ell}\right)^{k_\ell}\mathsf{R}\Big|_{\underline{x}=0,\underline{t}=1}\in\mathbb{Q}(y)(q)$$

for arbitrary $\underline{k}$.

Proof. Proof is identical to the proof of Lemma 24. ☐

This chapter is, in part, being prepared for submission for publication. The dissertation author was the collaborator and the coauthor for the material below.

- Noah Arbesfeld, Drew Johnson, Woonam Lim, Dragos Oprea, Rahul Pandharipande "The virtual K-theory of Quot schemes of surfaces"

Chapter 8

Virtual Segre/Verlinde series

Let S be a smooth projective surface and $\alpha \in K^0(S)$ be a K-theory class. We define the virtual Segre and Verlinde series, respectively, as

$$\mathsf{S}_{S,N,\beta}(q\,|\,\alpha) := \sum_{n\in\mathbb{Z}} q^n \int_{\left[\mathsf{Quot}_S(\mathbb{C}^N,\beta,n)\right]^{\mathrm{vir}}} s(\alpha^{[n]}), \tag{8.1}$$

$$\mathsf{V}_{S,N,\beta}(q\,|\,\alpha) := \sum_{n\in\mathbb{Z}} q^n \cdot \chi^{\mathrm{vir}}(\mathsf{Quot}_S(\mathbb{C}^N,\beta,n), \det \alpha^{[n]}). \tag{8.2}$$

Motivated from the parallel theory for non-virtual geometry of Hilbert schemes of points, we study the virtual Segre/Verlinde correspondence and algebraicity.

In section 8.1, we prove the virtual Segre/Verlinde correspondence up to change of variables for Seiberg-Witten classes of length N. The correspondence is simpler than the (non-virtual) Segre/Verlinde correspondence for Hilbert schemes of points in the following sense. First, it involves the same rank of the input α for both virtual Segre and Verlinde series. Second, the change of variables are significantly simpler. For example, it it implies the perfect match of the virtual Segre and Verlinde series for puntual Quot schemes up to a sign.

In section 8.2, we prove that the virtual Segre and Verlinde series are algebraic functions for Seiberg-Witten classes of length N. As a result, we conjecture the algebraicity without the assumption on the curve class.

8.1 Virtual Segre/Verlinde correspondence

For this section, suppose that β is a curve class of Seiberg-Witten length N. Note that the Segre series (8.1) corresponds to the general series (2.1) with the choice of

$$\ell = 1, \quad f(z) = \frac{1}{1+z}, \quad g(z) = 1.$$

Therefore the structural formula (4.10) reads

$$\mathsf{S}_{S,N,\beta}(q,\underline{w}\,|\,\alpha) = q^{-\beta.K_S} \cdot \mathsf{A}^{K_S^2} \cdot \mathsf{B}^{K_S.c_1(\alpha)} \cdot \sum_{\substack{\beta=\beta_1+\cdots+\beta_N \\ \mathrm{vd}_{\beta_i}=0}} \mathrm{SW}(\underline{\beta}) \cdot \prod_i \mathsf{U}_i^{\beta_i.K_S} \cdot \prod_i \mathsf{V}_i^{\beta_i.c_1(\alpha)} \cdot \prod_{i<j} \mathsf{W}_{i,j}^{\beta_i.\beta_j} \tag{8.3}$$

where

$$\begin{aligned}
\mathsf{A} &= \prod_{i=1}^{N} \frac{(1+w_i)^r}{(1-H_i)^r} \cdot \prod_{i,k\in[N]} (H_i+w_k) \cdot \prod_{\substack{i_1\neq i_2 \\ i_1,i_2\in[N]}} \frac{1}{H_{i_1}-H_{i_2}} \cdot \prod_{i=1}^{N} \left(\frac{-r}{1-H_i} + \sum_{k=1}^{N} \frac{1}{H_i+w_k} \right), \\
\mathsf{U}_i &= (1-H_i)^r \cdot \prod_{k=1}^{N} \frac{1}{H_i+w_k} \cdot \prod_{\substack{i'\neq i \\ i'\in[N]}} (H_{i'}-H_i)(H_i-H_{i'}) \cdot \left(\frac{-r}{1-H_i} + \sum_{k=1}^{N} \frac{1}{H_i+w_k} \right)^{-1}, \\
\mathsf{W}_{i,j} &= \frac{1}{H_i-H_j} \cdot \frac{1}{H_j-H_i}, \\
\mathsf{B} &= \prod_{k=1}^{N} \frac{1-H_k}{1+w_k}, \\
\mathsf{V}_i &= \frac{1}{1-H_i},
\end{aligned} \tag{8.4}$$

for change of variables

$$(-1)^N q = (1-H_i)^r \cdot \prod_{k=1}^{N} (H_i+w_k) \quad \text{with} \quad H_i(q=0) = -w_i. \tag{8.5}$$

Recall that r is the rank of α in the above formulas.

Now we look at virtual Verlinde series (8.2) which corresponds to the general series (2.1) with the choice of

$$\ell = 1, \quad f(z) = e^z, \quad g(z) = \frac{z}{1-e^{-z}}$$

by application of virtual Hirzebruch-Riemann-Roch formula. In this case, the structural formula (4.10) reads

$$\mathsf{V}_{S,N,\beta}(\widetilde{q},\underline{\widetilde{w}}\,|\,\alpha) = \widetilde{q}^{\,-\beta.K_S}\cdot \widetilde{\mathsf{A}}^{K_S^2}\cdot \widetilde{\mathsf{B}}^{K_S.c_1(\alpha)}\cdot \sum_{\substack{\beta=\beta_1+\cdots+\beta_N\\ \mathrm{vd}_{\beta_i}=0}} \mathrm{SW}(\underline{\beta})\cdot \prod_i \widetilde{\mathsf{U}}_i^{\beta_i.K_S}\cdot \prod_i \widetilde{\mathsf{V}}_i^{\beta_i.c_1(\alpha)}\cdot \prod_{i<j} \widetilde{\mathsf{W}}_{i,j}^{\beta_i.\beta_j} \tag{8.6}$$

