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PARAMETER ESTIMATION FOR HYBRID DYNAMICAL SYSTEMS

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Notation

$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}$	The set of all real numbers
$\mathbb{R}_{\geq 0}$	The set of all nonnegative real numbers
$\mathbb{R}_{>0}$	The set of all positive real numbers
$\mathbb{Z}$	The set of all integers
$\mathbb{N}$	The set of all positive integers including zero, i.e., $\{0, 1, 2, \cdots\}$
$\mathbb{B}$	The closed unit ball, of appropriate dimension, in the Euclidean norm
I	The identity matrix, of appropriate dimension
$\lfloor x \rfloor$	The floor of $x \in \mathbb{R}$ is $\lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leq x\}$
$\lceil x \rceil$	The ceiling of $x \in \mathbb{R}$ is $\lceil x \rceil = \min\{m \in \mathbb{Z} : m \geq x\}$
(v, w)	Given vectors $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$, we write $[v^\top, w^\top]^\top$ as (v, w)
$\langle v, w \rangle$	The inner product of vectors $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$, namely, $v^\top w$
$\lvert v \rvert$	The Euclidean vector norm $\lvert v \rvert = \sqrt{v^\top v}$ and the associated induced matrix norm
$\lvert v \rvert_\infty$	The infinity vector norm $\lvert v \rvert_\infty = \max_i \lvert v_i \rvert$ for $v = [v_1, v_2, \cdots, v_n]^\top$
$\mathrm{eig}(A)$	The set of all eigenvalues of a square matrix A
$\lambda_{\min}(A)$	The minimum eigenvalue of A is $\lambda_{\min}(A) = \min\{\lambda/2 : \lambda \in \mathrm{eig}(A + A^\top)\}$
$\lambda_{\max}(A)$	The maximum eigenvalue of A is $\lambda_{\max}(A) = \max\{\lambda/2 : \lambda \in \mathrm{eig}(A + A^\top)\}$
$\mathrm{tr}(A)$	The trace of a matrix $A \in \mathbb{R}^{n \times n}$ is $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$

$\mathrm{cond}(A)$	The condition number of an invertible matrix A is $\mathrm{cond}(A) = \lvert A^{-1}\rvert \lvert A\rvert$
$\lvert A\rvert_{\mathrm{F}}$	The Frobenius matrix norm $\lvert A\rvert_{\mathrm{F}} = \sqrt{\mathrm{tr}(A^{\top}A)}$
$\lvert x\rvert_S$	The distance of a point $x \in \mathbb{R}^n$ to a set $S \subset \mathbb{R}^n$, i.e., $\lvert x\rvert_S = \inf_{y\in S} \lvert y - x\rvert$
$\mathrm{cl}\, S$	The closure of a set S
$\mathrm{int}\, S$	The interior of a set S
$S \setminus U$	The set difference between S and U is $S \setminus U = \{x \in S : x \notin U\}$
$\sup S$	The supremum of a set $S \subset \mathbb{R}$ is the smallest $y \in \mathbb{R}$ such that $x \leq y$ for all $x \in S$
$\operatorname{ess\,sup} f$	The essential supremum of a measurable function $f : M \to \mathbb{R}$, where M is a measure space, is the smallest $c \in \mathbb{R}$ such that the set $\{x \in M : f(x) > c\}$ has measure zero
$\lVert f\rVert_\infty$	The infinity norm of a function $f : M \to \mathbb{R}^n$ is $\operatorname{ess\,sup}\{\lvert f(x)\rvert : x \in M\}$
$\mathcal{L}_\infty$	A function $f : M \to \mathbb{R}$ is a class-$\mathcal{L}_\infty$ function, also written $f \in \mathcal{L}_\infty$, if $\operatorname{ess\,sup} f$ is finite
$\mathcal{K}_\infty$	A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}_\infty$ function, also written $f \in \mathcal{K}_\infty$, if f is zero at zero, continuous, and strictly increasing, and unbounded
$\mathcal{KL}$	A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function, also written $\beta \in \mathcal{KL}$, if β is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r\to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$
$\mathrm{dom}\, H$	The domain of $H : \mathbb{R}^m \to \mathbb{R}^n$ is $\mathrm{dom}\, H = \{x \in \mathbb{R}^m : H(x) \neq \varnothing\}$
$\mathrm{rge}\, H$	The range of $H : \mathbb{R}^m \to \mathbb{R}^n$ is $\mathrm{rge}\, H = \{y \in \mathbb{R}^n : \exists x \in \mathbb{R}^m \text{ s.t. } y = H(x)\}$
$\mathrm{gph}\, H$	The graph of $H : \mathbb{R}^m \to \mathbb{R}^n$ is $\mathrm{gph}\, H = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : y = H(x)\}$

List of Figures

To my mother, Lynne, and my father, Michael.

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Abstract

Parameter Estimation for Hybrid Dynamical Systems

by

Ryan S. Johnson

Estimating the unknown parameters of a system is critical in many engineering applications, such as the control of power electronics, motion planning for autonomous cars, flight controllers for aircraft, and rendezvous and docking controllers for spacecraft. While modern continuous-time and discrete-time estimation algorithms have found widespread use throughout engineering, the recent rise of hybrid modeling paradigms highlights their limitations. Hybrid systems are characterized by state variables that may evolve continuously (flow) and, at times, evolve discretely (jump). When applied to hybrid systems, the parameter estimation error for purely continuous-time or purely discrete-time estimation algorithms may fail to converge to zero. Motivated by these shortcomings, this dissertation focuses on developing novel parameter estimation algorithms for hybrid dynamical systems.

The first algorithm, developed using hybrid systems tools, is inspired by the continuous-time and discrete-time gradient descent algorithms. Our proposed hybrid parameter estimation algorithm operates during both the flows and jumps of a hybrid system, and guarantees convergence of the parameter estimate to the true value under a notion of hybrid persistence of excitation that relaxes the classical continuous-time and discrete-time persistence of excitation conditions. Key properties of the algorithm are established, including exponential stability, convergence rate, and robustness to measurement noise.

The second algorithm is inspired by the recently proposed integral concurrent learning algorithm. Our proposed hybrid algorithm selectively stores measurements

of the inputs and outputs of a hybrid system during flows and jumps. The algorithm guarantees convergence of the parameter estimate to the true value if the stored data satisfies a (hybrid) richness condition. Key properties of the algorithm are established, including exponential stability, convergence rate, and robustness to measurement noise.

The third algorithm uses hybrid systems tools to estimate in finite-time the unknown parameters of a class of continuous-time systems. Our proposed hybrid algorithm ensures convergence of the parameter estimate to the true value when the system inputs are exciting over only a finite interval of time. As a result, the algorithm can also be employed to estimate unknown parameters of hybrid systems if the inputs are sufficiently exciting over a single interval of flow. Key properties of the algorithm are established, including time to convergence and robustness to measurement noise.

The fourth algorithm estimates unknown parameters for hybrid systems whose jump times are known only approximately. By solving an optimization problem to estimate the jump times of the system, our proposed algorithm ensures convergence of the parameter estimate to the true value, except possibly during the intervals wherein the detection of jumps is delayed. Key properties of the algorithm are established, including stability and robustness to perturbations.

The contributions of this dissertation are not limited to the theory of parameter estimation for hybrid systems, as they have implications in adaptive control algorithms for practically relevant engineering control systems. For such systems, we develop methods for the design of algorithms that learn and adapt using real-time data to cope with unknown parameters and features in the environment, to enable autonomous systems to perform near optimal conditions, with robustness. Numerical results validate the findings.

Chapter 1

Introduction and Motivation

1.1 Overview of the Work

The estimation of unknown parameters in dynamical systems has been an active research area for years [54]. Parameter estimation algorithms typically rely on exploiting information about the structure of the system along with the available input and output signals to compute online an estimate of the unknown parameters. One of the most popular estimation problems is recursive linear regression, for which the estimation scheme is often based on the gradient descent algorithm [37, 54]. For dynamical systems, control strategies leveraging estimation algorithms, such as model-reference adaptive control, are used in many engineering applications [19, 20].

More recently, there has been a growing interest in hybrid dynamical systems. Much work has been done on parameter estimation and system identification for specific sub-classes of hybrid systems, such as switched systems [21, 26, 42, 43] and piecewise-affine systems [4, 8]. However, these systems exhibit nonsmooth but continuous evolution of the state variables, rather than jumps in the state variables. Hence, such results are not applicable to a general class of hybrid systems.

In this dissertation, we develop algorithms for estimating unknown parameters in a general class of hybrid dynamical systems. These algorithms leverage information provided during both the flows and jumps of a hybrid system to adapt the parameter estimate. Using hybrid systems tools [22, 45], we establish the stability and robustness properties induced by our proposed algorithms.

1.2 Properties of Time-Varying Hybrid Systems

The parameter estimation algorithms that we develop in this dissertation result in the parameter estimate having (hybrid) time-varying dynamics. Thus, in order to analyze the properties induced by these algorithms, we first establish properties induced by a general class of time-varying hybrid systems.

1.2.1 Related Work

Much work has been done on analyzing the properties induced by time-varying continuous-time and discrete-time systems [3, 31, 57, 63]. Recently, for hybrid systems, the paper [48] studied a general class of hybrid systems in order to establish the stability properties induced by an algorithm for hybrid linear regression.

1.2.2 Contributions

The main contributions of the forthcoming Chapter 3 are as follows.

1. *Exponential stability under hybrid persistence of excitation*: In Section 3.2, inspired by [48], we establish that a general class of time varying hybrid systems induces global pre-exponential stability of a closed set under a notion of hybrid persistence of excitation (PE).

2. *Input-to-state stability under hybrid persistence of excitation*: In Section 3.3, we study the robustness properties of a general class of hybrid systems that includes hybrid disturbances. We construct an input-to-state stability (ISS) Lyapunov function by extending ISS results for continuous-time and discrete-time systems [9, 12, 31, 53, 52, 25].

1.3 Parameter Estimation for Hybrid Systems via Hybrid Gradient Descent

1.3.1 Motivation

While continuous-time and discrete-time estimation algorithms have seen widespread use throughout engineering, the recent rise of hybrid modeling paradigms highlights their limitations. Indeed, when employed to estimate the parameters of a hybrid system, the parameter estimation error for such algorithms may fail to converge to zero. Denoting the vector of parameters to be estimated as θ, and the estimate of θ as $\hat{\theta}$, Figure 1.1 shows the estimation error $\tilde{\theta} := \theta - \hat{\theta}$ that results from employing a continuous-time and a discrete-time estimation algorithm to estimate unknown parameters in a hybrid system (see the example in Section 4.5.1 for details).[1] The estimation error for both the continuous-time and discrete-time algorithm fails to converge to zero, as shown in blue and green, respectively. This finding motivates developing the hybrid parameter estimation algorithm that we propose in Chapter 4, which successfully estimates θ by leveraging the information available during both flows and jumps, as shown in orange in Figure 1.1.

[1] Code at `https://github.com/HybridSystemsLab/HybridGD_Motivation`

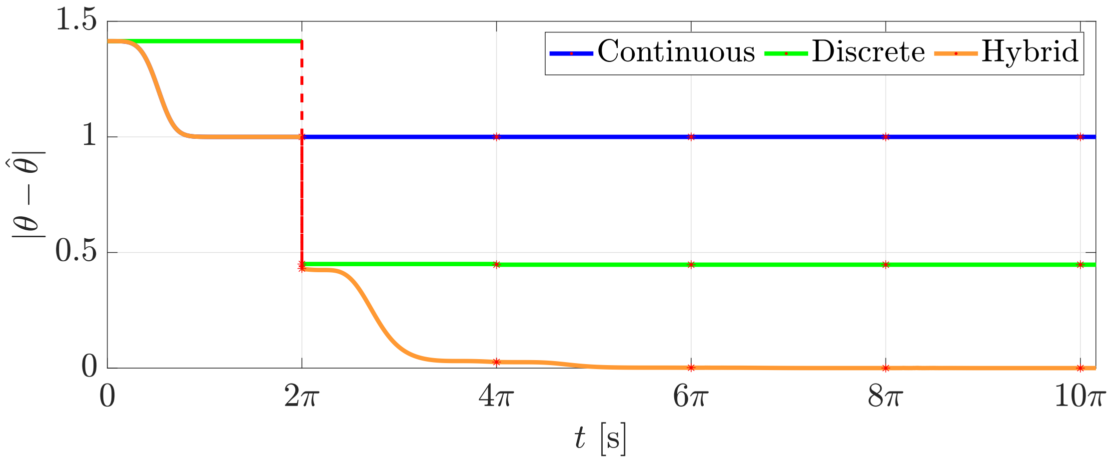

Figure 1.1: The projection onto t of the norm of the parameter estimation error for continuous-time and discrete-time estimation algorithms, and our proposed hybrid algorithm. The continuous-time and discrete-time algorithms produce nonzero steady-state error, whereas the error for our algorithm converges to zero.

1.3.2 Related Work

Many extensions to the classical gradient descent (GD) algorithm [37] have been proposed. The method of stochastic gradient descent [44] was proposed to reduce the computational burden associated with evaluating the gradient for large datasets. Accelerated gradient algorithms such as the heavy ball method [41] and Nesterov's accelerated gradient algorithm [38] have been proposed to improve the convergence speed compared to the classical gradient algorithm.

Parameter estimation for hybrid systems has been studied in several recent works. The paper [48] proposed a hybrid algorithm for estimating unknown parameters in linear regression models whose input and output signals are hybrid. The authors established global exponential convergence of the parameter estimate under a notion of hybrid persistence of excitation. The paper [60] develops a discretized hybrid algorithm for hybrid linear regression with sampled hybrid signals. A notion of hybrid PE is exploited in the recent work [47] to establish uniform exponential stability for a general class of time-varying hybrid dynamical systems and estimation problems. Our recent paper [27] proposed a hybrid algorithm for estimating unknown parameters in a class of hybrid systems with linear dynamics.

1.3.3 Contributions

The main contributions of the forthcoming Chapter 4 are as follows.

1. *A hybrid algorithm for estimating unknown parameters*: In Section 4.2, we propose a hybrid algorithm capable of leveraging information during flows and jumps of a hybrid system to adapt the parameter estimate. In particular, we augment the state vector of the estimation algorithm with components that allow us to express the unknown parameter vector in terms of a hybrid linear regression model plus an exponentially convergent term. We then adapt the parameter estimate during flows and jumps using dynamics inspired by the continuous-time and discrete-time gradient descent algorithms.

2. *Estimation under hybrid persistence of excitation*: In Section 4.3, we show that the error dynamics of our proposed estimation algorithm belong to the class of hybrid systems studied in Section 3.2. We establish that our proposed algorithm guarantees exponential convergence of the parameter estimate to the true value under a notion of hybrid persistence of excitation.

3. *Estimator robustness to measurement noise*: In Section 4.4, we establish that our proposed algorithm is ISS with respect to bounded hybrid noise on the measurements of the plant state.

1.4 Parameter Estimation for Hybrid Systems using Data

1.4.1 Motivation

While the hybrid PE condition that we impose in Chapter 4 relaxes the classical continuous-time and discrete-time PE conditions, it is possible to further relax this hybrid PE condition by using stored data. For example, Figure 1.2 shows the parameter estimation error that results from employing the hybrid algorithm proposed in Chapter 4, denoted by hybrid GD, to a system whose input and output signals do not satisfy hybrid PE (see the example in Section 5.7.1 for details).[2] The estimation error for the hybrid GD algorithms fails to converge to zero, as shown in blue in Figure 1.2. This finding motivates developing the hybrid ICL algorithm that we propose in Chapter 5, which successfully estimates the unknown parameters by leveraging stored data alongside current measurements to adapt the parameter estimate, as shown in green in Figure 1.2.

1.4.2 Related Work

The concurrent learning algorithm was first proposed in [14] to estimate unknown parameters in linear regression models. The algorithm uses stored data alongside current measurements to ensure convergence of the parameter estimate to the true value under relaxed excitation conditions. The concurrent learning approach was extended to estimate unknown parameters of time-invariant linear dynamical systems in [36], provided that the derivative of the state vector is known. If the state derivative is not available, smoothing techniques such as those in [15] can be used to generate estimates of the state derivatives. In such

[2] Code at `https://github.com/HybridSystemsLab/HybridICL_Motivation`

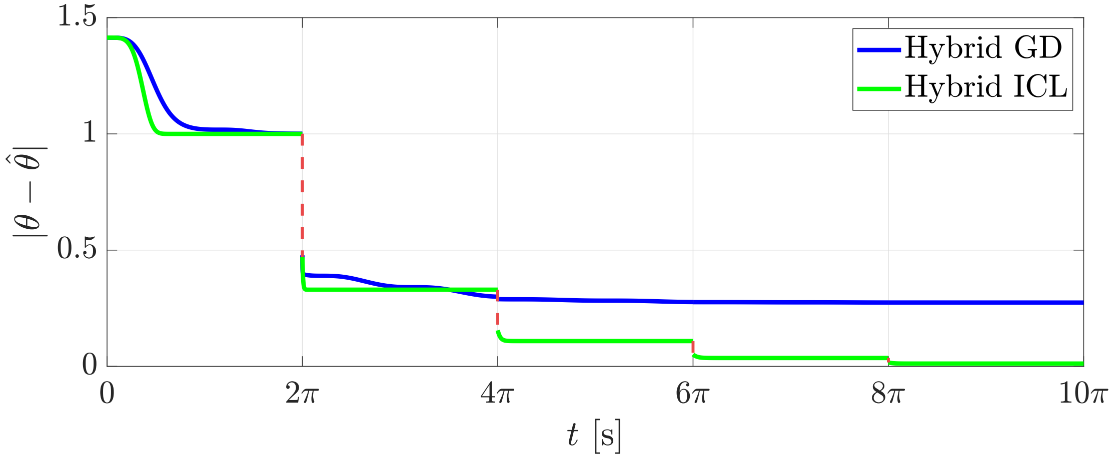

Figure 1.2: The projection onto t of the estimation error for the hybrid GD algorithm proposed in Chapter 4 and the hybrid ICL algorithm proposed in Chapter 5. The hybrid GD algorithm produces nonzero steady state error, whereas the error for our hybrid ICL algorithm converges to zero.

cases, the purging algorithm proposed in [30] can be applied to remove erroneous data recorded prior to convergence of the state derivate estimate. The integral concurrent learning (ICL) algorithm, which is the inspiration for our proposed hybrid ICL algorithm, is proposed in [40] in order to estimate unknown parameters for a class of continuous-time systems without requiring knowledge of the derivative of the state vector. For hybrid systems, our recent work [28] proposed a hybrid concurrent learning algorithm for hybrid linear regression.

1.4.3 Contributions

The main contributions of the forthcoming Chapter 5 are as follows.

1. *A hybrid algorithm for estimating unknown parameters using data*: In Section 5.2, we propose an algorithm for estimating unknown parameters in hybrid dynamical systems. The algorithm integrates the input and output signals of the plant during flows, and sums the signals at jumps, in order to express the unknown parameter vector in terms of hybrid linear regres-

sion models. We sample and store the resulting signals during flows and jumps, and use the stored data alongside current measurements to adapt the parameter estimate using dynamics inspired by the continuous-time ICL algorithm.

2. *Estimation without hybrid persistence of excitation*: In Section 5.3, we show that the dynamics of our proposed hybrid ICL algorithm belong to the class of hybrid systems studied in Section 3.2. We establish that our proposed algorithm guarantees exponential convergence of the parameter estimate to the true value when the stored data satisfied a (hybrid) richness condition.

3. *Estimator robustness to measurement noise*: In Section 5.4, we establish that our hybrid ICL algorithm is ISS with respect to bounded hybrid noise on the measurements of the plant state.

4. *A hybrid algorithm for finite-time estimation of unknown parameters using data*: In Section 5.5, we show that, by augmenting the dynamics of our proposed hybrid ICL algorithm, the stored data can be used to achieve finite-time convergence of the parameter estimate to the true value if the stored data satisfies a richness condition.

5. *Algorithms for selecting data for storage during flows and jumps*: In Section 5.6, we propose algorithms for selecting data for storage during flows and jumps of a hybrid system. Data is stored with the objective of satisfying the richness conditions established in Sections 5.4 and 5.5.

1.5 Finite-Time Parameter Estimation

1.5.1 Motivation

Classical estimation algorithms such as gradient descent typically guarantee convergence of the estimation error to zero only under a stringent condition of persistence of excitation [31, 37]. The PE condition is often difficult to verify online since it requires the regressor to be sufficiently exciting for all future time. For certain problems, PE can be assured by adding perturbations to an external reference input [10, 11]. However, enforcing PE through an exogenous reference is not always feasible, and such approaches may degrade control performance. These findings motivate developing the hybrid estimation algorithm that we propose in Chapter 6, which ensures finite-time convergence of the parameter estimate to the true value when the regressor is exciting over only a finite time interval. Moreover, such an estimation algorithm can also be employed to estimate unknown parameters of hybrid systems when the regressor is sufficiently exciting over a single interval of flow.

1.5.2 Related Work

The parameter estimation algorithm we propose in Chapter 6 is related to the ones in [23, 29, 49] – all results provide a finite-time estimator using a hybrid systems framework. The work [49] deals with a linear regression model while we deal with a dynamical model. The approaches in [23] and our recent work [29] rely on a persistence of excitation condition to ensure convergence of the parameter estimate to the true value. In contrast, the algorithm that we propose in Chapter 6 guarantees finite-time convergence of the parameter estimate to the true value when the regressor is exciting over only a finite time interval.

1.5.3 Contributions

The main contributions of the forthcoming Chapter 6 are as follows.

1. *A hybrid algorithm for finite-time parameter estimation*: In Section 6.2, we propose a hybrid algorithm for estimating unknown parameters in finite time. In particular, we augment the state vector of the estimation algorithm with components that allow us to express the unknown parameter vector in terms of a linear regression model. We design the algorithm to jump only when the regressor is sufficiently exciting, and we reset the parameter estimate at jumps so that, after each jump, the estimate of the unknown parameter vector is equal to the true parameter vector.

2. *Estimation under finite excitation*: In Section 6.3, we show that our proposed algorithm guarantees convergence of the parameter estimate to the true value when the regressor is exciting over only a finite interval of time.

3. *Estimator robustness to measurement noise*: In Section 6.4, we show that our proposed algorithm is ISS with respect to noise on the measurements of the plant state when the regressor is persistently exciting. Next, we show that, if the regressor is exciting over only a finite time interval, then the parameter estimation error for our algorithm is bounded by a function of the integral of the measurement noise.

1.6 Parameter Estimation for Hybrid Systems with Approximately Known Jump Times

1.6.1 Motivation

The parameter estimation algorithms that we propose in Chapters 4 and 5 assume that jumps in the plant state are detected instantaneously, which allows jumps of the estimators to be aligned with jumps in the plant state. In such cases, under sufficient excitation conditions, these estimation algorithms ensure convergence of the parameter estimation error to zero as shown in blue in Figure 1.3 (see the example in Section 7.5 for details).[3] However, in practice, the detection of jumps in the plant state is often delayed due to sensing, signal transmission, and computation delays. Moreover, if estimation algorithms such as those in Chapters 4 and 5 are employed in the presence of delays in the detection of jumps in the plant state, the estimation error may fail to converge to zero, as shown in green in Figure 1.3. This finding motivates developing a method of estimating unknown parameters in hybrid systems whose jump times are known only approximately.

1.6.2 Related Work

The recent work [35] proposes an algorithm for system identification of a class of hybrid systems with linear dynamics and unknown jump times. The authors impose a global Lipschitz continuity condition during flows. We avoid imposing such a condition in Chapter 7 by instead assuming that the jump times of the plant are approximately known. Hybrid systems with approximately known jump times were also studied in the recent works [7, 6] in the context of designing state

[3]Code at `https://github.com/HybridSystemsLab/ApproximatelyKnownJumpTimes_BouncingBall`

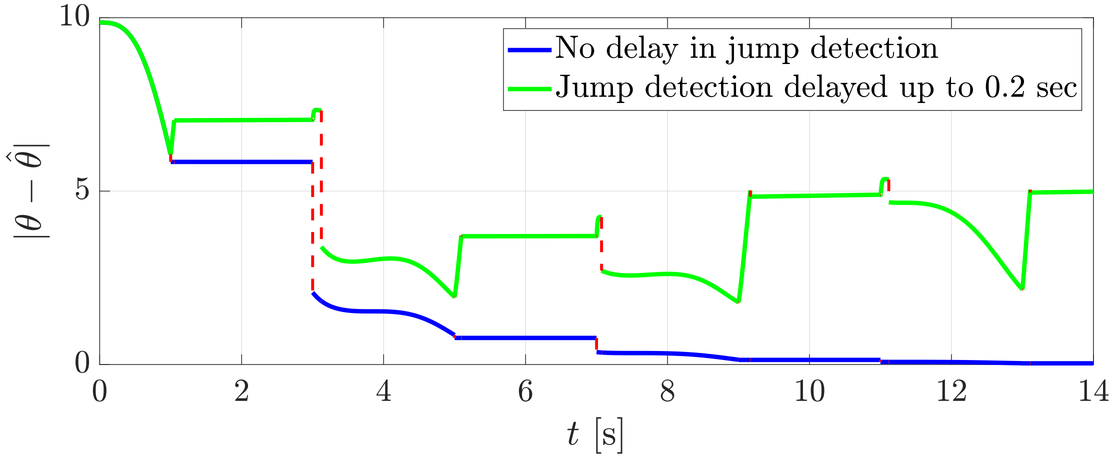

Figure 1.3: The projection onto t of the estimation error for the hybrid GD algorithm proposed in Chapter 4 with no delay in jump detection (blue) and a delay of up to 0.2 seconds (green). When detection of jumps in the plant state is delayed, the parameter estimate for the hybrid GD algorithm fails to converge.

observers.

1.6.3 Contributions

The main contributions of the forthcoming Chapter 7 are as follows.

1. *A hybrid algorithm for estimating unknown parameters*: In Section 7.2, we propose a method of modifying a hybrid estimation algorithm that is designed to jump coincident with jumps in the plant state – such as the algorithms proposed in Chapters 4 and 5 – to allow such algorithms to operate under delays in the detection of jumps in the plant state. Our proposed algorithm operates by storing samples of the plant state, and solving an optimization problem to estimate the jump times of the plant state.

2. *Estimation stability analysis*: In Section 7.3, we prove that the estimation algorithm that results from our proposed modification preserves the stability properties induced by the unmodified version, except possibly during the

delays in detection of jumps in the plant state.

3. *Estimator robustness analysis*: In Section 7.4, we establish that our proposed estimation algorithm is robust to vanishing state perturbations.

Chapter 2

Preliminaries

In this chapter, we present the hybrid systems framework and its basic properties. We then present the classical continuous-time and discrete-time gradient descent algorithms, and the continuous-time integral concurrent learning algorithm. Finally, we define excitation conditions from the literature.

2.1 Hybrid Systems

In this dissertation, we use the hybrid systems framework in [22, 45] to design our proposed algorithms since such a framework allows for the combination of continuous-time and discrete-time behavior. A hybrid system $\mathcal{H}$ has data (C, F, D, G) and is defined as [22, Definition 2.2]

$$\mathcal{H} = \begin{cases} \dot{x} = F(x) & x \in C \\ x^+ = G(x) & x \in D, \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $F : C \to \mathbb{R}^n$ is the flow map defining the continuous dynamics, and $C \subset \mathbb{R}^n$ defines the flow set on which flow is permitted. The mapping $G : D \to \mathbb{R}^n$ is the jump map defining the law resetting x at jumps, and $D \subset \mathbb{R}^n$ is the jump set on which jumps are permitted.

A solution x to $\mathcal{H}$ is a *hybrid arc* [45] that is parameterized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t is the elapsed ordinary time and j is the number of jumps that have occurred. The domain of x, denoted by $\operatorname{dom} x \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$, is a *hybrid time domain*, in the sense that for every $(T, J) \in \operatorname{dom} x$, there exists a nondecreasing sequence $\{t_j\}_{j=0}^{J+1}$ with $t_0 = 0$ such that

$$\operatorname{dom} x \cap ([0, T] \times \{0, 1, \ldots, J\}) = \bigcup_{j=0}^{J} ([t_j, t_{j+1}] \times \{j\}) .$$

The operations $\sup_t \operatorname{dom} x$ and $\sup_j \operatorname{dom} x$ return the supremum of the t and j coordinates, respectively, of points in $\operatorname{dom} x$. The length of $\operatorname{dom} x$ is $\operatorname{length}(\operatorname{dom} x) := \sup_t \operatorname{dom} x + \sup_j \operatorname{dom} x$. A hybrid arc x is said to be

- *nontrivial* if $\operatorname{dom} x$ contains more than one point;
- *complete* if $\operatorname{dom} x$ is unbounded;
- *continuous* if nontrivial and $\operatorname{dom} x \subset \mathbb{R}_{\geq 0} \times \{0\}$;
- *eventually continuous* if $J := \sup_j \operatorname{dom} x < \infty$ and $\operatorname{dom} x \cap (\mathbb{R}_{\geq 0} \times \{J\})$ contains at least two points;
- *discrete* if nontrivial and $\operatorname{dom} x \subset \{0\} \times \mathbb{N}$;
- *eventually discrete* if $T := \sup_t \operatorname{dom} x < \infty$ and $\operatorname{dom} x \cap (\{T\} \times \mathbb{N})$ contains at least two points;
- *Zeno* if it is complete and $\sup_t \operatorname{dom} x < \infty$.

We denote the supremum norm of a hybrid arc x from $(0, 0)$ to $(t, j) \in \operatorname{dom} x$ as

$$\|x\|_{(t,j)} := \max \left\{ \operatorname*{ess\,sup}_{\substack{(s,k) \in \operatorname{dom} x \setminus \Upsilon(\operatorname{dom} x), \\ (s,k) \preceq (t,j)}} |x(s,k)|, \sup_{\substack{(s,k) \in \Upsilon(\operatorname{dom} x), \\ (s,k) \preceq (t,j)}} |x(s,k)| \right\}$$

where

$$\Upsilon(\operatorname{dom} x) := \{(t,j) \in \operatorname{dom} x : (t, j+1) \in \operatorname{dom} x\}. \tag{2.2}$$

We employ the following notion of solutions to a hybrid system [22].

Definition 2.1: A hybrid arc x is a *solution* to a hybrid system $\mathcal{H}$ as in (2.1) if $x(0,0) \in \mathrm{cl}(C) \cup D$ and

- for all $j \in \mathbb{N}$ such that $I^j := \{t : (t,j) \in \operatorname{dom} x\}$ has nonempty interior,

$$\begin{aligned} &x(t,j) \in C && \text{for all} && t \in \mathrm{int} I^j, \\ &\dot{x}(t,j) = F(x(t,j)) && \text{for almost all} && t \in I^j; \end{aligned}$$

- for all $(t,j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

$$\begin{aligned} x(t,j) &\in D, \\ x(t,j+1) &= G(x(t,j)). \end{aligned}$$

A solution x to $\mathcal{H}$ is called *maximal* if it cannot be extended – that is, if there does not exist another solution x' to $\mathcal{H}$ such that $\operatorname{dom} x$ is a proper subset of $\operatorname{dom} x'$ and $x(t,j) = x'(t,j)$ for all $(t,j) \in \operatorname{dom} x$.

The following definitions and results will be used in the analysis of the hybrid closed-loop systems that are obtained with the proposed hybrid estimation algorithms [22, 45].

Definition 2.2: A hybrid system $\mathcal{H}$ is said to satisfy the *hybrid basic conditions* if its data (C, F, D, G) is such that

1. C and D are closed subsets of $\mathbb{R}^n$;

2. $F : \mathbb{R}^n \to \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to C, $C \subset \operatorname{dom} F$, and $F(x)$ is convex for every $x \in C$;

3. $G : \mathbb{R}^n \to \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to D, and $D \subset \operatorname{dom} G$.

Theorem 2.3: [22, Theorem 6.30] If a hybrid system $\mathcal{H}$ satisfies Definition 2.2, then it is well-posed as in [22, Definition 6.29].

Definition 2.4: The *tangent cone* to a set $S \subset \mathbb{R}^n$ at at point $x \in \mathbb{R}^n$, denoted by $T_S(x)$, is the set of all vectors $w \in \mathbb{R}^n$ for which there exist $x_i \in S$, $\tau_i > 0$ with $x \to x_i$, $\tau_i \searrow 0$, and

$$w = \lim_{i\to\infty} \frac{x_i - x}{\tau_i}.$$

Proposition 2.5: [22, Proposition 6.10] Let $\mathcal{H} = (C, F, D, G)$ satisfy the hybrid basic conditions in Definition 2.2. Take an arbitrary $\zeta \in C \cup D$. If $\zeta \in D$ or

(VC) there exists a neighborhood U of ζ such that for every $x \in U \cap C$,

$$F(x) \cap T_C(x) \neq \varnothing,$$

then there exists a nontrivial solution x to $\mathcal{H}$ with $x(0,0) = \zeta$. If (VC) holds for every $\zeta \in C \setminus D$, then there exists a nontrivial solution to $\mathcal{H}$ from every initial point in $C \cup D$, and every maximal solution x to $\mathcal{H}$ satisfies exactly one of the following conditions:

(a) x is complete;

(b) $\operatorname{dom} x$ is bounded and the interval I^J, where $J = \sup_j \operatorname{dom} \phi$ has nonempty interior and $t \mapsto x(t, J)$ is a maximal solution to $\dot{x} = F(x)$, in fact $\lim_{t\to T} |x(t, J)| = \infty$, where $T = \sup_t \operatorname{dom} x$;

(c) $x(T, J) \notin C \cup D$, where $(T, J) = \sup \operatorname{dom} x$.

Furthermore, if $G(D) \subset C \cup D$, then (c) above does not occur.

Definition 2.6: Given a set $S \subset \mathbb{R}^n$, a hybrid system $\mathcal{H}$ is *pre-forward complete* from S if every maximal solution x to $\mathcal{H}$ from $x(0,0) \in S$ is either bounded or complete.

We employ the following notion of closeness for two hybrid arcs [22, 45].

Definition 2.7: Given $\tau, \varepsilon > 0$, two hybrid arcs x_1 and x_2 are (τ, ε)-*close* if

1. for all $(t,j) \in \operatorname{dom} x_1$ with $t + j \leq \tau$, there exists s such that $(s,j) \in \operatorname{dom} \phi_2$ and $|t - s| < \varepsilon$, and
$$|x_1(t,j) - x_2(s,j)| < \varepsilon;$$

2. for all $(t,j) \in \operatorname{dom} x_2$ with $t + j \leq \tau$, there exists s such that $(s,j) \in \operatorname{dom} \phi_1$ and $|t - s| < \varepsilon$, and
$$|x_2(t,j) - x_1(s,j)| < \varepsilon.$$

Inspired by [5], we define a j-reparameterization of a hybrid arc as follows.[4]

Definition 2.8: Given a hybrid arc x, a hybrid arc x^r is a j-*reparameterization* of x if there exists a function $\eta : \mathbb{N} \to \mathbb{N}$ such that

$$\eta(0) = 0, \qquad \qquad \eta(j+1) - \eta(j) \in \{0, 1\} \quad \forall j \in \mathbb{N}$$

and

$$x^r(t, \eta(j)) = x(t,j) \quad \forall (t,j) \in \operatorname{dom} x.$$

We employ the following notions of stability [22, 45].

[4]Note that j-reparameterization in [5] adds trivial jumps to a hybrid arc. In contrast, j-reparameterization is used in this dissertation to remove trivial jumps from a hybrid arc.

