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Consecutive matches in permutations and cycles

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Consecutive Matches in Permutations and Cycles

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy

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Mathematics

by

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The dissertation of Miles Eli Jones is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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ABSTRACT OF THE DISSERTATION

Consecutive Matches in Permutations and Cycles

by

Miles Eli Jones

Doctor of Philosophy in Mathematics

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This dissertation studies 3 different methods to compute generating functions of the following forms

$$NM_\tau(t,x,y) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\tau)} x^{\mathrm{LRMin}(\sigma)} y^{1+\mathrm{des}(\sigma)}$$

$$NCM_\tau(t,x,y) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NCM}_n(\tau)} x^{\mathrm{cyc}(\sigma)} y^{\mathrm{cdes}(\sigma)}$$

where τ is a permutation pattern, $\mathcal{NM}_n(\tau)$ is the set of permutations in the symmetric group S_n with no τ-matches, and for any permutation $\sigma \in S_n$, $\mathrm{LRMin}(\sigma)$ is the number of left-to-right minima of σ and $\mathrm{des}(\sigma)$ is the number of descents of σ, $\mathcal{NCM}_n(\tau)$ is the set of permutations in the symmetric group S_n with no

cycle-τ-matches, and for $\sigma \in S_n$, $\text{cyc}(\sigma)$ is the number of cycles of σ and $\text{cdes}(\sigma)$ is the number of cycle descents of σ.

The first method uses the theory of exponential structures to study the generating function $NCM_\tau(t,x,y)$ by only looking at single cycles and developing recursions for the coefficients. The recursions lead to differential equations that can be solved to compute $NCM_\tau(t,x,y)$.

The second method does not compute $NM_\tau(t,x,y)$ directly, but assumes that for patterns τ that start with 1,

$$NM_\tau(t,x,y) = \frac{1}{(U_\tau(t,y))^x}$$

where $U_\tau(t,y) = \sum_{n\geq 0} U_{\tau,n}(y)\frac{t^n}{n!}$ so that $U_\tau(t,y) = \frac{1}{NM_\tau(t,1,y)}$. We then use the so-called homomorphism method and the combinatorial interpretation of $NM_\tau(t,1,y)$ to develop recursions for the coefficient of $U_\tau(t,y)$.

The third method uses a bijection from brick tabloids to cycles to compute the generating function $NCM_\Upsilon(t,1,1)$ for collections of patterns Υ.

Chapter 1

Introduction

The notion of patterns in permutations and words has proved to be a useful language in a variety of seemingly unrelated problems including the theory of Kazhdan-Lusztig polynomials, singularities of Schubert varieties, Chebyshev polynomials, rook polynomials for Ferrers boards, and various sorting algorithms including sorting stacks and permutations. The study of occurrences of patterns in words and permutations is a new, but rapidly growing, branch of combinatorics which has its roots in the works by Rotem, Rogers, and Knuth in the 1970s and early 1980s. The first systematic study of permutation patterns was not undertaken until the paper by Simion and Schmidt [38] which appeared in 1985. The field has experienced explosive growth since 1992.

1.1 Classical Pattern Avoidance

First we recall the basic definitions for pattern occurrence and consecutive pattern occurrence in permutations. Given a sequence $\sigma = \sigma_1 \ldots \sigma_n$ of distinct integers, let $\text{red}(\sigma)$ be the permutation found by replacing the i^{th} largest integer that appears in σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $\text{red}(\sigma) = 1\ 4\ 3\ 2$. Given a pattern $\tau = \tau_1 \ldots \tau_j$ in the symmetric group S_j and a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we say that τ *occurs* in σ if there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that $\text{red}(\sigma_{i_1} \cdots \sigma_{i_j}) = \tau$. For example, the permutation $\sigma = 6472351$ has an occurrence of the pattern 132 given by the subsequence 475. We say that σ

avoids τ if there are no occurrences of τ in σ. We say that there is a *consecutive occurrence* of τ in σ or equivalently we say that σ has a τ-*match* at position i provided $\text{red}(\sigma_i \ldots \sigma_{i+j-1}) = \tau$. Let τ-mch(σ) be the number of τ-matches in the permutation σ. For example let $\tau = 3412$ then $\sigma = 6472351$ has a τ-match starting at position 2.

These definitions can naturally be extended to sets of permutations. That is, given a set of permutations Υ in the symmetric group S_j, define a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ to have a Υ-*match starting at position* i provided $\text{red}(\sigma_i \ldots \sigma_{i+j-1}) \in \Upsilon$. Let Υ-mch(σ) be the number of Υ-matches in the permutation σ. Similarly, we say that Υ *occurs* in σ if there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that $\text{red}(\sigma_{i_1} \cdots \sigma_{i_j}) \in \Upsilon$. We say that σ *avoids* Υ if there are no occurrences of Υ in σ.

Much work has been done on enumerating permutations that avoid certain patterns. For a pattern τ, let $S_n(\tau)$ be the number of permutations $\sigma \in S_n$ such that σ avoids τ. For example for any pattern $\tau \in S_3$, we have a well known beautiful result [5].

Theorem 1. *Let* $\tau \in S_3$. *Then*

$$S_n(\tau) = C_n = \frac{\binom{2n}{n}}{n+1}.$$

The 24 patterns of length 4 can be divided into three classes such that all the elements τ of a class share the same value for $S_n(\tau)$ for all n. These classes are the following [5].

Ira Gessel proved the following result for the pattern 1234 [17].

Theorem 2.

$$S_n(1234) = 2\sum_{k=0}^{n} \binom{2k}{k}\binom{n}{k}^2 \frac{3k^2+2k+1-n-2nk}{(k+1)^2(k+2)(n-k+1)}.$$

The closed form result of $S_n(1342)$ can be found in [5].

Theorem 3.

$$S_n(1342) = (-1)^{n-1}\frac{7n^2-3n-2}{2} + 3\sum_{i=2}^{n}(-1)^{n-i}\cdot 2^{i+1}\frac{(2i-4)!}{i!(i-2)!}\binom{n-i+2}{2}.$$

There is no known closed form result for $S_n(1324)$.

Table 1.1: The Wilf-equivalency classes of patterns of length 4

1234	1324	1342
1243	4231	1423
1432		2314
2134		2413
2143		2431
2341		3124
3214		3142
3412		3241
3421		4132
4123		4213
4312		
4321		

1.2 Consecutive Pattern Occurrences

There have been a number of recent publications on consecutive occurrences of patterns or in other words pattern matching. There have been some recent results on the distribution of τ-matches in permutations. See, for example, [12, 20, 21]. Let τ-nlap(σ) be the maximum number of non-overlapping τ-matches in σ where two τ-matches are said to overlap if they contain any of the same integers. Then Kitaev [20, 21] proved the following.

Theorem 4.

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\sigma)} = \frac{A_\tau(t)}{(1-x)+x(1-t)A_\tau(t)} \tag{1.1}$$

where $A_\tau(t) = \sum_{n\geq 0} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}|$.

In other words, if the exponential generating function for the number of permutations in S_n without any τ-matches is known, then so is the exponential generating function for the entire distribution of the statistic τ-nlap(σ).

However, it is not always easy to find the generating function A_τ. For example, Goulden and Jackson [16] proved that when $\tau = 1\ 2 \cdots k$, then

$$A_\tau = \frac{1}{\sum_{i\geq 0} \frac{x^{ki}}{(ki)!} - \frac{x^{ki+1}}{(ki+1)!}} \tag{1.2}$$

Elizalde and Noy [12] proved a number of results about A_τ and the more general generating function

$$P_\tau(u,t) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} u^{\tau\text{-mch}(\sigma)}. \tag{1.3}$$

For example, they showed that

$$\begin{aligned}
A_{123}(t) &= \frac{\sqrt{3}}{2}\frac{e^{t/2}}{\cos(\sqrt{3}t/2+\pi/6)}, \\
A_{1234}(t) &= \frac{2}{\cos(t)-\sin(t)+e^t}, \\
P_{132}(t,u) &= \frac{1}{1-\int_0^t e^{\frac{(u-1)z^2}{2}}dz}, \text{ and} \\
P_{1342}(t,u) &= \frac{1}{1-\int_0^t e^{\frac{(u-1)z^3}{6}}dz}.
\end{aligned}$$

Elizalde and Noy [12] were also able to prove some implicit formulas for $P_\tau(u,t)$ as specialization of solutions to differential equations. For example, they proved the following theorem.

Theorem 5. *Let m and a be positive integers with $a \leq m$ and let τ be any permutation of the form*

$$\tau = 1\ 2 \cdots a-1\ a\ \sigma\ a+1$$

where σ is any permutation of the elements $\{a+2, a+3, \ldots, m+2\}$. Then $P_\tau(u,t) = 1/\omega(u,z)$ where ω is the solution of

$$w^{(a+1)} + (1-u)\frac{t^{m-a-1}}{(m-a-1)!}\omega' = 0 \tag{1.4}$$

where $\omega(0)=1$, $\omega'(0)=-1$, and $\omega^{(k)}(0)=0$ for $2\leq k\leq a$. In particular, $P_\tau(u,t)$ does not depend on σ.

Kitaev [21] showed that an inclusion exclusion argument could give an explicit generating function for $P_\tau(0,t) = A_\tau(t)$ for the τ's in Theorem 5. That is, Kitaev proved the following theorem [21]

Theorem 6. *Let* $\tau = 12\cdots a\sigma(a+1)$, *where* σ *is a permutation of* $\{a+2, a+3, \ldots, k+1\}$, *then*

$$A_\tau(t) = \frac{1}{1 - t + \sum_{i\geq 1} \frac{(-1)^{i+1}x^{ki+1}}{(ki+1)!} \prod_{j=2}^{i} \binom{jk-a}{k-a}}. \tag{1.5}$$

Mendes and Remmel [30] developed a general method for computing $A_\tau(t)$. They proved $A_\tau(t)$ is of the form

$$A_\tau(t) = \frac{1}{1 - t + \sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w\in J_\tau, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\tau|}. \tag{1.6}$$

where $||w|| = |\tau| + w_1 + \cdots + w_n$, J_τ is a certain collection of words associated with τ, and $\mathcal{P}_w^\tau$ is a certain collection of permutations $\sigma \in S_{||w||}$ if $w = w_1 \ldots w_n$. The exact definitions of J_τ and $\mathcal{P}_w^\tau$ are quite complicated and will not be discussed in this thesis.

Liese and Remmel [26] were able to explicitly compute

$$\sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w\in J_\tau, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\tau|$$

in several cases. For example, if $\tau = 1324$ then they showed that

$$\sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w\in J_\tau, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\tau| = \Gamma\left(\frac{t(1 + 2t^2 + 2t^3 - \sqrt{1+4t^2})}{2(1 + t + 2t^2 + t^3)}\right) \tag{1.7}$$

where Γ is the operator on formal power series $f(t) = \sum_{n\geq 0} f_n t^n$ such that $\Gamma(f(t)) = \sum_{n\geq 0} f_n \frac{t^n}{n!}$. This can be used to compute coefficients of the generating function $A_\tau(t)$. In Chapter 3 of this thesis, there is another method to compute these coefficients.

Recently, Duane and Remmel found a general formula for the generating function of τ-matches for a special class of permutations τ called minimal overlapping permutations.

We say that a permutation $\tau \in S_j$ where $j \geq 3$ has the *minimal overlapping property* if the smallest i such that there is a permutation $\sigma \in S_i$ with τ-mch$(\sigma) = 2$ is $2j-1$. Again this means that in any permutation $\sigma = \sigma_1 \ldots \sigma_n$, any two τ-matches in σ can share at most one letter which must be at the end of the first

τ-match and the start of the second τ-match. For example $\tau = 123$ does not have the minimal overlapping property since the 1234-mch$(=)2$ and the τ-match starting at position 1 and the τ-match starting at position 2 share two letters, namely, 2 and 3. However, it is easy to see that the permutation $\tau = 132$ does have the minimal overlapping property. That is, the fact that there is an ascent starting at position 1 and descent starting at position 2 means that there cannot be two τ-matches in a permutation $\sigma \in S_n$ which share 2 or more letters.

If $\tau \in S_j$ has the minimal overlapping property, the shortest permutations σ such that σ-mch$(=)n$ have length $n(j-1)+1$. Thus we let $\mathcal{MP}_{\tau,n(j-1)+1}$ equal the set of permutations $\sigma \in S_{n(j-1)+1}$ such that σ-mch$(=)n$. Duane and Remmel called the permutations in $\mathcal{MP}_{n,n(j-1)+1}$ as maximum packings for τ. Then let $mp_{\tau,n(j-1)+1} = |\mathcal{MP}_{\tau,n(j-1)+1}|$ and

$$mp_{\tau,n(j-1)+1}(p,q) = \sum_{\sigma \in \mathcal{MP}_{\tau,n(j-1)+1}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)}.$$

In general, it is a difficult problem to compute $mp_{\tau,n(j-1)+1}$ or $mp_{\tau,n(j-1)+1}(p,q)$, but we can compute these in the case that τ starts either ends or starts with 1 or ends or starts with j. For example, Duane and Remmel [10] proved following theorem.

Theorem 7. *Suppose that $\tau = \tau_1 \dots \tau_j$ where $\tau_1 = 1$ and $\tau_j = s$, then*

$$mp_{\tau,(n+1)(j-1)+1}(p,q) = p^{\mathrm{coinv}(\tau)} q^{inv(\tau)} p^{(s-1)n(j-1)} \begin{bmatrix} (n+1)(j-1)+1-s \\ j-s \end{bmatrix}_{p,q} mp_{\tau,n(j-1)+1}(p,q)$$

so that

$$mp_{\tau,(n+1)(j-1)+1}(p,q) = \left(p^{\mathrm{coinv}(\tau)} q^{inv(\tau)}\right)^{n+1} p^{(s-1)(j-1)\binom{n+1}{2}} \prod_{i=1}^{n+1} \begin{bmatrix} i(j-1)+1-s \\ j-s \end{bmatrix}_{p,q}. \tag{1.8}$$

Note that if $\tau = \tau_1 \dots \tau_j \in S_j$ has the minimal overlapping property, then the reverse of τ, $\tau^r = \tau_j \dots \tau_1$, and the complement of τ, $\tau^c = (j+1-\tau_1)\dots(j+1-\tau_j)$, also have the minimal overlapping property. Thus one can use Theorem 7 to compute $mp_{\tau,(n+1)(j-1)+1}(p,q)$ in the case where τ either ends or starts with j or ends with 1.

Duane and Remmel proved the following theorem.

Theorem 8. *If $\tau \in S_j$ has the minimal overlapping property, then*

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} = \frac{1}{1-(t+\sum_{n\geq 1} \frac{t^{n(j-1)+1}}{[n(j-1)+1]_{p,q}!}(x-1)^n mp_{\tau,n(j-1)+1}(p,q))}. \tag{1.9}$$

It follows from Theorems 7 and 8 that if $j \geq 3$ and $\tau = \tau_1 \ldots \tau_j \in S_j$ has the minimal overlapping property, $\tau_1 = 1$ and $\tau_j = s$, then

$$P_\tau(t,x,p,q) = \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} = \left(1-(t+\sum_{n\geq 1} \frac{(x-1)^n t^{n(j-1)+1}}{[n(j-1)+1]_{p,q}!} \cdot \left(p^{\text{coinv}(\tau)} q^{inv(\tau)}\right)^{(n+1)} p^{(s-1)(j-1)\binom{n+1}{2}} \prod_{i=1}^{n+1} \begin{bmatrix} i(j-1)+1-s \\ j-s \end{bmatrix}_{p,q})\right)^{-1}. \tag{1.10}$$

This results allowed Duane and Remmel [10] to refine several of the results of Elizalde and Noy and Kitaev mentioned above. For example, both 132 and 1342 have the minimal overlapping property.

By Theorem 7, we have

$$mp_{132,2n+1}(p,q) = (p^2q)^n p^{2\binom{n}{2}} \prod_{i=1}^{n} [2i-1]_{p,q}. \tag{1.11}$$

Plugging equation (1.27) into (**??**), we get the following p,q-analogue of Elizalde and Noy's result for $P_{132}(t,x)$.

$$\begin{aligned} P_{132}(t,x,p,q) &= \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{132\text{-mch}(\sigma)} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} \\ &= \frac{1}{1-(t+\sum_{n\geq 1} \frac{(x-1)^n t^{2n+1}}{[2n+1]_{p,q}!} p^{n^2+n} q^n \prod_{i=1}^{n} [2i-1]_{p,q})} \\ &= \frac{1}{1-\sum_{n\geq 0} \frac{p^{n^2+n} q^n (x-1)^n t^{2n+1}}{[2n+1]_{p,q} \prod_{i=1}^{n} [2i]_{p,q}}}. \end{aligned}$$

Similarly, by Theorem 7, we have

$$mp_{1342,2n+1} = (p^3q^2)^n p^{3\binom{n}{2}} \prod_{i=1}^{n} \begin{bmatrix} 3n+1-2 \\ 2 \end{bmatrix}_{p,q}$$
$$= p^{(3n^2+3n)/2} q^{2n} \prod_{i=1}^{n} \frac{[3n-1]_{p,q}[3n-2]_{p,q}}{[2]_{p,q}}. \quad (1.12)$$

Plugging equation (1.12) into (**??**), we get the following p,q-analogue of Elizalde and Noy's result for $P_{1342}(t,x)$.

$$P_{1342}(t,x,p,q) = \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{1342\text{-mch}(\sigma)} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)}$$
$$= \frac{1}{1-(t+\sum_{n\geq 1} \frac{(x-1)^n t^{3n+1}}{[3n+1]_{p,q}!} p^{(3n^2+3n)/2} q^{2n} \frac{1}{[2]_{p,q}^n} \prod_{i=1}^n [3i-1]_{p,q}[3i-2]_{p,q})}$$
$$= \frac{1}{1-\sum_{n\geq 0} \frac{p^{(3n^2+3n)/2} q^{2n} (x-1)^n t^{3n+1}}{[3n+1]_{p,q}[2]_{p,q}^n \prod_{i=1}^n [3i]_{p,q}}}.$$

Now if $\tau = 12\ldots a\sigma a+1$ where σ is a permutation of $\{a+2,\ldots,k+1\}$, then τ has the minimal overlapping property. Note that

$$\text{coinv}(\tau) = \binom{a+1}{2} + a(k-a) + \text{coinv}(\sigma) \text{ and}$$
$$\text{inv}(\tau) = (k-a) + \text{inv}(\sigma).$$

Using Theorems 7 and 8 , Duane and Remmel [10] proved the following generalization of Kitaev's Theorem 6.

Theorem 9. *Let $\tau = 12\ldots a\sigma a+1$ where σ is a permutation of $\{a+2,\ldots,k+1\}$. Then*

$$\sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} =$$
$$\Bigg(1-(t+\sum_{i\geq 0} \frac{(x-1)^{i+1} t^{ik+1}}{[ik+1]_{p,q}!} \cdot$$
$$(p^{a(k-a)+\text{coinv}(\sigma)+\binom{a+1}{2}} q^{k-a+\text{inv}(\sigma)})^{(i+1)} p^{ak\binom{i+1}{2}} \prod_{j=2}^{i} \begin{bmatrix} jk-a \\ k-a \end{bmatrix}_{p,q}\Bigg)^{-1}. \quad (1.13)$$

1.3 Our Goals

One of the main goals is to give generating functions for the joint distributions of left-to-right minima and descents over the set of permutations of S_n which either have no occurrences or no matches for certain patterns or sets of patterns. That is, given a permutation $\sigma = \sigma_1, \ldots, \sigma_n$, we let we let $\mathrm{des}(\sigma) = |\{i : \sigma_i > \sigma_{i+1}\}|$. Thus for example if $\sigma = 3756142$, then $\mathrm{des}(\sigma) = 3$ which counts the descent pairs 75, 61, and 42 as we read σ from left to right. We say that σ_j is a *left-to-right minimum* of σ if $\sigma_j < \sigma_i$ for all $i < j$. We let $\mathrm{LRMin}(\sigma)$ denote the number of left-to-right minima of σ. Our generating functions can be viewed as refinements of generating functions for the number of permutations of S_n which have no classical matches of τ.

We may define similar matching conditions within the cycle structure of a permutation. Suppose that $\tau = \tau_1 \ldots \tau_j$ is a permutation in S_j and σ is a permutation in S_n with k cycles $C_1 \ldots C_k$. We shall always write cycles in the form $C_i = (c_{0,i}, \ldots, c_{p_i-1,i})$ where $c_{0,i}$ is the smallest element in C_i and p_i is the length of C_i and assume that we have arranged the cycles by decreasing smallest elements. That is, we arrange the cycles of σ so that $c_{0,1} > \cdots > c_{0,k}$. Then we say that σ has a *cycle τ-match* (c-τ-match) if there is an i such that $C_i = (c_{0,i}, \ldots, c_{p_i-1,i})$ where $p_i \geq j$ and an r such that $\mathrm{red}(c_{r,i}c_{r+1,i} \ldots c_{r+j-1,i}) = \tau$ where we take indices of the form $r + s \mod p_i$. Let $c\text{-}\tau\text{-}\mathrm{mch}(\sigma)$ be the number of cycle τ-matches in the permutation σ. For example, if $\tau = 2\ 1\ 3$ and $\sigma = (1, 10, 9)(2, 3)(4, 7, 5, 8, 6)$, then 9 1 10 is a cycle τ-match in the first cycle and 7 5 8 and 6 4 7 are cycle τ-matches in the third cycle so that $c\text{-}\tau\text{-}\mathrm{mch}(\sigma) = 3$. Similarly, we say that τ *cycle occurs* in σ if there exists an i such that $C_i = (c_{0,i}, \ldots, c_{p_i-1,i})$ where $p_i \geq j$ and there is an r with $0 \leq r \leq p_i - 1$ and indices $0 \leq i_1 < \cdots < i_{j-1} \leq p_i - 1$ such that $\mathrm{red}(c_{r,i}c_{r+i_1,i} \ldots c_{r+i_{j-1},i}) = \tau$ where the indices $r + i_s$ are taken mod p_i. We say that σ *cycle avoids* τ if there are no cycle occurrences of τ in σ. For example, if $\tau = 1\ 2\ 3$ and $\sigma = (1, 10, 9)(2, 3)(4, 8, 5, 7, 6)$, then 4 5 7, 4 5 6, 5 6 8, and 5 7 8 are cycle occurrences of τ in the third cycle.

We can extend the notion of cycle matches and cycle occurrences to sets of permutations in the obvious fashion. That is, suppose that Υ is a set of permutations in S_j and σ is a permutation in S_n with k cycles $C_1 \dots C_k$. Then we say that σ has a *cycle* Υ*-match* (c-Υ-match) if there is an i such that $C_i = (c_{0,i}, \dots, c_{p_i-1,i})$ where $p_i \geq j$ and an r such that $\text{red}(c_{r,i} \dots c_{r+j-1,i}) \in \Upsilon$ where we take indices of the form $r+s$ modulo p_i. Let c-Υ-mch(σ) be the number of cycle Υ-matches in the permutation σ. Similarly, we say that Υ *cycle occurs* in σ if there exists an i such that $C_i = (c_{0,i}, \dots, c_{p_i-1,i})$ where $p_i \geq j$ and there is an r with $0 \leq r \leq p_i - 1$ and indices $0 \leq i_1 < \cdots < i_{j-1} \leq p_i - 1$ such that $\text{red}(c_{r,i}c_{r+i_1,i} \dots c_{r+i_{j-1},i}) \in \Upsilon$ where the indices $r + i_s$ are taken mod p_i. We say that σ *cycle avoids* Υ if there are no cycle occurrences of Υ in σ.

The study of patterns in cycle structures in not entirely new. That is, Callan [8] and Vella [41] studied cycle pattern avoidance in n-cycles in S_n. For example, they independently proved that the number of n-cycles in S_n which cycle avoid 1324 is the Fibonacci number F_{2n-3}, the number of n-cycles in S_n which cycle avoid 1342 is $2^{n-1} - (n-1)$, and the number of n-cycles in S_n which cycle avoid 1234 is $2^n + 1 - 2n - \binom{n}{3}$. However, neither Callan or Vella considered the more general problem of cycle avoidance in general cycle structures of permutations. We shall see below that one can use the theory of exponential structures to reduce the problem of finding certain generating functions on the cycle of structures of permutations in S_n to finding certain corresponding generating functions on n-cycles in S_n. Thus it is not difficult to extend the results of Callan and Vella to cycle structure of permutations in S_n. This idea was used by Deutsch and Elizalde [9] to study various generating functions for the analogue of up-down permutations relative to cycle structures of permutations in S_n. We shall see below that for even cycles, their definition of up-down cycles is equivalent to having no cycle $\{123, 321\}$-matches.

Another goal is to give generating functions for the joint distributions of cycles and cycle descents over the set of permutations of S_n which either have no cycle occurrences or no cycle matches for certain patterns or sets of patterns. That is, given a cycle $C = (c_0, \dots, c_{p-1})$ where c_0 is the smallest element in the cycle, we let $\text{cdes}(C) = 1 + \text{des}(c_0 \dots c_{p-1})$. Thus $\text{cdes}(C)$ counts the number of descent pairs as we traverse once around the cycle because the extra factor of 1 counts the descent pair $c_{p-1} > c_0$. For example if $C = (1, 5, 3, 7, 2)$, then $\text{cdes}(C) = 3$ which counts the descent pairs 53, 72, and 21 as we traverse once around C. By convention, if $C = (c_0)$ is one-cycle, we let $\text{cdes}(C) = 1$. If σ is a permutation in S_n with k cycles $C_1 \dots C_k$, then we define $\text{cdes}(\sigma) = \sum_{i=1}^{k} \text{cdes}(C_i)$. We let $\text{cyc}(\sigma)$ denote the number of cycles of σ. Our generating functions for such joint distributions are new. However, in the case where the pattern τ starts with 1, then our generating functions can be viewed as refinements of generating functions for the number of permutations of S_n which have no classical matches of τ.

1.3.1 Notation

Given $\Upsilon \subseteq S_j$, we let $\mathcal{S}_n(\Upsilon)$ (resp. $\mathcal{CAV}_n(\Upsilon)$) denote the set of permutations of S_n which avoid (resp. cycle avoid) Υ and $S_n(\Upsilon) = |\mathcal{S}_n(\Upsilon)|$ (resp. $CAV_n(\Upsilon) = |\mathcal{CAV}_n(\Upsilon)|$). Similarly, we let $\mathcal{NM}_n(\Upsilon)$ (resp. $\mathcal{NCM}_n(\Upsilon)$) denote the set of permutations of S_n which have no Υ-matches (resp. no cycle Υ-matches) and $NM_n(\Upsilon) = |\mathcal{NM}_n(\Upsilon)|$ (resp. $NCM_n(\Upsilon) = |\mathcal{NCM}_n(\Upsilon)|$). Throughout this thesis, when $\Upsilon = \{\tau\}$ is a singleton, we shall just write the τ rather than $\{\tau\}$. Thus for example, we shall write $\mathcal{S}_n(\tau)$ for $\mathcal{S}_n(\Upsilon)$ when $\Upsilon = \{\tau\}$.

Given α and β in S_j, we say that α and β are *Wilf equivalent* if $S_n(\alpha) = S_n(\beta)$ for all n. We say that α and β are *matching Wilf equivalent* (m-Wilf equivalent) if $NM_n(\alpha) = NM_n(\beta)$ for all n. For any permutation $\sigma = \sigma_1 \dots \sigma_n$, we let σ^r be the reverse of σ and σ^c be the complement of σ. That is, $\sigma^r = \sigma_n \dots \sigma_1$ and $\sigma^c = (n+1-\sigma_1) \dots (n+1-\sigma_n)$. It is well known that Wilf equivalence classes and m-Wilf equivalence classes are closed under reverse and complementation. We say that α and β are *cycle avoidance Wilf equivalent* (ca-Wilf equivalent) if $CAV_n(\alpha) = CAV_n(\beta)$ for all n and we say that α and β are *cycle matching Wilf equivalent* (cm-Wilf equivalent) if $NCM_n(\alpha) = NCM_n(\beta)$ for all n. If α and β are cycle avoidance Wilf equivalent, we shall write $\alpha \sim_{ca} \beta$. If α and β are cycle matching Wilf equivalent, we shall write $\alpha \sim_{cm} \beta$. Similarly, for sets of permutations Γ and Δ in S_j, we say that Γ and Δ are cycle avoidance Wilf equivalent (ca-Wilf equivalent) if $CAV_n(\Gamma) = CAV_n(\Delta)$ for all n and we say that Γ and Δ are cycle matching Wilf equivalent (cm-Wilf equivalent) if $NCM_n(\Gamma) = NCM_n(\Delta)$

for all n.

Callan [8] and Vella [41] observed that for n-cycles, ca-Wilf equivalence classes are closed under reverse and complement. This is also true for both ca-Wilf equivalence and cm-Wilf equivalence for general cycle structures. That is, let σ be a permutation in S_n with k cycles $C_1 \ldots C_k$. Then we let the *cycle reverse of* σ, denoted by σ^{cr}, be the permutation which arises from σ by replacing each cycle $C_i = (c_{0,i}, c_{1,i}, \ldots, c_{p_i-1,i})$ by its reverse cycle $C_i^{cr} = (c_{0,i}, c_{p_i-1,i}, \ldots c_{1,i})$. For example, if $\sigma = (1,10,9)(2,3)(4,7,5,8,6)$, then $\sigma^{cr} = (1,9,10)(2,3)(4,6,8,5,7)$. We let the cycle complement of σ, denoted by σ^{cc}, be the permutation that results from σ by replacing each element i in the cycle structure of σ by $n+1-i$. For example, if $\sigma = (1,10,9)(2,3)(4,7,5,8,6)$, then $\sigma^{cr} = (10,1,2)(9,8)(7,4,6,3,5) = (1,2,10)(3,5,7,4,6)(8,9)$. Note that in general σ^r, σ^c, σ^{cr} and σ^{cc} are all distinct. For example, if $\sigma = 2\ 3\ 1\ 5\ 4$ so that it cycle structure is $(1,2,3)(4,5)$, then

$$\begin{aligned} \sigma^r &= 4\ 5\ 1\ 3\ 2, \\ \sigma^c &= 4\ 3\ 5\ 1\ 2, \\ \sigma^{cr} &= (1,3,2)(4,5) = 3\ 1\ 2\ 5\ 4, \text{ and} \\ \sigma^{cc} &= (5,4,3)(2,1) = 2\ 1\ 5\ 3\ 4. \end{aligned}$$

It is easy to see that for any permutation $\sigma \in S_n$,

1. σ has a cycle τ-match if and only if σ^{cr} has a cycle τ^r-match,

2. σ has a cycle τ-match if and only if σ^{cc} has a cycle τ^c-match,

3. σ has a cycle τ occurrence if and only if σ^{cr} has a cycle τ^r occurrence, and

4. σ has a cycle τ occurrence if and only if σ^{cc} has a cycle τ^c occurrence.

It then easily follows that for all permutations τ, $NCM_n(\tau) = NCM_n(\tau^r) = NCM_n(\tau^c)$ so that τ, τ^r, and τ^c are all cycle matching Wilf equivalent. Similarly, $CAV_n(\tau) = CAV_n(\tau^r) = CAV_n(\tau^c)$ so that τ, τ^r, and τ^c are all cycle avoidance Wilf equivalent. Finally we observe that our definitions also ensure that for any $\tau = \tau_1 \ldots \tau_j \in S_j$, any cyclic rearrangement of τ, $\tau^{(i)} = \tau_i \ldots \tau_j \tau_1 \ldots \tau_{i-1}$ also has the property that for any $\sigma \in S_n$, τ cycle occurs in σ if and only if $\tau^{(i)}$ cycle occurs in σ. Thus for all $j \geq 1$, $CAV_n(\tau) = CAV_n(\tau^{(i)})$ so that τ and $\tau^{(i)}$ are cycle avoidance Wilf equivalent.

The generating functions we will study are

$$S_\Upsilon(t,x,y) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{S}_n(\Upsilon)} x^{\mathrm{LRMin}(\sigma)} y^{1+\mathrm{des}(\sigma)}, \tag{1.14}$$

$$CAV_\Upsilon(t,x,y) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{CAV}_n(\Upsilon)} x^{\mathrm{cyc}(\sigma)} y^{\mathrm{cdes}(\sigma)}, \tag{1.15}$$

$$NM_\Upsilon(t,x,y) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\Upsilon)} x^{\mathrm{LRMin}(\sigma)} y^{1+\mathrm{des}(\sigma)}, \tag{1.16}$$

and

$$NCM_\Upsilon(t,x,y) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NCM}_n(\Upsilon)} x^{\mathrm{cyc}(\sigma)} y^{\mathrm{cdes}(\sigma)} \tag{1.17}$$

for Υ a set of patterns.

1.4 Techniques

We have developed three different techniques that give results for the generating functions (1.15),(1.16), and (1.17).

1.4.1 Using the theory of exponential structures

First let's consider the generating functions (1.15) and (1.17). We can use the theory of exponential structures to reduce the problem down to studying pattern matching in n-cycles. That is, let $\mathcal{L}_m$ denote the set of m-cycles in S_m and let $L_m = |\mathcal{L}_m|$. Suppose that R is a ring and for each $m \geq 1$, we have a weight function $W_m : \mathcal{L}_m \to R$. We let $W(\mathcal{L}_m) = \sum_{C \in \mathcal{L}_m} W_m(C)$. Now suppose that $\sigma \in S_n$ and the cycles of σ are $C_1, \ldots, C_k$. If C_i is of size m, then we let $W(C_i) = W_m(\mathrm{red}(C_i))$. Then we define the weight of σ, $W(\sigma)$, by

$$W(\sigma) = \prod_{i=1}^{k} W(C_i).$$

We let $\mathcal{C}_{n,k}$ denote the set of all permutations of S_n with k cycles. This given, the following theorem easily follows from the theory of exponential structures, see [39].

Theorem 10.

$$1+\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{k=1}^{n}x^k\sum_{\sigma\in\mathcal{C}_{n,k}}W(\sigma)=e^{x\sum_{m\geq 1}\frac{W(\mathcal{L}_m)t^m}{m!}}. \tag{1.18}$$

Let $\Upsilon \subseteq S_j$. Then we will be most interested in the special case of weight functions W_m where $W_m(C) = 1$ if C cycle avoids a set of permutations and $W_m(C) = 0$ otherwise, or where $W_m(C) = 1$ if C has no cycle Υ-matches and $W_m(C) = 0$ otherwise. We let $\mathcal{CAV}_{n,k}(\Upsilon)$ denote the set of permutations σ of S_n with k cycles such that σ cycle avoids Υ and we let $CAV_{n,k}(\Upsilon) = |\mathcal{CAV}_{n,k}(\Upsilon)|$. We let $\mathcal{NCM}_{n,k}(\Upsilon)$ denote the set of permutations σ of S_n with k cycles such that σ has no cycle Υ-matches and $NCM_{n,k}(\Upsilon) = |\mathcal{NCM}_{n,k}(\Upsilon)|$. Similarly, we let $\mathcal{L}_m^{cav}(\Upsilon)$ be the set of m-cycles γ in S_m such that γ cycle avoids Υ, $L_m^{cav}(\Upsilon) = |\mathcal{L}_m^{cav}(\Upsilon)|$, $\mathcal{L}_m^{ncm}(\Upsilon)$ denote the set of m-cycles γ in S_m such that γ has no cycle Υ-matches, and $L_m^{ncm}(\Upsilon) = |\mathcal{L}_m^{ncm}(\Upsilon)|$. Then a special case of Theorem 10 is the following theorem.

Theorem 11.

$$CAV_{\Upsilon}(t,x,y)=1+\sum_{n\geq 1}\frac{t^n}{n!}\sum_{k=1}^{n}x^k\sum_{\sigma\in\mathcal{CAV}_{n,k}(\Upsilon)}y^{\text{cdes}(\sigma)}=e^{x\sum_{m\geq 1}\frac{t^m}{m!}\sum_{C\in\mathcal{L}_m^{cav}(\Upsilon)}y^{\text{cdes}(C)}}, \tag{1.19}$$

and

$$NCM_{\Upsilon}(t,x,y)=1+\sum_{n\geq 1}\frac{t^n}{n!}\sum_{k=1}^{n}x^k\sum_{\sigma\in\mathcal{NCM}_{n,k}(\Upsilon)}y^{\text{cdes}(\sigma)}=e^{x\sum_{m\geq 1}\frac{t^m}{m!}\sum_{C\in\mathcal{L}_m^{ncm}(\Upsilon)}y^{\text{cdes}(C)}}. \tag{1.20}$$

For example, suppose that $\tau = 1\ 2$. It is clear that any cycle of length k where $k \geq 2$ has both a cycle occurrence of τ and a cycle τ-match so that $L_m^{cav}(12) = L_m^{ncm}(12) = 0$ if $m \geq 2$. Since 1-cycles can not have any cycle occurrences of τ or any cycle τ-matches by definition, it follows that

$$y=\sum_{C\in\mathcal{L}_1^{cav}(12)}y^{\text{cdes}(C)}=\sum_{C\in\mathcal{L}_1^{ncm}(12)}y^{\text{cdes}(C)}.$$

Thus

$$CAV_{12}(t,x,y)=NCM_{12}(t,x,y)=e^{xyt}.$$

Next consider $\tau = 1\ 2\ 3$. It was observed by both Callan [8] and Vella [41] that for $k \geq 3$, the only k-cycle which cycle avoids τ is the cycle $(1, k, k-1, \ldots, 2)$. Let

$$A_m(y) = \sum_{C \in \mathcal{L}_m^{cav}(123)} y^{\mathrm{cdes}(C)},$$

then clearly $A_1(y) = y$ since $\mathrm{cdes}((1)) = 1$, and for $k \geq 2$, $A_k(y) = y^{k-1}$ since $\mathrm{cdes}((1, k, \ldots, 2)) = k - 1$. Thus

$$CAV_{123}(t, x, y) = e^{x\left(yt + \sum_{m \geq 2} \frac{y^{m-1}t^m}{m!}\right)} = e^{x\left(yt + \frac{1}{y}(e^{yt} - 1 - yt)\right)}.$$

It turns out that if $\tau \in S_j$ is a permutation that starts with 1, then we can reduce the problem of finding $NCM_\tau(t, x, y)$ to the usual problem of finding the generating function of permutations that have no τ-matches. That is, we consider the following well-known bijection described in [9]. Suppose we are given a permutation $\sigma \in S_n$ with k cycles $C_1 \cdots C_k$. Assume we have arranged the cycles so that the smallest element in each cycle is on the left and we arrange the cycles by decreasing smallest elements. Then we let $\bar{\sigma}$ be the permutation that arises from $C_1 \cdots C_k$ by erasing all the parentheses and commas. For example, if $\sigma = (7, 10, 9, 11)\ (4, 8, 6)\ (1, 5, 3, 2)$, then $\bar{\sigma} = 7\ 10\ 9\ 11\ 4\ 8\ 6\ 1\ 5\ 3\ 2$. It is easy to see that the minimal elements of the cycles correspond to left-to-right minima in $\bar{\sigma}$. It is also easy to see that under our bijection $\sigma \to \bar{\sigma}$, $\mathrm{cdes}(\sigma) = \mathrm{des}(\bar{\sigma}) + 1$ since every left-to-right minima is part of a descent pair in $\bar{\sigma}$. For example, if $\sigma = (7, 10, 9, 11)\ (4, 8, 6)\ (1, 5, 3, 2)$ so that $\bar{\sigma} = 7\ 10\ 9\ 11\ 4\ 8\ 6\ 1\ 5\ 3\ 2$, $\mathrm{cdes}((7, 10, 9, 11)) = 2$, $\mathrm{cdes}((4, 8, 6)) = 2$, and $\mathrm{cdes}((1, 5, 3, 2)) = 3$ so that $\mathrm{cdes}(\sigma) = 2 + 2 + 3 = 7$ while $\mathrm{des}(\bar{\sigma}) = 6$. This given, we have the following lemma.

Lemma 12. *If $\tau \in S_j$ and τ starts with 1, then for any $\sigma \in S_n$,*

1. *σ has k cycles if and only if $\bar{\sigma}$ has k left-to-right minima,*
2. *$\mathrm{cdes}(\sigma) = 1 + \mathrm{des}(\bar{\sigma})$, and*
3. *σ has no cycle-τ-matches if and only if $\bar{\sigma}$ has no τ-matches.*

Proof. For (3), suppose that $\bar{\sigma} = \bar{\sigma}_1 \ldots \bar{\sigma}_n$ and $\bar{\sigma}_i = 1$. Since τ starts with 1, it is easy to see that any τ-match in $\bar{\sigma}$ must either occur weakly to the right of $\bar{\sigma}_i$ or strictly to the left of $\bar{\sigma}_i$. That is, 1 can be part of τ-match in $\bar{\sigma}$ only if the τ-match starts at position i. If a τ-match occurred weakly to the right of $\bar{\sigma}_i$, then that τ-match would correspond to a cycle-τ-match in C_k in σ.

Next suppose that the τ-match occurred strictly to the left of $\bar{\sigma}_i = 1$. Then we claim that we can make a similar argument with respect to the cycles $C_1 \cdots C_{k-1}$. That is, suppose that C_{k-1} starts with m. Then m must be the smallest element among $\bar{\sigma}_1 \ldots \bar{\sigma}_{i-1}$. Suppose that $\bar{\sigma}_s = m$ where $1 \leq s < i$. Then again we can argue that any τ-match in $\bar{\sigma}_1 \ldots \bar{\sigma}_{i-1}$ must occur either weakly to the right of $\bar{\sigma}_s$ or strictly to left of $\bar{\sigma}_s$. If the τ-match in $\bar{\sigma}_1 \ldots \bar{\sigma}_{i-1}$ occurs weakly to the right of $\bar{\sigma}_s$, then it would correspond to a cycle-τ-match in C_{k-1}. Continuing on in this way, we see that any τ-match in $\bar{\sigma}$ must correspond to a cycle τ-match in C_j for some j.

Vice versa, it is easy to see that since τ starts with 1, the only way that a cycle-τ-match in C_i can involve the smallest element $c_{0,i}$ in the cycle C_i is if $c_{0,i}$ corresponds to the 1 in τ in cycle match. But this easily implies that any τ-cycle match in C_i must also correspond to a τ-match in the elements of $\bar{\sigma}$ corresponding to C_i.

Thus we have proved that for any σ, σ has cycle-τ-match if only if $\bar{\sigma}$ has a τ-match. □

We should note that if a permutation τ does not start with 1, then it may be that case that $NCM_n(\tau) \neq NM_n(\tau)$. The pattern $\tau = 3142$ is an example such that neither τ, τ^r, τ^c, nor $(\tau^r)^c$ starts with one. Even though we do not know how to compute closed forms for $NCM_\tau(t)$ and $NM_\tau(t)$, we have computed the following table. Let $\mathcal{L}_n^{ncm}(\tau)$ be the set of all cycles of length n and let $L_n^{ncm}(\tau) = |\mathcal{L}_n^{ncm}(\tau)|$.

It was conjectured by the author that for a set of patterns Υ, $NCM_n(\Upsilon) \neq NM_n(\Upsilon)$ if and only if Υ can *cover* a cycle. For a cycle $C = (c_0, \ldots, c_{n-1})$ in S_n is covered by Υ if each consecutive pair of elements $\{c_i, c_{i+1}\}, 0 \leq i \leq n-1$ is part of a cycle-Υ-match (this includes the consecutive pair $\{c_{n-1}, c_0\}$.) For example, if $\tau = 3142$, then τ covers the cycle $(1, 4, 2, 5, 3)$. There are two

Table 1.2: Coefficients for GFs involving 3142

n	$L_n^{ncm}(3142)$	$NCM_n(3142)$	$NM_n(3142)$
1	1	1	1
2	1	2	2
3	2	6	6
4	5	23	23
5	20	111	110
6	92	638	632
7	532	4278	4237
8	3565	32784	32465

cycle-τ-matches, namely 3142 and 4253 and every consecutive pair of elements $\{1,4\},\{4,2\},\{2,5\},\{5,3\},\{3,1\}$ belongs to one of these cycle-τ-matches.

Remmel and Tiefenbruck [36] were able to use the involution principle of Garsia and Milne [15] to show one direction of the conjecture.

Theorem 13. *(Tiefenbruck [36]) If Υ does not cover any cycle then*

$$NCM_n(\Upsilon) = NM_n(\Upsilon).$$

One consequence of Lemma 12 is that we can automatically obtain refinements of generating functions for the number of permutations with no τ-matches when τ starts with 1. We have the following corollary of Lemma 12.