where

$$\begin{aligned}
\widetilde{\mathsf{A}} &= \prod_{i=1}^N \frac{e^{-r\widetilde{H}_i}}{e^{r\widetilde{w}_i}}\cdot \prod_{i,k\in[N]}(1-e^{-\widetilde{H}_i-\widetilde{w}_k})\cdot \prod_{\substack{i_1\neq i_2\\ i_1,i_2\in[N]}} \frac{1}{1-e^{-\widetilde{H}_{i_1}+\widetilde{H}_{i_2}}}\cdot \prod_{i=1}^N\left(r+\sum_{k=1}^N \frac{-e^{\widetilde{H}_i+\widetilde{w}_k}}{1-e^{\widetilde{H}_i+\widetilde{w}_k}}\right),\\
\widetilde{\mathsf{U}}_i &= e^{r\widetilde{H}_i}\cdot \prod_{k=1}^N \frac{1}{1-e^{-\widetilde{H}_i-\widetilde{w}_k}}\cdot \prod_{\substack{i'\neq i\\ i'\in[N]}} (1-e^{-\widetilde{H}_{i'}+\widetilde{H}_i})(1-e^{-\widetilde{H}_i+\widetilde{H}_{i'}})\cdot \left(r+\sum_{k=1}^N \frac{-e^{\widetilde{H}_i+\widetilde{w}_k}}{1-e^{\widetilde{H}_i+\widetilde{w}_k}}\right)^{-1},\\
\widetilde{\mathsf{W}}_{i,j} &= \frac{1}{1-e^{-\widetilde{H}_i+\widetilde{H}_j}}\cdot \frac{1}{1-e^{-\widetilde{H}_j+\widetilde{H}_i}},\\
\widetilde{\mathsf{B}} &= \prod_{k=1}^N \frac{e^{\widetilde{w}_k}}{e^{-\widetilde{H}_k}},\\
\widetilde{\mathsf{V}}_i &= e^{-\widetilde{H}_i},
\end{aligned} \tag{8.7}$$

for change of variables

$$\widetilde{q} = e^{r\widetilde{H}_i}\cdot \prod_{k=1}^N (1-e^{\widetilde{H}_i+\widetilde{w}_k}) \quad \text{with} \quad \widetilde{H}_i(\widetilde{q}=0) = -\widetilde{w}_i. \tag{8.8}$$

We used tilde notations to distinguish them from the virtual Segre series. To simplify the formulas and compare them to their Segre analogues, we use the following exponential change of variables:

$$e^{\widetilde{H}_i} = 1 - H_i, \quad \text{and} \quad e^{-\widetilde{w}_k} = 1 + w_k. \tag{8.9}$$

According to these new variables, we can rewrite the universal series in (8.7) as

$$\begin{aligned}
\widetilde{\mathsf{A}} &= \prod_{i=1}^{N} \frac{(1+w_i)^r}{(1-H_i)^r} \cdot \prod_{i,k\in[N]} \frac{H_i + w_k}{-(1-H_i)} \cdot \prod_{\substack{i_1\neq i_2\\ i_1,i_2\in[N]}} \frac{-(1-H_{i_1})}{H_{i_1}-H_{i_2}} \cdot \prod_{i=1}^{N}\left(r + \sum_{k=1}^{N} \frac{-(1-H_i)}{H_i+w_k}\right),\\
\widetilde{\mathsf{U}}_i &= (1-H_i)^r \cdot \prod_{k=1}^{N} \frac{-(1-H_i)}{H_i+w_k} \cdot \prod_{\substack{i'\neq i\\ i'\in[N]}} \frac{H_{i'}-H_i}{-(1-H_{i'})} \cdot \frac{H_i - H_{i'}}{-(1-H_i)} \cdot \left(r + \sum_{k=1}^{N} \frac{-(1-H_i)}{H_i+w_k}\right)^{-1},\\
\widetilde{\mathsf{W}}_{i,j} &= \frac{1-H_i}{H_i-H_j} \cdot \frac{1-H_j}{H_j-H_i}, \qquad\qquad (8.10)\\
\widetilde{\mathsf{B}} &= \prod_{k=1}^{N} \frac{1-H_k}{1+w_k},\\
\widetilde{\mathsf{V}}_i &= \frac{1}{1-H_i},
\end{aligned}$$

and the change of variables (8.8) as

$$\prod_{k=1}^{N}(1+w_k)\cdot \widetilde{q} = (1-H_i)^r \cdot \prod_{k=1}^{N}(H_i+w_k) \quad \text{with} \quad H_i(\widetilde{q}=0) = -w_i. \tag{8.11}$$

Based on the above computations, we compare the virtual Segre series and virtual Verlinde series. Note that we intentionally abused the notations w_k, H_i to denote the variables for Segre side and the variables for Verlinde side after the exponential change of variables (8.9). To do this more precisely, we first need to identify the equivariant variables

$$w_k^{\mathsf{Segre}} = w_k^{\mathsf{Verlinde}} =: e^{-\widetilde{w}_k} - 1$$

and use w_k ambiguously for both. Second, we further equate the counting parameters

$$(-1)^N q = \prod_{k=1}^{N}(1+w_k)\cdot \widetilde{q}. \tag{8.12}$$

From the change of variable formulas (8.5) and (8.11), these identifications then imply

$$H_i^{\mathsf{Segre}} = H_i^{\mathsf{Verlinde}} =: 1 - e^{\widetilde{H}_i},$$

hence we use H_i ambiguously for both. Under these identifications, the formulas in (8.4) and (8.10) imply that some of the universal series are in perfect match

$$\mathsf{A} = \widetilde{\mathsf{A}}, \quad \mathsf{B} = \widetilde{\mathsf{B}}, \quad \mathsf{V}_i = \widetilde{\mathsf{V}}_i.$$