Definition 2.9: Given a hybrid system $\mathcal{H}$ with data as in (2.1), a nonempty closed set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *globally pre-exponentially stable*[5] for $\mathcal{H}$ if there exist $\kappa, \lambda > 0$ such that each solution x to $\mathcal{H}$ satisfies

$$|x(t,j)|_{\mathcal{A}} \leq \kappa \mathrm{e}^{-\lambda(t+j)} |x(0,0)|_{\mathcal{A}} \quad \forall (t,j) \in \operatorname{dom} x.$$

Definition 2.10: Given a hybrid system $\mathcal{H}$ with data as in (2.1), a nonempty closed set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *semiglobally pre-exponentially stable* for $\mathcal{H}$ if, for each compact set $K \subset \mathbb{R}^n$, there exist $\kappa, \lambda > 0$ such that each solution x to $\mathcal{H}$ from $x(0,0) \in K$ satisfies

$$|x(t,j)|_{\mathcal{A}} \leq \kappa \mathrm{e}^{-\lambda(t+j)} |x(0,0)|_{\mathcal{A}} \quad \forall (t,j) \in \operatorname{dom} x.$$

Definition 2.11: Given a hybrid system $\mathcal{H}$ with data as in (2.1), a nonempty closed set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *semiglobally $\mathcal{KL}$ pre-asymptotically stable* for $\mathcal{H}$ if for each compact set $K \subset \mathbb{R}^n$, there exists a function $\beta \in \mathcal{KL}$ such that each solution x to $\mathcal{H}$ from $x(0,0) \in K$ satisfies

$$|x(t,j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, t+j) \quad \forall (t,j) \in \operatorname{dom} x.$$

2.2 Gradient Descent Algorithms

2.2.1 Continuous-Time Gradient Descent

Consider a continuous-time linear regression model

$$y(t) = \theta^\top \phi(t) \qquad \forall t \geq 0 \tag{2.3}$$

[5]The term "pre-exponential," as opposed to "exponential," indicates the possibility of a maximal solution that is not complete. This allows for separating the conditions for completeness from the conditions for stability and attractivity.

where $t \mapsto y(t) \in \mathbb{R}$ is the measured output, $t \mapsto \phi(t) \in \mathbb{R}^p$ is the known regressor, $\theta \in \mathbb{R}^p$ is an unknown vector of constant parameters, and $p \in \mathbb{N}$.

The classical gradient descent algorithm [37] estimates the vector θ via an estimator of the form

$$\hat{y}(t) = \hat{\theta}(t)^\top \phi(t)$$

where $t \mapsto \hat{y}(t) \in \mathbb{R}$ is the estimated output and $t \mapsto \hat{\theta}(t) \in \mathbb{R}^p$ is the estimate of the unknown parameter θ. The error between the measured and the estimated outputs is

$$e(t) := y(t) - \hat{y}(t) = \tilde{\theta}(t)^\top \phi(t)$$

where $\tilde{\theta} := \theta - \hat{\theta}$ is the parameter estimation error. The gradient algorithm assigns dynamics to $\hat{\theta}$ so that it converges to θ based on measurements of y and ϕ. To do so, the following cost function is introduced:

$$J(e) := \frac{1}{2}e^2.$$

Then, $\hat{\theta}$ has dynamics

$$\dot{\hat{\theta}} = -\gamma \nabla_{\hat{\theta}} J(e) = \gamma \phi(t)(y(t) - \phi(t)^\top \hat{\theta}) \tag{2.4}$$

where $\gamma > 0$ is a design parameter that modifies the adaptation rate.

2.2.2 Discrete-Time Gradient Descent

Consider a discrete-time linear regression model

$$y(j) = \theta^\top \phi(j) \qquad \forall j \in \mathbb{N} \tag{2.5}$$

where $j \mapsto y(j) \in \mathbb{R}^n$ the measured output, $j \mapsto \phi(j) \in \mathbb{R}^p$ is the known regressor, and $\theta \in \mathbb{R}^p$ is an unknown vector of constant parameters.

By following a similar derivation as in the continuous-time case, the classical discrete-time gradient descent algorithm [54] is

$$\hat{\theta}(j+1) = \hat{\theta}(j) + \frac{\phi(j)}{\gamma + |\phi(j)|^2}\big(y(j) - \phi(j)^\top \hat{\theta}(j)\big) \tag{2.6}$$

where $\gamma > 0$ is a design parameter that modifies the adaptation rate.

2.3 Integral Concurrent Learning

Consider the continuous-time system

$$\dot{x} = f(x, u(t)) + \phi(t)\theta \qquad \forall t \geq 0 \tag{2.7}$$

where $x \in \mathbb{R}^n$ is the known state vector, $u \in \mathbb{R}^m$ is the known input, $t \mapsto \phi(t) \in \mathbb{R}^{n\times p}$ is the known regressor, $(x, u) \mapsto f(x, u)$ is a known continuous function, $\theta \in \mathbb{R}^p$ is a vector of unknown constant parameters, and $n, m, p \in \mathbb{N}$.

The integral concurrent learning algorithm was proposed in the recent paper [40] to estimate θ in (2.7). We first explain the ICL approach in words before formally defining the algorithm. Given $t \mapsto x(t)$, $t \mapsto u(t)$, and $t \mapsto \phi(t)$ satisfying (2.7), the ICL algorithm computes signals $t \mapsto \mathcal{Y}(t)$ and $t \mapsto \Phi(t)$ such that $\mathcal{Y}$, Φ, and θ are related via a linear regression model; that is, $\mathcal{Y}(t) = \Phi(t)\theta$ for all $t \geq 0$. Next, samples of the signals $\mathcal{Y}$ and Φ are stored, and the stored data is used alongside current measurement to adapt the parameter estimate. Samples are selected to ensure that the parameter estimation error converges to zero.

Given $t \mapsto x(t)$, $t \mapsto u(t)$, and $t \mapsto \phi(t)$ defined on $\mathbb{R}_{\geq 0}$, the ICL algorithm is implemented as follows. Over each time interval of length $\Lambda > 0$, where Λ is a design parameter, we integrate f and ϕ, and compute the change in x as

$$\begin{aligned}\mathcal{X}(t) &:= x(t) - x(\max\{t-\Lambda, 0\}) \\ \mathcal{F}(t) &:= \int_{\max\{t-\Lambda,0\}}^{t} f(x(s), u(s))ds \\ \Phi(t) &:= \int_{\max\{t-\Lambda,0\}}^{t} \phi(t)ds\end{aligned} \tag{2.8}$$

for all $t \geq 0$. Since θ is constant, it follows from the Fundamental Theorem of Calculus that

$$\mathcal{X}(t) = \mathcal{F}(t) + \Phi(t)\theta \qquad \forall t \geq 0.$$

By defining $\mathcal{Y}(t) := \mathcal{X}(t) - \mathcal{F}(t)$, we obtain

$$\mathcal{Y}(t) = \Phi(t)\theta \qquad \forall t \geq 0. \tag{2.9}$$

Though perhaps not intuitive, the definitions in (2.8) conveniently lead to this linear regression model.

Next, we sample $t \mapsto \mathcal{Y}(t)$ and $t \mapsto \Phi(t)$ at time instants $\{\tilde{t}_i\}_{i=1}^{S(t)}$ satisfying $0 \leq \tilde{t}_1 < \tilde{t}_2 < \cdots \leq t$, where $t \mapsto S(t)$ indicates a time-dependent number of samples, which is to be designed. Samples are stored in the time-varying matrices

$$\begin{aligned}Y(t) &:= \begin{bmatrix} Y_1(t), & Y_2(t), & \cdots, & Y_N(t) \end{bmatrix} \in \mathbb{R}^{n \times N} \\ Z(t) &:= \begin{bmatrix} Z_1(t), & Z_2(t), & \cdots, & Z_N(t) \end{bmatrix} \in \mathbb{R}^{n \times p \times N}\end{aligned}$$

where $N \in \mathbb{N} \setminus \{0\}$ is a design parameter satisfying $N \geq \lceil p/n \rceil$ that represents the maximum number of samples that can be stored. The elements of Y and Z are initially empty (zero), and are populated by the samples of $\mathcal{Y}$ and Φ, respectively. In [40], samples are stored in empty elements or, if no empty element is available, by replacing the data in the element that maximizes the minimum eigenvalue of

$\sum_{\ell=1}^{N} Z_\ell^\top Z_\ell$. Thus, the elements of Y and Z are piecewise constant right-continuous signals that change only at the sample times. Moreover, since θ is constant, it follows from (2.9) that, for all $t \in \mathbb{R}_{\geq 0}$,[6]

$$Y_\ell(t) = Z_\ell(t)\theta \qquad \forall\, \ell \in \{1, 2, \cdots, p\}. \tag{2.10}$$

The ICL algorithm [40] for $\hat{\theta}$ is

$$\dot{\hat{\theta}} = \Gamma\phi(t)^\top(x(t) - \hat{x}(t)) + \rho\Gamma\sum_{\ell=1}^{N} Z_\ell(t)^\top(Y_\ell(t) - Z_\ell(t)\hat{\theta}) \tag{2.11}$$

where $\Gamma \in \mathbb{R}^{p\times p}$ and $\rho \in \mathbb{R}_{>0}$ are positive definite design parameters and $t \mapsto \hat{x}(t)$ is a filtered version of the plant state x, generated by

$$\dot{\hat{x}} = f(x(t), u(t)) + \phi(t)\hat{\theta}(t) + K(x(t) - \hat{x}) \tag{2.12}$$

where $K \in \mathbb{R}^{n\times n}$ is a positive definite design parameter. In [40], samples are chosen so that $\sum_{\ell=1}^{N} Z_\ell^\top Z_\ell$ is uniformly positive definite, which ensures exponential convergence of $\hat{\theta}$ to θ.

2.4 Excitation Conditions

We employ the following notion of persistence of excitation [37].

Definition 2.12: A bounded signal $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ is *persistently exciting* if there exist constants $T, \mu > 0$ such that, for all $t_0 \geq 0$,

$$\int_{t_0}^{t_0+T} \phi(t)\phi(t)^\top dt \geq \mu I. \tag{2.13}$$

[6] Note that the index ℓ in (2.10) increments over the second dimension – that is, the columns – of Y, but increments over the third dimension of Z. We refer to the third dimension of Z as the "layers" of Z.

We employ the following notion of excitation over a finite interval [37].

Definition 2.13: Given $t_0 \geq 0$ and $T > 0$, a bounded signal $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ is *exciting over the finite interval* $[t_0, t_0 + T]$ if there exists $\mu > 0$ such that

$$\int_{t_0}^{t_0+T} \phi(t)\phi(t)^\top dt \geq \mu I. \tag{2.14}$$

Similarly, in discrete-time, we rely on the following notions of excitation [54].

Definition 2.14: A bounded signal $\phi : \mathbb{N} \to \mathbb{R}^p$ is *persistently exciting* if there exist $J \in \mathbb{N} \setminus \{0\}$ and $\mu > 0$ such that, for all $j_0 \in \mathbb{N}$,

$$\sum_{j=j_0}^{j_0+J} \phi(j)\phi(j)^\top \geq \mu I. \tag{2.15}$$

Definition 2.15: Given $j_0 \in \mathbb{N}$ and $J \in \mathbb{N} \setminus \{0\}$, a bounded signal $\phi : \mathbb{N} \to \mathbb{R}^p$ is *exciting over the finite interval* $\{j_0, j_0 + 1, \cdots, j_0 + J\}$ if there exists $\mu > 0$ such that

$$\sum_{j=j_0}^{j_0+J} \phi(j)\phi(j)^\top \geq \mu I. \tag{2.16}$$

Although the notions of excitation above are defined for vector signals, they can be extended to apply for matrix signals as well [37, 31].

Chapter 3

Properties of Time-Varying Hybrid Systems

In this chapter, we establish stability and robustness properties induced by a class of time-varying hybrid systems. These results will be used in subsequent chapters to prove the properties induced by our proposed hybrid parameter estimation algorithms.

3.1 Hybrid Model

Given $A : E \to \mathbb{R}^{p\times p}$ and $B : E \to \mathbb{R}^{p\times p}$, where $E := \operatorname{dom} A = \operatorname{dom} B$ is a hybrid time domain[7] and $p \in \mathbb{N}$, consider the hybrid system, denoted by $\mathcal{H}_0$, with

[7] Note that A and B are defined on a hybrid time domain, but they are not necessarily hybrid arcs. That is, they do not need to be locally absolutely continuous during flows – see [22] for details.

state $\xi := (\vartheta, \tau, k) \in \mathcal{X} := \mathbb{R}^p \times E$ and dynamics

$$\mathcal{H}_0 : \begin{cases} \begin{bmatrix} \dot{\vartheta} \\ \dot{\tau} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} -A(\tau, k)\vartheta \\ 1 \\ 0 \end{bmatrix} =: F_0 & \xi \in C_0 \\ \begin{bmatrix} \vartheta^+ \\ \tau^+ \\ k^+ \end{bmatrix} = \begin{bmatrix} \vartheta - B(\tau, k)\vartheta \\ \tau \\ k+1 \end{bmatrix} =: G_0 & \xi \in D_0 \end{cases} \tag{3.1}$$

The flow and jump sets of $\mathcal{H}_0$ are defined so that the system flows when A flows, and jumps when B jumps. Namely,

$$C_0 := \mathrm{cl}(\mathcal{X} \setminus D_0), \qquad D_0 := \{\xi \in \mathcal{X} \ : \ (\tau, k+1) \in E \ \}. \tag{3.2}$$

The state components τ and k correspond to t and j, respectively, from the hybrid time domain E. Including τ and k in ξ allows A and B to be part of the definitions of F_0 and G_0, rather than modeled as inputs to $\mathcal{H}_0$. Thus, we can express $\mathcal{H}_0$ as an autonomous hybrid system, which allows us to leverage recent results on stability and robustness properties for such systems [22, 45].

3.2 Stability Analysis

In this section, we establish sufficient conditions on A and B that ensure the hybrid system $\mathcal{H}_0$ induces global pre-exponential stability of the set

$$\mathcal{A} := \{\xi \in \mathcal{X} \ : \ \vartheta = 0 \ \}. \tag{3.3}$$

Global pre-exponential stability of $\mathcal{A}$ implies that, for each solution ξ to $\mathcal{H}_0$, the distance from ξ to the set $\mathcal{A}$ is bounded above by an exponentially decreasing function of the initial condition – see Definition 2.9. As a consequence, for each complete solution ξ to $\mathcal{H}_0$, the state component ϑ converges exponentially to zero.

We impose on A and B the following structural properties, which are similar to those imposed in the design of continuous-time and discrete-time gradient algorithms.

Assumption 3.1: Given $A, B : E \to \mathbb{R}^{p\times p}$, where $E := \operatorname{dom} A = \operatorname{dom} B$ is a hybrid time domain, suppose that A and B are symmetric, positive definite, and bounded as follows:

1. $A(t,j) = A(t,j)^\top \geq 0$ for all $(t,j) \in E$;
2. $B(t,j) = B(t,j)^\top \geq 0$ for all $(t,j) \in \Upsilon(E)$;
3. there exists $a_M > 0$ such that $\operatorname{ess\,sup}\,\{|A(t,j)| : (t,j) \in E\} \leq a_M$;
4. $|B(t,j)| < 1$ for all $(t,j) \in \Upsilon(E)$,

with Υ as in (2.2).

We impose the following hybrid PE condition [48], which is inspired by the continuous-time and discrete-time PE conditions (see Section 2.4).

Assumption 3.2: Given $A, B : E \to \mathbb{R}^{p\times p}$, where $E := \operatorname{dom} A = \operatorname{dom} B$ is a hybrid time domain, there exist $\Delta, \mu_0 \in \mathbb{R}_{>0}$ such that, for each $(t',j'), (t^*,j^*) \in E$ satisfying[8]

$$\Delta \leq (t^* - t') + (j^* - j') < \Delta + 1, \tag{3.4}$$

the following holds:

$$\sum_{j=j'}^{j^*} \int_{\max\{t',t_j\}}^{\min\{t^*,t_{j+1}\}} A(s,j)ds + \frac{1}{2}\sum_{j=j'}^{j^*-1} B(t_{j+1},j) \geq \mu_0 I \tag{3.5}$$

where $t_{J+1} := T$, with $J := \sup_j E$ and $T := \sup_t E$.

[8]The hybrid time instants (t',j') and (t^*,j^*) are the beginning and the end, respectively, of a hybrid time interval with length satisfying (3.4), over which (3.5) holds.

Inspired by [48], we establish sufficient conditions that ensure the hybrid system $\mathcal{H}_0$ induces global pre-exponential stability of the set $\mathcal{A}$ in (3.3).

Theorem 3.3: Given the hybrid system $\mathcal{H}_0$ in (3.1)*, suppose that Assumptions 3.1 and 3.2 hold. Then, each solution ξ to $\mathcal{H}_0$ satisfies*

$$|\xi(t,j)|_{\mathcal{A}} \leq \kappa_0 \mathrm{e}^{-\lambda_0(t+j-s-k)} |\xi(s,k)|_{\mathcal{A}} \tag{3.6}$$

for all $(t,j),(s,k) \in \operatorname{dom}\xi$ satisfying $t+j \geq s+k$, where

$$\begin{aligned} \kappa_0 &:= \sqrt{\frac{1}{1-\sigma}}, \qquad\qquad \lambda_0 := -\frac{\ln(1-\sigma)}{2(\Delta+1)}, \\ \sigma &:= \frac{2\mu_0}{\left(1+\sqrt{(a_M+2)(\Delta+2)^3(a_M(\Delta+2)+1/2)}\right)^2} \end{aligned} \tag{3.7}$$

with a_M from Assumptions 3.1 and μ_0, Δ from Assumption 3.2.

Proof. This proof is in Appendix A. □

3.3 Robustness Analysis

In this section, we study the robustness properties induced by $\mathcal{H}_0$ with respect to bounded hybrid disturbances on the state component ϑ. Given $A, B : E \to \mathbb{R}^{p\times p}$, consider disturbances $d_c : E \to \mathbb{R}^p$ and $d_d : E \to \mathbb{R}^p$ added to the dynamics of ϑ during flows and jumps, respectively. We denote the hybrid system $\mathcal{H}_0$ in (3.1) under the effect of these disturbances as $\mathcal{H}$, with state $\xi = (x, \tau, k) \in \mathcal{X}$ and dynamics

$$\mathcal{H} : \begin{cases} \begin{bmatrix} \dot{\vartheta} \\ \dot{\tau} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} -A(\tau,k)\vartheta + d_c(\tau,k) \\ 1 \\ 0 \end{bmatrix} =: F & \xi \in C \\ \begin{bmatrix} \vartheta^+ \\ \tau^+ \\ k^+ \end{bmatrix} = \begin{bmatrix} \vartheta - B(\tau,k)\vartheta + d_d(\tau,k) \\ \tau \\ k+1 \end{bmatrix} =: G & \xi \in D \end{cases} \tag{3.8}$$

where $C := C_0$ and $D := D_0$, with C_0, D_0 as in (3.2).

To analyze the robustness properties induced by $\mathcal{H}$, we first require the following result for the hybrid system $\mathcal{H}_0$ in (3.1).

Lemma 3.4: Given the hybrid system $\mathcal{H}_0$ *in* (3.1)*, suppose that Assumptions 3.1 and 3.2 hold and let the hybrid time domain* E *come from these assumptions. Then, for each* $q_M \geq q_m > 0$ *and each symmetric matrix function* $Q : E \to \mathbb{R}^{p\times p}$ *satisfying*

$$q_m I \leq Q(t,j) \leq q_M I \quad \forall (t,j) \in E, \tag{3.9}$$

there exists a symmetric matrix function $P : E \to \mathbb{R}^{p\times p}$ *satisfying*

$$p_m I \leq P(t,j) \leq p_M I \quad \forall (t,j) \in E, \tag{3.10}$$

where

$$p_m := q_m, \quad p_M := q_m + \frac{q_M \kappa_0^2}{2\lambda_0} + \frac{q_M \kappa_0^2 \mathrm{e}^{2\lambda_0}}{\mathrm{e}^{2\lambda_0} - 1}, \tag{3.11}$$

with κ_0 *and* λ_0 *from Theorem 3.3. Moreover, for each* $j \in \mathbb{N}$ *and for almost all* $t \in I^j := \{t : (t,j) \in E\}$, $(t,j) \mapsto P(t,j)$ *satisfies*

$$\frac{d}{dt}P(t,j) - P(t,j)A(t,j) - A(t,j)^\top P(t,j) \leq -Q(t,j) \tag{3.12}$$

and, for all $(t,j) \in \Upsilon(E)$*, with* Υ *as in* (2.2),

$$(I - B(t,j))^\top P(t,j+1)(I - B(t,j)) - P(t,j) \leq -Q(t,j). \tag{3.13}$$

Proof. This proof is in Appendix B.1. □

Then, we establish the following ISS result for $\mathcal{H}$.

Theorem 3.5: Given the hybrid system $\mathcal{H}$ in (3.8), let Assumptions 3.1 and 3.2 hold. Then, for each $q_M \geq q_m > 0$ and each $\zeta \in (0,1)$, each solution ξ to $\mathcal{H}$ satisfies

$$|\xi(t,j)|_{\mathcal{A}} \leq \beta(|\xi(0,0)|_{\mathcal{A}}, t+j) + \rho|d|_{(t,j)} \tag{3.14}$$

for all $(t,j) \in \operatorname{dom}\xi$, where

$$d(t,j) := \begin{cases} d_c(t,j) & \text{if} \quad (t,j) \in E \setminus \Upsilon(E) \\ d_d(t,j) & \text{if} \quad (t,j) \in \Upsilon(E) \end{cases} \tag{3.15}$$

and

$$\beta(s,r) := \sqrt{\frac{p_M}{p_m}}\mathrm{e}^{-\omega r}s, \quad \rho := \sqrt{\frac{2p_M^3}{q_m p_m \zeta}\left(\frac{2p_M}{q_m}+1\right)}$$
$$\omega := \frac{1}{2}\min\left\{\frac{q_m}{2p_M}(1-\zeta), -\ln\left(1-\frac{q_m}{2p_M}(1-\zeta)\right)\right\},$$

with $\kappa_0, \lambda_0, \sigma$ as in Theorem 3.3, p_m, p_M as in Lemma 3.4, Υ as in (2.2), and a_M, μ_0, Δ from Assumptions 3.1 and 3.2.

Proof. The proof is in Appendix B.2. □

When the disturbances d_c and d_d converge exponentially to zero, we establish the following result for $\mathcal{H}$.

Theorem 3.6: Given the hybrid system $\mathcal{H}$ in (3.8), suppose that Assumptions 3.1 and 3.2 hold, and that there exist $a, b > 0$ such that d in (3.15) satisfies

$$|d(t,j)| \leq a\mathrm{e}^{-b(t+j)}|d(0,0)| \tag{3.16}$$

for all $(t,j) \in E$. Then, for each $q_M \geq q_m > 0$ and each $\zeta \in (0,1)$, each solution ξ to $\mathcal{H}$ satisfies

$$|\xi(t,j)|_{\mathcal{A}} \leq \kappa\mathrm{e}^{-\lambda(t+j)}\big(|\xi(0,0)|_{\mathcal{A}} + |d(0,0)|\big) \tag{3.17}$$

for all $(t,j) \in \operatorname{dom} \xi$*, where*

$$\kappa := 2 \max \left\{ \frac{p_M}{p_m},\ a\rho \sqrt{\frac{p_M}{p_m}} \right\}, \qquad \lambda := \frac{1}{2} \min \{\omega, b\} \tag{3.18}$$

with $\kappa_0, \lambda_0, \sigma$ *as in Theorem 3.3,* p_m, p_M *as in Lemma 3.4,* ρ, ω *as in Theorem 3.5,* Υ *as in* (2.2)*, and* a_M, μ_0, Δ *from Assumptions 3.1 and 3.2.*

Proof. This proof is in Appendix B.3. □

Chapter 4

Parameter Estimation for Hybrid Systems via Hybrid Gradient Descent

In this chapter, we propose a parameter estimation algorithm for a class of hybrid dynamical systems. The algorithm leverages the information provided during both the flows and jumps of the hybrid system, and updates the parameter estimate with dynamics inspired by the continuous-time and discrete-time gradient descent algorithms in Section 2.2.

4.1 Problem Statement

Motivated by the limitations of the continuous-time and discrete-time estimation algorithms highlighted in Section 1.3.1, we develop a hybrid algorithm for estimating parameters in hybrid dynamical systems of the form

$$\begin{aligned} \dot{x} &= f_c(x, u(t,j)) + \phi_c(t,j)\theta \quad (x, u(t,j)) \in C_P \\ x^+ &= g_d(x, u(t,j)) + \phi_d(t,j)\theta \quad (x, u(t,j)) \in D_P, \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the known state vector and $(x,u) \mapsto f_c(x,u) \in \mathbb{R}^n$ and $(x,u) \mapsto g_d(x,u) \in \mathbb{R}^n$ are known continuous functions. The regressors $(t,j) \mapsto \phi_c(t,j) \in \mathbb{R}^{n\times p}$ and $(t,j) \mapsto \phi_d(t,j) \in \mathbb{R}^{n\times p}$ and the input $(t,j) \mapsto u(t,j) \in \mathbb{R}^m$ are known, and are defined on hybrid time domains as described in Section 2.1, but are not necessarily hybrid arcs.Note that ϕ_c plays no role in the dynamics of (4.1) at jumps and ϕ_d plays no role in the dynamics of (4.1) during flows.

Our goal is to estimate the parameter vector θ in (4.1). Since ϕ_c and ϕ_d may exhibit both flows and jumps, it is important to update the parameter estimate $\hat{\theta}$ continuously whenever ϕ_c flows, and discretely each time ϕ_d jumps, which is possible when jumps are detected instantaneously. Hence, we propose to estimate θ using a hybrid algorithm, denoted $\mathcal{H}_g$, of the form

$$\mathcal{H}_g : \begin{cases} \dot{\xi} = F_g(\xi) & \xi \in C_g \\ \xi^+ = G_g(\xi) & \xi \in D_g, \end{cases} \tag{4.2}$$

with data designed to solve the following problem.

Problem 4.1: Design the data (C_g, F_g, D_g, G_g) of $\mathcal{H}_g$ in (4.2) and determine conditions on ϕ_c and ϕ_d that ensure the parameter estimate $\hat{\theta}$ converges to the unknown parameter vector θ in (4.1).

Next, we present our solution to this estimation problem.

4.2 Problem Solution

Given $\phi_c, \phi_d : E \to \mathbb{R}^{n\times p}$ and $u : E \to \mathbb{R}^m$, where $E := \operatorname{dom}\phi_c = \operatorname{dom}\phi_d = \operatorname{dom} u$ is a hybrid time domain, we define the state ξ of $\mathcal{H}_g$ as $\xi := (x, \hat{\theta}, \psi, \eta, \tau, k) \in \mathcal{X}_g := \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^{n\times p} \times \mathbb{R}^n \times E$, where x is the state of the plant in (4.1), $\hat{\theta}$ is the estimate of θ, and ψ, η are auxiliary state variables. The state components

τ and k have dynamics such that they evolve as t and j, respectively, from the hybrid time domain E. Including τ and k in ξ allows ϕ_c, ϕ_d, and u to be part of the definitions of F_g and G_g, rather than modeled as inputs to $\mathcal{H}_g$. Thus, we can express $\mathcal{H}_g$ as an autonomous hybrid system, which allows us to leverage recent results on stability and robustness properties [22, 45].

The flow and jump sets of $\mathcal{H}_g$ are defined so that the algorithm flows when ϕ_c flows, and jumps when ϕ_d jumps. Since $\operatorname{dom}\phi_c = \operatorname{dom}\phi_d = E$,

$$C_g := \operatorname{cl}(\mathcal{X}_g \setminus D_g), \quad D_g := \{\xi \in \mathcal{X}_g : (\tau, k+1) \in E\}. \tag{4.3}$$

Remark 4.2: We assume for simplicity that the plant state x has the same hybrid time domain as ϕ_c, ϕ_d, and u. As a result, the flow set C_P and jump set D_P of the plant are not part of the construction of $\mathcal{H}_g$. Our algorithm can be extended to the case where x, ϕ_c, ϕ_d, and u have different hybrid time domains by considering the flow and jump sets in (4.1). In this case, we can reparameterize the domains of ϕ_c, ϕ_d, and u to express x, ϕ_c, ϕ_d, and u on a common hybrid time domain. See, e.g., [5].

4.2.1 Parameter Estimation During Flows

During flows of $\mathcal{H}_g$, we transform the flow map of (4.1) into a form similar to a linear regression model using the state variables ψ and η, with dynamics [39]

$$\begin{aligned} \dot{\psi} &= -\lambda_c \psi + \phi_c(\tau, k) \\ \dot{\eta} &= -\lambda_c(x + \eta) - f_c(x, u(\tau, k)), \end{aligned} \tag{4.4}$$

where $\lambda_c > 0$ is a design parameter. Defining $\varepsilon := x + \eta - \psi\theta$ and $y := x + \eta$, it follows that ψ and ε are related via

$$y = \psi\theta + \varepsilon. \tag{4.5}$$

Since θ is constant, differentiating ε along trajectories of (4.1), (4.4) yields $\dot{\varepsilon} = -\lambda_c \varepsilon$ during flows. Thus, ε converges exponentially to zero during flows. Moreover, we have the following equivalences:

$$\varepsilon \;\to\; 0 \iff x+\eta \;\to\; \psi\theta \iff y \;\to\; \psi\theta.$$

Hence, y, ψ, and θ are related via a linear regression model plus an exponentially convergent term.

During flows, we update $\hat{\theta}$ continuously with dynamics inspired by the continuous-time gradient algorithm in (2.4). Namely,

$$\dot{\hat{\theta}} = \gamma_c \psi^\top (y - \psi\hat{\theta}), \tag{4.6}$$

where $\gamma_c > 0$ is a design parameter. Hence, the flow map for $\mathcal{H}_g$ in (4.2) is

$$F_g(\xi) := \begin{bmatrix} f_c(x, u(\tau,k)) + \phi_c(\tau,k)\theta \\ \gamma_c \psi^\top (y - \psi\hat{\theta}) \\ -\lambda_c \psi + \phi_c(\tau,k) \\ -\lambda_c (x+\eta) - f_c(x, u(\tau,k)) \\ 1 \\ 0 \end{bmatrix} \qquad \forall \xi \in C_g.$$

4.2.2 Parameter Estimation at Jumps

At jumps of $\mathcal{H}_g$, using similar reasoning as during flows, we update $\hat{\theta}$ discretely using a reset map inspired by the discrete-time gradient algorithm in (2.6). Namely, at each $(\tau,k) \in E$ such that $(\tau,k+1) \in E$,

$$\hat{\theta}(\tau,k+1) = \hat{\theta}(\tau,k) + \frac{\psi(\tau,k+1)^\top}{\gamma_d + |\psi(\tau,k+1)|^2}(y(\tau,k+1) - \psi(\tau,k+1)\hat{\theta}(\tau,k)) \tag{4.7}$$

with

$$\begin{aligned} \psi(\tau,k+1) &= (1-\lambda_d)\psi(\tau,k) + \phi_d(\tau,k) \\ \eta(\tau,k+1) &= (1-\lambda_d)(x(\tau,k) + \eta(\tau,k)) - g_d(x(\tau,k), u(\tau,k)) \\ y(\tau,k+1) &= x(\tau,j+1) + \eta(\tau,k+1), \end{aligned} \tag{4.8}$$

where $\gamma_d > 0$, $\lambda_d \in (0,2)$ are design parameters.

Remark 4.3: To compute the update law for $\hat{\theta}$ in (4.7), we require measurements of x for two consecutive hybrid time instants, (τ, k) and $(\tau, k+1)$. Moreover, two computational steps are required to update $\hat{\theta}$ at time $(\tau, k) \in E$. The first step computes $\psi(\tau, k+1)$, $\eta(\tau, k+1)$, and $y(\tau, k+1)$ in (4.8), and the second step computes $\hat{\theta}(\tau, j+1)$ in (4.7). For simplicity, we omit the first computational step in (4.7).

For readability, we denote $\psi(\tau, k+1)$ as ψ^+, $\eta(\tau, k+1)$ as η^+, and $y(\tau, k+1)$ as y^+. Then, the jump map for $\mathcal{H}_g$ in (4.2) is

$$G_g(\xi) := \begin{bmatrix} g_d(x, u(\tau, k)) + \phi_d(\tau, k)\theta \\ \hat{\theta} + \frac{\psi^{+\top}}{\gamma_d + |\psi^+|^2}(y^+ - \psi^+\hat{\theta}) \\ (1-\lambda_d)\psi + \phi_d(\tau, k) \\ (1-\lambda_d)(x+\eta) - g_d(x, u(\tau, k)) \\ \tau \\ k+1 \end{bmatrix} \qquad \forall \xi \in D_g. \tag{4.9}$$

Remark 4.4: For simplicity, the hybrid algorithm $\mathcal{H}_g$ in (4.2) is expressed such that jumps in the parameter estimate coincide with jumps in x. This results in $\mathcal{H}_g$ being noncausal since measurements of x^+ are not available until after a jump. We can remove the simplification at the price of letting the algorithm jump twice for each jump in x, as follows. Immediately before a jump in x, the algorithm jumps once to reset the values of ψ, η, and k per the jump map (4.9). Immediately after a jump in x, the algorithm jumps a second time to update the parameter estimate using the current value of x. A logic variable ensures that, after the second jump, the algorithm flows or jumps in accordance with the hybrid time domain E. Since θ in (4.1) is constant, the stability properties induced by $\mathcal{H}_g$ in (4.2) are equivalent to the stability properties induced by the causal modification, after we reparameterize the domain of solutions to $\mathcal{H}_g$ to match the domain of solutions to the causal system. Hence, for simplicity, we focus our analysis on (4.2).

4.3 Stability Analysis

In this section, we establish sufficient conditions that that ensure the hybrid system $\mathcal{H}_g$ in (4.2) induces semiglobal pre-exponential stability of the set

$$\mathcal{A}_g := \left\{ \xi \in \mathcal{X}_g \;:\; \hat{\theta} = \theta,\ \varepsilon = 0 \right\}, \tag{4.10}$$

where

$$\varepsilon := x + \eta - \psi\theta. \tag{4.11}$$

Semiglobal pre-exponential stability of $\mathcal{A}_g$ implies that, given any compact set of initial conditions, for each solution ξ to $\mathcal{H}_g$ from such compact set, the distance from ξ to the set $\mathcal{A}_g$ is bounded above by an exponentially decreasing function of the initial condition – see Definition 2.10. As a consequence, for each complete solution ξ to $\mathcal{H}_g$, the parameter estimate $\hat{\theta}$ converges exponentially to θ, and ε converges exponentially to zero.

We impose the following assumptions.

Assumption 4.5: Given $\phi_c, \phi_d : E \to \mathbb{R}^{n\times p}$, where $E := \operatorname{dom}\phi_c = \operatorname{dom}\phi_d$ is a hybrid time domain, there exists $\phi_M > 0$ such that $|\phi_c(t,j)|_{\mathrm{F}} \le \phi_M$ for all $(t,j) \in E$ and $|\phi_d(t,j)|_{\mathrm{F}} \le \phi_M$ for all $(t,j) \in \Upsilon(E)$, with Υ as in (2.2).

Assumption 4.6: Given $\gamma_c, \lambda_c, \gamma_d > 0$, $\lambda_d \in (0,2)$, and $\phi_c, \phi_d : E \to \mathbb{R}^{n\times p}$, where $E := \operatorname{dom}\phi_c = \operatorname{dom}\phi_d$ is a hybrid time domain, there exist $\Delta, \mu \in \mathbb{R}_{>0}$ such that, for all $(t', j'), (t^*, j^*) \in E$ satisfying

$$\Delta \le (t^* - t') + (j^* - j') < \Delta + 1, \tag{4.12}$$

the following hybrid PE condition holds:

$$\begin{aligned}&\sum_{j=j'}^{j^*}\int_{\max\{t',t_j\}}^{\min\{t^*,t_{j+1}\}}\psi(s,j)^\top\psi(s,j)ds\\&+\sum_{j=j'}^{j^*-1}\psi(t_{j+1},j+1)^\top\psi(t_{j+1},j+1)\geq\mu I\end{aligned}\tag{4.13}$$

where $\{t_j\}_{j=0}^{J}$ is the sequence defining E as in Section 2.1, $t_{J+1}:=T$, with $J:=\sup_j E$ and $T:=\sup_t E$, and $(t,j)\mapsto\psi(t,j)$ is generated by (4.2).