Corollary 14. *If $\tau \in S_j$ and τ starts with 1, then*

$$NCM_\tau(t,x,y) \quad = \quad NM_\tau(t,x,y). \tag{1.21}$$

Then by Theorem 11 and Lemma 12, if $\tau \in S_j$ and τ starts with 1, we have that

$$\begin{aligned} NM_\tau(t,1,1) &= NCM_\tau(t,1,1) \\ &= e^{\sum_{m\geq 1} L_m^{ncm}(\tau)\frac{t^m}{m!}} \end{aligned}$$

so that

$$\ln(NM_\tau(t,1,1)) = \sum_{m\geq 1} L_m^{ncm}(\tau)\frac{t^m}{m!}. \tag{1.22}$$

But then

$$\begin{aligned} NM_\tau(t,x,1) &= NCM_\tau(t,x,1) & (1.23)\\ &= e^{x\sum_{m\geq 1} L_m^{ncm}(\tau)\frac{t^m}{m!}} \\ &= e^{x\ln(NM_\tau(t,1,1))} = (NM_\tau(t,1,1))^x. & (1.24)\end{aligned}$$

Thus if we can compute $NM_\tau(t,1,1)$ for a permutation $\tau \in S_j$ that starts with 1, we automatically can compute $NM_\tau(t,x,1)$. For example, Goulden and Jackson [16] proved that when $\tau = 1\ 2 \dots k$, then

$$NM_\tau(t,1,1) = \frac{1}{\sum_{i\geq 0} \frac{t^{ki}}{(ki)!} - \frac{t^{ki+1}}{(ki+1)!}}. \tag{1.25}$$

Hence, we automatically have the following refinement of Goulden and Jackson's result.

Theorem 15. *If $\tau = 1\ 2 \dots k$ where $k \geq 2$, then*

$$NM_\tau(t,x,1) = \left(\frac{1}{\sum_{i\geq 0} \frac{t^{ki}}{(ki)!} - \frac{t^{ki+1}}{(ki+1)!}}\right)^x. \tag{1.26}$$

An example, where one can use the full power of Theorem 10 is the following. In Section 2, we shall show that

$$\sum_{n\geq 1} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(132)} y^{\mathrm{cdes}(C)} = \ln\left(\frac{1}{1-y\int_0^t e^{(1-y)s-y\frac{s^2}{2}}ds}\right). \tag{1.27}$$

Then it follows that

$$\begin{aligned} NCM_{132}(t,x,y) &= \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma\in S_n} x^{\mathrm{cyc}(\sigma)} y^{\mathrm{cdes}(\sigma)} & (1.28)\\ &= \sum_{n\geq 0} \frac{t^n}{n!} \sum_{k=1}^{n} x^k \sum_{\sigma\in \mathcal{NCM}_{n,k}(\tau)} y^{\mathrm{cdes}(\sigma)} \\ &= e^{x\ln\left(\frac{1}{1-y\int_0^t e^{(1-y)s-y\frac{s^2}{2}}ds}\right)} \\ &= \left(\frac{1}{1-y\int_0^t e^{(1-y)s-y\frac{s^2}{2}}ds}\right)^x. \end{aligned}$$

This result is a refinement of theorem of Elizalde and Noy [12].

Our plan is to consider n-cycles and directly compute $\sum_{C\in\mathcal{L}_m^{cav}(\Upsilon)} y^{\mathrm{cdes}(C)}$ and $\sum_{C\in\mathcal{L}_m^{ncm}(\Upsilon)} y^{\mathrm{cdes}(C)}$. Then we can use Theorem 10 to compute $CAV_\tau(t,x,y)$ and $NCM_\tau(t,x,y)$. Furthermore, if τ starts with 1, we can use Corollary 14 to compute $NM_\tau(t,x,y)$ from $NCM_\tau(t,x,y)$.

1.4.2 The reciprocal method

Our second method uses the function $U_\tau(t,y)$ defined by

$$NM_\tau(t,x,y) = \sum_{n\geq 0} NM_{\tau,n}(x,y)\frac{t^n}{n!} = \frac{1}{(U_\tau(t,y))^x}.$$

It follows that

$$U_\tau(t,y) = \frac{1}{NM_\tau(t,1,y)} = \frac{1}{\sum_{n\geq 0} NM_{\tau,n}(1,y)\frac{t^n}{n!}}. \tag{1.29}$$

Remmel and his coauthors [3, 25, 29, 30, 31, 32, 35, 42] developed a method called the homomorphism method to show that many generating functions involving permutation statistics can be applied to simple symmetric function identities such as

$$H(t) = 1/E(-t) \tag{1.30}$$

where

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_{i\geq 1} \frac{1}{1-x_i t} \tag{1.31}$$

is the generating function of the homogeneous symmetric functions h_n in infinitely many variables $x_1, x_2, \ldots$ and

$$E(t) = \sum_{n\geq 0} e_n t^n = \prod_{i\geq 1} 1 + x_i t \tag{1.32}$$

is the generating function of the elementary symmetric functions e_n in infinitely many variables $x_1, x_2, \ldots$. Now if we define a homomorphism θ and the ring of symmetric functions so that

$$\theta(e_n) = \frac{(-1)^n}{n!} NM_{\tau,n}(1,y),$$

then

$$\theta(E(-t)) = \frac{1}{\sum_{n\geq 0} NM_{\tau,n}(1,y)\frac{t^n}{n!}}.$$

Thus $\theta(H(t))$ should equal $U_\tau(t,y)$. One can then use the combinatorial methods associated with the homomorphism method to develop recursions for the coefficient of $U_\tau(t,y)$. Note that if τ starts with 1 then we can use Corollary 14 to compute $NCM_\tau(t,x,y)$ from $NM_\tau(t,x,y)$.

1.4.3 Using the bijection between cycles and brick tabloids

The third method we shall use to compute $NCM_\Upsilon(t)$ is completely different than the other two approaches. This method involves defining a certain bijection between the set of cycles and certain fillings of brick tabloids. That bijection allows one to compute generating functions for the number cycles that have no cycle Υ-matches by applying an appropriate ring homomorphism defined on the ring of symmetric functions Λ in infinitely many variables $x_1, x_2, \dots$ to certain simple symmetric function identities as described above. This approach is generally much more complicated than the first two approaches. However, it allows us to compute $NCM_\Upsilon(t)$ for a number of sets of permutations Υ which seem beyond the other techniques employed in this thesis. For example, one can show that

$$NCM_\Upsilon(t) = \frac{e^{t+t^2/2+t^4/12}}{1-\sum_{n\geq 3}(n-1)\frac{t^n}{n!}} = \frac{2e^{t^2/2}e^{t^4/12}}{2-2t+t^2e^{-t}}$$

where Υ is the set of patterns that contains 1234 and all patterns $\tau = \tau_1\tau_2\tau_3\tau_4\tau_5$ such that $\tau_1 < \tau_2 > \tau_3 < \tau_4 > \tau_5$.

1.5 Results

1.5.1 Results using the theory of exponential structures

In this subsection, we will state several of the results that we proved using the theory of exponential structures. Since cycle avoidance Wilf equivalence is closed under cyclic rearrangements, it follows that $1\ 2\ 3 \sim_{ca} 2\ 3\ 1$ and each pattern of length 3 is a reverse, complement or reverse complement of one of these patterns which means that all permutations of length three are cycle avoidance Wilf equivalent.

Theorem 16. *For all patterns $\tau \in S_3$,*

$$CAV_\tau(t) = CAV_{123}(t) = e^{e^t-1}$$

$$CAV_\tau(t,x,1) = e^{x\sum_{m\geq 1} L_m^{cav}(\tau)\frac{t^m}{m!}} = e^{x(e^t-1)}$$

for $\tau \in \{123, 312, 231\}$,

$$CAV_\tau(t,x,y) = e^{x\left(yt+\frac{1}{y}(e^{yt}-1-yt)\right)}$$

and for $\tau \in \{132, 213, 321\}$

$$CAV_\tau(t,x,y) = e^{x\left(\sum_{m\geq 1}\frac{yt^m}{m!}\right)} = e^{xy(e^t-1)}.$$

Theorem 17. ([30]) *If $\tau = j \ldots 2\ 1$ where $j \geq 2$, then*

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NM}_n(\tau)} y^{\mathrm{des}(\sigma)} = \left(\sum_{n\geq 0}\frac{t^n}{n!}\sum_{i\geq 0}(-1)^i\mathcal{R}_{n-1,i,j-1}y^i\right)^{-1}$$

where $\mathcal{R}_{n,i,j}$ is the number of rearrangements of i zeroes and $n-i$ ones such that j zeroes never appear consecutively.

Theorem 18. *For $j \geq 2$ and $\tau = 1\ 2 \ldots j$,*

$$\begin{aligned} NM_\tau(t,x,y) = NCM_\tau(t,x,y) &= \sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NCM}_n(\tau)} x^{\mathrm{cyc}(\sigma)}y^{\mathrm{cdes}(\sigma)} \\ &= e^{x\ln\left(\frac{1}{\sum_{n\geq 0}\frac{t^n}{n!}\sum_{i\geq 0}(-1)^i\mathcal{R}_{n-1,i,j-1}y^{n-i}}\right)} \\ &= \left(\frac{1}{\sum_{n\geq 0}\frac{t^n}{n!}\sum_{i\geq 0}(-1)^i\mathcal{R}_{n-1,i,j-1}y^{n-i}}\right)^x. \end{aligned}$$

where $\mathcal{R}_{n,i,j}$ is the number of rearrangements of i zeroes and $n-i$ ones such that j zeroes never appear consecutively. In particular

$$NCM_{123}(t,x,y) = e^{\frac{xyt}{2}}\left(\frac{1}{\left(\cosh\left(\frac{t\sqrt{y^2-4y}}{2}\right) - \frac{\sqrt{y}}{\sqrt{y-4}}\sinh\left(\frac{t\sqrt{y^2-4y}}{2}\right)\right)}\right)^x.$$

Theorem 19.

$$\begin{aligned} NCM_{132}(t,x,y) &= e^{x\ln\left(\frac{1}{1-y\int_0^t e^{(1-y)s-ys^2/2}ds}\right)} \\ &= \frac{1}{\left(1-y\int_0^t e^{(1-y)s-ys^2/2}ds\right)^x}. \end{aligned} \tag{1.33}$$

Theorem 20. *For* $\Upsilon = \{123, 321\}$,

$$CAV_\Upsilon(t,x,y) = e^{x\left(yt + \frac{yt^2}{2}\right)},$$

$$NCM_\Upsilon(t,x,y) = e^{xyt}\sec(t\sqrt{y})^x.$$

Theorem 21. *For* $\Upsilon = \{123, 231\}$,

$$NCM_\Upsilon(t,x,y) = e^{\frac{xyt^2}{2}}\left(\frac{1}{\left(1 - \frac{e^{-\frac{y}{2}}\sqrt{2\pi y}}{2}\operatorname{erfi}\left(\sqrt{\frac{y}{2}}\right) - \frac{e^{-\frac{y}{2}}\sqrt{2\pi y}}{2}\operatorname{erfi}\left((t-1)\sqrt{\frac{y}{2}}\right)\right)}\right)^x.$$

Theorem 22. *Let* $\tau = \tau_1 \ldots \tau_j \in S_j$ *where* $j \geq 3$ *and* $\tau_1 = 1$ *and* $\tau_j = 2$. *Then*

$$NCM_\tau(t,x,y) = NM_\tau(t,x,y) = \frac{1}{\left(1 - \int_0^t e^{(y-1)s - \frac{y^{\mathrm{des}(\tau)}s^{j-1}}{(j-1)!}}ds\right)^x}.$$

Theorem 23. *Suppose that* $\tau = 1\ 2\ \ldots j-1\ \gamma\ j$ *where* γ *is a permutation of* $j+1, \ldots, j+p$ *and* $j \geq 3$. *Then*

$$NCM_\tau(t,x,y) = \frac{1}{(U_\tau(t,y))^x}$$

where

$$U_\tau(t,y) = \sum_{n\geq 0} U_{n,\tau}(y)\frac{t^n}{n!}$$

and

$$U_{n+j,\tau}(y) = (1-y)U_{n+j-1,\tau}(y) - y^{\mathrm{des}(\tau)}\binom{n}{p}U_{n-p+1,\tau}(y).$$

1.5.2 Results using the reciprocal method

In this subsection, we shall state several results that we proved using the reciprocal method.

Theorem 24. *Let* $\tau = 1324\ldots p$ *where* $p \geq 5$. *Then* $NM_\tau(t,x,y) = \left(\frac{1}{U_\tau(t,y)}\right)^x$ *where* $U_\tau(t,y) = 1 + \sum_{n\geq 1} U_{\tau,n}(y)\frac{t^n}{n!}$, $U_{\tau,1}(y) = -y$, *and for* $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=1}^{\lfloor\frac{n-2}{p-2}\rfloor}(-y)U_{\tau,n-(k(p-2)+1)}(y).$$

Theorem 25. *Let $\tau = 1324$. Then*

$$NCM_\tau(t,x,y) = \left(\frac{1}{U_\tau(t,y)}\right)^x \text{ where } U_\tau(t,y) = 1 + \sum_{n\geq 1} U_{\tau,n}(y)\frac{t^n}{n!}$$

and $U_{\tau,1}(y) = -y$ and for $n > 1$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=2}^{\lfloor \frac{n}{2} \rfloor} (-y)^{k-1} C_{k-1} U_{\tau,n-2k+1}(y)$$

where C_k is the k^{th} Catalan number.

Theorem 26. *Let $\tau = 1p23\ldots(p-1)$ where $p \geq 4$. Then*

$$NM_\tau(t,x,y) = \left(\frac{1}{U_\tau(t,y)}\right)^x \text{ where } U_\tau(t,y) = 1 + \sum_{n\geq 1} U_{\tau,n}(y)\frac{t^n}{n!},$$

$U_{\tau,1}(y) = -y$, and for $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=1}^{\lfloor \frac{n-2}{p-2} \rfloor} (-y)^k \binom{n-k(p-3)-2}{k} U_{\tau,n-(k(p-2)+1)}(y).$$

Theorem 27. *Let $\tau = 13\ldots(p-1)2p$ where $p \geq 4$. Then*

$$NM_\tau(t,x,y) = \left(\frac{1}{U_\tau(t,y)}\right)^x \text{ where } U_\tau(t,y) = 1 + \sum_{n\geq 1} U_{\tau,n}(y)\frac{t^n}{n!},$$

$U_{\tau,1}(y) = -y$ and for $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=1}^{\lfloor \frac{n-2}{p-2} \rfloor} (-y)^k \frac{1}{(p-2)k+1} \binom{k(p-1)}{k} U_{\tau,n-(k(p-2)+1)}(y)$$

Theorem 28. *Let $\tau = 145\ldots p23$ where $p \geq 5$. Then*

$$NM_\tau(t,x,y) = \left(\frac{1}{U_\tau(t,y)}\right)^x \text{ where } U_\tau(t,y) = 1 + \sum_{n\geq 1} U_{\tau,n}(y)\frac{t^n}{n!},$$

$U_{\tau,1}(y) = -y$, and for $n > 1$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=2}^{\lfloor \frac{n-2}{p-2} \rfloor + 1} (-y)^{k-1} \binom{n-k-1}{(p-3)(k-1)} \prod_{i=1}^{k-1} \binom{i(p-3)-1}{p-4} U_{\tau,n-((k-1)(p-2)+1)}(y).$$

1.5.3 Using the bijection between cycles and brick tabloids

In this subsection we shall state several results that we proved using our bijection between cycles and brick tabloids.

For Theorems 29, 30, and 31, let Γ be the set of all patterns $\tau = \tau_1\tau_2\tau_3\tau_4\tau_5$ such that

$$\tau_1 < \tau_2 > \tau_3 < \tau_4 > \tau_5.$$

Theorem 29. *Let* $\Upsilon_1 = \Gamma \cup \{1234\}$ *then*

$$NCM_{\Upsilon_1}(t) = \frac{2e^{t^2/2}e^{t^4/12}}{2 - 2t + t^2e^{-t}}.$$

Theorem 30. *Let* $\Upsilon_2 = \{132, 1234, 35241, 45231, 34251\}$ *then*

$$NCM_{\Upsilon_2}(t) = \frac{2e^te^{t^2/2}}{4 - 2e^t + t^2 + 2t}.$$

Theorem 31. *Let* $\Upsilon_3 = \{231, 1234, 13254, 14253, 15243\}$ *then*

$$NCM_{\Upsilon_3}(t) = \frac{e^te^{t^2/2}}{-1 - t + 2e^t - te^t}.$$

Chapter 2

Generating functions for cycle matches using the theory of exponential structures

In this chapter, we shall describe how we can use the theory of exponential structures to find generating functions for cycle matches in permutations. The results of this chapter are based on the results appearing in a paper by Jones and Remmel [18].

2.1 Patterns of length 3

In this section, we study $CAV_\tau(t,x,y)$ and $NCM_\tau(t,x,y)$ for $\tau \in S_3$.

First we consider $CAV_\tau(t,x)$ for $\tau \in S_3$. It follows from our remarks in the introduction that both cycle avoidance Wilf equivalence and cycle matching Wilf equivalence are closed under the operation of reverse and complement. Thus

1. $1\ 2\ 3 \sim_{ca} 3\ 2\ 1$ and $1\ 2\ 3 \sim_{cm} 3\ 2\ 1$ and

2. $1\ 3\ 2 \sim_{ca} 2\ 3\ 1 \sim_{ca} 2\ 1\ 3 \sim_{ca} 3\ 1\ 2$ and $1\ 3\ 2 \sim_{cm} 2\ 3\ 1 \sim_{cm} 2\ 1\ 3 \sim_{cm} 3\ 1\ 2$.

Now since cycle avoidance Wilf equivalence is closed under cyclic rearrangements, it follows that $1\ 2\ 3 \sim_{ca} 2\ 3\ 1$ which means that all permutations of length three are cycle avoidance Wilf equivalent. Thus for all permutations τ of length three, we have from Theorem 16

$$CAV_\tau(t) = CAV_{123}(t) = e^{e^t-1}$$

which is also the exponential generating function for Bell numbers B_n that count the number of partitions of a set with n elements. But since

$$CAV_\tau(t) = e^{\sum_{m\geq 1} L_m^{cav}(\tau)\frac{t^m}{m!}}$$

for all $\tau \in S_3$, it must be the case that

$$\sum_{m\geq 1} L_m^{cav}(\tau)\frac{t^m}{m!} = e^t - 1$$

for all $\tau \in S_3$ and, hence also from Theorem 16,

$$CAV_\tau(t,x) = e^{x\sum_{m\geq 1} L_m^{cav}(\tau)\frac{t^m}{m!}} = e^{x(e^t-1)}$$

for all $\tau \in S_3$. However it is not the case that the generating functions $CAV_\tau(t,x,y)$ are equal for all $\tau \in S_3$ as stated before in Theorem 16. That is, suppose that α is a cyclic rearrangement of β. Then it is easy to see that $\mathcal{L}_m^{cav}(\alpha) = \mathcal{L}_m^{cav}(\beta)$ for all $m \geq 1$ so that

$$\sum_{C\in\mathcal{L}_m^{cav}(\alpha)} y^{\text{cdes}(C)} = \sum_{C\in\mathcal{L}_m^{cav}(\beta)} y^{\text{cdes}(C)}. \tag{2.1}$$

But then it follows from Theorem 11 that we must have $CAV_\alpha(t,x,y) = CAV_\beta(t,x,y)$. It thus follows from our results in the introduction that

$$CAV_{123}(t,x,y) = CAV_{312}(t,x,y) = CAV_{231}(t,x,y) = e^{x\left(yt+\frac{1}{y}(e^{yt}-1-yt)\right)}.$$

Next consider $\tau = 1\ 3\ 2$. It is easy to see that for $k \geq 3$, the only k-cycle which cycle avoids τ is the cycle $(1, 2, \ldots, k)$. Thus

$$\sum_{C\in\mathcal{L}_m^{cav}(132)} y^{\text{cdes}(C)} = y,$$

for all $k \geq 1$. Hence

$$CAV_{132}(t,x,y) = CAV_{213}(t,x,y) = CAV_{321}(t,x,y) = e^{x\left(\sum_{m\geq 1}\frac{yt^m}{m!}\right)} = e^{xy(e^t-1)}.$$

Next we shall consider the generating functions $NCM_\tau(t,x,y)$ for $\tau \in S_3$. We claim that is enough to compute $NCM_{123}(t,x,y)$ and $NCM_{132}(t,x,y)$. That is, for any $j \geq 2$ and $\tau \in S_j$, we can compute $NCM_{\tau^r}(t,x,y)$ and $NCM_{\tau^c}(t,x,y)$ from $NCM_\tau(t,x,y)$. Note that it follows from Theorem 11 that

$$\sum_{n\geq 1} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(C)} = \ln\left(NCM_\tau(t,1,y)\right). \tag{2.2}$$

Since $\sum_{C\in\mathcal{L}_1^{ncm}(\tau)} y^{\mathrm{cdes}(C)} = y$, it follows that

$$\sum_{n\geq 2} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(C)} = \ln\left(NCM_\tau(t,1,y)\right) - yt. \tag{2.3}$$

Given any n-cycle C in S_n, recall C^{cr} denotes its cycle-reverse and C^{cc} denotes its cycle-complement. Then $C \in \mathcal{L}_n^{ncm}(\tau)$ if and only if $C^{cr} \in \mathcal{L}_n^{ncm}(\tau^r)$ and $C \in \mathcal{L}_n^{ncm}(\tau)$ if and only if $C^{cc} \in \mathcal{L}_n^{ncm}(\tau^c)$. Now if $n \geq 2$, then it is easy to see that $n - \mathrm{cdes}(C) = \mathrm{cdes}(C^{cr}) = \mathrm{cdes}(C^{cc})$. That is, each descent as we read once around the cycle C becomes a rise as we read around the cycles of C^{cr} and C^{cc} and each rise as we read once around the cycle C becomes a descent as we read around the cycles of C^{cr} and C^{cc}. Note, however, that if C is a one-cycle, then $C^{cr} = C^{cc} = C$ and $\mathrm{cdes}(C) = \mathrm{cdes}(C^{cr}) = \mathrm{cdes}(C^{cc}) = 1$ so that it is not the case that $\mathrm{cdes}(C^{cr}) = \mathrm{cdes}(C^{cr}) = 1 - \mathrm{cdes}(C)$. Thus we have to treat the one-cycles separately. Thus we have that

$$\begin{aligned}\sum_{n\geq 2} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{n-\mathrm{cdes}(C)} &= \sum_{n\geq 2} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\tau^r)} y^{\mathrm{cdes}(C)} \\ &= \sum_{n\geq 2} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\tau^c)} y^{\mathrm{cdes}(C)}.\end{aligned}$$

It follows that if $\tau \in S_j$ where $j \geq 2$ and

$$G(t,y) = \sum_{n\geq 2} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(C)}, \tag{2.4}$$

then

$$G(ty,y^{-1}) = \sum_{n\geq 2} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\tau^r)} y^{\mathrm{cdes}(C)} = \sum_{n\geq 2} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\tau^c)} y^{\mathrm{cdes}(C)}. \tag{2.5}$$

Thus by (2.3), we have that

$$\begin{aligned}\ln\left(NCM_\tau(ty,1,y^{-1})\right) - t &= \sum_{n\geq 2}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau^r)} y^{\mathrm{cdes}(C)} \\ &= \sum_{n\geq 2}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau^c)} y^{\mathrm{cdes}(C)}\end{aligned}$$

so that

$$\begin{aligned}ty - t + \ln\left(NCM_\tau(ty,1,y^{-1})\right) &= \sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau^r)} y^{\mathrm{cdes}(C)} \\ &= \sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau^c)} y^{\mathrm{cdes}(C)}.\end{aligned}$$

Then we can apply Theorem 11 to obtain the following result.

Theorem 32. *Let $\tau \in S_j$ where $j \geq 2$. Then*

$$NCM_{\tau^r}(t,x,y) = NCM_{\tau^c}(t,x,y) = e^{x(yt-t+\ln(NCM_\tau(ty,1,y^{-1})))}. \tag{2.6}$$

Next we shall show that we can find an explicit expression $NCM_\tau(t,x,y)$ where $\tau = 1\ 2\ldots j$ for any $j \geq 3$ using some results of Mendes and Remmel [30]. Suppose that we want to compute the generating function

$$\begin{aligned}NCM_\tau(t,x,y) &= \sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NCM}_n(\tau)} x^{\mathrm{cyc}(\sigma)}y^{\mathrm{cdes}(\sigma)} \\ &= e^{x\sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(\sigma)}}\end{aligned} \tag{2.7}$$

in the case where τ starts with 1. Then by Corollary 14, we know that

$$NCM_\tau(t,x,y) = NM_\tau(t,x,y) = \sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NM}_n(\tau)} x^{\mathrm{LRMin}(\sigma)}y^{1+\mathrm{des}(\sigma)}. \tag{2.8}$$

Now suppose that we can compute

$$NM_\tau(t,1,y) = \sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NM}_n(\tau)} y^{1+\mathrm{des}(\sigma)}. \tag{2.9}$$

Then we know that

$$NM_\tau(t,1,y) = e^{\sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(\sigma)}}$$

so that

$$\sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(\sigma)} = \ln\left(NM_\tau(t,1,y)\right).$$

But then it follows that

$$NCM_\tau(t,x,y) = NM_\tau(t,x,y) = e^{x\ln(NM_\tau(t,1,y))}. \tag{2.10}$$

Thus we need only compute (2.9). However, Mendes and Remmel [30] proved the following theorem.

Theorem 17. ([30]) *If $\tau = j\ldots 2\ 1$ where $j\geq 2$, then*

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NM}_n(\tau)} y^{\mathrm{des}(\sigma)} = \left(\sum_{n\geq 0}\frac{t^n}{n!}\sum_{i\geq 0}(-1)^i\mathcal{R}_{n-1,i,j-1}y^i\right)^{-1} \tag{2.11}$$

where $\mathcal{R}_{n,i,j}$ is the number of rearrangements of i zeroes and $n-i$ ones such that j zeroes never appear consecutively.

Replacing y by $1/y$ and then replacing t by yt in (2.11) yields

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NM}_n(\tau)} y^{n-\mathrm{des}(\sigma)} = \left(\sum_{n\geq 0}\frac{t^n}{n!}\sum_{i\geq 0}(-1)^i\mathcal{R}_{n-1,i,j-1}y^{n-i}\right)^{-1}. \tag{2.12}$$

It is easy to see that if $\sigma\in S_n$ has no $j\ldots 2\ 1$-matches, then the reverse of σ, σ^r has no $1\ 2\ldots j$-matches and that $n-\mathrm{des}(\sigma)$ equals $1+\mathrm{des}(\sigma^r)$. Thus it follows that if $\alpha = 1\ 2\ldots j$, then

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NM}_n(\alpha)} y^{1+\mathrm{des}(\sigma)} = \left(\sum_{n\geq 0}\frac{t^n}{n!}\sum_{i\geq 0}(-1)^i\mathcal{R}_{n-1,i,j-1}y^{n-i}\right)^{-1}. \tag{2.13}$$

Thus we have the first part of Theorem 18.

Theorem 18. *For $j\geq 2$ and $\tau = 1\ 2\ldots j$,*

$$\begin{aligned} NCM_\tau(t,x,y) &= \sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NCM}_n(\tau)} x^{\mathrm{cyc}(\sigma)}y^{\mathrm{cdes}(\sigma)} \\ &= e^{x\ln\left(\frac{1}{\sum_{n\geq 0}\frac{t^n}{n!}\sum_{i\geq 0}(-1)^i\mathcal{R}_{n-1,i,j-1}y^{n-i}}\right)} \\ &= \left(\frac{1}{\sum_{n\geq 0}\frac{t^n}{n!}\sum_{i\geq 0}(-1)^i\mathcal{R}_{n-1,i,j-1}y^{n-i}}\right)^x. \end{aligned} \tag{2.14}$$

where $\mathcal{R}_{n,i,j}$ is the number of rearrangements of i zeroes and $n-i$ ones such that j zeroes never appear consecutively. In particular

$$NCM_{123}(t,x,y) = e^{\frac{xyt}{2}} \left(\frac{1}{\left(\cosh\left(\frac{t\sqrt{y^2-4y}}{2}\right) - \frac{\sqrt{y}}{\sqrt{y-4}}\sinh\left(\frac{t\sqrt{y^2-4y}}{2}\right)\right)} \right)^x. \quad (2.15)$$

For the second part of Theorem 18, in the case where $\tau = 1\ 2\ 3$, we can obtain an explicit formula $NCM_{123}(t,x,y)$ by another method. We start with a general observation. Suppose $\tau = \tau_1 \dots \tau_j \in S_j$ where $\tau_1 = 1$. We can write any n-cycle C in the form $C = (\alpha_1, \dots, \alpha_n)$ where $\alpha_1 = 1$. It is easy to see that the only cycle τ-match in C that can involve $\alpha_1 = 1$ is $\alpha_1\ \alpha_2 \dots \alpha_j$. This means that the only possible cycle τ-matches in C must be of the form $\alpha_i\ \alpha_{i+1} \dots \alpha_{i+j-1}$ where $i \leq n-j+1$. Thus the problem of finding n-cycles with no cycle τ-matches is equivalent to the problem of finding permutations $\sigma = \sigma_1 \dots \sigma_n$ where $\sigma_1 = 1$ and σ has no τ-matches. Let S_n^1 denote the set of all permutations $\sigma = \sigma_1 \dots \sigma_n \in S_n$ such that $\sigma_1 = 1$ and let $\mathcal{NM}_n^1(\tau) = S_n^1 \cap \mathcal{NM}_n(\tau)$ be the set of permutations of S_n^1 with no τ-matches. Then

$$A_{n,\tau}(y) = \sum_{\sigma \in \mathcal{NM}_n^1(\tau)} y^{1+\text{des}(\sigma)} = \sum_{C \in \mathcal{L}_n^{ncm}(\tau)} y^{\text{cdes}(C)}. \quad (2.16)$$

It turns out that in many cases we can find recurrences for $A_{n,\tau}(y)$ by classifying the permutations $\sigma = \sigma_1 \dots \sigma_n \in S_n$ such that $\sigma_1 = 1$ according to the position of 2 in σ. Let $\mathcal{E}_{n,k,\tau}$ denote the set of permutations $\sigma = \sigma_1 \dots \sigma_n \in \mathcal{NM}_n^1(\tau)$ such that $\sigma_k = 2$.

Now fix $\tau = 1\ 2\ 3$ and let $A_m(y) = A_{m,\tau}(y)$ and $\mathcal{E}_{n,k} = \mathcal{E}_{n,k,\tau}$. Our goal is to compute $A(t,y) = \sum_{m \geq 1} \frac{A_m(y)t^m}{m!}$. Now $A_1(y) = A_2(y) = y$ since the permutation 1 has no τ-matches, $1 + \text{des}(1) = 1$, the permutation 1 2 has no τ-matches, and $1+\text{des}(12) = 1$. There are two permutations in S_3 that start with 1, namely, 1 2 3 and 1 3 2 and only 1 3 2 has no τ-matches so that $A_3(y) = y^2$ since $1+\text{des}(132) = 2$. Now suppose that $n \geq 3$. Every permutation in $\mathcal{E}_{n,2}$ is of the form $1\ 2\ \sigma_3 \dots \sigma_n$. Clearly, $1\ 2\ \sigma_3$ is a 1 2 3-match so that the elements in $\mathcal{E}_{n,2}$ do not contribute to $A_n(y)$. For $3 \leq k \leq n$, the elements of the $\mathcal{E}_{n,k}$ are of the form

$$1\ \sigma_2 \dots \sigma_{k-1}\ 2\ \sigma_{k+1} \dots \sigma_n.$$

In such a case, the only way that 2 can be part of a 1 2 3-match is if the τ-match is 2 σ_{k+1} σ_{k+2}. It follows that an element of $\mathcal{E}_{n,k}$ contributes to $A_n(y)$ only if there is no τ-match in $\sigma_1 \ldots \sigma_{k-1}$ and there is no τ-match in 2 $\sigma_{k+1} \ldots \sigma_n$. Note that since $(\sigma_{k-1}, 2)$ is a descent pair,

$$1 + \mathrm{des}(1\ \sigma_2 \ldots \sigma_{k-1}\ 2\ \sigma_{k+1} \ldots \sigma_n) = 1 + \mathrm{des}(1\ \sigma_2 \ldots \sigma_{k-1}) + 1 + \mathrm{des}(2\ \sigma_{k+1} \ldots \sigma_n).$$

Hence the contribution of $\mathcal{E}_{n,k}$ to $A_n(y)$ is just $\binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y)$ since there are $\binom{n-2}{k-2}$ ways to choose the elements which make up $\sigma_2, \ldots, \sigma_{k-1}$.

It then follows that for $n \geq 3$,

$$A_n(y) = \sum_{k=3}^{n} \binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y). \tag{2.17}$$

Multiplying both sides of (2.17) by $\frac{t^{n-2}}{(n-2)!}$ and summing for $n \geq 3$, we see that

$$\begin{aligned} \frac{\partial^2 A(t,y)}{\partial t^2} - y &= \sum_{n \geq 3} t^{n-2} \sum_{k=3}^{n} \frac{A_{k-1}(y)}{(k-2)!} \frac{A_{n-k+1}(y)}{(n-k)!} \\ &= \frac{\partial A(t,y)}{\partial t} \left(\frac{\partial A(t,y)}{\partial t} - y \right). \end{aligned}$$

Thus thinking of $A(t,y)$ as a function of t, we see that $A(t,y)$ satisfies the differential equation

$$A''(t,y) - (A'(t,y))^2 + yA'(t,y) - y = 0 \tag{2.18}$$

where $A(0,y) = 0$ and $A'(0,y) = y$. If we let $A(y,t) = -\ln(U(t,y))$, then thinking of $U(t,y)$ as a function of t, it follows that

$$\begin{aligned} A'(t,y) &= -\frac{U'(t,y)}{U(t,y)} \text{ and} \\ A''(t,y) &= -\frac{U''(t,y)U(t,y) - (U'(t,y))^2}{(U(t,y))^2} = -\frac{U''(t,y)}{U(t,y)} + \left(\frac{U'(t,y)}{U(t,y)}\right)^2. \end{aligned}$$

Substituting these expressions into (2.18), one can easily show that $U(t,y)$ satisfies the differential equation

$$U''(t,y) + yU'(t,y) + yU(t,y) = 0 \tag{2.19}$$

where $U(0,y) = 1$ and $U'(0,y) = -y$. One can use Mathematica to solve this differential equation to conclude that

$$U(t,y) = e^{-\frac{yt}{2}} \left(\cosh\left(\frac{t\sqrt{y^2-4y}}{2}\right) - \frac{\sqrt{y}}{\sqrt{y-4}} \sinh\left(\frac{t\sqrt{y^2-4y}}{2}\right)\right)$$

so that

$$A(t,y) = -\ln\left(e^{-\frac{yt}{2}} \cosh\left(\frac{t\sqrt{y^2-4y}}{2}\right) - \frac{\sqrt{y}}{\sqrt{y-4}} \sinh\left(\frac{t\sqrt{y^2-4y}}{2}\right)\right). \quad (2.20)$$

We can then apply Theorem 11 to obtain the second part of Theorem 18.

$$NCM_{123}(t,x,y) = e^{\frac{xyt}{2}} \left(\frac{1}{\left(\cosh\left(\frac{t\sqrt{y^2-4y}}{2}\right) - \frac{\sqrt{y}}{\sqrt{y-4}} \sinh\left(\frac{t\sqrt{y^2-4y}}{2}\right)\right)}\right)^x. \quad (2.21)$$

One can use Mathematica to compute that

$$\begin{aligned} A(y,t) \quad = \quad & yt + y\frac{t^2}{2!} + y^2\frac{t^3}{3!} + (2y^2+y^3)\frac{t^4}{4!} + \\ & (8y^3+y^4)\frac{t^5}{5!} + (16y^3+22y^4+y^5)\frac{t^6}{6!} \\ & (136y^4+52y^5+y^6)\frac{t^7}{7!} + (272y^4+720y^5+114y^6+y^7)\frac{t^8}{8!} + \\ & (3968y^5+3072y^6+240y^7+y^8)\frac{t^9}{9!} + \cdots \end{aligned}$$

and

$$\begin{aligned} & NCM(t,x,y) = 1 + xyt + xy(1+xy)\frac{t^2}{2!} + xy^2(1+3x+x^2y)\frac{t^3}{3!} + \\ & xy^2(2+3x+y+4xy+6x^2y+x^3y^2)\frac{t^4}{4!} + \\ & xy^3(8+20x+15x^2+y+5xy+10x^2y+10x^3y+x^4y^2)\frac{t^5}{5!} + \\ & xy^3\left(16+30x+15x^2+22y+73xy+90x^2y+45x^3y+y^2+\right. \\ & \left. 6xy^2+15x^2y^2+20x^3y^2+15x^4y^2+x^5y^3\right)\frac{t^6}{6!} + \\ & xy^4\left(136+350x+315x^2+105x^3+52y+210xy+343x^2y+280x^3y+\right. \\ & \left. 105x^4y+y^2+7xy^2+21x^2y^2+35x^3y^2+35x^4y^2+21x^5y^2+x^6y^3\right)\frac{t^7}{7!} + \cdots \end{aligned}$$

We note that the sequence $\{A_n(t)\}_{n\geq 1}$ starts out

$$1, 1, 1, 3, 9, 39, 189, 1107, 7281, \ldots$$

which is sequence A080635 in the OIES and counts the number of permutations in S_n without double falls and without an initial fall. Bergeron, Flajolet, and Salvy [4] showed that the exponential generating function of this sequence which starts $1 + t + \frac{t^2}{2} + \cdots$ is

$$\frac{1}{2}\left(1+\sqrt{3}\tan\left(\frac{1}{2}\left(\frac{\pi}{3}+\sqrt{3}t\right)\right)\right).$$

Thus our generating function $A(y, t)$ which starts at t can be viewed as a refinement of the result of Begeron, Flajolet, and Salvy.

We can use the same method to compute $NCM_{132}(t, x, y)$. In this case, we will directly compute

$$B(t,y) = \sum_{m\geq 1} \frac{t^m}{m!} \sum_{C \in \mathcal{L}^{ncm}_m(132)} y^{\text{cdes}(C)}. \tag{2.22}$$

Let $B_m(y) = B_{m,132}(y)$ and $\mathcal{E}_{n,k} = \mathcal{E}_{n,k,132}$. Our goal is to compute $B(t, y) = \sum_{m\geq 1} \frac{B_m(y)t^m}{m!}$. Now $B_1(y) = B_2(y) = y$ since the permutation 1 has no τ-matches, $1 + \text{des}(1) = 1$, the permutation 1 2 has no τ-matches, and $1 + \text{des}(12) = 1$. There are two permutations in S_3 that start with 1, namely, 1 2 3 and 1 3 2, and only 1 2 3 has no τ-matches so that $B_3(y) = y$ since $1 + \text{des}(123) = 1$. Now suppose that $n \geq 4$. Every permutation in $\mathcal{E}_{n,2}$ is of the form $1\ 2\ \sigma_3 \ldots \sigma_n$. Clearly, the only τ-matches must be of the form $\sigma_i\ \sigma_{i+1}\ \sigma_{i+2}$ where $i \geq 2$ so that $\mathcal{E}_{n,2}$ contributes $B_{n-1}(y)$ to $B_n(y)$. Every permutation in $\mathcal{E}_{n,3}$ is of the form $1\ \sigma_2\ 2 \ldots \sigma_n$ where $\sigma_2 \geq 3$. Thus all such permutations have a τ-match so that $\mathcal{E}_{n,3}$ contributes nothing to $B_n(y)$. For $4 \leq k \leq n$, the elements of the $\mathcal{E}_{n,k}$ are of the form

$$1\ \sigma_2 \ldots \sigma_{k-1}\ 2\ \sigma_{k+1} \ldots \sigma_n.$$

In such a case, the only way that 2 can be part of a τ-match is if the τ-match is $2\ \sigma_{k+1}\ \sigma_{k+2}$. It follows that an element of $\mathcal{E}_{n,k}$ contributes to $B_n(y)$ only if there is no τ-match in $\sigma_1 \ldots \sigma_{k-1}$ and there is no τ-match in $2\ \sigma_{k+1} \ldots \sigma_n$. Note that since $(\sigma_{k-1}, 2)$ is a descent pair,

$$1 + \text{des}(1\ \sigma_2 \ldots \sigma_{k-1}\ 2\ \sigma_{k+1} \ldots \sigma_n) = 1 + \text{des}(1\ \sigma_2 \ldots \sigma_{k-1}) + 1 + \text{des}(2\ \sigma_{k+1} \ldots \sigma_n).$$

Hence the contribution of $\mathcal{E}_{n,k}$ to $B_n(y)$ is just $\binom{n-2}{k-2}B_{k-1}(y)B_{n-k+1}(y)$ since there are $\binom{n-2}{k-2}$ ways to choose the elements which make up $\sigma_2, \ldots, \sigma_{k-1}$. Thus for $n \geq 4$,

$$B_n(y) = B_{n-1}(y) + \sum_{k=4}^{n} \binom{n-2}{k-2} B_{k-1}(y) B_{n-k+1}(y). \tag{2.23}$$

Dividing both sides of (2.23) by $(n-2)!$, we obtain that for all $n \geq 4$,

$$\frac{B_n(y)}{(n-2)!} = \frac{B_{n-1}(y)}{(n-2)!} + \sum_{k=2}^{n-2} \frac{B_{k+1}(y)}{k!} \frac{B_{n-k-1}(y)}{(n-2-k)!}. \tag{2.24}$$

If we multiply both sides of (2.24) by t^{n-2} and sum, we obtain the differential equation

$$\frac{\partial^2 B(t,y)}{\partial t^2} - y - yt = \frac{\partial B(t,y)}{\partial t} - y - yt + \left(\frac{\partial B(t,y)}{\partial t} - y - yt \right) \frac{\partial B(t,y)}{\partial t}$$

Let $B'(t,y) = \frac{\partial B(t,y)}{\partial t}$, then $B(t,y)$ satisfies the differential equation

$$B''(t,y) = B'(t,y)(1 - y - yt + B'(t,y)) \tag{2.25}$$

with initial conditions $B_0(y) = 0$ and $B_1(y) = y$. One can check that the solution to (2.25) is

$$B(t,y) = \ln \left(\frac{1}{1 - y \int_0^t e^{(1-y)s - ys^2/2} ds} \right). \tag{2.26}$$

Hence

$$\begin{aligned} L_{132}(t,y) &= \sum_{m \geq 1} \frac{t^m}{m!} \sum_{C \in \mathcal{L}_m^{ncm}(132)} y^{\mathrm{cdes}(C)}. \\ &= \ln \left(\frac{1}{1 - y \int_0^t e^{(1-y)s - ys^2/2} ds} \right). \end{aligned} \tag{2.27}$$

We can then apply Theorem 11 to obtain the following theorem.

Theorem 19.

$$\begin{aligned} NCM_{132}(t,x,y) &= e^{x \ln \left(\frac{1}{1 - y \int_0^t e^{(1-y)s - ys^2/2} ds} \right)} \\ &= \frac{1}{\left(1 - y \int_0^t e^{(1-y)s - ys^2/2} ds \right)^x}. \end{aligned} \tag{2.28}$$

Table 2.1: Coefficients for GFs involving 132

n	$L_n^{ncm}(132)$	$NCM_n(132)$
1	1	1
2	1	2
3	1	5
4	2	16
5	7	63
6	28	296
7	131	1623
8	720	10176
9	4513	71793
10	31824	562848

We note that specialization

$$NCM_{132}(t,1,1) = \frac{1}{1 - \int_0^t e^{-s^2/2} ds}$$

has been proved by Elizalde and Noy [12].

One can use our generating functions for $NCM_{132}(t,x,y)$ to compute the initial values of $L_n^{ncm}(132)$ and $NCM_n(132)$.

If one looks in the OEIS [37], then both the sequences for $L_n^{ncm}(132)$ and $NCM_n(132)$ occur. The sequence for $L_n^{ncm}(132)$ is sequence A052319 which counts the number of increasing rooted trimmed trees with n nodes. Here an increasing tree is a tree labeled with $1,\ldots,n$ where the numbers increase as you move away from the root. A tree with a forbidden limb of length k is a tree where the path from any leaf inward hits a branching node or another leaf within k steps. A trimmed tree is a tree with a forbidden limb of length 2. The sequence for $NCM_n(132)$ is the number of permutations that have no 1 3 2-matches as expected.