However, not all universal series match as we have

$$\frac{\mathsf{U}_i}{\widetilde{\mathsf{U}}_i} = (-1)^{N-1} \cdot \prod_{i' \neq i} (1 - H_{i'}), \quad \frac{\mathsf{W}_{i,j}}{\widetilde{\mathsf{W}}_{i,j}} = \frac{1}{(1-H_i)(1-H_j)}$$

that are not equal to 1 if $N > 1$. We can rewrite these discrepancies as relations between universal series:

$$\begin{aligned} \frac{\mathsf{U}_i}{\widetilde{\mathsf{U}}_i} &= (-1)^{N-1} \cdot \prod_{k=1}^{N} (1 + w_k) \cdot \mathsf{B} \cdot \mathsf{V}_i, \\ \frac{\mathsf{W}_{i,j}}{\widetilde{\mathsf{W}}_{i,j}} &= \mathsf{V}_i \cdot \mathsf{V}_j. \end{aligned}$$

This proves the virtual Segre/Verlinde correspondence for Seiberg-Witten classes of length N.

For punctual Quot schemes, the only universal series involved in are A and B. Therefore we have

$$\mathsf{S}_{S,N,\beta=0}(q, \underline{w} \,|\, \alpha) = \mathsf{V}_{S,N,\beta=0}(\widetilde{q}, \widetilde{\underline{w}} \,|\, \alpha).$$

Note that both A and B have no poles at $\underline{w} = 0$ and are symmetric in $H_1, \ldots, H_N$. By taking the limit $\underline{w}$=0, or equivalently $\widetilde{\underline{w}} = 0$, we obtain the equality for non-equivariant series

$$\mathsf{S}_{S,N,\beta=0}(q \,|\, \alpha) = \mathsf{V}_{S,N,\beta=0}(\widetilde{q} \,|\, \alpha).$$

Here, the counting parameters are simply related by

$$(-1)^N q = \widetilde{q}$$

from the formula (8.12).

8.2 Algebraicity of virtual Segre/Verlinde series

In this section, we prove the algebraicity of the virtual Segre and Verlinde series for Seiberg-Witten classes of length N. Since we have

$$\mathsf{A}\,,\ \widetilde{\mathsf{A}}\,,\ \mathsf{B}\,,\ \widetilde{\mathsf{B}}\,,\ \mathsf{U}_i\,,\ \widetilde{\mathsf{U}}_i\,,\ \mathsf{V}_i\,,\ \widetilde{\mathsf{V}}_i\,,\ \mathsf{W}_{i,j}\,,\ \widetilde{\mathsf{W}}_{i,j} \in\ \mathbb{Q}(w_1,\ldots,w_N)(H_1,\ldots,H_N),$$

we only need to check that the elementary symmetric polynomials in $H_1,\ldots,H_N$ are algebraic functions in the q variable after taking the limit $\underline{w}=0$. Therefore, the lemma below proves the claimed algebraicity.

Lemma 26. *Let $z_1,\ldots,z_N$ be power series in the q variable with coefficients in $\mathbb{Q}(\underline{w})$ defined recursively by*

$$q = (1-z_i)^r \cdot \prod_{k=1}^{N}(z_i + w_k) \quad \textit{with} \quad z_i(q=0) = -w_i.$$

Let $e_1,\ldots,e_N$ be the elementary symmetric polynomial in $z_1,\ldots,z_N$. Then

$$e_i\Big|_{\underline{w}=0},\ldots,e_N\Big|_{\underline{w}=0} \in \mathbb{Q}[\![q]\!]$$

are given by algebraic functions in q.

Proof. For simplicity of the notations involving signs, we modify the problem by changing the equation into

$$q = (z_i-1)^r \cdot \prod_{k=1}^{N}(z_i + w_k) \quad \text{with} \quad z_i(q=0) = -w_i$$

and replacing e_i by a signed elementary symmetric polynomial $(-1)^i e_i$. These changes do not affect the claimed algebraicity. We divide the proof into four cases:

$$r > 0, \quad -N < r \leq 0, \quad r = -N, \quad r < -N.$$

First assume that $r > 0$. Denote

$$\begin{aligned}
\mathsf{P}(z) &:= (z-1)^r = z^r + p_1 z^{r-1} + \cdots + p_r, \\
\mathsf{Q}(z) &:= \prod_{k=1}^{N} (z + w_k) = z^N + q_1 z^{N-1} + \cdots + q_N.
\end{aligned}$$

Since $z_1, \ldots, z_N \in \mathbb{Q}(\underline{w})[\![q]\!]$ are roots of the polynomial equation $\mathsf{P}(z) \cdot \mathsf{Q}(z) - q = 0$, we obtain the factorization

$$\begin{aligned}
\mathsf{P}(z) \cdot \mathsf{Q}(z) - q &= (z^r + f_1 z^{r-1} + \cdots + f_r) \cdot \prod_{i=1}^{N} (z - z_i) \\
&= (z^r + f_1 z^{r-1} + \cdots + f_r)(z^N + e_1 z^{N-1} + \cdots + e_N) \qquad (8.13)
\end{aligned}$$

for some $f_1, \ldots, f_r \in \mathbb{Q}(\underline{w})[\![q]\!]$. The second equality follows from the definition of e_i's.