Theorem 4.7: Given the hybrid system $\mathcal{H}_g$ *in* (4.2), $\gamma_c,\lambda_c,\gamma_d>0$, *and* $\lambda_d\in(0,2)$, *suppose that Assumptions 4.5 and 4.6 hold. Then, for each* $\psi_0\geq 0$, $q_M\geq q_m>0$, *and each* $\zeta\in(0,1)$, *each solution* ξ *to* $\mathcal{H}_g$ *from* $\xi(0,0)\in\mathcal{X}_0:=\{\xi\in\mathcal{X}_g:|\psi|_{\mathrm{F}}\leq\psi_0\}$ *satisfies*

$$|\xi(t,j)|_{\mathcal{A}_g}\leq\kappa_g\mathrm{e}^{-\lambda_g(t+j)}|\xi(0,0)|_{\mathcal{A}_g}\tag{4.14}$$

for all $(t,j) \in \operatorname{dom} \xi$*, where* $\kappa_g := \sqrt{3}\kappa$*,* $\lambda_g := \min\{\lambda, b\}$*, and*

$$\begin{aligned}
\kappa &:= 2\max\left\{\frac{p_M}{p_m},\ a\rho\sqrt{\frac{p_M}{p_m}}\right\}, \qquad \lambda := \frac{1}{2}\min\left\{\omega, b\right\}, \\
a &:= \max\left\{\gamma_c\psi_M, \frac{1}{2\sqrt{\gamma_d}}\right\}, \\
b &:= \frac{1}{2}\min\left\{\lambda_c,\ -\ln\left(1-\lambda_d(2-\lambda_d)\right)\right\}, \\
p_m &:= q_m, \qquad p_M := q_m + \frac{q_M\kappa_0^2}{2\lambda_0} + \frac{q_M\kappa_0^2 \mathrm{e}^{2\lambda_0}}{\mathrm{e}^{2\lambda_0}-1}, \\
\rho &:= \sqrt{\frac{2p_M^3}{q_m p_m \zeta}\left(\frac{2p_M}{q_m}+1\right)}, \\
\omega &:= \frac{1}{2}\min\left\{\frac{q_m}{2p_M}(1-\zeta), -\ln\left(1-\frac{q_m}{2p_M}(1-\zeta)\right)\right\}, \\
\kappa_0 &:= \sqrt{\frac{1}{1-\sigma}}, \qquad \lambda_0 := -\frac{\ln(1-\sigma)}{2(\Delta+1)}, \\
\sigma &:= \frac{2\mu_0}{\left(1+\sqrt{(a_M+2)(\Delta+2)^3(a_M(\Delta+2)+1/2)}\right)^2}, \\
\mu_0 &:= \left\{\gamma_c, \frac{1}{2(\gamma_d+\psi_M^2)}\right\}\mu, \qquad a_M := \gamma_c\psi_M^2, \\
\psi_M &:= \psi_0 + \max\left\{\frac{1}{\lambda_c}, \frac{\sqrt{2\lambda_d(2-\lambda_d)+16}}{\lambda_d(2-\lambda_d)}\right\}\phi_M.
\end{aligned}$$

Theorem 4.7 states that, if $|\phi_c|_{\mathrm{F}}$ and $|\phi_d|_{\mathrm{F}}$ are uniformly bounded above and the hybrid PE condition (4.13) is satisfied, then the set $\mathcal{A}_g$ in (4.10) is semiglobally pre-exponentially stable for $\mathcal{H}_g$. The hybrid PE condition (4.13) reduces to the continuous-time PE condition (2.13) if ψ is continuous, and reduces to the discrete-time PE condition (2.13) if ψ is discrete. Hence, in such cases, we recover the results established in [37, 54].[9]

[9]In fact, if ψ is continuous and ϕ_c, $\frac{d}{dt}\phi_c \in \mathcal{L}_\infty$, then it follows from [50, Lemma 2.6.7] that the ψ component of each solution ξ to $\mathcal{H}_g$ from $\mathcal{X}_0$ is PE as in (2.13) if ϕ_c is PE. Given such ϕ_c, the excitation parameters for ψ – μ and T in (2.13) – depend on the initial condition of ψ. However, since $\xi(0,0) \in \mathcal{X}_0$, the initial condition of ψ lies in a compact set, and therefore we can find these parameters independent of the initial condition. If ψ is discrete, then a similar persistence of excitation property holds for ψ if ϕ_d is PE as in (2.15).

4.3.1 Proof of Theorem 4.7

To prove Theorem 4.7, we use that the error dynamics of $\mathcal{H}_g$ belong to the class of hybrid systems studied in Chapter 3. We denote the hybrid system resulting from expressing $\mathcal{H}_g$ in error coordinates $\tilde{\theta} := \theta - \hat{\theta}$ as $\widetilde{\mathcal{H}}_g$, with state $\xi = (x, \tilde{\theta}, \psi, \eta, \tau, k) \in \mathcal{X}_g$ and dynamics

$$\widetilde{\mathcal{H}}_g : \begin{cases} \dot{\xi} = \widetilde{F}_g(\xi) & \xi \in \widetilde{C}_g \\ \xi^+ = \widetilde{G}_g(\xi) & \xi \in \widetilde{D}_g, \end{cases} \tag{4.15}$$

where $\widetilde{C}_g := C_g$ and $\widetilde{D}_g := D_g$, with C_g, D_g in (4.3), and

$$\widetilde{F}_g(\xi) := \begin{bmatrix} f_c(x, u(\tau,k)) + \phi_c(\tau,k)\theta \\ -\gamma_c \psi^\top \psi \tilde{\theta} - \gamma_c \psi^\top \varepsilon \\ -\lambda_c \psi + \phi_c(\tau,k) \\ -\lambda_c(x+\eta) - f_c(x, u(\tau,k)) \\ 1 \\ 0 \end{bmatrix} \quad \forall \xi \in \widetilde{C}_g$$

$$\widetilde{G}_g(\xi) := \begin{bmatrix} g_d(x, u(\tau,k)) + \phi_d(\tau,k)\theta \\ \tilde{\theta} - \frac{\psi^{+\top}\psi^+}{\gamma_d + |\psi^+|^2}\tilde{\theta} - \frac{\psi^{+\top}}{\gamma_d + |\psi^+|^2}\varepsilon^+ \\ (1-\lambda_d)\psi + \phi_d(\tau,k) \\ (1-\lambda_d)(x+\eta) - g_d(x, u(\tau,k)) \\ \tau \\ k+1 \end{bmatrix} \quad \forall \xi \in \widetilde{D}_g,$$

with ε as in (4.11) and $\varepsilon^+ = x^+ + \eta^+ - \psi^+\theta$, where x^+ gives the plant state x after a jump per (4.1).

The hybrid system $\mathcal{H}$ in (3.8) reduces to $\widetilde{\mathcal{H}}_g$ in (4.15) when $\vartheta = \tilde{\theta}$, and

$$A(\tau,k) = \gamma_c \psi(\tau,k)^\top \psi(\tau,k) \tag{4.16a}$$

$$d_c(\tau,k) = -\gamma_c \psi(\tau,k)^\top \varepsilon(\tau,k) \tag{4.16b}$$

for all $(\tau, k) \in E$, and[10]

$$B(\tau, k) = \frac{\psi(\tau, k+1)^\top \psi(\tau, k+1)}{\gamma_d + |\psi(\tau, k+1)|^2} \tag{4.16c}$$

$$d_d(\tau, k) = -\frac{\psi(\tau, k+1)^\top}{\gamma_d + |\psi(\tau, k+1)|^2} \varepsilon(\tau, k+1) \tag{4.16d}$$

for all $(\tau, k) \in \Upsilon(E)$, with Υ as in (2.2), where $\varepsilon = x + \eta - \psi\theta$ is a hybrid disturbance, and x, η, ψ are generated by the dynamics in (4.15).

Then, to prove Theorem 4.7, we require the following results for $\widetilde{\mathcal{H}}_g$.

Lemma 4.8: Given the hybrid system $\widetilde{\mathcal{H}}_g$ *in* (4.15)*, for each* $\lambda_c > 0$, $\lambda_d \in (0, 2)$*, and each solution* $\xi = (x, \tilde{\theta}, \psi, \eta, \tau, k)$ *to* $\widetilde{\mathcal{H}}_g$, $(t, j) \mapsto \varepsilon(t, j) := x(t, j) + \eta(t, j) - \psi(t, j)\theta$ *in* (4.11) *satisfies*

$$|\varepsilon(t, j)| \leq \mathrm{e}^{-b(t+j)} |\varepsilon(0, 0)| \quad \forall (t, j) \in \operatorname{dom} \xi, \tag{4.17}$$

where $b := \frac{1}{2} \min \{2\lambda_c, \ -\ln(1 - \lambda_d(2 - \lambda_d))\}$.

Proof. This proof is in Appendix C.1. □

Lemma 4.9: Given the hybrid system $\widetilde{\mathcal{H}}_g$ *in* (4.15)*, suppose that Assumption 4.5 holds and let* $\phi_M > 0$ *come from that assumption. Then, for each* $\psi_0 \geq 0$, $\lambda_c > 0$, $\lambda_d \in (0, 2)$*, the* ψ *component of each solution* ξ *to* $\widetilde{\mathcal{H}}_g$ *from* $\xi(0, 0) \in \mathcal{X}_0 := \{\xi \in \mathcal{X}_g : |\psi|_{\mathrm{F}} \leq \psi_0\}$ *satisfies*

$$|\psi(t, j)| \leq \psi_M \tag{4.18}$$

for all $(t, j) \in \operatorname{dom} \xi$*, where*

$$\psi_M := \psi_0 + \max \left\{ \frac{1}{\lambda_c}, \frac{\sqrt{2\lambda_d(2 - \lambda_d) + 16}}{\lambda_d(2 - \lambda_d)} \right\} \phi_M.$$

[10]Note that B is evaluated only at jump times in (3.8), and $B(\tau, k)$ in (4.16c) is well defined for all $(\tau, k) \in E$ such that $(\tau, k+1) \in E$. Furthermore, the expression for d_d in (4.16d) includes the value of the disturbance ε after a jump, which results in a noncausal algorithm – see Remark 4.4.

Proof. This proof is in Appendix C.2. □

We now have all the ingredients to prove Theorem 4.7.

Proof of Theorem 4.7: To prove Theorem 4.7, we show that the error dynamics of $\mathcal{H}_g$ – that is, $\widetilde{\mathcal{H}}_g$ in (4.15) – satisfy the conditions of Theorem 3.6 with A, B, d_c, d_d in (4.16). Beginning with Assumption 3.1, since A, B in (4.16) are symmetric and $\gamma_c, \gamma_d > 0$, it follows that they are positive semidefinite. Hence, items 1 and 2 of Assumption 3.1 hold. Next, we show that items 3 and 4 of Assumption 3.1 hold. Since, by Assumption 4.5, the conditions of Lemma 4.9 are satisfied, it follows that, for each solution ξ to $\widetilde{\mathcal{H}}_g$ from $\mathcal{X}_0$, the ψ component of ξ satisfies (4.18). Thus, $|A(t,j)| \leq \gamma_c |\psi(t,j)|^2 \leq \gamma_c \psi_M^2$ for all $(t,j) \in E$, with ψ_M from Lemma 4.9, and $|B(t,j)| = \frac{|\psi(t,j+1)|^2}{\gamma_d + |\psi(t,j)|^2} < 1$ for all $(t,j) \in \Upsilon(E)$. Hence, items 3 and 4 of Assumption 3.1 hold with a_M in Theorem 4.7.

Next, using Lemma 4.9 and Assumption 4.6, we show that Assumption 3.2 holds with A, B in (4.16). Substituting A, B into (3.5), we have that, for all $(t', j'), (t^*, j^*) \in E$ satisfying (4.12),

$$\begin{aligned}
&\sum_{j=j'}^{j^*} \int_{\max\{t', t_j\}}^{\min\{t^*, t_{j+1}\}} \gamma_c \psi(s,j)^\top \psi(s,j) ds \\
&\quad + \frac{1}{2} \sum_{j=j'}^{j^*-1} \frac{\psi(t_{j+1}, j+1)^\top \psi(t_{j+1}, j+1)}{\gamma_d + |\psi(t_{j+1}, j+1)|^2} \\
&\geq \min\left\{\gamma_c, \frac{1}{2(\gamma_d + \psi_M^2)}\right\} \mu I.
\end{aligned}$$

Hence, Assumption 3.2 holds with μ_0 and Δ from Theorem 4.7.

Finally, we show that (3.16) is satisfied with d in (3.15) and d_c, d_d in (4.16). By Assumption 4.5, it follows from Lemmas 4.8 and 4.9 that, for each solution ξ to $\widetilde{\mathcal{H}}_g$ from $\mathcal{X}_0$, $|d_c(t,j)| \leq \gamma_c |\psi(t,j)||\varepsilon(t,j)| \leq \gamma_c \psi_M \mathrm{e}^{-b(t+j)} |\varepsilon(0,0)|$ for all $(t,j) \in \operatorname{dom} \xi$, with ψ_M from Lemma 4.9 and b from Lemma 4.8. Furthermore, using that

$\frac{|\psi(t,j+1)|}{\gamma_d+|\psi(t,j+1)|^2} \leq \frac{1}{2\sqrt{\gamma_d}}$ for all $(t,j) \in \Upsilon(\operatorname{dom}\xi)$, we have that $|d_d(t,j)| \leq \frac{1}{2\sqrt{\gamma_d}}|\varepsilon(t,j+1)| \leq \frac{1}{2\sqrt{\gamma_d}}\mathrm{e}^{-b(t+j+1)}|\varepsilon(0,0)| \leq \frac{1}{2\sqrt{\gamma_d}}\mathrm{e}^{-b(t+j)}|\varepsilon(0,0)|$ for all $(t,j) \in \Upsilon(\operatorname{dom}\xi)$. Thus, we conclude that (3.16) holds with a and b from Theorem 4.7 and $|d(0,0)| = |\varepsilon(0,0)|$. Hence, the conditions of Theorem 3.6 hold and, from the equivalence between the data of $\widetilde{\mathcal{H}}_g$ in (4.15) and $\mathcal{H}$ in (3.8) with A, B, d_c, d_d in (4.16),[11] we have from Theorem 3.6 that the $\tilde{\theta}$ component of each solution ξ to $\widetilde{\mathcal{H}}_g$ from $\mathcal{X}_0$ satisfies

$$|\tilde{\theta}(t,j)| \leq \kappa \mathrm{e}^{-\lambda(t+j)}\big(|\tilde{\theta}(0,0)| + |\varepsilon(0,0)|\big) \tag{4.19}$$

for all $(t,j) \in \operatorname{dom}\xi$, with κ, λ in (3.18), and p_m, p_M, ρ, ω from Theorem 4.7.

To conclude the proof, using the definition of $\mathcal{A}_g$ in (4.10), we rewrite $|\xi|_{\mathcal{A}_g}$ for all $(t,j) \in \operatorname{dom}\xi$ as $|\xi(t,j)|_{\mathcal{A}_g} = \sqrt{|\tilde{\theta}(t,j)|^2 + |\varepsilon(t,j)|^2}$. Substituting the bounds in (4.17) and (4.19) and using that $\kappa \geq 1$ and, for any $\alpha, \beta \in \mathbb{R}$, $\alpha\beta \leq \frac{1}{2}(\alpha^2 + \beta^2)$, we conclude that, for all $(t,j) \in \operatorname{dom}\xi$,

$$|\xi(t,j)|_{\mathcal{A}_g} \leq \sqrt{3}\kappa \mathrm{e}^{-\min\{\lambda,b\}(t+j)}\sqrt{|\tilde{\theta}(0,0)|^2 + |\varepsilon(0,0)|^2}.$$

Hence, (4.14) holds with κ_g, λ_g as in Theorem 4.7. □

4.4 Robustness Analysis

In this section, we study the robustness properties induced by $\mathcal{H}_g$ with respect to bounded (hybrid) noise on the state measurements.

Given $\phi_c, \phi_d : E \to \mathbb{R}^{n \times p}$ and $u : E \to \mathbb{R}^m$, where $E := \operatorname{dom}\phi_c = \operatorname{dom}\phi_d = \operatorname{dom} u$ is a hybrid time domain, consider additive noise $\nu : E \to \mathbb{R}^n$ in the measurements of the plant state x in (4.1).[12] We denote the hybrid system

[11] In other words, by substituting A, B, d_c, d_d in (4.16) into (3.8) and treating ψ as a given hybrid signal and ε as hybrid disturbance satisfying (4.15), we obtain a hybrid system with dynamics that are equivalent to $\widetilde{\mathcal{H}}_g$ in (4.15).

[12] By definition of a solution pair, the measurement noise ν has the same hybrid time domain as x, ϕ_c, ϕ_d, and u.

$\mathcal{H}$ in (4.2) under the effect of the measurement noise ν as $\mathcal{H}_\nu$, with state $\xi = (x, \hat{\theta}, \psi, \eta, \tau, k) \in \mathcal{X}_g$ and dynamics

$$\mathcal{H}_\nu : \begin{cases} \dot{\xi} = F_\nu(\xi) & \xi \in C_\nu \\ \xi^+ = G_\nu(\xi) & \xi \in D_\nu \end{cases} \tag{4.20}$$

where

$$F_\nu(\xi) := \begin{bmatrix} f_c(x, u(\tau,k)) + \phi_c(\tau,k)\theta \\ \gamma_c \psi^\top (y_\nu - \psi\hat{\theta}) \\ -\lambda_c \psi + \phi_c(\tau,k) \\ -\lambda_c(x + \nu(\tau,k) + \eta) - f_c(x + \nu(\tau,k), u(\tau,k)) \\ 1 \\ 0 \end{bmatrix}$$

$$G_\nu(\xi) := \begin{bmatrix} g_d(x, u(\tau,k)) + \phi_d(\tau,k)\theta \\ \hat{\theta} + \frac{\psi^{+\top}}{\gamma_d + |\psi^+|^2}(y_\nu^+ - \psi^+\hat{\theta}) \\ (1-\lambda_d)\psi + \phi_d(\tau,k) \\ (1-\lambda_d)(x+\nu(\tau,k)+\eta) - g_d(x+\nu(\tau,k), u(\tau,k)) \\ \tau \\ k+1 \end{bmatrix}$$

where $C_\nu := C_g$, $D_\nu := D_g$, with C_g, D_g in (4.3), and we define $y_\nu := x + \nu(\tau,k) + \eta$ and $y_\nu^+ := x^+ + \nu(\tau, k+1) + \eta^+$, where x^+ gives the plant state x after a jump per (4.1).

For analyzing the effect of the noise, we make the following Lipschitz continuity assumption.

Assumption 4.10: Given the hybrid plant in (4.1), there exist $L_c, L_d > 0$ such that, for all $x_1, x_2 \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$|f_c(x_1, u) - f_c(x_2, u)| \le L_c |x_1 - x_2|,$$
$$|g_d(x_1, u) - g_d(x_2, u)| \le L_d |x_1 - x_2|.$$

We now establish our main robustness result stating conditions that ensure $\mathcal{A}_g$ in (4.10) is ISS for $\mathcal{H}_\nu$.

Theorem 4.11: Given the hybrid system $\mathcal{H}_\nu$ in (4.20), $\gamma_c, \lambda_c, \gamma_d > 0$, and $\lambda_d \in (0, 2)$, suppose that Assumptions 4.5, 4.6, and 4.10 hold. Then, for each $\psi_0 \geq 0$, $q_M \geq q_m > 0$, and each $\zeta \in (0, 1)$, each solution ξ to $\mathcal{H}_\nu$ from $\xi(0,0) \in \mathcal{X}_0$ satisfies

$$|\xi(t,j)|_{\mathcal{A}_g} \leq \kappa_\nu \mathrm{e}^{-\lambda_\nu (t+j)} |\xi(0,0)|_{\mathcal{A}_g} + \rho_\nu d_\nu(t,j) \tag{4.21}$$

for all $(t,j) \in \operatorname{dom} \xi$, where

$$\begin{aligned}
\kappa_\nu &:= \sqrt{\frac{2p_M}{p_m}}, \quad \lambda_\nu := \min\{\omega, \lambda_\varepsilon\}, \quad \rho_\nu := \sqrt{2} \max\{\rho, \rho_\varepsilon\} \\
\lambda_\varepsilon &:= \frac{1}{2} \min\left\{ \lambda_c(1-\zeta), -\ln\left(1 - \frac{\lambda_d}{2}(2-\lambda_d)(1-\zeta)\right)\right\} \\
\rho_\varepsilon &:= \max\left\{ \frac{2}{\lambda_c \sqrt{\zeta}}, \frac{\sqrt{2\lambda_d(2-\lambda_d)+16}}{\lambda_d(2-\lambda_d)\sqrt{\zeta}} \right\}
\end{aligned}$$

with ρ, ω from Theorem 3.5, p_m, p_M from Theorem 4.7, $d_\nu(t,j) := \sqrt{\|d\|^2_{(t,j)} + \|d_\varepsilon\|^2_{(t,j)}}$, with d as in (3.15) and

$$d_\varepsilon(t,j) := \begin{cases} \alpha_c(t,j) & \text{if} \quad (t,j) \in E \setminus \Upsilon(E) \\ \alpha_d(t,j) & \text{if} \quad (t,j) \in \Upsilon(E), \end{cases} \tag{4.22}$$

where, for all $(t,j) \in E$,

$$d_c(t,j) := -\gamma_c \psi(t,j)^\top (\varepsilon(t,j) + \nu(t,j)) \tag{4.23a}$$

$$\begin{aligned}
\alpha_c(t,j) := {}& -\lambda_c \nu(t,j) + f_c(x(t,j), u(t,j)) \\
& - f_c(x(t,j) + \nu(t,j), u(t,j))
\end{aligned} \tag{4.23b}$$

and, for all $(t,j) \in \Upsilon(E)$,

$$\begin{aligned}
d_d(t,j) := {}& -\frac{\psi(t,j+1)^\top}{\gamma_d + |\psi(t,j+1)|^2} \big(\varepsilon(t,j+1) \\
& + \nu(t,j+1)\big),
\end{aligned} \tag{4.23c}$$

$$\begin{aligned}
\alpha_d(t,j) := {}& (1-\lambda_d)\nu(t,j) + g_d(x(t,j), u(t,j)) \\
& - g_d(x(t,j) + \nu(t,j), u(t,j))
\end{aligned} \tag{4.23d}$$

with ε *as in* (4.11)*. Moreover, for all* $(t,j) \in E$*,*

$$
\begin{aligned}
|d_c(t,j)| &\le \gamma_c \psi_M \big(\mathrm{e}^{-\lambda_\varepsilon (t+j)} |\varepsilon(0,0)| \\
&\qquad + (\rho_\varepsilon \max\{\lambda_c + L_c, 1 - \lambda_d + L_d\} + 1)|\nu(t,j)|\big) \\
|\alpha_c(t,j)| &\le (\lambda_c + L_c)|\nu(t,j)|
\end{aligned}
$$

and for all $(t,j) \in \Upsilon(E)$*,*

$$
\begin{aligned}
|d_d(t,j)| &\le \frac{1}{2\sqrt{\gamma_d}} \big(\mathrm{e}^{-\lambda_\varepsilon (t+j+1)} |\varepsilon(0,0)| \\
&\qquad + (\rho_\varepsilon \max\{\lambda_c + L_c, 1 - \lambda_d + L_d\} + 1)|\nu(t,j+1)|\big) \\
|\alpha_d(t,j)| &\le (1 - \lambda_d + L_d)|\nu(t,j)|
\end{aligned}
$$

with ψ_M *from Assumption 4.5,* L_c*,* L_d *from Assumption 4.10, and* $\varepsilon(0,0) = x(0,0) + \eta(0,0) - \psi(0,0)\theta$*.*

To prove Theorem 4.11, we require the following result.

Lemma 4.12: Given the hybrid system $\mathcal{H}_\nu$ *in* (4.20)*, suppose that Assumption 4.10 holds. Then, for each* $\lambda_c > 0$*,* $\lambda_d \in (0,2)$*,* $\zeta \in (0,1)$*, and each solution* $\xi = (x, \hat{\theta}, \psi, \eta, \tau, k)$ *to* $\mathcal{H}_\nu$*,* $(t,j) \mapsto \varepsilon(t,j) := x(t,j) + \eta(t,j) - \psi(t,j)\theta$ *in* (4.11) *satisfies, for all* $(t,j) \in \operatorname{dom} \xi$*,*

$$
|\varepsilon(t,j)| \le \mathrm{e}^{-\lambda_\varepsilon (t+j)} |\varepsilon(0,0)| + \rho_\varepsilon \|d_\varepsilon\|_{(t,j)} \tag{4.24}
$$

with $\lambda_\varepsilon, \rho_\varepsilon > 0$ *and* $(t,j) \mapsto d_\varepsilon(t,j)$ *from Theorem 4.11.*

Proof. This proof is given in Appendix C.3. □

We now have all the ingredients to prove Theorem 4.11.

Proof of Theorem 4.11: Using the same arguments as in the proof of Theorem 4.7, we conclude that, by Assumptions 4.5 and 4.6, the conditions of

Theorem 3.5 are satisfied with μ_0 and a_M from Theorem 4.7. It can be shown that, under Assumption 4.10, the hybrid system that is obtained by expressing $\mathcal{H}_\nu$ in error coordinates is equivalent to $\mathcal{H}$ in (3.8) with A, B in (4.16) and d_c, d_d in (4.23). Hence, it follows from Theorem 3.5 that, for each solution ξ to $\mathcal{H}_\nu$ from $\mathcal{X}_0$, the parameter estimation error $\tilde{\theta} = \theta - \hat{\theta}$ satisfies, for all $(t,j) \in \operatorname{dom}\xi$,

$$|\tilde{\theta}(t,j)| \leq \sqrt{\frac{p_M}{p_m}} \mathrm{e}^{-\omega(t+j)} |\tilde{\theta}(0,0)| + \rho \|d\|_{(t,j)}, \tag{4.25}$$

with $(t,j) \mapsto d(t,j)$ as in (3.15) and ρ, ω from Theorem 3.5, with p_m, p_M substituted by p_m, p_M from Theorem 4.7.

Using the definition of $\mathcal{A}_g$ in (4.10), we rewrite $|\xi|_{\mathcal{A}_g}$ for all $(t,j) \in \operatorname{dom}\xi$ as $|\xi(t,j)|_{\mathcal{A}_g} = \sqrt{|\tilde{\theta}(t,j)|^2 + |\varepsilon(t,j)|^2}$. Since, by Assumption 4.10, the conditions of Lemma 4.12 are satisfied, we substitute the bounds in (4.24) and (4.25). Using that, for any $\alpha, \beta \in \mathbb{R}$, $\alpha\beta \leq \frac{1}{2}(\alpha^2 + \beta^2)$, we obtain $|\xi(t,j)|_{\mathcal{A}_g} \leq \sqrt{\frac{2p_M}{p_m}} \mathrm{e}^{-\min\{\omega,\lambda_\varepsilon\}(t+j)} |\xi(0,0)|_{\mathcal{A}_g} + \sqrt{2}\max\{\rho, \rho_\varepsilon\} \sqrt{\|d\|^2_{(t,j)} + \|d_\varepsilon\|^2_{(t,j)}}$ for all $(t,j) \in \operatorname{dom}\xi$. Hence, (4.21) holds.

To conclude the proof, we upper bound d_c, d_d, α_c, and α_d for all $(t,j) \in \operatorname{dom}\xi$. The bounds for α_c and α_d in Theorem 4.11 follow directly from Assumption 4.10 and the definitions of α_c and α_d in (4.23). Moreover, since, by Assumptions 4.5 and 4.10, the conditions of Lemmas 4.9 and 4.12 are satisfied, we have from (4.23) that, for each solution ξ to $\mathcal{H}_\nu$ from $\mathcal{X}_0$,

$$\begin{aligned} |d_c(t,j)| &\leq \gamma_c |\psi(t,j)| (|\varepsilon(t,j)| + |\nu(t,j)|) \\ &\leq \gamma_c \psi_M (\mathrm{e}^{-\lambda_\varepsilon(t+j)} |\varepsilon(0,0)| \\ &\quad + (\rho_\varepsilon \max\{\lambda_c + L_c, 1 - \lambda_d + L_d\} + 1) |\nu(t,j)|) \end{aligned}$$

for all $(t,j) \in \operatorname{dom}\xi$, with λ_ε and ρ_ε from Theorem 4.11, where the last inequality follows from (4.24) the definition of d_ε in (4.22). Next, using that $\frac{|\psi(t,j+1)|}{\gamma_d + |\psi(t,j+1)|^2} \leq$

$\frac{1}{2\sqrt{\gamma_d}}$ for all $(t,j) \in \Upsilon(\operatorname{dom}\xi)$, we have that, for all $(t,j) \in \Upsilon(\operatorname{dom}\xi)$,

$$\begin{aligned} |d_d(t,j)| &\le \frac{1}{2\sqrt{\gamma_d}}\big(\varepsilon(t,j+1) + \nu(t,j+1)\big) \\ &\le \frac{1}{2\sqrt{\gamma_d}}(\mathrm{e}^{-\lambda_\varepsilon(t+j+1)}|\varepsilon(0,0)| \\ &\quad +(\rho_\varepsilon \max\{\lambda_c + L_c, 1-\lambda_d + L_d\} + 1)|\nu(t,j+1)|). \end{aligned}$$ □

Remark 4.13: A similar ISS result as in Theorem 4.11 can be developed without Assumption 4.10 by constraining the range of the plant state x and the input u to a compact set. Under such conditions, it follows from the continuity of f_c and g_d that d_c, d_d, α_c, α_d in (4.23) can be upper bounded by functions of only ν. Then, ISS follows from similar arguments as in the proof of Theorem 4.11.

4.5 Numerical Examples

4.5.1 Comparison with Continuous-Time and Discrete-Time Gradient Descent

Consider the hybrid arcs $\phi_c, \phi_d : E \to \mathbb{R}^{2\times 2}$ with hybrid time domain

$$E = \bigcup_{k=0}^{\infty} \big([2\pi k,\ \pi(2k+2)] \times \{k\}\big). \tag{4.26}$$

The values of ϕ_c and ϕ_d are

$$\phi_c(t,j) = \begin{bmatrix} \sin(t) & 0 \\ 0 & 0 \end{bmatrix}, \qquad \phi_d(t,j) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

for all $(t,j) \in E$. For such ϕ_c and ϕ_d, consider a hybrid system as in (2.1) with an added input[13] $u : E \to \mathbb{R}$, state $x = (x_1, x_2) \in \mathbb{R}^2$, and dynamics

$$\begin{aligned} \dot{x} &= \phi_c(t,j)\theta \qquad (x, u(t,j)) \in C_P \\ x^+ &= \phi_d(t,j)\theta \qquad (x, u(t,j)) \in D_P, \end{aligned} \tag{4.27}$$

[13] See [45] for details on hybrid systems with inputs.

where $\theta = [1\ \ 1]^\top$ is a vector of unknown parameters. The flow and jump sets are $C_P = (\mathbb{R}^2 \times \mathbb{R}) \setminus D_p$ and $D_P = \{(x, u) \in \mathbb{R}^2 \times \mathbb{R} : u \geq 2\pi\}$, respectively. The input $u(t, j) = t - 2\pi j$ for all $(t, j) \in E$ is a sawtooth function that periodically ramps to a value of 2π and then resets to zero.[14]

Given a solution x to (4.27) from $x(0, 0) \in (3, 6)$, we wish to estimate the parameter vector θ. To do so, we first separately analyze the flows and jumps of the input and output signals. We define the continuous-time signals

$$\bar{x}_c(t) := \begin{bmatrix} 4 - \cos(t) \\ 6 \end{bmatrix}, \qquad \bar{\phi}_c(t) := \begin{bmatrix} \sin(t) & 0 \\ 0 & 0 \end{bmatrix}$$

for all $t \geq 0$, which are obtained by neglecting the resets of x and ϕ_c, respectively, at jumps. The signals $\bar{x}_c$ and $\bar{\phi}_c$ are solutions to the continuous-time system $\dot{\bar{x}}_c(t) = \bar{\phi}_c(t)\theta$ for all $t \geq 0$. Next, we define the discrete-time signals

$$\bar{x}_d(j) := \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \qquad \phi_d(j) := \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

for all $j \in \mathbb{N}$, which are obtained by neglecting the evolution of x and ϕ_d, respectively, during flows. The signals $\bar{x}_d$ and $\bar{\phi}_d$ are solutions to the discrete-time system $\bar{x}_d(j) = \bar{\phi}_d(j)\theta$ for all $j \in \mathbb{N}$. Finally, using the transformations in Section 4.2, we express $\bar{x}_c$ and $\bar{x}_d$ as the outputs of linear regressions models and apply the continuous-time and discrete-time estimation algorithms (4.6) and (4.7), respectively, to estimate the unknown parameter θ. Both algorithms fail to converge, as shown in blue and green in Figure 1.1.

To see why the continuous-time algorithm fails to converge, note that for $\bar{\phi}_c$, the value of $t \mapsto \psi(t)$ in (4.4) is

$$\psi(t) = \mathrm{e}^{-\lambda_c t}\psi(0) + \int_0^t \mathrm{e}^{-\lambda_c(t-s)}\bar{\phi}_c(s)ds$$

[14]With C_P, D_P, and u given below (4.27), the hybrid time domain of each maximal solution to the hybrid system in (4.27) is equal to the hybrid time domain E of ϕ_c and ϕ_d.

for all $t \geq 0$. Since $e^{-\lambda_c t}\psi(0)$ converges exponentially to zero and the second column of $\bar{\phi}_c$ is zero, it follows that $t \mapsto \psi(t)$ does not satisfy the continuous-time PE condition (2.13) for any $T > 0$. Similarly for $\bar{\phi}_d$, the value of $j \mapsto \psi(j)$ in (4.8) is

$$\psi(j) = (1 - \lambda_d)^j \psi(0) + \sum_{i=0}^{j-1} (1 - \lambda_d)^{(j-i-1)} \bar{\phi}_d(i)$$

for all $j \in \mathbb{N}$. Since $(1-\lambda_d)^j \psi(0)$ converges exponentially to zero and $\bar{\phi}_d$ is constant and not full rank, it follows that $j \mapsto \psi(j)$ does not satisfy the discrete-time PE condition (2.15) for any $J \in \mathbb{N} \setminus \{0\}$.

We now employ $\mathcal{H}_g$ in (4.2) to estimate θ in (4.27). The system (4.27) can be written as (4.1) by setting f_c, g_d in (4.1) to zero. The algorithm is simulated for $\gamma_c = 1$, $\lambda_c = 0.1$, $\gamma_d = 1$, and $\lambda_d = 0.5$ alongside the continuous-time and discrete-time estimation algorithms from Section 2.2 with the same parameters, where applicable. To illustrate the robustness of our algorithm, we also simulate $\mathcal{H}_g$ with additive noise $(t, j) \mapsto \nu(t, j) = \sin(2t)$ in the measurements of x. It can be shown that the solution component ψ satisfies the hybrid PE condition (4.13) with $\Delta = 2\pi + 1$ and $\mu = 5.1$.