We end this section with some results on $CAV_\Upsilon(t,x,y)$ and $NCM_\Upsilon(t,x,y)$ where $\Upsilon \subseteq S_3$. For certain Υ's, this problem is uninteresting. For example, if Υ contains both 1 2 3 and 1 3 2, then any k-cycle $C = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ where $\sigma_1 = 1$ and $k \geq 3$ will have a cycle Υ-match since $\sigma_1\ \sigma_2\ \sigma_3$ must be either a cycle 1 2 3-match or a cycle 1 3 2-match. Thus in this case $\mathcal{L}_1^{ca}(\Upsilon) = \mathcal{L}_1^{ncm}(\Upsilon) = \{(1)\}$, $\mathcal{L}_2^{ca}(\Upsilon) = \mathcal{L}_2^{ncm}(\Upsilon) = \{(1,2)\}$, and $\mathcal{L}_k^{ca}(\Upsilon) = \mathcal{L}_k^{ncm}(\Upsilon) = \emptyset$ for $k \geq 3$. It then follows from Theorem 11 that

$$CAV_{\Upsilon}(t,x,y) = NCM_{\Upsilon}(t,x,y) = e^{x\left(yt+\frac{yt^2}{2}\right)}.$$

Similarly, suppose that $\Upsilon = \{123, 213\}$. Then we claim that $\mathcal{L}_k^{ncm}(\Upsilon) = \emptyset$ for $k \geq 3$. That is, for $k \geq 3$, a k-cycle $C = (1, c_2, \ldots, c_k)$ has no cycle $\{123, 213\}$-matches. Then consider the possible positions for k in $c_2, \ldots, c_k$. Clearly, we can not have $k = c_2$ since then c_k 1 k would be a cycle match of 2 1 3. We can not have $k = c_3$ since then 1 c_2 k would be a cycle match of 1 2 3. Now suppose that $k = c_i$ where $i > 3$. Then we either have (i) $c_{i-2} < c_{i-1} < k$ or (ii) $c_{i-2} > c_{i-1} < k$. But in case (i), C would contain a cycle 1 2 3-match and in case (ii), C would contain a cycle 2 1 3-match. Thus such a C can not exist and we can conclude that

$$NCM_{\{123,213\}}(t,x,y) = e^{x\left(yt+\frac{yt^2}{2}\right)}.$$

A more interesting case is when $\Upsilon = \{123, 321\}$. First observe that since any cycle contains a cycle occurrence of 1 3 2 if and only if it contains a cycle occurrence of 3 2 1, then it is the case that any k-cycle C where $k \geq 3$ must have a cycle occurrence of either 1 2 3 or 3 2 1. Thus

$$CAV_{\Upsilon}(t,x,y) = e^{x\left(yt+\frac{yt^2}{2}\right)}.$$

Next consider the case of computing $NCM_{\Upsilon}(t,x,y)$. Let $C = (\sigma_1, \ldots, \sigma_n)$ be an n-cycle such that $\sigma_1 = 1$. If $n \geq 3$, then we must have $\sigma_2 > \sigma_3$ since otherwise there will be a cycle 1 2 3-match. But then we must have $\sigma_3 < \sigma_4$ since otherwise there would be cycle 3 2 1-match. Continuing on in this way, we see that $\sigma_2 \ldots \sigma_n$ must be an alternating permutation. That is, we must have

$$\sigma_2 > \sigma_3 < \sigma_4 > \sigma_5 < \sigma_6 > \sigma_7 \cdots .$$

However, this means that if $n = 2k + 1 \geq 3$, then there are no n cycles which have no cycle Υ-matches since we are forced to have $\sigma_{2k} > \sigma_{2k+1} > \sigma_1$ which is a cycle 3 2 1-match. If $n = 2k$ and $\sigma_2 \ldots \sigma_n$ is alternating, then C will have no cycle Υ-matches. For such σ it is easy to see that $1 + \mathrm{des}(\sigma) = k$. Thus in this case, $L_{2k+1}^{ncm}(\Upsilon) = 0$ for $k \geq 1$ and $L_{2k}^{ncm}(\Upsilon)$ is just the number of odd alternating permutations of length $2k-1$ for $k \geq 1$. Deutsch and Elizalde [9] called n-cycles $(1, \sigma_2, \ldots, \sigma_n)$ such that

$$\sigma_2 > \sigma_3 < \sigma_4 > \sigma_5 < \sigma_6 > \sigma_7 \cdots ,$$

cycle up-down permutations. One can follow the methods of [9] to find an explicit formula for $NCM_\Upsilon(t,x,y)$. That is, we let Alt_n denote the number of down-up permutations of length n, then André [1, 2] proved that

$$\sum_{n\geq 0} Alt_{2n+1} \frac{t^{2n+1}}{(2n+1)!} = \frac{\sin(t)}{\cos(t)}. \tag{2.29}$$

Thus

$$\begin{aligned} \sum_{n\geq 1} L_{2n}^{ncm}(\Upsilon)\frac{t^{2n}}{(2n)!} &= \sum_{n\geq 1} Alt_{2n-1}\frac{t^{2n}}{(2n)!} \\ &= \int_0^t \frac{\sin(z)}{\cos(z)} dz = -\ln|\cos(t)|. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} \sum_{C\in\mathcal{L}_{2n}^{ncm}(\Upsilon)} y^{\mathrm{cdes}(C)} &= \sum_{n\geq 1} y^n L_{2n}^{ncm}(\Upsilon)\frac{t^{2n}}{(2n)!} \\ &= -\ln|\cos(t\sqrt{y})|. \end{aligned}$$

and

$$\sum_{n\geq 1} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\Upsilon)} y^{\mathrm{cdes}(C)} = ty - \ln|\cos(t\sqrt{y})|. \tag{2.30}$$

Thus it follows from Theorem 11

Theorem 20.

$$NCM_\Upsilon(t,x,y) = e^{x(ty-\ln|\cos(t\sqrt{y})|)} = \frac{e^{xyt}}{\cos(t\sqrt{y})^x} = e^{xyt}\sec(t\sqrt{y})^x.$$

We end section with another non-trivial example which is the case where $\Gamma = \{123, 231\}$. Let

$$G_n(y) = \sum_{C\in\mathcal{L}_n^{ncm}(\Gamma)} y^{\mathrm{cdes}(C)} \tag{2.31}$$

and

$$G(t,y) = \sum_{n\geq 1} G_n(y)\frac{t^n}{n!}. \tag{2.32}$$

Note that $\mathcal{L}_1^{ncm}(\Gamma) = \{(1)\}$ and $\mathcal{L}_2^{ncm}(\Gamma) = \{(1,2)\}$ so that $G_1(y) = G_2(y) = y$.

Now suppose that $n \geq 3$ and $C = (1, \sigma_2, \ldots, \sigma_n)$ is an n-cycle in S_n which has no cycle Γ-matches. Then it cannot be the case that $\sigma_{n-1} < \sigma_n$ since otherwise $\sigma_{n-1}\ \sigma_n\ 1$ would be a cycle 2 3 1-match in C. Thus it must be the case that $\sigma_{n-1} > \sigma_n$. It cannot be that $\sigma_i = 2$ since then $1\ \sigma_2\ \sigma_3$ would be a cycle 1 2 3-match in C and it cannot be that $\sigma_{n-1} = 2$ since then $\sigma_{n-1}\ \sigma_n\ 1$ would be a cycle 2 3 1-match in C. If $\sigma_n = 2$, it easy to see that $C = (1, \sigma_2, \ldots, \sigma_{n-1}, 2)$ has no cycle Γ-matches if and only if $C' = (1, \sigma_2 - 1, \ldots, \sigma_{n-1} - 1)$ has no cycle Γ-matches. Note $\text{cdes}(C) = \text{cdes}(C') + 1$ so that the n-cycles of the form $C = (1, \sigma_2, \ldots, \sigma_{n-1}, 2)$ contribute $yG_{n-1}(y)$ to $G_n(y)$. Thus consider the cases where $\sigma_k = 2$ where $3 \leq k \leq n-2$. We claim that it must be the case that neither $(1, \sigma_2, \ldots, \sigma_{k-1})$ nor $(2, \sigma_{k+1}, \ldots, \sigma_n)$ have any cycle Γ-matches. That is, it is easy to see that the only possible cycle Γ-match in $(1, \sigma_2, \ldots, \sigma_{k-1})$ that does not occur in C is if $k-1 \geq 3$ and $\sigma_{k-2}\ \sigma_{k-1}\ 1$ is a cycle 2 3 1-match. But in that case, $\sigma_{k-2}, \sigma_{k-1} > 2$ so that $\sigma_{k-2}\ \sigma_{k-1}\ 2$ would have been a cycle 2 3 1-match in C. Similarly, the only possible cycle Γ-match in $(2, \sigma_{k+1}, \ldots, \sigma_n)$ that does not occur in C is if $\sigma_{n-1}\ \sigma_n\ 2$ is cycle 2 3 1-match. But in that case, $\sigma_{n-1}\sigma_n 1$ would have been a cycle 231-match in C. Vice versa, it is easy to see that if $\sigma_k = 2$ where $3 \leq k \leq n-2$ and neither $(1, \sigma_2, \ldots, \sigma_{k-1})$ nor $(2, \sigma_{k+1}, \ldots, \sigma_n)$ have any cycle Γ-matches, then C does not have any cycle Γ-matches. That is, the only possible cycle Γ-match in C that does not occur in either $(1, \sigma_2, \ldots, \sigma_{k-1})$ nor $(2, \sigma_{k+1}, \ldots, \sigma_n)$ is if $\sigma_{k-2}\ \sigma_{k-1}\ 2$ is a cycle 2 3 1-match. This is not possible if $k = 3$ since in that case $\sigma_{k-2} = 1$. Similarly if $3 < k \leq n-2$ and $\sigma_{k-2}\ \sigma_{k-1}\ 2$ is a cycle 2 3 1-match in C, then $\sigma_{k-2}\ \sigma_{k-1}\ 1$ would be a cycle 2 3 1-match in $(1, \sigma_2, \ldots, \sigma_{k-1})$. Note that it is also the case that

$$\text{cdes}((1, \sigma_2, \ldots, \sigma_{k-1})) + \text{cdes}((2, \sigma_{k+1}, \ldots, \sigma_n)) = \text{cdes}(C).$$

Thus for $3 \leq k \leq n-2$, the cycles of the form $C = (1, \sigma_2, \ldots, \sigma_n)$ where $\sigma_k = 2$ contribute $\binom{n-2}{k-2} G_{k-1}(y) G_{n-k+1}(y)$ to $G_n(y)$ since we have $\binom{n-2}{k-2}$ ways to choose the elements $\sigma_2, \ldots, \sigma_{k-1}$.

It then follows that for $n \geq 3$,

$$G_n(y) = yG_{n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} G_{k-1}(y)G_{n-k+1}(y). \tag{2.33}$$

Multiplying both sides of (2.33) by $\frac{t^{n-2}}{(n-2)!}$ and summing for $n \geq 3$, we see that

$$\begin{aligned}
\frac{\partial^2 G(t,y)}{\partial t^2} - y &= \sum_{n\geq 3} G_n(t,y)\frac{t^{n-2}}{(n-2)!} \\
&= y\sum_{n\geq 3} G_{n-1}(y)\frac{t^{n-2}}{(n-2)!} + \sum_{n\geq 3} t^{n-2} \sum_{i=3}^{n} \frac{G_{i-1}(y)}{(i-2)!}\frac{G_{n-i+1}(y)}{(n-2)!} \\
& y\left(\frac{\partial G(t,y)}{\partial t} - y\right) + \left(\frac{\partial G(t,y)}{\partial t} - y\right)\left(\frac{\partial G(t,y)}{\partial t} - y - yt\right).
\end{aligned}$$

Thus thinking of $G(t, y)$ as a function of t, we see that $G(t, y)$ satisfies the differential equation

$$G''(t,y) - (G'(t,y))^2 + (y + yt)yG'(t,y) - (y + y^2t) = 0 \tag{2.34}$$

where $G(0, y) = 0$ and $G'(0, y) = y$. If we let $G(y, t) = -\ln(U(t, y))$, then thinking of $U(t, y)$ as a function of t, one can easily show that $U(t, y)$ satisfies the differential equation

$$U''(t,y) + (y + yt)U'(t,y) + (y + y^2t)U(t,y) = 0 \tag{2.35}$$

where $U(0, y) = 1$ and $U'(0, y) = -y$. We used Mathematica to solve this differential equation which gave the formula

$$U(t,y) = e^{-\frac{yt^2}{2}}\left(1 - \frac{e^{-\frac{y}{2}}\sqrt{2\pi y}}{2}\mathrm{erfi}\left(\sqrt{\frac{y}{2}}\right) - \frac{e^{-\frac{y}{2}}\sqrt{2\pi y}}{2}\mathrm{erfi}\left((t-1)\sqrt{\frac{y}{2}}\right)\right)$$

so that

$$G(t,y) = -ln\left(e^{-\frac{yt^2}{2}}\left(1 - \frac{e^{-\frac{y}{2}}\sqrt{2\pi y}}{2}\mathrm{erfi}\left(\sqrt{\frac{y}{2}}\right) - \frac{e^{-\frac{y}{2}}\sqrt{2\pi y}}{2}\mathrm{erfi}\left((t-1)\sqrt{\frac{y}{2}}\right)\right)\right) \tag{2.36}$$

where erfi(z) is imaginary error function defined by the series

$$\mathrm{erfi}(z) = \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{z^{2k+1}}{k!(2k+1)}.$$

We can then apply Theorem 11 to obtain the following theorem.

Theorem 21.

$$NCM_{\{123,231\}}(t,x,y) = e^{xG(t,y)} \tag{2.37}$$
$$= e^{\frac{xyt^2}{2}} \left(\frac{1}{\left(1 - \frac{e^{-\frac{y}{2}}\sqrt{2\pi y}}{2}\text{erfi}\left(\sqrt{\frac{y}{2}}\right) - \frac{e^{-\frac{y}{2}}\sqrt{2\pi y}}{2}\text{erfi}\left((t-1)\sqrt{\frac{y}{2}}\right)\right)} \right)^x.$$

One can use Mathematica to compute that

$$\begin{aligned} G(y,t) &= yt + y\frac{t^2}{2!} + y^2\frac{t^3}{3!} + (y^3)\frac{t^4}{4!} + \\ &(3y^3+y^4)\frac{t^5}{5!} + (13y^4+y^5)\frac{t^6}{6!} + (15y^4+38y^5+y^6)\frac{t^7}{7!} + \\ &(183y^5+94y^6+y^7)\frac{t^8}{8!} + (105y^5+1205y^6+213y^7+y^8)\frac{t^9}{9!} + \cdots \end{aligned}$$

and

$$\begin{aligned} &NCM_{\{123,231\}}(t,x,y) = 1 + xyt + xy(1+xy)\frac{t^2}{2!} + xy^2(1+3x+x^2y)\frac{t^3}{3!} + \\ &xy^2(3x+y+4xy+6x^2y+x^3y^2)\frac{t^4}{4!} + \\ &xy^3(3+10x+15x^2++y+5xy+10x^2y+10x^3y+x^4y^2)\frac{t^5}{5!} + \\ &xy^3(15x^2+13y+43xy+60x^2y+45x^3y+y^2+6xy^2+15x^2y^2+20x^3y^2+ \\ &15x^4y^2+x^5y^3)\frac{t^6}{6!} + \\ &xy^4\left(15+63x+105x^2+105x^3+38y+147xy+238x^2y+210x^3y+ \right.\\ &\left.105x^4y+y^2+7xy^2+21x^2y^2+35x^3y^2+35x^4y^2+21x^5y^2+x^6y^3\right)\frac{t^7}{7!} + \cdots \end{aligned}$$

Neither the sequences $\{G_n(1)\}_{n\geq 1}$ nor the sequences $\{NCM_n(\{123,231\})\}_{n\geq 1}$ appear in the OEIS.

2.2 General results

In this section, we shall describe how we can compute $NCM_\tau(t,x,y)$ for certain general classes of permutations τ. We start by considering permutations $\tau = \tau_1 \ldots \tau_j$ where $\tau_1 = 1$ and $\tau_j = 2$. In that case, we have the following theorem.

Theorem 22. *Let* $\tau = \tau_1 \ldots \tau_j \in S_j$ *where* $j \geq 3$ *and* $\tau_1 = 1$ *and* $\tau_j = 2$. *Then*

$$NCM_\tau(t,x,y) = \frac{1}{\left(1 - \int_0^t e^{(y-1)s - \frac{y^{\mathrm{des}(\tau)} s^{j-1}}{(j-1)!}} ds\right)^x} \tag{2.38}$$

Proof. Note that in the special case where $j = 3$, the only permutation satisfying the hypothesis of the theorem is $\tau = 1\ 3\ 2$. In that special case, the result follows from Theorem 19. Thus assume that we fix a $\tau = \tau_1 \ldots \tau_j \in S_j$ where $\tau_1 = 1$ and $\tau_j = 2$ and $j \geq 4$.

Our first goal is to compute

$$A(t,y) = \sum_{n \geq 1} A_n(y) \frac{t^n}{n!} \tag{2.39}$$

where $A_n(y) = \sum_{\sigma \in \mathcal{NM}_n^1(\tau)} y^{\mathrm{des}(\sigma)+1}$. Now it is easy to see that

$$A_n(y) = \sum_{\sigma \in S_n^1} y^{\mathrm{des}(\sigma)+1}$$

for $1 \leq n \leq j-1$. Thus

$$\begin{aligned} A(t,y) &= yt + y\frac{t^2}{2} + (y+y^2)\frac{t^3}{3!} + \cdots \\ \frac{\partial A(t,y)}{\partial t} &= y + yt + (y+y^2)\frac{t^2}{2!} + \cdots \text{ and} \\ \frac{\partial^2 A(t,y)}{\partial t^2} &= y + (y+y^2)t + \cdots . \end{aligned}$$

For $n \geq j$, we shall prove a recursive formula for $A_n(y)$. We consider three cases for $\sigma = \sigma_1 \ldots \sigma_n \in \mathcal{NM}_n^1(\tau)$ depending on the position of 2 in σ.

Case 1. $\sigma_2 = 2$.

In this case because $j \geq 4$, the only possible τ-matches in σ must occur in $\sigma_2 \ldots \sigma_n$. Since $\mathrm{des}(\sigma) + 1 = \mathrm{des}(\sigma_2 \ldots \sigma_n) + 1$, it follows that the contribution of the permutations in this case to $A_n(y)$ is just $A_{n-1}(y)$.

Case 2. $\sigma_k = 2$ where $k \notin \{2, j\}$.

In this case, we have $\binom{n-2}{k-2}$ ways to choose the elements D_k that will constitute

$\sigma_2 \ldots \sigma_{k-1}$. Once we have chosen D_k, we have to consider the ways in which we can arrange the elements of D_k to form $\sigma_2 \ldots \sigma_{k-1}$ and the ways that we can arrange $[n] - (D_k \cup \{1,2\})$ to form $\sigma_{k+1} \ldots \sigma_n$ so that

$$\sigma = 1\ \sigma_2 \ldots \sigma_{k-1}\ 2\ \sigma_{k+1} \ldots \sigma_n \tag{2.40}$$

has no τ-matches. However it is easy to see that since $k \notin \{2, j\}$ the only τ-matches for σ of the form (2.40) can occur either entirely in $1\ \sigma_2 \ldots \sigma_{k-1}$ or entirely in $2\ \sigma_{k+1} \ldots \sigma_n$. Moreover it is the case that

$$\mathrm{des}(\sigma) + 1 = \mathrm{des}(1\ \sigma_2 \ldots \sigma_{k-1}) + 1 + \mathrm{des}(2\ \sigma_{k+1} \ldots \sigma_n) + 1$$

since $\sigma_{k-1} > 2$. Thus the contribution to $A_n(y)$ of the permutations in this case is

$$\binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y).$$

Case 3. $\sigma_j = 2$.

In this case, we have $\binom{n-2}{j-2}$ ways to choose the elements D_j that will constitute $\sigma_2 \ldots \sigma_{j-1}$. Once we have chosen D_j, we have to consider the ways in which we can arrange the elements of D_j to form $\sigma_2 \ldots \sigma_{j-1}$ and we can arrange $[n]-(D_j \cup \{1,2\})$ to form $\sigma_{j+1} \ldots \sigma_n$ so that

$$\sigma = 1\ \sigma_2 \ldots \sigma_{j-1}\ 2\ \sigma_{j+1} \ldots \sigma_n \tag{2.41}$$

has no τ-matches. Unlike Case 2, it is not enough just to ensure that $1\ \sigma_2 \ldots \sigma_{j-1}$ and $2\ \sigma_{j+1} \ldots \sigma_n$ have no τ-matches. That is, we must also ensure that

$$\mathrm{red}(\sigma_2 \ldots \sigma_{j-1}) \neq \mathrm{red}(\tau_2 \ldots \tau_{j-1})$$

since otherwise $1\ \sigma_2 \ldots \sigma_{j-1}\ 2$ would be a τ-match. Note that in such a situation $\mathrm{des}(1\ \sigma_2 \ldots \sigma_{j-1}) + 1 = \mathrm{des}(\tau)$. Thus the contributions to $A_n(y)$ of the permutations in this case is

$$\binom{n-2}{j-2} (A_{j-1}(y) - y^{\mathrm{des}(\tau)}) A_{n-j+1}(y).$$

It follows that for $n \geq j$,

$$A_n(y) = A_{n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y) - \binom{n-2}{j-2} y^{\mathrm{des}(\tau)} A_{n-j+1}(y) \tag{2.42}$$

or, equivalently,

$$\frac{A_n(y)}{(n-2)!} = \frac{A_{n-1}(y)}{(n-2)!} + \left(\sum_{k=3}^{n} \frac{A_{k-1}(y)}{(k-2)!} \frac{A_{n-k+1}(y)}{(n-k)!} \right) - \frac{y^{\mathrm{des}(\tau)}}{(j-2)!} \frac{A_{n-j+1}(y)}{(n-j)!}. \tag{2.43}$$

For any formal power series $f(t) = \sum_{n \geq 1} f_n t^n$, let $f(t)|_{t \leq j}$ denote $f_0 + f_1 t + \cdots + f_j t^j$. Multiplying both sides of (2.43) by t^{n-2} and summing, we obtain the differential equation where $A'(t,y) = \frac{\partial A(t,y)}{\partial t}$

$$\begin{aligned} &A''(t,y) - A''(t,y)|_{t \leq j-3} \\ &= A'(t,y) - A'(t,y)|_{t \leq j-3} + \\ &\quad (A'(t,y) - y)\, A'(t,y) - \left((A'(t,y) - y)\, A'(t,y) \right)|_{t \leq j-3} - \frac{y^{\mathrm{des}(\tau)}}{(j-2)!} A'(t,y). \end{aligned}$$

Thus

$$\begin{aligned} A''(t,y) &= (1 - y - y^{\mathrm{des}(\tau)}) A'(t,y) + (A'(t,y))^2 + \\ &\quad (A''(t,y)|_{t \leq j-3}) - (A'(t,y)|_{t \leq j-3}) - \left((A'(t,y) - y)\, A'(t,y) \right)|_{t \leq j-3}. \end{aligned}$$

We claim that

$$0 = (A''(t,y)|_{t \leq j-3}) - \left(\frac{\partial A(t,y)}{\partial t}|_{t \leq j-3} \right) - \left(\left(\frac{\partial A(t,y)}{\partial t} - y \right) \frac{\partial A(t,y)}{\partial t} \right)|_{t \leq j-3}$$

or, equivalently, that

$$A''(t,y)|_{t \leq j-3} = \left(A'(t,y) + (A'(t,y) - y)\, A'(t,y) \right)|_{t \leq j-3}. \tag{2.44}$$

If we take the coefficient of t^s where $0 \leq s \leq t^{j-3}$ on both sides of (2.44), then we must show that

$$\begin{aligned} \frac{A_{s+2}(y)}{s!} &= \frac{A_{s+1}(y)}{s!} + \sum_{k=1}^{s} \frac{A_{k+1}(y)}{k!} \frac{A_{s-k+1}(y)}{(s-k)!} \\ &= \frac{A_{s+1}(y)}{s!} + \sum_{k=3}^{s+2} \frac{A_{k-1}(y)}{(k-2)!} \frac{A_{s+2-(k-1)}(y)}{(s+2-k)!}. \end{aligned}$$

Thus if we multiply both sides by $s!$, we see that we must show that for $0 \leq s \leq j-3$,

$$A_{s+2}(y) = A_{s+1}(y) + \sum_{k=3}^{s+2} \binom{s+2}{k-2} A_{k-1}(y) A_{s+2-(k-1)}(y). \tag{2.45}$$

However this follows from our analysis of Cases 1, 2, and 3 above for the recursion of $A_{s+2}(y)$. That is, since $s+2 \leq j-1$, Case 2 does not apply so that we only get the contributions from Cases 1 and 3 which is exactly (2.45).

Thus we have shown that $A(y,t)$ satisfies the partial differential equation

$$A''(t,y) = (1-y-y^{\mathrm{des}(\tau)})A'(t,y) + (A'(t,y))^2 \tag{2.46}$$

with initial conditions that $A(y,0)=0$, $A(y,t)|_t = y$, and $A(y,t)|_{\frac{t^2}{2!}} = y$. It is then easy to check that the solution to this differential equation is

$$A(y,t) = \ln\left(\frac{1}{1-\int_0^t e^{(1-y)s+y^{\mathrm{des}(\tau)}\frac{s^{j-1}}{(j-1)!}}ds}\right). \tag{2.47}$$

Thus

$$\begin{aligned} A(y,t) &= \sum_{n\geq 1} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(C)} \\ &= \ln\left(\frac{1}{1-\int_0^t e^{(1-y)s+y^{\mathrm{des}(\tau)}\frac{s^{j-1}}{(j-1)!}}ds}\right). \end{aligned} \tag{2.48}$$

But then we know by Theorem 11, that

$$\begin{aligned} NCM_\tau(t,x,y) &= e^{x\sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(C)}} \\ &= e^{x\ln\left(\frac{1}{1-\int_0^t e^{(1-y)s+y^{\mathrm{des}(\tau)}\frac{s^{j-1}}{(j-1)!}}ds}\right)} \\ &= \left(\frac{1}{1-\int_0^t e^{(1-y)s+y^{\mathrm{des}(\tau)}\frac{s^{j-1}}{(j-1)!}}ds}\right)^x \end{aligned}$$

which is what we wanted to prove. □

We end this section by showing how one can compute $NCM_\tau(t,x,y)$ where $\tau \in S_m$ is of the form $\tau = 1\ 2\ \ldots\ (j-1)\ \gamma\ j$ where γ is a permutation of the elements $j+1,\ldots,m$ where $m \geq j+1$. We let $p = m-j$ so that $\text{red}(\gamma) \in S_p$. We shall assume that $j \geq 3$ since we have already dealt with permutations that start with 1 and end with 2.

Using our previous theorems as a guide, we shall assume that $NCM_\tau(t,x,y)$ is of the form

$$NCM_\tau(t,x,y) = e^{x\sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\text{cdes}(C)}} = \frac{1}{(U_\tau(t,y))^x}$$

where

$$U_\tau(t,y) = \sum_{n\geq 0} U_{n,\tau}(y)\frac{t^n}{n!}. \tag{2.49}$$

We have been unable to find a closed form for $U_\tau(t,y)$. However, we can show that the coefficients of $U_{n,\tau}(y)$ satisfy a simple recursion. That is, we shall prove the following.

Theorem 23. *Suppose that* $\tau = 1\ 2\ \ldots j-1\ \gamma\ j$ *where* γ *is a permutation of* $j+1,\ldots,j+p$ *and* $j \geq 3$. *Then*

$$NCM_\tau(t,x,y) = \frac{1}{(U_\tau(t,y))^x}$$

where

$$U_\tau(t,y) = \sum_{n\geq 0} U_{n,\tau}(y)\frac{t^n}{n!} \tag{2.50}$$

and

$$U_{n+j,\tau}(y) = (1-y)U_{n+j-1,\tau}(y) - y^{\text{des}(\tau)}\binom{n}{p}U_{n-p+1,\tau}(y). \tag{2.51}$$

Proof. Taking the natural logarithm of both sides (2.49) and using (2.16), we see

$$-\ln(U_\tau(t,y)) = \sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\text{cdes}(C)} = \sum_{n\geq 1}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NM}_n^1(\tau)} y^{\text{des}(\sigma)+1}. \tag{2.52}$$

Before proceeding, we need to establish some notation. Fix τ of the form $1\ 2\ \ldots j-1\gamma j$ where $j \geq 3$. For any $\sigma \in S_n^1$, we let $\tau\text{-imch}(\sigma)$ be the indicator function that the initial segment of size m in σ is a τ-match. Thus $\tau\text{-imch}(\sigma) = 1$ if $\text{red}(\sigma_1\ldots\sigma_m) = \tau$ and $\tau\text{-imch}(\sigma) = 0$ otherwise. For $i = 1,\ldots,j-1$, we let $\tau^{(i)} = \text{red}(i\ i+1\ \ldots j-1\ \gamma\ j)$. Our first goal is to compute

$$A(t,y) = \sum_{n\geq 1} A_n(y)\frac{t^n}{n!} \tag{2.53}$$

where

$$A_n(y) = \sum_{\sigma \in \mathcal{NM}_n^1(\tau)} y^{1+\text{des}(\sigma)}.$$

For $i = 2, \ldots, k-1$, we shall also need the following functions

$$B_i(t,y) = 1 + \sum_{n \geq 1} B_{i,n}(y) \frac{t^n}{n!} \tag{2.54}$$

where

$$B_{i,n}(y) = \sum_{\substack{\sigma \in S_n^1 \\ \tau\text{-mch}(\sigma)=0 \\ \tau^{(2)}\text{imch}(\sigma)=0 \\ \tau^{(3)}\text{imch}(\sigma)=0 \\ \vdots \\ \tau^{(i)}\text{imch}(\sigma)=0}} y^{1+\text{des}(\sigma)}.$$

Thus $B_{i,n}(y)$ is the sum of $y^{1+\text{des}(\sigma)}$ over all permutation σ in S_n^1 such that σ has no τ-matches and σ does not start with a $\tau^{(j)}$-match for $j = 2, \ldots, i$.

First we develop recursions for $A_n(y)$ for $n \geq 2$. Let $\mathcal{E}_{n,k,\tau}$ denote the set of all $\sigma = \sigma_1 \ldots \sigma_n \in \mathcal{NM}_n^1(\tau)$ such that $\sigma_k = 2$. We then consider two cases for $\sigma \in \mathcal{NM}_n^1(\tau)$ depending on which $\mathcal{E}_{n,k,\tau}$ contains σ.

Case 1. $\sigma \in \mathcal{E}_{n,2,\tau}$.

Thus $\sigma = 1\ 2\ \sigma_3 \ldots \sigma_n$. To ensure that σ has no τ-matches, we must ensure that there are no τ matches in $2\ \sigma_3 \ldots \sigma_n$ and that σ does not start with a τ-match which is equivalent to ensuring that $2\ \sigma_3 \ldots \sigma_n$ does not start with a $\tau^{(2)}$-match. Thus in this case, the permutations of $\mathcal{E}_{n,2,\tau}$ contribute $B_{2,n-1}(y)$ to $A_n(y)$.

Case 2 $\sigma \in \mathcal{E}_{n,k,\tau}$ where $3 \leq k \leq n$.

In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_k \ldots \sigma_n$ or in $\sigma_1 \ldots \sigma_{k-1}$. Thus we have $\binom{n-2}{k-2}$ ways to choose the elements that will constitute $\sigma_2 \ldots \sigma_{k-1}$ and $A_{k-1}(1)$ ways to order them so that there are no τ-matches in $\sigma_1 \ldots \sigma_{k-1}$. Once we have picked $\sigma_2 \ldots \sigma_{k-1}$, there are $A_{n-k+1}(1)$ ways to order the remaining elements so that there are no τ-matches in $\sigma_k \ldots \sigma_n$. Having picked σ, we have that

$$\text{des}(\sigma) + 1 = \text{des}(\sigma_1 \ldots \sigma_{k-1}) + 1 + \text{des}(\sigma_k \ldots \sigma_n) + 1$$

since $\sigma_{k-1} > 2$. Hence in this case, the permutations in $\mathcal{E}_{n,k,\tau}$ will contribute

$$\binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y)$$

to $A_n(y)$.

It follows that for $n \geq 2$,

$$A_n(y) = B_{2,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y). \tag{2.55}$$

We can develop similar recursions for $B_{2,n}(y)$ for $n \geq 2$. However we have to consider the cases $j = 3$ and $j > 3$ separately.

First consider, the case where $j = 3$. Note in this case $\tau^{(2)} = \text{red}(2\ \gamma\ 3) = 1\ \alpha\ 2$ where α is a permutation of $3, \ldots, p+2$ such that $\text{red}(\alpha) = \text{red}(\gamma)$. We then consider three cases for $\sigma \in \mathcal{NM}_n^1(\tau)$ depending on which $\mathcal{E}_{n,k,\tau}$ contains σ.

Case 1. $\sigma \in \mathcal{E}_{n,2,\tau}$.

Thus $\sigma = 1\ 2\ \sigma_3 \ldots \sigma_n$. To guarantee that σ has no τ-matches, we must ensure there are no τ matches in $2\ \sigma_3 \ldots \sigma_n$ and that σ does not start with a τ-match which is equivalent to ensuring that $2\ \sigma_3 \ldots \sigma_n$ does not start with a $\tau^{(2)}$-match. It might seem that to ensure σ does not start with a $\tau^{(2)}$-match then we must ensure that $2\ \sigma_3 \ldots \sigma_n$ does start a $\tau^{(3)}$-match. However, in this case $\tau^{(3)} = \text{red}(\gamma\ 3)$ does not start with 1 so then it is automatically true that $2\ \sigma_3 \ldots \sigma_n$ does start with a $\tau^{(3)}$-match. Thus the permutations in $\mathcal{E}_{n,2,\tau}$ contribute $B_{2,n-1}(y)$ to $B_{2,n}(y)$.

Case 2. $\sigma \in \mathcal{E}_{n,p+2,\tau}$.

In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_{p+1} \ldots \sigma_n$ or in $\sigma_1 \ldots \sigma_p$. Now we have $\binom{n-2}{p}$ ways to choose the elements that will constitute $\sigma_2 \ldots \sigma_{p+1}$. We can order these elements in any way that we want except that we cannot have $\text{red}(\sigma_2 \ldots \sigma_{p+1}) = \text{red}(\gamma)$ since otherwise σ would start with a $\tau^{(2)}$ match. Note that $B_{2,p+1}(y) = \sum_{\beta \in S_{p+1}^1} y^{\text{des}(\beta)+1}$ since no permutation of length $p+1$ can contain a τ-match or start with a $\tau^{(2)}$-match. Since

$$\text{des}(1\ \sigma_2 \ldots \sigma_{p+1}) + 1 + \text{des}(2\ \sigma_{p+2} \ldots \sigma_n) + 1 = \text{des}(\sigma)$$

and $\mathrm{des}(1\ \gamma)+1=\mathrm{des}(\tau)$, the permutations in $\mathcal{E}_{n,p+2,\tau}$ will contribute

$$\binom{n-2}{p}(B_{2,p+1}(y)-y^{\mathrm{des}(\tau)})A_{n-p-1}(y)$$

to $B_{2,n}(y)$.

Case 3. $\sigma \in \mathcal{E}_{n,k,\tau}$ where $3 \leq k \leq n$ and $k \notin \{2, p+2\}$.
In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_k \ldots \sigma_n$ or in $\sigma_1 \ldots \sigma_{k-1}$. Thus we have $\binom{n-2}{k-2}$ ways to choose that elements that will constitute $\sigma_2 \ldots \sigma_{k-1}$ and $B_{2,k-1}(1)$ ways to order them so that there are no τ-matches in $\sigma_1 \ldots \sigma_{k-1}$ and $\sigma_1 \ldots \sigma_{k-1}$ does not start with a $\tau^{(2)}$ match and $A_{n-k+1}(1)$ ways to order $\sigma_k \ldots \sigma_n$ so it contains no τ-match. It follows that the permutations in $\mathcal{E}_{n,k,\tau}$ will contribute $\binom{n-2}{k-2}B_{2,k-1}(y)A_{n-k+1}(y)$ to $B_{2,n}(y)$.

Thus if $n \geq p+2$, we have the recursion

$$\begin{aligned} &B_{2,n}(y) = \\ &B_{2,n-1}(y) + \left(\sum_{k=3}^{n}\binom{n-2}{p-2}B_{2,k-1}(y)A_{n-k+1}(y)\right) - \binom{n-2}{p}y^{\mathrm{des}(\tau)}A_{n-p-1}(y). \end{aligned} \tag{2.56}$$

For $2 \leq n \leq p+1$, Case 2 does not apply so that we have the recursion

$$B_{2,n}(y) = B_{2,n-1}(y) + \left(\sum_{k=3}^{n}\binom{n-2}{p-2}B_{2,k-1}(y)A_{n-k+1}(y)\right). \tag{2.57}$$

Before considering the case where $j > 3$, we shall show how we can derive a recursion (2.51) for the $U_{n,\tau}(y)$s in this case. We have shown that for all $n \geq 2$,

$$\begin{aligned} A_n(y) &= B_{2,n-1}(y) + \sum_{k=3}^{n}\binom{n-2}{k-2}A_{k-1}(y)A_{n-k+1}(y) \text{ and} \\ B_{2,n}(y) &= B_{2,n-1}(y) + \left(\sum_{k=3}^{n}\binom{n-2}{p-2}B_{2,k-1}(y)A_{n-k+1}(y)\right) - \\ &\qquad \chi(n \geq p+2)y^{\mathrm{des}(\tau)}\binom{n-2}{p}A_{n-p-1}(y) \end{aligned}$$

where for any statement A, we let $\chi(A)$ equal 1 if A is true and equal 0 if A is false. Thus we have that for all $n \geq 2$,

$$\begin{aligned}
\frac{A_n(y)}{(n-2)!} &= \frac{B_{2,n-1}(y)}{(n-2)!} + \sum_{k=3}^{n} \frac{A_{k-1}(y)}{(k-2)!}\frac{A_{n-k+1}(y)}{(n-k)!} \text{ and} \\
\frac{B_{2,n}(y)}{(n-2)!} &= \frac{B_{2,n-1}(y)}{(n-2)!} + \left(\sum_{k=3}^{n} \frac{B_{2,k-1}(y)}{(k-2)!}\frac{A_{n-k+1}(y)}{(n-k)!}\right) - \\
&\quad \chi(n \geq p+2)\frac{y^{\mathrm{des}(\tau)}}{p!}\frac{A_{n-p-1}(y)}{(n-p)!}.
\end{aligned}$$

Multiplying by t^{n-2} and summing, we obtain the following differential equations when we think of $A = A(t,y)$ and $B_2 = B_2(t,y)$ as just functions of t:

$$\begin{aligned}
A'' &= B_2' + (A' - y)A' \text{ and} \\
B_2'' &= B_2' + (B_2' - y)A' - \frac{y^{\mathrm{des}(\tau)}t^p}{p!}A'.
\end{aligned}$$

Now if $U = U(t,y) = U_\tau(t,y)$, then $A = -ln(U)$. Thus

$$A' = \frac{-U'}{U} \text{ and} \tag{2.58}$$

$$A'' = \frac{-U''}{U} + \left(\frac{U'}{U}\right)^2. \tag{2.59}$$

Making these substitutions in our first differential equation and solving for B_2', we see that

$$B_2' = -\frac{U'' + yU'}{U}. \tag{2.60}$$

Thus

$$B_2'' = -\frac{U''' + yU''}{U} + \frac{(U'' + yU')U'}{U^2}. \tag{2.61}$$

Substituting these expressions into our second differential equation and simplifying, we obtain the following differential equation for U,

$$U''' = (1-y)U'' - \frac{y^{\mathrm{des}(\tau)}t^p}{p!}U'. \tag{2.62}$$

Taking the coefficient of $\frac{t^n}{n!}$ on both sides of (2.62), we see that

$$U_{n+3,\tau}(y) = (1-y)U_{n+2}(y) - \binom{n}{p}y^{\mathrm{des}(\tau)}U_{n-p+1}(y) \tag{2.63}$$

in the case where $\tau = 1\ 2\ \gamma\ 3$ and γ is a permutation of $4, \ldots, 3+p$.

Now consider the recursion for $B_{2,n}(y)$ where $j > 3$. We then consider two cases for $\sigma \in \mathcal{NM}_n^1(\tau)$ depending on which set $\mathcal{E}_{n,k,\tau}$ contains σ.

Case 1. $\sigma \in \mathcal{E}_{n,2,\tau}$.

Thus $\sigma = 1\ 2\ \sigma_3 \ldots \sigma_n$. To ensure that σ has no τ-matches, we must ensure that there are no τ matches in $2\ \sigma_3 \ldots \sigma_n$ and that σ does not start with a τ-match which is equivalent to ensuring that $2\ \sigma_3 \ldots \sigma_n$ does not start with a $\tau^{(2)}$-match. However in this case, we must also ensure that σ does not start with a $\tau^{(2)}$ which means that $2\ \sigma_3 \ldots \sigma_n$ must not start with a $\tau^{(3)}$-match. Thus in this case, the $\sigma \in \mathcal{E}_{n,2,\tau}$ contribute $B_{3,n-1}(y)$ to $B_{2,n}(y)$.

Case 2 $\sigma \in \mathcal{E}_{n,k,\tau}$ where $3 \leq k \leq n$.

In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_k \ldots \sigma_n$ or in $\sigma_1 \ldots \sigma_{k-1}$. Thus we have $\binom{n-2}{k-2}$ ways to choose the elements that will constitute $\sigma_2 \ldots \sigma_{k-1}$ and $B_{2,k-1}(1)$ ways to order them so that there are no τ-matches in $\sigma_1 \ldots \sigma_{k-1}$ and $\sigma_1 \ldots \sigma_{k-1}$ does not start with a $\tau^{(2)}$ match and there are $A_{n-k+1}(1)$ ways to order $\sigma_k \ldots \sigma_n$ so that there is no τ-match. It follows that the permutations in $\mathcal{E}_{n,k}$ will contribute $\binom{n-2}{k-2} B_{2,k-1}(y) A_{n-k+1}(y)$ elements to $B_{2,n}(y)$.

It follows that if $j \geq 3$, then for $n \geq 2$,

$$B_{2,n}(y) = B_{3,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{2,k-1}(y) A_{n-k+1}(y). \tag{2.64}$$

One can repeat this type of argument to show that in general, for $2 \leq i \leq j-2$

$$B_{i,n}(y) = B_{i+1,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{i,k-1}(y) A_{n-k+1}(y). \tag{2.65}$$

The recursion for $B_{j-1,n}(y)$ is similar to the recursion for $B_{2,n}(y)$ when $j = 3$. That is, $\tau^{(j-1)} = \text{red}(j-1\ \gamma\ j) = 1\ \alpha\ 2$, where α is a permutation of $3, \ldots, p+2$ and $\text{red}(\gamma) = \text{red}(\alpha)$. Then we have to consider three cases depending on which set $\mathcal{E}_{n,k,\tau}$ contains σ.

Case 1. $\sigma \in \mathcal{E}_{n,2,\tau}$.

Thus $\sigma = 1\ 2\ \sigma_3 \dots \sigma_n$. To ensure that σ has no τ-matches and does not start with a $\tau^{(i)}$-match for $i = 2, \dots, j-1$, we clearly have to ensure that $2\ \sigma_3 \dots \sigma_n$ has no τ-matches and does not start with a $\tau^{(i)}$-match for $i = 2, \dots, j-1$. However, we do not have to worry about $2\ \sigma_3 \dots \sigma_n$ starting with a $\tau^{(j)} = \text{red}(\sigma\ j)$ since $\tau^{(j)}$ does not start with its least element. Thus in this case, the permutations in $\mathcal{E}_{n,2,\tau}$ contribute $B_{j-1,n-1}(y)$ to $B_{j-1,n}(y)$.