Denote coefficients of q^m with superscript notations, for example $f_i^{(m)}, e_j^{(m)}$. We prove by induction on m that

$$e_1^{(m)}, \ldots, e_N^{(m)}, f_1^{(m)}, \ldots, f_r^{(m)} \in \mathbb{Q}(\underline{w})$$

satisfy the following condition:

$$\text{rational function whose denominators are powers of } \Delta := \prod_{k=1}^{N} \mathsf{P}(-w_k). \qquad (8.14)$$

For the base case of the induction, we substitute $q = 0$ for the equation (8.13):

$$\mathsf{P}(z) \cdot \mathsf{Q}(z) = (z^r + f_1^{(0)} z^{r-1} + \cdot + f_r^{(0)})(z^N + e_1^{(0)} z^{N-1} + \cdots + e_N^{(0)}).$$

By the initial conditions $z_i(q=0) = -w_i$, it is straightforward to check that

$$z^N + e_1^{(0)} z^{N-1} + \cdots + e_N^{(0)} = \prod_{k=1}^{N} (z + w_k) = \mathsf{Q}(z),$$

hence

$$z^r + f_1^{(0)} z^{r-1} + \cdot + f_r^{(0)} = \mathsf{P}(z).$$

The $m=0$ case of the claim directly follows from these calculations. This also shows that f_i and e_j are naturally a q-deformation of p_i and q_j, respectively.

For the inductive step, we compare the $q^{m+1} z^{N+r-n}$ coefficients of both sides of the equation (8.13) for $m \geq 0$ and $1 \leq n \leq N+r$. Clearly, we have

$$[q^{m+1} z^{N+r-n}] \Big(\mathsf{P}(z) \cdot \mathsf{Q}(z) - q \Big) = -1 \text{ or } 0 \tag{8.15}$$

which satisfies the condition (8.14) trivially. On the other hand, we have

$$\begin{aligned}
[q^{m+1} z^{N+r-n}] & \Big((z^r + f_1 z^{r-1} + \cdots + f_r)(z^N + e_1 z^{N-1} + \cdots + e_N) \Big) \\
&= [q^{m+1}] \sum_{i+j=n} f_i \cdot e_j \\
&= \Big(\sum_{i+j=n} f_i^{(m+1)} q_j + e_j^{(m+1)} p_i \Big) + \text{ lower order terms}
\end{aligned}$$

where we define $p_0 = f_0 = q_0 = e_0 = 1$ and $p_{>r} = f_{>r} = q_{>N} = e_{>N} = 0$. Note that the lower order terms in the above equality is a polynomial in $f_i^{(k)}, e_j^{(k)}$ for some i, j and $k \leq m$. By induction hypothesis, the lower order terms satisfy the condition (8.14). Therefore, we conclude that

$$c_n := \sum_{i+j=n} f_i^{(m+1)} q_j + e_j^{(m+1)} p_i$$

satisfies the condition (8.14) for all $1 \leq n \leq N+r$. Since $f_0^{(m+1)} = e_0^{(m+1)} = 0$, we can regard the above conditions as a linear system in

$$f_1^{(m+1)}, \ldots, f_r^{(m+1)}, e_1^{(m+1)}, \ldots, e_N^{(m+1)} \tag{8.16}$$

whose coefficient matrix is given by a transpose of the Sylvester matrix associated to polynomials $\mathsf{Q}(z)$ and $\mathsf{P}(z)$. By means of Cramer's rule, we conclude that any elements in the list (8.16) are given by a polynomial in $c_1, \ldots, c_{N+r}$ with the denominator given by the determinant of the coefficient matrix. On the other hand, determinant of the transpose of the Sylvester matrix is a resultant of the two polynomials

$$\mathrm{Res}(\mathsf{Q}, \mathsf{P}) = \prod_{k=1}^{N} \mathsf{P}(-w_k),$$

hence equal to Δ. This completes the induction.

Since Δ specializes to non-zero at $\underline{w} = 0$ limit, the q^m coefficients of f_i and e_j are rational functions in $\mathbb{Q}(\underline{w})$ without poles at the specialization $\underline{w} = 0$. Therefore we can take $\underline{w} = 0$ limit for the equation (8.13) and obtain

$$(z-1)^r \cdot z^N - q = (z^r + f_1\Big|_{\underline{w}=0} z^{r-1} + \cdots + f_r\Big|_{\underline{w}=0})(z^N + e_1\Big|_{\underline{w}=0} z^{N-1} + \cdots + e_N\Big|_{\underline{w}=0}).$$

This clearly shows that

$$e_1\Big|_{\underline{w}=0}, \ldots, e_N\Big|_{\underline{w}=0} \in \mathbb{Q}[\![q]\!]$$

are algebraic functions in the q variable because they are polynomials in the roots of the equation $(z-1)^r \cdot z^N - q = 0$. This completes the proof of the first case.

Assume that $-N < r \leq 0$. Then similar method as above shows that

$$\prod_{k=1}^{N} (z + w_k) - q(z-1)^{-r} = (z^N + e_1 z^{N-1} + \cdots + e_N).$$

Therefore $e_1, \ldots, e_N$ have no poles at $\underline{w} = 0$ and the specializations are given by the formula

$$z^N - q(z-1)^{-r} = z^N + e_1\Big|_{\underline{w}=0} z^{N-1} + \cdots + e_N\Big|_{\underline{w}=0}$$

and they are polynomials in q variable.

Assume that $r = -N$. Then we have

$$\frac{z^N - q(z-1)^N}{1-q} = z^N + e_1\Big|_{\underline{w}=0} z^{N-1} + \cdots + e_N\Big|_{\underline{w}=0}$$

which implies that $e_i\Big|_{\underline{w}=0}$ are rational functions in q.