The simulation is performed from two separate initial conditions: one with $\varepsilon(0, 0) = 0$ and one with $\varepsilon(0, 0) \neq 0$. In particular, $x(0, 0) = (3, 6)$, $\hat{\theta}(0, 0) = (0, 0)$, and

1. $\psi(0, 0) = 0,\ \eta(0, 0) = -(3, 6) \implies \varepsilon(0, 0) = (0, 0)$

2. $\psi(0, 0) = 0,\ \eta(0, 0) = -(1.5, 3) \implies \varepsilon(0, 0) = (1.5, 3)$

producing the results in Figure 4.1. When no noise is present, $|\xi|_{\mathcal{A}_g}$ converges exponentially to zero in accordance with Theorem 4.7, as shown in blue for the case with $\varepsilon(0, 0) = 0$ and in green for the case with $\varepsilon(0, 0) \neq 0$. When noise is present, $|\xi|_{\mathcal{A}_g}$ remains bounded in accordance with Theorem 4.11, as shown in orange in Figure 4.1.

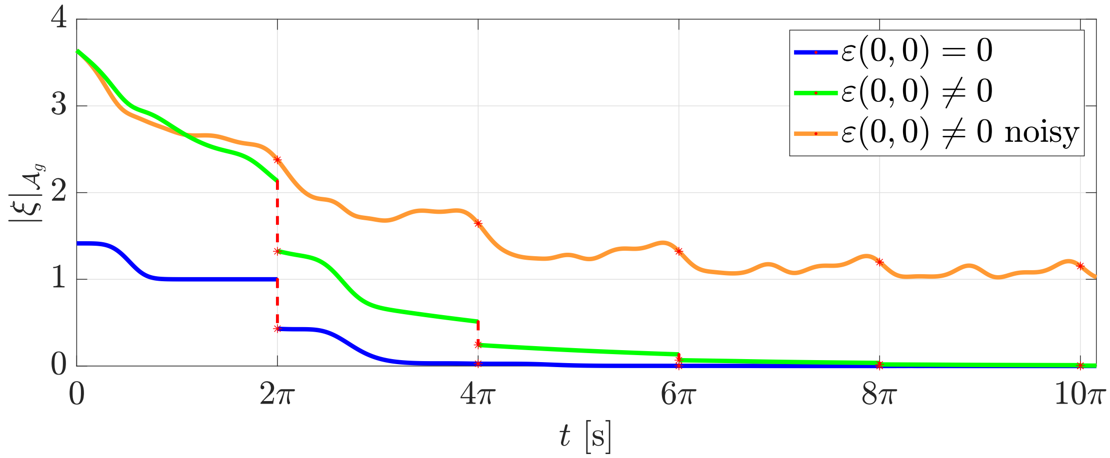

Figure 4.1: The projection onto t of $|\xi|_{\mathcal{A}_g}$ for $\mathcal{H}_g$.

4.5.2 Spacecraft Bias Torque Estimation

Consider the problem of estimating a constant disturbance torque applied to a spacecraft, controlled by reaction wheels (RW) and reaction control system (RCS) thrusters. Such bias torques may arise in practice due to aerodynamic effects, gravity gradients, or solar radiation pressure differentials. For simplicity, we consider the dynamics of a spacecraft rotating about only a single principle axis of inertia, although our approach can be extended to three-axis rotation. In the following, we derive the closed-loop dynamics of the spacecraft when controlled by RW and RCS thrusters separately, and then combine the results into a single hybrid model.

The dynamics of a spacecraft rotating along a principle axis of inertia under the effect of RW are [51]

$$J_s \ddot{z} = -J_w \dot{\Omega} + \theta, \tag{4.28}$$

where $z \in \mathbb{R}$ is the known pointing angle of the spacecraft, $\Omega \in \mathbb{R}$ is the known rotational velocity of the RW, $J_s > 0$ is the known spacecraft moment of inertia, $J_w > 0$ is the known RW moment of inertia, and $\theta \in \mathbb{R}$ is an unknown bias torque.

Suppose RW control the attitude to a pointing angle, $z_{\text{des}} \in \mathbb{R}$. The dynamics of the reaction wheel are [51]

$$J_w \dot{\Omega} = \alpha(t), \tag{4.29}$$

where $t \mapsto \alpha(t) \in \mathbb{R}$ is the RW motor torque that is designed to maintain the spacecraft pointing angle. Substituting (4.29) into (4.28), we obtain

$$J_s \ddot{z} = -\alpha(t) + \theta. \tag{4.30}$$

When the bias torque is nonzero, the industry-standard proportional-derivative (PD) control scheme for the RW motor fails to yield zero pointing error in steady-state. In this case, a feedfoward term is added that compensates for the effect of the bias torque using an estimate of the bias, denoted by $\hat{\theta}$ [51]. Hence, the RW torque is

$$-\alpha(t) = K_P(z_{\text{des}} - z(t)) - K_D \dot{z}(t) - \hat{\theta}(t), \tag{4.31}$$

where $K_P, K_D > 0$ are design parameters. From (4.29), (4.30), (4.31), the dynamics of the closed-loop system are

$$\ddot{z} = \frac{-\alpha(t) + \theta}{J_s}, \qquad \dot{\Omega} = \frac{\alpha(t)}{J_w}. \tag{4.32}$$

The spacecraft pointing angle can be maintained only if an equivalent RW torque is delivered to counteract the bias torque. If the bias torque is nonzero, the angular velocity of the RW constantly increases in order to counteract the disturbance and the RW motor eventually reaches its maximum angular velocity. In order to avoid the RW motor from becoming saturated, "momentum dumping" is applied to decrease the angular velocity of the RW [51]. This procedure involves firing the

RCS thrusters to generate a torque that is compensated by the attitude controller by actions that cause the RW to reduce their angular momentum.

The dynamics of a spacecraft rotating along a principle axis of inertia under the effect of RCS thrusters are [51]

$$J_s \ddot{z} = M + \theta, \tag{4.33}$$

where $M \in \mathbb{R}$ is the known RCS thruster torque. For simplicity, we assume that the velocity of the RW is constant for the duration of each thruster firing. As a result, the RW dynamics do not play a role in (4.33).

Suppose that, at time $t \geq 0$, the thrusters are fired for $\delta > 0$ seconds. Integrating (4.33) over the time interval $[t, t + \delta]$ yields $\dot{z}(t + \delta) = \dot{z}(t) + \frac{\delta}{J_s}(M + \theta)$. If the thruster firing duration δ is negligibly small compared to the other time scales of the system, which is appropriate due to the slow spacecraft attitude maneuvering, we model the thruster firing as an instantaneous jump in the angular velocity of the spacecraft, given by

$$\dot{z}^+ = \dot{z} + \frac{\delta}{J_s}(M + \theta). \tag{4.34}$$

To avoid chatter, a timer, denoted by τ_s, is used to briefly inhibit the RCS thrusters after each thruster firing. Each time the thrusters are fired, the timer is reset to zero.

By combining the expression in (4.32) and (4.34), we express the closed-loop dynamics of the spacecraft as a hybrid system as in (4.1). Given an input $u := (z_{\text{des}}, \hat{\theta})$, where $z_{\text{des}} \in \mathbb{R}$ is the desired constant spacecraft pointing angle and $\hat{\theta} \in \mathbb{R}$ is an estimate of the unknown bias torque, the hybrid model of the spacecraft has state $x = (z, \dot{z}, \Omega, \tau_s) \in \mathbb{R}^4$ and data

$$f_c(x, u(t,j)) := \begin{bmatrix} \dot{z} \\ -\frac{1}{J_s}\alpha(x, u(t,j)) \\ \frac{1}{J_w}\alpha(x, u(t,j)) \\ 1 \end{bmatrix}, \phi_c(t,j) := \begin{bmatrix} 0 \\ \frac{1}{J_s} \\ 0 \\ 0 \end{bmatrix}$$

$$g_d(x, u(t,j)) := \begin{bmatrix} z \\ \dot{z} + \frac{\delta}{J_s}M \\ \Omega \\ 0 \end{bmatrix}, \qquad \phi_d(t,j) := \begin{bmatrix} 0 \\ \frac{\delta}{J_s} \\ 0 \\ 0 \end{bmatrix}$$

where $\alpha(x, u(t,j)) := -K_P(z_{\text{des}} - z) + K_D\dot{z} + \hat{\theta}(t,j)$. The flow and jump sets of the hybrid spacecraft model implement the momentum dumping procedure. The system jumps each time the angular velocity of the RW exceeds a design parameter $\Omega_{\max} > 0$ and the timer τ_s exceeds a design parameter $\tau^* > 0$, and flows otherwise, as

$$C_P := \{x \in \mathbb{R}^4 : \Omega \leq \Omega_{\max}\} \cup \{x \in \mathbb{R}^4 : \tau_s \leq \tau^*\}$$
$$D_P := \{x \in \mathbb{R}^4 : \Omega \geq \Omega_{\max},\ \tau_s \geq \tau^*\}.$$

We employ $\mathcal{H}_g$ to estimate the unknown bias torque. The closed-loop system is simulated[15] with initial conditions $x(0,0) = (0,0,0,0)$, $\hat{\theta}(0,0) = 0$, $\psi(0,0) = 0$, and $\eta(0,0) = -x(0,0)$. The hybrid spacecraft model has parameters $z_{\text{des}} = 0$ rad, $\Omega_{\max} = 10000$ RPM, $J_s = 5000$ kg-m^2, $J_w = 0.1$ kg-m^2, $M = -10$ N-m, $\delta = 9.5$ sec, $\tau^* = 10$ sec, $K_p = 10$, $K_d = 1200$, and with an unknown bias torque of $\theta = 0.005$ N-m. Our algorithm $\mathcal{H}_g$ has parameters $\gamma_c = 0.0012$, $\lambda_c = 0.001$, $\gamma_d = 0.01$, and $\lambda_d = 0.5$. With the initial conditions and design parameters given above, it can be shown numerically that the conditions of Theorem 4.7 hold.

The bias torque estimation error from $\mathcal{H}_g$ converges exponentially to zero in accordance with Theorem 4.7, as shown in Figure 4.2. The spacecraft pointing

[15]Code at `https://github.com/HybridSystemsLab/HybridGD_SpacecraftBiasTorque`

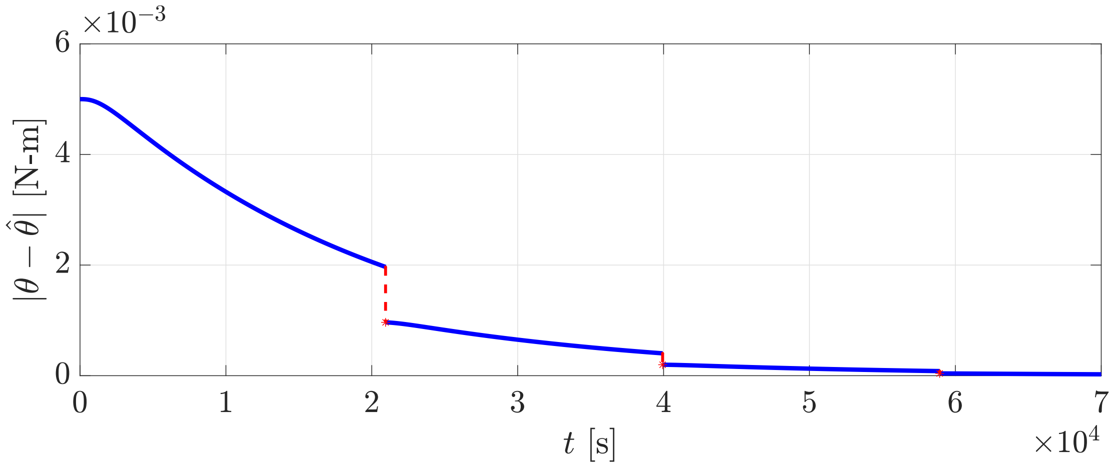

Figure 4.2: The projection onto t of the bias torque estimation error for $\mathcal{H}_g$.

angle error and RW angular velocity are shown in the top and bottom plots, respectively, in Figure 4.3, where the control performance resulting from our hybrid algorithm is compared against an industry-standard PID control scheme that is tuned to achieve a similar pointing error convergence rate during flows. For the PID controller, we inhibit accumulation of the integrator during each thruster firing, otherwise the spacecraft pointing angle fails to converge to the set point. With the exception of the transients caused by the thruster firings, the pointing error converges to zero for both controllers. However, our hybrid algorithm converges faster due to our estimator's ability to leverage information during both flows and jumps to estimate the unknown bias torque.

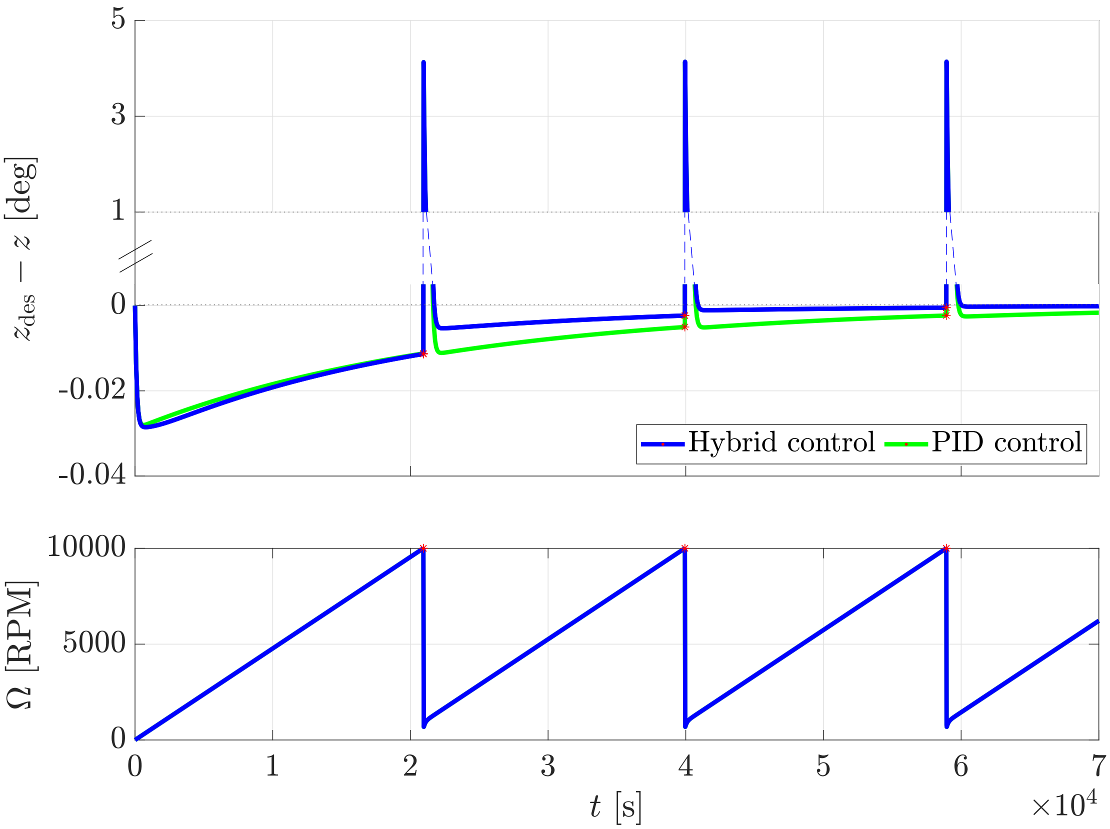

Figure 4.3: The projection onto t of the spacecraft pointing angle error (top) and the RW angular velocity (bottom).

Chapter 5

Parameter Estimation for Hybrid Systems using Data

Inspired by the integral concurrent learning algorithm in Section 2.3, in this chapter we propose a hybrid algorithm to estimate unknown parameters for a class of hybrid dynamical systems. The algorithm stores measurements during flows and jumps of the hybrid system in order to satisfy a (hybrid) richness condition that guarantees convergence of the parameter estimate to the true value.

5.1 Problem Statement

Inspired by the continuous-time integral concurrent learning algorithm [40], we develop a hybrid algorithm for estimating parameters of hybrid dynamical systems as in (4.1), recalled below:

$$\begin{aligned} \dot{x} &= f_c(x, u(t,j)) + \phi_c(t,j)\theta \quad (x, u(t,j)) \in C_P \\ x^+ &= g_d(x, u(t,j)) + \phi_d(t,j)\theta \quad (x, u(t,j)) \in D_P \end{aligned} \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the known state vector, $(t,j) \mapsto u(t,j) \in \mathbb{R}^m$ is the known input, and $(t,j) \mapsto \phi_c(t,j) \in \mathbb{R}^{n\times p}$ and $(t,j) \mapsto \phi_d(t,j) \in \mathbb{R}^{n\times p}$ are known regressors,

$(x, u) \mapsto f_c(x, u)$ and $(x, u) \mapsto g_d(x, u)$ are known continuous functions, $\theta \in \mathbb{R}^p$ is a vector of unknown constant parameters, and $n, m, p \in \mathbb{N}$. The flow set is $C_P \subset \mathbb{R}^n \times \mathbb{R}^m$ and the jump set is $D_P \subset \mathbb{R}^n \times \mathbb{R}^m$. The regressors ϕ_c and ϕ_d and the input u are defined on hybrid time domains as described in Section 2.1, but are not necessarily hybrid arcs.[16]

Our goal is to estimate the parameter vector θ in (5.1). Since ϕ_c and ϕ_d may exhibit both flows and jumps, it is important to update the parameter estimate $\hat{\theta}$ continuously whenever ϕ_c flows, and to update $\hat{\theta}$ discretely each time ϕ_d jumps, which is possible when jumps are detected instantaneously. Hence, we propose to estimate θ using a hybrid algorithm, denoted $\mathcal{H}$, of the form

$$\mathcal{H} : \begin{cases} \dot{\xi} = F(\xi) & \xi \in C \\ \xi^+ = G(\xi) & \xi \in D \end{cases} \tag{5.2}$$

with data designed to solve the following problem.

Problem 5.1: Design the data (C, F, D, G) of $\mathcal{H}$ in (5.2) and determine conditions that ensure the parameter estimate $\hat{\theta}$ converges to the unknown parameter vector θ in (5.1).

Next, we present our solution to this estimation problem.

5.2 Problem Solution

Our proposed hybrid ICL algorithm is inspired by the continuous-time ICL algorithm. We begin by explaining the approach in words before formally defining the algorithm. Let $(t, j) \mapsto \phi_c(t, j)$, $(t, j) \mapsto \phi_d(t, j)$, and $(t, j) \mapsto u(t, j)$ be hybrid signals satisfying (5.1) for all $(t, j) \in E := \operatorname{dom} \phi_c = \operatorname{dom} \phi_d = \operatorname{dom} u$. During

[16] In other words, ϕ_c, ϕ_d, and u do not need to be locally absolutely continuous during flows — see [45] for details.

flows, we compute signals $(t,j) \mapsto \mathcal{Y}_c(t,j)$ and $(t,j) \mapsto \Phi_c(t,j)$ such that $\mathcal{Y}_c$, Φ_c, and θ are related via a linear regression model; that is, $\mathcal{Y}_c(t,j) = \Phi_c(t,j)\theta$ for all $(t,j) \in E$. Similarly, at jumps, we compute signals $(t,j) \mapsto \mathcal{Y}_d(t,j)$ and $(t,j) \mapsto \Phi_d(t,j)$ such that $\mathcal{Y}_d$, Φ_d, and θ are related via a linear regression model; that is, $\mathcal{Y}_d(t,j) = \Phi_d(t,j)\theta$ for all $(t,j) \in E$. Next, samples of the signals $\mathcal{Y}_c$, Φ_c, $\mathcal{Y}_d$, and Φ_d are stored, and the stored data is used alongside current measurement to adapt the parameter estimate. Samples are selected to ensure that the parameter estimation error converges to zero.

Given $\phi_c, \phi_d : E \to \mathbb{R}^{n\times p}$ and $u : E \to \mathbb{R}^m$, where $E := \operatorname{dom}\phi_c = \operatorname{dom}\phi_d = \operatorname{dom} u$ is a hybrid time domain, we define the state ξ of $\mathcal{H}$ as $\xi = (x, \hat{\theta}, \tau, k) \in \mathcal{X} := \mathbb{R}^n \times \mathbb{R}^p \times E$, where x is the state of the plant in (5.1), $\hat{\theta}$ is the estimate of θ, and τ and k correspond to t and j, respectively, from the hybrid time domain E. Including τ and k in ξ allows ϕ_c, ϕ_d, and u to be part of the definitions of F and G, rather than modeled as inputs to $\mathcal{H}$. Thus, we can express $\mathcal{H}$ as an autonomous hybrid system, which allows us to leverage recent results on stability and robustness properties [22, 45]. The flow and jump sets of $\mathcal{H}$ are defined so that $\mathcal{H}$ flows when ϕ_c flows and jumps when ϕ_d jumps. Since $E := \operatorname{dom}\phi_c = \operatorname{dom}\phi_d$,

$$C := \operatorname{cl}\big(\mathcal{X} \setminus D\big), \quad D := \{\xi \in \mathcal{X} : (\tau, k+1) \in E\}. \tag{5.3}$$

Remark 5.2: We assume for simplicity that the plant state x in (5.1) has the same hybrid time domain as ϕ_c, ϕ_d, and u. As a result, the flow set C_P and jump set D_P of the plant are not part of the construction of $\mathcal{H}$. Our algorithm can be extended to the case where x, ϕ_c, ϕ_d, and u have different hybrid time domains by considering C_P and D_P in (5.1). In this case, we can reparameterize the domains of ϕ_c, ϕ_d, and u to express x, ϕ_c, ϕ_d, and u on a common hybrid time domain. See, e.g., [5].

To implement our proposed hybrid ICL algorithm, we first define the symbols $\underline{t} \in \mathbb{R}_{\geq 0}$ and $\underline{j} \in \mathbb{N}$. Given a design parameter $\Lambda > 0$, for each $(t, j) \in E$:

- $\underline{t}(t, j)$ gives the largest $t' \in \mathbb{R}_{\geq 0}$ such that there exists $j' \in \mathbb{N}$ such that $(t', j') \in E$ and the hybrid time interval $E \cap ([t', t] \times \{j', j' + 1, \cdots, j\})$ has length greater than or equal to Λ. If no such t' exists, then $\underline{t} = 0$.

- $\underline{j}(t, j)$ gives the largest $j' \in \mathbb{N}$ such that there exists $t' \in \mathbb{R}_{\geq 0}$ such that $(t', j') \in E$ and the hybrid time interval $E \cap ([t', t] \times \{j', j' + 1, \cdots, j\})$ has length greater than or equal to Λ. If no such j' exists, then $\underline{j} = 0$.

Formally, $\underline{t}$ and $\underline{j}$ are defined as follows. Given $(t, j) \in E$, if $t + j < \Lambda$, then $\underline{t} := 0$ and $\underline{j} := 0$; and if $t + j \geq \Lambda$, then

$$\begin{aligned} \underline{t}(t, j) &:= \sup\{t' \in \mathbb{R}_{\geq 0} : \exists j' \in \mathbb{N} \text{ s.t. } (t', j') \in E,\ t' + j' \leq t + j - \Lambda\}, \\ \underline{j}(t, j) &:= \sup\{j' \in \mathbb{N} : \exists t' \in \mathbb{R}_{\geq 0} \text{ s.t. } (t', j') \in E,\ t' + j' \leq t + j - \Lambda\}. \end{aligned} \tag{5.4}$$

For simplicity of notation, we subsequently omit the arguments of $\underline{t}$ and $\underline{j}$.

5.2.1 Parameter Estimation during Flows

Following the approach outlined in Section 2.3, during flows of each solution to $\mathcal{H}$, we compute signals $(t, j) \mapsto \mathcal{Y}_c(t, j)$ and $(t, j) \mapsto \Phi_c(t, j)$ such that $\mathcal{Y}_c$, Φ_c, and θ are related via a linear regression model. For each $(t, j) \in E$, we integrate f_c and ϕ_c during flows and compute the change in x due to flow over the preceding hybrid time interval of length Λ as

$$\begin{aligned} \mathcal{X}_c(t, j) &:= \sum_{i=\underline{j}}^{j} x(\min\{t, t_{i+1}\}, i) - x(\max\{\underline{t}, t_i\}, i) \\ \mathcal{F}_c(t, j) &:= \sum_{i=\underline{j}}^{j} \int_{\max\{\underline{t}, t_i\}}^{\min\{t, t_{i+1}\}} f_c(x(s, i), u(s, i)) ds \\ \Phi_c(t, j) &:= \sum_{i=\underline{j}}^{j} \int_{\max\{\underline{t}, t_i\}}^{\min\{t, t_{i+1}\}} \phi_c(s, i) ds \end{aligned} \tag{5.5}$$

for all $(t,j) \in E$, where $\{t_j\}_{j=0}^{J}$ is the sequence defining E as in Section 2.1, and $t_{J+1} := T$, with $J := \sup_j E$ and $T := \sup_t E$. Since θ is constant, it follows from (5.1) that

$$\mathcal{X}_c(t,j) = \mathcal{F}_c(t,j) + \Phi_c(t,j)\theta \qquad \forall (t,j) \in E. \tag{5.6}$$

Then, by defining $\mathcal{Y}_c(t,j) := \mathcal{X}_c(t,j) - \mathcal{F}_c(t,j)$ we rewrite (5.6) as a hybrid linear regression model, namely,

$$\mathcal{Y}_c(t,j) = \Phi_c(t,j)\theta \qquad \forall (t,j) \in E \tag{5.7}$$

We sample $\mathcal{Y}_c$ and Φ_c at hybrid time instants $\{(\tilde{t}_i, \tilde{j}_i)\}_{i=1}^{S_c(t,j)}$ satisfying $0 \leq \tilde{t}_1 < \tilde{t}_2 < \cdots \leq t$, where $(t,j) \mapsto S_c(t,j)$ indicates a time-dependent number of samples, which is to be designed. To store the samples, we define, for all $(t,j) \in E$, the history stacks

$$\begin{aligned} Y_c(t,j) &:= \Big[Y_{c,1}(t,j), \quad Y_{c,2}(t,j), \quad \cdots, \quad Y_{c,N_c}(t,j) \Big] \in \mathbb{R}^{n \times N_c} \\ Z_c(t,j) &:= \Big[Z_{c,1}(t,j), \quad Z_{c,2}(t,j), \quad \cdots, \quad Z_{c,N_c}(t,j) \Big] \in \mathbb{R}^{n \times p \times N_c} \end{aligned} \tag{5.8}$$

where $N_c \in \mathbb{N} \setminus \{0\}$ is a design parameter satisfying $N_c \geq \lceil p/n \rceil$ that represents the maximum number of samples that can be stored during flows. The elements of Y_c and Z_c are initially empty (zero), and are populated by the samples of $\mathcal{Y}_c$ and Φ_c, respectively. Thus, during flows, the elements of Y_c and Z_c are piecewise constant right-continuous signals, with values changing only at the sample times and, for all $(t,j) \in E$,

$$Y_{c,\ell}(t,j) = Z_{c,\ell}(t,j)\theta \qquad \forall\, \ell \in \{1, 2, \cdots, N_c\}. \tag{5.9}$$

During flows, we update the parameter estimate continuously with dynamics inspired by the continuous-time ICL algorithm in (2.11). However, in contrast to (2.11), we do not use a filtered version of the plant state to inject current measurements into the dynamics of the parameter estimate. Instead, current measurements are injected by employing a gradient descent estimation scheme [48] on the hybrid signals $\mathcal{Y}_c$ and Φ_c in (5.7). This modification allows us to show exponential convergence of $\hat{\theta}$ to θ in the hybrid case. By employing a gradient descent scheme to estimate θ in (5.7), we obtain the estimator dynamics

$$\dot{\hat{\theta}} = \gamma_c \Phi_c(\tau, k)^\top (\mathcal{Y}_c(\tau, k) - \Phi_c(\tau, k)\hat{\theta})$$

where $\gamma_c > 0$ is a design parameter. Then, as in (2.11), we add a term that captures the information stored in Z_c and Y_c, and obtain

$$\begin{aligned} \dot{\hat{\theta}} &= \gamma_c \Phi_c(\tau, k)^\top (\mathcal{Y}_c(\tau, k) - \Phi_c(\tau, k)\hat{\theta}) \\ &\quad + \rho_c \sum_{\ell=1}^{N_c} \Big(Z_{c,\ell}(\tau, k)^\top (Y_{c,\ell}(\tau, k) - Z_{c,\ell}(\tau, k)\hat{\theta}) \Big) =: \alpha_c(\xi) \end{aligned} \tag{5.10}$$

where $\rho_c > 0$ is a design parameter. Hence, the flow map of $\mathcal{H}$ in (5.2) is

$$F(\xi) := \begin{bmatrix} f_c(x, u(\tau, k)) + \phi_c(\tau, k)\theta \\ \alpha_c(\xi) \\ 1 \\ 0 \end{bmatrix} \qquad \forall \xi \in C. \tag{5.11}$$

5.2.2 Parameter Estimation at Jumps

Each time a solution to $\mathcal{H}$ jumps, we compute signals $(t, j) \mapsto \mathcal{Y}_d(t, j)$ and $(t, j) \mapsto \Phi_d(t, j)$ such that $\mathcal{Y}_d$, Φ_d, and θ are related via a linear regression model. For each $(t, j) \in E$, we sum g_d and ϕ_d before each jump, and sum x after each jump, over the preceding hybrid time interval of length Λ as

$$\begin{aligned}
\mathcal{X}_d(t,j) &:= \sum_{i=\underline{j}}^{j-1} x(t_{i+1}, i+1) \\
\mathcal{G}_d(t,j) &:= \sum_{i=\underline{j}}^{j-1} g_d(x(t_{i+1}, i), u(t_{i+1}, i)) \\
\Phi_d(t,j) &:= \sum_{i=\underline{j}}^{j-1} \phi_d(t_{i+1}, i)
\end{aligned} \tag{5.12}$$

for all $(t,j) \in E$. Since θ is constant, it follows from (5.1) that

$$\mathcal{X}_d(t,j) = \mathcal{G}_d(t,j) + \Phi_d(t,j)\theta \qquad \forall (t,j) \in E \tag{5.13}$$

Then, by defining $\mathcal{Y}_d(t,j) := \mathcal{X}_d(t,j) - \mathcal{G}_d(t,j)$, we rewrite (5.13) as a hybrid linear regression model, namely,

$$\mathcal{Y}_d(t,j) = \Phi_d(t,j)\theta \qquad \forall (t,j) \in E. \tag{5.14}$$

We sample $\mathcal{Y}_d$ and Φ_d after the jump at hybrid time instants $\{(t_{\tilde{j}_i+1}, \tilde{j}_i+1)\}_{i=1}^{S_d(t,j)}$ satisfying $0 \leq \tilde{j}_1 < \tilde{j}_2 < \cdots \leq j$, where $(t,j) \mapsto S_d(t,j)$ indicates a time-dependent number of samples, which is to be designed. To store the samples, we define, for all $(t,j) \in E$, the history stacks

$$\begin{aligned}
Y_d(t,j) &:= \begin{bmatrix} Y_{d,1}(t,j), & Y_{d,2}(t,j), & \cdots, & Y_{d,N_d}(t,j) \end{bmatrix} \in \mathbb{R}^{n \times N_d} \\
Z_d(t,j) &:= \begin{bmatrix} Z_{d,1}(t,j), & Z_{d,2}(t,j), & \cdots, & Z_{d,N_d}(t,j) \end{bmatrix} \in \mathbb{R}^{n \times p \times N_d}
\end{aligned} \tag{5.15}$$

where $N_d \in \mathbb{N} \setminus \{0\}$ is a design parameter satisfying $N_d \geq \lceil p/n \rceil$ that represents the maximum number of samples that can be stored at jumps. The elements of Y_d and Z_d are initially empty, and are populated by the samples of $\mathcal{Y}_c$ and Φ_c, respectively. Thus, the elements of Y_d and Z_d change only after jumps associated with the sample times and, for all $(t,j) \in E$,

$$Y_{d,\ell}(t,j) = Z_{d,\ell}(t,j)\theta \qquad \forall\, \ell \in \{1, 2, \cdots, N_d\}. \tag{5.16}$$

We update the parameter estimate $\hat{\theta}$ discretely at jumps. To incorporate current measurements, we employ a discrete-time gradient descent estimation scheme [54] to estimate θ in (5.14). Namely,

$$\hat{\theta}^+ = \hat{\theta} + \Gamma \frac{\Phi_d(\tau, k+1)^\top (\mathcal{Y}_d(\tau, k+1) - \Phi_d(\tau, k+1)\hat{\theta})}{\gamma_d + |\Phi_d(\tau, k+1)|^2}$$

where $\gamma_d > 0$ and $\Gamma \in (0, 1/2]$ are design parameters. Then, as during flows, we add a term that captures the information stored in Z_d and Y_d, to obtain

$$\begin{aligned} \hat{\theta}^+ = \hat{\theta} &+ \Gamma \frac{\Phi_d(\tau, k+1)^\top (\mathcal{Y}_d(\tau, k+1) - \Phi_d(\tau, k+1)\hat{\theta})}{\gamma_d + |\Phi_d(\tau, k+1)|^2} \\ &+ \Gamma \frac{\sum_{\ell=1}^{N_d} \Big(Z_{d,\ell}(\tau, k+1)^\top \big(Y_{d,\ell}(\tau, k+1) - Z_{d,\ell}(\tau, k+1)\hat{\theta} \big) \Big)}{\rho_d + \sum_{\ell=1}^{N_d} |Z_{d,\ell}(\tau, k+1)|^2} =: \alpha_d(\xi) \end{aligned} \tag{5.17}$$

where $\rho_d > 0$ is a design parameter. Hence, the jump map of $\mathcal{H}$ in (5.2) is

$$G(\xi) := \begin{bmatrix} g_d(x, u(\tau, k)) + \phi_d(\tau, k)\theta \\ \alpha_d(\xi) \\ \tau \\ k+1 \end{bmatrix} \qquad \forall \xi \in D. \tag{5.18}$$

Remark 5.3: For simplicity, $\mathcal{H}$ in (5.2) is expressed such that jumps in the parameter estimate coincide with jumps in x. This construction results in $\mathcal{H}$ being noncausal since, due to the definition of $\mathcal{X}_d$ in (5.12), the term $\mathcal{Y}_d(\tau, k+1)$ in (5.17) depends on the value of $x(\tau, k+1)$ – that is, the value of x after the jump at hybrid time (τ, k). We can remove this simplification at the price of letting the algorithm jump twice for each jump in x, as follows. Immediately before a jump in x, the algorithm computes $\mathcal{G}_d$ and Φ_d in (5.12), and jumps to increment the value of k to $k+1$. Immediately after a jump in x, the algorithm computes $\mathcal{X}_d$ in (5.12) and jumps a second time to update the parameter estimate as in (5.17). A logic variable ensures that, after the second jump, the algorithm flows or jumps in accordance with the hybrid time domain E. Since θ in (5.1) is constant, the stability properties induced by $\mathcal{H}$ in (5.2) are equivalent to the stability properties induced by the causal modification, after we reparameterize the domain of solutions to $\mathcal{H}$ to match the domain of solutions to the causal system. Hence, for simplicity, we focus our analysis on (5.2).

5.3 Stability Analysis

We now establish sufficient conditions that ensure our proposed estimation algorithm induces global pre-exponential stability[17] of the set

$$\mathcal{A} := \left\{ \xi \in \mathcal{X} \ : \ \hat{\theta} = \theta \right\}. \tag{5.19}$$

Global pre-exponential stability of $\mathcal{A}$ implies that, for each solution ξ to $\mathcal{H}$, the distance from ξ to the set $\mathcal{A}$ is bounded above by an exponentially decreasing function of the initial condition – see Definition 2.9. As a consequence, for each complete solution ξ to $\mathcal{H}$, the parameter estimate $\hat{\theta}$ converges exponentially to θ.