Case 2. $\sigma \in \mathcal{E}_{n,p+2,\tau}$ In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_{p+1} \dots \sigma_n$ or in $\sigma_1 \dots \sigma_p$. Now we have $\binom{n-2}{p}$ ways to choose that elements that will constitute $\sigma_2 \dots \sigma_{p+1}$. We can order these elements in any way that we want except that we cannot have $\text{red}(\sigma_2 \dots \sigma_{p+1}) = \text{red}(\gamma)$ since otherwise σ would start with at $\tau^{(j-1)}$ match. Note that $B_{j-1,p+1}(y) = \sum_{\beta \in S^1_{p+1}} y^{\text{des}(\beta)+1}$ since no permutation of length $p+1$ can contain a τ-match or start with a $\tau^{(i)}$-match for $i = 2, \dots j-1$. Thus since

$$\text{des}(1\ \sigma_2 \dots \sigma_{p+1}) + 1 + \text{des}(2\ \sigma_{p+2} \dots \sigma_n) + 1 = \text{des}(\sigma)$$

and $\text{des}(1\ \gamma) + 1 = \text{des}(\tau)$, the permutations in $\mathcal{E}_{n,p+2,\tau}$ will contribute

$$\binom{n-2}{p}(B_{j-1,p+1}(y) - y^{\text{des}(\tau)})A_{n-p+1}(y)$$

to $B_{j-1,n}(y)$.

Case 3. $\sigma \in \mathcal{E}_{n,k,\tau}$ where $3 \leq k \leq n$ and $k \notin \{2, p+2\}$.

In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_k \dots \sigma_n$ or in $\sigma_1 \dots \sigma_{k-1}$. Thus we have $\binom{n-2}{k-2}$ ways to choose that elements that will constitute $\sigma_2 \dots \sigma_{k-1}$ and $B_{j-1,k-1}(1)$ ways to order them so that there are no τ-matches in $\sigma_1 \dots \sigma_{k-1}$ and $\sigma_1 \dots \sigma_{k-1}$ does not start with a $\tau^{(i)}$-match for $i = 2, \dots, j-1$ and there are $A_{n-k+1}(1)$ ways to order $\sigma_k \dots \sigma_n$ so that there is no τ-match. Thus the permutations in $\mathcal{E}_{n,k,\tau}$ will contribute $\binom{n-2}{k-2}B_{j-1,k-1}(y)A_{n-k+1}(y)$ to $B_{j-1,n}(y)$.

It follows that for $n \geq 2$,

$$B_{j-1,n}(y) = B_{j-1,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{j-1,k-1}(y) A_{n-k+1}(y) - \chi(n \geq p+2) \binom{n-2}{p} y^{\mathrm{des}(\tau)} A_{n-p-1}(y). \quad (2.66)$$

Thus for all $n \geq 2$, we have proved that in general

$$\begin{aligned}
A_n(y) &= B_{2,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y) \\
B_{2,n}(y) &= B_{3,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{2,k-1}(y) A_{n-k+1}(y) \\
B_{3,n}(y) &= B_{4,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{3,k-1}(y) A_{n-k+1}(y) \\
&\vdots \\
B_{j-2,n}(y) &= B_{j-1,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{j-2,k-1}(y) A_{n-k+1}(y) \\
B_{j-1,n}(y) &= B_{j-1,n-1}(y) + \left(\sum_{k=3}^{n} \binom{n-2}{k-2} B_{j-1,k-1}(y) A_{n-k+1}(y) \right) - \\
&\quad \chi(n \geq p+2) \binom{n-2}{p} y^{\mathrm{des}(\tau)} A_{n-p-1}(y).
\end{aligned}$$

As in the case for $j = 3$, if we multiply everything by $\frac{t^{n-2}}{(n-2)!}$ and then sum over $n \geq 2$ we get the following system of differential equations where we think of $A(t,y)$ and $B_i(t,y)$ for $i = 2, \ldots, j-1$ as functions of t.

$$\begin{aligned}
&(D_1) \quad A'' = B_2' + A'^2 - yA' \\
&(D_2) \quad B_2'' = B_3' + B_2'A' - yA' \\
&(D_3) \quad B_3'' = B_4' + B_3'A' - yA' \\
&\vdots \\
&(D_{j-2}) \; B_{j-2}'' = B_{j-1}' + B_{j-2}'A' - yA' \\
&(D_{j-1}) \; B_{j-1}'' = B_{j-1}' + B_{j-1}'A' - yA' - \frac{t^p}{(p)!} y^{\mathrm{des}(\tau)} A'.
\end{aligned}$$

As in the case $j = 3$, we let $A(t, y) = -\log(U(t, y))$ so that $A' = \frac{-U'}{U}$ and $A'' = \frac{-U''}{U} + \frac{U'^2}{U^2}$. Thus under this substitution, the first differential equation becomes

$$\frac{-U''}{U} + \frac{U'^2}{U^2} = B_2' + \frac{U'^2}{U^2} + y\frac{U'}{U}$$

so that

$$B_2' = \frac{-U'' - yU'}{U}. \tag{2.67}$$

In fact, we have the following lemma.

Lemma 33. *For $2 \leq i \leq j-1$,*

$$B_i' = \frac{-U^{(i)} - y\sum_{k=1}^{i-1} U^{(k)}}{U}. \tag{2.68}$$

Proof. We proceed by induction on i. We have already shown that (2.68) in the case where $i = 2$. Now suppose that

$$B_i' = \frac{-U^{(i)} - y\sum_{k=1}^{i-1} U^{(k)}}{U}. \tag{2.69}$$

Then we must show that

$$B_{i+1}' = \frac{-U^{(i+1)} - y\sum_{k=1}^{i} U^{(k)}}{U}. \tag{2.70}$$

Taking the derivative of both sides of (2.69) with respect to t, we see that

$$B_i'' = \frac{-U^{(i+1)} - y\sum_{k=2}^{i} U^{(k)}}{U} + \left(\frac{U^{(i)} + y\sum_{k=1}^{i-1} U^{(k)}}{U} \cdot \frac{U'}{U}\right).$$

Plugging our expression for B_i'' and B_i' into the differential equation

$$(D_i)\ B_i'' = B_{i+1}' + B_i'A' - yA',$$

we see that

$$\frac{-U^{(i+1)} - y\sum_{k=2}^{i} U^{(k)}}{U} + \left(\frac{U^{(i)} + y\sum_{k}^{i-1} U^{(k)}}{U} \cdot \frac{U'}{U}\right)$$
$$= B_{i+1}' + \left(\frac{-U^{(i)} - y\sum_{k=1}^{i-1} U^{(k)}}{U} \cdot \frac{-U'}{U}\right) - \left(y \cdot \frac{-U'}{U}\right).$$

Solving for B_{i+1}' we see that

$$B_{i+1}' = \frac{-U^{(i+1)} - y\sum_{k=1}^{i} U^{(k)}}{U}.$$

□

By the Lemma, we know that

$$B'_{j-1} = \frac{-U^{(j-1)} - y\sum_{k=1}^{j-2} U^{(k)}}{U},$$

and, hence,

$$B''_{j-1} = \frac{-U^{(j)} - y\sum_{k=2}^{j-1} U^{(k)}}{U} + \left(\frac{U^{(j-1)} + y\sum_{k}^{j-2} U^{(k)}}{U} \cdot \frac{U'}{U}\right).$$

Thus plugging these expressions into the differential equation (D_{j-1}), we obtain that

$$\frac{-U^{(j)} - y\sum_{k=2}^{j-1} U^{(k)}}{U} + \left(\frac{U^{(j-1)} + y\sum_{k}^{j-2} U^{(k)}}{U} \cdot \frac{U'}{U}\right)$$
$$= \frac{-U^{(j-1)} - y\sum_{k=1}^{j-2} U^{(k)}}{U} +$$
$$\left(\frac{-U^{(j-1)} - y\sum_{k=1}^{j-2} U^{(k)}}{U} \cdot \frac{-U'}{U}\right) - \left(y \cdot \frac{-U'}{U}\right) - \left(\frac{t^p}{p!} y^{\text{des}(\tau)} \cdot \frac{-U'}{U}\right).$$

Simplifying this expression yields that

$$U^{(j)} = (1-y)U^{(j-1)} - \frac{t^p}{p!} y^{\text{des}(\tau)} U'. \tag{2.71}$$

Then taking the coefficient of $\frac{t^n}{n!}$ on both side of (2.71) gives that

$$U_{n+j} = (1-y)U_{n+j-1} + y^{\text{des}(\tau)} \binom{n}{p} U_{n-p+1}$$

which is what we wanted to prove. □

We end this section with an example of the use of Theorem 23. Let $\tau = 1243$ and

$$A_{n,\tau}(t,y) = \sum_{n\geq 1} A_{n,\tau}(y)\frac{t^n}{n!} = \sum_{n\geq 1}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NM}_n^1(\tau)} y^{\text{des}(\sigma)+1} = \sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\text{cdes}(\sigma)}.$$

It is easy to check that $A_{1,\tau}(y) = y$, $A_{2,\tau}(y) = y$, $A_{3,\tau}(y) = y + y^2$, and $A_{4,\tau}(y) = y + 3y^2 + y^3$. Now

$$U_\tau(t,y) = \sum_{n\geq 0} U_{n,\tau}(y) = e^{-A_\tau(t,y)}$$

so that one can use Mathematica to compute that $U_{0,\tau}(y) = 1$, $U_{1,\tau}(y) = -y$, $U_{2,\tau}(y) = -y + y^2 y$, $U_{3,\tau}(y) = -y + 2y^2 - y^3$, and $U_{4,\tau}(y) = -y + 4y^2 - 3y^3 + y^4$.

By Theorem 23, we know that we have the recursion that

$$U_{n+3,\tau}(y) = (1-y)U_{n+2,\tau}(y) - yU_{n,\tau}(y).$$

Thus we can use this recursion to compute that
$U_{5,\tau}(y) = -y + 6y^2 - 8y^3 + 4y^4 - y^5$,
$U_{6,\tau}(y) = -y + 8y^2 - 16y^3 + 13y^4 - 5y^5 + y^6$,
$U_{7,\tau}(y) = -y + 10y^2 - 28y^3 + 32y^4 - 19y^5 + 6y^6 - y^7$, and
$U_{8,\tau}(y) = -y + 12y^2 - 44y^3 + 68y^4 - 55y^5 + 26y^6 - 7y^7 + y^8$.

But then we know that $NCM_\tau(t,x,y) = \frac{1}{(U_\tau(t,y))^x}$. Let $NCM_{\tau,n}(x,y)$ be the coefficient of $\frac{t^n}{n!}$ in $NCM_\tau(t,x,y)$. That is, let

$$NCM_\tau(t,x,y) = \sum_{n\geq 0} NCM_{\tau,n}(x,y)\frac{t^n}{n!}.$$

Thus one can use Mathematica to compute the polynomials $NCM_{\tau,n}(x,y)$ where

$NCM_{\tau,0}(x,y) = 1$,

$NCM_{\tau,1}(x,y) = xy$,

$NCM_{\tau,2}(x,y) = xy + x^2y^2$,

$NCM_{\tau,3}(x,y) = xy + xy^2 + 3x^2y^2 + x^3y^3$,

$NCM_{\tau,4}(x,y) = xy + 3xy^2 + 7x^2y^2 + xy^3 + 4x^2y^3 + 6x^3y^3 + x^4y^4$,

$NCM_{\tau,5}(x,y) = xy + 9xy^2 + 15x^2y^2 + 8xy^3 + 25x^2y^3 + 25x^3y^3 + xy^4 + 5x^2y^4 + 10x^3y^4 + 10x^4y^4 + x^5y^5$,

$NCM_{\tau,6}(x,y) = xy + 23xy^2 + 31x^2y^2 + 45xy^3 + 119x^2y^3 + 90x^3y^3 + 20xy^4 + 73x^2y^4 + 105x^3y^4 + 65x^4y^4 + xy^5 + 6x^2y^5 + 15x^3y^5 + 20x^4y^5 + 15x^5y^5 + x^6y^6$,

$NCM_{\tau,7}(x,y) = xy + 53xy^2 + 63x^2y^2 + 217xy^3 + 490x^2y^3 + 301x^3y^3 + 192xy^4 + 623x^2y^4 + 749x^3y^4 + 350x^4y^4 + 47xy^5 + 196x^2y^5 + 343x^3y^5 + 315x^4y^5 + 140x^5y^5 + xy^6 + 7x^2y^6 + 21x^3y^6 + 35x^4y^6 + 35x^5y^6 + 21x^6y^6 + x^7y^7$, and

$NCM_{\tau,8}(x,y) = xy + 115xy^2 + 127x^2y^2 + 916xy^3 + 1838x^2y^3 + 966x^3y^3 + 1500xy^4 + 4333x^2y^4 + 4466x^3y^4 + 1701x^4y^4 + 765xy^5 + 2810x^2y^5 + 4214x^3y^5 + 3164x^4y^5 + 1050x^5y^5 + 105xy^6 + 495x^2y^6 + 1008x^3y^6 + 1148x^4y^6 + 770x^5y^6 + 266x^6y^6 + xy^7 + 8x^2y^7 + 28x^3y^7 + 56x^4y^7 + 70x^5y^7 + 56x^6y^7 + 28x^7y^7 + x^8y^8$.

Chapter 3

The reciprocal method

In this chapter, we will present several results based on the reciprocal method introduced by Jones and Remmel [19].

3.1 Introduction

Recall the definitions

$$NM_{\Upsilon}(t,x,y) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\Upsilon)} x^{\text{LRMin}(\sigma)} y^{1+\text{des}(\sigma)}, \tag{1.16}$$

and

$$NCM_{\Upsilon}(t,x,y) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NCM}_n(\Upsilon)} x^{\text{cyc}(\sigma)} y^{\text{cdes}(\sigma)} \tag{1.17}$$

This chapter will be focusing on the generating function $NM_{\tau}(t,x,y)$ for single patterns that start with 1. Recall that in this case by Corollary 14, $NM_{\tau}(t,x,y) = NCM_{\tau}(t,x,y)$.

This method does not compute $NM_{\tau}(t,x,y)$ directly. Instead, we assume that

$$NM_{\tau}(t,x,y) = \left(\frac{1}{U_{\tau}(t,y)}\right)^x \text{ where } U_{\tau}(t,y) = 1 + \sum_{n\geq 1} U_{\tau,n}(y)\frac{t^n}{n!}. \tag{3.1}$$

Thus

$$U_{\tau}(t,y) = \frac{1}{1+\sum_{n\geq 1} NM_{\tau,n}(1,y)\frac{t^n}{n!}} \tag{3.2}$$

Our method gives a combinatorial interpretation the right-hand side of (3.2) and then uses that combinatorial interpretation to develop simple recursions on the coefficients $U_{\tau,n}(y)$.

It follows from Theorems 11 and Corollary 14 that if $\tau \in S_j$ and τ starts with 1, then

$$NM_\tau(t,x,y) = F(t,y)^x \tag{3.3}$$

for some function $F(t,y)$. Thus our assumption that

$$NM_\tau(t,x,y) = \left(\frac{1}{U_\tau(t,y)}\right)^x \tag{3.4}$$

is fully justified in the case when τ starts with 1. Now we review the background in symmetric functions that is needed to give combinatorial interpretation to the coefficients $U_{\tau,n}(y)$ from (3.1).

3.2 Symmetric functions.

In this section, we give the necessary background on symmetric functions that will be needed for our proofs.

Given a partition $\lambda = (\lambda_1, \ldots \lambda_\ell)$ where $0 < \lambda_1 \leq \cdots \leq \lambda_\ell$, we let $\ell(\lambda)$ be the number of nonzero integers in λ. If the sum of these integers is equal to n, then we say λ is a partition of n and write $\lambda \vdash n$.

Let Λ denote the ring of symmetric functions in infinitely many variables $x_1, x_2, \ldots$. The n^{th} elementary symmetric function $e_n = e_n(x_1, x_2, \ldots)$ and n^{th} homogeneous symmetric function $h_n = h_n(x_1, x_2, \ldots)$ are defined by the generating functions given in (1.31) and (1.32). For any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$ and $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$. It is well known that $\{e_\lambda : \lambda \text{ is a partition}\}$ is a basis for Λ. In particular, $e_0, e_1, \ldots$ is an algebraically independent set of generators for Λ and, hence, a ring homomorphism θ on Λ can be defined by simply specifying $\theta(e_n)$ for all n.

A key element of our proofs is the combinatorial description of the coefficients of the expansion of h_n in terms of the elementary symmetric functions e_λ given by Eğecioğlu and Remmel in [13]. They defined a λ-brick tabloid of shape

(n) to be a rectangle of height 1 and length n chopped into "bricks" of lengths found in the partition λ. For example, Figure 3.1 shows one brick $(2,3,7)$-tabloid of shape (12).

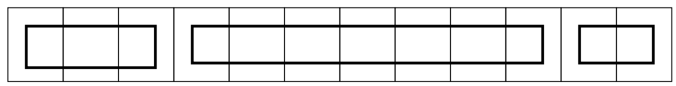


Figure 3.1: A $(2,3,7)$-brick tabloid of shape (12).

Let $\mathcal{B}_{\lambda,n}$ denote the set of λ-brick tabloids of shape (n) and let $B_{\lambda,n}$ be the number of λ-brick tabloids of shape (n). If $B \in \mathcal{B}_{\lambda,n}$ we will write $B = (b_1, \ldots, b_{\ell(\lambda)})$ if the lengths of the bricks in B, reading from left to right, are $b_1, \ldots, b_{\ell(\lambda)}$. Through simple recursions, Eğecioğlu and Remmel [13] proved that

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} e_\lambda. \tag{3.5}$$

3.3 Combinatorial interpretation of $U_{\tau,n}(y)$

Suppose that $\tau \in S_j$ which starts with 1 and $\mathrm{des}(\tau) = 1$. Let us also assume that τ has only one descent. Consider the generating function

$$U_\tau(t,y) = \frac{1}{NM_\tau(t,1,y)} = \frac{1}{1 + \sum_{n\geq 1} \frac{t^n}{n!} NM_{\tau,n}(1,y)} \tag{3.6}$$

where $NM_{\tau,n}(1,y) = \sum_{\sigma \in \mathcal{NM}_n(\tau)} y^{1+\mathrm{des}(\sigma)}$.

To this end, we define a ring homomorphism θ_τ on the ring of symmetric functions Λ by setting $\theta_\tau(e_0) = 1$ and

$$\theta_\tau(e_n) = \frac{(-1)^n}{n!} NM_{\tau,n}(1,y) \text{ for } n \geq 1. \tag{3.7}$$

It follows that

$$\begin{aligned} \theta_\tau(H(t)) &= \sum_{n\geq 0} \theta_\tau(h_n) t^n = \frac{1}{\theta_\tau(E(-t))} = \frac{1}{1 + \sum_{n\geq 1} (-t)^n \theta_\tau(e_n)} \\ &= \frac{1}{1 + \sum_{n\geq 1} \frac{t^n}{n!} NM_{\tau,n}(1,y)} \end{aligned}$$

which is what we want to compute.

By (3.5), we have that

$$\begin{aligned} n!\theta_\tau(h_n) &= n!\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}B_{\mu,n}\theta_\tau(e_\mu) \\ &= n!\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}\sum_{(b_1,\ldots,b_{\ell(\mu)})\in\mathcal{B}_{\mu,n}}\prod_{i=1}^{\ell(\mu)}\frac{(-1)^{b_i}}{b_i!}NM_{\tau,b_i}(1,y) \\ &= \sum_{\mu\vdash n}(-1)^{\ell(\mu)}\sum_{(b_1,\ldots,b_{\ell(\mu)})\in\mathcal{B}_{\mu,n}}\binom{n}{b_1,\ldots,b_{\ell(\mu)}}\prod_{i=1}^{\ell(\mu)}NM_{\tau,b_i}(1,y). \quad (3.8)\end{aligned}$$

Our next goal is to give a combinatorial interpretation to the right-hand side of (3.8). If we are given a brick tabloid $B=(b_1,\ldots,b_{\ell(\mu)})$, then we can interpret the multinomial coefficient $\binom{n}{b_1,\ldots,b_\mu}$ as all ways to assign sets $S_1,\ldots,S_{\ell(\mu)}$ to the bricks of B in such a way that $|S_i|=b_i$ for $i=1,\ldots,\ell(\mu)$ and the sets $S_1,\ldots,S_{\ell(\mu)}$ form a set partition of $\{1,\ldots,n\}$. Next for each brick b_i, we use the factor

$$NM_{\tau,b_i}(1,y)=\sum_{\sigma\in S_{b_i},\tau\text{-mch}(\sigma)=0}y^{\mathrm{des}(\sigma)+1}$$

to pick a rearrangement $\sigma^{(i)}$ of S_i which has no τ-matches to put in cells of b_i and then we place a label of y on each cell that starts a descent in $\sigma^{(i)}$ plus a label of y on the last cell of b_i. Finally, we use the term $(-1)^{\ell(\mu)}$ to turn each label y at the end of brick to a $-y$. We let $\mathcal{O}_{\tau,n}$ denote the set of all objects created in this way. For each element $O\in\mathcal{O}_{\tau,n}$, we define the weight of O, $W(O)$, to be the product of y labels and the sign of O, $sgn(O)$, to be $(-1)^{\ell(\mu)}$. For example, if $\tau=13245$, then such an object O constructed from the brick tabloid $B=(2,8,3)$ is pictured in Figure 3.2 where $W(O)=y^7$ and $sgn(O)=(-1)^3$. It follows that

$$n!\theta_\tau(h_n)=\sum_{O\in\mathcal{O}_{\tau,n}}sgn(O)W(O). \quad (3.9)$$

Next we define a weight-preserving sign-reversing involution I_τ on $\mathcal{O}_{\tau,n}$. Given an element $O\in\mathcal{O}_{\tau,n}$, scan the cells of O from left to right looking for the first cell c such that either

(i) c is labeled with a y or

	−y			y		y		y	−y	y		−y
4	8	7	10	11	5	12	3	9	6	2	1	13

Figure 3.2: An element of $\mathcal{O}_{13245,13}$.

(ii) c is a cell at the end of a brick b_i, the number in cell c is greater than the number in the first cell of the next brick b_{i+1}, and there is no τ-match in the cells of bricks b_i and b_{i+1}.

In case (i), if c is a cell in brick b_j, then we split b_j in to two bricks b'_j and b''_j where b'_j contains all the cells of b_j up to an including cell c and b''_j consists of the remaining cells of b_j and we change to label on cell c from y to $-y$. In case (ii), we combine the two bricks b_i and b_{i+1} into a single brick b and change the label on cell c from $-y$ to y. For example, consider the element $O \in \mathcal{O}_{13245,13}$ pictured in Figure 3.2. Note that even though the number in the last cell of brick 1 is greater than the the number in the first cell of brick 2, we can not combine these two bricks because the numbers 4 8 7 10 11 would be a 13245-match. Thus the first place that we can apply the involution is on cell 5 which is labeled with a y so that $I_\tau(O)$ is the object pictured in Figure 3.3.

	−y			−y		y		y	−y	y		−y
4	8	7	10	11	5	12	3	9	6	2	1	13

Figure 3.3: $I_\tau(O)$ for O in Figure 3.2.

We claim that I_τ is an involution so that I_τ^2 is the identity. To see this, consider case (i) where we split a brick b_j at cell c which is labeled with a y. In that case, we let a be the number in cell c and a' be the number in cell $c+1$ which must also be in brick b_j. It must be the case that there is no cell labeled y before cell c since otherwise we would not use cell c to define the involution. However, we have to consider the possibility that when we spilt b_j into b'_j and b''_j that we might then be able to combine the brick b_{j-1} with b'_j because the number in that last cell of b_{j-1} is greater than the number in the first cell of b'_j and there is no

τ-match in the cells of b_{j-1} and b'_j. Since we always take an action on the left most cell possible when defining I, we know that we cannot combine b_{j-1} and b_j so that there must be a τ-match in the cells of b_{j-1} and b_j. Clearly, that match must have involved the number a' and the number in cell d which is the last cell in brick b_{j-1}. But that is impossible because then there would be two descents among the numbers between cell d and cell $c+1$ which would violate our assumption that τ has only one descent. Thus whenever we apply case (i) to define $I_\tau(O)$, the first action that we can take is combine bricks b'_j and b''_j so that $I_\tau^2(O) = O$.

If we are in case (ii), then again we can assume that there are no cells labeled y that occur before cell c. When we combine brick b_i and b_{i+1}, then we will label cell c with a y. It is clear that combining the cells of b_i and b_{i+1} cannot help us combine the resulting brick b with an earlier brick since it will be harder to have no τ-matches with the larger brick b. Thus the first place cell c where we can apply the involution will again be cell c which is now labeled with a y so that $I_\tau^2(O) = O$ if we are in case (ii).

It is clear from out definitions that if $I_\tau(O) \neq O$, then $sgn(O)W(O) = -sgn(I_\tau(O))W(I_\tau(O))$. Hence it follows from (3.9) that

$$n!\theta_\tau(h_n) = \sum_{O \in \mathcal{O}_{\tau,n}} sgn(O)W(O) = \sum_{O \in \mathcal{O}_{\tau,n}, I_\tau(O)=O} sgn(O)W(O). \tag{3.10}$$

Thus we must examine the fixed points of I_τ. So assume that O is a fixed point of I_τ. First it is easy to see that there can be no cells which are labeled with y so that numbers in each brick of O must be increasing. Second we cannot combine two consecutive bricks b_i and b_{i+1} in O which means either that there is an increase between the bricks b_i and b_{i+1} or there is a decrease between the bricks b_i and b_{i+1} but there is a τ-match in the cells of the bricks b_i and b_{i+1}. We claim that, in addition, the numbers in the first cells of the bricks must form an increasing sequence, reading from left to right. That is, suppose that b_i and b_{i+1} are two consecutive bricks in a fixed point O of I_τ and that $a > a'$ where a is the number in the first cell of b_i and a' is the number in the first cell of b_{i+1}. Then clearly the number in the last cell of b_i must be greater than a' so that it must be the case that there is a τ-match in the cells of b_i and b_{i+1}. However a' is the least number that resides in cells of of b_i and b_{i+1} which means that the only way that a' could be part of a τ-match that occurs in the cells of b_i and b_{i+1} is to have a' play the role of 1. But since we are assuming that τ starts with 1, this would mean that if a' is part of a τ-match, then that τ-match must be entirely contained in b_{i+1} which is impossible. Thus a' cannot be part of any τ-match that occurs in the cells of b_i and b_{i+1}. However, this would mean that the τ-match that occurs in the cells of b_i and b_{i+1} must either be contained entirely in the cells of b_i or entirely in the cells of b_{i+1} which again is impossible. Hence it must be the case that $a < a'$.

Thus we have proved the following.

Lemma 34. *Suppose that $\tau \in S_j$, τ starts with 1, and* $\mathrm{des}(\tau) = 1$*. Let $\theta_\tau : \Lambda \to \mathbb{Q}(y)$ where $\mathbb{Q}(y)$ is the set of rational functions in the variable y over the rationals $\mathbb{Q}$ defined by $\theta_\tau(e_0) = 1$ and $\theta_\tau(e_n) = \frac{(-1)^n}{n!} NM_{\tau,n}(1, y)$ for $n \geq 1$. Then*

$$n!\theta_\tau(h_n) = \sum_{O \in \mathcal{O}_{\tau,n}, I_\tau(O)=O} sgn(O)W(O) \tag{3.11}$$

where $\mathcal{O}_{\tau,n}$ is the set of objects and I_τ is the involution defined above. Moreover, every fixed point O of I_τ has the following three properties.

1. *There are no cells labeled with y in O so that the elements in each brick of O are increasing,*

2. *the numbers in the first cell of each brick of O form an increasing sequence, reading from left to right, and*

3. *b_i and b_{i+1} are two consecutive bricks in O, then either (a) there is increase between b_i and b_{i+1} or (b) there is a decrease between b_i and b_{i+1} but there is τ-match in the cells of b_i and b_{i+1}.*

3.4 Special cases

3.4.1 $\tau = 1324\ldots p$ for $p \geq 5$

Now we specialize to the case where $\tau = 1324\ldots p$ where $p \geq 5$. In this case, we have

$$\begin{aligned}\theta_\tau(H(t)) &= \sum_{n\geq 0} \theta_\tau(h_n)t^n = \frac{1}{\theta_\tau(E(-t))} = \frac{1}{1+\sum_{n\geq 1}(-t)^n\theta_\tau(e_n)} \\ &= \frac{1}{1+\sum_{n\geq 1}\frac{t^n}{n!}NM_{\tau,n}(1,y)} = 1+\sum_{n\geq 1} U_{\tau,n}(y)\frac{t^n}{n!}.\end{aligned}$$

Thus $\theta(h_n) = \frac{U_{\tau,n}(y)}{n!}$ or, equivalently, $n!\theta(h_n) = U_{\tau,n}(y)$. Hence

$$U_{\tau,n}(y) = \sum_{O\in\mathcal{O}_{\tau,n}, I_\tau(O)=O} sgn(O)W(O).$$

In this situation, we can make a finer analysis of the fixed points of I_τ. Let O be a fixed point of I_τ. By Lemma 34, we know that 1 is in the first cell of O. We claim that 2 must be in cell the second or third cell of O. That is, suppose that 2 is in cell c where $c > 3$. Then since there are no descents within any brick 2 must be the first cell of a brick. Moreover, since the minimal numbers in the bricks of O form an increasing sequence, reading from left to right, 2 must be in the first cell of the second brick. Thus if b_1 and b_2 are the first two bricks in O, then 1 is in the first cell of b_1 and 2 is in the first cell of b_2. But then we claim that there is no τ-match in the cells of b_1 and b_2. That is, since $c > 3$, b_1 has at least three cells so that O starts with an increasing sequence of length 3. But this means that 1 can not be part of a $1324\ldots p$-match. Similarly, no other cell of b_1 can be part of $1324\ldots p$-match because the 2 in cell c is less than any of the remaining numbers of b_1. Thus if there is a $1324\ldots p$-match among the cells of b_1 and b_2, it would have to be entirely contained in b_2 which is impossible. But this would mean that we could apply case (ii) of the definition of I_τ to b_1 and b_2 which would violate our assumption that O is a fixed point of I_τ. Thus, we have two cases.

Case 1. 2 is in cell 2 of O.

In this case there are two possibilities, namely, either (i) 1 and 2 are both in the first brick b_1 of O or (ii) brick b_1 is a single cell filled with 1 and 2 is in the first cell of the second brick b_2 of O. In either case, it is easy to see that 1 is not part of a $1324\ldots p$-match in O and if we remove cell 1 from O and subtract 1 from the numbers in the remaining cells, we would end up with a fixed point O' of I_τ in $\mathcal{O}_{\tau,n-1}$. Now in case (i), it is easy to see that $sgn(O)W(O) = sgn(O')W(O')$ and in case (ii) since b_1 will have a label $-y$ on the first cell, $sgn(O)W(O) = (-y)sgn(O')W(O')$. It follows that fixed points in Case 1 will contribute $(1-y)U_{\tau,n-1}(y)$ to $U_{\tau,n}(y)$.

Case 2. 2 is in cell 3 of O.

Let $O(i)$ denote the number in cell i of O and $b_1, b_2, \ldots$ be the bricks of O, reading from left to right. Since there are no descents within bricks in O, we know that 2 is in the first cell of a brick. We claim that it must be the case that b_1 is has two cells and b_2 has at least $p-2$ cells. That is, it cannot be that b_1 and b_2 both only have one cell each. Otherwise, it would be the case that 2 is the least number in the cells of bricks b_2 and b_3 so that there would be a decrease between bricks b_2 and b_3 and there could be no τ-match in cells of b_2 and b_3. But then we could combine bricks b_2 and b_3 according to the definition of I_τ and O would not be a fixed point of I_τ. Thus b_1 has two cells. But then b_2 must have at least $p-2$ cells since otherwise, there could be no τ-match contained in the cells of b_1 and b_2 and we could combine bricks b_1 and b_2 which again would mean that O is not a fixed point of I_τ. Thus b_1 is a brick with two cells and b_2 is brick with at least $p-2$ cells. But then the only reason that we could not combine bricks b_1 and b_2 is that there is a τ-match in the cells of b_1 and b_2 which could only start at position 1.

Next we claim that $O(p-1) = p-1$. That is, since there is a τ-match starting at position 1 and $p \geq 5$, we know that all the numbers in the first $p-2$ cells of O are strictly less than $O(p-1)$. Thus $O(p-1) \geq p-1$. Now if $O(p-1) > p-1$, then let i be least number in the set $\{1, \ldots, p-1\}$ that is not contained in bricks b_1 and b_2. Since the numbers in each brick are increasing and the minimal numbers of the bricks are increasing, the only possible position for i is the first cell of brick

b_3. But then it follows that there is a decrease between bricks b_2 and b_3. Since O is a fixed point of I_τ, this must mean that there is a τ-match in the cells of b_2 and b_3. But since τ has only one descent, this τ-match can only start at the cell c which is the second to the last cell of b_2. Thus c could be $p-1$ if b_2 has size $p-2$ or $c > p-1$ if b_2 has size $> p-2$. In either case, $p-1 < O(p-1) \leq O(c) < O(c+1) > O(c+2) = i$. But this is impossible since to have a τ-match starting at cell c, we must have $O(c) < O(c+2)$. Thus it must be the case that $O(p-1) = p-1$ and $\{O(1), \ldots, O(p-1)\} = \{1, \ldots, p-1\}$.

We now have two subcases.

Case 2.a. There is no τ-match in O starting at cell $p-1$.

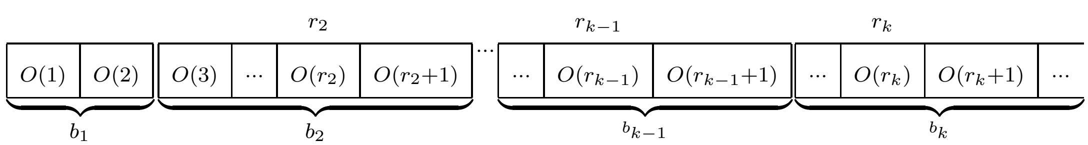
Then we claim that $O(p) = p$. That is, if $O(p) \neq p$, then p cannot be in b_2 so that p must be the first cell of the brick b_3. But then we claim that we could combine bricks b_2 and b_3. That is, there will be a decrease between bricks b_2 and b_3 since $p < O(p)$ and $O(p)$ is in b_2. Since there is no τ-match in O starting at cell $p-1$, the only possible τ-match among the cells of b_2 and b_3 would have start at a cell $c \neq p-1$. But it can't be that $c < p-1$ since then it would be the case that $O(c) < O(c+1) < O(c+2)$. Similarly, it cannot be that $c > p-1$ since then $O(c) > p$ and p has to be part of the τ-match which is impossible since $O(c)$ must play the role of 1 in the τ-match. Thus it must be the case that $O(p) = p$. It then follows that if let O' be the result of removing the first $p-1$ cells from p and subtracting $p-1$ from the remaining numbers, then O' will be a fixed point of I_τ in $\mathcal{O}_{\tau,n-(p-1)}$. Note that if b_2 has $p-2$ cells, then O' will start with a brick with one cell and if b_2 has more than $p-2$ cells, then O' will start with a brick with at least two cells. Since there is $-y$ coming from the brick b_1, it is easy to see that the fixed points in Case 2.a will contribute $-yU_{\tau,n-(p-1)}(y)$ to $U_{\tau,n}(y)$.

Case 2.b. There is a τ-match starting a $p-1$ in O.

In this case, it must be that $O(p-1) < O(p) > O(p+1)$ so that b_2 must have

$p-2$ cells and brick b_3 starts at cell $p+1$. We claim that b_3 must have at least $p-2$ cells. That is, if b_3 has less than $p-2$ cells, then there could be no τ-match among the cells of b_2 and b_3 so then we could combine b_2 and b_3 violating the fact that O is a fixed point of I_τ.

In the general case, assume that in O, the bricks $b_2, \ldots, b_{k-1}$ all have $(p-2)$ cells. Then let $r_1 = 1$ and for $j = 2, \ldots, k-1$, let $r_j = 1 + (j-1)(p-2)$. Thus r_j is the position of the second to last cell of brick b_j for $1 \leq j \leq k-1$. Furthermore, assume that there is a τ-match starting at cell r_j for $1 \leq j \leq k-1$. It follows that $O(r_{k-1}) < O(r_{k-1}+1) > O(r_{k-1}+2)$ so that brick b_k must start at cell $r_{k-1}+2$ and there is a decrease between bricks b_{k-1} and b_k. But then it must be the case that b_k has at least $p-2$ cells since if b_k has less than $p-2$ cells, we could combine bricks b_{k-1} and b_k violating the fact that O is a fixed point of I_τ. Let $r_k = 1 + k(p-2)$. We shall also assume that O does not have a τ-match starting at position r_k. Thus we have the situation pictured below.



First we claim $O(r_j) = r_j$ and $\{1, \ldots, r_j\} = \{O(1), \ldots, O(r_j)\}$ for $j = 1, \ldots, k$. We have shown that $O(1) = 1$ and that $O(r_2) = O(p-1) = p-1$ and $\{O(1), \ldots, O(p-1)\} = \{1, \ldots, p-1\}$. Thus assume by induction, $O(r_{j-1}) = r_{j-1}$ and $\{1, \ldots, r_{j-1}\} = \{O(1), \ldots, O(r_{j-1})\}$. Since there is a τ-match that starts at cell r_{j-1} and $p \geq 5$, we know that all the numbers

$$O(r_{j-1}), O(r_{j-1}+1), \ldots, O(r_{j-1}+p-3)$$

are less than $O(r_j) = O(r_{j-1}+p-2)$. Since $\{1, \ldots, r_{j-1}\} = \{O(1), \ldots, O(r_{j-1})\}$, it follows that $O(r_j) \geq r_j$. Next suppose that $O(r_j) > r_j$. Then let i be the least number that does not lie in the bricks $b_1, \ldots, b_j$. Because the numbers in each brick increase and the minimal numbers in the bricks are increasing, it must be the case that i is in the first cell of the next brick b_{j+1}. Now it cannot be that $j < k$ because then we have that $i = O(r_j+2) \leq r_j < O(r_j) < O(r_{j+1})$

which would violate the fact that there is a τ-match in O starting at cell r_j. If $j = k$, then it follows that there is a decrease between bricks b_k and b_{k+1} since b_{k+1} starts with $i \leq r_k < O(r_k)$. Since O is a fixed point of I_τ, this must mean that there is a τ-match in the cells of b_k and b_{k+1}. But since τ has only one descent, this τ-match can only start at the cell c which is the second to the last cell of b_k. Thus c must be greater than r_k because by hypothesis there cannot be a τ-match starting at cell r_k. So b_{k+1} must have more than $p-2$ cells. In this case, we have that $i \leq r_k < O(r_k) \leq O(c) < O(c+1) > O(c+2) = i$. But this can not be since to have a τ-match starting at cell c, we must have $O(c) < O(c+2)$. Thus it must be the case that $O(r_j) = r_j$. But then it must be the case that $r_{j-1} = O(r_{j-1}) < O(c) < O(r_j) = r_j$ for $r_{j-1} < c < r_j$ so that $\{O(1), \ldots, O(r_j)\} = \{1, \ldots, r_j\}$ as desired. Thus we have proved by induction that $O(r_j) = r_j$ and $\{1, \ldots, r_j\} = \{O(1), \ldots, O(r_j)\}$ for $j = 1, \ldots, k$.

This means that the sequence $O(1), \ldots, O(r_k)$ is completely determined. Next we claim that since there is no τ-match starting at position r_k, it must be the case that $O(r_k+1) = r_k+1$. That is, if $O(r_k+1) \neq r_k+1$, then r_k+1 cannot be in brick b_k so then r_k+1 must be in the first cell of the brick b_{k+1}. But then we claim that we could combine bricks b_k and b_{k+1}. That is, there will be a decrease between bricks b_k and b_{k+1} since $r_k+1 < O(r_k+1)$ and $O(r_k+1)$ is in b_k. Since there is no τ-match starting in O at cell r_k, the only possible τ-match among the cells of b_k and b_{k+1} would have to start at a cell $c \neq r_k$. Now it cannot be that $c < r_k$ since then $O(c) < O(c+1) < O(c+2)$. But it cannot be that $c > r_k$ since then $O(c) > r_k+1$ and r_k+1 would have to be part of the τ-match which means that $O(c)$ could not play the role of 1 in the τ-match. Thus it must be the case that $O(r_k+1) = r_k+1$. It then follows that if we let O' be the result of removing the first r_k cells from O and subtracting r_k from each number in the remaining cells, then O' will be a fixed point I_τ in $\mathcal{O}_{\tau,n-r_k}$. Note that if b_k has $p-2$ cells, then the first brick of O' will have one cell and if b_k has more than $p-2$ cells, then the first brick of O' will have at least two cells. Since there is a factor $-y$ coming from each of the bricks $b_1, \ldots, b_{k-1}$, it is easy to see that the fixed points in Case 2.b will contribute $\sum_{k\geq 3}(-y)^{k-1}U_{\tau,n-((k-1)(p-2)+1)}(y)$ to $U_{\tau,n}(y)$.

Thus we have proved the following theorem.