Lastly, assume that $r < -N$. Put $R := -r$ and $d := R - N > 0$. Then we have

$$(z-1)^R - q^{-1}\prod_{k=1}^{N}(z+w_k) = (z^d + f_1 z^{d-1} + \cdots + f_{d-1}z + \widetilde{f_d})(z^N + e_1 z^{N-1} + \cdots + e_N).$$

By comparing coefficients of $z^R, z^{R-1}, \ldots, z^{N+1}$ on the both sides, we show that

$$\widetilde{f_d} = -q^{-1} + f_d$$

where

$$f_1, \ldots, f_d \in \mathbb{Q}(\underline{w})[\![q]\!].$$

We prove by induction that

$$e_1^{(m)}, \ldots, e_N^{(m)}, f_1^{(m-1)}, \ldots, f_d^{(m-1)}$$

are rational functions in $\underline{w}$ without poles at $\underline{w} = 0$ for all $m \geq 0$. Base case is trivial. The induction step is similar to the case with $r > 0$ except that the coefficient matrix is given by the lower triangle matrix with a diagonal $(1, \ldots, 1, -1, \ldots, -1)$ of d number of 1's and N number of

(-1)'s. Hence, we may take the $\underline{w} = 0$ limit to obtain

$$(z-1)^R - q^{-1}z^N = (z^d + f_1\Big|_{\underline{w}=0} z^{d-1} + \cdots + f_d\Big|_{\underline{w}=0} - q^{-1})(z^N + e_1\Big|_{\underline{w}=0} z^{N-1} + \cdots + e_N\Big|_{\underline{w}=0}).$$

This proves the claimed algebraicity and completes the proof. □

This chapter is, in part, being prepared for submission for publication. The dissertation author was the collaborator and the coauthor for the material below.

- Noah Arbesfeld, Drew Johnson, Woonam Lim, Dragos Oprea, Rahul Pandharipande "The virtual K-theory of Quot schemes of surfaces"

Appendix A

Multiplicative characteristic of line bundle twist

In this appendix, we prove lemmas regarding universal structure of the multiplicative characteristic of line bundle twist. Let $f(x) \in F[[x]]$ be an invertible power series with coefficients in a field F. Recall that this defines a multiplicative characteristic

$$f : K^0(M) \to \Big(H^*(M,F)\Big)^\times$$

which is functorial in pullbacks.

Lemma 27. *Let $f \in F[[x]]$ be an invertible formal power series. Then there exist a power series*

$$\widetilde{f}(h \,|\, c_1, c_2, \dots) \in F[[h, c_1, c_2, \dots]]$$

with a following property: Let V be a K-theory class of rank r with the Chern classes $\{c_k\}_{k \geq 1}$ and L be a line bundle with the first Chern class h. Then we have

$$f(V \otimes L) = f(h)^r \cdot \widetilde{f}(h \,|\, c_1, c_2, \dots).$$

Furthermore, $\widetilde{f}$ only depends on the reduced power series $f^{\mathrm{red}}(x) := \frac{f(x)}{f(0)}$.

Proof. Let V and L be as in the lemma. As a K-theory class, V can be written as a difference of two vector bundles E and F. Denote the Chern roots of E and F by $\{u_i\}_{i\in I}$ and $\{v_j\}_{j\in J}$, respectively. Then virtual Chern roots of V consist of

$$\{u_i\}_{i\in I}, \quad \{\ominus v_j\}_{j\in J}.$$

Let $\{c_k\}_{k\geq 1}$ denotes the Chern class of V. Chern roots and Chern class of V are related by the equation

$$\sum_{k\geq 0} c_k t^k = \frac{\prod_i (1+u_i t)}{\prod_j (1+v_j t)} \tag{A.1}$$

where we set $c_0 = 1$. To be more precise, we can express c_k using $\{u_i\}_{i\in I}$ and $\{v_j\}_{j\in J}$ by computing the coefficient of t^k on the right hand side:

$$\begin{aligned}
&t^1 \text{ coefficient: } c_1 = \sum_i u_i + \sum_j (-v_j),\\
&t^2 \text{ coefficient: } c_2 = \frac{1}{2}\sum_{i_1\neq i_2} u_{i_1}u_{i_2} + \frac{1}{2}\sum_{j_1\neq j_2}(-v_{j_1})(-v_{j_2}) + \sum_{i,j} u_i(-v_j) + \sum_j (-v_j)^2,\\
&\qquad\qquad\qquad\qquad\vdots
\end{aligned}$$

We can rewrite these relations by taking logarithm of (A.1):

$$\begin{aligned}
\log(1+c_1t+c_2t^2+\cdots) &= \sum_i \log(1+u_it) - \sum_j \log(1+v_jt)\\
&= \sum_{n\geq 1}\left(\frac{(-1)^n t^n}{n}\Big(\sum_i u_i^n - \sum_j v_j^n\Big)\right).
\end{aligned}$$

By taking t^n coefficient on the both sides, we obtain

$$\sum_i u_i^n - \sum_j v_j^n = (-1)^n \cdot n \cdot [t^n] \log(1+c_1t+c_2t^2+\cdots), \quad n \geq 1 \tag{A.2}$$

where the bracket extracts the coefficient of the specified monomials. For example, we have

$$\begin{aligned} n=1: &\ \sum_i u_i - \sum_j v_j = c_1, \\ n=2: &\ \sum_i u_i^2 - \sum_j v_j^2 = -2c_2 + c_1^2, \\ &\ \vdots \end{aligned}$$