Convergence of $\hat{\theta}$ to θ is achieved when the parameter estimation error $\tilde{\theta} = \theta - \hat{\theta}$ converges to zero. In such error coordinates, using (5.7) and (5.9), the dynamics

[17] Since each solution ξ to $\mathcal{H}$ inherits the hybrid time domain of ϕ_c, ϕ_d, and u, the use of "pre-exponential," as opposed to "exponential," stability means that these signals do not need to be complete. Note that the continuous-time ICL algorithm in [40] assumes completeness of maximal solutions.

of the parameter estimate in (5.10) become

$$\dot{\tilde{\theta}} = -\gamma_c \Phi_c(\tau,k)^\top \Phi_c(\tau,k)\tilde{\theta} - \rho_c \mathcal{Z}_c(\tau,k)\tilde{\theta} \tag{5.20}$$

where

$$\mathcal{Z}_c(\tau,k) := \sum_{\ell=1}^{N_c} Z_{c,\ell}(\tau,k)^\top Z_{c,\ell}(\tau,k) \in \mathbb{R}^{p\times p}. \tag{5.21}$$

Furthermore, using (5.14) and (5.16), the reset map for the parameter estimate in (5.17) becomes

$$\tilde{\theta}^+ = \tilde{\theta} - \Gamma \frac{\Phi_d(\tau,k+1)^\top \Phi_d(\tau,k+1)}{\gamma_d + |\Phi_d(\tau,k+1)|^2}\tilde{\theta} - \Gamma \frac{\mathcal{Z}_d(\tau,k+1)}{\rho_d + \sum_{\ell=1}^{N_d} |Z_{d,\ell}(\tau,k+1)|^2}\tilde{\theta} \tag{5.22}$$

where

$$\mathcal{Z}_d(\tau,k) := \sum_{\ell=1}^{N_d} Z_{d,\ell}(\tau,k)^\top Z_{d,\ell}(\tau,k) \in \mathbb{R}^{p\times p}. \tag{5.23}$$

We impose the following boundedness condition on ϕ_c and ϕ_d, whose values are collected in Φ_c and Φ_d in (5.5) and (5.12), respectively.

Assumption 5.4: Given $\phi_c, \phi_d : E \to \mathbb{R}^{n\times p}$, where $E := \operatorname{dom}\phi_c = \operatorname{dom}\phi_d$ is a hybrid time domain, there exists $\phi_M > 0$ such that $|\phi_c(t,j)| \leq \phi_M$ for all $(t,j) \in E$ and $|\phi_d(t,j)| \leq \phi_M$ for all $(t,j) \in \Upsilon(E)$, with Υ as in (2.2).

Moreover, we impose the following hybrid persistence of excitation condition.

Assumption 5.5: Given $\Lambda > 0$, $N_c, N_d \in \mathbb{N}$, $\phi_c, \phi_d : E \to \mathbb{R}^{n\times p}$, $Z_c : E \to \mathbb{R}^{n\times p\times N_c}$, and $Z_d : E \to \mathbb{R}^{n\times p\times N_d}$, where $E := \operatorname{dom}\phi_c = \operatorname{dom}\phi_d = \operatorname{dom} Z_c = \operatorname{dom} Z_d$ is a hybrid time domain, there exist $\Delta, \mu > 0$ such that, for all $(t',j'), (t^*,j^*) \in E$ satisfying

$$\Delta \leq (t^* - t') + (j^* - j') < \Delta + 1, \tag{5.24}$$

the following condition holds:

$$\begin{aligned}&\sum_{j=j'}^{j^*}\int_{\max\{t',t_j\}}^{\min\{t^*,t_{j+1}\}}\Big(\Phi_c(s,j)^\top\Phi_c(s,j)+\mathcal{Z}_c(s,j)\Big)ds\\&+\sum_{j=j'}^{j^*-1}\Big(\Phi_d(t_{j+1},j+1)^\top\Phi_d(t_{j+1},j+1)+\mathcal{Z}_d(t_{j+1},j+1)\Big)\geq\mu I\end{aligned}\tag{5.25}$$

with Φ_c as in (5.5), Φ_d as in (5.12), $\mathcal{Z}_c$ as in (5.21), and $\mathcal{Z}_d$ as in (5.23), where $\{t_j\}_{j=0}^{J}$ is the sequence defining E as in Section 2.1 and $t_{J+1} := T$, with $J := \sup_j E$ and $T := \sup_t E$.

Remark 5.6: The excitation condition in Assumption 5.5 is similar to the hybrid persistence of excitation condition in Assumption 4.6. However, in contrast to Assumption 4.6, which imposes the excitation condition on only ϕ_c and ϕ_d (via the solution component ψ), Assumption 5.5 incorporates the information provided by ϕ_c and ϕ_d (via Φ_c and Φ_d, respectively) and from the history stacks Z_c and Z_d. Hence, it is possible that Assumption 5.5 holds when ϕ_c and ϕ_d are not hybrid PE in the sense of Assumption 4.6. An example of such a case is given in Section 5.7.

We now establish conditions that ensure $\mathcal{H}$ induces global pre-exponential stability of $\mathcal{A}$.

Theorem 5.7: Given $\Lambda > 0$, $N_c, N_d \in \mathbb{N}$, $\phi_c, \phi_d : E \to \mathbb{R}^{n\times p}$, $u : E \to \mathbb{R}^m$, $Z_c : E \to \mathbb{R}^{n\times p\times N_c}$, $Z_d : E \to \mathbb{R}^{n\times p\times N_d}$, $Y_c : E \to \mathbb{R}^{n\times N_c}$, and $Y_d : E \to \mathbb{R}^{n\times N_d}$ defining the hybrid system $\mathcal{H}$ in (5.2), where $E := \operatorname{dom}\phi_c = \operatorname{dom}\phi_d = \operatorname{dom} u = \operatorname{dom} Z_c = \operatorname{dom} Z_d = \operatorname{dom} Y_c = \operatorname{dom} Y_d$ is a hybrid time domain, suppose that Assumptions 5.4 and 5.5 hold. Then, for each $\gamma_c, \gamma_d, \rho_c, \rho_d > 0$ and each $\Gamma \in (0, 1/2]$, the parameter estimation error $(t,j) \mapsto \tilde{\theta}(t,j) := \theta - \hat{\theta}(t,j)$ for each solution ξ to $\mathcal{H}$ satisfies

$$|\tilde{\theta}(t,j)| \leq \kappa e^{-\lambda(t+j)}|\tilde{\theta}(0,0)| \tag{5.26}$$

for all $(t,j) \in \operatorname{dom}\xi$, *where*

$$\kappa := \sqrt{\frac{1}{1-\sigma}}, \qquad \lambda := \frac{1}{2(\Delta+1)} \ln\left(\frac{1}{1-\sigma}\right)$$
$$\sigma := \frac{2\mu_0}{\left(1+\sqrt{(a_M+2)(\Delta+2)^3(a_M(\Delta+2)+1/2)}\right)^2}, \tag{5.27}$$

and

$$\mu_0 := \mu \min\left\{\gamma_c,\ \rho_c,\ \frac{\Gamma}{2(\gamma_d+(\Lambda+1)^2\phi_M^2)},\ \frac{\Gamma}{2(\rho_d+N_d(\Lambda+1)^2\phi_M^2)}\right\}$$
$$a_M := (\gamma_c+\rho_c N_c)(\Lambda+1)^2\phi_M^2 \tag{5.28}$$

with ϕ_M *from Assumption 5.4 and* Δ, μ *from Assumption 5.5.*

5.3.1 Proof of Theorem 5.7

To prove Theorem 5.7, we use that the error dynamics of $\mathcal{H}$ belong to the class of hybrid systems studied in Chapter 3. Using (5.20) and (5.22), the hybrid system, denoted by $\widetilde{\mathcal{H}}$, resulting from expressing $\mathcal{H}$ in error coordinates $\tilde{\theta} = \theta - \hat{\theta}$ has state $\xi := (\tilde{\theta}, \tau, k) \in \widetilde{\mathcal{X}} := \mathbb{R}^p \times E$ and dynamics

$$\widetilde{\mathcal{H}} : \begin{cases} \dot{\xi} = \widetilde{F}(\xi) & \xi \in \widetilde{C} \\ \xi^+ = \widetilde{G}(\xi) & \xi \in \widetilde{D}, \end{cases} \tag{5.29}$$

where $\widetilde{C} := C$ and $\widetilde{D} := D$, with C, D in (5.3), and

$$\widetilde{F}(\xi) := \begin{bmatrix} -\gamma_c \Phi_c(\tau,k)^\top \Phi_c(\tau,k)\tilde{\theta} - \rho_c \mathcal{Z}_c(\tau,k)\tilde{\theta} \\ 1 \\ 0 \end{bmatrix} \qquad \forall \xi \in \widetilde{C}$$
$$\widetilde{G}(\xi) := \begin{bmatrix} \tilde{\theta} - \Gamma \frac{\Phi_d(\tau,k+1)^\top \Phi_d(\tau,k+1)}{\gamma_d+|\Phi_d(\tau,k+1)|^2}\tilde{\theta} - \Gamma \frac{\mathcal{Z}_d(\tau,k+1)}{\rho_d+\sum_{\ell=1}^{N_d}|Z_{d,\ell}(\tau,k+1)|^2}\tilde{\theta} \\ \tau \\ k+1 \end{bmatrix} \qquad \forall \xi \in \widetilde{D}.$$

The hybrid system $\mathcal{H}$ in (3.1) reduces to $\widetilde{\mathcal{H}}$ in (5.29) when $\vartheta = \tilde{\theta}$,

$$A(\tau,k) := \gamma_c \Phi_c(\tau,k)^\top \Phi_c(\tau,k) + \rho_c \mathcal{Z}_c(\tau,k) \tag{5.30a}$$

for all $(\tau,k) \in E$, and

$$B(\tau,k) := \Gamma \frac{\Phi_d(\tau,k+1)^\top \Phi_d(\tau,k+1)}{\gamma_d + |\Phi_d(\tau,k+1)|^2} + \Gamma \frac{\mathcal{Z}_d(\tau,k+1)}{\rho_d + \sum_{\ell=1}^{N_d} |Z_{d,\ell}(\tau,k+1)|^2} \tag{5.30b}$$

for all $(\tau,k) \in \Upsilon(E)$, with Υ as in (2.2). Hence, we use Theorem 3.3 to prove Theorem 5.7.

Proof of Theorem 5.7: To prove Theorem 5.7, we show that, under the conditions of Theorem 5.7, A, B in (5.30) satisfy Assumptions 3.1 and 3.2. Beginning with Assumption 3.1, the matrix functions A, B in (5.30) are symmetric by construction since $\mathcal{Z}_c$ in (5.21) and $\mathcal{Z}_d$ in (5.23) are symmetric and, since $\gamma_c, \gamma_d, \rho_c, \rho_d > 0$ and $\Gamma \in (0, 1/2]$, it follows that A, B are positive semidefinite. Hence, items 1 and 2 of Assumption 3.1 hold. Next, we show that items 3 and 4 of Assumption 3.1 hold. Since, by Assumption 5.4, $|\phi_c(t,j)| \leq \phi_M$ for all $(t,j) \in E$, it follows from the definition of Φ_c in (5.5) that

$$|\Phi_c(t,j)| \leq \sum_{i=\underline{j}}^{j} \int_{\max\{\underline{t},t_i\}}^{\min\{t,t_{i+1}\}} |\phi_c(s,i)| ds \leq (\Lambda+1)\phi_M \tag{5.31}$$

for all $(t,j) \in \operatorname{dom}\xi$, where the last inequality follows from the fact that, by the definition of $\underline{t}$ and $\underline{j}$ in (5.4), $(t-\underline{t}) + (j-\underline{j}) \leq \Lambda+1$ for all $(t,j) \in \operatorname{dom}\xi$, and thus

$$\begin{aligned}\sum_{i=\underline{j}}^{j} \big(\min\{t,t_{i+1}\} - \max\{\underline{t},t_i\}\big) &= \min\{t,t_{j+1}\} - \max\{\underline{t},t_{\underline{j}}\} \\ &= t - \underline{t} \leq \Lambda+1 \end{aligned}\tag{5.32}$$

for all $(t, j) \in \operatorname{dom} \xi$. Finally, since Z_c in (5.8) stores samples of Φ_c, it follows that

$$|Z_{c,\ell}(t, j)| \leq |\Phi_c(t, j)| \leq (\Lambda + 1)\phi_M \tag{5.33}$$

for all $\ell \in \{1, 2, \cdots, N_c\}$ and all $(t, j) \in \operatorname{dom} \xi$. Then, from (5.31) and the definition of $\mathcal{Z}_c$ in (5.21), we have

$$|\mathcal{Z}_c(t, j)| \leq \sum_{\ell=1}^{N_c} |Z_{c,\ell}(t, j)|^2 \leq N_c(\Lambda + 1)^2\phi_M^2 \tag{5.34}$$

for all $(t, j) \in \operatorname{dom} \xi$. Hence, from (5.31) and (5.34), it follows that

$$\begin{aligned} |A(t, j)| &\leq \gamma_c|\Phi_c(t, j)|^2 + \rho_c|\mathcal{Z}_c(t, j)| \\ &\leq \gamma_c(\Lambda + 1)^2\phi_M^2 + \rho_c N_c(\Lambda + 1)^2\phi_M^2 \end{aligned}$$

for all $(t, j) \in E$. Thus, item 3 of Assumption 3.1 holds with a_M in (5.28). Moreover, since $\gamma_d, \rho_d > 0$ and $\Gamma \in (0, 1/2]$, it follows that

$$|B(t, j)| \leq \Gamma \frac{|\Phi_d(t, j + 1)|^2}{\gamma_d + |\Phi_d(t, j + 1)|^2} + \Gamma \frac{\sum_{\ell=1}^{N_d} |Z_{d,\ell}(t, j + 1)|^2}{\rho_d + \sum_{\ell=1}^{N_d} |Z_{d,\ell}(t, j + 1)|^2} < 1$$

for all $(t, j) \in \Upsilon(E)$. Thus, item 4 of Assumption 3.1 holds.

To complete the proof, we show that Assumption 3.2 holds. Substituting A, B

in (5.30) into (3.5), we have that, for all $(t',j'),(t^*,j^*)\in E$ satisfying (5.24),

$$\begin{aligned}
&\sum_{j=j'}^{j^*}\int_{\max\{t',t_j\}}^{\min\{t^*,t_{j+1}\}}\Big(\gamma_c\Phi_c(s,j)^\top\Phi_c(s,j)+\rho_c\mathcal{Z}_c(s,j)\Big)ds\\
&\quad+\frac{1}{2}\sum_{j=j'}^{j^*-1}\left(\Gamma\frac{\Phi_d(t_{j+1},j+1)^\top\Phi_d(t_{j+1},j+1)}{\gamma_d+|\Phi_d(t_{j+1},j+1)|^2}+\Gamma\frac{\mathcal{Z}_d(t_{j+1},j+1)}{\rho_d+\sum_{\ell=1}^{N_d}|Z_{d,\ell}(t_{j+1},j+1)|^2}\right)\\
&\geq\min\left\{\gamma_c,\,\rho_c,\,\frac{\Gamma}{2(\gamma_d+(\Lambda+1)^2\phi_M^2)},\,\frac{\Gamma}{2(\rho_d+N_d(\Lambda+1)^2\phi_M^2)}\right\}\\
&\quad\times\left(\sum_{j=j'}^{j^*}\int_{\max\{t',t_j\}}^{\min\{t^*,t_{j+1}\}}\Big(\Phi_c(s,j)^\top\Phi_c(s,j)+\mathcal{Z}_c(s,j)\Big)ds\right.\\
&\qquad\left.+\sum_{j=j'}^{j^*-1}\Big(\Phi_d(t_{j+1},j+1)^\top\Phi_d(t_{j+1},j+1)+\mathcal{Z}_d(t_{j+1},j+1)\Big)\right)\\
&\geq\mu\min\left\{\gamma_c,\,\rho_c,\,\frac{\Gamma}{2(\gamma_d+(\Lambda+1)^2\phi_M^2)},\,\frac{\Gamma}{2(\rho_d+N_d(\Lambda+1)^2\phi_M^2)}\right\}>0
\end{aligned}$$

with $\mu>0$ from Assumption 5.5. Thus, Assumption 3.2 holds with Δ,μ_0 from Theorem 5.7, and the conditions of Theorem 3.3 are satisfied. Then, from the equivalence between $|\tilde{\theta}|$ and $|\xi|_{\mathcal{A}}$, namely, since $\mathcal{A}$ in (5.19) is the set of points $\{\xi\in\mathcal{X}:\hat{\theta}=\theta\}$, the result follows from Theorem 3.3. □

5.3.2 Point-wise Excitation Conditions

Theorem 5.7 provides conditions on Φ_c, Φ_d, Z_c, and Z_d that ensure $\mathcal{H}$ induces global pre-exponential stability of $\mathcal{A}$. However, Assumption 5.5 may be difficult to verify online since (5.25) requires integrating Φ_c and $\mathcal{Z}_c$, and summing Φ_d and $\mathcal{Z}_d$, over each hybrid time interval in E with length satisfying (5.24). Hence, in this section, we establish point-wise conditions on the history stacks Z_c and Z_d that ensure global pre-exponential stability of $\mathcal{A}$ for $\mathcal{H}$.

For our analysis, we partition hybrid time into two phases. During the initial phase, insufficient data has been collected to satisfy a richness condition on the history stacks. In Theorem 5.10, we show that the parameter estimate remains bounded for all hybrid time despite the lack of persistently exciting data. Next, we assume that after a finite period of hybrid time, the system transitions to the second phase where the history stacks are sufficiently rich and, in Theorem 5.11, we show that $\hat{\theta}$ converges exponentially to θ. To guarantee that the transition to the second phase occurs in finite hybrid time, we require the history stacks to satisfy the following assumption.

Assumption 5.8: Given $\Lambda > 0$, $N_c, N_d \in \mathbb{N}$, $Z_c : E \to \mathbb{R}^{n \times p \times N_c}$, and $Z_d : E \to \mathbb{R}^{n \times p \times N_d}$, where $E := \operatorname{dom} Z_c = \operatorname{dom} Z_d$ is a hybrid time domain, there exist $\Delta > 1$, $\Omega > 0$, and $(t_o, j_o) \in E$ such that, for all $(t', j'), (t^*, j^*) \in E$ satisfying $t^* + j^* \geq t' + j' \geq t_o + j_o$ and

$$(t^* - t') + (j^* - j') \geq \Delta \tag{5.35}$$

the following holds:

$$\min\{t^* - t', j^* - j'\} \geq \Omega. \tag{5.36}$$

Furthermore, there exist $\mu_o > 0$ such that, for all $(t, j) \in E$ satisfying $t + j \geq t_o + j_o$,

$$\mathcal{Z}_c(t, j) + \mathcal{Z}_d(t, j) \geq \mu_o I \tag{5.37}$$

and

$$\mathcal{Z}_c(t, j) \geq \mathcal{Z}_c(t_o, j_o), \qquad \mathcal{Z}_d(t, j) \geq \mathcal{Z}_d(t_o, j_o), \tag{5.38}$$

with $\mathcal{Z}_c$ as in (5.21) and $\mathcal{Z}_d$ as in (5.23).

Remark 5.9: The inequalities in (5.35) and (5.36) mean that each hybrid time interval in E with length greater than or equal to Δ contains at least Ω seconds of flow, and jumps at least $\lceil \Omega \rceil$ times. Note that (5.36) implies that Z_c and Z_d are neither continuous, eventually continuous, discrete, eventually discrete, or Zeno. The inequality in (5.37) means that the sum of $\mathcal{Z}_c$ and $\mathcal{Z}_d$ is uniformly positive definite, and (5.38) means that the minimum eigenvalues values of $\mathcal{Z}_c$ and $\mathcal{Z}_d$ are non-decreasing.

When there is insufficient data to satisfy Assumption 5.8, we have the following result.

Theorem 5.10: Given $\Lambda > 0$, $N_c, N_d \in \mathbb{N}$, $\phi_c, \phi_d : E \to \mathbb{R}^{n\times p}$, $u : E \to \mathbb{R}^m$, $Z_c : E \to \mathbb{R}^{n\times p\times N_c}$, $Z_d : E \to \mathbb{R}^{n\times p\times N_d}$, $Y_c : E \to \mathbb{R}^{n\times N_c}$, *and* $Y_d : E \to \mathbb{R}^{n\times N_d}$ *defining the hybrid system* $\mathcal{H}$ *in* (5.2), *where* $E := \operatorname{dom}\phi_c = \operatorname{dom}\phi_d = \operatorname{dom} u = \operatorname{dom} Z_c = \operatorname{dom} Z_d = \operatorname{dom} Y_c = \operatorname{dom} Y_d$ *is a hybrid time domain, for each* $\gamma_c, \gamma_d, \rho_c, \rho_d > 0$, $\Gamma \in (0, 1/2]$, *the parameter estimation error* $(t,j) \mapsto \tilde{\theta}(t,j) := \theta - \hat{\theta}(t,j)$ *for each solution* ξ *to* $\mathcal{H}$ *satisfies*

$$|\tilde{\theta}(t,j)| \le |\tilde{\theta}(0,0)| \quad \forall (t,j) \in \operatorname{dom}\xi. \tag{5.39}$$

Proof. Consider the Lyapunov function

$$V(\xi) := \frac{1}{2}\tilde{\theta}^\top\tilde{\theta} \qquad \forall \xi \in C \cup D \tag{5.40}$$

where $\tilde{\theta} = \theta - \hat{\theta}$. From (5.20), we have that, for all $\xi \in C$,

$$\langle \nabla V(\xi), F(\xi)\rangle = -\gamma_c \tilde{\theta}^\top \Phi_c(\tau,k)^\top \Phi_c(\tau,k)\tilde{\theta} - \rho_c \tilde{\theta}^\top \mathcal{Z}_c(\tau,k)\tilde{\theta} \le 0 \tag{5.41}$$

where the inequality follows from the fact that $\mathcal{Z}_c$ is positive semidefinite and $\gamma_c, \rho_c > 0$. Next, from (5.22), for all $\xi \in D$,

$$\begin{aligned} V(G(\xi)) - V(\xi) \le &-\frac{\Gamma}{2}\tilde{\theta}^\top \frac{\Phi_d(\tau,k+1)^\top \Phi_d(\tau,k+1)}{\gamma_d + |\Phi_d(\tau,k+1)|^2}\tilde{\theta} \\ &-\frac{\Gamma}{2}\tilde{\theta}^\top \frac{\mathcal{Z}_d(\tau,k+1)}{\rho_d + \sum_{\ell=1}^{N_d} |Z_{d,\ell}(\tau,k+1)|^2}\tilde{\theta} \le 0 \end{aligned} \tag{5.42}$$

where the inequalities follow from the fact that $\mathcal{Z}_d$ is positive semidefinite, $\gamma_d, \rho_d > 0$, and $\Gamma \in (0, 1/2]$. Hence, for each solution ξ to $\mathcal{H}$, by integration using the bounds above, we conclude that

$$V(\xi(t,j)) \leq V(\xi(0,0))$$

for all $(t,j) \in \operatorname{dom} \xi$. Using the definition of V in (5.40), we obtain (5.39). □

When the history stacks are sufficiently rich as in Assumption 5.8, we establish the following stability result.

Theorem 5.11: Given $\Lambda > 0$, $N_c, N_d \in \mathbb{N}$, $\phi_c, \phi_d : E \to \mathbb{R}^{n \times p}$, $u : E \to \mathbb{R}^m$, $Z_c : E \to \mathbb{R}^{n \times p \times N_c}$, $Z_d : E \to \mathbb{R}^{n \times p \times N_d}$, $Y_c : E \to \mathbb{R}^{n \times N_c}$, *and* $Y_d : E \to \mathbb{R}^{n \times N_d}$ *defining the hybrid system* $\mathcal{H}$ *in* (5.2), *where* $E := \operatorname{dom} \phi_c = \operatorname{dom} \phi_d = \operatorname{dom} u = \operatorname{dom} Z_c = \operatorname{dom} Z_d = \operatorname{dom} Y_c = \operatorname{dom} Y_d$ *is a hybrid time domain, suppose that Assumptions 5.4 and 5.8 hold and let* $\Delta, \Omega > 0$ *and* $(t_o, j_o) \in E$ *come from Assumption 5.8. Then, for each* $\gamma_c, \gamma_d, \rho_c, \rho_d > 0$, $\Gamma \in (0, 1/2]$, *the parameter estimation error* $(t,j) \mapsto \tilde{\theta}(t,j) := \theta - \hat{\theta}(t,j)$ *for each solution* ξ *to* $\mathcal{H}$ *satisfies*

$$|\tilde{\theta}(t,j)| \leq \kappa \mathrm{e}^{-\lambda(t+j-t_o-j_o)} |\tilde{\theta}(t_o, j_o)|$$

for all $(t,j) \in \operatorname{dom} \xi$ *satisfying* $t \geq t_o$ *and* $j \geq j_o$, *with* κ, λ *from Theorem 5.7 and* μ *in* (5.27) *substituted by* $\mu := \Omega \mu_o$.

Proof. To prove Theorem 5.11, we show that the conditions of Theorem 5.7 hold for all $(t,j) \in E$ satisfying $t \geq t_o$ and $j \geq j_o$. Beginning with Assumption 5.5, we evaluate the left-hand side of (5.25) and obtain that, for all $(t', j'), (t^*, j^*) \in E$

satisfying $t^* + j^* \geq t' + j' \geq t_o + j_o$ and (5.35),

$$\begin{aligned}
&\sum_{j=j'}^{j^*} \int_{\max\{t',t_j\}}^{\min\{t^*,t_{j+1}\}} \Big(\Phi_c(s,j)^\top \Phi_c(s,j) + \mathcal{Z}_c(s,j)\Big) ds \\
&\qquad + \sum_{j=j'}^{j^*-1} \Big(\Phi_c(t_{j+1}, j+1)^\top \Phi_c(t_{j+1}, j+1) + \mathcal{Z}_d(t_{j+1}, j+1)\Big) \\
&\qquad \geq \sum_{j=j'}^{j^*} \int_{\max\{t',t_j\}}^{\min\{t^*,t_{j+1}\}} \mathcal{Z}_c(s,j) ds + \sum_{j=j'}^{j^*-1} \mathcal{Z}_d(t_{j+1}, j+1) \\
&\qquad \geq (t^* - t')\mathcal{Z}_c(t_o, j_o) + (j^* - j')\mathcal{Z}_d(t_o, j_o) \\
&\qquad \geq \min\{t^* - t', j^* - j'\}(\mathcal{Z}_c(t_o, j_o) + \mathcal{Z}_d(t_o, j_o)) \\
&\qquad \geq \Omega\mu_o
\end{aligned}$$

where the second line follows from (5.38) and the last line follows from (5.36) and (5.37). Hence, Assumption 5.5 holds with $\mu := \Omega\mu_o$. Then, since Assumption 5.4 holds, it follows that the conditions of Theorem 5.7 are satisfied for all $(t, j) \in E$ satisfying $t + j \geq t_o + j_o$, and the result follows from (5.26) with κ and λ from Theorem 5.7. □

We can remove the constraints (5.35) and (5.36) on the hybrid time domain E at the price of requiring $\mathcal{Z}_c$ and $\mathcal{Z}_d$ to each be uniformly positive definite, as in the following result.

Theorem 5.12: Given $\Lambda > 0$, $N_c, N_d \in \mathbb{N}$, $\phi_c, \phi_d : E \to \mathbb{R}^{n\times p}$, $u : E \to \mathbb{R}^m$, $Z_c : E \to \mathbb{R}^{n\times p\times N_c}$, $Z_d : E \to \mathbb{R}^{n\times p\times N_d}$, $Y_c : E \to \mathbb{R}^{n\times N_c}$, and $Y_d : E \to \mathbb{R}^{n\times N_d}$ defining the hybrid system $\mathcal{H}$ in (5.2), where $E := \operatorname{dom}\phi_c = \operatorname{dom}\phi_d = \operatorname{dom} u = \operatorname{dom} Z_c = \operatorname{dom} Z_d = \operatorname{dom} Y_c = \operatorname{dom} Y_d$ is a hybrid time domain, suppose that Assumption 5.4 holds. Furthermore, suppose that there exist $\mu_c, \mu_d > 0$ and $(t_o, j_o) \in E$ such that, for all $(t, j) \in E$ satisfying $t + j \geq t_o + j_o$,

$$\mathcal{Z}_c(t, j) \geq \mu_c I \tag{5.43a}$$

$$\mathcal{Z}_d(t, j) \geq \mu_d I \tag{5.43b}$$

with $\mathcal{Z}_c$ *as in* (5.21) *and* $\mathcal{Z}_d$ *as in* (5.23). *Then, for each* $\gamma_c, \gamma_d, \rho_c, \rho_d > 0$, $\Gamma \in (0, 1/2]$, *the parameter estimation error* $(t,j) \mapsto \tilde{\theta}(t,j) := \theta - \hat{\theta}(t,j)$ *for each solution* ξ *to* $\mathcal{H}$ *satisfies*

$$|\tilde{\theta}(t,j)| \leq \mathrm{e}^{-\lambda_0(t+j-t_o-j_o)}|\tilde{\theta}(t_o,j_o)| \tag{5.44}$$

for all $(t,j) \in \operatorname{dom} \xi$ *satisfying* $t + j \geq t_o + j_o$, *where*

$$\lambda_0 := \min\left\{\rho_c\mu_c,\ -\frac{1}{2}\ln\left(1 - \frac{\Gamma\mu_d}{\rho_d + N_d(\Lambda+1)^2\phi_M^2}\right)\right\}, \tag{5.45}$$

with ϕ_M *from Assumption 5.4.*

Proof. Consider the Lyapunov function

$$V(\xi) := \frac{1}{2}\tilde{\theta}^\top\tilde{\theta} \qquad \forall \xi \in C \cup D \tag{5.46}$$

where $\tilde{\theta} = \theta - \hat{\theta}$. From (5.20), we have that, for all $\xi \in C$ such that $\tau + k \geq t_o + j_o$,

$$\langle \nabla V(\xi), F(\xi)\rangle \leq -\rho_c\tilde{\theta}^\top\mathcal{Z}_c(\tau,k)\tilde{\theta} \leq -2\rho_c\mu_c V(\xi) \tag{5.47}$$

where the second inequality follows from (5.43a) and the fact that $\rho_c > 0$. Next, from (5.22), we have that, for all $\xi \in D$ such that $\tau + k \geq t_o + j_o$,

$$\begin{aligned} V(G(\xi)) - V(\xi) &\leq -\frac{\Gamma}{2}\tilde{\theta}^\top\frac{\mathcal{Z}_d(\tau,k+1)}{\rho_d + \sum_{\ell=1}^{N_d}|Z_{d,\ell}(\tau,k+1)|^2}\tilde{\theta} \\ &\leq -\frac{\Gamma\mu_d}{\rho_d + N_d(\Lambda+1)^2\phi_M^2}V(\xi) \end{aligned} \tag{5.48}$$

where the second inequality follows from (5.43b) and the fact that $\rho_d > 0$ and $\Gamma \in (0, 1/2]$. Combining the expressions in (5.47) and (5.48), we obtain

$$\begin{aligned} \langle \nabla V(\xi), F(\xi)\rangle &\leq -2\rho_c\mu_c V(\xi) && \forall \xi \in C \cap S \\ V(G(\xi)) - V(\xi) &\leq -\frac{\Gamma\mu_d}{\rho_d + N_d(\Lambda+1)^2\phi_M^2}V(\xi) && \forall \xi \in D \cap S \end{aligned}$$

where $S := \{\xi \in C \cup D : \tau + k \geq t_o + j_o\}$. Hence, for each solution ξ to $\mathcal{H}$, by integration using the bounds above, we conclude that, for all $(t, j) \in \operatorname{dom} \xi$ satisfying $t + j \geq t_o + j_o$,

$$V(\xi(t,j)) \leq \exp\Bigg\{\Bigg(-2\rho_c\mu_c(t-t_o) + \ln\left(1 - \frac{\Gamma\mu_d}{\rho_d + N_d(\Lambda+1)^2\phi_M^2}\right)(j - j_o)\Bigg)\Bigg\} V(\xi(t_o, j_o))$$

Using the definition of V in (5.46), we conclude that (5.44) holds with λ_0 in (5.45). □

5.4 Robustness Analysis

In this section, we study the robustness properties induced by our proposed algorithm with respect to bounded hybrid noise on measurements of the plant state.

Given $\phi_c, \phi_d : E \to \mathbb{R}^{n\times p}$ and $u : E \to \mathbb{R}^m$, where $E := \operatorname{dom} \phi_c = \operatorname{dom} \phi_d = \operatorname{dom} u$ is a hybrid time domain, consider additive noise $\nu : E \to \mathbb{R}^n$ in the measurements of the plant state x in (5.1).[18] We employ $\mathcal{H}$ in (5.2) to estimate the unknown parameter θ in (5.1). The measurement noise ν affects the dynamics of the parameter estimate as follows. Let $\mathcal{X}_c'$ and $\mathcal{F}_c'$ denote $\mathcal{X}_c$ and $\mathcal{F}_c$, respectively, in (5.5) under the effect of the measurement noise ν. Hence, $\mathcal{X}_c'$ is

$$\mathcal{X}_c'(t,j) := \mathcal{X}_c(t,j) + \sum_{i=\underline{j}}^{j} \nu(\min\{t, t_{i+1}\}, i) - \nu(\max\{\underline{t}, t_i\}, i)$$

[18]By definition of a solution pair, the measurement noise ν has the same hybrid time domain as x, ϕ_c, ϕ_d, and u.

and $\mathcal{F}'_c$ is

$$\begin{aligned}\mathcal{F}'_c(t,j) &:= \sum_{i=\underline{j}}^{j} \int_{\max\{\underline{t},t_i\}}^{\min\{t,t_{i+1}\}} f_c(x(s,i)+\nu(s,i),u(s,i))ds \\ &= \mathcal{F}_c(t,j) + \sum_{i=\underline{j}}^{j} \int_{\max\{\underline{t},t_i\}}^{\min\{t,t_{i+1}\}} \Big(f_c(x(s,i)+\nu(s,i),u(s,i)) - f_c(x(s,i),u(s,i))\Big)ds\end{aligned}$$

for all $(t,j) \in E$. Denoting $\mathcal{Y}'_c$ as the term $\mathcal{Y}_c$ in (5.7) under the effect of the measurement noise, it follows that

$$\mathcal{Y}'_c(t,j) := \mathcal{X}'_c(t,j) - \mathcal{F}'_c(t,j) = \mathcal{Y}_c(t,j) + \mathcal{V}_c(t,j) \tag{5.49}$$

where

$$\begin{aligned}\mathcal{V}_c(t,j) :=& \sum_{i=\underline{j}}^{j} \nu(\min\{t,t_{i+1}\},i) - \nu(\max\{\underline{t},t_i\},i) \\ &- \sum_{i=\underline{j}}^{j} \int_{\max\{\underline{t},t_i\}}^{\min\{t,t_{i+1}\}} \Big(f_c(x(s,i)+\nu(s,i),u(s,i)) - f_c(x(s,i),u(s,i))\Big)ds.\end{aligned} \tag{5.50}$$

for all $(t,j) \in E$. As in Section 5.2, the hybrid signals $\mathcal{Y}'_c$ and Φ_c are sampled and stored during flows. We denote the matrix used to store the samples of $\mathcal{Y}'_c$ as $(t,j) \mapsto Y'_c(t,j) \in \mathbb{R}^{n\times N_c}$. From (5.49), it follows that the columns of Y'_c, denoted by $Y'_{c,\ell}$ for all $\ell \in \{1,2,\cdots,N_c\}$, can be decomposed into "nominal" and "noise" terms as

$$Y'_{c,\ell}(t,j) = Y_{c,\ell}(t,j) + V_{c,\ell}(t,j) \tag{5.51}$$

for all $(t,j) \in E$, where Y^c_ℓ excludes the effect of the measurement noise, and $V_{c,\ell}$ gives the effect of the measurement noise – that is, the value of $(t,j) \mapsto \mathcal{V}_c(t,j)$ when a sample of $\mathcal{Y}'_c$ was stored in $Y'_{c,\ell}$.