Theorem 24. *Let $\tau = 1324\dots p$ where $p \geq 5$. Then* $NM_\tau(t,x,y) = \left(\frac{1}{U_\tau(t,y)}\right)^x$ *where* $U_\tau(t,y) = 1 + \sum_{n\geq 1} U_{\tau,n}(y)\frac{t^n}{n!}$, $U_{\tau,1}(y) = -y$, *and for* $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=1}^{\lfloor \frac{n-2}{p-2} \rfloor} (-y)U_{\tau,n-(k(p-2)+1)}(y).$$

For example, we have computed the following. $U_{13245,1}(y) = -y$,
$U_{13245,2}(y) = -y + y^2$,
$U_{13245,3}(y) = -y + 2y^2 - y^3$,
$U_{13245,4}(y) = -y + 3y^2 - 3y^3 + y^4$,
$U_{13245,5}(y) = -y + 5y^2 - 6y^3 + 4y^4 - y^5$,
$U_{13245,6}(y) = -y + 7y^2 - 12y^3 + 10y^4 - 5y^5 + y^6$,
$U_{13245,7}(y) = -y + 9y^2 - 21y^3 + 23y^4 - 15y^5 + 6y^6 - y^7$,
$U_{13245,8}(y) = -y + 11y^2 - 34y^3 + 47y^4 - 39y^5 + 21y^6 - 7y^7 + y^8$,
$U_{13245,9}(y) = -y + 13y^2 - 51y^3 + 88y^4 - 90y^5 + 61y^6 - 28y^7 + 8y^8 - y^9$,
$U_{13245,10}(y) = -y + 15y^2 - 72y^3 + 153y^4 - 189y^5 + 156y^6 - 90y^7 + 36y^8 - 9y^9 + y^{10}$

$U_{132456,1}(y) = -y$,
$U_{132456,2}(y) = -y + y^2$,
$U_{132456,3}(y) = -y + 2y^2 - y^3$,
$U_{132456,4}(y) = -y + 3y^2 - 3y^3 + y^4$,
$U_{132456,5}(y) = -y + 4y^2 - 6y^3 + 4y^4 - y^5$,
$U_{132456,6}(y) = -y + 6y^2 - 10y^3 + 10y^4 - 5y^5 + y^6$,
$U_{132456,7}(y) = -y + 8y^2 - 17y^3 + 20y^4 - 15y^5 + 6y^6 - y^7$,
$U_{132456,8}(y) = -y + 10y^2 - 27y^3 + 38y^4 - 35y^5 + 21y^6 - 7y^7 + y^8$,
$U_{132456,9}(y) = -y + 12y^2 - 40y^3 + 68y^4 - 74y^5 + 56y^6 - 28y^7 + 8y^8 - y^9$,
$U_{132456,10}(y) = -y + 14y^2 - 57y^3 + 114y^4 - 146y^5 + 131y^6 - 84y^7 + 36y^8 - 9y^9 + y^{10}$

$U_{1324567,1}(y) = -y$,
$U_{1324567,2}(y) = -y + y^2$,

$$U_{1324567,3}(y) = -y + 2y^2 - y^3,$$
$$U_{1324567,4}(y) = -y + 3y^2 - 3y^3 + y^4,$$
$$U_{1324567,5}(y) = -y + 4y^2 - 6y^3 + 4y^4 - y^5,$$
$$U_{1324567,6}(y) = -y + 5y^2 - 10y^3 + 10y^4 - 5y^5 + y^6,$$
$$U_{1324567,7}(y) = -y + 7y^2 - 15y^3 + 20y^4 - 15y^5 + 6y^6 - y^7,$$
$$U_{1324567,8}(y) = -y + 9y^2 - 23y^3 + 35y^4 - 35y^5 + 21y^6 - 7y^7 + y^8,$$
$$U_{1324567,9}(y) = -y + 11y^2 - 34y^3 + 59y^4 - 70y^5 + 56y^6 - 28y^7 + 8y^8 - y^9,$$
$$U_{1324567,10}(y) = -y + 13y^2 - 48y^3 + 96y^4 - 130y^5 + 126y^6 - 84y^7 + 36y^8 - 9y^9 + y^{10}$$

Of course, one can use these initial values of the $U_{1324\ldots p,n}(y)$ to compute the initial values of $NM_{1324\ldots p}(t,x,y)$. For example, we have used Mathematica to compute the following initial terms of $NM_{13245}(t,x,y)$.

$$
\begin{aligned}
&NM_{13245}(t,x,y) = 1 + xyt + \frac{1}{2}\left(xy + x^2y^2\right)t^2 + \\
&\frac{1}{6}\left(xy + xy^2 + 3x^2y^2 + x^3y^3\right)t^3 + \\
&\frac{1}{24}\left(xy + 4xy^2 + 7x^2y^2 + xy^3 + 4x^2y^3 + 6x^3y^3 + x^4y^4\right)t^4 + \\
&\frac{1}{120}\left(xy + 10xy^2 + 15x^2y^2 + 11xy^3 + 30x^2y^3 + 25x^3y^3 + \right.\\
&\left. xy^4 + 5x^2y^4 + 10x^3y^4 + 10x^4y^4 + x^5y^5\right)t^5 + \\
&\frac{1}{720}\left(xy + 24xy^2 + 31x^2y^2 + 62xy^3 + 140x^2y^3 + 90x^3y^3 + 26xy^4 + 91x^2y^4 + \right.\\
&\left. 120x^3y^4 + 65x^4y^4 + xy^5 + 6x^2y^5 + 15x^3y^5 + 20x^4y^5 + 15x^5y^5 + x^6y^6\right)t^6 + \\
&\frac{1}{5040}\left(xy + 54xy^2 + 63x^2y^2 + 273xy^3 + 553x^2y^3 + 301x^3y^3 + 292xy^4 + \right.\\
&840x^2y^4 + 875x^3y^4 + 350x^4y^4 + 57xy^5 + 238x^2y^5 + 406x^3y^5 + 350x^4y^5 + \\
&\left. 140x^5y^5 + xy^6 + 7x^2y^6 + 21x^3y^6 + 35x^4y^6 + 35x^5y^6 + 21x^6y^6 + x^7y^7\right)t^7 + \\
&\frac{1}{40320}\left(xy + 116xy^2 + 127x^2y^2 + 1068xy^3 + 2000x^2y^3 + 966x^3y^3 + 2228xy^4 + \right.\\
&5726x^2y^4 + 5152x^3y^4 + 1701x^4y^4 + 1171xy^5 + 4016x^2y^5 + 5474x^3y^5 + \\
&3640x^4y^5 + 1050x^5y^5 + 120xy^6 + 575x^2y^6 + 1176x^3y^6 + 1316x^4y^6 + 840x^5y^6 + \\
&266x^6y^6 + xy^7 + 8x^2y^7 + 28x^3y^7 + 56x^4y^7 + 70x^5y^7 + \\
&\left. 56x^6y^7 + 28x^7y^7 + x^8y^8\right)t^8 + \cdots
\end{aligned}
$$

We note that there are many terms in these expansions which are easily explained. For example, we claim that for any $p \geq 4$, the coefficient of $x^k y^k$ in $NM_{1324\ldots p,n}(x,y)$ is always the Stirling number $S(n,k)$ which is the number of set partitions of $\{1,\ldots,n\}$ into k parts. That is, a permutation $\sigma \in S_n$ that contributes to the coefficient $x^k y^k$ in $NM_{1324\ldots p,n}(x,y)$ must have k left-to-right minima and $k-1$ descents. Since each left-to-right minima of σ which is not the first element is always the second element of descent pair, it follows that if $1 = i_1 < i_2 < i_3 < \cdots < i_k$ are the positions of the left to right minima, then σ must be increasing in each of the intervals $[1,i_2),[i_2,i_3),\ldots,[i_{k-1},i_k),[i_k,n]$. It is then easy to see that

$$\{\sigma_1,\ldots,\sigma_{i_2-1}\},\{\sigma_{i_2},\ldots,\sigma_{i_3-1}\},\ldots,\{\sigma_{i_{k-1}},\ldots,\sigma_{i_k-1}\},\{\sigma_{i_k},\ldots,\sigma_n\}$$

is just a set partition of $\{1,\ldots,n\}$ ordered by decreasing minimal elements. Moreover, it is easy to see that no such permutation can have a $1324\ldots p$-match for any $p \geq 4$. Vice versa, if $A_1,\ldots,A_k$ is a set partition of $\{1,\ldots,n\}$ such that $min(A_1) > \cdots > min(A_k)$, then the permutation $\sigma = A_k \uparrow A_{k-1} \uparrow \ldots A_1 \uparrow$ is a permutation with k left-to-right minima and $k-1$ descents where for any set $A \subseteq \{1,\ldots,n\}$, $A \uparrow$ is the list of the element of A in increasing order. It follows that for any $p \geq 4$,

1. $NM_{1324\ldots p,n}(x,y)|_{xy} = S(n,1) = 1$,
2. $NM_{1324\ldots p,n}(x,y)|_{x^2y^2} = S(n,2) = 2^{n-1}-1$,
3. $NM_{1324\ldots p,n}(x,y)|_{x^ny^n} = S(n,n) = 1$, and
4. $NM_{1324\ldots p,n}(x,y)|_{x^ny^n} = S(n,n-1) = \binom{n}{2}$.

We claim that

$$NM_{1324\ldots p,n}(x,y)|_{xy^2} = \begin{cases} 2^{n-1}-n & \text{if } n < p \\ 2^{n-1}-n-(n-(p-1)) = 2^{n-1}-2n+p-1 & \text{if } n \geq p. \end{cases}$$

That is, suppose that $\sigma \in S_n$ contributes to $NM_{1324\ldots p,n}(x,y)|_{xy^2}$. Then σ must have 1 left-to-right minima and one descent. It follows that σ must start with 1 and have one descent. Now if A is any subset of $\{2,\ldots,n\}$ and $B=\{2,\ldots,n\}-A$, then we let σ_A be the permutation $\sigma_A = 1\ A\uparrow B\uparrow$. The only choices of A that do not give rise to a permutation with one descent are $\emptyset$ and $\{2,\ldots,i\}$ for $i=2,\ldots,n$. It follows that there $2^{n-1}-n$ permutations that start with 1 and have 1 descent. Next consider when such a σ_A could have a $1324\ldots p$-match. If the $1324\ldots p$-match starts at position i, then it must be the case that $\text{red}(\sigma_i\sigma_{i+1}\sigma_{i+2}\sigma_{i+3}) = 1324$. This means that the only descent is at position $i+1$ and all the elements σ_j for $j \geq i+3$ are greater than or equal to σ_{i+3}. But this means that all the elements between 1 and σ_{i+2} must appear in increasing order in $\sigma_2\ldots\sigma_{i-1}$. It follows that σ_A is of the form $1\ldots(q-2)q(q+2)(q+1)(q+2)\ldots n$. There are no such permutations if $n \leq p-1$ and there are $n-(p-1)$ such permutations if $n \geq p$ as q can range from 1 to $n-(p-1)$.

3.4.2 $\tau = 1324$

Suppose $\tau = 1324$, the analysis of the fixed points of I_{1324} is a bit different from when $\tau = 1324...p$ for $p \geq 5$. Let O be a fixed point of I_{1324}. By Lemma 34, we know that 1 is in the first cell of O. Again, we claim that 2 must be in the second or third cell of O. That is, suppose that 2 is in cell c where $c > 3$. Then since there are no descents within any brick, 2 must be in the first cell of a brick. Moreover, since the minimal numbers in the bricks of O form an increasing sequence, reading from left to right, 2 must be in the first cell of the second brick. Thus if b_1 and b_2 are the first two bricks in O, then 1 is in the first cell of b_1 and 2 is in the first cell of b_2. But then we claim that there is no 1324-match in the elements of b_1 and b_2. That is, since $c > 3$, b_1 has at least three cells so that O starts with an increasing sequence of length 3. But this means that 1 can not be part of a 1324-match. Similarly, no other cell of b_1 can be part of 1324-match because the 2 in cell c is smaller than any of the remaining numbers of b_1. But this would mean that we could apply case (ii) of the definition of I_{1324} to b_1 and b_2 which would violate our assumption that O is a fixed point of I_{1324}. Thus, we have two cases.

Case I. 2 is in cell 2 of O.

In this case there are two possibilities, namely, either (i) 1 and 2 lie in the first brick b_1 of O or (ii) brick b_1 has one cell and 2 is the first cell of the second brick b_2 of O. In either case, it is easy to see that 1 is not part of a 1324-match and if we remove cell 1 from O and subtract 1 from the elements in the remaining cells, we would end up with a fixed point O' of I_{1324} in $\mathcal{O}_{1324,n-1}$. Now in case (i), it is easy to see that $sgn(O)W(O) = sgn(O')W(O')$ and in case (ii) since b_1 will have a label $-y$ on the first cell, $sgn(O)W(O) = (-y)sgn(O')W(O')$. It follows that fixed points in Case 1 will contribute $(1-y)U_{1324,n-1}(y)$ to $U_{1324,n}(y)$.

Case II. 2 in cell 3 of O.

Let $O(i)$ denote the element in i cell of O and $b_1, b_2, \ldots$ be the bricks of O, reading from left to right. Since there are no descents within bricks in O and the minimal elements in the bricks are increasing, we know that 2 is in the first cell of a brick b_2. Thus b_1 has two cells. But then b_2 must have at least two cells since if b_2 has one cell, there could be no 1324-match contained in the cells of b_1 and b_2 and we could combine bricks b_1 and b_2 which would mean that O is not a fixed point of I_{1324}. Thus b_1 has two cells and b_2 has at least two cells. But then the only reason that we could not combine bricks b_1 and b_2 is that there is a 1324-match in the cells of b_1 and b_2 which could only start at the first cell.

We now have two subcases.

Case II.a. There is no 1324-match in O starting at cell 3.

Then we claim that $\{O(1), O(2), O(3), O(4)\} = \{1, 2, 3, 4\}$. That is, if

$$\{O(1), O(2), O(3), O(4)\} \neq \{1, 2, 3, 4\},$$

then let $i = min(\{1, 2, 3, 4\} - \{O(1), O(2), O(3), O(4)\})$. Since there is a 1324-match starting at position 1, it follows that $O(4) > 4$ since $O(4)$ is the fourth largest element in $\{O(1), O(2), O(3), O(4)\}$. Since the minimal elements of the bricks of O are increasing, it must be that i is the first element in brick b_3. But then we claim that we could combine bricks b_2 and b_3. That is, there will be a decrease between bricks b_2 and b_3 since $i < O(4)$ and $O(4)$ is in b_2. Since there is no 1324-match in O starting at cell 3, the only possible 1324-match among the elements in b_2 and b_3 would have start at a cell $c > 3$. But then $O(c) > i$, which is impossible since it would have to play the role of 1 in the 1324-match and i would have to play the role of 2 in the 1324-match since i occupies the first cell of b_3. Thus it must be the case that $O(1) = 1$, $O(2) = 3$, $O(3) = 2$, and $O(4) = 4$.

It then follows that if we let O' be the result of removing the first 3 cells from O and subtracting 3 from the remaining elements, then O' will be a fixed point I_{1324} in $\mathcal{O}_{1324,n-3}$. Since there is $-y$ coming from the brick b_1, it is easy to see that the fixed points in Case II.a will contribute $-yU_{1324,n-3}(y)$ to $U_{1324,n}(y)$.

Case II.b. There is a 1324-match starting a 3 in O.

In this case, it must be that $O(3) < O(4) > O(5)$ so that b_2 must have two cells and brick b_3 starts at cell 5. We claim that b_3 must have at least two cells. That is, if b_3 has one cell, then there could be no 1324-match among the cells of b_2 and b_3 so that we could combine b_2 and b_3 violating the fact that O is a fixed point of I_{1324}.

In the general case, assume that in O, the bricks $b_2, \ldots, b_{k-1}$ all have two cells and there are 1324-matches starting at cells $1, 3, \ldots, 2k-3$ but there is no 1324-match starting at cell $2k-1$ in O. Then we know that b_k has least two cells. Let $c_i < d_i$ be the numbers in the first two cells of brick b_i for $i = 1, \ldots, k$. Then we have that $\text{red}(c_i d_i c_{i+1} d_{i+1}) = 1324$ for $1 \leq i \leq k-1$. This means that $c_i < c_{i+1} < d_i < d_{i+1}$.

First we claim that it must be the case that $\{O(1), \ldots, O(2k)\} = \{1, \ldots, 2k\}$. If not there is a number greater than $2k$ that occupies one of the first $2k$ cells. Let M be the greatest such number. If M occupies one of the first $2k$ cells then there must be a number less than $2k$ that occupies one of the last $n-2k$ cells. Let m be the least such number. Since numbers in bricks are increasing, M must occupy the last cell in one of the first $k-1$ bricks or occupy cell $2k$. If M occupies the last cell in one of the first $k-1$ bricks, then M is part of a τ-match

$$\cdots \boxed{c_i \mid M} \boxed{c_{i+1} \mid d_{i+1}} \cdots.$$

But then $\mathrm{red}(c_i\ M\ c_{i+1}\ d_{i+1}) = 1\ 3\ 2\ 4$ implies that $M < d_{i+1}$ which contradicts our choice of M as the greatest number in the first $2k$ cells. Thus M cannot occupy the last cell in one of the first $k-1$ bricks. This means that M must occupy cell $2k$ in O.

Since numbers in bricks are increasing, m must occupy the first cell of b_{k+1}. But then there is a descent between bricks b_k and b_{k+1} so that m must be part of a 1324-match. But the only way this can happen is if in the 1324-match involving m, m plays the role of 2 and the numbers in the last two cells of brick b_k play the role of 1 3. Since, we are assuming that a 1324-match does not start at cell $2k-1$ which is the cell that the number c_k occupies, the numbers in the last two cells of brick b_k must be greater than or equal to $d_k = M$ which is impossible since $m < M$. Thus it must be the case that $\{O(1), \ldots, O(2k)\} = \{1, \ldots, 2k\}$ and that $d_k = 2k$. It now follows that if we remove the first $2k-1$ cells from O and replace each remaining number i in O by $i-(2k-1)$, then we will end up with a fixed point in O' of I_{1324} in $\mathcal{O}_{n-(2k-1)}$. Thus each such fixed point O will contribute $(-y)^{k-1}U_{n-2k+1}(y)$ to $U_n(y)$.

The only thing left to do is to count the number of such fixed points O. That is, we must count the number of sequences $c_1d_1c_2d_2\ldots c_kd_k$ such that (i) $c_1 = 1$, (ii) $c_2 = 2$, (iii) $d_k = 2k$, (iv) $\{c_1, d_1, \ldots, c_k, d_k\} = \{1, 2, \ldots, 2k\}$, and (v) $\mathrm{red}(c_id_ic_{i+1}d_{i+1}) = 1324$ for each $1 \leq i \leq k-1$. We claim that there are C_{k-1} such sequences where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catlan number. It is well known that C_{k-1} counts the number of Dyck paths of length $2k-2$. A Dyck path of length $2k-2$ is a path that starts at $(0,0)$ and ends at $(2k-2,0)$ and consists of either *up-steps* (1,1) or *down-steps* (1,-1) in such a way that the path never goes below the x-axis. Thus we will give a bijection ϕ between the set of Dyck paths of length $2k-2$ and the set of sequences $c_1, d_1, \ldots, c_k, d_k$ satisfying conditions (i)-(v). The map ϕ is quite simple. That is, suppose that we start with a Dyck path $P = (p_1, p_2, \ldots, p_{2k-2})$ of length $2k-2$. First, label the segments $p_1, \ldots, p_{2k-2}$ with

$2, \ldots, 2k-1$, respectively. Then $\phi(P)$ is the sequence $c_1 d_1 \ldots c_k d_k$ where $c_1 = 1$ and $c_2 \ldots c_k$ are the labels of the up-steps of P, reading from left to right, $d_1 \ldots d_{k-1}$ are the labels of the down steps, reading from left to right, and $d_{2k} = 2k$. We have pictured an example in Figure 3.4 of the bijection ϕ in the case where $k = 6$.

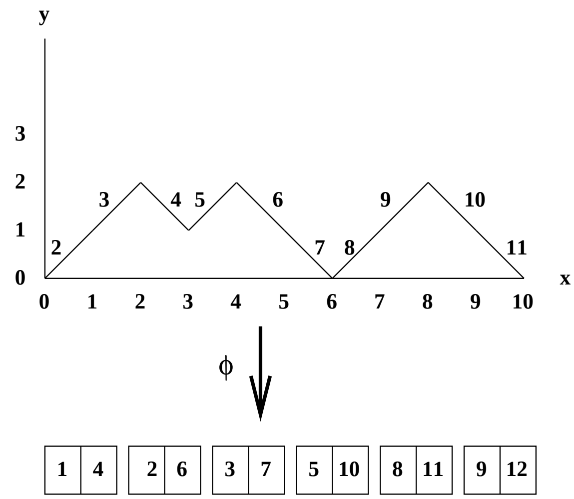


Figure 3.4: The bijection ϕ.

It is easy to see by construction that if P is a Dyck path of length $2k-2$ and $\phi(P) = c_1 d_1 \ldots c_k d_k$, then $c_1 < c_2 < \cdots < c_k$ and $d_1 < d_2 < \cdots < d_k$. Moreover, since each Dyck path must start with an up-step, we have that $c_2 = 2$. Clearly $c_1 = 1$, $d_k = 2k$, and $\{c_1, d_1, \ldots, c_k, d_k\} = \{1, \ldots, 2k\}$ by construction. Thus $c_1 d_1 \ldots c_k d_k$ satisfies conditions (i)-(iv). For condition (v), note that $c_1 = 1 < d_1 > 2 = c_2 < d_2$ so that $\text{red}(c_1 d_1 c_2 d_2) = 1\ 3\ 2\ 4$. If $2 \leq i \leq k-1$, then note that c_i equals the label of the $(i-1)$st up-step, c_{i+1} equals the label of the i-th up-step, and d_i is the label of i-th down-step. Since in a Dyck path, the i-th down-step must occur after the i-th up-step, it follows that $c_i < c_{i+1} < d_i < d_{i+1}$ so that $\text{red}(c_i d_i c_{i+1} d_{i+1}) = 1\ 3\ 2\ 4$. Vice versa, if we start with a sequence $c_1 d_1 \ldots c_k d_k$ satisfying conditions (i)-(v) and create a path $P = (p_1, \ldots, p_{2k-2})$ with labels $2, \ldots, 2k-1$ such that p_j is an up-step if $j+1 \in \{c_2, \ldots, c_k\}$ and p_j is an down-step if $j+1 \in \{d_1, \ldots, d_{k-1}\}$, then condition (iii) ensures P starts with an up-step and condition (v) ensures that the i-th up-step occurs before the i-th down step so that P will be a Dyck path. Thus ϕ is a bijection between the set of Dyck paths of length $2k-2$ and the set of sequence $c_1, d_1, \ldots, c_k, d_k$ satisfying conditions (i)-(v).

It follows that fixed points O of I_{1324} where the bricks $b_1, b_2, \ldots, b_{k-1}$ are of size 2 and there are 1324-matches starting at positions $1, 3, \ldots, 2k-3$ in O, but there is no 1324-match starting at position $2k-1$ in O contribute to $U_{\tau,n}(y)$,

$$C_{k-1}(-y)^{k-1}U_{\tau,n-2k+1}(y).$$

Thus we have proved the following theorem.

Theorem 25. *Let $\tau = 1324$. Then*

$$NCM_\tau(t,x,y) = \left(\frac{1}{U_\tau(t,y)}\right)^x \text{ where } U_\tau(t,y) = 1 + \sum_{n\geq 1} U_{\tau,n}(y)\frac{t^n}{n!}$$

and $U_{\tau,1}(y) = -y$ and for $n > 1$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=2}^{\lfloor \frac{n}{2} \rfloor}(-y)^{k-1}C_{k-1}U_{\tau,n-2k+1}(y)$$

where C_k is the k^{th} Catalan number.

In this case, one can easily compute that

$U_{1324,1}(y) = -y,$
$U_{1324,2}(y) = -y + y^2,$
$U_{1324,3}(y) = -y + 2y^2 - y^3,$
$U_{1324,4}(y) = -y + 4y^2 - 3y^3 + y^4,$
$U_{1324,5}(y) = -y + 6y^2 - 8y^3 + 4y^4 - y^5,$
$U_{1324,6}(y) = -y + 8y^2 - 18y^3 + 13y^4 - 5y^5 + y^6,$
$U_{1324,7}(y) = -y + 10y^2 - 32y^3 + 36y^4 - 19y^5 + 6y^6 - y^7,$
$U_{1324,8}(y) = -y + 12y^2 - 50y^3 + 85y^4 - 61y^5 + 26y^6 - 7y^7 + y^8,$
$U_{1324,9}(y) = -y + 14y^2 - 72y^3 + 166y^4 - 170y^5 + 94y^6 - 34y^7 + 8y^8 - y^9,$
$U_{1324,10}(y) = -y + 16y^2 - 98y^3 + 287y^4 - 412y^5 + 296y^6 - 136y^7 + 43y^8 - 9y^9 + y^{10}$

This, in turn, allows us to compute the first few terms of the generating function $NM_{1324}(t,x,y)$. That is, one can use Mathematica to compute that

$$\begin{aligned}
&NM_{1324}(t,x,y) = \\
&1 + txy + \frac{1}{2}t^2\left(xy + x^2y^2\right) + \frac{1}{6}t^3\left(xy + xy^2 + 3x^2y^2 + x^3y^3\right) + \\
&\frac{1}{24}t^4\left(xy + 3xy^2 + 7x^2y^2 + xy^3 + 4x^2y^3 + 6x^3y^3 + x^4y^4\right) + \\
&\frac{1}{120}t^5\left(xy + 9xy^2 + 15x^2y^2 + 8xy^3 + 25x^2y^3 + 25x^3y^3 + xy^4 + \right.\\
&\left.5x^2y^4 + 10x^3y^4 + 10x^4y^4 + x^5y^5\right) + \\
&\frac{1}{720}t^6\left(xy + 23xy^2 + 31x^2y^2 + 47xy^3 + 119x^2y^3 + 90x^3y^3 + 20xy^4 + 73x^2y^4 + \right.\\
&\left.105x^3y^4 + 65x^4y^4 + xy^5 + 6x^2y^5 + 15x^3y^5 + 20x^4y^5 + 15x^5y^5 + x^6y^6\right) + \\
&\frac{1}{5040}t^7\left(xy + 53xy^2 + 63x^2y^2 + 221xy^3 + 490x^2y^3 + 301x^3y^3 + 202xy^4 + \right.\\
&637x^2y^4 + 749x^3y^4 + 350x^4y^4 + 47xy^5 + 196x^2y^5 + 343x^3y^5 + 315x^4y^5 + \\
&\left.140x^5y^5 + xy^6 + 7x^2y^6 + 21x^3y^6 + 35x^4y^6 + 35x^5y^6 + 21x^6y^6 + x^7y^7\right) + \\
&\frac{1}{40320}\left(xy + 115xy^2 + 127x^2y^2 + 922xy^3 + 1838x^2y^3 + 966x^3y^3 + 1571xy^4 + \right.\\
&4421x^2y^4 + 4466x^3y^4 + 1701x^4y^4 + 795xy^5 + 2890x^2y^5 + 4270x^3y^5 + \\
&3164x^4y^5 + 1050x^5y^5 + 105xy^6 + 495x^2y^6 + 1008x^3y^6 + 1148x^4y^6 + \\
&770x^5y^6 + 266x^6y^6 + xy^7 + 8x^2y^7 + 28x^3y^7 + 56x^4y^7 + \\
&\left.70x^5y^7 + 56x^6y^7 + 28x^7y^7 + x^8y^8\right)t^8 + \cdots
\end{aligned}$$

We note that there are other methods to compute $NM_{1324}(t,1,1)$. That is, Elizalde [11] developed recursive techniques to find the coefficients of the series $NM_{1324}(t,1,1)$.

3.4.3 $\tau = 1p23\ldots(p-1)$ for $p \geq 4$

Now we specialize to the case where $\tau = 1p23\ldots(p-1)$ and $p \geq 4$. In this case, we can make a finer analysis of the fixed points of I_τ. Let O be a fixed point of I_τ. By Lemma 34, we know that 1 is in the first cell of O. We claim that 2 must be in the second or third cell of O. That is, suppose that 2 is in cell c where $c > 3$. Then since there are no descents within any brick, 2 must be the first cell of a brick. Moreover, since the minimal numbers in the bricks of O form an increasing sequence, reading from left to right, 2 must be in the first cell of the second brick. Thus if b_1 and b_2 are the first two bricks in O, then 1 is in the first cell of b_1 and 2 is in the first cell of b_2. But then we claim that there is no τ-match in the cells of b_1 and b_2. That is, since $c > 3$, b_1 has at least three cells so that O starts with an increasing sequence of length 3. But this means that 1 can not be part of a $1p23\ldots(p-1)$-match. Similarly, no other cell of b_1 can be part of $1p23\ldots(p-1)$-match because the 2 in cell c is less than any of the remaining numbers of b_1. Thus if there is a $1p23\ldots(p-1)$-match among the cells of b_1 and b_2, it would have to be entirely contained in b_2 which is impossible. But this would mean that we could apply case (ii) of the definition of I_τ to b_1 and b_2 which would violate our assumption that O is a fixed point of I_τ. Thus, we have two cases.

Case 1. 2 is in cell 2 of O.

In this case there are two possibilities, namely, either (i) 1 and 2 are both in the first brick b_1 of O or (ii) brick b_1 is a single cell filled with 1 and 2 is in the first cell of the second brick b_2 of O. In either case, it is easy to see that 1 is not part of a $1p23\ldots(p-1)$-match in O and if we remove cell 1 from O and subtract 1 from the numbers in the remaining cells, we would end up with a fixed point O' of I_τ in $\mathcal{O}_{\tau,n-1}$. Now in case (i), it is easy to see that $sgn(O)W(O) = sgn(O')W(O')$ and in case (ii) since b_1 will have a label $-y$ on the first cell, $sgn(O)W(O) = (-y)sgn(O')W(O')$. It follows that fixed points in Case 1 will contribute $(1-y)U_{\tau,n-1}(y)$ to $U_{\tau,n}(y)$.

Case 2. 2 is in cell 3 of O.

Let $O(i)$ denote the number in cell i of O and $b_1, b_2, \ldots$ be the bricks of O, reading from left to right. Since there are no descents within bricks in O and the minimal elements of the bricks are increasing, reading from left to right, it must be the case that 2 is in the first cell of brick b_2. Thus b_1 has two cells. But then b_2 must have at least $p-2$ cells since otherwise, there could be no τ-match contained in the cells of b_1 and b_2 and we could combine bricks b_1 and b_2 which again would mean that O is not a fixed point of I_τ. Thus b_1 is a brick with two cells and b_2 is brick with at least $p-2$ cells. But then the only reason that we could not combine bricks b_1 and b_2 is that there is a τ-match in the cells of b_1 and b_2 which could only start at the very first cell.

Next we claim that $O(p-1) = p-2$. That is, since there is a τ-match starting at the first cell and $p \geq 4$, we know that $O(p-1)$ is greater than all elements of the set $\{O(1), \ldots, O(p-2)\} - \{O(2)\}$ and $O(2) > O(p-1) > O(p-2)$. Thus $O(p-1)$ is greater than $p-3$ other numbers so $O(p-1) \geq p-2$. Now if $O(p-1) > p-2$, then let i be least number in the set $\{1, \ldots, p-2\}$ that is not contained in bricks b_1 and b_2. Since the numbers in each brick are increasing and the minimal numbers of the bricks are increasing, the only possible position for i is the first cell of brick b_3. But then it follows that there is a decrease between bricks b_2 and b_3. Since O is a fixed point of I_τ, this must mean that there is a τ-match in the cells of b_2 and b_3. But since τ has only one descent, this τ-match can only start at the cell c which is the second to the last cell of b_2. Thus c could be $p-1$ if b_2 has $p-2$ cells or $c > p-1$ if b_2 has more than $p-2$ cells. In either case, $p-1 \leq O(p-1) \leq O(c) < O(c+1) > O(c+2) = i$. But this is impossible since to have a τ-match starting at cell c, we must have $O(c) < O(c+2)$. Thus it must be the case that $O(p-1) = p-2$ and $\{O(1), \ldots, O(p-1)\} - \{O(2)\} = \{1, \ldots, p-2\}$.

We now have two subcases.

Case 2.a. There is no τ-match in O starting at cell $p-1$.

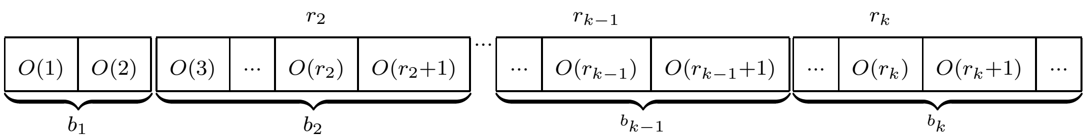
Then we claim that $O(p) = p-1$. That is, if $O(p) \neq p-1$, then $O(p) > p-1$. This means that $p-1$ cannot be in brick b_2. Similarly, $p-1$ can not be $O(2)$ since the fact that there is a $1p2\ldots(p-1)$-match starting at cell 1 means that $O(2) > O(p)$. Thus $p-1$ must be the first cell of the brick b_3. But then we claim that we could combine bricks b_2 and b_3. That is, there will be a decrease between bricks b_2 and b_3 since $p-1 < O(p)$ and $O(p)$ is in b_2. Since there is no τ-match in O starting at cell $p-1$, the only possible τ-match among the cells of b_2 and b_3

would have start at a cell $c \neq p-1$. But it cannot be that $c < p-1$ since then it would be the case that $O(c) < O(c+1) < O(c+2)$. Similarly, it cannot be that $c > p-1$ since then $O(c) > p-1$ and $p-1$ has to be part of the τ-match which is impossible since $O(c)$ must play the role of 1 in the τ-match. Thus it must be the case that $O(p) = p-1$. It then follows that if we let O' be the result of removing the first $p-1$ cells from O and renumbering the remaining cells in such a way that we keep the same relative order but use the numbers $1, \ldots n-(p-1)$, then O' will be a fixed point of I_τ in $\mathcal{O}_{\tau,n-(p-1)}$. Note that if b_2 has $p-2$ cells, then O' will start with a brick with one cell and if b_2 has more than $p-2$ cells, then O' will start with a brick with at least two cells. The τ-match starts at the first cell, so $O(2) > O(p) = p-1$. Since there are $n-(p-1)$ such numbers to choose from and since there is $-y$ coming from the brick b_1, it is easy to see that the fixed points in Case 2.a will contribute $(-y)(n-(p-1))U_{\tau,n-(p-1)}(y)$ to $U_{\tau,n}(y)$.

Case 2.b. There is a τ-match starting at $p-1$ in O.

In this case, it must be that $O(p-1) < O(p) > O(p+1)$ so that b_2 must have $p-2$ cells and brick b_3 starts at cell $p+1$. We claim that b_3 must have at least $p-2$ cells. That is, if b_3 has less than $p-2$ cells, then there could be no τ-match among the cells of b_2 and b_3 so then we could combine b_2 and b_3 violating the fact that O is a fixed point of I_τ.

In the general case, assume that in O, the bricks $b_2, \ldots, b_{k-1}$ all have $p-2$ cells. Then let $r_1 = 1$ and for $j = 2, \ldots, k-1$, let $r_j = 1 + (j-1)(p-2)$. Thus r_j is the position of the second to last cell of brick b_j for $1 \leq j \leq k-1$. Furthermore, assume that there is a τ-match starting at cell r_j for $1 \leq j \leq k-1$. It follows that $O(r_{k-1}) < O(r_{k-1}+1) > O(r_{k-1}+2)$ so that brick b_k must start at cell $r_{k-1}+2$ and there is a decrease between bricks b_{k-1} and b_k. But then it must be the case that b_k has at least $p-2$ cells since if b_k has less than $p-2$ cells, we could combine bricks b_{k-1} and b_k violating the fact that O is a fixed point of I_τ. Let $r_k = 1 + (k-1)(p-2)$ and assume that O does not have a τ-match starting at position r_k. Thus we have the situation pictured below.



First we claim $O(r_j) = r_j - (j-1)$ and

$$\{1, \ldots, r_j - (j-1)\} = \{O(1), \ldots, O(r_j)\} - \{O(r_i + 1)\}_{i=1\ldots j-1}$$

for $j = 1, \ldots, k$. We have shown that $O(1) = 1$ and that $O(r_2) = O(p-1) = p-2$ and $\{O(1), \ldots, O(p-1)\} - \{O(2)\} = \{1, \ldots, p-2\}$. Thus assume by induction, $O(r_{j-1}) = r_{j-1} - (j-2)$ and $\{1, \ldots, r_{j-1} - (j-2)\} = \{O(1), \ldots, O(r_{j-1})\} - \{O(r_i + 1)\}_{i=1\ldots j-2}$. Since there is a τ-match that starts at cell r_{j-1} and $p \geq 4$, we know that all the numbers

$$\{O(r_{j-1}), O(r_{j-1}+2), \ldots, O(r_{j-1}+p-3)\} - \{O(r_{j-1}+1)\}$$

are less than $O(r_j) = O(r_{j-1} + p - 2)$. Since

$$\{1, \ldots, r_{j-1} - (j-2)\} = \{O(1), \ldots, O(r_{j-1})\} - \{O(r_i + 1)\}_{i=1\ldots j-2},$$

it follows that $O(r_j) \geq r_{j-1} - (j-2) + (p-3) = r_j - (j-1)$. Next suppose that $O(r_j) > r_j - (j-1)$. Then let i be the least number which is in

$$\{1, \ldots, r_j - (j-1)\} - (\{O(1), \ldots, O(r_j)\} - \{O(r_i + 1)\}_{i=1\ldots j-1}).$$

It is that case that $O(r_1 + 1) > O(r_2 + 1) > \ldots > O(r_j + 1)$ so that i does not lie in the bricks $b_1, \ldots, b_j$. Because the numbers in each brick increase and the minimal numbers in the bricks are increasing, it must be the case that i is in the first cell of the next brick b_{j+1}. Now it cannot be that $j < k$ because then we have that $i = O(r_j + 2) \leq r_j - (j-1) < O(r_j) < O(r_{j+1})$ which would violate the fact that there is a τ-match in O starting at cell r_j. If $j = k$, then it follows that there is a decrease between bricks b_k and b_{k+1} since b_{k+1} starts with $i \leq r_k - (k-1) < O(r_k)$. Since O is a fixed point of I_τ, this must mean that there is a τ-match in the cells of b_k and b_{k+1}. But since τ has only one descent, this τ-match can only start at the cell c which is the second to the last cell of b_k.

Thus c must be greater than r_k because we are assuming that there is no τ-match starting at cell r_k. So b_{k+1} must have more than $p-2$ cells. In this case, we have that $i \leq r_k - (k-1) < O(r_k) \leq O(c) < O(c+1) > O(c+2) = i$. But this cannot be since to have a τ-match starting at cell c, we must have $O(c) < O(c+2)$. Thus it must be the case that $O(r_j) = r_j - (j-1)$. Finally since

1. $\{1, \ldots, r_{j-1} - (j-2)\} = \{O(1), \ldots, O(r_{j-1})\} - \{O(r_i + 1)\}_{i=1\ldots j-2}$ and
2. $O(r_{j-1}), O(r_{j-1}+2), \ldots, O(r_{j-1}+p-3) < O(r_j)$,

it must be the case that

$$\{1, \ldots, r_j - (j-1)\} = \{O(1), \ldots, O(r_j)\} - \{O(r_i + 1)\}_{i=1\ldots j-1}$$

as desired. Thus we have proved by induction that $O(r_j) = r_j - (j-1)$ and $\{1, \ldots, r_j - (j-1)\} = \{O(1), \ldots, O(r_j)\} - \{O(r_i+1)\}_{i=1\ldots j-1}$ for $j = 1, \ldots, k$.

This means that the set $\{O(1), \ldots, O(r_k)\} - \{O(r_i+1)\}_{i=1\ldots k-1}$ is completely determined. Next we claim that since there is no τ-match starting at position r_k, it must be the case that $O(r_k+1) = r_k - (k-1) + 1 = r_k - k + 2$. That is, since there is a τ-match starting at each of the cells r_j for $j = 1, \ldots, k-1$, it must be the case that $O(r_1+1) > O(r_2+1) > \cdots > O(r_{k-1}+1) > O(r_k+1)$. If $O(r_k+1) \neq r_k - k + 2$, then $O(r_k+1) > r_k - k + 2$ and, hence, $r_k - k + 2$ cannot be in any of the bricks $b_1, \ldots, b_k$. Thus $r_k - k + 2$ must be in the first cell of the brick b_{k+1}. But then we claim that we could combine bricks b_k and b_{k+1}. That is, there will be a decrease between bricks b_k and b_{k+1} since $r_k - k + 2 < O(r_k+1)$ and $O(r_k+1)$ is in b_k. Since there is no τ-match starting in O at cell r_k, the only possible τ-match among the cells of b_k and b_{k+1} would have to start at a cell $c \neq r_k$. But it cannot be that $c < r_k$ since then $O(c) < O(c+1) < O(c+2)$. Similarly, it cannot be that $c > r_k$ since $O(c) > r_k - k + 2$ and $r_k - k + 2$ would have to be part of the τ-match which means that $O(c)$ could not play the role of 1 in the τ-match. Thus it must be the case that $O(r_k+1) = r_k - k + 2$. It then follows that if we let O' be the result of removing the first r_k cells from O and renumbering the remaining cells in such a way that we keep the same relative order but use the numbers $1, \ldots, n - r_k$, then O' will be a fixed point I_τ in $\mathcal{O}_{\tau, n-r_k}$. Note that if b_k

has $p-2$ cells, then the first brick of O' will have one cell and if b_k has more than $p-2$ cells, then the first brick of O' will have at least two cells. Since there is a τ-match starting at each of the cells r_j for $j = 1, \ldots, k-1$, it must be the case that $O(r_1+1) > O(r_2+1) > \cdots > O(r_{k-1}+1) > O(r_k+1) = r_k - k + 2$. Hence we can choose any $k-1$ numbers from $n-(r_k-k+2) = n-((k-1)(p-2)+1-k+2) = n-(k-1)(p-3)-2$ and place them in cells $r_1+1, r_2+1, \ldots, r_{k-1}+1$ in decreasing order, reading from left to right, to produce an O which has τ-matches starting at cells $1, r_2, \ldots, r_{k-1}$. Thus we have $\binom{n-(k-1)(p-3)-2}{k-1}$ ways to assign numbers to $O(r_1+1), O(r_2+1), \ldots, O(r_{k-1}+1)$. Since there is a factor $-y$ coming from each of the bricks $b_1, \ldots, b_{k-1}$, it follows that the fixed points in Case 2.b will contribute $\sum_{k\geq 3}(-y)^{k-1}\binom{n-(k-1)(p-3)-2}{k-1}U_{\tau,n-((k-1)(p-2)+1)}(y)$ to $U_{\tau,n}(y)$.

Hence we have proved Theorem 26.

Theorem 26. *Let $\tau = 1p23\ldots(p-1)$ where $p \geq 4$. Then*

$$NM_\tau(t,x,y) = \left(\frac{1}{U_\tau(t,y)}\right)^x \text{ where } U_\tau(t,y) = 1 + \sum_{n\geq 1} U_{\tau,n}(y)\frac{t^n}{n!},$$

$U_{\tau,1}(y) = -y$, and for $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=1}^{\lfloor \frac{n-2}{p-2} \rfloor} (-y)^k \binom{n-k(p-3)-2}{k} U_{\tau,n-(k(p-2)+1)}(y).$$

We have computed the following.

$U_{1423,1}(y) = -y$
$U_{1423,2}(y) = -y + y^2$
$U_{1423,3}(y) = -y + 2y^2 - y^3$
$U_{1423,4}(y) = -y + 4y^2 - 3y^3 + y^4$
$U_{1423,5}(y) = -y + 7y^2 - 9y^3 + 4y^4 - y^5$
$U_{1423,6}(y) = -y + 11y^2 - 23y^3 + 16y^4 - 5y^5 + y^6$
$U_{1423,7}(y) = -y + 16y^2 - 53y^3 + 54y^4 - 25y^5 + 6y^6 - y^7$
$U_{1423,8}(y) = -y + 22y^2 - 110y^3 + 165y^4 - 105y^5 + 36y^6 - 7y^7 + y^8$
$U_{1423,9}(y) = -y + 29y^2 - 208y^3 + 457y^4 - 400y^5 + 181y^6 - 49y^7 + 8y^8 - y^9$
$U_{1423,10}(y) = -y + 37y^2 - 364y^3 + 1151y^4 - 1391y^5 + 826y^6 - 287y^7 + 64y^8 - 9y^9 + y^{10}$

$$U_{15234,1}(y) = -y$$
$$U_{15234,2}(y) = -y + y^2$$
$$U_{15234,3}(y) = -y + 2y^2 - y^3$$
$$U_{15234,4}(y) = -y + 3y^2 - 3y^3 + y^4$$
$$U_{15234,5}(y) = -y + 5y^2 - 6y^3 + 4y^4 - y^5$$
$$U_{15234,6}(y) = -y + 8y^2 - 13y^3 + 10y^4 - 5y^5 + y^6$$
$$U_{15234,7}(y) = -y + 12y^2 - 27y^3 + 26y^4 - 15y^5 + 6y^6 - y^7$$
$$U_{15234,8}(y) = -y + 17y^2 - 52y^3 + 65y^4 - 45y^5 + 21y^6 - 7y^7 + y^8$$
$$U_{15234,9}(y) = -y + 23y^2 - 97y^3 + 150y^4 - 130y^5 + 71y^6 - 28y^7 + 8y^8 - y^9$$
$$U_{15234,10}(y) = -y + 30y^2 - 174y^3 + 337y^4 - 346y^5 + 231y^6 - 105y^7 + 36y^8 - 9y^9 + y^{10}$$

$$U_{162345,1}(y) = -y$$
$$U_{162345,2}(y) = -y + y^2$$
$$U_{162345,3}(y) = -y + 2y^2 - y^3$$
$$U_{162345,4}(y) = -y + 3y^2 - 3y^3 + y^4$$
$$U_{162345,5}(y) = -y + 4y^2 - 6y^3 + 4y^4 - y^5$$
$$U_{162345,6}(y) = -y + 6y^2 - 10y^3 + 10y^4 - 5y^5 + y^6$$
$$U_{162345,7}(y) = -y + 9y^2 - 18y^3 + 20y^4 - 15y^5 + 6y^6 - y^7$$
$$U_{162345,8}(y) = -y + 13y^2 - 33y^3 + 41y^4 - 35y^5 + 21y^6 - 7y^7 + y^8$$
$$U_{162345,9}(y) = -y + 18y^2 - 58y^3 + 86y^4 - 80y^5 + 56y^6 - 28y^7 + 8y^8 - y^9$$
$$U_{162345,10}(y) = -y + 24y^2 - 97y^3 + 174y^4 - 186y^5 + 141y^6 - 84y^7 + 36y^8 - 9y^9 + y^{10}$$

$U_{1723456,1}(y) = -y$
$U_{1723456,2}(y) = -y + y^2$
$U_{1723456,3}(y) = -y + 2y^2 - y^3$
$U_{1723456,4}(y) = -y + 3y^2 - 3y^3 + y^4$
$U_{1723456,5}(y) = -y + 4y^2 - 6y^3 + 4y^4 - y^5$
$U_{1723456,6}(y) = -y + 5y^2 - 10y^3 + 10y^4 - 5y^5 + y^6$
$U_{1723456,7}(y) = -y + 7y^2 - 15y^3 + 20y^4 - 15y^5 + 6y^6 - y^7$
$U_{1723456,8}(y) = -y + 10y^2 - 24y^3 + 35y^4 - 35y^5 + 21y^6 - 7y^7 + y^8$
$U_{1723456,9}(y) = -y + 14y^2 - 40y^3 + 62y^4 - 70y^5 + 56y^6 - 28y^7 + 8y^8 - y^9$
$U_{1723456,10}(y) = -y + 19y^2 - 66y^3 + 114y^4 - 136y^5 + 126y^6 - 84y^7 + 36y^8 - 9y^9 + y^{10}$

There are a number of coefficients of $U_{1p2\ldots(p-1),n}(y)$ that are easily explained. For example, it is clear that $U_{1p2\ldots(p-1),n}(y)|_y = -1$ for all $p \geq 4$ since there is only one fixed point of $I_{1p2\ldots(p-1)}$ with one brick since the elements in each brick of a fixed point of $I_{1p2\ldots(p-1)}$ are increasing. Similarly, $U_{1p2\ldots(p-1),n}(y)|_{y^n} = (-1)^n$ for all $p \geq 4$. That is, if a fixed point of $I_{1p2\ldots(p-1)}$ has n bricks, then the underlying permutation must be the identity since the minimal element in bricks are increasing from left to right. For any $p \geq 4$, we claim that

$$U_{1p2\ldots(p-1),n}(y)|_{y^2} = \begin{cases} n-1 & \text{if } 2 \leq n \leq p-1 \text{ and} \\ \binom{n-p+3}{2} + p - 3 & \text{if } n \geq p. \end{cases} \tag{3.12}$$

That is, we want to consider the fixed points O of $I_{1p2\ldots(p-1)}$ which have precisely 2 bricks, B_1 of size b_1 followed by B_2 of size b_2. Now if there is an increase between B_1 and B_2, then the underlying permutation must be the identity and hence there are $n-1$ such fixed points as b_1 can range from 1 to $n-1$. Now if O is such that there is a decrease between B_1 and B_2, then there must be a $1p2\ldots(p-1)$-match starting at the second to last cell of B_1. This can only happen if $n \geq p$. Now suppose that $O(b_1-1) = x$. Then we know that $x < O(b_2)+1$. Since the elements in B_2 are increasing, it follows that $1, \ldots, x-1$ must be in brick B_1. As $O(b_1) > x$, it follows that $x = b_1 - 1$. But then $x+1, \ldots, x+(p-2)$ must be the first $p-2$ elements of B_2. We then have $n - (x + (p-2))$ choices for $O(b_1)$. As x can vary from 1 to $n-p+1$, it follows that the number of fixed points O where there is a decrease between bricks B_1 and B_2 is

$$\sum_{x=1}^{n-p+1} n-(x+(p-2)) = \binom{n-p-2}{2}.$$

It then follows that if $n \geq p$, the the fixed points O of $I_{1p2\ldots(p-1)}$ which have precisely 2 bricks is

$$\binom{n-p-2}{2} + n - 1 = \binom{n-p+3}{2} + p - 3.$$

This establishes (3.12).