If we denote $c_1(L) = h$, then virtual Chern roots of $V \otimes L$ consist of

$$\{u_i + h\}_{i \in I}, \quad \{\ominus(v_j + h)\}_{j \in J}.$$

Straightforward computation shows

$$f(V \otimes L) = \big(f(0)\big)^r \cdot f^{\text{red}}(V \otimes L).$$

Write the logarithm of f^{red} as

$$\log f^{\text{red}}(x) = \sum_{n \geq 1} a_n x^n.$$

Then we have

$$\begin{aligned} \log f^{\text{red}}(V \otimes L) &= \sum_i \log f(u_i + h) - \sum_j \log f(v_j + h) \\ &= \sum_{n \geq 1} a_n \Big(\sum_i (u_i + h)^n - \sum_j (v_j + h)^n \Big) \\ &= \sum_{n \geq 1} a_n \sum_{k=0}^{n} \binom{n}{k} h^{n-k} \Big(\sum_i u_i^k - \sum_j v_j^k \Big) \\ &= r \sum_{n \geq 1} a_n h^n + \sum_{n \geq 1} a_n \sum_{k=1}^{n} \binom{n}{k} h^{n-k} \Big(\sum_i u_i^k - \sum_j v_j^k \Big) \\ &= r \log f^{\text{red}}(h) + \sum_{n \geq 1} a_n \sum_{k=1}^{n} \binom{n}{k} h^{n-k} \Big(\sum_i u_i^k - \sum_j v_j^k \Big), \end{aligned}$$

and so

$$f(V\otimes L) = \Big(f(0)f^{\mathrm{red}}(h)\Big)^r \cdot \exp\left(\sum_{n\geq 1} a_n \sum_{k=1}^{n}\binom{n}{k} h^{n-k}\Big(\sum_i u_i^k - \sum_j v_j^k\Big)\right).$$

Clearly, we have $f(0)f^{\mathrm{red}}(h) = f(h)$. Note that equation (A.2) can be used to rewrite the exponential part universally in terms of $h, c_1, c_2, \dots$ which only depends on $\{a_n\}_{n\geq 1}$. This proves the lemma. $\square$

If one specializes to the case $f(x) = 1+x$, then one obtains the Chern class formula of line bundle twist for a K-theory class.

Lemma 28. *Let V be a K-theory class of rank r and L be a line bundle with the first Chern class h. Then we have*

$$c_n(V\otimes L) = \sum_{k=0}^{n}\binom{r-n+k}{k} h^k \cdot c_{n-k}(V).$$

Proof. Let $\{u_i\}_{i\in I}$ and $\{\ominus v_j\}_{j\in J}$ be a set of virtual Chern roots of V. Straightforward computation shows

$$\frac{c(V\otimes L)}{(1+h)^r} = \frac{1}{(1+h)^r}\cdot\frac{\prod_i(1+h+u_i)}{\prod_j(1+h+v_j)} = \frac{\prod_i(1+\frac{u_i}{1+h})}{\prod_j(1+\frac{v_j}{1+h})} = c_{\frac{1}{1+h}}(V).$$

Therefore we have

$$c(V\otimes L) = \sum_{n\geq 0}(1+h)^{r-n}c_n(V).$$

By taking cohomological degree n components, we obtain

$$c_n(V\otimes L) = \sum_{k=0}^{n}\binom{r-n+k}{k} h^k \cdot c_{n-k}(V).$$

$\square$

Now we prove a different version of Lemma 27 involving the equivariant weight.

Lemma 29. *Let $f \in F[\![x]\!]$ be an invertible formal power series. Then there exist unique set of associated power series*

$$f^{(k)}(r\,|\,w\,|\,c_1,c_2,\dots) \in F[r][\![w,c_1,c_2,\dots]\!], \quad k \geq 0$$

with a following property: Let V be a K-theory class of rank r with the Chern classes $\{c_k\}_{k\geq 1}$ and L be a line bundle with the first Chern class h. Let w be a $\mathbb{C}^*$-weight. Then we have

$$f(V \otimes L[w]) = f(w)^r \cdot \sum_{k\geq 0} h^k \cdot f^{(k)}(r\,|\,w\,|\,c_1,c_2,\dots).$$

Furthermore, $f^{(k)}$ only depends on k and the reduced power series $f^{\mathrm{red}}(x) = \frac{f(x)}{f(0)}$.

Proof. Lemma 27 reads

$$f(V \otimes L[w]) = f(w)^r \cdot \widetilde{f}(w\,|\,c_1(V\otimes L), c_2(V\otimes L),\cdots),$$

if we choose a K-theory class $V \otimes L$ and an equivariant line bundle $\mathscr{O}[w]$. Denote the power series $\widetilde{f}$ as

$$\widetilde{f}(w\,|\,c_1,c_2,\dots) = \sum_{\substack{n\geq 0\\ n_1,n_2,\dots\geq 0}} a_{n,\{n_i\}} \cdot w^n c_1^{n_1} c_2^{n_2} \cdots .$$

By using Lemma 28, we have

$$\widetilde{f}(w\,|\,c_1(V\otimes L), c_2(V\otimes L),\cdots) = \sum a_{n,\{n_i\}} \cdot w^n \cdot \prod_{i=1}^{\infty}\prod_{j=1}^{n_i} \binom{r-i+k_{ij}}{k_{ij}} \cdot h^{k_{ij}} \cdot c_{i-k_{ij}}(V)$$

where summation takes over all possible choices of

$$n \geq 0, \quad n_i \geq 0 \ (i=1,2,\dots), \quad 0 \leq k_{ij} \leq i \ (i=1,2,\dots,j=1,2,\dots,n_i)$$

with only finitely many non-zero n_i's. Note that

$$\prod_{i=1}^{\infty}\prod_{j=1}^{n_i}\binom{r-i+k_{ij}}{k_{ij}}$$

is a polynomial in r of degree $\sum k_{ij}$ that is bounded by the exponent of h. This completes the proof. $\square$

Lastly, we have the following basic lemma about the equivariant Euler characteristic.

Lemma 30. *Let V be a K-theory class of rank r with the Chern classes $\{c_k\}_{k\geq 1}$ and L be a line bundle with the first Chern class h. Let w be a non-zero $\mathbb{C}^*$-weight. Then we have*

$$e(V\otimes L[w]) = w^r \cdot \sum_{k\geq 0} h^k \left(\sum_{m\geq 0}\binom{r-m}{k} w^{-k-m}\cdot c_m\right).$$

In particular, $e(V\otimes L[w])$ is invertible in equivariant cohomology.