Using (5.49) and (5.51), the flow map for $\hat{\theta}$ in (5.10) becomes

$$\begin{aligned}\dot{\hat{\theta}} &= \gamma_c \Phi_c(t,j)^\top (\mathcal{Y}_c(t,j) - \Phi_c(t,j)\hat{\theta}) \\ &\quad + \rho_c \sum_{\ell=1}^{N_c} \Big(Z_{c,\ell}(t,j)^\top (Y_{c,\ell}(t,j) - Z_{c,\ell}(t,j)\hat{\theta}) \Big) \\ &\quad + \gamma_c \Phi_c(t,j)^\top \mathcal{V}_c(t,j) + \rho_c \sum_{\ell=1}^{N_c} Z_{c,\ell}(t,j)^\top V_{c,\ell}(t,j) \\ &= \alpha_c(\xi) - d_c(t,j),\end{aligned}$$

with α_c as in (5.10) and, for all $(t,j) \in E$,

$$d_c(t,j) := -\gamma_c \Phi_c(t,j)^\top \mathcal{V}_c(t,j) - \rho_c \sum_{\ell=1}^{N_c} Z_{c,\ell}(t,j)^\top V_{c,\ell}(t,j). \tag{5.52}$$

Next, let $\mathcal{X}_d'$ and $\mathcal{G}_d'$ denote $\mathcal{X}_d$ and $\mathcal{G}_d$, respectively, in (5.12) under the effect of the measurement noise ν. Hence, $\mathcal{X}_d'$ is

$$\mathcal{X}_d'(t,j) := \mathcal{X}_d(t,j) + \sum_{i=\underline{j}}^{j-1} \nu(t_{i+1}, i+1)$$

and $\mathcal{G}_d'$ is

$$\begin{aligned}\mathcal{G}_d'(t,j) &:= \sum_{i=\underline{j}}^{j-1} g_d(x(t_{i+1},i) + \nu(t_{i+1},i), u(t_{i+1},i)) \\ &= \mathcal{G}_d(t,j) + \sum_{i=\underline{j}}^{j-1} \Big(g_d(x(t_{i+1},i) + \nu(t_{i+1},i), u(t_{i+1},i)) - g_d(x(t_{i+1},i), u(t_{i+1},i)) \Big)\end{aligned}$$

for all $(t,j) \in E$. Denoting $\mathcal{Y}_d'$ as the term $\mathcal{Y}_d$ in (5.14) under the effect of the measurement noise, it follows that

$$\mathcal{Y}_d'(t,j) := \mathcal{X}_d'(t,j) - \mathcal{G}_d'(t,j) = \mathcal{Y}_d(t,j) + \mathcal{V}_d(t,j) \tag{5.53}$$

where

$$\mathcal{V}_d(t,j) := \sum_{i=\underline{j}}^{j-1} \nu(t_{i+1}, i+1) \\ - \sum_{i=\underline{j}}^{j-1} \Big(g_d(x(t_{i+1}, i) + \nu(t_{i+1}, i), u(t_{i+1}, i)) - g_d(x(t_{i+1}, i), u(t_{i+1}, i)) \Big) \tag{5.54}$$

for all $(t,j) \in E$. As in Section 5.2, the hybrid signals $\mathcal{Y}'_d$ and Φ_d are sampled and stored at jumps. We denote the matrix used to store the samples of $\mathcal{Y}'_d$ as $(t,j) \mapsto Y'_d(t,j) \in \mathbb{R}^{n \times N_d}$. From (5.53), it follows that the columns of Y'_d, denoted by $Y'_{d,\ell}$ for all $\ell \in \{1, 2, \cdots, N_d\}$, can be decomposed into "nominal" and "noise" terms as

$$Y'_{d,\ell}(t,j) = Y_{d,\ell}(t,j) + V_{d,\ell}(t,j) \tag{5.55}$$

for all $(t,j) \in E$, where $Y_{d,\ell}$ excludes the effect of the measurement noise, and $V_{d,\ell}$ gives the effect of the measurement noise – that is, the value of $(t,j) \mapsto \mathcal{V}_d(t,j)$ when a sample of $\mathcal{Y}'_d$ was stored in $Y'_{d,\ell}$.

Using (5.53) and (5.55), the jump map for $\hat{\theta}$ in (5.17) becomes

$$\begin{aligned} \hat{\theta}^+ &= \hat{\theta} + \Gamma \frac{\Phi_d(t,j+1)^\top (\mathcal{Y}_d(t,j+1) - \Phi_d(t,j+1)\hat{\theta})}{\gamma_d + |\Phi_d(t,j+1)|^2} \\ &\quad + \Gamma \frac{\sum_{\ell=1}^{N_d} \Big(Z_{d,\ell}(t,j+1)^\top \big(Y_{d,\ell}(t,j+1) - Z_{d,\ell}(t,j+1)\hat{\theta} \big) \Big)}{\rho_d + \sum_{\ell=1}^{N_d} |Z_{d,\ell}(t,j+1)|^2} \\ &\quad + \Gamma \frac{\Phi_d(t,j+1)^\top \mathcal{V}_d(t,j+1)}{\gamma_d + |\Phi_d(t,j+1)|^2} + \Gamma \frac{\sum_{\ell=1}^{N_d} Z_{d,\ell}(t,j+1)^\top V_{d,\ell}(t,j+1)}{\rho_d + \sum_{\ell=1}^{N_d} |Z_{d,\ell}(t,j+1)|^2} \\ &= \alpha_d(\xi) - d_d(t,j), \end{aligned}$$

with α_d as in (5.17) and, for all $(t,j) \in \Upsilon(E)$,

$$\begin{aligned} d_d(t,j) := &- \Gamma \frac{\Phi_d(t,j+1)^\top \mathcal{V}_d(t,j+1)}{\gamma_d + |\Phi_d(t,j+1)|^2} \\ &- \Gamma \frac{\sum_{\ell=1}^{N_d} Z_{d,\ell}(t,j+1)^\top V_{d,\ell}(t,j+1)}{\rho_d + \sum_{\ell=1}^{N_d} |Z_{d,\ell}(t,j+1)|^2}. \end{aligned} \tag{5.56}$$

We denote the hybrid system $\mathcal{H}$ under the effect of the state measurements noise ν as $\mathcal{H}_\nu$, with state $\xi := (x, \hat{\theta}, \tau, k) \in \mathcal{X} := \mathbb{R}^n \times \mathbb{R}^p \times E$, and dynamics

$$\mathcal{H}_\nu : \begin{cases} \dot{\xi} = \begin{bmatrix} f_c(x, u(\tau,k)) + \phi_c(\tau,k)\theta \\ \alpha_c(\xi) - d_c(\tau,k) \\ 1 \\ 0 \end{bmatrix} =: F_\nu(\xi) & \xi \in C_\nu \\ \xi^+ = \begin{bmatrix} g_d(x, u(\tau,k)) + \phi_d(\tau,k)\theta \\ \alpha_d(\xi) - d_d(\tau,k) \\ \tau \\ k+1 \end{bmatrix} =: G_\nu(\xi) & \xi \in D_\nu \end{cases} \tag{5.57}$$

where $C_\nu := C$ and $D_\nu := D$, with C and D as in (5.3).

To enable analysis on the noise effect, we make the following Lipschitz continuity assumption.

Assumption 5.13: Given the hybrid plant in (5.1), there exist $L_c, L_d > 0$ such that, for all $x_1, x_2 \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$|f_c(x_1, u) - f_c(x_2, u)| \leq L_c |x_1 - x_2|,$$
$$|g_d(x_1, u) - g_d(x_2, u)| \leq L_d |x_1 - x_2|.$$

Then, we establish the following input-to-state stability result for $\mathcal{H}_\nu$.

Theorem 5.14: Given $\Lambda > 0$, $N_c, N_d \in \mathbb{N}$, $\phi_c, \phi_d : E \to \mathbb{R}^{n \times p}$, $u : E \to \mathbb{R}^m$, $\nu : E \to \mathbb{R}^n$, $Z_c : E \to \mathbb{R}^{n \times p \times N_c}$, $Z_d : E \to \mathbb{R}^{n \times p \times N_d}$, $Y_c : E \to \mathbb{R}^{n \times N_c}$, and $Y_d : E \to \mathbb{R}^{n \times N_d}$ defining the hybrid system $\mathcal{H}_\nu$ in (5.57), where $E := \operatorname{dom} \phi_c = \operatorname{dom} \phi_d = \operatorname{dom} u = \operatorname{dom} \nu = \operatorname{dom} Z_c = \operatorname{dom} Z_d = \operatorname{dom} Y_c = \operatorname{dom} Y_d$ is a hybrid time domain, suppose that Assumptions 5.4, 5.5, and 5.13 hold. Then, for each $\gamma_c, \gamma_d, \rho_c, \rho_d > 0$, $\Gamma \in (0, 1/2]$, $q_M \geq q_m > 0$, $\zeta \in (0,1)$, the parameter estimation error $(t,j) \mapsto \tilde{\theta}(t,j) := \theta - \hat{\theta}(t,j)$ for each solution ξ to $\mathcal{H}_\nu$ satisfies

$$|\tilde{\theta}(t,j)| \leq \beta(|\tilde{\theta}(0,0)|, t+j) + \alpha \|d\|_{(t,j)} \tag{5.58}$$

for all $(t,j) \in \operatorname{dom}\xi$*, where*

$$\beta(s,r) := \sqrt{\frac{p_M}{p_m}} \mathrm{e}^{-\omega r} s, \quad \alpha := \sqrt{\frac{2p_M^3}{q_m p_m \zeta}\left(\frac{2p_M}{q_m}+1\right)}$$

$$d(t,j) := \begin{cases} d_c(t,j) & \text{if} \quad (t,j) \in E \setminus \Upsilon(E) \\ d_d(t,j) & \text{if} \quad (t,j) \in \Upsilon(E) \end{cases}$$

with d_c *as in* (5.52)*,* d_d *as in* (5.56)*,* Υ *as in* (2.2)*, and*

$$\omega := \frac{1}{2}\min\left\{\frac{q_m}{2p_M}(1-\zeta), -\ln\left(1-\frac{q_m}{2p_M}(1-\zeta)\right)\right\}$$

$$p_m := q_m, \qquad p_M := q_m + \frac{q_M\kappa^2}{2\lambda} + \frac{q_M\kappa^2\mathrm{e}^{2\lambda}}{\mathrm{e}^{2\lambda}-1}$$

where κ *and* λ *are defined in Theorem 5.7. Moreover, for all* $(t,j) \in E$*,*

$$|d_c(t,j)| \leq (\gamma_c + \rho_c N_c)\phi_M\big(2(\Lambda+1)(\Lambda+2) + L_c(\Lambda+1)^2\big)\|\nu\|_{(t,j)} \tag{5.59}$$

and, for all $(t,j) \in \Upsilon(E)$*,*

$$|d_d(t,j)| \leq \frac{\Gamma(1+L_d)(\Lambda+1)}{2}\left(\frac{1}{\sqrt{\gamma_d}} + \sqrt{\frac{N_d}{\rho_d}}\right)\|\nu\|_{(t,j+1)} \tag{5.60}$$

with L_c *and* L_d *from Assumption 5.13.*

Proof. Using the same arguments as in the proof of Theorem 5.7, we conclude that, by Assumptions 5.4 and 5.5, the conditions of Theorem 3.5 are satisfied with υ, Δ, and a_M from Assumptions 5.4 and 5.5. It can be shown that the hybrid system that is obtained by expressing $\mathcal{H}_\nu$ in error coordinates is equivalent to $\mathcal{H}$ in (3.8) with A, B in (5.30) and d_c in (5.52), and d_d in (5.56). Hence, it follows from Theorem 3.5 that, for each solution ξ to $\mathcal{H}_\nu$, the parameter estimation error $\tilde{\theta}$ satisfies (3.14) for all $(t,j) \in \operatorname{dom}\xi$, with $(t,j) \mapsto d(t,j)$ as in (3.15) and ρ, ω from Theorem 3.5, with p_m, p_M substituted by p_m, p_M from Theorem 5.14.

To complete the proof, we show that the bounds on d_c and d_d in (5.59) and (5.60) hold. To bound d_c, we use that, from (5.50),

$$|\mathcal{V}_c(t,j)| \le \sum_{i=\underline{j}}^{j} 2\|\nu\|_{(t,j)} + \sum_{i=\underline{j}}^{j} \int_{\max\{\underline{t},t_i\}}^{\min\{t,t_{i+1}\}} L_c\|\nu\|_{(t,j)} ds$$
$$\le (\Lambda+2)2\|\nu\|_{(t,j)} + (\Lambda+1)L_c\|\nu\|_{(t,j)} \tag{5.61}$$

for all $(t,j) \in E$, with L_c from Assumption 5.13, where the inequality follows from (5.32) and the fact that, by the definition of $\underline{t}$ and $\underline{j}$ in (5.4), $(t-\underline{t})+(j-\underline{j}) \le \Lambda+1$ for all $(t,j) \in \operatorname{dom}\xi$, and thus $j - \underline{j} + 1 \le \Lambda + 2$. Moreover, since $V_{c,\ell}$ gives the value of $\mathcal{V}_c$ when samples of $\mathcal{Y}'_c$ are stored in column ℓ of Y'_c, it follows from (5.61) that

$$|V_{c,\ell}(t,j)| \le |\mathcal{V}_c(t,j)| \le (\Lambda+2)2\|\nu\|_{(t,j)} + (\Lambda+1)L_c\|\nu\|_{(t,j)}$$

for all $(t,j) \in \operatorname{dom}\xi$ and all $\ell \in \{1,2,\cdots N_c\}$. Then, we have from (5.52), (5.31), and (5.33) that, for all $(t,j) \in E$,

$$\begin{aligned}
|d_c(t,j)| &\le \gamma_c|\Phi_c(t,j)||\mathcal{V}_c(t,j)| + \rho_c\sum_{\ell=1}^{N_c}|Z_{c,\ell}(t,j)||V_{c,\ell}(t,j)| \\
&\le \gamma_c\phi_M\big(2(\Lambda+1)(\Lambda+2) + L_c(\Lambda+1)^2\big)\|\nu\|_{(t,j)} \\
&\quad + \rho_c\sum_{\ell=1}^{N_c}\phi_M\big(2(\Lambda+1)(\Lambda+2) + L_c(\Lambda+1)^2\big)\|\nu\|_{(t,j)} \\
&\le (\gamma_c+\rho_c N_c)\phi_M\big(2(\Lambda+1)(\Lambda+2) + L_c(\Lambda+1)^2\big)\|\nu\|_{(t,j)}
\end{aligned}$$

Hence, (5.59) holds.

To bound d_d, we use that

$$\frac{|\Phi_d(t,j+1)|}{\gamma_d + |\Phi_d(t,j+1)|^2} \le \frac{1}{2\sqrt{\gamma_d}}$$

for all $(t,j) \in \Upsilon(E)$ and, from (5.54),

$$|\mathcal{V}_d(t,j)| \leq \sum_{i=\underline{j}}^{j-1} \|\nu\|_{(t,j+1)} + \sum_{i=\underline{j}}^{j-1} L_d \|\nu\|_{(t,j)} \leq (1+L_d)(\Lambda+1)\|\nu\|_{(t,j+1)}$$

for all $(t,j) \in E$, with L_d from Assumption 5.13, where the inequality follows from the fact that, by the definition of $\underline{t}$ and $\underline{j}$ in (5.4), $(t-\underline{t}) + (j-\underline{j}) \leq \Lambda + 1$ for all $(t,j) \in \operatorname{dom}\xi$, and thus $j - \underline{j} \leq \Lambda + 1$. Moreover, since $V_{d,\ell}$ gives the value of $\mathcal{V}_d$ when samples of $\mathcal{Y}'_d$ are stored in column ℓ of Y'_d, it follows that

$$|V_{d,\ell}(t,j)| \leq |\mathcal{V}_d(t,j)| \leq (1+L_d)(\Lambda+1)\|\nu\|_{(t,j+1)}$$

for all $(t,j) \in \Upsilon(E)$ and all $\ell \in \{1, 2, \cdots N_d\}$. Furthermore, from the Cauchy-Schwarz inequality,

$$\begin{aligned}
\frac{\sum_{\ell=1}^{N_d} |Z_{d,\ell}(t,j+1)|}{\rho_d + \sum_{\ell=1}^{N_d} |Z_{d,\ell}(t,j+1)|^2} &\leq \frac{\sum_{\ell=1}^{N_d} |Z_{d,\ell}(t,j+1)|}{\rho_d + \frac{1}{N_d}\left(\sum_{\ell=1}^{N_d} |Z_{d,\ell}(t,j+1)|\right)^2} \\
&\leq \frac{N_d \sum_{\ell=1}^{N_d} |Z_{d,\ell}(t,j+1)|}{\rho_d N_d + \left(\sum_{\ell=1}^{N_d} |Z_{d,\ell}(t,j+1)|\right)^2} \\
&\leq \frac{N_d}{2\sqrt{\rho_d N_d}} = \frac{1}{2}\sqrt{\frac{N_d}{\rho_d}}.
\end{aligned}$$

Thus, we have from (5.56) that, for all $(t,j) \in \Upsilon(E)$,

$$\begin{aligned}
|d_d(t,j)| &\leq \Gamma \frac{|\Phi_d(t,j+1)||\mathcal{V}_d(t,j+1)|}{\gamma_d + |\Phi_d(t,j+1)|^2} + \Gamma \frac{\sum_{\ell=1}^{N_d} |Z_{d,\ell}(t,j+1)||V_{d,\ell}(t,j+1)|}{\rho_d + \sum_{\ell=1}^{N_d} |Z_{d,\ell}(t,j+1)|^2} \\
&\leq \frac{\Gamma(1+L_d)(\Lambda+1)}{2\sqrt{\gamma_d}}\|\nu\|_{(t,j+1)} + \frac{\Gamma(1+L_d)(\Lambda+1)}{2}\sqrt{\frac{N_d}{\rho_d}}\|\nu\|_{(t,j+1)}.
\end{aligned}$$

Hence, (5.60) holds. □

Remark 5.15: Using similar arguments as in the proof of Theorem 5.14, ISS results can also be established for $\mathcal{H}_\nu$ under the point-wise excitation conditions in Theorems 5.11 and 5.12.

5.5 Finite-Time Parameter Estimation

In this section we show that, by augmenting the dynamics of $\mathcal{H}$ in (5.2), the data stored in Z_c and Z_d can be used to achieve finite-time convergence of the parameter estimate to the true value.

Given $Y_c : E \to \mathbb{R}^{n\times N_c}$ and $Z_c : E \to \mathbb{R}^{n\times p\times N_c}$ satisfying (5.9), and $Y_d : E \to \mathbb{R}^{n\times N_d}$ and $Z_d : E \to \mathbb{R}^{n\times p\times N_d}$ satisfying (5.16), we pre-multiply (5.9) by ${Z_{c,\ell}}^\top$ to obtain

$$Z_{c,\ell}(t,j)^\top Y_{c,\ell}(t,j) = Z_{c,\ell}(t,j)^\top Z_{c,\ell}(t,j)\theta$$

for all $(t,j)\in E$ and all $\ell\in\{1,2,\cdots,N_c\}$. Similarly, we pre-multiply (5.16) by ${Z_{d,\ell}}^\top$ to obtain

$$Z_{d,\ell}(t,j)^\top Y_{d,\ell}(t,j) = Z_{d,\ell}(t,j)^\top Z_{d,\ell}(t,j)\theta$$

for all $(t,j)\in E$ and all $\ell\in\{1,2,\cdots,N_d\}$. Since the expressions above hold for all $(t,j)\in E$, we sum them to obtain

$$\begin{aligned}
&\sum_{\ell=1}^{N_c} Z_{c,\ell}(t,j)^\top Y_{c,\ell}(t,j) + \sum_{\ell=1}^{N_d} Z_{d,\ell}(t,j)^\top Y_{d,\ell}(t,j) \\
&\quad= \left(\sum_{\ell=1}^{N_c} Z_{c,\ell}(t,j)^\top Z_{c,\ell}(t,j) + \sum_{\ell=1}^{N_d} Z_{d,\ell}(t,j)^\top Z_{d,\ell}(t,j)\right)\theta \\
&\quad= \big(\mathcal{Z}_c(t,j) + \mathcal{Z}_d(t,j)\big)\theta
\end{aligned}$$

for all $(t,j)\in E$, with $\mathcal{Z}_c$ as in (5.21) and $\mathcal{Z}_d$ as in (5.23). Then, by defining

$$\begin{aligned}
\mathcal{W}(t,j) &:= \sum_{\ell=1}^{N_c} Z_{c,\ell}^\top(t,j) Y_{c,\ell}(t,j) + \sum_{\ell=1}^{N_d} Z_{d,\ell}^\top(t,j) Y_{d,\ell}(t,j) \\
\mathcal{Z}(t,j) &:= \mathcal{Z}_c(t,j) + \mathcal{Z}_d(t,j)
\end{aligned} \tag{5.62}$$

it follows that

$$\mathcal{W}(t,j) = \mathcal{Z}(t,j)\theta \quad \forall (t,j) \in E.$$

Hence, if there exists $(t,j) \in E$ such that $\mathcal{Z}(t,j)$ is invertible, the value of θ is

$$\theta = \mathcal{Z}(t,j)^{-1}\mathcal{W}(t,j). \tag{5.63}$$

Motivated by the analysis above, we modify our hybrid ICL algorithm in (5.2) to estimate θ in finite-time by including an additional jump set and associated jump maps. The new jump maps compute θ as in (5.63), and the new jump set ensures that the inverse of $\mathcal{Z}$ is well defined. We denote this hybrid algorithm as $\mathcal{H}_{\mathrm{FT}}$, with state $\xi = (x, \hat{\theta}, \tau, k) \in \mathcal{X}$ and dynamics

$$\begin{aligned} \dot{\xi} &= F_{\mathrm{FT}}(\xi) & \quad \xi \in C_{\mathrm{FT}} \\ \xi^+ &= G_{\mathrm{FT}}(\xi) & \quad \xi \in D_{\mathrm{FT}} \end{aligned} \tag{5.64}$$

where

$$F_{\mathrm{FT}}(\xi) := \begin{bmatrix} f_c(x, u(\tau,k)) + \phi_c(\tau,k)\theta \\ \alpha_c(\xi) \\ 1 \\ 0 \end{bmatrix} \tag{5.65}$$

$$G_{\mathrm{FT}}(\xi) := \begin{cases} \begin{bmatrix} g_d(x, u(\tau,k)) + \phi_d(\tau,k)\theta \\ \alpha_d(\xi) \\ \tau \\ k+1 \end{bmatrix} =: G^1_{\mathrm{FT}}(\xi) & \xi \in D^1_{\mathrm{FT}} \setminus D^2_{\mathrm{FT}} \\ \begin{bmatrix} g_d(x, u(\tau,k)) + \phi_d(\tau,k)\theta \\ \mathcal{Z}(\tau,k)^{-1}\mathcal{W}(\tau,k) \\ \tau \\ k+1 \end{bmatrix} =: G^2_{\mathrm{FT}}(\xi) & \xi \in D^1_{\mathrm{FT}} \cap D^2_{\mathrm{FT}} \\ \begin{bmatrix} x \\ \mathcal{Z}(\tau,k)^{-1}\mathcal{W}(\tau,k) \\ \tau \\ k \end{bmatrix} =: G^3_{\mathrm{FT}}(\xi) & \xi \in D^2_{\mathrm{FT}} \setminus D^1_{\mathrm{FT}} \end{cases}$$

with f as in (5.10), g as in (5.17), $D_{\mathrm{FT}} := D^1_{\mathrm{FT}} \cup D^2_{\mathrm{FT}}$, and

$$\begin{aligned} C_{\mathrm{FT}} &:= \mathrm{cl}\left(\mathcal{X} \setminus D_{\mathrm{FT}}\right) \\ D^1_{\mathrm{FT}} &:= \{\xi \in \mathcal{X} : (\tau, k+1) \in E\} \\ D^2_{\mathrm{FT}} &:= \{\xi \in \mathcal{X} : \mathrm{cond}(\mathcal{Z}(\tau, k)) \leq \sigma\} \end{aligned}$$

where $\mathrm{cond}(\mathcal{Z}(\tau, k)) := |\mathcal{Z}(\tau, k)^{-1}||\mathcal{Z}(\tau, k)|$ gives the condition number of the matrix $\mathcal{Z}(\tau, k)$, and $\sigma \in [1, \infty)$ is a design parameter.

The jump set D^2_{FT} and the design parameter σ ensure that the inverse of $\mathcal{Z}$ is well defined at jumps per the jump maps G^2_{FT} and G^3_{FT}. Each time $\mathcal{H}_{\mathrm{FT}}$ jumps based on the jump maps G^2_{FT} or G^3_{FT}, the history stacks Z_c, Y_c, Z_d, and Y_d are reset to zero, thereby ensuring that $\mathcal{H}_{\mathrm{FT}}$ subsequently flows or jumps in accordance with the hybrid time domain E.

We establish the following result stating conditions that ensure each complete solution ξ to $\mathcal{H}$ converges in finite-time to the set $\mathcal{A}$.

Theorem 5.16: Given $\Lambda > 0$, $N_c, N_d \in \mathbb{N}$, $\sigma \in [1, \infty)$, $\phi_c, \phi_d : E \to \mathbb{R}^{n \times p}$, $u : E \to \mathbb{R}^m$, $Z_c : E \to \mathbb{R}^{n \times p \times N_c}$, $Z_d : E \to \mathbb{R}^{n \times p \times N_d}$, $Y_c : E \to \mathbb{R}^{n \times N_c}$, *and* $Y_d : E \to \mathbb{R}^{n \times N_d}$ *defining the hybrid system* $\mathcal{H}_{\mathrm{FT}}$ *in* (5.64), *where* $E := \mathrm{dom}\, \phi_c = \mathrm{dom}\, \phi_d = \mathrm{dom}\, u = \mathrm{dom}\, Z_c = \mathrm{dom}\, Z_d = \mathrm{dom}\, Y_c = \mathrm{dom}\, Y_d$ *is a hybrid time domain, suppose that there exists* $(t^*, j^*) \in E$ *such that* $\mathrm{cond}(\mathcal{Z}(t^*, j^*)) \leq \sigma$, *with* $\mathcal{Z}$ *as in* (5.62). *Then, for each* $\gamma_c, \gamma_d, \rho_c, \rho_d > 0$ *and each* $\Gamma \in (0, 1/2]$, *the parameter estimation error* $(t, j) \mapsto \tilde{\theta}(t, j) := \theta - \hat{\theta}(t, j)$ *for each solution* ξ *to* $\mathcal{H}_{\mathrm{FT}}$ *satisfies*

$$|\tilde{\theta}(t, j)| \leq |\tilde{\theta}(0, 0)| \tag{5.66}$$

for all $(t, j) \in \mathrm{dom}\, \xi$, *and the parameter estimate satisfies*

$$|\hat{\theta}(t, j)| = \theta \tag{5.67}$$

for all $(t, j) \in \mathrm{dom}\, \xi$ *satisfying* $t \geq t^*$ *and* $j \geq j^* + 1$.

Proof. Consider the Lyapunov function

$$V(\xi) := \frac{1}{2}\tilde{\theta}^\top \tilde{\theta} \qquad \forall \xi \in C_{\mathrm{FT}} \cup D_{\mathrm{FT}} \tag{5.68}$$

where $\tilde{\theta} = \theta - \hat{\theta}$. Using the same arguments as in the proof of Theorem 5.10, we conclude that, for all $\xi \in C_{\mathrm{FT}}$,

$$\langle \nabla V(\xi), F_{\mathrm{FT}}(\xi) \rangle \leq 0 \tag{5.69}$$

and for all $\xi \in D^1_{\mathrm{FT}} \setminus D^2_{\mathrm{FT}}$,

$$V(G^1_{\mathrm{FT}}(\xi)) - V(\xi) \leq 0. \tag{5.70}$$

Hence, from the definition of V in (5.69), it follows that $|\xi|_{\mathcal{A}}$ is nonincreasing during flows, and at jumps from $D^1_{\mathrm{FT}} \setminus D^2_{\mathrm{FT}}$.

To complete the proof, pick a maximal solution ξ to $\mathcal{H}_{\mathrm{FT}}$. By assumption, there exists $(t^*, j^*) \in \operatorname{dom} \xi$ such that $\operatorname{cond}(\mathcal{Z}(t^*, j^*)) \leq \sigma$ and thus $\xi(t^*, j^*) \in D^2_{\mathrm{FT}}$. Since ξ is maximal, the solution jumps per the jump map G_{FT} in (5.65) and it follows from (5.63) that

$$\hat{\theta}(t^*, j^* + 1) = \theta. \tag{5.71}$$

Then, from the bounds in (5.69) and (5.70) and from the definition of V in (5.69), it follows that $\hat{\theta}(t, j) = \theta$ for all $(t, j) \in \operatorname{dom} \xi$ such that $t \geq t^*$ and $j \geq j^* + 1$. Hence (5.67) holds. Finally, in view of (5.69), (5.70), and (5.71) we conclude that (5.66) holds. □

5.6 Data Recording

In this section, we propose algorithms to select data for storage during flows and jumps, with the objective of satisfying the conditions of Theorem 5.11 and Theorem 5.12. In particular,

1. (5.38) means that the minimum eigenvalues of $\mathcal{Z}_c$ and $\mathcal{Z}_d$ are non-decreasing;

2. (5.43) means that $\mathcal{Z}_c$ and $\mathcal{Z}_d$ are uniformly positive definite;

3. (5.37) means that $\mathcal{Z}_c + \mathcal{Z}_d$ is uniformly positive definite.

From the proof of Theorem 5.12, we have that the convergence rate of the parameter estimate is proportional to $\lambda_{\min}(\mathcal{Z}_c)$ during flows, and $\lambda_{\min}(\mathcal{Z}_d)/(\rho_d+\lambda_{\max}(\mathcal{Z}_d))$ at jumps. Hence, we store data with the primary objective of maximizing $\lambda_{\min}(\mathcal{Z}_c)$ during flows, and maximizing $\lambda_{\min}(\mathcal{Z}_d)/(\rho_d + \lambda_{\max}(\mathcal{Z}_d))$ at jumps. However, in contrast to [16, 18], we do not assume that the data can be sampled so that $\mathcal{Z}_c$ and $\mathcal{Z}_d$ are full rank – see Remark 5.6. In such cases, we select data for storage with the secondary objective of maximizing $\lambda_{\min}(\mathcal{Z}_c + \mathcal{Z}_d)$.

Our proposed data recording algorithms are inspired by [16, 18]. In words, the algorithms select data for storage based on the following criteria. Given measurements of Φ_c and $\mathcal{Y}_c$ during flows,

1. if Z_c has empty (zero) layers[19] and Φ_c is nonzero, then Φ_c is stored in an empty layer of Z_c;

2. if $\mathcal{Z}_c$ is full rank and Φ_c increases $\lambda_{\min}(\mathcal{Z}_c)$, then Φ_c is stored in the layer of Z_c that maximizes $\lambda_{\min}(\mathcal{Z}_c)$;

3. if $\mathcal{Z}_c$ is not full rank and Φ_c increases the rank$(\mathcal{Z}_c)$, then Φ_c is stored in the layer of Z_c that maximizes rank$(\mathcal{Z}_c)$;

4. if $\mathcal{Z}_c$ is not full rank and Φ_c increases $\lambda_{\min}(\mathcal{Z}_c + \mathcal{Z}_d)$, then Φ_c is stored in the layer of Z_c that maximizes $\lambda_{\min}(\mathcal{Z}_c + \mathcal{Z}_d)$;

5. if none of the items above are satisfied, Φ_c is not stored.

[19] Recall that we refer to the third dimension of Z_c as the "layers" of Z_c.

Whenever Φ_c is stored in a layer of Z_c, the current value of the hybrid signal $\mathcal{Y}_c$ is stored in the corresponding column of Y_c. Similar criteria are applied to select measurements for storage at jumps. We implement this logic using Algorithm 2 during flows and Algorithm 3 at jumps – see Appendix D.

5.7 Numerical Examples

5.7.1 Comparison with Hybrid Gradient Descent

Consider the hybrid signals $\phi_c, \phi_d : E \to \mathbb{R}^{2\times 2}$ with hybrid time domain

$$E = \bigcup_{k=0}^{\infty} \left(\Big[2\pi k, \quad \pi(2k+2)\Big] \times \{k\}\right) \tag{5.72}$$

The values of ϕ_c and ϕ_d are

$$\phi_c(t,j) = \mathrm{e}^{-j} \begin{bmatrix} \sin(t) & 0 \\ 0 & 0 \end{bmatrix}, \qquad\qquad \phi_d(t,j) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \tag{5.73}$$

for all $(t,j) \in E$. For the given hybrid signals ϕ_c and ϕ_d, consider a hybrid system as in (2.1) with an added input $u : E \to \mathbb{R}$, and with state $x = (x_1, x_2) \in \mathbb{R}^2$ and dynamics

$$\begin{aligned} \dot{x} &= \phi_c(t,j)\theta \qquad (x, u(t,j)) \in C_P \\ x^+ &= \phi_d(t,j)\theta \qquad (x, u(t,j)) \in D_P \end{aligned} \tag{5.74}$$

where $\theta = [1 \;\; 1]^\top$ is a vector of unknown parameters. The flow and jump sets are

$$\begin{aligned} C_P &= \{(x,u) \in \mathbb{R}^2 \times \mathbb{R} : u \leq 2\pi\} \\ D_P &= \{(x,u) \in \mathbb{R}^2 \times \mathbb{R} : u \geq 2\pi\} \end{aligned}$$

and the input is $u(t,j) = t - 2\pi j$ for all $(t,j) \in E$. Hence, u is a sawtooth function that periodically ramps to a value of 2π and then resets to zero.[20]

[20]With C_P, D_P, and u given below (5.74), the hybrid time domain of each maximal solution x to the hybrid system in (5.74) is equal to the hybrid time domain of ϕ_c and ϕ_d in (5.72).

Suppose our goal is to estimate θ. We first apply the hybrid parameter estimation algorithm proposed in Chapter 4, which we refer to as hybrid GD. The parameter estimation error for the hybrid GD algorithm, shown in blue in Figure 1.2, fails to converge to zero since ϕ_c and ϕ_d do not satisfy the hybrid PE condition in Assumption 4.6. To see why this condition is not satisfied, note that, for the given hybrid arcs in (5.73), $\phi_d(t, j)$ is constant and singular for all $(t, j) \in E$ and $\phi_c(t, j)$ goes to zero as j goes to infinity. On the other hand, by leveraging stored data alongside current measurements to adapt the parameter estimate, our proposed hybrid ICL algorithm successfully estimate θ as shown in green in Figure 1.2.