Of course, one can use these initial values of $U_{1p2\ldots,(p-1),n}(y)$ to compute the initial terms of $NM_{1p2\ldots(p-1)}(t,x,y)$ for small values of p. For example, we have computed the following.

$$\begin{aligned}
&NM_{1423}(t,x,y) = 1 + xyt + \frac{1}{2}\left(xy + x^2y^2\right)t^2 + \\
&\frac{1}{6}\left(xy + xy^2 + 3x^2y^2 + x^3y^3\right)t^3 + \\
&\frac{1}{24}\left(xy + 3xy^2 + 7x^2y^2 + xy^3 + 4x^2y^3 + 6x^3y^3 + x^4y^4\right)t^4 + \\
&\frac{1}{120}\left(xy + 8xy^2 + 15x^2y^2 + 9xy^3 + 25x^2y^3 + 25x^3y^3 + xy^4 + 5x^2y^4 + \right. \\
&\quad \left. 10x^3y^4 + 10x^4y^4 + x^5y^5\right)t^5 + \\
&\frac{1}{720}\left(xy + 20xy^2 + 31x^2y^2 + 46xy^3 + 113x^2y^3 + 90x^3y^3 + 23xy^4 + 79x^2y^4 + \right. \\
&\quad \left. 105x^3y^4 + 65x^4y^4 + xy^5 + 6x^2y^5 + 15x^3y^5 + 20x^4y^5 + 15x^5y^5 + x^6y^6\right)t^6 + \\
&\frac{1}{5040}\left(xy + 47xy^2 + 63x^2y^2 + 200xy^3 + 448x^2y^3 + 301x^3y^3 + 219xy^4 + \right. \\
&\quad 651x^2y^4 + 728x^3y^4 + 350x^4y^4 + 53xy^5 + 217x^2y^5 + 364x^3y^5 + 315x^4y^5 + \\
&\quad \left. 140x^5y^5 + xy^6 + 7x^2y^6 + 21x^3y^6 + 35x^4y^6 + 35x^5y^6 + 21x^6y^6 + x^7y^7\right)t^7 + \\
&\frac{1}{40320}\left(xy + 105xy^2 + 127x^2y^2 + 794xy^3 + 1650x^2y^3 + 966x^3y^3 + 1547xy^4 + \right. \\
&\quad 4225x^2y^4 + 4214x^3y^4 + 1701x^4y^4 + 919xy^5 + 3166x^2y^5 + 4410x^3y^5 + \\
&\quad 3108x^4y^5 + 1050x^5y^5 + 115xy^6 + 543x^2y^6 + 1092x^3y^6 + \\
&\quad 1204x^4y^6 + 770x^5y^6 + 266x^6y^6 + xy^7 + 8x^2y^7 + \\
&\quad \left. 28x^3y^7 + 56x^4y^7 + 70x^5y^7 + 56x^6y^7 + 28x^7y^7 + x^8y^8\right)t^8 + \cdots
\end{aligned}$$

We note that there are many terms in these expansions which are easily explained. For example, we claim that the coefficient of $x^k y^k$ in $NM_{1p2\ldots(p-1),n}(x,y)$ is always the Stirling number $S(n,k)$ which is the number of set partitions of $\{1,\ldots,n\}$ into k parts. That is, a permutation $\sigma \in S_n$ that contributes to the coefficient $x^k y^k$ in $NM_{1p2\ldots(p-1),n}(x,y)$ must have k left-to-right minima and $k-1$ descents. Since each left-to-right minima of σ which is not the first element is always the second element of descent pair, it follows that if $1 = i_1 < i_2 < i_3 < \cdots < i_k$ are the positions of the left to right minima, then σ must be increasing in each of the intervals $[1,i_2), [i_2,i_3), \ldots, [i_{k-1},i_k), [i_k,n]$. It is then easy to see that

$$\{\sigma_1,\ldots,\sigma_{i_2-1}\},\{\sigma_{i_2},\ldots,\sigma_{i_3-1}\},\ldots,\{\sigma_{i_{k-1}},\ldots,\sigma_{i_k-1}\},\{\sigma_{i_k},\ldots,\sigma_n\}$$

is just a set partition of $\{1,\ldots,n\}$ ordered by decreasing minimal elements. Moreover, it is easy to see that no such permutation can have a $1p2\ldots(p-1)$-match for any $p \geq 4$. Vice versa, if $A_1,\ldots,A_k$ is a set partition of $\{1,\ldots,n\}$ such that $min(A_1) > \cdots > min(A_k)$, then the permutation $\sigma = A_k \uparrow A_{k-1} \uparrow \ldots A_1 \uparrow$ is a permutation with k left-to-right minima and $k-1$ descents where for any set $A \subseteq \{1,\ldots,n\}$, $A\uparrow$ is the list of the elements of A in increasing order. It follows that for any $p \geq 4$,

1. $NM_{1p2\ldots(p-1),n}(x,y)|_{xy} = S(n,1) = 1$,
2. $NM_{1p2\ldots(p-1)}(x,y)|_{x^2y^2} = S(n,2) = 2^{n-1}-1$,
3. $NM_{1p2\ldots(p-1)}(x,y)|_{x^ny^n} = S(n,n) = 1$, and
4. $NM_{1p2\ldots(p-1)}(x,y)|_{x^{n-1}y^{n-1}} = S(n,n-1) = \binom{n}{2}$.

We claim that

$$NM_{1p2\ldots(p-1),n}(x,y)|_{xy^2} = \begin{cases} 2^{n-1}-n & \text{if } n<p \text{ and} \\ 2^{n-1}-n-\binom{n-p+2}{2} & \text{if } n \geq p. \end{cases} \tag{3.13}$$

That is, suppose that $\sigma \in S_n$ contributes to $NM_{1p2\ldots(p-1),n}(x,y)|_{xy^2}$. Then σ must have 1 left-to-right minima and one descent. It follows that σ must start with 1 and have one descent. Now if A is any subset of $\{2,\ldots,n\}$ and $B = \{2,\ldots,n\} - A$, then we let σ_A be the permutation $\sigma_A = 1\ A \uparrow B \uparrow$. The only choices of A that do not give rise to a permutation with one descent are $\emptyset$ and $\{2,\dots,i\}$ for $i = 2,\dots,n$. It follows that there $2^{n-1} - n$ permutations that start with 1 and have 1 descent. Next consider when such a σ_A could have a $1p2\dots(p-1)$-match. If the $1p2\dots(p-1)$-match starts at position i, then it must be the case that $\sigma_i < \sigma_{i+1} > \sigma_{i+2}$. Thus it follows that $\sigma_1,\dots,\sigma_{i+1}$ and $\sigma_{i+2},\dots,\sigma_n$ are increasing sequences. But the fact that there is a $1p2\dots(p-1)$-match starting at position i also implies that $\sigma_i < \sigma_{i+2}$. It follows that $1,\dots,\sigma_i - 1$ must preceed σ_i which implies that $\sigma_i = i$. But since σ_{i+1} is greater than $\sigma_{i+2},\dots,\sigma_{i+p-1}$, it follows that

$$\sigma_{i+2} = i+1, , \sigma_{i+3} = i+2, \dots, \sigma_{i+p-1} = i+p-2.$$

But then we have $n-(i+(p-2))$ choices for σ_{i+1}. As i can vary from 1 to $n-p+1$, it follows that there are $\sum_{i=1}^{n-p+1} n-(i+(p-2)) = \binom{n-p+2}{2}$ such σ_A which have a $1p2\dots(p-1)$-match. It is then easy to see that this implies (3.13).

3.4.4 $\tau = 13\dots(p-1)2p$ for $p \geq 4$

Next we analyze the fixed points of I_τ for $\tau = 13\dots(p-1)2p$ where $p \geq 4$. By Lemma 34, we know that 1 is in the first cell of O. We claim that 2 must be in cell 2 or $p-1$ in O. That is, suppose that 2 is in cell c where $c \notin \{2, p-1\}$. Since there are no descents within any brick, 2 must be in the first cell of a brick. Moreover, since the minimal elements in the bricks of O form an increasing sequence, reading from left to right, 2 must be in the first cell of the second brick. Thus if b_1 and b_2 are the first two bricks in O, then 1 is at the start of b_1 and 2 is at the start of b_2. But then we claim that there is no τ-match in the elements of b_1 and b_2. That is, if $2 < c < p-1$, then 1 cannot be the start of a τ-match in O because 2 is too close to 1. But if 1 is not part of a τ match, then no other element in brick b_1 can play the role of the 1 in a τ-match so that the τ-match in bricks b_1 and b_2 must be entirely contained in b_2 which is impossible if O is a fixed point of I_τ. Similarly, if $p-1 < c$, then again 1 cannot be the start of a τ-match in O since 2 is too far away from 1. But then no other element in brick b_1 can play the role of 1 in a τ-match in O. Thus, we have two cases.

Case 1. 2 is in cell 2 of O.

In this case there are two possibilities, namely, either (i) 1 and 2 are both in the first brick b_1 of O or (ii) brick b_1 is a single cell filled with 1 and 2 is in the first cell of the second brick b_2 of O. In either case, it is easy to see that 1 is not part of a $13\ldots(p-1)2p$-match in O and if we remove cell 1 from O and subtract 1 from the numbers in the remaining cells, we would end up with a fixed point O' of I_τ in $\mathcal{O}_{\tau,n-1}$. Now in case (i), it is easy to see that $sgn(O)W(O) = sgn(O')W(O')$ and in case (ii) since b_1 will have a label $-y$ on the first cell, $sgn(O)W(O) = (-y)sgn(O')W(O')$. It follows that fixed points in Case 1 will contribute $(1-y)U_{\tau,n-1}(y)$ to $U_{\tau,n}(y)$.

Case 2. 2 is in cell $p-1$ of O.

Let $O(i)$ denote the element in cell i of O and $b_1, b_2, \ldots$ be the bricks of O, reading from left to right. Since there are no descents within bricks in O and the minimal elements of the bricks are increasing, reading from left to right, it must be the case that 2 is at the start of a brick b_2. Thus b_1 is a brick of size $p-2$. But then b_2 must have at least two cells since otherwise, there could be no τ-match contained in the cells of b_1 and b_2 and we could combine bricks b_1 and b_2 which would mean that O is not a fixed point of I_τ. But then the only reason that we could not combine bricks b_1 and b_2 is that there is a τ-match in the cells of b_1 and b_2 which could only start at position 1.

We now have two subcases.

Case 2.a. There is no τ-match in O starting at cell $p-1$.

Then we claim that $O(p) = p$. First observe that $O(p)$ must be greater than or equal to $O(1), O(2), \ldots, O(p-1)$ since there is a τ-match starting at position 1 in O. Thus $O(p) \geq p$. Thus if $O(p) \neq p$, then it must be the case that $O(p) > p$.

Hence p cannot be in brick b_2. Since brick b_1 has $p-2$ cells and 1 is in b_1, we can not have all of the elements $3,\ldots,p$ in b_1 so let i be the least element in $\{3,\ldots,p\}$ which is not in b_1. Since our assumption is that $O(p) > p$, we know i cannot be in brick b_2. Since the minimal elements in the bricks are increasing, it must be the case that i starts brick b_3 and there is a descrease between brick b_2 and brick b_3. But then we claim that we could combine bricks b_2 and b_3 which violates the fact that O is a fixed point of I_τ. That is, since 2 is the first element of b_2 and, by assumption, it does not start a τ-match in O, then no element in b_2 can start a τ-match in O since i would have to be part of any such τ-match and i is smaller than all of the remaining elements in b_2. Thus it must be the case that $O(p) = p$. But since 1 starts a τ-match in O, then we know that $O(2),\ldots,O(p-2)$ are all less than $O(p)$ so that $O(2),\ldots,O(p-2)$ must be the sequence $3,\ldots,p-1$.

It then follows that if we let O' be the result of removing the first $p-1$ cells from O and subtracting $p-1$ from the remaining numbers, then O' will be a fixed point of I_τ in $\mathcal{O}_{\tau,n-(p-1)}$. Note that if b_2 has 2 cells, then O' will start with a brick with one cell and if b_2 has more than 2 cells, then O' will start with a brick with at least two cells. Since there is $-y$ coming from the brick b_1, it is easy to see that the fixed points in Case 2.a will contribute $-yU_{\tau,n-(p-1)}(y)$ to $U_{\tau,n}(y)$.

Case 2.b. There is a τ-match starting at $p-1$ in O.

In this case, it must be that $O(p-1) < O(2p-3) < O(p) < \cdots < O(2p-4)$. But then it must be the case b_2 must be of size $p-2$. Clearly, b_2 has at most $p-2$ cells since the elements in each brick are increasing and $O(2p-4) > O(2p-3)$. Now if b_2 has less than $p-2$ cells, then $O(2p-3)$ must start some brick b_k and brick b_{k-1} would have less than $p-2$ cells. But then we could combine bricks b_{k-1} and b_k since that would mean that all the elements in b_{k-1} are strictly bigger than the first element of b_k so that it would not be possible to have a τ-match contained in the bricks b_{k-1} and b_k. Thus brick b_2 has $p-2$ cells which, in turn, implies that brick b_3 must have at least 2 cells. That is, if b_3 has less than 2 cells, there could be no τ-match among the cells of b_2 and b_3 so that we could combine b_2 and b_3

violating the fact that O is a fixed point of I_τ.

In the general case, assume that in O, the bricks $b_1, \ldots, b_{k-1}$ all have $p-2$ cells and the first elements of each of these bricks start τ-matches in O. . Let $r_j = 1 + (j-1)(p-2)$ for $j = 1, \ldots, k-1$ so that r_j is the position of the first cell in brick b_j for $1 \leq j \leq k-1$. Furthermore, assume that there is no τ-match starting at position $r_k = (k-1)(p-2)+1$. First we claim that brick b_k must have at least 2 cells. That is, since there is a τ-match starting at position r_{k-1} in O, we know that $O(r_k - 1) > O(r_k)$. Thus if b_k has 1 cell, then we could combine brick b_{k-1} and b_k which would violate the fact O is a fixed point of I_τ.

Next we claim that $O(r_k + 1) = r_k + 1$ and $\{O(1), \ldots, O(r_k), O(r_k + 1)\} = \{1, \ldots, r_k + 1\}$. That is, since there are τ-matches starting at positions $r_1, r_2, \ldots, r_{k-1}$, we have that $O(r_j), \ldots, O(r_{j+1}) < O(r_{j+1} + 1)$ and for each $1 \leq j \leq k-1$. It follows that $O(r_k + 1)$ is greater than $O(i)$ for $i = 1, \ldots, r_k$ so that $O(r_k+1) \geq r_k+1$. For a contradiction, assume that $O(r_k+1) > r_k+1$. It then follows that there is at least one $i \in \{1, \ldots, r_k+1\}$ which does not occupy any of the positions $1, \ldots, r_k$ so let j be the least element in $\{1, \ldots, r_k+1\} - \{O(1), \ldots, O(r_k)\}$. Then j cannot lie in brick b_k because $j < O(r_k+1)$ so that j must be the first element in brick b_{k+1}. Thus there is a decrease between bricks b_k and b_{k+1}. But then we claim that there can be no τ-match contained in the cells of b_k and b_{k+1}. That is, we are assuming that there is no τ-match starting at position r_k in O. Thus if there is a τ-match contained in the cells of b_k and b_{k+1}, it must start after position r_k and involve j. But j is smaller than all the elements in brick b_k that appear after position r_k which means that none of them can play the role of 1 in τ-match. But this would mean that we could combine bricks b_k and b_{k+1} which would violate the fact that O is a fixed point of I_τ. Thus it must be the case that $O(r_k+1) = r_k+1$. Since $O(r_k+1)$ is greater than $O(i)$ for $i = 1, \ldots, r_k$ so that $O(r_k+1) \geq r_k+1$, it automatically follows that $\{O(1), \ldots, O(r_k+1)\} = \{1, \ldots, r_k+1\}$. Thus if we let O' be the result of removing the first r_k cells from O and subtracting r_k from the remaining elements, then O' will be a fixed point of I_τ in $\mathcal{O}_{\tau,n-((k-1)(p-2)+1)}$. Moreover, since each of the first $k-1$ bricks contributes a factor of $-y$ to $sgn(O)W(O)$, we have that $sgn(O')W(O') = (-y)^{k-1} sgn(O)W(O)$. Let $\alpha^O = O(1) \ldots O(r_k+1)$

be the permutation of S_{r_k+1} determined by the first r_k+1 cells of O. It is easy to see that α^O has τ-matches starting at positions $1, r_2, \ldots, r_{k-1}$. We claim that if we place any permutation $\sigma \in S_{r_k+1}$ such that there are τ-matches in σ starting at positions $1, r_2, \ldots, r_{k-1}$, then we will create another fixed point of I_τ. That is, if $\sigma \in S_{r_k+1}$ is such that there are τ-matches in σ starting at positions $1, r_2, \ldots, r_{k-1}$, then this forces that

1. $\sigma_{r_1} < \cdots < \sigma_{r_k}$ which means that the minimal in the first k bricks are increasing,
2. σ is increasing within each of the bricks $b_1, \ldots, b_{k-1}$, and
3. $\sigma_{r_k+1} = r_k + 1$.

It follows that the contribution of such fixed points to $U_{\tau,n}(y)$ is

$$(-y)^{k-1} D_{\tau,r_k+1} U_{\tau,n-(k-1)(p-2)+1)}(y)$$

where D_{τ,r_k+1} is the number of $\sigma \in S_{r_k+1}$ such that there are τ-matches in σ starting at positions $1, r_2, \ldots, r_{k-1}$.

Fortunately, Harmse in his thesis [**?**] has already found a formula for D_{τ,r_k+1} for any $\tau = 13\ldots(p-1)2p$ where $p \geq 4$ in a different context. In particular, this formula was needed for the study of column strict fillings of rectangular shapes initiated by Harmse and Remmel [**?**]. That is, Harmse and Remmel [**?**] defined $\mathcal{F}_{n,k}$ to be the set of all fillings of a $k \times n$ rectangular array with the integers $1, \ldots, kn$ such that that the elements increase from bottom to top in each column. We let (i,j) denote the cell in the i-th row from the bottom and the j-th column from the left of the $k \times n$ rectangle and we let $F(i,j)$ denote the element in cell (i,j) of $F \in \mathcal{F}_{n,k}$.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ where $0 < \lambda_1 \leq \cdots \leq \lambda_k$, we let F_λ denote the Ferrers diagram of λ, i.e. F_λ is the set of left-justified rows of squares where the size of the i-th row is λ_i. Thus a $k \times n$ rectangular array corresponds to the Ferrers diagram corresponding to n^k. If $F \in \mathcal{F}_{n,k}$ and the integers are increasing in each row, reading from left to right, then F is a standard tableau of shape n^k. We let St_{n^k} denote the set of all standard tableaux of shape n^k and let

$st_{n^k} = |St_{n^k}|$. One can use the Frame-Robinson-Thrall hook formula [14] to show that

$$st_{n^k} = \frac{(kn)!}{\prod_{i=0}^{k-1}(i+n)\downarrow_n} \tag{3.14}$$

where $(n)\downarrow_0 = 1$ and $(n)\downarrow_k = n(n-1)\cdots(n-k+1)$ for $k > 0$.

If F is any filling of a $k \times n$-rectangle with distinct positive integers such that elements in each column increase, reading from bottom to top, then we let red(F) denote the element of $\mathcal{F}_{n,k}$ which results from F by replacing the i-th smallest element of F by i. For example, Figure 3.5 demonstrates a filling, F, with its corresponding reduced filling, red(F).

F

12	16	22
8	15	17
6	10	13
1	7	5

red(F)

7	10	12
5	9	11
3	6	8
1	4	2

Figure 3.5: An example of $F \in \mathcal{F}_{3,4}$ and red(F).

If $F \in \mathcal{F}_{n,k}$ and $1 \le c_1 < \cdots < c_j \le n$, then we let $F[c_1, \ldots, c_j]$ be the filling of the $k \times j$ rectangle where the elements in column a of $F[c_1, \ldots, c_j]$ equal the elements in column c_a in F for $a = 1, \ldots, j$. Let P be an element of $\mathcal{F}_{j,k}$ and $F \in \mathcal{F}_{n,k}$ where $j \le n$. Then we say there is a **P-match in F starting at position** i if red$(F[i, i+1, \ldots, i+j-1]) = P$. We let P-mch(F) denote the number of P-matches in F.

If $P \in \mathcal{F}_{2,s}$, then we define $\mathcal{MP}_{P,n}$ to be the set of $F \in \mathcal{F}_{n,s}$ such that P-mch$(F) = n-1$, i.e. the set of $F \in \mathcal{F}_{n,k}$ such that there is a P-match in F starting at positions $1, 2, \ldots, n-1$. Elements of $\mathcal{MP}_{P,n}$ are called maximum packings for P. We let $mp_{P,n} = |\mathcal{MP}_{P,n}|$ and use the convention that $mp_{P,1} = 1$. For example, if P is the element of $\mathcal{F}_{2,k}$ that has the integers $1, \ldots, s$ in the first column and the integers $s+1, \ldots, 2s$ in the second column, then it is easy to see that $mp_{P,n} = 1$ for all $n \ge 1$, since the only element of $F \in \mathcal{F}_{n,k}$ with P-mch$(F) = n-1$ has the integers $(i-1)s+1, \ldots, (i-1)s+s$ in the i-th column, for $i = 1, \ldots, n$. Harmse and Remmel [**?**] proved that the computation of the generating function for the number of P-matches in $F_{n,k}$ can be reduced to computing $mp_{P,n}$ for all n so that they computed $mp_{P,n}$ for various $P \in \mathcal{F}_{2,k}$. In particular, let $P_s \in St_{2^s}$ be the standard tableau which has $1, 3, 4, \ldots, s+1$ in the first column and $2, s+1, s+2, \ldots, 2s$ in the second column. For example, P_5 is pictured in Figure 3.6.

6	10
5	9
4	8
3	7
1	2

Figure 3.6: The standard tableau P_5.

Then Harmse proved that for $s, n \geq 2$,

$$mp_{P_s,n-1} = \frac{1}{(s-1)n+1}\binom{sn}{n} \tag{3.15}$$

Now suppose that $s, n \geq 2$ and $F \in \mathcal{MP}_{P_s,n}$. It is easy to see that in F that the top $s-1$ elements of column i are larger than any of the elements in columns $i-1$ and are greater than or equal to $F(1,i)$. It follows that the top $s-1$ elements in column n are greater than all the remaining elements in F so that they must be $s(n-1)+2, s(n-1)+3, \ldots, sn$ reading from bottom to top. Given such an F, we let σ_F be the permutation in $S_{k(n-1)+2}$ where

$$\sigma_F = F(1,1)F(2,1)\ldots F(s,1)\ldots F(1,n-1)F(2,n-1)\ldots F(s,n-1)F(1,n)F(2,n).$$

For example, if F is the element of $\mathcal{MP}_{P_5,4}$ pictured at the top of Figure 3.7, then σ_F is pictured at the bottom of Figure 3.7.

F=

8	12	16	20
6	11	15	19
5	10	14	18
4	9	13	17
1	2	3	7

$\sigma_F =$ 1 4 5 6 8 2 9 10 11 12 3 13 14 15 16 7 17

Figure 3.7: An example of σ_F.

It is then easy to see that if $F \in \mathcal{MP}_{P_s,n}$, then σ_F is a permutation in $S_{k(n-1)+2}$ which has $1\,3\ldots(s-1)\,2\,s$-matches starting at positions $1+(s-2)(j-1)$ for $j = 1,\ldots,n-1$. Vice versa, if $\sigma \in S_{s(n-1)+2}$ is a permutation which has $1\,3\ldots(s-1)\,2\,s$-matches starting at positions $1+(s-2)(j-1)$ for $j = 1,\ldots,n-1$, then we can create a filling of $F_\sigma \in \mathcal{MP}_{P_s,n}$ by letting r^{th} column of F consist of $\sigma_{s(r-1)+1},\ldots,\sigma_{s(r-1)+s}$, reading from bottom to top, for $r = 1,\ldots,n-1$ and letting the n^{th} column of $\sigma_{s(n-1)+1}\sigma_{s(n-1)+2}, s(n-1)+3,\ldots,sn$. It then follows from (3.15) that the number of permutations $\sigma \in S_{(k-1)(p-1)+2}$ that have $1\;3\ldots(p-1)\;2\;p$-matches starting at positions $1+(p-2)(j-1)$ for $j = 1,\ldots,k-1$ is $\frac{1}{(p-2)k+1}\binom{k(p-1)}{k}$. Hence if $\tau = 1\;3\ldots(p-1)\;2\;p$, then

$$D_{\tau,r_k+1} = \frac{1}{(p-2)k+1}\binom{k(p-1)}{k}.$$

It then follows that the contribution to $U_{\tau,n}$ of those fixed points O such that the bricks $b_1,\ldots,b_{k-1}$ all have $p-2$ cells and there is a τ-match starting at cell r_j for $1 \leq j \leq k-1$, but there is no τ-match starting at position $r_k = (k-1)(p-2)+1$ is

$$(-y)^{k-1}\frac{1}{(p-2)(k-1)+1}\binom{(k-1)(p-1)}{k-1}U_{\tau,n-((k-1)(p-2)+1)}(y).$$

Hence we have shown that if if $\tau = 13\ldots(p-1)2p$ where $p \geq 4$, then $U_{\tau,1}(y) = -y$ and for $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=1}^{\lfloor\frac{n-2}{p-2}\rfloor}(-y)^k\frac{1}{(p-2)k+1}\binom{k(p-1)}{k}U_{\tau,n-(k(p-2)+1)}(y) \tag{3.16}$$

This proves Theorem 27.

Theorem 27. *Let $\tau = 13\ldots(p-1)2p$ where $p \geq 4$. Then*

$$NM_\tau(t,x,y) = \left(\frac{1}{U_\tau(t,y)}\right)^x \text{ where } U_\tau(t,y) = 1 + \sum_{n\geq 1}U_{\tau,n}(y)\frac{t^n}{n!},$$

$U_{\tau,1}(y) = -y$ and for $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=1}^{\lfloor\frac{n-2}{p-2}\rfloor}(-y)^k\frac{1}{(p-2)k+1}\binom{k(p-1)}{k}U_{\tau,n-(k(p-2)+1)}(y)$$

Again, it is not difficult to compute some initial values of $U_{13\ldots(p-1)2p,n}(y)$ for small p. For example, we have computed the following.

$U_{1324,1}(y) = -y$
$U_{1324,2}(y) = -y + y^2$
$U_{1324,3}(y) = -y + 2y^2 - y^3$
$U_{1324,4}(y) = -y + 4y^2 - 3y^3 + y^4$
$U_{1324,5}(y) = -y + 6y^2 - 8y^3 + 4y^4 - y^5$
$U_{1324,6}(y) = -y + 8y^2 - 19y^3 + 13y^4 - 5y^5 + y^6$
$U_{1324,7}(y) = -y + 10y^2 - 34y^3 + 38y^4 - 19y^5 + 6y^6 - y^7$
$U_{1324,8}(y) = -y + 12y^2 - 53y^3 + 98y^4 - 64y^5 + 26y^6 - 7y^7 + y^8$
$U_{1324,9}(y) = -y + 14y^2 - 76y^3 + 194y^4 - 196y^5 + 98y^6 - 34y^7 + 8y^8 - y^9$
$U_{1324,10}(y) = -y + 16y^2 - 103y^3 + 334y^4 - 531y^5 + 337y^6 - 141y^7 + 43y^8 - 9y^9 + y^{10}$

$U_{13425,1}(y) = -y$
$U_{13425,2}(y) = -y + y^2$
$U_{13425,3}(y) = -y + 2y^2 - y^3$
$U_{13425,4}(y) = -y + 3y^2 - 3y^3 + y^4$
$U_{13425,5}(y) = -y + 5y^2 - 6y^3 + 4y^4 - y^5$
$U_{13425,6}(y) = -y + 7y^2 - 12y^3 + 10y^4 - 5y^5 + y^6$
$U_{13425,7}(y) = -y + 9y^2 - 21y^3 + 23y^4 - 15y^5 + 6y^6 - y^7$
$U_{13425,8}(y) = -y + 11y^2 - 37y^3 + 47y^4 - 39y^5 + 21y^6 - 7y^7 + y^8$
$U_{13425,9}(y) = -y + 13y^2 - 57y^3 + 94y^4 - 90y^5 + 61y^6 - 28y^7 + 8y^8 - y^9$
$U_{13425,10}(y) = -y + 15y^2 - 81y^3 + 171y^4 - 198y^5 + 156y^6 - 90y^7 + 36y^8 - 9y^9 + y^{10}$

$U_{134526,1}(y) = -y$
$U_{134526,2}(y) = -y + y^2$
$U_{134526,3}(y) = -y + 2y^2 - y^3$
$U_{134526,4}(y) = -y + 3y^2 - 3y^3 + y^4$
$U_{134526,5}(y) = -y + 4y^2 - 6y^3 + 4y^4 - y^5$
$U_{134526,6}(y) = -y + 6y^2 - 10y^3 + 10y^4 - 5y^5 + y^6$

$$U_{134526,7}(y) = -y + 8y^2 - 17y^3 + 20y^4 - 15y^5 + 6y^6 - y^7$$
$$U_{134526,8}(y) = -y + 10y^2 - 27y^3 + 38y^4 - 35y^5 + 21y^6 - 7y^7 + y^8$$
$$U_{134526,9}(y) = -y + 12y^2 - 40y^3 + 68y^4 - 74y^5 + 56y^6 - 28y^7 + 8y^8 - y^9$$
$$U_{134526,10}(y) = -y + 14y^2 - 61y^3 + 114y^4 - 146y^5 + 131y^6 - 84y^7 + 36y^8 - 9y^9 + y^{10}$$

$$U_{1345627,1}(y) = -y$$
$$U_{1345627,2}(y) = -y + y^2$$
$$U_{1345627,3}(y) = -y + 2y^2 - y^3$$
$$U_{1345627,4}(y) = -y + 3y^2 - 3y^3 + y^4$$
$$U_{1345627,5}(y) = -y + 4y^2 - 6y^3 + 4y^4 - y^5$$
$$U_{1345627,6}(y) = -y + 5y^2 - 10y^3 + 10y^4 - 5y^5 + y^6$$
$$U_{1345627,7}(y) = -y + 7y^2 - 15y^3 + 20y^4 - 15y^5 + 6y^6 - y^7$$
$$U_{1345627,8}(y) = -y + 9y^2 - 23y^3 + 35y^4 - 35y^5 + 21y^6 - 7y^7 + y^8$$
$$U_{1345627,9}(y) = -y + 11y^2 - 34y^3 + 59y^4 - 70y^5 + 56y^6 - 28y^7 + 8y^8 - y^9$$
$$U_{1345627,10}(y) = -y + 13y^2 - 48y^3 + 96y^4 - 130y^5 + 126y^6 - 84y^7 + 36y^8 - 9y^9 + y^{10}$$

Again, there are a number of coefficients of $U_{13\ldots(p-1)2p,n}(y)$ that are easily explained. For example, as in the case of $U_{1p2\ldots(p-1),n}(y)$, it is easy to see that $U_{13\ldots(p-1)2p,n}(y)|_y = -1$ and $U_{13\ldots(p-1)2p,n}(y)|_{y^n} = (-1)^n$ for all $p \geq 4$. We also claim that for any $p \geq 4$,

$$U_{13\ldots(p-1)2p,n}(y)|_{y^2} = \begin{cases} n-1 & \text{if } 2 \leq n \leq p-1 \text{ and} \\ 2n-p & \text{if } n \geq p. \end{cases} \tag{3.17}$$

That is, we want to consider the fixed points O of $I_{13\ldots(p-1)2p}$ which have precisely 2 bricks B_1 of size b_1 followed by B_2 of size b_2. Now if there is an increase between B_1 and B_2, then the underlying permutation must be the identity and hence there are $n-1$ such fixed points as b_1 can range from 1 to $n-1$. Now if O is such that there is a decrease between B_1 and B_2, then there must be a $13\ldots(p-1)2p$-match starting at position $b_1-(p-2)+1$. This can only happen if $n \geq p$. Now suppose that $O(b_1-(p-2)+1) = x$. Then we know that $x < O(b_1)+1$. Since the elements in B_2 are increasing, it follows that $1,\ldots,x-1$ must be in brick

B_1. Thus $x = b_1 - (p-2) + 1$. Note that for $O(b_1+2) > O(j) > O(b_1+1)$ for $j = b_1 + (p-2) + 2, \ldots, b_1$ so that it must be the case that $O(b_1+1) = x+1$, $O(b_1+2) = x+p-1$, and $\{O(j) : j = b_1+(p-2)+2, \ldots, b_1\} = \{x+2, \ldots, x+(p-2)\}$. As x can vary from 1 to $n-p+1$, it follows that the number of fixed points O where there is a decrease between bricks B_1 and B_2 is $n-p+1$. This establishes (3.17).

Of course, one can use the initial values of the $U_{134\ldots(p-1)2p,n}(y)$ to compute the initial values of $NM_{134\ldots(p-1)2p}(t,x,y)$. For example, we have computed that

$$
\begin{aligned}
&NM_{13425}(t,x,y) = 1 + xyt + \frac{1}{2}\left(xy + x^2y^2\right)t^2 + \\
&\frac{1}{6}\left(xy + xy^2 + 3x^2y^2 + x^3y^3\right)t^3 + \\
&\frac{1}{24}\left(xy + 4xy^2 + 7x^2y^2 + xy^3 + 4x^2y^3 + 6x^3y^3 + x^4y^4\right)t^4 + \\
&\frac{1}{120}\left(xy + 10xy^2 + 15x^2y^2 + 11xy^3 + 30x^2y^3 + 25x^3y^3 + xy^4 + 5x^2y^4\right. \\
&\quad \left.+10x^3y^4 + 10x^4y^4 + x^5y^5\right)t^5 + \\
&\frac{1}{720}\left(xy + 24xy^2 + 31x^2y^2 + 62xy^3 + 140x^2y^3 + 90x^3y^3 + 26xy^4 + 91x^2y^4 + \right. \\
&\quad \left. 120x^3y^4 + 65x^4y^4 + xy^5 + 6x^2y^5 + 15x^3y^5 + 20x^4y^5 + 15x^5y^5 + x^6y^6\right)t^6 + \\
&\frac{1}{5040}\left(xy + 54xy^2 + 63x^2y^2 + 273xy^3 + 553x^2y^3 + 301x^3y^3 + 292xy^4 + \right. \\
&\quad 840x^2y^4 + 875x^3y^4 + 350x^4y^4 + 57xy^5 + 238x^2y^5 + 406x^3y^5 + 350x^4y^5 + \\
&\quad \left. 140x^5y^5 + xy^6 + 7x^2y^6 + 21x^3y^6 + 35x^4y^6 + 35x^5y^6 + 21x^6y^6 + x^7y^7\right)t^7 + \\
&\frac{1}{40320}\left(xy + 116xy^2 + 127x^2y^2 + 1071xy^3 + 2000x^2y^3 + 966x^3y^3 + 2228xy^4 + \right. \\
&\quad 5726x^2y^4 + 5152x^3y^4 + 1701x^4y^4 + 1171xy^5 + 4016x^2y^5 + 5474x^3y^5 + \\
&\quad 3640x^4y^5 + 1050x^5y^5 + 120xy^6 + 575x^2y^6 + 1176x^3y^6 + 1316x^4y^6 + \\
&\quad 840x^5y^6 + 266x^6y^6 + xy^7 + 8x^2y^7 + 28x^3y^7 + 56x^4y^7 + \\
&\quad \left. 70x^5y^7 + 56x^6y^7 + 28x^7y^7 + x^8y^8\right)t^8 + \cdots
\end{aligned}
$$

Again, it is easy to explain several of these coefficients. For example, the same argument that we used to prove that $NM_{1p2\ldots,(p-1),n}(x,y)|_{x^ky^k} = S(n,k)$ will prove that

$$NM_{13\ldots,(p-1)2p,n}(x,y)|_{x^ky^k} = S(n,k).$$

We claim that for $p \geq 4$,

$$NM_{13\ldots(p-1)2p,n}(x,y)|_{xy^2} = \begin{cases} 2^{n-1} - n & \text{if } n < p \text{ and} \\ 2^{n-1} - 2n + p - 1 & \text{if } n \geq p. \end{cases} \quad (3.18)$$

That is, suppose that $\sigma \in S_n$ contributes to $NM_{13\ldots(p-1)2p,n}(x,y)|_{xy^2}$. Then σ must have 1 left-to-right minima and one descent. It follows that σ must start with 1 and have one descent. Again, there are $2^{n-1} - n$ permutations that start with 1 and have 1 descent. Next consider when such a σ which starts with 1 and has 1 descent can have a $13\ldots(p-1)2p$-match. If the $13\ldots(p-1)2p$-match starts at position i, then it must be the case that $\sigma_{i+p-3} > \sigma_{i+p-2}$. Thus it follows that $\sigma_1, \ldots, \sigma_{i+p-3}$ and $\sigma_{i+p-2}, \ldots, \sigma_n$ are increasing sequences. But the fact that there is a $13\ldots(p-1)$-match starting at position i also implies that $\sigma_i < \sigma_{i+p-2}$. It follows that $1, \ldots, \sigma_i - 1$ must preceed σ_i which implies that $\sigma_i = i$. But since σ_{i+p-1} is greater than $\sigma_{i+1}, \ldots, \sigma_{i+p-3}$, it follows that $\sigma_{i+p-2} = i+1$ and

$$\sigma_{i+1} = i+2, , \sigma_{i+3} = i+3, \ldots, \sigma_{i+p-3} = i+p-2.$$

Thus there is only one such σ which has $13\ldots(p-1)2p$-match starting at position i As i can vary from 1 to $n-p+1$, it follows that there are $n-p+1$ permutations σ which starts with 1 and have 1 descent and contain a $13\ldots(p-1)2p$-match. It is then easy to see that this implies (3.18).

3.4.5 $\tau = 145\ldots p23$ for $p \geq 5$

We will start by analyzing the fixed points of I_τ for $\tau = 145\ldots p23$ where $p \geq 5$. By Lemma 34, we know that 1 is in the first cell of O. We claim that 2 must be in cell 2 or $p-1$ in O. That is, suppose that 2 is in cell c where $c \notin \{2, p-1\}$. Since there are no descents within any brick, 2 must be in the first cell of a brick. Moreover, since the minimal elements in the bricks of O form an increasing sequence, reading from left to right, 2 must be in the first cell of the second brick. Thus if b_1 and b_2 are the first two bricks in O, then 1 is at the start of b_1 and 2 is at the start of b_2. But then we claim that there is no τ-match in the elements of b_1 and b_2. That is, if $2 < c < p-1$, then 1 cannot be the start of a τ-match in O because 2 is too close to 1. But if 1 is not part of a τ match, then no other element in brick b_1 can play the role of the 1 in a τ-match so that the τ-match in bricks b_1 and b_2 must be entirely contained in b_2 which is impossible if O is a fixed point of I_τ. Similarly, if $p-1<c$, then again 1 cannot be the start of a τ-match in O since 2 is too far away from 1. But then no other element in brick b_1 can play the role of 1 in a τ-match in O. Thus, we have two cases.

Case 1. 2 is in cell 2 of O.

In this case there are two possibilities, namely, either (i) 1 and 2 are both in the first brick b_1 of O or (ii) brick b_1 is a single cell filled with 1 and 2 is in the first cell of the second brick b_2 of O. In either case, it is easy to see that 1 is not part of a $145\ldots p23$-match in O and if we remove cell 1 from O and subtract 1 from the numbers in the remaining cells, we would end up with a fixed point O' of I_τ in $\mathcal{O}_{\tau,n-1}$. Now in case (i), it is easy to see that $sgn(O)W(O)=sgn(O')W(O')$ and in case (ii) since b_1 will have a label $-y$ on the first cell, $sgn(O)W(O)=(-y)sgn(O')W(O')$. It follows that fixed points in Case 1 will contribute $(1-y)U_{\tau,n-1}(y)$ to $U_{\tau,n}(y)$.

Case 2. 2 is in cell $p-1$ of O.

Let $O(i)$ denote the element in cell i of O and $b_1,b_2,\ldots$ be the bricks of O, reading from left to right. Since there are no descents within bricks in O and the minimal elements of the bricks are increasing, reading from left to right, it must be the case that 2 is at the start of a brick b_2. Thus b_1 is a brick of size $p-2$. But then b_2 must have at least two cells since otherwise, there could be no τ-match contained in the cells of b_1 and b_2 and we could combine bricks b_1 and b_2 which would mean that O is not a fixed point of I_τ. But then the only reason that we could not combine bricks b_1 and b_2 is that there is a τ-match in the cells of b_1 and b_2 which could only start at position 1.

We now have two subcases.

Case 2.a. There is no τ-match in O starting at cell $p-1$.

Then we claim that $O(p)=3$. To show this, suppose 3 is in a cell c such that $c \notin \{1, p-1, p\}$. If $1 < c < p-1$ then since there are no descents within bricks, $c=2$ or c is the first cell of a brick. Because there is a τ-match starting at the first cell, c cannot be equal to 2 because $O(2)$ must be at least 4 to be part of a τ-match. Furthermore, c cannot be the first cell of a brick because the minimal elements in bricks are increasing but $(O(1), O(c), O(p-1)) = (1,3,2)$. Now suppose that $c > p$. Then since there are no descents within bricks and minimal elements of bricks are increasing, c must be the first cell of brick b_3. There is a decrease between bricks b_2 and b_3 so there must be a τ-match within these bricks. This means that $O(c)=3$ must play the role of 2 in the τ-match. This is impossible because 2 does not start a τ-match therefore there are no numbers left to play the role of 1 in the τ-match that contains 3. Thus $O(p)=3$.