Proof. Denote the virtual Chern roots of V as

$$\{u_i\}_{i\in I}, \quad \{\ominus v_j\}_{j\in J}.$$

Then we have

$$e(V[w]) = \frac{\prod_i(w+u_i)}{\prod_j(w+v_j)} = w^r\cdot\frac{\prod_i(1+w^{-1}u_i)}{\prod_j(1+w^{-1}v_j)} = w^r\cdot c_{w^{-1}}(V).$$

With the line bundle twist, we use Lemma 28 to obtain

$$\begin{aligned}
e(V \otimes L[w]) &= w^r \cdot c_{w^{-1}}(V \otimes L). \\
&= w^r \cdot \sum_{n \geq 0} w^{-n} c_n(V \otimes L) \\
&= w^r \cdot \sum_{n \geq k \geq 0} w^{-n} \binom{r-n+k}{k} h^k \cdot c_{n-k} \\
&= w^r \cdot \sum_{m,k \geq 0} \binom{r-m}{k} w^{-k-m} h^k c_m.
\end{aligned}$$

This completes the proof. $\square$

Remark 31. For later use, we denote the power series appearing as a coefficient of h^k in Lemma 30 by

$$e^{(k)}(r \,|\, w \,|\, c_1, c_2, \dots) := \sum_{m \geq 0} \binom{r-m}{k} w^{-k-m} \cdot c_m.$$

Since coefficient of c_m is polynomial in r and rational in w, we have

$$e^{(k)}(r \,|\, w \,|\, c_1, c_2, \dots) \in \mathbb{Q}[r](w)[[c_1, c_2, \dots]].$$

Bibliography

[AJLOP] N. Arbesfeld, D. Johnson, W. Lim, D. Oprea, R. Pandharipande, *The virtual K-theory of Quot schemes of surfaces*, `arXiv:2008.10661`.

[BF] K. Behrend, B. Fantechi, *The intrinsic normal cone,* Invent. Math. **128** (1997), 601 – 617.

[BL] J. Bryan, C. Leung, *The enumerative geometry of K3 surfaces and modular forms*, J. Amer. Math. Soc. **13** (2000), 371 – 410.

[CK] H. Chang, Y. Kiem, *Poincaré invariants are Seiberg-Witten invariants*, Geom. Topol. **17** (2013), 1149 – 1163.

[CFK] I Ciocan-Fontanine, M Kapranov, *Virtual fundamental classes via dg-manifolds*, Geom. Topol. **13** (2009), 1779 – 1804.

[DKO] M. Dürr, A. Kabanov, C. Okonek, *Poincaré invariants*, Topology **46** (2007), 225 – 294.

[EGL] G. Ellingsrud, L. Göttsche, M. Lehn, *On the cobordism class of the Hilbert scheme of a surface*, J. Algebraic Geom. **10** (2001), 81 – 100.

[FG] B. Fantechi, L. Göttsche, *Riemann-Roch theorems and elliptic genus for virtually smooth schemes*, Geom. Topol. **14** (2010), 83 – 115.

[F] J. Fogarty, *Algebraic Families on an Algebraic Surface*, Amer. J. Math. **10** (1968), 511 – 521.

[Ge] I. Gessel, *A combinatorial proof of the multivariable Lagrange inversion formula*, J. Comb. Theory, Ser. A, **45** (1987), 178 – 195.

[GSY1] A. Gholampour, A. Sheshmani, S. Yau, *Localized Donaldson-Thomas theory of surfaces*, Amer. J. Math. **142** (2020), 405 – 442.

[GSY2] A. Gholampour, A. Sheshmani, S. Yau, *Nested Hilbert schemes on surfaces: Virtual fundamental class*, Adv. Math. **365** (2020), 50 pp.

[GT1] A. Gholampour, R. P. Thomas, *Degeneracy loci, virtual cycles and nested Hilbert schemes I*, Tunis. J. Math. **2** (2020), 633 – 665.

[GT2] A. Gholampour, R. P. Thomas, *Degeneracy loci, virtual cycles and nested Hilbert schemes II*, Compos. Math. **156** (2020), 1623 – 1663.

[G] L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. **286** (1990), 193 – 207.

[GK1] L. Göttsche, M. Kool, *Refined $SU(3)$ Vafa-Witten invariants and modularity*, Pure and Appl. Math. Quart. **14** (2018), 467 – 513.

[GK2] L. Göttsche, M. Kool, *Sheaves on surfaces and virtual invariants*, `arXiv:2007.12730`.

[GK3] L. Göttsche, M. Kool, *Virtual refinements of the Vafa-Witten formula*, Commun. Math. Phys. **376** (2020), 1 – 49.

[GK4] L. Göttsche, M. Kool, *Virtual Segre and Verlinde numbers of projective surfaces*, `arXiv:2007.11631`.

[GKW] L Göttsche, M Kool, R. A. Williams, *Verlinde formulae on complex surfaces I: K-theoretic invariants*, arXiv:1903.03869.

[GNY] L. Göttsche, H. Nakajima, K. Yoshioka, *Donaldson = Seiberg-Witten from Mochizuki's formula and instanton counting*, Publ. Res. Inst. Math. Sci. **47** (2011), 307 – 359.

[GP] T. Graber, R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), 487 – 518.

[GS] L. Göttsche, V. Shende, *The χ_{-y}-genera of relative Hilbert schemes for linear systems on Abelian and K3 surfaces*, Algebr. Geom. **2** (2015), 405 – 421.

[GSo] L. Göttsche, W. Soergel, *Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces*, Math. Ann. **296** (1993), 235 – 245.