5.7.2 Finite-time Estimation for the Bouncing Ball

Consider the problem of estimating the acceleration due to gravity and the restitution coefficient for a bouncing ball. The ball has state $x := (x_1, x_2) \in \mathbb{R}^2$, where x_1 is the height above the ground and x_2 is the vertical velocity. The bouncing ball has dynamics [22, Example 1.1]

$$\begin{aligned} \dot{x} &= \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} && x \in C_P \\ x^+ &= \begin{bmatrix} 0 \\ -\lambda x_2 \end{bmatrix} && x \in D_P \end{aligned}$$

where $\gamma > 0$ is the acceleration due to gravity, $\lambda \in (0, 1]$ is the restitution coefficient, $C_P := \{x \in \mathbb{R}^2 : x_1 \geq 0\}$, and $D_P := \{x \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0\}$. The problem of estimating γ and λ for the bouncing ball can be written as the problem of estimating θ for the system in (5.1), where $\theta = (\gamma, \lambda)$ and

$$f_c(x) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \qquad \phi_c(t) = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

$$g_d(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \phi_d(t) = \begin{bmatrix} 0 & 0 \\ 0 & -x_2(t) \end{bmatrix}.$$

We simulate the bouncing ball with $\gamma = 9.81$ and initial condition $x(0,0) = (4.91, 0)$. The trajectory of the state component x_1 when $\lambda = 1$ is shown in blue in Figure 5.1, and the trajectory when $\lambda = 0.7$ is shown in green in Figure 5.1. Note that the bouncing ball exhibits Zeno behavior when $\lambda = 0.7$. That is, the number of jumps in $\operatorname{dom} x$ goes to infinity while the amount of flow remains finite. We employ our proposed hybrid ICL algorithm $\mathcal{H}$ in (5.2) to estimate θ, with design parameters $\gamma_c = \rho_c = 0.1$, $\gamma_d = \rho_d = 10$, $\Gamma = 0.5$, $N_c = N_d = 1$, and $\Lambda = 3$, and initial condition $\hat{\theta}(0,0) = (0,0)$. When $\lambda = 1$, it can be shown that Assumptions 5.4 and 5.5 are satisfied. Thus, $\hat{\theta}$ converges exponentially to θ in accordance with Theorem 5.7, as shown in Figure 5.2.[21]

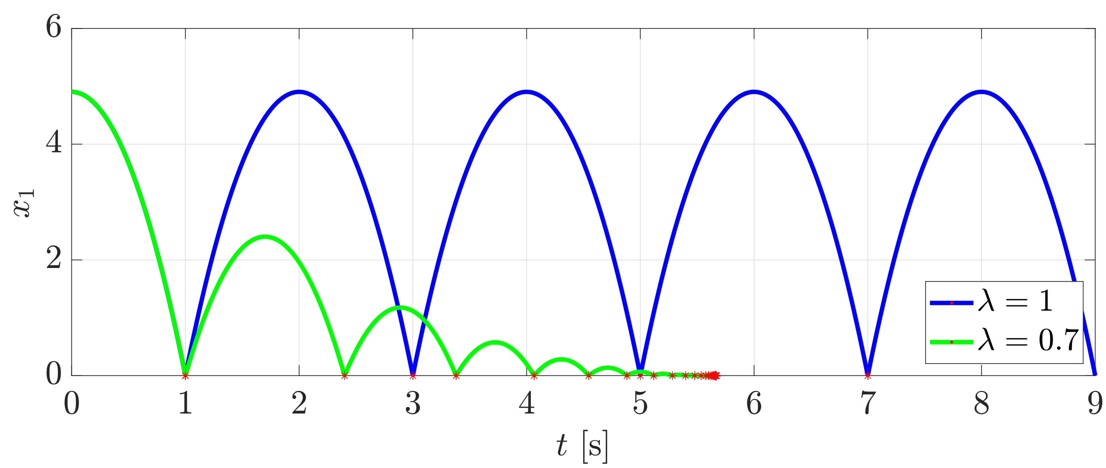

Figure 5.1: The projection onto t of x_1 for the bouncing ball with $\lambda = 1$ and $\lambda = 0.7$.

[21] Code at `https://github.com/HybridSystemsLab/HybridICL_BouncingBall`

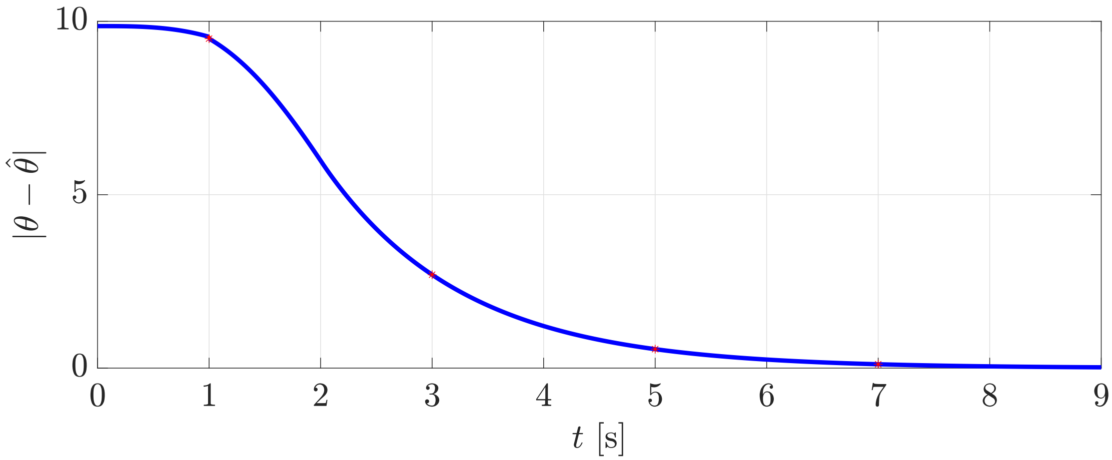

Figure 5.2: The projection onto t of the parameter estimation error for $\mathcal{H}$ with $\lambda = 1$.

Next, consider the case when $\lambda = 0.7$. For all $(t, j) \in E$ satisfying $t + j > \Lambda$, $\mathcal{Z}_c$ and $\mathcal{Z}_d$ have values

$$\mathcal{Z}_c(t, j) = \begin{bmatrix} 3.91 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathcal{Z}_d(t, j) = \begin{bmatrix} 0 & 0 \\ 0 & 96.2 \end{bmatrix}.$$

Hence, $\mathcal{Z}_c(t, j) + \mathcal{Z}_d(t, j)$ is uniformly positive definite for all $(t, j) \in E$ satisfying $t + j > \Lambda$. However, Assumption 5.5 is not satisfied since the duration of each interval of flow in E goes to zero as the number of jumps goes to infinity. Similarly, while $\mathcal{Z}_c$ and $\mathcal{Z}_d$ satisfy (5.37) and (5.38), Assumption 5.8 does not hold since the hybrid time domain E fails to satisfy (5.36). In this case, the parameter estimate $\hat{\theta}$ for $\mathcal{H}$ fails to converge to θ as shown in blue in Figure 5.3. We now employ our proposed finite-time estimation algorithm $\mathcal{H}_{\mathrm{FT}}$ in (5.64) to estimate θ, with the same design parameters and initial conditions as $\mathcal{H}$, and with $\sigma = 25$. It can be shown that $\mathcal{Z}$ in (5.62) satisfies $\mathrm{cond}(\mathcal{Z}(2, 1)) \leq \sigma$. Hence, the conditions of Theorem 5.16 hold with $(t^*, j^*) = (2, 1)$. The parameter estimate for $\mathcal{H}_{\mathrm{FT}}$ converges in finite-time to θ in accordance with Theorem 5.16, as shown in green in Figure 5.3.

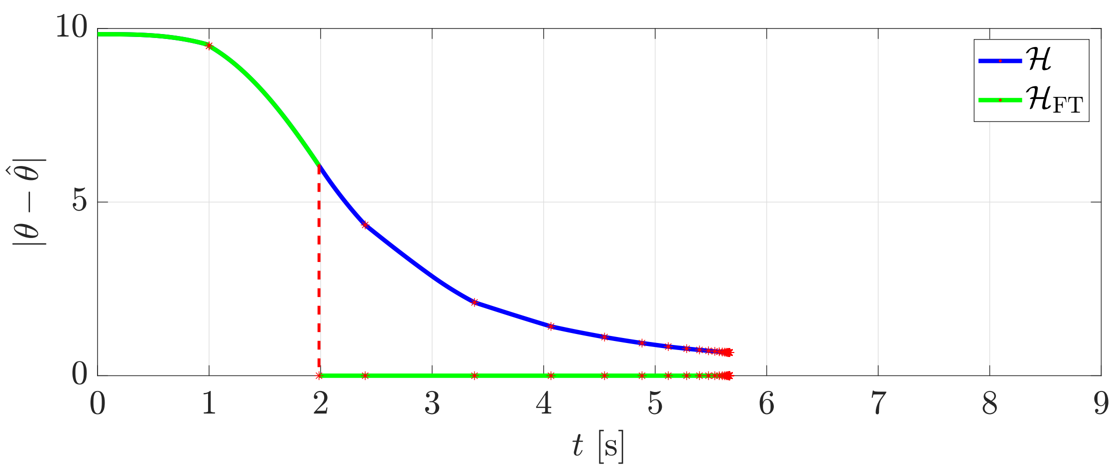

Figure 5.3: The projection onto t of the parameter estimation error for $\mathcal{H}$ and $\mathcal{H}_{\mathrm{FT}}$ with $\lambda = 0.7$.

Chapter 6

Finite-Time Parameter Estimation

In this chapter, we propose a hybrid parameter estimation algorithm for finite-time estimation of unknown parameters for a class of continuous-time systems. The algorithm guarantees convergence of the parameter estimate to the true value when the system input signals are exciting over only a finite interval of time.

6.1 Problem Statement

Consider a continuous-time system of the form

$$\dot{x} = f(x, u(t)) + \phi(t)\theta \tag{6.1}$$

where $x \in \mathbb{R}^n$ is the known state vector, $t \mapsto u(t) \in \mathbb{R}^r$ is the known input, $t \mapsto \phi(t) \in \mathbb{R}^{n\times p}$ is the known regressor, $(x, u) \mapsto f(x, u) \in \mathbb{R}^n$ is a known continuous function, $\theta \in \mathbb{R}^p$ is a vector of unknown constant parameters, and $n, r, p \in \mathbb{N}$.

We propose to estimate θ using a hybrid algorithm, denoted by $\mathcal{H}$ as in (2.1), with data designed to solve the following problem.

Problem 6.1: Design the data (C, F, D, G) of $\mathcal{H}$ and determine conditions on ϕ that ensure the parameter estimate $\hat{\theta}$ converges in finite time to the unknown parameter vector θ in (6.1).

6.2 Problem Solution

Given $t \mapsto \phi(t) \in \mathbb{R}^{n\times p}$ and $t \mapsto u(t) \in \mathbb{R}^r$, we define the state ξ of $\mathcal{H}$ as $\xi = (x, \hat{\theta}, \psi, \eta, \Phi, Q, m, \tau) \in \mathcal{X} := \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^{n\times p} \times \mathbb{R}^n \times \mathbb{R}^{p\times p} \times \mathbb{R}^{p\times p} \times \mathbb{R}^p \times \mathbb{R}_{\geq 0}$, where x is the state of the plant in (6.1), $\hat{\theta}$ is the estimate of θ, and ψ, η, Φ, Q, m are auxiliary state variables. The state component τ has dynamics such that it evolves as ordinary time t. Including τ in ξ allows ϕ and u to be part of the definitions of F and G, rather than modeled as inputs to $\mathcal{H}$. Thus, we can express $\mathcal{H}$ as an autonomous hybrid system, which allows us to leverage recent results on stability and robustness properties for such systems [22, 45].

The dynamics of $\mathcal{H}$ are designed so that, each time $\mathcal{H}$ jumps, the parameter estimate $\hat{\theta}$ is equal to θ after the jump. To explain this construction, we begin by defining the dynamics of ξ during flows. In particular, during flows, we convert the plant dynamics (6.1) into a form similar to a linear regression model using the state variables ψ and η with dynamics [39]

$$\begin{aligned} \dot{\psi} &= -\lambda\psi + \phi(t) \\ \dot{\eta} &= -\lambda(x + \eta) - f(x, u(t)) \end{aligned} \tag{6.2}$$

where $\lambda > 0$ is a design parameter. Defining

$$\varepsilon := x + \eta - \psi\theta, \qquad\qquad y := x + \eta \tag{6.3}$$

it follows that ψ and ε are related via

$$y = \psi\theta + \varepsilon. \tag{6.4}$$

Since θ is constant, differentiating ε along trajectories of (6.1), (6.2) yields $\dot{\varepsilon} = -\lambda\varepsilon$. Thus, ε converges exponentially to zero during flows. Moreover, we have the following series of equivalences: $\varepsilon \to 0 \iff x + \eta \to \psi\theta \iff y \to \psi\theta$. Hence, y, ψ, and θ are related via a linear regression model plus an exponentially convergent term. We estimate θ using the classical gradient descent algorithm [37], namely,

$$\dot{\hat{\theta}} = \gamma\psi^\top(y - \psi\hat{\theta})$$

where $\gamma > 0$ is a design parameter. Defining the parameter estimation error as $\tilde{\theta} := \theta - \hat{\theta}$, we obtain the error dynamics

$$\dot{\tilde{\theta}} = -\gamma\psi^\top(\psi\tilde{\theta} + \varepsilon). \tag{6.5}$$

Let $(t, j) \mapsto \xi(t, j)$ be a solution to $\mathcal{H}$ – hence, defined on a hybrid time domain – and consider the initial interval of flow $I^0 := \{t : (t, 0) \in \operatorname{dom}\xi\}$, with initial conditions $\hat{\theta}(0, 0) = m(0, 0)$, $\psi(0, 0) = 0$, $\eta(0, 0) = -x(0, 0)$, and $\Phi(0, 0) = I$. From such initial conditions, we have

$$\begin{aligned}\tilde{\theta}(t, 0) &= \Phi(t, 0)\tilde{\theta}(0, 0) - \gamma\int_0^t \Phi(s, 0)\psi^\top(s, 0)\varepsilon(s, 0)ds \\ &= \Phi(t, 0)\tilde{\theta}(0, 0)\end{aligned} \tag{6.6}$$

for all $t \in I^0$, where Φ is the state transition matrix for (6.5), generated by

$$\dot{\Phi} = -\gamma\psi^\top\psi\Phi$$

and the second equality in (6.6) follows from the fact that, since $\varepsilon(0, 0) = x(0, 0) + \eta(0, 0) - \psi(0, 0)\theta = 0$ and $\dot{\varepsilon} = -\lambda\varepsilon$, $\varepsilon(t, 0) = 0$ for all $t \in I^0$.

Then, if there exists a time $t_1 \in I^0$ such that the matrix $\Phi(t_1, 0) - I$ is invertible, we reset $\hat{\theta}$ to the value of the function

$$R(\xi) := K_1(\xi)\hat{\theta} + K_2(\xi)m \tag{6.7}$$

where

$$K_1(\xi) := -\big(\Phi - I\big)^{-1}, \qquad\qquad K_2(\xi) := I - K_1(\xi) \tag{6.8}$$

and m is a memory state that stores the initial condition of $\hat{\theta}$. For readability, we define $\tilde{m} := \theta - m$ and omit the argument ξ of K_1 and K_2. Then, after the reset at hybrid time $(t_1, 0)$, we obtain

$$\begin{aligned}
\hat{\theta}(t_1, 1) &= R(\xi(t_1, 0)) = K_1\hat{\theta}(t_1, 0) + K_2 m(t_1, 0) \\
&= K_1(\theta - \tilde{\theta}(t_1, 0)) + K_2(\theta - \tilde{m}(t_1, 0)) \\
&= -K_1(\tilde{\theta}(t_1, 0) - \tilde{m}(t_1, 0)) - \tilde{m}(t_1, 0) + (K_1 + K_2)\theta \\
&= -K_1(\Phi(t_1, 0)\tilde{\theta}(0, 0) - \tilde{m}(0, 0)) - \tilde{m}(t_1, 0) + \theta \\
&= -K_1(\Phi(t_1, 0) - I)\tilde{m}(0, 0) - \tilde{m}(0, 0) + \theta \\
&= \tilde{m}(0, 0) - \tilde{m}(0, 0) + \theta \\
&= \theta
\end{aligned} \tag{6.9}$$

where the fourth line follows from (6.6), (6.8), and the fact that $m(t, 0) = m(0, 0) = \hat{\theta}(0, 0)$ for all $t \in I^0$, and the fifth line follows from the fact that $\tilde{\theta}(0, 0) = \tilde{m}(0, 0)$. Hence, we have finite-time convergence of $\hat{\theta}$ to θ.

In Theorem 6.6 below, we establish that if the integral of $\psi^\top\psi$ is positive definite, then there exists a time $t_1 \in I^0$ such that $\Phi(t_1, 0) - I$ is invertible. Hence, the dynamics of the solution component Q are defined to compute this integral online, as

$$\dot{Q} = \psi^\top\psi.$$

The data of $\mathcal{H}$ is defined to implement the estimation scheme outlined above.

Hence, $\mathcal{H}$ has state $\xi = (x, \hat{\theta}, \psi, \eta, \Phi, Q, m, \tau) \in \mathcal{X}$ and dynamics

$$\mathcal{H}: \begin{cases} \dot{\xi} = \begin{bmatrix} f(x, u(\tau)) + \phi(\tau)\theta \\ \gamma\psi^\top(x + \eta - \psi\hat{\theta}) \\ -\lambda\psi + \phi(\tau) \\ -\lambda(x + \eta) - f(x, u(\tau)) \\ -\gamma\psi^\top\psi\Phi \\ \psi^\top\psi \\ 0 \\ 1 \end{bmatrix} =: F(\xi) \quad \xi \in C \\ \xi^+ = (x, R(\xi), 0, -x, I, 0, R(\xi), \tau) =: G(\xi) \quad \xi \in D \end{cases} \tag{6.10}$$

with R as in (6.7) and

$$\begin{aligned} C &:= \{\xi \in \mathcal{X} : \lambda_{\min}(Q) \leq \alpha\} \\ D &:= \{\xi \in \mathcal{X} : \lambda_{\min}(Q) \geq \alpha\} \end{aligned} \tag{6.11}$$

where $\alpha > 0$ is a design parameter.

The jump map G in (6.10) computes θ as in (6.9) by resetting $\hat{\theta}$ to the value of R in (6.7). The state components ψ and η are reset as $\psi^+ = 0$ and $\eta^+ = -x$ which, from (6.3), implies that $\varepsilon^+ = x^+ + \eta^+ - \psi^+\theta = 0$. The state component Φ is reset to the identity, which ensures that, during flows, Φ is the state transition matrix for the estimation error system (6.5). The state component Q is reset to zero, which ensures that Q gives the value of the integral of $\psi^\top\psi$ during each interval of flow. The state component m is reset to the value of R so that m stores the value of $\hat{\theta}$ after the jump. The flow and jump sets of $\mathcal{H}$ in (6.11) are defined so that the algorithm jumps only when the minimum eigenvalue of Q is greater than the design parameter α.

Remark 6.2: The jump map G of $\mathcal{H}$ in (6.10) requires computation of the inverse of $\Phi - I$ to evaluate the function R – see (6.7) and (6.8). Similarly, the estimation algorithms proposed in [23, 48, 29] all compute a matrix inverse in order to achieve finite-time convergence of the parameter estimate to the true value. However, in contrast to (6.10), the algorithms in [23, 48, 29] use the determinant to evaluate whether the matrix inverse is well defined, which may result in the matrix being numerically ill-conditioned for inversion. For example, given $d > 0$, consider the matrix

$$M := \frac{\sqrt{d}}{3a - a^2} \begin{bmatrix} 1 + (1-a)^2 & 4 - 3a \\ 4 - 3a & 4 + (2-a)^2 \end{bmatrix}$$

where $a \in (0, 3)$. For any such a, the determinant of M is equal to d, but the condition number of M, that is, $\mathrm{cond}(M)$, grows unbounded as a approaches 0 or 3. Recall that the condition number measures how sensitive matrix inversion is to numerical errors. This fact motivates using a method other than the determinant to evaluate whether a matrix is invertible. For $\mathcal{H}$ in (6.10), we impose a lower bound on the minimum eigenvalue of the solution component Q in the jump set, which allows us to upper-bound the condition number of $\Phi - I$ at jumps – see Theorem 6.6 for details.

6.3 Stability Analysis

We now establish our main convergence result stating conditions ensuring that each maximal solution ξ to $\mathcal{H}$ with initial conditions in a closed set converges in finite time to the set

$$\mathcal{A} := \left\{ \xi \in \mathcal{X} \ : \ \hat{\theta} = \theta \right\}. \tag{6.12}$$

We impose the following boundedness condition on ϕ.

Assumption 6.3: Given $t \mapsto \phi(t) \in \mathbb{R}^{n \times p}$, there exists $\phi_M > 0$ such that $|\phi(t)|_{\mathrm{F}} \leq \phi_M$ for all $t \geq 0$.

Furthermore, we impose the following excitation condition over a finite time interval.

Assumption 6.4: Given $t \mapsto \phi(t) \in \mathbb{R}^{n\times p}$ and $\lambda > 0$, there exist $t' \geq 0$ and $T, \mu > 0$ such that

$$\int_{t'}^{t'+T} \Psi(s,t')^\top \Psi(s,t') ds \geq \mu I \tag{6.13}$$

where

$$\Psi(s,t') := \int_{t'}^{s} \mathrm{e}^{-\lambda(s-v)} \phi(v) dv. \tag{6.14}$$

Remark 6.5: The excitation condition in Assumption 6.4 relaxes the classical persistence of excitation condition [37], which requires $t \mapsto \phi(t)$ to satisfy $\int_t^{t+T} \phi(s)^\top \phi(s) ds \geq \mu I$ for all $t \geq 0$. In contrast, Assumption 6.4 imposes (6.13) over only a finite time interval $[t', t'+T]$, for some $t' \geq 0$ and $T > 0$.

Theorem 6.6: Given the hybrid system $\mathcal{H}$ in (6.10) and $\lambda, \alpha > 0$, suppose that Assumptions 6.3 and 6.4 hold with $t' \geq 0$, $T, \phi_M > 0$, and $\mu \geq \alpha$. Then, for each $\gamma > 0$, the parameter estimation error $(t,j) \mapsto \tilde{\theta}(t,j) := \theta - \hat{\theta}(t,j)$ for each solution ξ to $\mathcal{H}$ from $\xi(0,0) \in \mathcal{X}_0 := \{\xi \in \mathcal{X} : \hat{\theta} = m,\ \psi = 0,\ \eta = -x,\ \Phi = I,\ Q = 0,\ \tau = 0\}$ satisfies

$$|\tilde{\theta}(t,j)| \leq |\tilde{\theta}(0,0)| \tag{6.15}$$

for all $(t,j) \in \operatorname{dom} \xi$, and the parameter estimate satisfies

$$\hat{\theta}(t,j) = \theta \tag{6.16}$$

for all $(t,j) \in \operatorname{dom} \xi$ satisfying $t \geq t' + T$ and $j \geq 1$. Moreover, for each maximal solution ξ to $\mathcal{H}$, the condition number of $\Phi - I$ immediately before the first jump at hybrid time $(t_1, 0)$ satisfies

$$\operatorname{cond}(\Phi(t_1,0) - I) \leq \frac{1 + \sigma(\gamma,\alpha)}{1 - \sigma(\gamma,\alpha)} \tag{6.17}$$

where $t_1 \in (0, t' + T]$ *and*

$$\sigma(\gamma, \alpha) := \sqrt{1 - \frac{2\gamma\alpha}{(1 + \gamma\psi_M^2 T)^2}}, \qquad\qquad \psi_M := \frac{\phi_M}{\lambda}. \tag{6.18}$$

Remark 6.7: In practice, the integral in (6.13) does not need to be computed a priori in order to implement our proposed estimation scheme. This is due to the fact that, for each solution ξ to $\mathcal{H}$ from $\xi(0,0) \in \mathcal{X}_0$, the value of the solution component ψ is $\psi(t,0) = \Psi(t,0)$ for all $t \in I^0 := \{t : (t,0) \in \operatorname{dom}\xi\}$, with Ψ as in (6.14). Thus, the value of solution component Q is $Q(t,0) = \int_0^t \Psi(s,0)^\top \Psi(s,0) ds$ for all $t \in I^0$, and the jump set D in (6.11) triggers a jump only once $Q(t,0) \geq \alpha I$. Hence, $\mathcal{H}$ jumps opportunistically when reconstruction of θ is possible through the conditions imposed in Assumptions 6.3 and 6.4.

6.3.1 Proof of Theorem 6.6

To prove Theorem 6.6, we require the following results.

Lemma 6.8: Given $t \mapsto \phi(t) \in \mathbb{R}^{n\times p}$*, suppose that Assumption 6.3 holds with* $\phi_M > 0$*. Then, for each* $t' \geq 0$ *and each* $\lambda > 0$*, the solution* $\psi : [t', \infty) \to \mathbb{R}^{n\times p}$ *to the system*

$$\dot{\psi} = -\lambda\psi + \phi(t), \qquad \psi(t') = 0 \tag{6.19}$$

satisfies

$$|\psi(t)| \leq \frac{\phi_M}{\lambda} =: \psi_M \qquad \forall t \geq t'. \tag{6.20}$$

Proof. Consider the Lyapunov function

$$W(\psi) := \frac{1}{2}\operatorname{tr}(\psi^\top\psi) = \frac{1}{2}|\psi|_{\mathrm{F}}^2 \quad \forall \psi \in \mathbb{R}^{n\times p}. \tag{6.21}$$

Differentiating W along solutions to (6.19), we obtain

$$\begin{aligned}\dot{W}(\psi) &= \frac{1}{2}\mathrm{tr}(\dot{\psi}^\top \psi + \psi^\top \dot{\psi}) \\ &= \mathrm{tr}(-\lambda\psi^\top\psi + \psi^\top\phi(t)) \\ &= -2\lambda W(\psi) + \mathrm{tr}(\psi^\top\phi(t))\end{aligned}$$

where the second equality follows from the flow map for ψ in (6.10), and the third equality follows from the definition of W in (6.21). Applying the Cauchy-Schwarz inequality on $\mathrm{tr}(\psi^\top\phi(t))$ yields

$$\dot{W}(\psi) \le -2\lambda W(\psi) + \sqrt{2W(\psi)}|\phi(t)|_{\mathrm{F}}.$$

Hence,

$$\dot{W}(\psi) \le 0 \quad \forall \psi \in \mathbb{R}^{n\times p} : W(\psi) \ge \frac{1}{2\lambda^2}|\phi(t)|_{\mathrm{F}}^2.$$

Let ψ be the solution to (6.19). By integration using the bounds above, we conclude that

$$W(\psi(t)) \le W(\psi(t')) + \frac{1}{2\lambda^2}\phi_M^2 = \frac{1}{2\lambda^2}\phi_M^2$$

for all $t \ge t'$, where the first inequality follows from Assumption 6.3, and the last equality follows from the fact that $\psi(t') = 0$. Using the definition of W in (6.21), we obtain

$$|\psi(t)| \le |\psi(t)|_F = \sqrt{2W(\psi(t))} \le \sqrt{2\frac{\phi_M^2}{2\lambda^2}} = \frac{\phi_M}{\lambda}$$

for all $t \ge t'$. Hence, (6.20) holds. □

Lemma 6.9: Given $t \mapsto \phi(t) \in \mathbb{R}^{n\times p}$, consider the system

$$\dot{\vartheta} = -\gamma\psi^\top\psi\vartheta \tag{6.22a}$$

$$\dot{\psi} = -\lambda\psi + \phi(t), \qquad \psi(t') = 0 \tag{6.22b}$$

where $\gamma, \lambda > 0$ and $t' \geq 0$, and let $\Omega : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{p \times p}$ be the state transition matrix of (6.22a)*. Then, for each $t', t \geq 0$ and each $\gamma, \lambda > 0$,*

$$|\Omega(t' + t, t')| \leq 1. \tag{6.23}$$

Moreover, given $\lambda, \alpha > 0$, suppose that Assumptions 6.3 and 6.4 hold with $t' \geq 0$, $T, \phi_M > 0$, and $\mu \geq \alpha$. Then,

$$|\Omega(t' + T, t')| \leq \sigma(\gamma, \alpha) < 1 \tag{6.24}$$

where

$$\sigma(\gamma, \alpha) := \sqrt{1 - \frac{2\gamma\alpha}{(1 + \gamma\psi_M^2 T)^2}} \tag{6.25}$$

with ψ_M as in (6.20)*. Additionally, the matrix $\Omega(t' + T, t') - I$ is invertible and satisfies*

$$|(\Omega(t' + T, t') - I)^{-1}| \leq \frac{1}{1 - \sigma(\gamma, \alpha)} \tag{6.26}$$

and the condition number of $\Omega(t' + T, t') - I$ satisfies

$$\operatorname{cond}(\Omega(t' + T, t') - I) \leq \frac{1 + \sigma(\gamma, \alpha)}{1 - \sigma(\gamma, \alpha)}. \tag{6.27}$$

Remark 6.10: The property $\sigma(\gamma, \alpha) < 1$ follows from the fact that, since Assumption 6.4 holds with $\mu \geq \alpha$, for each solution (ϑ, ψ) to (6.22), $\alpha I \leq \mu I \leq \int_{t'}^{t'+T} \psi(s)^\top \psi(s) ds \leq \psi_M^2 T I$, with ψ_M as in (6.20). Thus,

$$0 < \frac{2\gamma\alpha}{(1 + \gamma\psi_M^2 T)^2} \leq \frac{2\gamma\psi_M^2 T}{(1 + \gamma\psi_M^2 T)^2} \leq \frac{1}{2}.$$

To prove Lemma 6.9, we first recall the following result from [24].

Lemma 6.11: Given a matrix $A \in \mathbb{R}^{p \times p}$, if $|A| < 1$ then $I - A$ is invertible and $|(I - A)^{-1}| \leq (1 - |A|)^{-1}$.

Proof of Lemma 6.9: Consider the following Lyapunov function:

$$V(\vartheta) := \frac{1}{2\gamma}\vartheta^\top \vartheta \quad \forall \vartheta \in \mathbb{R}^p. \tag{6.28}$$

We first show that (6.23) holds. Differentiating V along trajectories of (6.22), we obtain

$$\dot{V}(\vartheta) = -\vartheta^\top \psi^\top \psi \vartheta = -\left|\psi\vartheta\right|^2 \leq 0. \tag{6.29}$$

Let (ϑ, ψ) be the solution to (6.22). By integration using the bound above, it follows that

$$V(\vartheta(t'+t)) \leq V(\vartheta(t')) \quad \forall t', t \geq 0.$$

Using the definition of V in (6.28), we obtain that $|\vartheta(t'+t)| \leq |\vartheta(t')|$ for all $t', t \geq 0$, which we rewrite as

$$|\Omega(t'+t, t')\vartheta(t')| \leq |\vartheta(t')| \quad \forall t', t \geq 0$$

which, if $|\vartheta(t')| \neq 0$, implies that

$$\frac{|\Omega(t'+t, t')\vartheta(t')|}{|\vartheta(t')|} \leq 1.$$

Since this inequality holds for any $\vartheta(t') \in \mathbb{R}^p \setminus \{0\}$, it follows that

$$|\Omega(t'+t, t')| = \sup_{s \in \mathbb{R}^p \setminus \{0\}} \frac{|\Omega(t'+t, t')s|}{|s|} \leq 1$$

for all $t', t \geq 0$. Hence, (6.23) holds.

To show that (6.24) holds, we follow an approach inspired by [48]. Since Assumption 6.4 holds with $t' \geq 0$, $T > 0$, and $\mu \geq \alpha$, we integrate V between times t' and $t' + T$, using (6.29), to obtain the following:

$$
\begin{aligned}
&V(\vartheta(t'+T)) - V(\vartheta(t')) \\
&\quad = -\int_{t'}^{t'+T} |\psi(s)\vartheta(s)|^2 \, ds \\
&\quad = -\int_{t'}^{t'+T} \left|\psi(s)\left(\vartheta(t') - \gamma \int_{t'}^{s} \psi^\top(u)\psi(u)\vartheta(u)du\right)\right|^2 ds \\
&\quad \leq -\frac{\varrho}{1+\varrho}\int_{t'}^{t'+T} |\psi(s)\vartheta(t')|^2 \, ds + \varrho \int_{t'}^{t'+T} \left|\gamma\psi(s)\int_{t'}^{s} \psi^\top(u)\psi(u)\vartheta(u)du\right|^2 ds \\
&\quad \leq -\frac{\varrho\alpha}{1+\varrho}\big|\vartheta(t')\big|^2 + \varrho\gamma^2\psi_M^4 T^2 \int_{t'}^{t'+T} |\psi(u)\vartheta(u)|^2 \, du \\
&\quad \leq -\frac{2\gamma\varrho\alpha}{1+\varrho} V(\vartheta(t')) - \varrho\gamma^2\psi_M^4 T^2 \big(V(\vartheta(t'+T)) - V(\vartheta(t'))\big)
\end{aligned}
$$

where the second equality comes from the fact that the solution $t \mapsto \vartheta(t)$ to (6.22a) is $\vartheta(t) = \vartheta(t') - \gamma \int_{t'}^{t} \psi^\top(u)\psi(u)\vartheta(u)du$ for all $t \geq t' \geq 0$, the first inequality follows from the fact that, for any $a, b \in \mathbb{R}$ and any $\varrho > 0$, $(a-b)^2 \geq \frac{\varrho}{1+\varrho}a^2 - \varrho b^2$, the second inequality comes from (6.20), from the triangle and Cauchy-Schwartz inequalities, from the fact that Assumption 6.4 holds with $\mu \geq \alpha$, and from the fact that $\psi(t) = \Psi(t, t')$ for all $t \geq t'$, with Ψ as in (6.14). Finally, the last inequality follows from the definition of V in (6.28). Hence, we have that

$$V(\vartheta(t'+T)) \leq (1-\beta)V(\vartheta(t'))$$

where

$$\beta := \frac{2\varrho\gamma\alpha}{(1+\varrho)(1+\varrho\gamma^2\psi_M^4 T^2)}.$$

By choosing $\varrho = \frac{1}{\gamma\psi_M^2 T}$, we obtain

$$\beta = \frac{2\gamma\alpha}{(1+\gamma\psi_M^2 T)^2}$$

and, since Assumption 6.4 holds with $\mu \geq \alpha$, it follows that

$$\alpha I \leq \mu I \leq \int_{t'}^{t'+T} \psi(s)^\top \psi(s) ds \leq T\psi_M^2 I$$

and thus $\beta \in (0,1)$. Using the definition of V in (6.28), it follows that $|\vartheta(t'+T)| \leq \sqrt{1-\beta}|\vartheta(t')|$, which we rewrite as

$$|\Omega(t'+T,t')\vartheta(t')| \leq \sqrt{1-\beta}|\vartheta(t')|$$

which, if $|\vartheta(t')| \neq 0$, implies that

$$\frac{|\Omega(t'+T,t')\vartheta(t')|}{|\vartheta(t')|} \leq \sqrt{1-\beta}.$$

Since this inequality holds for any $\vartheta(t') \in \mathbb{R}^p \setminus \{0\}$, it follows that

$$|\Omega(t'+T,t')| = \sup_{s\in\mathbb{R}^p\setminus\{0\}} \frac{|\Omega(t'+T,t')s|}{|s|} \leq \sqrt{1-\beta} < 1$$

where the last inequality follows from the fact that $\beta \in (0,1)$. Hence, (6.24) holds and, in view of Lemma 6.11, the matrix $I - \Omega(t'+T,t')$ is invertible, and thus $\Omega(t'+T,t') - I$ is invertible.