It then follows that if we let O' be the result of removing the first $p-1$ cells from O and subtracting $p-1$ from the remaining numbers, then O' will be a fixed point of I_τ in $\mathcal{O}_{\tau,n-(p-1)}$. Note that if b_2 has 2 cells, then O' will start with a brick with one cell and if b_2 has more than 2 cells, then O' will start with a brick with at least two cells. The τ-match starts at the first cell so all of the numbers $\{O(2), \ldots, O(p-2)\}$ are greater than $O(p)=3$. Since there are $n-3$ numbers greater than 3 occupying $p-3$ cells and there is $-y$ coming from the brick b_1, it is easy to see that the fixed points in Case 2.a will contribute $-y\binom{n-3}{p-3}U_{\tau,n-(p-1)}(y)$ to $U_{\tau,n}(y)$.

Case 2.b. There is a τ-match starting at $p-1$ in O.

In this case, it must be that

$$O(p-1) < O(2p-3) < O(2p-2) < O(p) < \cdots < O(2p-4).$$

But then it must be the case that b_2 must have $p-2$ cells. Clearly, b_2 has at most $p-2$ cells since the elements in each brick are increasing and $O(2p-4) > O(2p-3)$. Now if b_2 has less than $p-2$ cells, then $O(2p-3)$ must start some brick b_k and brick b_{k-1} would have less than $p-2$ cells. But then we could combine bricks b_{k-1} and b_k since that would mean that all the elements in b_{k-1} are strictly bigger than the first element of b_k so that it would not be possible to have a τ-match contained in the bricks b_{k-1} and b_k. Thus brick b_2 has $p-2$ cells which, in turn, implies that brick b_3 must have at least 2 cells. That is, if b_3 has less than 2 cells, there could be no τ-match among the cells of b_2 and b_3 so then we could combine b_2 and b_3 violating the fact that O is a fixed point of I_τ.

In the general case, assume that in O, the bricks $b_1, \ldots, b_{k-1}$ all have $p-2$ cells and the first elements of each of these bricks start τ-matches in O. Let $r_j = 1 + (j-1)(p-2)$ for $j = 1, \ldots, k$ so that r_j is the position of the first cell in brick b_j for $1 \leq j \leq k$. Furthermore, assume that there is no τ-match starting at position $r_k = (k-1)(p-2)+1$. First we claim that brick b_k must have at least 2 cells. That is, since there is a τ-match starting at position r_{k-1} in O, we know that $O(r_k - 1) > O(r_k)$. Thus if b_k has 1 cell, then we could combine brick b_{k-1} and b_k which would violate the fact O is a fixed point of I_τ.

Next, we claim that $O(r_j) = j$ for $1 \leq j \leq k$. We have shown that $O(r_1) = O(1) = 1$ and $O(r_2) = O(p-1) = 2$. Thus, assume by induction that $O(r_{j-1}) = j-1$. Notice that since there is a τ-match starting at cell r_{j-1}, we have that $O(r_j) > O(r_{j-1})$ so $O(r_j) \geq j$. Now let's figure out where the number j could be. Let's assume that $O(r_j) \neq j$, so $O(r_j) > j$.

Now, suppose j is in cell c such that $c < r_j$ and $c \notin \{r_1, \ldots, r_{j-1}\}$. If $r_{j-1} < c < r_j$, then since there is a τ-match starting at r_{j-1}, $O(c) > O(r_j)$ which cannot happen because $O(c) = j$ and $O(r_j) > j$. Next suppose that $r_i < c < r_{i+1}$ for $1 \leq i \leq j-2$. Then since there is a τ-match starting at cell r_i, $O(c) > O(r_{i+1}+1)$ but $O(r_{i+1}+1)$ cannot be less than j because all of the numbers $1, 2, \ldots j-1$ occupy the cells $r_1, r_2, \ldots, r_{j-1}$. Therefore j must occupy a cell c such that $c \geq r_j$.

Suppose $c > r_j$, then since there are no descents within bricks and $O(r_j) > j$, cell c must be the first cell in a brick. But then cell c cannot be the first cell of a brick because minimal elements of bricks increase and $O(r_j) > O(c)$. Therefore $c = r_j$ or in other words, $O(r_j) = j$.

Next we claim that $O(r_k + 1) = k + 1$. Since the numbers $1, \ldots, k$ already occupy the cells $r_1, r_2, \ldots, r_k$ we have that $O(r_k+1) \geq k+1$. Let's assume that $O(r_k+1) \neq k+1$.

First suppose that $k+1$ is in cell c such that $c < r_k+1$ and $c \notin \{r_1, \ldots, r_k\}$. In other words, suppose $r_i < c < r_{i+1}$ for $1 \leq i \leq k-1$. Then since there is a τ-match starting at cell r_i, $k+1 = O(c) > O(r_{i+1}+1)$ but $O(r_{i+1}+1)$ cannot be less than $k+1$ because all of the numbers $\{1, 2, \ldots k\}$ occupy the cells $\{r_1, r_2, \ldots, r_k\}$. Therefore $k+1$ must occupy a cell c such that $c \geq r_k+1$.

Suppose $c > r_k+1$, then since there are no descents within bricks and $O(r_k+1) > k+1$, cell c must be the first cell in a brick. But then cell c cannot be the first cell of a brick because minimal elements of bricks increase and $O(r_k+1) > O(c)$. Therefore $c = r_k+1$ or in other words, $O(r_k+1) = k+1$.

If we let O' be the result of removing the first r_k cells from O and renumbering the remaining cells in such a way that we keep the same relative order but use the numbers $1, \ldots, n-r_k$, then O' will be a fixed point I_τ in $\mathcal{O}_{\tau,n-r_k}$. Moreover, since each of the first $k-1$ bricks contributes a factor of $-y$ to $sgn(O)W(O)$, we have that $sgn(O')W(O') = (-y)^{k-1} sgn(O)W(O)$. Since we have shown that the numbers $\{1, 2, \ldots, k, k+1\}$ occupy the cells $\{r_1, r_2, \ldots, r_k, r_k+1\}$, it must be the case that the numbers within the cells in between r_i and r_{i+1} for $1 \leq i \leq k-1$ are all greater than $k+1$. So there are $n-(k+1)$ numbers to choose from that must occupy $r_k - k = (k-1)(p-3)$ cells. Thus there are $\binom{n-k-1}{(k-1)(p-3)}$ ways to choose these numbers. Let $\alpha^O = \text{red}(O(1)\ldots O(r_k+1))$ be the permutation of S_{r_k+1} determined by the first r_k+1 cells of O after renumbering. It is easy to see that α^O has τ-matches starting at positions $1, r_2, \ldots, r_{k-1}$. We claim that if we place any permutation $\sigma \in S_{r_k+1}$ of the numbers $\{O(1)\ldots O(r_k+1)\}$ such that there are τ-matches in σ starting at positions $r_1, r_2, \ldots, r_{k-1}$, then we will create another fixed point of I_τ. It follows that the contribution of such fixed points to $U_{\tau,n}(y)$ is $(-y)^{k-1}\binom{n-k-1}{(k-1)(p-3)} F_{\tau,r_k+1} U_{\tau,n-(k-1)(p-2)+1)}(y)$ where F_{τ,r_k+1} is the number of $\sigma \in S_{r_k+1}$ such that there are τ-matches in σ starting at positions $r_1, r_2, \ldots, r_{k-1}$.

Suppose $\sigma \in S_{r_k+1}$ such that there are τ-matches in σ starting at positions $r_1, r_2, \ldots, r_{k-1}$ and let $\sigma(i)$ be the number that occupies position i. Then we already know from above that $1, \ldots, k+1$ occupy positions $r_1, r_2, \ldots, r_k, r_k + 1$. Now we must assign numbers to the positions between r_i and r_{i+1} for $1 \leq i \leq k-1$. Since there are $r_{i+1} - r_i - 1 = p - 3$ numbers in between each pair of positions r_i, r_{i+1}, then there are $(p-3)(k-1)$ numbers left to assign. Since there is a τ-match starting at position r_i for $1 \leq i \leq k-1$, then $\sigma(r_i + 1) < \sigma(r_i + 2) < \cdots < \sigma(r_i + p - 3) = \sigma(r_{i+1} - 1)$. So each value $\sigma(r_i + 1)$ is less than the following $p-4$ numbers and also $\sigma(r_i+1) > \sigma(r_i + p - 1) = \sigma(r_{i+1}+1)$ which means that $\sigma(r_1 + 1) > \sigma(r_2 + 1) > \cdots > \sigma(r_{k-1} + 1) > \sigma(r_k + 1) = k + 1$. This implies that $\sigma(r_{k-1} + 1)$ is less than $(p-4)(k-1) + (k-2) = (p-3)(k-1) - 1$ numbers so it must be the least of the numbers left to assign. Then we may choose from the remaining $(p-3)(k-1)-1$ numbers to fill in the positions $r_{k-1}+2, \ldots, r_{k-1}+p-3$ so there are $\binom{(p-3)(k-1)-1}{p-4}$ ways to do this.

Now we must assign the remaining $(p-3)(k-2)$ numbers to the positions between r_i and r_{i+1} for $1 \leq i \leq k-2$. By the same argument above, $\sigma(r_{k-2}+1)$ is less than $(p-4)(k-2) + (k-3) = (p-3)(k-2) - 1$ numbers so it must be the least of the numbers left to assign. Then we may choose from the remaining $(p-3)(k-2)-1$ numbers to fill in the positions $r_{k-2}+2, \ldots, r_{k-2}+p-3$ so there are $\binom{(p-3)(k-2)-1}{p-4}$ possible ways to do this.

We repeat this process by arguing that after assigning numbers to positions $r_{k-(i-1)}+1, \ldots r_{k-(i-1)}+p-3$, the number $\sigma(r_{k-i})$ must be the least of the numbers left to assign and then we may choose from the remaining $(p-3)(k-i)-1$ numbers to fill in the positions $r_{k-i}+2, \ldots, r_{k-i}+p-3$ so there are $\binom{(p-3)(k-i)-1}{p-4}$ possible ways to do this and we take i from 1 up to $k-1$.

Therefore, F_{τ, r_k+1}, the number of $\sigma \in S_{r_k+1}$ such that there are τ-matches in σ starting at positions $r_1, r_2, \ldots, r_{k-1}$ is given by

$$F_{\tau, r_k+1} = \prod_{i=1}^{k-1} \binom{(p-3)(k-i)-1}{p-4}. \tag{3.19}$$

Thus we have proved Theorem 28

Theorem 28. *Let* $\tau = 145 \ldots p23$ *where* $p \geq 5$. *Then*

$$NM_\tau(t, x, y) = \left(\frac{1}{U_\tau(t, y)}\right)^x \text{ where } U_\tau(t, y) = 1 + \sum_{n \geq 1} U_{\tau, n}(y) \frac{t^n}{n!},$$

$U_{\tau,1}(y) = -y$, *and for* $n > 1$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=2}^{\lfloor \frac{n-2}{p-2} \rfloor + 1} (-y)^{k-1} \binom{n-k-1}{(p-3)(k-1)} \prod_{i=1}^{k-1} \binom{i(p-3)-1}{p-4} U_{\tau,n-((k-1)(p-2)+1)}(y).$$

We have computed the following coefficients.

$U_{14523,1}(y) = -y$
$U_{14523,2}(y) = -y + y^2$
$U_{14523,3}(y) = -y + 2y^2 - y^3$
$U_{14523,4}(y) = -y + 3y^2 - 3y^3 + y^4$
$U_{14523,5}(y) = -y + 5y^2 - 6y^3 + 4y^4 - y^5$
$U_{14523,6}(y) = -y + 9y^2 - 14y^3 + 10y^4 - 5y^5 + y^6$
$U_{14523,7}(y) = -y + 16y^2 - 35y^3 + 30y^4 - 15y^5 + 6y^6 - y^7$
$U_{14523,8}(y) = -y + 27y^2 - 84y^3 + 95y^4 - 55y^5 + 21y^6 - 7y^7 + y^8$
$U_{14523,9}(y) = -y + 43y^2 - 201y^3 + 284y^4 - 210y^5 + 91y^6 - 28y^7 + 8y^8 - y^9$
$U_{14523,10}(y) = -y + 65y^2 - 478y^3 + 869y^4 - 749y^5 + 406y^6 - 140y^7 + 36y^8 - 9y^9 + y^{10}$

$U_{145623,1}(y) = -y$
$U_{145623,2}(y) = -y + y^2$
$U_{145623,3}(y) = -y + 2y^2 - y^3$
$U_{145623,4}(y) = -y + 3y^2 - 3y^3 + y^4$
$U_{145623,5}(y) = -y + 4y^2 - 6y^3 + 4y^4 - y^5$
$U_{145623,6}(y) = -y + 6y^2 - 10y^3 + 10y^4 - 5y^5 + y^6$
$U_{145623,7}(y) = -y + 11y^2 - 20y^3 + 20y^4 - 15y^5 + 6y^6 - y^7$
$U_{145623,8}(y) = -y + 22y^2 - 51y^3 + 50y^4 - 35y^5 + 21y^6 - 7y^7 + y^8$
$U_{145623,9}(y) = -y + 43y^2 - 133y^3 + 161y^4 - 105y^5 + 56y^6 - 28y^7 + 8y^8 - y^9$
$U_{145623,10}(y) = -y + 79y^2 - 326y^3 + 504y^4 - 406y^5 + 196y^6 - 84y^7 + 36y^8 - 9y^9 + y^{10}$

$U_{1456723,1}(y) = -y$
$U_{1456723,2}(y) = -y + y^2$
$U_{1456723,3}(y) = -y + 2y^2 - y^3$

$U_{1456723,4}(y) = -y + 3y^2 - 3y^3 + y^4$
$U_{1456723,5}(y) = -y + 4y^2 - 6y^3 + 4y^4 - y^5$
$U_{1456723,6}(y) = -y + 5y^2 - 10y^3 + 10y^4 - 5y^5 + y^6$
$U_{1456723,7}(y) = -y + 7y^2 - 15y^3 + 20y^4 - 15y^5 + 6y^6 - y^7$
$U_{1456723,8}(y) = -y + 13y^2 - 27y^3 + 35y^4 - 35y^5 + 21y^6 - 7y^7 + y^8$
$U_{1456723,9}(y) = -y + 29y^2 - 70y^3 + 77y^4 - 70y^5 + 56y^6 - 28y^7 + 8y^8 - y^9$
$U_{1456723,10}(y) = -y + 65y^2 - 204y^3 + 252y^4 - 182y^5 + 126y^6 - 84y^7 + 36y^8 - 9y^9 + y^{10}$

3.5 Permutations with no $1324\ldots p$-matches and one or two descents

In this section, we will show how we can use Theorem 24 and Theorem 25 to find the generating function for the number of permutations $\sigma \in S_n$ which have no $1324\ldots p$-matches and have exactly k descents for $k = 1, 2$ and $p \geq 4$.

That is, fix $p \geq 4$ and let $d_{n,p}^{(i)}$ denote the number of $\sigma \in S_n$ such that $1324\ldots p\text{-mch}(\sigma) = 0$ and $\text{des}(\sigma) = i$. Our goal is to compute

$$D_p^{(i)}(t) = \sum_{n \geq 0} d_{n,p}^{(i)} \frac{t^n}{n!} = NM_{1324\ldots p}(t, 1, y)|_{y^{i+1}}$$

for $i = 1$ and $i = 2$.

To this end, we first want to compute $U_{1324\ldots p,n}(y)|_y$, $U_{1324\ldots p,n}(y)|_{y^2}$, and $U_{1324\ldots p,n}(y)|_{y^3}$. That is, we want to compute the number of fixed points of $I_{1324\ldots p}$ that have either 1, 2, or 3 bricks. Clearly there is only one fixed point of $I_{1324\ldots p}$ of length n which has just one brick since in that case, the underlying permutation must be the identity. In such a situation, the last cell of the brick is labeled with $-y$ so that for all $n \geq 1$ and all $p \geq 4$, $U_{1324\ldots p,n}(y)|_y = -1$. Hence

$$U_{1324\ldots p}(t, y)|_y = 1 - e^t. \tag{3.20}$$

Next we consider the fixed points of $I_{1324\ldots p}$ which are of length n and consists of two bricks, a brick B_1 of length b_1 followed by a brick B_2 of length b_2. Note that in this case, the last cells of B_1 and B_2 are labeled with $-y$ so that the weight of all such fixed points is y^2. Suppose the underlying permutation is $\sigma = \sigma_1 \ldots \sigma_n$. Then there are two cases.

Case 1. There is an increase between the two bricks, i.e. $\sigma_{b_1} < \sigma_{b_1+1}$.

In this case, it easy to see that σ must be the identity permutation and, hence, there are $n-1$ fixed points in case 1 since b_1 can range from 1 to $n-1$.

Case 2. There is an decrease between the two bricks, i.e. $\sigma_{b_1} > \sigma_{b_1+1}$.

In this case, there must be a $1324\ldots p$-match in the elements in the bricks of B_1 and B_2 which means that it must be the case that $\mathrm{red}(\sigma_{b_1-1}\sigma_{b_1}\sigma_{b_1+1}\ldots\sigma_{b_1+p-2}) = 1324\ldots p$. Now suppose that $\sigma_{b_1-1} = x$. Since σ_{b_1+1} is the smallest element in brick B_2 and the elements in brick B_2 increase and $\sigma_{b_1} > \sigma_{b_1+1}$, it must be the case that $1,\ldots,x-1$ must lie in brick B_1. It cannot be that $\sigma_{b_1} = x+1$ since $\sigma_{b_1} > \sigma_{b_1+1}$. Thus it must be the case that $\sigma_{b_1+1} = x+1$ and $\sigma_{b_1} = x+2$ since $\sigma_{b_1} < \sigma_{b_1+2}$. Thus brick B_1 consists of the elements $1,\ldots,x,x+2$. Hence there are $n-p+1$ possibilities in case 2 if $n \geq p$ and no possibilities in case 2 if $n < p$.

It follows that

$$U_{1324\ldots p,n}|_{y^2} = \begin{cases} 0 & \text{if } n = 0,1 \\ n-1 & \text{if } 2 \leq n \leq p-1 \text{ and} \\ 2n-p & \text{if } n \geq p. \end{cases} \tag{3.21}$$

Note that

$$\begin{aligned} \sum_{n\geq p}(2n-p)\frac{t^n}{n!} &= 2t\sum_{n\geq p}\frac{t^{n-1}}{(n-1)!} - p\sum_{n\geq p}\frac{t^n}{n!} \\ &= 2t(e^t - \sum_{n=0}^{p-2}\frac{t^n}{n!}) - p(e^t - \sum_{n=0}^{p-1}\frac{t^n}{n!}) \\ &= (2t-p)e^t + p + \sum_{n=1}^{p-1}(p-2n)\frac{t^n}{n!}. \end{aligned}$$

Thus

$$\begin{aligned} U_{1324\ldots p}(t,y)|_{y^2} &= (2t-p)e^t + p + \sum_{n=1}^{p-1}(p-2n)\frac{t^n}{n!} + \sum_{n=2}^{p-1}(n-1)\frac{t^n}{n!} \\ &= (2t-p)e^t + p + \sum_{n=1}^{p-1}(p-n-1)\frac{t^n}{n!}. \end{aligned} \tag{3.22}$$

Next we consider a fixed point of $I_{1324\ldots p}$ which has 3 bricks, B_1 of size b_1 followed by B_2 of size b_2 followed by B_3 of size b_3. Let $\sigma = \sigma_1 \ldots \sigma_n$ be the underlying permutation. The weight of all such fixed points is $-y^3$. We then have 4 cases.

Case a. There are increases between B_1 and B_2 and between B_2 and B_3, i.e. $\sigma_{b_1} < \sigma_{b_1+1}$ and $\sigma_{b_1+b_2} < \sigma_{b_1+b_2+1}$.

In this case, it is easy to see that σ must be the identity permutation so that there are $\binom{n-1}{2}$ possibilities in case 1 if $n \geq 3$.

Case b. There is an increase between B_1 and B_2 and a decrease between B_2 and B_3, i.e. $\sigma_{b_1} < \sigma_{b_1+1}$ and $\sigma_{b_1+b_2} > \sigma_{b_1+b_2+1}$.

In this case, it must be the case that $\sigma_1 < \cdots < \sigma_{b_1+b_2}$ and

$$\text{red}(\sigma_{b_1+b_2-1}\sigma_{b_1+b_2}\sigma_{b_1+b_2+1}\ldots\sigma_{b_1+b_2+p-2}) = 1324\ldots p.$$

Then we can argue exactly as in case 2 above that there must exist an x such that $\sigma_{b_1+b_2-1} = x$ and $1,\ldots,x-1$ must occur to the left of $\sigma_{b_1+b_2-1}$, $\sigma_{b_1+b_2} = x+2$ and $\sigma_{b_1+b_2+1} = x+1$. Then for any fixed $x \geq 2$, we have $x-1$ choices for the length of B_1 so that we have $\sum_{x=2}^{n-p+1}(x-1) = \binom{n-p+1}{2}$ possibilities if $n \geq p+1$ and no possibilities if $n \leq p$.

Case c. There is a decrease between B_1 and B_2 and an increase between B_2 and B_3, i.e. $\sigma_{b_1} > \sigma_{b_1+1}$ and $\sigma_{b_1+b_2} < \sigma_{b_1+b_2+1}$.

In this case, it must be the case that $\sigma_{b_1+1} < \cdots < \sigma_n$ and $\text{red}(\sigma_{b_1-1}\sigma_{b_1}\sigma_{b_1+1}\ldots\sigma_{b_1+p-2}) = 1324\ldots p$. Again we can argue as in case 2 above that there must exist an x such that $\sigma_{b_1-1} = x$ and $1,\ldots,x-1$ must occur to the left of σ_{b_1-1}, $\sigma_{b_1} = x+2$ and $\sigma_{b_1+1} = x+1$. Then for any fixed x, we have $n-1-(x+p-1)$ choices for the length of B_2 so that we have $\sum_{x=1}^{n-p} n-x-p-1 = \binom{n-p+1}{2}$ possibilities if $n \geq p+1$ and no possibilities if $n \leq p$.

Case d. There are decreases between B_1 and B_2 and between B_2 and B_3, i.e. $\sigma_{b_1} > \sigma_{b_1+1}$ and $\sigma_{b_1+b_2} > \sigma_{b_1+b_2+1}$.

In this case, there are two subcases.

Subcase d.1 $p = 4$.

Now we must have $\text{red}(\sigma_{b_1-1}\sigma_{b_1}\sigma_{b_1+1}\sigma_{b_1+2}) = 1324$. First suppose that $b_2 = 2$. Then we also have that $\text{red}(\sigma_{b_1+1}\sigma_{b_1+2}\sigma_{b_1+3}\sigma_{b_1+4}) = 1324$. It follows that $x = \sigma_{b_1-1} < \sigma_{b_1+1} < \sigma_{b_1+3}$. Since σ_{b_1+1} is the smallest element in brick B_2 and σ_{b_1+3} is the smallest element in brick B_3, it must be the case that $1,\ldots x-1$ lie in brick B_1 and that $\sigma_{b_1+1} = x+1$. It also must be the case that $\sigma_{b_1} < \sigma_{b_1+2} < \sigma_{b_1+4}$ so that $\sigma_{b_1}, \sigma_{b_1+2} \in \{x+2, x+3\}$ and $\sigma_{b_1+4} = x+4$. Thus there are two possibilities for each x. As x can vary between 1 and $n-5$ in this case, we have $2(n-5)$ possibilities if $b_2 = 2$ and $n \geq 6$ and no possibilities if $n < 6$.

Next consider the case where $b_2 \geq 3$. Again we must have $\text{red}(\sigma_{b_1-1}\sigma_{b_1}\sigma_{b_1+1}\sigma_{b_1+2}) = 1324$. Similarly we must have $\text{red}(\sigma_{b_1+b_2-1}\sigma_{b_1+b_2}\sigma_{b_1+b_2+1}\sigma_{b_1+b_2+2}) = 1324$, but this condition does not involve σ_{b_1+1}. Nevertheless, these conditions still force that $\sigma_{b_1} < \sigma_{b_1+b_2+1}$ so that if $\sigma_{b_1-1} = x$, then x is less than the least elements in bricks B_2 and B_3 so that $1,\ldots,x-1$ must be in brick B_1 and $\sigma_{b_1+1} = x+1$. However in this case, $\text{red}(\sigma_{b_1+b_2-1}\sigma_{b_1+b_2}\sigma_{b_1+b_2+1}\sigma_{b_1+b_2+2}) = 1324$ ensures that σ_{b_1+2} is also less than the least element of B_3 and since $\sigma_{b_1} < \sigma_{b_1+2}$, we must have $\sigma_{b_1} = x+2$ and $\sigma_{b_1+2} = x+3$. If we then remove the first $x+2$ cells which contain the numbers $1,\ldots,x+2$, then we must be left with a fixed point which has two bricks on $n-x-2$ cells. Then by our analysis of case 2, there are $n-x-2-3$ possibilities for B_3 so that we have a total of $\sum_{x=1}^{n-6} n-x-5 = \binom{n-5}{2}$ possibilities.

It follows that in subcase d.1 where $\tau = 1324$, we have $2(n-5) + \binom{n-5}{2}$ possibilities if $n \geq 7$ and no possibilities if $n < 7$.

Subcase d.2. $p \geq 5$.

In this case, we must have a $1324\ldots p$-match among the elements of bricks B_1 and B_2 which can only happen if $\text{red}(\sigma_{b_1-1}\sigma_{b_1}\sigma_{b_1+1}\ldots\sigma_{b_1+p-2}) = 1324\ldots p$ and $b_2 \geq p-2$. Similarly we must have have $1324\ldots p$-match among the elements of bricks B_2 and B_3 which can only happen if $\text{red}(\sigma_{b_1+b_2-1}\sigma_{b_1+b_2}\sigma_{b_1+b_2+1}\ldots\sigma_{b_1+b_2+p-2}) = 1324\ldots p$ which is a condition that does not involve σ_{b_1+1}. Nevertheless, these conditions still force that $\sigma_{b_1}, \sigma_{b_1+1} < \sigma_{b_1+b_2+1}$ so that if $\sigma_{b_1-1} = x$, then x is less than the least elements in bricks B_2 and B_3 so that $1, \ldots, x-1$ must be in brick B_1 and $\sigma_{b_1+1} = x+1$. We also have that $\sigma_{b_1} < \sigma_{b_1+2}$ and that σ_{b_2+2} must be less than the least element in brick B_3 which is $\sigma_{b_1+b_2+1}$. It follows that it must be the case that $\sigma_{b_1} = x+2$ and $\sigma_{b_1+2} = x+3$. If we then remove the first $x+p-3$ cells which contain the numbers $1, \ldots, x+p-3$, then we will be left with a fixed point which has two bricks on $n-x-p+3$ cells. Then by our analysis of case 2, there are $n-x-p+3-p+1$ possibilities for B_3 so that we have a total of $\sum_{x=1}^{n-2(p-2)-1} n-x-2p+4 = \binom{n-2p+4}{2}$ possibilities if $n \geq 2p-2$ and no possibilities if $n < 2p-2$.

It follows that

$$U_{1324,n}(y)|_{y^3} = \begin{cases} 0 & \text{if } n = 0,1,2 \\ \binom{n-1}{2} & \text{if } n = 3,4 \text{ and} \\ \binom{n-1}{2} + 2\binom{n-3}{2} + \binom{n-5}{2} + 2(n-5) = 2(n-3)^2 & \text{if } n \geq 5. \end{cases} \tag{3.23}$$

Similarly, if $p \geq 5$, then

$$U_{1324\ldots p,n}(y)|_{y^3} = \begin{cases} 0 & \text{if } n = 0,1,2 \\ \binom{n-1}{2} & \text{if } n = 3 \leq n \leq p \\ \binom{n-1}{2} + 2\binom{n-p+1}{2} & \text{if } n = p+1 \leq \ n \leq 2p-3 \text{ and} \\ \binom{n-1}{2} + 2\binom{n-p+1}{2} + \binom{n-2p+4}{2} & \text{if } n \geq 2p-2. \end{cases} \tag{3.24}$$

Note that $2(n-3)^2 = 2n(n-1) - 10n + 18$ so that

$$\begin{aligned} U_{1324}(t,y)|_{y^3} &= \frac{t^3}{3!} + 3\frac{t^4}{4!} + \sum_{n\geq 5} 2(n-3)^2 \frac{t^n}{n!} \\ &= \frac{t^3}{3!} + 3\frac{t^4}{4!} + \sum_{n\geq 5} 2n(n-1)\frac{t^n}{n!} - \sum_{n\geq 5} 10n\frac{t^n}{n!} + + \sum_{n\geq 5} 18\frac{t^n}{n!} \\ &= \frac{t^3}{3!} + 3\frac{t^4}{4!} + 2t^2(e^t - \sum_{n=0}^{2} \frac{t^n}{n!}) - 10t(e^t - \sum_{n=0}^{3} \frac{t^n}{n!}) + 18(e^t - \sum_{n=0}^{4} \frac{t^n}{n!}) \\ &= (2t^2 - 10t + 18)e^t - 18 - 8t - t^2 + \frac{t^3}{3!} + \frac{t^4}{4!}. \end{aligned}$$

For $p \geq 5$, note that

$$\binom{n-1}{2} + 2\binom{n-p+1}{2} + \binom{n-2p+4}{2} = \\ 2n(n-1) + (5-4p)n + 3p^2 - 8p + 7$$

so that

$$\sum_{n\geq 2p-2} \left(\binom{n-1}{2} + 2\binom{n-p+1}{2} + \binom{n-2p+4}{2} \right) \frac{t^n}{n!} = \\ \sum_{n\geq 2p-2} (2n(n-1) + (5-4p)n + 3p^2 - 8p + 7)\frac{t^n}{n!} = \\ 2t^2(e^t - \sum_{n=0}^{2p-5} \frac{t^n}{n!}) + (5-4p)t(e^t - \sum_{n=0}^{2p-4} \frac{t^n}{n!}) + (3p^2 - 8p + 7)(e^t - \sum_{n=0}^{2p-3} \frac{t^n}{n!}).$$

It follows that for $p \geq 5$,

$$\begin{aligned}
U_{1324\ldots p}(t,y)|_{y^3} &= (2t^2+(5-4p)t+3p^2-8p+7)e^t + \\
&\sum_{n=3}^{p}\binom{n-1}{2}\frac{t^n}{n!} + \sum_{n=p+1}^{2p-3}(\binom{n-1}{2}+2\binom{n-p+1}{2})\frac{t^n}{n!} - \\
&\left(2t^2\sum_{n=0}^{2p-5}\frac{t^n}{n!} + (5-4p)t\sum_{n=0}^{2p-4}\frac{t^n}{n!} + (3p^2-8p+7)\sum_{n=0}^{2p-3}\frac{t^n}{n!}\right).
\end{aligned}$$

One can then use Mathematica to show that

$$U_{1324\ldots p}(t,y)|_{y^3} = (2t^2-(5-4p)t+3p^2-8p+7)e^t + \sum_{n=0}^{2p-3} f(n,p) \tag{3.25}$$

where

$$f(n,p) = \begin{cases} -3p^2+8p-7 \text{ if } n=0, \\ -3p^2+12p-12 \text{ if } n=1, \\ -3p^2+16p-21 \text{ if } n=2, \\ -\frac{3n^2}{2}+(4p-\frac{9}{2})-3p^2+8p-6 \text{ if } 3 \leq n \leq p, \text{ and} \\ -\frac{n^2}{2}+(2p-\frac{7}{2})-3p^2+7p-6 \text{ if } p+1 \leq n \leq 2p-3. \end{cases} \tag{3.26}$$

Now for any τ, we can write

$$\begin{aligned}
NM_\tau(t,1,y) &= \frac{1}{U_\tau(t,y)} = \frac{1}{1-(A_\tau(t)y - B_\tau(t)y^2 + C_\tau(t)y^3 + O(y^4))} \\
&= 1 + \sum_{n\geq 1}(A_\tau(t)y - B_\tau(t)y^2 + C_\tau(t)y^3 + O(y^4))^n. \qquad (3.27)
\end{aligned}$$

It then follows that

$$\begin{aligned}
NM_\tau(t,1,y)|_y &= A_\tau(t), \\
NM_\tau(t,1,y)|_{y^2} &= (A_\tau(t))^2 - B_\tau(t), \text{ and} \\
NM_\tau(t,1,y)|_{y^3} &= (A_\tau(t))^2 - 2A_\tau(t)B_\tau(t) + C_\tau(t).
\end{aligned}$$

We have shown that

$$\begin{aligned}
A_{1324}(t) &= e^t - 1, \\
B_{1324}(t) &= (2t-4)e^t + 4 + 2t + \frac{t^2}{2}, \text{ and} \\
C_{1324}(t) &= (2t^2-10t+18)e^t - 18 - 8t - t^2 + \frac{t^3}{3!} + \frac{t^4}{4!}.
\end{aligned}$$

One can then use Mathematica to compute that

$$D_4^{(1)}(t) = NM_{1324}(t,1,y)|_{y^2} = e^{2t} - (2t-2)e^t - 3 - 2t + \frac{t^3}{6}. \tag{3.28}$$

It follows that for $n \geq 4$,

$$d_{n,4}^{(1)} = 2^n - 2n + 2. \tag{3.29}$$

This is easy to explain directly. That is, if $\sigma \in NM_{1324,n}$ and has one descent, then σ has either one or two left-right-minima. Thus for $p \geq 4$,

$$\begin{aligned} d_{n,4}^{(1)} &= NM_{1324,n}(x,y)|_{xy^2} + NM_{1324,n}(x,y)|_{x^2y^2} \\ &= 2^{n-1} - 2n + 4 - 1 + 2^{n-1} - 1 \\ &= 2^n - 2n + 2. \end{aligned}$$

One can also use Mathematica or Maple to compute that

$$D_4^{(2)}(t) = NM_{1324}(t,1,y)|_{y^3} = e^{3t} + (5-4t)e^{2t} + (t^2 - 10t + 5)e^t - 11 - 4t + \frac{t^3}{6} + \frac{t^4}{24}. \tag{3.30}$$

It then follows that for $n \geq 5$,

$$d_{n,4}^{(2)} = 3^n + (5-2n)2^n + n^2 - 11n + 5. \tag{3.31}$$

We do not know of a simple direct proof of this result.

We have shown that for $p \geq 5$,

$$\begin{aligned} A_{1324\ldots p}(t) &= e^t - 1, \\ B_{1324\ldots p}(t) &= (2t-p)e^t + p + \sum_{n=1}^{p-2}(p-n-1)\frac{t^n}{n!}, \text{ and} \\ C_{1324\ldots p}(t) &= (2t^2 + (5-4p)t + 3p^2 - 8p + 7)e^t + \sum_{n=0}^{2p-3} f(n,p)\frac{t^n}{n!}. \end{aligned}$$

One can then use Mathematica to compute that

$$D_p^{(1)}(t) = NM_{1324}(t,1,y)|_{y^2} = e^{2t} - (2t-p+2)e^t - \sum_{n=0}^{p-2}(p-1-n)\frac{t^n}{n!}. \tag{3.32}$$

It follows that for $n \geq p-1$,

$$d_{n,p}^{(1)} = 2^n - 2n - p + 2. \tag{3.33}$$

One can easily modify the direct argument that we used to prove $d^{(1)}_{n,4} = 2^n - 2n+2$ for $n \geq 4$ to give a direct proof of this result.

One can also use Mathematica to compute that

$$D^{(2)}_p(t) = NM_{1324\dots p}(t,1,y)|_{y^3} = e^{3t} + (2p-3-4t)e^{2t} + \left(3p^2 - 12p + 10 + (13-6p)t + (5-p)t^2 - \sum_{n=3}^{p-2}(2(p-n-1)\frac{t^n}{n!}\right)e^t + 2p - 1 + \sum_{n=3}^{p-2}(2(p-n-1)\frac{t^n}{n!} + \sum_{n=0}^{2p-3} f(n,p)\frac{t^n}{n!} \quad (3.34)$$

where $f(n,p)$ is defined as in (3.26). For example, one can compute that

$$\begin{aligned} D^{(2)}_5(t) &= e^{3t} + (7-4t)e^{2t} + (25 - 17t - \frac{2t^3}{3!})e^t - \\ &\quad 33 - 21t - 6t^2 - t^3 - \frac{3t^4}{4!} - \frac{t^5}{5!}, \\ D^{(2)}_6(t) &= e^{3t} + (9-4t)e^{2t} + (46 - 23t - t^2 - \frac{4t^3}{3!} - \frac{2t^4}{4!})e^t \\ &\quad -56 - 40t - \frac{27t^2}{2} - \frac{17t^3}{3!} - \frac{3t^4}{4!} - \frac{6t^5}{5!} - \frac{3t^6}{6!} - \frac{t^7}{7!}, \text{ and} \\ D^{(2)}_7(t) &= e^{3t} + (11-4t)e^{2t} + (73 - 29t - 2t^2 - t^3 - \frac{4t^4}{4!} - \frac{2t^5}{5!})e^t \\ &\quad -85 - 65t - \frac{48t^2}{2} - \frac{34t^3}{3!} - \frac{23t^4}{4!} - \frac{15t^5}{5!} - \frac{10t^6}{6!} - \frac{6t^7}{7!} - \frac{3t^8}{8!} - \frac{t^9}{9!}. \end{aligned}$$

It then follows that for $n \geq 2p-2$,

$$d^{(2)}_{n,p} = 3^n + (2p-3-2n)2^n + 3p^2 - 12p + 10 + (13-6p)n + (5-p)n(n-1) - \sum_{k=3}^{p-2} \frac{2(p-k-1)}{k!} n(n-1)\cdots(n-k+1). \quad (3.35)$$

For example, for $n \geq 8$,

$$d^{(2)}_{n,5} = 3^n + (7-2n)2^n + 25 - \frac{53n}{3} + n^2 - \frac{n^3}{3}.$$

For $n \geq 10$,

$$d^{(2)}_{n,6} = 3^n + (9-2n)2^n + 46 - \frac{136n}{6} + \frac{n^2}{12} - \frac{n^3}{6} - \frac{n^4}{12}.$$

For $n \geq 12$,

$$d_{n,7}^{(2)} = 3^n + (11 - 2n)2^n + 73 - \frac{142n}{5} + \frac{7n^3}{12} - \frac{n^5}{60}.$$

Chapter 4

The Bijection Between Cycles and Brick Tabloids

In this chapter, we will define a bijection between cycles and brick tabloids that leads to results about permutations that have no cycle-matches for unusual groups of patterns.

4.1 Symmetric Functions

Let $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ denote the set of integers and $\mathbb{P} = \{1, 2, 3, \ldots\}$ denote the set of positive integers. Recall that h_n and e_n are given by (1.31) and (1.32). Given a $f : \mathbb{P} \to \mathbb{P}$, we define a ring homorphism $\Theta_f : \Lambda \to \mathbb{Z}$ by $\Theta_f(e_0) = 1$ and $\Theta_f(e_n) = \frac{(-1)^{n-1} f(n)}{n!}$ for $n > 0$. This given, we want to give a combinatorial interpretation for $n!\Theta(h_n)$. Using (3.5), we see that

$$\begin{aligned} n!\Theta(h_n) &= n! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} B_{\lambda,n} \Theta(e_\mu) \\ &= n! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \sum_{(b_1,\ldots,b_{\ell(\mu)}) \in \mathcal{B}_{\mu,n}} \prod_{i=1}^{\ell(\mu)} \frac{(-1)^{b_i - 1} f(b_i)}{b_i!} \\ &= \sum_{\mu \vdash n} \sum_{(b_1,\ldots,b_{\ell(\mu)}) \in \mathcal{B}_{\mu,n}} \binom{n}{b_1, \ldots, b_{\ell(\mu)}} f(b_1) \ldots f(b_{\ell(\mu)}). \end{aligned} \tag{4.1}$$

We can interpret $\binom{n}{b_1,\ldots,b_{\ell(\mu)}}$ as picking the number of ways of placing the numbers $1,\ldots,n$ in the brick tabloid $(b_1,\ldots,b_{\ell(\mu)})$ so that the numbers increase from left to right in each brick. We let $\mathcal{F}_n$ denote the set of all pairs (T,σ) where $T = (b_1,\ldots,b_k)$ is a brick tabloid in $\mathcal{B}_n$ and $\sigma \in S_n$ which is increasing in each brick. We then define the f-weight of (T,σ), $w_f(T,\sigma)$, to be $f(b_1)\cdots f(b_k)$. Thus we have shown that

$$n!\Theta_f(h_n) = \sum_{(T,\sigma)\in\mathcal{F}_n} w_f(T,\sigma). \tag{4.2}$$

We can now apply Θ_f to (1.30), we see that

$$\begin{aligned} 1+\sum_{n\geq 1}\frac{t^n}{n!}\sum_{(T,\sigma)\in\mathcal{F}_n} w_f(T,\sigma) &= \Theta_f(H(t)) \\ &= \Theta_f(E(-t)) \\ &= \frac{1}{1+\sum_{n\geq 1}(-t)^n\frac{(-1)^{n-1}f(n)}{n!}} \\ &= \frac{1}{1-\sum_{n\geq 1}\frac{f(n)}{n!}t^n}. \end{aligned} \tag{4.3}$$

Let's use the theory of exponential structures to describe other objects dependent on the function $f(n)$. Consider an object $(T,\sigma) \in \mathcal{F}_n$. Notice that since the bricks increase from left to right, the left-to-right minima will always be the first element of a brick. So we can now use the inverse map described on page 15 where we group "cycles" by starting a new cycle at each left-to-right minimum. Notice that the grouping will never break apart a brick. Then by the theory of exponential structures, Theorem 10 we can think about this generating function as

$$\frac{1}{1-\sum_{n\geq 1}\frac{f(n)}{n!}t^n} = \exp\left(\sum_{n\geq 1}\frac{t^n}{n!}g_f(n)\right) \tag{4.4}$$

where $g_f(n)$ counts cycles of length n that are weighted by the following function.

Let $C = (c_0,\ldots,c_{n-1}) \in S_n$ be a cycle with $c_0 = 1$, then let $B(C)$ be the set of all brick tabloids (T,C) such that C starting with 1 is the underlying permutation and the numbers in each brick are increasing.

Then let $v_f(C) = \sum_{(T,C)\in B(C)} w_f(T,C)$ be a weight function based on f

and let $\mathcal{L}_n$ be the set of all n-cycles. Then we have that

$$g_f(n) = \sum_{C \in \mathcal{L}_n} v_f(C). \tag{4.5}$$

We can interpret $g_f(n)$ as the number of brick tabloids $T = (b_1, \ldots, b_k)$ filled with the numbers $1, \ldots, n$ so that the numbers increase from left to right in each brick and such that the first element of the first brick is 1. Furthermore the brick tabloid is weighted by the function $w_f(T, C) = f(b_1) \cdots f(b_k)$. Now we can express (4.4) as

$$\sum_{n \geq 1} \frac{t^n}{n!} g_f(n) = \log \left(\frac{1}{1 - \sum_{n \geq 1} \frac{f(n)}{n!} t^n} \right) \tag{4.6}$$

4.2 The bijection between cycles and filled circled brick tabloids

We start off by considering the generating function (4.3) where $f(n) = n-1$ for all $n > 1$. Then the right hand side of (4.3) becomes

$$\begin{aligned} \frac{1}{1 - \sum_{n \geq 1} \frac{n-1}{n!} t^n} &= \frac{1}{1 + \sum_{n \geq 1} \frac{1}{n!} t^n - t \sum_{n \geq 1} \frac{1}{(n-1)!} t^{n-1}} \\ &= \frac{1}{e^t - te^t} = \frac{e^{-t}}{1-t}. \end{aligned} \tag{4.7}$$

And so we have that

$$\begin{aligned} \sum_{n \geq 1} \frac{t^n}{n!} g_f(n) &= \log \left(\frac{1}{1 - \sum_{n \geq 1} \frac{n-1}{n!} t^n} \right) \\ &= \log \left(\frac{e^{-t}}{1-t} \right) \\ &= -t - \log(1-t) \\ &= \sum_{n \geq 2} \frac{t^n}{n} = \sum_{n \geq 2} \frac{t^n}{n!} (n-1)! \end{aligned} \tag{4.8}$$

Notice that this is the EGF for cycles with length ≥ 2. Let $\mathcal{L}_n$ be the set of all cycles of length n and let $L_n = |\mathcal{L}_n|$. It follows that

$$L_n = v_f(n). \tag{4.9}$$

Since $f(n) = n - 1$, we can interpret $v_f(n)$ as the number of brick tabloids (T, C) that start with 1 and are made up of bricks that have size 2 or greater in which we circle one element in each brick which is not the final element in the brick. Let $\mathcal{F}_{C,n}$ be this set of objects. For example, Figure 4.1 shows two elements of $\mathcal{F}_{C,12}$

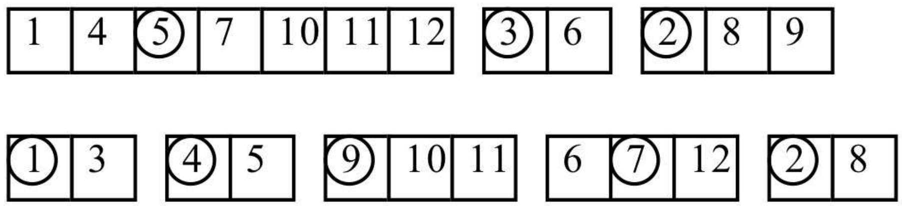
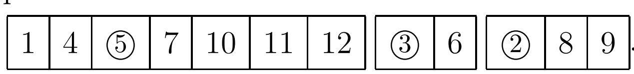


Figure 4.1: Elements of $\mathcal{F}_{C,12}$.