[HL] D. Huybrechts, M. Lehn, *The geometry of moduli spaces of sheaves*, Cambridge University Press (2010).

[J] D. Johnson, *Universal series for Hilbert schemes and Strange Duality*, Int. Math. Res. Not. **10** (2020), 3130 – 3152.

[JOP] D. Johnson, D. Oprea, R. Pandharipande, *Rationality of descendent series for Hilbert and Quot schemes of surfaces*, `arXiv:2002.08787`.

[KY] T. Kawai, K. Yoshioka, *String partition functions and infinite products*, Adv. Theor. Math. Phys. **4** (2000), 397 – 485.

[KL] Y. H. Kiem, J. Li, *Localizing Virtual Cycles by Cosections*, J. Amer. Math. Soc. **26** (2013), 1025 – 1050.

[K] M. Kool, *Stable pair invariants of surfaces and Seiberg-Witten invariants*, Q. J. Math. **67** (2016), 365 – 386.

[KT1] M. Kool, R. P. Thomas, *Reduced classes and curve counting on surfaces I: theory*, Algebr. Geom. **1** (2014), 334 – 383.

[KT2] M. Kool, R. P. Thomas, *Reduced classes and curve counting on surfaces II: calculations*, Algebr. Geom. **1** (2014), 384 – 399.

[La1] T. Laarakker, *Monopole contributions to refined Vafa-Witten invariants*, Geom. Topol. (to appear).

[La2] T. Laarakker, *Vertical Vafa-Witten invariants*, `arXiv:1906.01264`.

[Le] M. Lehn, *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces*, Invent. Math. **136** (1999), 157 – 207.

[Lee] Y.P. Lee, *Quantum K-theory. I. Foundations*, Duke Math. J. **121** (2004), 389 – 424.

[LT] J. Li, G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math. Soc. **11** (1998), 119 – 174.

[L] W. Lim, *Virtual χ_{-y}-genera of Quot schemes on surfaces*, `arXiv:2003.04429`.

[Lin] Y. Lin, *Moduli spaces of stable pairs*, Pacific J. Math. **294** (2018), 123 – 158.

[MO1] A. Marian, D. Oprea, *Virtual intersections on the Quot scheme and Vafa-Intriligator formulas*, Duke Math. Journal **136** (2007), 81 – 131.

[MO2] A. Marian, D. Oprea, *The level-rank duality for non-abelian theta functions*, Invent. Math. **168** (2007), 225 – 247.

[MO3] A. Marian, D. Oprea, *Counts of maps to Grassmannians and intersections on the moduli space of bundles*, J. Differential Geom. **76** (2007), 155 – 175.

[MOP1] A. Marian, D. Oprea, R. Pandharipande, *Segre classes and Hilbert schemes of points*, Ann. Sci. Éc. Norm. Supér. **50** (2017), 239 – 267.

[MOP2] A. Marian, D. Oprea, R. Pandharipande, *The combinatorics of Lehn's conjecture*, J. Math. Soc. Japan **71** (2019), 299 – 308.

[MOP3] A. Marian, D. Oprea, R. Pandharipande, *Higher rank Segre integrals over the Hilbert scheme of points*, J. Eur. Math. Soc. (to appear).

[MPT] D. Maulik, R. Pandharipande, R. P. Thomas, *Curves on K3 surfaces and modular forms*, J. Topol. **3** (2010), 937 – 996.

[Mo] T. Mochizuki, *Donaldson type invariants for algebraic surfaces*, Lecture Notes in Math. 1972, Springer-Verlag, Berlin (2009).

[OP] D. Oprea, R. Pandharipande, *Quot schemes of curves and surfaces: virtual classes, integrals, Euler characteristics*, Geom. Topol. (to appear).

[PT1] R. Pandharipande, R. P. Thomas, *Stable pairs and BPS invariants*, J. Amer. Math. Soc. **23** (2010), 267 – 297.

[PT2] R. Pandharipande, R. P. Thomas, *The Katz-Klemm-Vafa conjecture for K3 surfaces*, Forum Math. Pi **4** (2016), 111 pp.

[S] B. Siebert, *Virtual fundamental classes, global normal cones and Fulton's canonical classes*, in: Frobenius manifolds, ed. K. Hertling and M. Marcolli, Aspects Math. **36** Vieweg (2004), 341 – 358.

[Sh] J. Shen, *Cobordism invariants of the moduli space of stable pairs*, J. Lond. Math. Soc. **94** (2016), 427 – 446.

[TT1] Y. Tanaka, R. P. Thomas, *Vafa-Witten invariants for projective surfaces I: stable case*, J. Algebraic Geom. **29** (2020), 603 – 668.

[TT2] Y. Tanaka, R. P. Thomas, *Vafa-Witten invariants for projective surfaces II: semistable case*, Pure Appl. Math. Quart. **13** (2017), 517 – 562.

[Ta1] H. Taubes, *Gr $\Rightarrow$ SW: from pseudo-holomorphic curves to Seiberg-Witten solutions*, J. Diff. Geom. **51** (1999), 203 – 334.

[Ta2] H. Taubes, *SW $\Rightarrow$ Gr: from the Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. **9** (1996), 845 – 918.

[T] R. P. Thomas, *Equivariant K-theory and refined Vafa-Witten invariants*, Comm. Math. Phys. **378** (2020), 1451 – 1500.

[VW] C. Vafa, E. Witten, *A strong coupling test of S-duality*, Nuclear Phys. B **431** (1994), 3 – 77.

[V] C. Voisin, *Segre classes of tautological bundles on Hilbert schemes of points of surfaces*, Alg. Geom. **6** (2019), 186 – 195.

[YZ] S. Yau and E. Zaslow, *BPS states, string duality, and nodal curves on K3*, Nucl. Phys. B **457** (1995), 484 – 512.