It follows (4.9) that $|\mathcal{F}_{C,n}| = L_n$ for all $n \geq 2$ and, hence, there must exist a bijection $\Phi_n : \mathcal{F}_{C,n} \to \mathcal{L}_n$ for all $n \geq 2$. This best way to describe the bijection is a step by step process. We will use the first brick tabloid of Figure 4.1 as a example. Let $(T, \sigma) =$ .

Step 1. Arrange the elements in each brick so that the circled elements comes first followed by the remaining elements arranged in decreasing order.

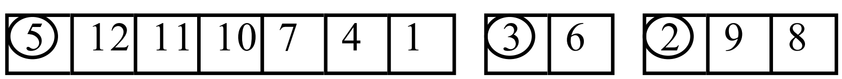


Figure 4.2: Step 1 of transforming (T, σ).

Step 2. Now simply erase the brick and think of the resulting configuration as a cycle $\Phi_n(T, \sigma)$.

$$\begin{aligned}\Phi_{12}((T, \sigma)) &= (5, 12, 11, 10, 7, 4, 1, 3, 6, 2, 9, 8) \\ &= (1, 3, 6, 2, 9, 8, 5, 12, 11, 10, 7, 4).\end{aligned}$$

For the inverse map, suppose that we are given a cycle σ in L_n. Then we can compute $\Phi_n^{-1}(\sigma)$ by the following four step process. For this example suppose

$\sigma = (1, 6, 11, 10, 5, 3, 2, 9, 13, 7, 8, 12, 14, 4)$

Step I. Arrange the cycle σ so that 1 is at the start of the cycle. (In our example, σ already starts with 1)

Step II. Read the entries of σ from right to left. We then circle the first element σ_{i_1} such that $\sigma_{i_1} < \sigma_{i_1+1}$. If i_1 is not the first entry, then we continue reading the elements from right to left until we find a cell i_2 in the brick such that $\sigma_{i_2} < \sigma_{i_2+1}$ and $i_2 + 1 < i_1$ in which case we circle σ_{i_2}. If i_2 is not the first entry, then we continue reading the elements from right to left until we find a cell i_3 in the brick such that $\sigma_{i_3} < \sigma_{i_2+1}$ and $i_3 + 1 < i_2$ in which case we circle σ_{i_3}, etc. In our example after Step II, the circled elements are shown below

$$(1, \textcircled{6}, 11, 10, 5, 3, 2, \textcircled{9}, 13, \textcircled{7}, 8, \textcircled{12}, 14, 4)$$

Step III. Next we break up σ from Step II into bricks by starting with each circled element and form a brick which consists of the circled element plus all elements to the right that come before the next circled element. For the last circled element in σ, the corresponding brick will contain all the elements to its right if 1 is circled or it will contain all the elements to its right with 1 added to the end if 1 is not circled. We may think about this last remark as wrapping around the cycle until you get to the next circled element. In our example, the bricks are shown below.

$$\boxed{\textcircled{6} \mid 11 \mid 10 \mid 5 \mid 3 \mid 2}\ \boxed{\textcircled{9} \mid 13}\ \boxed{\textcircled{7} \mid 8}\ \boxed{\textcircled{12} \mid 14 \mid 4 \mid 1}$$

Step IV. Finally cyclically arrange the bricks so that the brick that includes 1 is the first brick and arrange the numbers within each brick in increasing order. Here is the pre-image of our example $\Phi_{14}^{-1}(\sigma)$.

$$\boxed{1 \mid 4 \mid \textcircled{12} \mid 14}\ \boxed{2 \mid 3 \mid 5 \mid \textcircled{6} \mid 10 \mid 11}\ \boxed{\textcircled{9} \mid 13}\ \boxed{\textcircled{7} \mid 8}$$

To see that the procedure for Φ_n^{-1} is correct, suppose that $T = (b_1, \ldots, b_k)$ and consider these two cases.

Case 1. In (T,σ), 1 is circled in b_1.

Since in each brick of (T,σ), we put the circled element first and the remaining elements in decreasing order, $\Phi_n(T,\sigma)$ will directly produce a cycle $(c_1,\ldots,c_p)$ where $c_1 = 1$. We can find the circled element that corresponds to brick b_k because it will correspond to right-most rise in $c_1\cdots c_p$. If that element is c_r, then the circled element in brick b_{k-1} will correspond to the right-most rise in $c_1\cdots c_{r-1}$, etc.. Thus it is easy to see that Step II will recover the circled elements and Step III will recover the original bricks.

Case 2. In (T,σ), 1 is not circled in b.

This means that brick b_1 has length at least three. Under Φ_n, we would arrange the elements in b_1 so that $a_1 < a_2 > \cdots > a_{b_1-1} > 1$ where a_1 is the circled element in b_1. Thus the cycle produced by Φ_n from the elements in bricks $b_1,\ldots,b_k$ would be of the form $(c_1,\ldots,c_p)$ where $c_1 = 1$ and $c_{p-i} = a_{b_1-1-i}$ for $i = 0,\ldots,b_1-2$. This means that, we can recover the elements in the b_1 by looking for the the right-most rise in $c_1\ldots c_p$ and combining it with the element 1. Once we have recovered the elements in brick b_1, then, as in Case 1, we can find the circled element that corresponds to brick b_k because it will correspond to right-most rise in $c_2\cdots c_{p-b_1}$. If that element is c_r, then the circled element in brick b_{k-1} will correspond to the right-most rise in $c_2\cdots c_{r-1}$, etc.. Hence our procedure in Step II will properly recover the circled elements and Step III will recover the original bricks $b_1,\ldots,b_k$. Thus we have shown that $\Phi_n^{-1}\circ\Phi_n$ is the identity. The argument that $\Phi_n\circ\Phi_n^{-1}$ equals the identity is similar. Thus we have the following theorem.

Theorem 35. *Φ_n is a bijection from $\mathcal{F}_{C,n}$ onto $\mathcal{L}_n$.*

4.3 Restricting brick tabloids

Once a bijection is formed between two sets of objects that do not obviously have the same cardinality, it is natural to restrict the structure of one set and see what this restricted subset maps to under the bijection. The restrictions that we consider will be restrictions that come about by altering the function $f(n)$. If

$f(n) = n - 1$ produces the set of objects $\mathcal{F}_{C,n}$ then changing this function will produce subsets of $\mathcal{F}_{C,n}$.

4.3.1 No bricks of size 2

The first restriction that we consider is restricting the set $\mathcal{F}_{C,n}$ to include objects that do not have any bricks of size 2.

In this case, the function $f(n)$ in the generating function (4.3) can be defined as

$$f_1(n) = \begin{cases} 0 & \text{if } n = 1, 2 \\ n-1 & \text{if } n \geq 3 \end{cases}. \tag{4.10}$$

Theorem 36. *Let $\mathcal{F}^1_{C,n}$ be the subset of $\mathcal{F}_{C,n}$ restricted by only including objects that do not have any bricks of size 2.*

For $n \geq 5$, The map Φ_n maps $\mathcal{F}^1_{C,n}$ to the set of cycles that have no consecutive cycle occurrences of the patterns in the set $\Gamma \cup \{1234\}$ where Γ is the set of all patterns $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$ such that

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \sigma_5.$$

Proof. We will prove this by showing that the inverse map Φ_n is a bijection between the complement of $\mathcal{L}^{ncm}_n(\Gamma \cup \{1234\})$ and the complement of $\mathcal{F}^1_{C,n}$ First lets say that $\text{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234\}))$ is the complement of $\mathcal{L}^{ncm}_n(\Gamma \cup \{1234\})$ and $\text{comp}(\mathcal{F}^1_{C,n})$ is the complement of $\mathcal{F}^1_{C,n}$. We will show that the maps

$$\Phi_n \quad : \text{comp}(\mathcal{F}^1_{C,n}) \to \text{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234\})) \tag{4.11}$$

$$\Phi^{-1}_n \quad : \text{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234\})) \to \text{comp}(\mathcal{F}^1_{C,n}) \tag{4.12}$$

are both injections.

$\Phi_n : \text{comp}(\mathcal{F}^1_{C,n}) \to \text{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234\}))$:

Suppose $(T, \sigma) \in \text{comp}(\mathcal{F}^1_{C,n})$ with $T = (b_1, \ldots, b_k)$ then for some brick $|b_i| = 2$. Suppose $b_i = \boxed{ⓐ_1 \mid a_2}$ and suppose that the circled element of b_{i+1} is a_3

Case 1: $a_2 < a_3$

Suppose that the greatest element of b_{i+1} is a_4. From Step 1 of the map Φ_n, the numbers of $\Phi_n(T,\sigma)$ will be arranged so that a_1 is followed by a_2 then a_3 and then a_4. Then $a_1 < a_2 < a_3 < a_4$ so there is a cycle-1234-match in $\Phi_n(T,\sigma)$.

Case 2: $a_2 > a_3$

We must consider 2 subcases.

Subcase 1: All bricks are of size 2.

We will assume that there are no two consecutive bricks $\boxed{\textcircled{a_1} \mid a_2}\boxed{\textcircled{a_3} \mid a_4}$ such that $a_2 < a_3$ because this has been considered in Case 1. Specifically there will be three bricks in a row $\boxed{\textcircled{a_1} \mid a_2}\boxed{\textcircled{a_3} \mid a_4}\boxed{\textcircled{a_5} \mid a_6}$ such that $a_1 < a_2 > a_3 < a_4 > a_5 < a_6$ so when we apply the map $\Phi_n(T,\sigma)$, there will be a cycle pattern match in $\Phi_n(T,\sigma)$ for a pattern in Γ.

Subcase 2: There exists a brick that has size larger than 2.

Consider the brick of size 2 that comes directly before the brick of size larger than 2. For the larger brick, let a_3 be the circled element, let a_4 be the largest element and let a_5 be the second largest element (excluding the circled element.) Then after Step 1 of the map, the numbers of $\Phi(T,\sigma)$ will be arranged so that a_1 is followed by a_2, a_3, a_4, a_5. Then $a_1 < a_2 > a_3 < a_4 > a_5$ so there is a cycle pattern match in $\Phi_n(T,\sigma)$ for a pattern in Γ.

$\Phi_n^{-1} : \mathrm{comp}(\mathcal{L}_n^{ncm}(\Gamma \cup \{1234\})) \to \mathrm{comp}(\mathcal{F}^1_{C,n})$:

Suppose that $c \in \mathrm{comp}(\mathcal{L}_n^{ncm}(\Gamma \cup \{1234\}))$ has a cycle match of 1234 then somewhere in the cycle we have

$$(\ldots c_i < c_{i+1} < c_{i+2} < c_{i+3} > c_{i+4} \ldots).$$

From Step II of the inverse map, c_{i+3} and c_{i+1} are circled which will map to a brick tabloid (T,σ) that includes the brick $\boxed{\textcircled{c_{i+1}} \mid c_{i+2}}$ of size 2.

Now suppose that $c \in C_n$ has a cycle match of a pattern in Γ, then somewhere in the cycle we have

$$(\ldots c_i < c_{i+1} > c_{i+2} < c_{i+3} > c_{i+4} \ldots).$$

From Step II of the inverse map, c_{i+2} and c_i are circled which will map to a brick tabloid (T,σ) that includes the brick $\boxed{\textcircled{c_i} \mid c_{i+1}}$ of size 2.

Therefore we have shown that $\Phi_n : \text{comp}(\mathcal{F}^1_{C,n}) \to \text{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234\}))$ is a bijection, therefore $\Phi_n : \mathcal{F}^1_{C,n} \to \mathcal{L}^{ncm}_n(\Gamma \cup \{1234\})$ must also be a bijection. ☐

Corollary 37. *Let* $\Upsilon_1 = \Gamma \cup \{1234\}$ *then*

$$NCM_{\Upsilon_1}(t) = \frac{2e^{t^2/2}e^{t^4/12}}{2 - 2t + t^2 e^{-t}}.$$

Proof. Since $|\mathcal{F}^1_{C,n}| = |\mathcal{L}^{ncm}_n(\Upsilon_1)|$ for $n \geq 5$, and $|\mathcal{F}^1_{C,n}|$ is counted by the generating function $\sum_{n\geq 1} \frac{t^n}{n!} g_{f_1}(n)$ where

$$f_1(n) = \begin{cases} 0 & \text{if } n = 1, 2 \\ n-1 & \text{if } n \geq 3 \end{cases}$$

then we have that

$$\begin{aligned} \sum_{n\geq 5} \frac{t^n}{n!} |\mathcal{L}^{ncm}_n(\Upsilon_1)| &= \sum_{n\geq 5} \frac{t^n}{n!} g_{f_1}(n) \\ &= \log\left(\frac{1}{1 - \sum_{n\geq 1} \frac{f_1(n)}{n!} t^n}\right) - \sum_{n=1}^{4} \frac{t^n}{n!} g_{f_1}(n) \\ &= \log\left(\frac{1}{1 - \sum_{n\geq 3} \frac{n-1}{n!} t^n}\right) - \sum_{n=1}^{4} \frac{t^n}{n!} |\mathcal{F}^1_{C,n}| \end{aligned}$$

One can easily show that for $n = (1, 2, 3, 4)$, $|\mathcal{F}^1_{C,n}| = (0, 0, 2, 3)$ so this generating function becomes:

$$\begin{aligned} \sum_{n\geq 5} \frac{t^n}{n!} |\mathcal{L}^{ncm}_n(\Upsilon_1)| &= \log\left(\frac{1}{1 - \sum_{n\geq 3} \frac{n-1}{n!} t^n}\right) - 2\frac{t^3}{3!} - 3\frac{t^4}{4!} \\ &= \log\left(\frac{2}{2e^t - 2te^t + t^2}\right) - \frac{2t^3}{3!} - \frac{3t^4}{4!} \end{aligned}$$

Furthermore, one can easily show that for $n = (1, 2, 3, 4)$, $|\mathcal{L}^{ncm}_n(\Upsilon_1)| = (1, 1, 2, 5)$ so we can get the following generating function for $|\mathcal{L}^{ncm}_n(\Upsilon_1)|$

$$\begin{aligned} \sum_{n\geq 1} \frac{t^n}{n!} |\mathcal{L}^{ncm}_n(\Upsilon_1)| &= \log\left(\frac{2}{2e^t - 2te^t + t^2}\right) - \frac{2t^3}{3!} - \frac{3t^4}{4!} + t + \frac{t^2}{2!} + \frac{2t^3}{3!} + \frac{5t^4}{4!} \\ &= \log\left(\frac{2}{2e^t - 2te^t + t^2}\right) + t + \frac{t^2}{2!} + \frac{2t^4}{4!} \end{aligned}$$

Then by the theory of exponential structures (10) we can compute the generating function of $NCM_{\Upsilon_1}(t)$

$$\begin{aligned} NCM_{\Upsilon_1}(t) &= \exp\left(\sum_{n\geq 1}\frac{t^n}{n!}|\mathcal{L}_n^{ncm}(\Upsilon_1)|\right) \\ &= \exp\left(\log\left(\frac{2}{2e^t - 2te^t + t^2}\right) + t + \frac{t^2}{2!} + \frac{2t^4}{4!}\right) \\ &= \frac{2e^{t^2/2}e^{t^4/12}}{2-2t+t^2e^{-t}} \end{aligned}$$

□

4.3.2 No bricks of size 2 and circle the second to last element

In this case we consider the function

$$f_2(n) = \begin{cases} 0 & \text{if } \ n = 1,2 \\ 1 & \text{if } \ n \geq 3 \end{cases} . \tag{4.13}$$

Theorem 38. *Let $\mathcal{F}_{C,n}^2$ be the subset of $\mathcal{F}_{C,n}$ restricted by only including objects that do not have any bricks of size 2 and furthermore only the second to last element may be circled.*

For $n \geq 5$, The map Φ_n maps $\mathcal{F}_{C,n}^2$ to the set of cycles that have no consecutive cycle occurrences of the patterns in the set $\Gamma \cup \{1234, 132\}$ where Γ is the set of all patterns $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$ such that

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \sigma_5.$$

Proof. Just like in the proof of Theorem 36, we will show that

$$\Phi_n \quad : \mathrm{comp}(\mathcal{F}_{C,n}^2) \to \mathrm{comp}(\mathcal{L}_n^{ncm}(\Gamma \cup \{1234, 132\})) \tag{4.14}$$

$$\Phi_n^{-1} \quad : \mathrm{comp}(\mathcal{L}_n^{ncm}(\Gamma \cup \{1234, 132\})) \to \mathrm{comp}(\mathcal{F}_{C,n}^2) \tag{4.15}$$

are both injections.

$\Phi_n : \mathrm{comp}(\mathcal{F}_{C,n}^2) \to \mathrm{comp}(\mathcal{L}_n^{ncm}(\Gamma \cup \{1234, 132\}))$:

Case 1: $(T, \sigma) \in \text{comp}(\mathcal{F}^2_{C,n})$ has a brick of size 2.

From Theorem 36, we know that this will map to the set $\text{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234\}))$ which is a subset of $\text{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234, 132\}))$.

Case 2: $(T, \sigma) \in \text{comp}(\mathcal{F}^2_{C,n})$ does not have a brick of size 2.

This means that there exists a brick b_i in (T, σ) such that the circled element of b_i is not the second to last element. Then suppose for b_i, a_1 is the circled element, a_2 is the greatest element and a_3 is the second greatest element. Then by Step 2 of the map Φ_n, the elements of $\Phi_n(T, \sigma)$ will be arranged so that a_1 is followed by a_2 and a_3. But $\text{red}(a_1, a_2, a_3) = 1, 3, 2$ so the image under the map Φ_n maps to a cycle that has a cycle-132-match.

$\Phi_n^{-1} : \text{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234, 132\})) \rightarrow \text{comp}(\mathcal{F}^2_{C,n})$:

Case 1: Suppose $\sigma \in \text{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234, 132\}))$ has a cycle-Γ-match or a cycle-1234-match.

Then by Theorem 36, $\Phi_n^{-1}(\sigma)$ is in $\text{comp}(\mathcal{F}^1_{C,n})$ which is a subset of $\text{comp}(\mathcal{F}^2_{C,n})$

Case 2: Suppose $\sigma \in \text{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234, 132\}))$ has a cycle-132-match.

Then suppose the entries c_i, c_{i+1}, c_{i+2} have relative order 132. Then consider two subcases.

Subcase 1: c_{i+2} is circled in Step II

If this is the case then by Step II, c_i is also circled and therefore the pre-image $\Phi^{-1}(\sigma)$ has a brick of size 2 consisting of the entries c_i, c_{i+1} and therefore $\Phi^{-1}(\sigma) \in \text{comp}(\mathcal{F}^2_{C,n})$.

Subcase 2: c_{i+2} is not circled in Step II

If this is the case then then c_i is circled and the pre-image $\Phi^{-1}(\sigma)$ has a brick of size greater than 2 that includes the elements c_i, c_{i+1}, c_{i+2} and perhaps more elements. Since by Step III, the elements are arranged by increasing order in the bricks, and because $c_i < c_{i+2} < c_{i+1}$, it is clear that in this brick there are at least two elements greater than the circled element c_i and therefore this is a brick where the second to last element is not circled and therefore $\Phi^{-1}(\sigma) \in \text{comp}(\mathcal{F}^2_{C,n})$.

Therefore we have shown that $\Phi_n : \text{comp}(\mathcal{F}^2_{C,n}) \to \text{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234, 132\}))$ is a bijection, therefore $\Phi_n : \mathcal{F}^2_{C,n} \to \mathcal{L}^{ncm}_n(\Gamma \cup \{1234, 132\})$ is also bijection. $\square$

Corollary 39. *Let* $\Upsilon_2 = \{132, 1234, 35241, 45231, 34251\}$ *then*

$$NCM_{\Upsilon_2}(t) = \frac{2e^t e^{t^2/2}}{4 - 2e^t + t^2 + 2t}.$$

Proof. First notice that the only 3 patterns in Γ that do not have a 132-match are $\{35241, 45231, 34251\}$. Notice that if a permutation σ has no cycle-τ-matches for a pattern τ that has a 132-match then σ will have no cycle-132-matches. Therefore

$$\mathcal{L}^{ncm}_n(\Gamma \cup \{1234, 132\}) = \mathcal{L}^{ncm}_n(\{132, 1234, 35241, 45231, 34251\})$$

Since $|\mathcal{F}^2_{C,n}| = |\mathcal{L}^{ncm}_n(\Gamma \cup \{1234, 132\})| = |\mathcal{L}^{ncm}_n(\Upsilon_2)|$ for $n \geq 5$, and $|\mathcal{F}^2_{C,n}|$ is counted by the generating function $\sum_{n\geq 1} \frac{t^n}{n!} g_{f_2}(n)$ where

$$f_2(n) = \begin{cases} 0 & \text{if } \; n = 1, 2 \\ 1 & \text{if } \; n \geq 3 \end{cases}$$

then we have that

$$\begin{aligned} \sum_{n\geq 5} \frac{t^n}{n!} |\mathcal{L}^{ncm}_n(\Upsilon_2)| &= \sum_{n\geq 5} \frac{t^n}{n!} g_{f_2}(n) \\ &= \log\left(\frac{1}{1 - \sum_{n\geq 1} \frac{f_2(n)}{n!} t^n}\right) - \sum_{n=1}^{4} \frac{t^n}{n!} g_{f_2}(n) \\ &= \log\left(\frac{1}{1 - \sum_{n\geq 3} \frac{1}{n!} t^n}\right) - \sum_{n=1}^{4} \frac{t^n}{n!} |\mathcal{F}^2_{C,n}| \end{aligned}$$

One can easily show that for $n = (1, 2, 3, 4)$, $|\mathcal{F}^2_{C,n}| = (0, 0, 1, 1)$ so this generating function becomes:

$$\begin{aligned} \sum_{n\geq 5} \frac{t^n}{n!} |\mathcal{L}^{ncm}_n(\Upsilon_2)| &= \log\left(\frac{1}{1 - \sum_{n\geq 3} \frac{1}{n!} t^n}\right) - \frac{t^3}{3!} - \frac{t^4}{4!} \\ &= \log\left(\frac{2}{4 - 2e^t + t^2 + 2t}\right) - \frac{t^3}{3!} - \frac{t^4}{4!} \end{aligned}$$

Furthermore, one can easily show that for $n = (1,2,3,4)$, $|\mathcal{L}_n^{ncm}(\Upsilon_2)| = (1,1,1,1)$ so we can get the following generating function for $|\mathcal{L}_n^{ncm}(\Upsilon_2)|$

$$\begin{aligned}\sum_{n\geq 1}\frac{t^n}{n!}|\mathcal{L}_n^{ncm}(\Upsilon_2)| &= \log\left(\frac{2}{4-2e^t+t^2+2t}\right) - \frac{t^3}{3!} - \frac{t^4}{4!} + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} \\ &= \log\left(\frac{2}{4-2e^t+t^2+2t}\right) + t + \frac{t^2}{2!}\end{aligned}$$

Then by the theory of exponential structures (10) we can compute the generating function of $NCM_{\Upsilon_2}(t)$

$$\begin{aligned}NCM_{\Upsilon_2}(t) &= \exp\left(\sum_{n\geq 1}\frac{t^n}{n!}|\mathcal{L}_n^{ncm}(\Upsilon_2)|\right) \\ &= \exp\left(\log\left(\frac{2}{4-2e^t+t^2+2t}\right) + t + \frac{t^2}{2!}\right) \\ &= \frac{2e^t e^{t^2/2}}{4-2e^t+t^2+2t}\end{aligned}$$

□

4.3.3 No bricks of size 2 and do not circle the second to last element

In this case we consider the function

$$f_3(n) = \begin{cases} 0 & \text{if } n = 1,2 \\ n-2 & \text{if } n \geq 3 \end{cases}. \tag{4.16}$$

Theorem 40. *Let $\mathcal{F}_{C,n}^3$ be the subset of $\mathcal{F}_{C,n}$ restricted by only including objects that do not have any bricks of size 2 and furthermore the second to last element may not be circled.*

For $n \geq 5$, The map Φ_n maps $\mathcal{F}_{C,n}^3$ to the set of cycles that have no consecutive cycle occurrences of the patterns in the set $\Gamma \cup \{1234, 231\}$ where Γ is the set of all patterns $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$ such that

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \sigma_5.$$

Proof. Just like in the proof of Theorem 36, we will show that

$$\Phi_n \quad : \mathrm{comp}(\mathcal{F}^3_{C,n}) \to \mathrm{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234, 231\})) \tag{4.17}$$

$$\Phi_n^{-1} \quad : \mathrm{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234, 231\})) \to \mathrm{comp}(\mathcal{F}^3_{C,n}) \tag{4.18}$$

are both injections.

$\Phi_n : \mathrm{comp}(\mathcal{F}^3_{C,n}) \to \mathrm{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234, 231\}))$:

Case 1: $(T, \sigma) \in \mathrm{comp}(\mathcal{F}^3_{C,n})$ has a brick of size 2.

From Theorem 36, we know that this will map to the set $\mathrm{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234\}))$ which is a subset of $\mathrm{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234, 231\}))$.

Case 2: $(T, \sigma) \in \mathrm{comp}(\mathcal{F}^3_{C,n})$ does not have a brick of size 2.

This means that there exists a brick b_i in (T, σ) such that the circled element of b_i is the second to last element. Then suppose for b_i, a_1 is the circled element, a_2 is the greatest element and a_3 is the third greatest element. Then by Step 2 of the map Φ_n, the elements of $\Phi_n(T, \sigma)$ will be arranged so that a_1 is followed by a_2 and a_3. But $\mathrm{red}(a_1, a_2, a_3) = 2, 3, 1$ so the image under the map Φ_n maps to a cycle that has a cycle-231-match.

$\Phi_n^{-1} : \mathrm{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234, 231\})) \to \mathrm{comp}(\mathcal{F}^3_{C,n})$:

Case 1: Suppose $\sigma \in \mathrm{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234, 231\}))$ has a cycle-Γ-match or a cycle-1234-match.

Then by Theorem 36, $\Phi_n^{-1}(\sigma)$ is in $\mathrm{comp}(\mathcal{F}^1_{C,n})$ which is a subset of $\mathrm{comp}(\mathcal{F}^3_{C,n})$

Case 2: Suppose $\sigma \in \mathrm{comp}(\mathcal{L}^{ncm}_n(\Gamma \cup \{1234, 132\}))$ has a cycle-132-match.

Then suppose the entries c_i, c_{i+1}, c_{i+2} have relative order 231. Then consider two subcases.

Subcase 1: c_{i+2} is circled in Step II

If this is the case then by Step II, c_i is also circled and therefore the pre-image $\Phi^{-1}(\sigma)$ has a brick of size 2 consisting of the entries c_i, c_{i+1} and therefore $\Phi^{-1}(\sigma) \in \mathrm{comp}(\mathcal{F}^3_{C,n})$.

Subcase 2: c_{i+2} is not circled in Step II

If this is the case then then c_i is circled and the pre-image $\Phi^{-1}(\sigma)$ has a brick of size greater than 2 that includes the elements c_i, c_{i+1}, c_{i+2} and perhaps more elements. Since by Step III, the elements are arranged by increasing order in the bricks, and because $c_{i+2} < c_i < c_{i+1}$, it is clear that in this brick there is only one element greater than the circled element c_i and therefore this is a brick where the second to last element is circled and therefore $\Phi^{-1}(\sigma) \in \text{comp}(\mathcal{F}^3_{C,n})$.

Therefore we have shown that $\Phi_n : \text{comp}(\mathcal{F}^3_{C,n}) \to \text{comp}(\mathcal{L}_n^{ncm}(\Gamma \cup \{1234, 231\}))$ is a bijection, therefore $\Phi_n : \mathcal{F}^3_{C,n} \to \mathcal{L}_n^{ncm}(\Gamma \cup \{1234, 231\})$ is also bijection. ☐

Corollary 41. *Let* $\Upsilon_3 = \{231, 1234, 13254, 14253, 15243\}$ *then*

$$NCM_{\Upsilon_3}(t) = \frac{e^t e^{t^2/2}}{-1 - t + 2e^t - te^t}.$$

Proof. First notice that the only 3 patterns in Γ that do not have a 132-match are $\{13254, 14253, 15243\}$. Notice that if a permutation σ has no cycle-τ-matches for a pattern τ that has a 132-match then σ will have no cycle-132-matches. Therefore

$$\mathcal{L}_n^{ncm}(\Gamma \cup \{1234, 231\}) = \mathcal{L}_n^{ncm}(\{231, 1234, 13254, 14253, 15243\})$$

Since $|\mathcal{F}^3_{C,n}| = |\mathcal{L}_n^{ncm}(\Gamma \cup \{1234, 231\})| = |\mathcal{L}_n^{ncm}(\Upsilon_3)|$ for $n \geq 5$, and $|\mathcal{F}^3_{C,n}|$ is counted by the generating function $\sum_{n\geq 1} \frac{t^n}{n!} g_{f_3}(n)$ where

$$f_3(n) = \begin{cases} 0 & \text{if } n = 1, 2 \\ n-2 & \text{if } n \geq 3 \end{cases}$$

then we have that

$$\begin{aligned} \sum_{n\geq 5} \frac{t^n}{n!} |\mathcal{L}_n^{ncm}(\Upsilon_3)| &= \sum_{n\geq 5} \frac{t^n}{n!} g_{f_3}(n) \\ &= \log\left(\frac{1}{1 - \sum_{n\geq 1} \frac{f_3(n)}{n!} t^n}\right) - \sum_{n=1}^{4} \frac{t^n}{n!} g_{f_3}(n) \\ &= \log\left(\frac{1}{1 - \sum_{n\geq 3} \frac{n-2}{n!} t^n}\right) - \sum_{n=1}^{4} \frac{t^n}{n!} |\mathcal{F}^3_{C,n}| \end{aligned}$$

One can easily show that for $n = (1, 2, 3, 4)$, $|\mathcal{F}^3_{C,n}| = (0, 0, 1, 2)$ so this generating function becomes:

$$\begin{aligned}\sum_{n\geq 5} \frac{t^n}{n!}|\mathcal{L}^{ncm}_n(\Upsilon_3)| &= \log\left(\frac{1}{1-\sum_{n\geq 3}\frac{n-2}{n!}t^n}\right) - \frac{t^3}{3!} - \frac{2t^4}{4!} \\ &= \log\left(\frac{1}{-1-t+2e^t-te^t}\right) - \frac{t^3}{3!} - \frac{2t^4}{4!}\end{aligned}$$

Furthermore, one can easily show that for $n = (1, 2, 3, 4)$, $|\mathcal{L}^{ncm}_n(\Upsilon_3)| = (1, 1, 1, 2)$ so we can get the following generating function for $|\mathcal{L}^{ncm}_n(\Upsilon_3)|$

$$\begin{aligned}\sum_{n\geq 1} \frac{t^n}{n!}|\mathcal{L}^{ncm}_n(\Upsilon_3)| &= \log\left(\frac{1}{1-\sum_{n\geq 3}\frac{n-2}{n!}t^n}\right) - \frac{t^3}{3!} - \frac{2t^4}{4!} + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{2t^4}{4!} \\ &= \log\left(\frac{1}{-1-t+2e^t-te^t}\right) + t + \frac{t^2}{2!}\end{aligned}$$

Then by the theory of exponential structures (10) we can compute the generating function of $NCM_{\Upsilon_3}(t)$

$$\begin{aligned}NCM_{\Upsilon_3}(t) &= \exp\left(\sum_{n\geq 1}\frac{t^n}{n!}|\mathcal{L}^{ncm}_n(\Upsilon_3)|\right) \\ &= \exp\left(\log\left(\frac{1}{-1-t+2e^t-te^t}\right) + t + \frac{t^2}{2!}\right) \\ &= \frac{e^t e^{t^2/2}}{-1-t+2e^t-te^t}\end{aligned}$$

□

4.3.4 No bricks of size 2 or 3

In this case we consider the function

$$f_4(n) = \begin{cases} 0 & \text{if } n = 1, 2, 3 \\ n-1 & \text{if } n \geq 4 \end{cases}. \tag{4.19}$$

Theorem 42. *Let $\mathcal{F}^4_{C,n}$ be the subset of $\mathcal{F}_{C,n}$ restricted by only including objects that do not have any bricks of size 2 or 3.*

For $n \geq 6$, The map Φ_n maps $\mathcal{F}^4_{C,n}$ to the set of cycles that have no consecutive cycle occurrences of the patterns in the set $\Upsilon_4 = \{1234\} \cup \Gamma_{\uparrow\downarrow\uparrow} \cup \Gamma_{\uparrow\downarrow\downarrow\uparrow\downarrow}$ where

$\Gamma_{\uparrow\downarrow\uparrow}$ *is the set of all patterns* $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4$ *such that*

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4.$$

And $\Gamma_{\uparrow\downarrow\downarrow\uparrow\downarrow}$ *is the set of all patterns* $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6$ *such that*

$$\sigma_1 < \sigma_2 > \sigma_3 > \sigma_4 < \sigma_5 > \sigma_6.$$

Before we start the proof of this Theorem, consider the following lemma.

Lemma 43. *For* $n \geq 6$*,* $\mathrm{comp}(\mathcal{L}_n^{ncm}(\Gamma \cup \{1234\}))$ *is a subset of* $\mathrm{comp}(\mathcal{L}_n^{ncm}(\Upsilon_4))$

Proof. Suppose $\sigma \in \mathrm{comp}(\mathcal{L}_n^{ncm}(\Gamma \cup \{1234\}))$ has a cycle-1234-match, then certainly $\sigma \in \mathrm{comp}(\mathcal{L}_n^{ncm}(\Upsilon_4))$ because $1234 \in \Upsilon_4$.Now suppose σ has a cycle-τ-match for $\tau \in \Gamma$, $(\tau_1 < \tau_2 > \tau_3 < \tau_4 > \tau_5)$. Then the first three elements of this match would be a cycle-τ-match in σ for $\tau \in \Gamma_{\uparrow\downarrow\uparrow}$. ☐

Proof. of Theorem 42

Just like in the proof of Theorem 36, we will show that

$$\Phi_n : \mathrm{comp}(\mathcal{F}_{C,n}^4) \rightarrow \mathrm{comp}(\mathcal{L}_n^{ncm}(\Upsilon_4)) \quad (4.20)$$

$$\Phi_n^{-1} : \mathrm{comp}(\mathcal{L}_n^{ncm}(\Upsilon_4)) \rightarrow \mathrm{comp}(\mathcal{F}_{C,n}^4) \quad (4.21)$$

are both injections.

$\Phi_n : \mathrm{comp}(\mathcal{F}_{C,n}^4) \rightarrow \mathrm{comp}(\mathcal{L}_n^{ncm}(\Upsilon_4))$:

Case 1: $(T, \sigma) \in \mathrm{comp}(\mathcal{F}_{C,n}^4)$ has a brick of size 2.

From Theorem 36, we know that this will map to the set $\mathrm{comp}(\mathcal{L}_n^{ncm}(\Gamma \cup \{1234\}))$ which is a subset of $\mathrm{comp}(\mathcal{L}_n^{ncm}(\Upsilon_4))$ by Lemma 43.

Case 2: $(T, \sigma) \in \mathrm{comp}(\mathcal{F}_{C,n}^4)$ does not have a brick of size 2.

This means that there exists a brick of size 3. Suppose this brick consists of the elements a_1, a_2, a_3 written in increasing order. If a_1 is circled then the map Φ_n arranges the elements in the order a_1, a_3, a_2 and if a_2 is circled then the map arranges the elements in the order a_2, a_3, a_1, in either case there is an increase then a decrease. Without loss of generality, let's just consider the arrangement a_1, a_3, a_2 and the argument for a_2, a_3, a_1 will be similar. Suppose that the circled element

in the next brick is a_4, the greatest element of the next brick is a_5 and the next greatest element is a_6 (We are assuming that there are no bricks of size 2 so this brick must have at least 3 elements.) Then the map Φ_n arranges these elements in the order $a_1, a_3, a_2, a_4, a_5, a_6$. First assume that $a_2 > a_4$ so this arrangement follows the pattern $\uparrow\downarrow\downarrow\uparrow\downarrow$ so there is a cycle-τ-match for $\tau \in \Gamma_{\uparrow\downarrow\downarrow\uparrow\downarrow}$. Now suppose that $a_2 < a_4$ then this arrangement follows the pattern $\uparrow\downarrow\uparrow\uparrow\downarrow$ which means that the first three elements follow the pattern $\uparrow\downarrow\uparrow$ and there is a cycle-τ-match for $\tau \in \Gamma_{\uparrow\downarrow\uparrow}$.
$\Phi_n^{-1} : \text{comp}(\mathcal{L}_n^{ncm}(\Upsilon_4)) \rightarrow \text{comp}(\mathcal{F}_{C,n}^4)$:

Case 1: Suppose $\sigma \in \text{comp}(\mathcal{L}_n^{ncm}(\Upsilon_4))$ has a cycle-Γ-match (Recall that if $\tau \in \Gamma$ then $\tau_1 < \tau_2 > \tau_3 < \tau_4 > \tau_5$.) or a cycle-1234-match.
Then by Theorem 36, $\Phi_n^{-1}(\sigma)$ is in $\text{comp}(\mathcal{F}_{C,n}^1)$ which is a subset of $\text{comp}(\mathcal{F}_{C,n}^4)$
Case 2: Suppose $\sigma \in \text{comp}(\mathcal{L}_n^{ncm}(\Upsilon_4))$ does not have a cycle-τ-match for $\tau \in \Gamma$ or $\tau = 1234$. Then suppose that σ has a cycle-τ-match for $\tau_1 < \tau_2 > \tau_3 < \tau_4$. (Note that this pattern is of type $\uparrow\downarrow\uparrow$.) Because of the restrictions that have been made, there must be a match of type $\uparrow\downarrow\uparrow\uparrow\downarrow$. Suppose $a_1 < a_2 > a_3 < a_4 < a_5 > a_6$ is such a match in σ. Then by the map, the pre-image will have a_4 as a circled element of one brick and a_1 as the circled element of the brick before. The pre-image will include a brick of size three consisting of the elements a_1, a_2, a_3.
Case 3: Suppose $\sigma \in \text{comp}(\mathcal{L}_n^{ncm}(\Upsilon_4))$ has a cycle-τ-match for $\tau \in \Gamma_{\uparrow\downarrow\downarrow\uparrow\downarrow}$
Suppose $a_1 < a_2 > a_3 > a_4 < a_5 > a_6$ is such a match in σ. Then by the map, the pre-image will have a_4 as a circled element of one brick and a_1 as the circled element of the brick before. The pre-image will include a brick of size three consisting of the elements a_1, a_2, a_3. and therefore $\Phi^{-1}(\sigma) \in \text{comp}(\mathcal{F}_{C,n}^4)$.

Therefore we have shown that $\Phi_n : \text{comp}(\mathcal{F}_{C,n}^4) \rightarrow \text{comp}(\mathcal{L}_n^{ncm}(\Upsilon_4))$ is a bijection, therefore $\Phi_n : \mathcal{F}_{C,n}^4 \rightarrow \mathcal{L}_n^{ncm}(\Upsilon_4)$ is also a bijection. □

Corollary 44. *Let* $\Upsilon_4 = \{1234\} \cup \Gamma_{\uparrow\downarrow\uparrow} \cup \Gamma_{\uparrow\downarrow\downarrow\uparrow\downarrow}$ *then*

$$NCM_{\Upsilon_4}(t) = \frac{6e^t e^{t^2/2} e^{t^3/3}}{6(1-t)e^t + 2t^3 + 3t^2}.$$

Proof. Since $|\mathcal{F}^4_{C,n}| = |\mathcal{L}^{ncm}_n(\Upsilon_4)|$ for $n \geq 6$, and $|\mathcal{F}^4_{C,n}|$ is counted by the generating function $\sum_{n\geq 1} \frac{t^n}{n!} g_{f_4}(n)$ where

$$f_4(n) = \begin{cases} 0 & \text{if } n = 1,2,3 \\ n-1 & \text{if } n \geq 4 \end{cases}$$

then we have that

$$\begin{aligned}
\sum_{n\geq 6} \frac{t^n}{n!} |\mathcal{L}^{ncm}_n(\Upsilon_4)| &= \sum_{n\geq 6} \frac{t^n}{n!} g_{f_4}(n) \\
&= \log\left(\frac{1}{1-\sum_{n\geq 1}\frac{f_4(n)}{n!}t^n}\right) - \sum_{n=1}^{5} \frac{t^n}{n!} g_{f_4}(n) \\
&= \log\left(\frac{1}{1-\sum_{n\geq 4}\frac{n-1}{n!}t^n}\right) - \sum_{n=1}^{5} \frac{t^n}{n!} |\mathcal{F}^4_{C,n}|
\end{aligned}$$

One can easily show that for $n = (1,2,3,4,5)$, $|\mathcal{F}^4_{C,n}| = (0,0,0,3,4)$ so this generating function becomes:

$$\begin{aligned}
\sum_{n\geq 6} \frac{t^n}{n!} |\mathcal{L}^{ncm}_n(\Upsilon_4)| &= \log\left(\frac{1}{1-\sum_{n\geq 4}\frac{n-1}{n!}t^n}\right) - 3\frac{t^4}{4!} - 4\frac{t^5}{5!} \\
&= \log\left(\frac{6}{(1-t)e^t + 2t^3 + 3t^2}\right) - \frac{3t^4}{4!} - \frac{4t^5}{5!}
\end{aligned}$$

Furthermore, one can easily show that for $n = (1,2,3,4,5)$, $|\mathcal{L}^{ncm}_n(\Upsilon_4)| = (1,1,2,3,4)$ so we can get the following generating function for $|\mathcal{L}^{ncm}_n(\Upsilon_1)|$

$$\begin{aligned}
\sum_{n\geq 1} \frac{t^n}{n!} |\mathcal{L}^{ncm}_n(\Upsilon_4)| &= \log\left(\frac{6}{(1-t)e^t + 2t^3 + 3t^2}\right) - \\
&\quad \frac{3t^4}{4!} - \frac{4t^5}{5!} + t + \frac{t^2}{2!} + \frac{2t^3}{3!} + \frac{3t^4}{4!} + \frac{4t^5}{5!} \\
&= \log\left(\frac{6}{(1-t)e^t + 2t^3 + 3t^2}\right) + t + \frac{t^2}{2!} + \frac{2t^3}{3!}
\end{aligned}$$

Then by the theory of exponential structures (10) we can compute the generating function of $NCM_{\Upsilon_4}(t)$

$$\begin{aligned} NCM_{\Upsilon_4}(t) &= \exp\left(\sum_{n\geq 1} \frac{t^n}{n!}|\mathcal{L}_n^{ncm}(\Upsilon_4)|\right) \\ &= \exp\left(\log\left(\frac{6}{(1-t)e^t + 2t^3 + 3t^2}\right) + t + \frac{t^2}{2!} + \frac{2t^3}{3!}\right) \\ &= \frac{6e^t e^{t^2/2} e^{t^3/3}}{6(1-t)e^t + 2t^3 + 3t^2} \end{aligned}$$

□

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