To upper bound the condition number of $\Omega(t'+T,t') - I$, we first use the triangle inequality to obtain

$$|\Omega(t'+T,t') - I| \leq |\Omega(t'+T,t')| + |I| \leq \sigma(\gamma,\alpha) + 1. \tag{6.30}$$

Next, since $|\Omega(t'+T,t')| \leq \sigma(\gamma,\alpha) < 1$, it follows from Lemma 6.11 that

$$\begin{aligned} |(\Omega(t'+T,t') - I)^{-1}| &= |(I - (\Omega(t'+T,t'))^{-1}| \\ &\leq \frac{1}{1-|\Omega(t'+T,t')|} \leq \frac{1}{1-\sigma(\gamma,\alpha)}. \end{aligned} \tag{6.31}$$

Hence, (6.26) holds. Combining the expressions in (6.30) and (6.31), we conclude that (6.27) holds. □

Lemma 6.12: Given the hybrid system $\mathcal{H}$ in (6.10), for each $\lambda, \gamma > 0$ and each solution ξ to $\mathcal{H}$ from $\xi(0,0) \in \mathcal{X}_0$, $(t,j) \mapsto \varepsilon(t,j)$ in (6.3) satisfies

$$\varepsilon(t,j) = 0 \quad \forall (t,j) \in \operatorname{dom} \xi. \tag{6.32}$$

The proof of Lemma 6.12 is immediate since, for each solution ξ to $\mathcal{H}$ from $\xi(0,0) \in \mathcal{X}_0$, $\varepsilon(0,0) = 0$ and, from (6.10), $\dot{\varepsilon} = -\lambda\varepsilon$ during flows, and $\varepsilon^+ = 0$ at jumps.

We now have all the ingredients to prove Theorem 6.6.

Proof of Theorem 6.6: Let $\xi = (x, \hat{\theta}, \psi, \eta, \Phi, Q, m, \tau)$ be a maximal solution to $\mathcal{H}$ in (6.10) from $\xi(0,0) \in \mathcal{X}_0$ and consider the following function:

$$V(t,j) := \frac{1}{2}\tilde{\theta}(t,j)^\top \tilde{\theta}(t,j) = \frac{1}{2}|\xi(t,j)|_{\mathcal{A}}^2 \qquad \forall (t,j) \in \operatorname{dom} \xi \tag{6.33}$$

where $\tilde{\theta} = \theta - \hat{\theta}$ is the parameter estimation error. Since $\xi(0,0) \in \mathcal{X}_0$, it follows from Lemma 6.12 that $\varepsilon(t,j) = 0$ for all $(t,j) \in \operatorname{dom} \xi$. Hence, we have from (6.5) that

$$\begin{aligned} \frac{d}{dt}V(t,0) &= -\gamma\tilde{\theta}(t,0)^\top \psi(t,0)^\top \psi(t,0)\tilde{\theta}(t,0) \\ &= -\gamma|\psi(t,0)\tilde{\theta}(t,0)|^2 \leq 0 \end{aligned} \tag{6.34}$$

for almost all $t \in I^0 := \{t : (t,0) \in \operatorname{dom} \xi\}$. Thus, V is nonincreasing during the first interval of flow in ξ.

Next, we show that ξ converges in finite time to $\mathcal{A}$. Since $\xi(0,0) \in \mathcal{X}_0$, it follows that $\Phi(0,0) = I$ and, thus, (6.6) holds for all $t \in I^0$. Moreover, since $\psi(0,0) = 0$, it follows from the third component of the flow map F in (6.10) that

$$\psi(t,0) = \int_0^t \mathrm{e}^{-\lambda(t-s)}\phi(s)ds = \Psi(t,0)$$

for all $t \in I^0$, with Ψ as in (6.14). Using that $Q(0,0) = 0$, it follows from (6.10) that

$$Q(t,0) = \int_0^t \psi(s,0)^\top \psi(s,0) ds = \int_0^t \Psi(s,0)^\top \Psi(s,0) ds$$

for all $t \in I^0$. Then, since Assumption 6.4 holds with $\mu \geq \alpha$, it follows that there exists $t_1 \in (0, t' + T]$ such that $(t_1, 0) \in \operatorname{dom} \xi$ and

$$Q(t_1,0) = \int_0^{t_1} \Psi(s,0)^\top \Psi(s,0) ds \geq \alpha I \tag{6.35}$$

and thus $\xi(t_1, 0) \in D$. Since ξ is maximal, the solution jumps at hybrid time $(t_1, 0)$ and, after the jump according to the jump map G in (6.10), we have from (6.9) that

$$\hat{\theta}(t_1, 1) = \theta \tag{6.36}$$

From the definition of $\mathcal{A}$ in (6.12), we have $\xi(t_1, 1) \in \mathcal{A}$. Furthermore, since $\xi(t_1, 0) \in D$ and $\xi(t_1, 1) = G(\xi(t_1, 0))$, it follows that $\xi(t_1, 1) \in G(D)$, and thus $\xi(t_1, 1) \in \mathcal{A} \cap G(D)$.

We now show that each solution to $\mathcal{H}$ from $\mathcal{A} \cap G(D)$ cannot leave $\mathcal{A}$ via flows or jumps. Let $\xi^* = (x^*, \hat{\theta}^*, \psi^*, \eta^*, \Phi^*, Q^*, m^*, \tau^*)$ be a maximal solution to $\mathcal{H}$ from $\xi^*(0,0) \in \mathcal{A} \cap G(D)$ and recall the function V in (6.33). Since $\mathcal{A} \cap G(D) \subset \mathcal{X}_0$, it follows from Lemma 6.12 that $\varepsilon^*(t,j) = x^*(t,j) + \eta^*(t,j) - \psi^*(t,j)\theta = 0$ for all $(t,j) \in \operatorname{dom} \xi^*$. Hence, we have from (6.5) that, for all $j \in \mathbb{N}$ and almost all $t \in I^j$,

$$\begin{aligned} \frac{d}{dt} V(t,j) &= -\gamma \tilde{\theta}^*(t,j)^\top \psi^*(t,j)^\top \psi^*(t,j) \tilde{\theta}^*(t,j) \\ &= -\gamma |\psi^*(t,j) \tilde{\theta}^*(t,j)|^2 \leq 0. \end{aligned} \tag{6.37}$$

Thus, V is nonincreasing during flows, and ξ^* cannot leave $\mathcal{A}$ via flow. We now analyze the variation of V at jumps. Let $(t,j) \in \operatorname{dom} \xi^*$ be such that

$(t, j+1) \in \operatorname{dom} \xi^*$. Using again the fact that $\mathcal{A} \cap G(D) \subset \mathcal{X}_0$, it follows from the jump map G in (6.10) and from (6.9) that $\hat{\theta}(t, j+1) = \theta$. Hence,

$$V(t, j+1) - V(t, j) = -V(t, j) \leq 0. \tag{6.38}$$

Thus, V is nonincreasing at jumps, and ξ^* cannot leave $\mathcal{A}$ via jump.

In view of (6.36), (6.37), and (6.38), we conclude that, under Assumption 6.4, each solution ξ to $\mathcal{H}$ from $\xi(0,0) \in \mathcal{X}_0$ converges in finite time to $\mathcal{A} \cap G(D)$ and remains in $\mathcal{A}$ for all future hybrid time in $\operatorname{dom} \xi$. Hence, (6.16) holds for all $(t, j) \in \operatorname{dom} \xi$ satisfying $t \geq t' + T$ and $j \geq 1$. Moreover, by integration using the bounds in (6.34), (6.37), and (6.38), and from the definition of V in (6.33), for each solution ξ to $\mathcal{H}$ from $\xi(0,0) \in \mathcal{X}_0$,

$$|\xi(t,j)|_{\mathcal{A}}^2 = V(\xi(t,j)) \leq V(\xi(0,0)) = |\xi(0,0)|_{\mathcal{A}}^2$$

for all $(t, j) \in \operatorname{dom} \xi$. Hence, (6.15) holds for all $(t, j) \in \operatorname{dom} \xi$.

To conclude the proof, we bound the condition number of $\Phi - I$ as in (6.17). Since Assumptions 6.3 and 6.4 hold with $\mu \geq \alpha$, the conditions of Lemma 6.9 are satisfied. Hence, from Lemma 6.9, there exists $t_1 \in (0, T]$ such that $(t_1, 0) \in \operatorname{dom} \xi$ and

$$\operatorname{cond}(\Omega(t_1, 0) - I) \leq \frac{1 + \sigma(\gamma, \alpha)}{1 - \sigma(\gamma, \alpha)} \tag{6.39}$$

with σ as in (6.18), where Ω is the state transition matrix for (6.22a). Moreover, since $\Phi(0,0) = I$ and the dynamics during flows of the state component Φ are equivalent to the dynamics of Ω, it follows that $\Phi(t, 0) = \Omega(t, 0)$ for all $t \in I^0$, and (6.17) follows from (6.39). $\square$

6.4 Robustness Analysis

In this section, we analyze the robustness of our algorithm with respect to bounded noise on the state measurements. Consider additive noise $t \mapsto \nu(t) \in \mathbb{R}^n$

in the measurements of the plant state x. During flows, ν affects the dynamics of the state components $\hat{\theta}$ and η in (6.10). In particular, the dynamics of $\hat{\theta}$ become

$$\dot{\hat{\theta}} = \gamma\psi^\top(x + \nu(t) + \eta - \psi\hat{\theta})$$

and the dynamics of η become

$$\dot{\eta} = -\lambda(x + \nu(t) + \eta) - f(x + \nu(t), u(t)).$$

Similarly, at jumps, the reset map for η becomes

$$\eta^+ = -x - \nu(t)$$

which implies that ε in (6.3) is reset as $\varepsilon^+ = -\nu(t)$ when a jump occurs at hybrid time (t, j).

We denote the hybrid system $\mathcal{H}$ in (6.10) under the effect of the measurement noise ν as $\mathcal{H}_\nu$, with state $\xi := (x, \hat{\theta}, \psi, \eta, \Phi, Q, m, \tau) \in \mathcal{X}$ and dynamics

$$\mathcal{H}_\nu : \begin{cases} \dot{\xi} = F_\nu(\xi) & \xi \in C_\nu \\ \xi^+ = G_\nu(\xi) & \xi \in D_\nu \end{cases} \tag{6.40}$$

where

$$F_\nu(\xi) := \begin{bmatrix} f(x, u(\tau)) + \phi(\tau)\theta \\ \gamma\psi^\top(x + \nu(\tau) + \eta - \psi\hat{\theta}) \\ -\lambda\psi + \phi(\tau) \\ -\lambda(x + \nu(\tau) + \eta) - f(x + \nu(\tau), u(\tau)) \\ -\gamma\psi^\top\psi\Phi \\ \psi^\top\psi \\ 0 \\ 1 \end{bmatrix} \tag{6.41}$$

$$G_\nu(\xi) := (x, R(\xi), 0, -x - \nu(\tau), I, 0, R(\xi), \tau)$$

with R as in (6.7) and $C_\nu := C$, $D_\nu := D$, with C, D as in (6.10).

To enable analysis on the noise effect, we make the following Lipschitz continuity assumption.

Assumption 6.13: Given the system (6.1), there exists $L_x > 0$ such that, for all $x_1, x_2 \in \mathbb{R}^n$ and all $u \in \mathbb{R}^r$,

$$|f(x_1, u) - f(x_2, u)| \leq L_x |x_1 - x_2|.$$

6.4.1 Robustness Under Persistence of Excitation

We first analyze the robustness properties induced by $\mathcal{H}_\nu$ under the following persistence of excitation condition.

Assumption 6.14: Given $t \mapsto \phi(t) \in \mathbb{R}^{n \times p}$ and $\lambda > 0$, there exist $T, \mu > 0$ such that

$$\int_t^{t+T} \Psi(s,t)^\top \Psi(s,t) ds \geq \mu I \qquad \forall t \geq 0 \tag{6.42}$$

with Ψ as in (6.14).

Remark 6.15: Compared to Assumption 6.4, which requires that the inequality in (6.13) holds over only a finite time interval, Assumption 6.14 requires that (6.42) holds for all $t \geq 0$. Moreover, from [50, Lemma 2.6.7], we have that if ϕ, $\frac{d}{dt}\phi \in \mathcal{L}_\infty$ and ϕ satisfies the classical continuous-time persistence of excitation condition in Definition 2.12, that is, there exist $T, \mu > 0$ such that

$$\int_t^{t+T} \phi(s)^\top \phi(s) ds \geq \mu \qquad \forall t \geq 0, \tag{6.43}$$

then the solution $t \mapsto \psi(t)$ to (6.22b) is persistently exciting. Moreover, since $\Psi(s,t)$ in (6.14) gives the solution to (6.22b) for all $s \geq t \geq 0$, it follows that Assumption 6.14 holds if ϕ is persistently exciting as in (6.43) and ϕ, $\frac{d}{dt}\phi \in \mathcal{L}_\infty$.

We establish that, under the persistence of excitation condition in Assumption 6.14, the hybrid system $\mathcal{H}_\nu$ is such that solutions from a perturbed version of $\mathcal{X}_0$ in Theorem 6.6 satisfy an ISS bound for any essentially bounded noise ν in the measurements of x.

Theorem 6.16: Given the hybrid system $\mathcal{H}_\nu$ in (6.40) and $\lambda, \alpha > 0$, suppose that Assumptions 6.3, 6.13, and 6.14 hold with $T, \phi_M > 0$ and $\mu \geq \alpha$. Then, for each $\gamma > 0$ and each $\zeta \in (0,1)$, the parameter estimation error $(t,j) \mapsto \tilde{\theta}(t,j) := \theta - \hat{\theta}(t,j)$ for each solution ξ to $\mathcal{H}_\nu$ from $\xi(0,0) \in \mathcal{X}_0^\nu := \{\xi \in \mathcal{X} : \hat{\theta} = m,\ \psi = 0,\ \eta = -x - \nu(0),\ \Phi = I,\ Q = 0,\ \tau = 0\}$ satisfies

$$|\tilde{\theta}(t,j)| \leq q(j)|\tilde{\theta}(0,0)| + (1 - q(j))\kappa_1(\|\nu\|_\infty) + \kappa_2(\|\nu\|_\infty) \tag{6.44}$$

for all $(t,j) \in \operatorname{dom} \xi$, where $q(j) := 1$ if $j = 0$, $q(j) := 0$ if $j > 0$, and

$$\kappa_1(s) := \frac{\gamma \psi_M}{1 - \sigma(\gamma, \alpha)} \left(\frac{1}{\omega} + T(\rho + 1) \right) s \tag{6.45a}$$

$$\kappa_2(s) := \gamma \psi_M \left(\frac{1}{\omega} + T(\rho + 1) \right) s \tag{6.45b}$$

$$\omega := \frac{1}{2} \min\{\lambda(1 - \zeta), -\ln(\zeta)\} \tag{6.45c}$$

$$\rho := \max \left\{ \frac{2(L_x + \lambda)}{\lambda \sqrt{\zeta}}, \frac{1}{\sqrt{\zeta}} \right\}, \tag{6.45d}$$

with σ, ψ_M as in (6.18) and L_x from Assumption 6.13.

To prove Theorem 6.16, we require the following results.

Lemma 6.17: Given the hybrid system $\mathcal{H}_\nu$ in (6.40), for each solution ξ to $\mathcal{H}_\nu$ from $\xi(0,0) \in \mathcal{X}_0^\nu$, the parameter estimation error $(t,j) \mapsto \tilde{\theta}(t,j) := \theta - \hat{\theta}(t,j)$ satisfies

$$\tilde{\theta}(t,j) = \Phi(t,j)\tilde{\theta}(t_j, j) - \gamma \int_{t_j}^{t} \Phi(s,j)\psi(s,j)^\top (\varepsilon(s,j) + \nu(s)) ds \tag{6.46}$$

for each $j \in \mathbb{N}$ *and all* $t \in I^j := \{t : (t,j) \in \operatorname{dom}\xi\}$*, where* $\{t_j\}_{j=0}^{\sup_j \operatorname{dom}\xi}$ *is the sequence defining* $\operatorname{dom}\xi$ *as in Section 2.1, and* ε *is defined in* (6.3)*. Moreover, for each* $(t,j) \in \operatorname{dom}\xi$ *such that* $(t,j+1) \in \operatorname{dom}\xi$*,*

$$\tilde{\theta}(t,j+1) = -K_1(\xi(t,j))\gamma \int_{t_j}^{t} \Phi(s,j)\psi(s,j)^\top(\varepsilon(s,j)+\nu(s))ds \tag{6.47}$$

with K_1 *as in* (6.8)*.*

Proof. This proof is given in Appendix E.1. □

Lemma 6.18: Given the hybrid system $\mathcal{H}_\nu$ *in* (6.40)*, suppose that Assumption 6.13 holds. Then, for each* $\lambda > 0$*, the* ψ *component of each solution* ξ *to* $\mathcal{H}$ *from* $\xi(0,0) \in \mathcal{X}_0$ *satisfies*

$$|\psi(t,j)| \leq \psi_M \tag{6.48}$$

for all $(t,j) \in \operatorname{dom}\xi$*, with* ψ_M *as in* (6.18)*.*

The proof of Lemma 6.18 follows directly from Lemma 6.8 and the fact that $\psi(0,0) = 0$ and, at each jump, $\psi^+ = 0$.

Lemma 6.19: Given the hybrid system $\mathcal{H}_\nu$ *in* (6.40)*, suppose that Assumption 6.13 holds. Then, for each* $\lambda, \gamma > 0$*,* $\zeta \in (0,1)$*, and each solution* ξ *to* $\mathcal{H}_\nu$*,* $(t,j) \mapsto \varepsilon(t,j) = z(t,j) + \eta(t,j) - \psi(t,j)\theta$ *in* (6.3) *satisfies*

$$|\varepsilon(t,j)| \leq \mathrm{e}^{-\omega(t+j-t'-j')}|\varepsilon(t',j')| + \rho\|\nu\|_\infty \tag{6.49}$$

for all $(t,j),(t',j') \in \operatorname{dom}\xi$ *satisfying* $t+j \geq t'+j'$*, with* ω *as in* (6.45c) *and* ρ *as in* (6.45d)*.*

Proof. This proof is given in Appendix E.2. □

Lemma 6.20: Given $t \mapsto \phi(t) \in \mathbb{R}^{n\times p}$ and $\lambda > 0$, suppose that Assumption 6.14 holds. Then, for all $t' \geq 0$, Assumption 6.4 holds with T from Assumption 6.14.

The proof of Lemma 6.20 is immediate since, by Assumption 6.14, there exists $T, \mu > 0$ such that $\int_{t'}^{t'+T} \Psi(s,t')^\top \Psi(s,t') ds \geq \mu I$ for all $t' \geq 0$, with Ψ as in (6.14).

We now have all the ingredients to prove Theorem 6.16.

Proof of Theorem 6.16: Let $\xi = (x, \hat{\theta}, \psi, \eta, \Phi, Q, m, \tau)$ be a maximal solution to $\mathcal{H}_\nu$ in (6.40) from $\xi(0,0) \in \mathcal{X}_0^\nu$. We first bound the parameter estimation error $\tilde{\theta} = \theta - \hat{\theta}$ during the initial interval of flow $I^0 := \{t : (t,0) \in \operatorname{dom} \xi\}$. From Lemma 6.17, we have that for all $t \in I^0$,

$$\tilde{\theta}(t,0) = \Phi(t,0)\tilde{\theta}(0,0) - \gamma \int_0^t \Phi(s,0)\psi(s,0)^\top(\varepsilon(s,0) + \nu(s))ds.$$

Since Assumption 6.14 is satisfied, it follows from Lemma 6.20 that Assumption 6.4 holds over the time interval $[0,T]$. Hence, by Assumption 6.3, the conditions of Lemma 6.9 are satisfied with $t' = 0$. Moreover, since $\Phi(0,0) = I$ and the dynamics during flows of the state component Φ are equivalent to the dynamics of the state transition matrix Ω for (6.22a), it follows from (6.23) that $|\Phi(t,0)| \leq 1$ for all $t \in I^0$. Hence,

$$\begin{aligned}
|\tilde{\theta}(t,0)| &\leq |\Phi(t,0)||\tilde{\theta}(0,0)| + \gamma \int_0^t |\Phi(s,0)||\psi(s,0)|(|\varepsilon(s,0)| + |\nu(s)|)ds \\
&\leq |\theta(0,0)| + \gamma\psi_M \int_0^t \left(\mathrm{e}^{-\omega s}|\varepsilon(0,0)| + (\rho+1)\|\nu\|_\infty\right)ds \\
&\leq |\tilde{\theta}(0,0)| + \gamma\psi_M \left(\int_0^\infty \mathrm{e}^{-\omega s}|\varepsilon(0,0)|ds + t(\rho+1)\|\nu\|_\infty\right) \\
&\leq |\tilde{\theta}(0,0)| + \gamma\psi_M \left(\frac{1}{\omega}|\varepsilon(0,0)| + t(\rho+1)\|\nu\|_\infty\right) \\
&\leq |\tilde{\theta}(0,0)| + \gamma\psi_M \left(\frac{1}{\omega} + t(\rho+1)\right)\|\nu\|_\infty
\end{aligned} \tag{6.50}$$

for all $t \in I^0$, where the second line follows from Lemmas 6.18 and 6.19, the fourth line uses the fact that $\omega > 0$, and the last line follows from the fact that, since $\xi(0,0) \in \mathcal{X}_0^\nu$, $|\varepsilon(0,0)| = |\nu(0)| \leq \|\nu\|_\infty$.

Note that the measurement noise ν does not affect the dynamics of the ψ or Q components of the solution ξ of $\mathcal{H}_\nu$. Hence, by Assumptions 6.3 and 6.14 and Lemma 6.20, the conditions of Theorem 6.6 hold. Moreover, by the equivalence between the jump sets of $\mathcal{H}$ in (6.10) and $\mathcal{H}_\nu$ in (6.40), it follows from Theorem 6.6 that ξ jumps at hybrid time $(t_1, 0)$ with $t_1 \in (0, T]$, and thus $I^0 \subset [0, T]$. Hence, for all $t \in I^0$,

$$|\tilde{\theta}(t,0)| \leq |\tilde{\theta}(0,0)| + \kappa_2(\|\nu\|_\infty) \tag{6.51}$$

with κ_2 as in (6.45b).

Next, we bound the parameter estimation error after each jump in $\operatorname{dom}\xi$. Let $(t_j, j) \in \operatorname{dom}\xi$ be such that $(t_j, j-1) \in \operatorname{dom}\xi$. From Lemma 6.17, we have that

$$\tilde{\theta}(t_j, j) = -K_1(\xi(t_j, j-1))\gamma \int_{t_{j-1}}^{t_j} \Phi(s, j-1)\psi(s, j-1)^\top(\varepsilon(s, j-1) + \nu(s))ds$$

Since Assumption 6.14 is satisfied, it follows from Lemma 6.20 that Assumption 6.4 holds over the time interval $[t_{j-1}, t_{j-1} + T]$. Hence, by Assumption 6.3, the conditions of Lemma 6.9 are satisfied with $t' = t_{j-1}$. Moreover, since $\Phi(t_{j-1}, j-1) = I$ and the dynamics during flows of Φ in (6.40) are equivalent to the dynamics of the state transition matrix Ω for (6.22a), it follows from (6.23) that $|\Phi(t, j-1)| \leq 1$

for all $t \in I^{j-1}$. Thus,

$$
\begin{aligned}
|\tilde{\theta}(t_j, j)| &\leq |K_1(\xi(t_j, j-1))|\gamma \int_{t_{j-1}}^{t_j} |\Phi(s, j-1)||\psi(s, j-1)|(|\varepsilon(s, j-1)| + |\nu(s)|)ds \\
&\leq |K_1(\xi(t_j, j-1))|\gamma\psi_M \int_{t_{j-1}}^{t_j} \left(\mathrm{e}^{-\omega(s+j-1)}|\varepsilon(0,0)| + (\rho+1)\|\nu\|_\infty\right)ds \\
&\leq |K_1(\xi(t_j, j-1))|\gamma\psi_M \left(\int_0^\infty \mathrm{e}^{-\omega s}|\varepsilon(0,0)|ds + (t_j - t_{j-1})(\rho+1)\|\nu\|_\infty\right) \\
&\leq |K_1(\xi(t_j, j-1))|\gamma\psi_M \left(\frac{1}{\omega}|\varepsilon(0,0)| + (t_j - t_{j-1})(\rho+1)\|\nu\|_\infty\right) \\
&\leq |K_1(\xi(t_j, j-1))|\gamma\psi_M \left(\frac{1}{\omega} + (t_j - t_{j-1})(\rho+1)\right)\|\nu\|_\infty
\end{aligned}
$$

where the second line follows from Lemmas 6.18 and 6.19, the fourth line uses the fact that $\omega > 0$, and the last line follows from the fact that $|\varepsilon(0,0)| = |\nu(0)| \leq \|\nu\|_\infty$. Since, by Assumption 6.14 and Lemma 6.20, Assumption 6.4 holds over the time interval $[t_{j-1}, t_{j-1} + T]$, it follows from the same arguments as in the proof of Theorem 6.6 that $t_j - t_{j-1} \leq T$. Hence,

$$
\begin{aligned}
|\tilde{\theta}(t_j, j)| &\leq |K_1(\xi(t_j, j-1))|\gamma\psi_M \left(\frac{1}{\omega} + T(\rho+1)\right)\|\nu\|_\infty \\
&\leq \frac{\gamma\psi_M}{1-\sigma(\gamma,\alpha)} \left(\frac{1}{\omega} + T(\rho+1)\right)\|\nu\|_\infty = \kappa_1(\|\nu\|_\infty)
\end{aligned}
$$

with κ_1 as in (6.45a) and σ as in (6.18), where the last inequality follows from the definition of $K_1 = -(\Phi - I)^{-1}$ in (6.8), and we use the equivalence between the dynamics of the state component Φ in (6.40) and the state transition matrix Ω from Lemma 6.9 to upper bound $|(\Phi - I)^{-1}|$ using (6.26).

During the interval of flow after the jump at hybrid time (t_j, j), we have from Lemma 6.17 that, for all $t \in I^j$,

$$
\tilde{\theta}(t, j) = \Phi(t, j)\tilde{\theta}(t_j, j) - \gamma \int_{t_j}^{t} \Phi(s, j)\psi(s, j)^\top(\varepsilon(s, j) + \nu(s))ds.
$$

Since Assumption 6.14 is satisfied, it follows from Lemma 6.20 that Assumption 6.4 holds over the time interval $[t_j, t_j + T]$. Hence, by Assumption 6.3, the conditions of Lemma 6.9 are satisfied with $t' = t_j$. Moreover, since $\Phi(t_j, j) = I$ and the dynamics during flows of Φ in (6.40) are equivalent to the dynamics of the state transition matrix Ω for (6.22a), it follows from (6.23) that $|\Phi(t, j)| \leq 1$ for all $t \in I^j$. Using similar arguments as in (6.50), we obtain

$$|\tilde{\theta}(t, j)| \leq |\tilde{\theta}(t_j, j)| + \gamma\psi_M \left(\frac{1}{\omega} + (t - t_j)(\rho + 1)\right) \|\nu\|_\infty \tag{6.52}$$

for all $t \in I^j$, where the second line follows from Lemmas 6.18 and 6.19, and the last line follows from the fact that $|\varepsilon(0, 0)| = |\nu(0)| \leq \|\nu\|_\infty$. Since, by Assumption 6.14 and Lemma 6.20, Assumption 6.4 holds over the time interval $[t_j, t_j + T]$, it follows from the same arguments as in the proof of Theorem 6.6 that $t_{j+1} - t_j \leq T$. Hence, for all $t \in I^j$,

$$\begin{aligned} |\tilde{\theta}(t, j)| &\leq |\tilde{\theta}(t_j, j)| + \gamma\psi_M \left(\frac{1}{\omega} + T(\rho + 1)\right) \|\nu\|_\infty \\ &\leq \kappa_1(\|\nu\|_\infty) + \kappa_2(\|\nu\|_\infty) \end{aligned} \tag{6.53}$$

where the last line follows from substituting (6.52).

Since the bound in (6.53) does not depend on the choice of hybrid time instant (t_j, j), it follows that (6.53) holds for each $j \in \mathbb{N} \setminus \{0\}$ and all $t \in I^j$. Hence, we unify the bounds in (6.51) and (6.53) to obtain that, for all $(t, j) \in \operatorname{dom} \xi$,

$$|\tilde{\theta}(t, j)| \leq q(j)|\tilde{\theta}(0, 0)| + (1 - q(j))\kappa_1(\|\nu\|_\infty) + \kappa_2(\|\nu\|_\infty)$$

where $q(j) := 1$ if $j = 0$ and $q(j) := 0$ if $j > 0$. Hence, (6.44) holds. □

6.4.2 Robustness Under Finite Excitation

To analyze the robustness properties of $\mathcal{H}_\nu$ under the finite excitation condition in Assumption 6.4, we modify the algorithm to jump only once by augmenting the state vector with a new component $k \in \{0,1\}$. The resulting algorithm, denoted by $\mathcal{H}'_\nu$, has state $\xi := (x, \hat{\theta}, \psi, \eta, \Phi, Q, m, \tau, k) \in \mathcal{X}' := \mathcal{X} \times \{0,1\}$ and dynamics

$$\mathcal{H}'_\nu : \begin{cases} \dot{\xi} = \begin{bmatrix} F_\nu(\xi) \\ 0 \end{bmatrix} =: F'_\nu(\xi) & \xi \in C'_\nu \\ \xi^+ = \begin{bmatrix} G_\nu(\xi) \\ 1 \end{bmatrix} =: G'_\nu(\xi) & \xi \in D'_\nu \end{cases} \tag{6.54}$$

with F_ν, G_ν as in (6.41),

$$\begin{aligned} C'_\nu &:= \{\xi \in \mathcal{X}' : \lambda_{\min}(Q) \leq \alpha\} \cup \{\xi \in \mathcal{X}' : k = 1\} \\ D'_\nu &:= \{\xi \in \mathcal{X}' : \lambda_{\min}(Q) \geq \alpha,\, k = 0\}, \end{aligned} \tag{6.55}$$

and $\alpha > 0$ is a design parameter. The jump set D'_ν in (6.55) ensures that $\mathcal{H}'_\nu$ jumps only once, when $k = 0$. Note that the results in Lemmas 6.19 and 6.20 also apply to the hybrid system $\mathcal{H}'_\nu$.

We now show that, under the finite excitation condition in Assumption 6.4, the hybrid system $\mathcal{H}'_\nu$ is such that, for solutions from an augmented version of $\mathcal{X}_0^\nu$ defined above (6.44), the norm of the parameter estimation error is upper bounded by a function of the integral of the noise ν in the measurements of x.

Theorem 6.21: Given the hybrid system $\mathcal{H}'_\nu$ in (6.54) *and $\lambda, \alpha > 0$, suppose that Assumptions 6.3, 6.4, and 6.13 hold with $t' \geq 0$, $T, \phi_M > 0$, and $\mu \geq \alpha$. Then, for each $\gamma > 0$ and each $\zeta \in (0,1)$, the parameter estimation error $(t,j) \mapsto \tilde{\theta}(t,j) := \theta - \hat{\theta}(t,j)$ for each solution ξ to $\mathcal{H}'_\nu$ from $\xi(0,0) \in \mathcal{X}_0^{\nu'} := \mathcal{X}_0^\nu \times \{0\}$ satisfies*

$$\begin{aligned} |\tilde{\theta}(t,j)| \leq q(j)|\tilde{\theta}(0,0)| &+ (1 - q(j))\kappa'_1(\|\nu\|_\infty) \\ &+ \kappa'_2(\|\nu\|_\infty) + \int_0^t \kappa'_3(|\nu(s)|, \|\nu\|_\infty) ds \end{aligned} \tag{6.56}$$

for all $(t,j) \in \operatorname{dom}\xi$*, where* $q(j) := 1$ if $j = 0$, $q(j) := 0$ if $j > 0$*, and*

$$\begin{aligned}
\kappa_1'(s) &:= \frac{\gamma\psi_M}{1-\sigma(\gamma,\alpha)}\left(\frac{1}{\omega} + (t'+T)(\rho+1)\right)s \\
\kappa_2'(s) &:= \frac{\gamma\psi_M}{\omega}s \\
\kappa_3'(s_1,s_2) &:= \gamma\psi_M\left(s_1+\rho s_2\right)
\end{aligned} \tag{6.57}$$

with ψ_M *as in* (6.18) *and* ω, ρ *as in* (6.45).

Proof. Let $\xi = (x, \hat{\theta}, \psi, \eta, \Phi, Q, m, \tau, k)$ be a maximal solution to $\mathcal{H}_\nu'$ in (6.54) from $\xi(0,0) \in \mathcal{X}_0^{\nu'}$. We first bound the parameter estimation error $\tilde{\theta} = \theta - \hat{\theta}$ during the initial interval of flow $I^0 := \{t : (t,0) \in \operatorname{dom}\xi\}$. From Lemma 6.17, we have that for all $t \in I^0$,

$$\tilde{\theta}(t,0) = \Phi(t,0)\tilde{\theta}(0,0) - \gamma\int_0^t \Phi(s,0)\psi(s,0)^\top(\varepsilon(s,0)+\nu(s))ds.$$

Since $\Phi(0,0) = I$ and the dynamics during flows of Φ in (6.54) are equivalent to the dynamics of the state transition matrix Ω for (6.22a), it follows from (6.23) that $|\Phi(t,0)| \le 1$ for all $t \in I^0$. Thus,

$$\begin{aligned}
|\tilde{\theta}(t,0)| &\le |\Phi(t,0)||\tilde{\theta}(0,0)| + \gamma\int_0^t |\Phi(s,0)||\psi(s,0)|(|\varepsilon(s,0)|+|\nu(s)|)ds \\
&\le |\theta(0,0)| + \gamma\psi_M\int_0^t\left(\mathrm{e}^{-\omega s}|\varepsilon(0,0)| + \int_0^t\left(|\nu(s)|+\rho\|\nu\|_\infty\right)ds\right) \\
&\le |\tilde{\theta}(0,0)| + \gamma\psi_M\left(\int_0^\infty \mathrm{e}^{-\omega s}|\varepsilon(0,0)|ds + \int_0^t\left(|\nu(s)|+\rho\|\nu\|_\infty\right)ds\right) \\
&\le |\tilde{\theta}(0,0)| + \gamma\psi_M\left(\frac{1}{\omega}|\varepsilon(0,0)| + \int_0^t\left(|\nu(s)|+\rho\|\nu\|_\infty\right)ds\right) \\
&\le |\tilde{\theta}(0,0)| + \gamma\psi_M\left(\frac{1}{\omega}\|\nu\|_\infty + \int_0^t\left(|\nu(s)|+\rho\|\nu\|_\infty\right)ds\right) \\
&\le |\tilde{\theta}(0,0)| + \kappa_2'(\|\nu\|_\infty) + \int_0^t \kappa_3'(|\nu(s)|,\|\nu\|_\infty)ds
\end{aligned} \tag{6.58}$$

for all $t \in I^0$, with κ_2', κ_3' as in (6.57), where the second line follows from Lemmas 6.18 and 6.19, the fourth line uses the fact that $\omega > 0$, and the fifth line follows from the fact that $|\varepsilon(0,0)| = |\nu(0)| \leq \|\nu\|_\infty$.

Note that the measurement noise ν does not affect the dynamics of the ψ or Q components of the solution ξ of $\mathcal{H}_\nu$. Hence, by Assumptions 6.3 and 6.4, the conditions of Theorem 6.6 hold. Thus, it follows from Theorem 6.6 that the solution ξ jumps at hybrid time $(t_1, 0)$ with $t_1 \in (0, t' + T]$. After the jump, we have from Lemma 6.17 that

$$\tilde{\theta}(t_1, 1) = -K_1(\xi(t_1, 0))\gamma \int_0^{t_1} \Phi(s,0)\psi(s,0)^\top(\varepsilon(s,0) + \nu(s))ds$$

which is upper bounded by

$$\begin{aligned}
|\tilde{\theta}(t_1, 1)| &\leq |K_1(\xi(t_1, 0))|\gamma \int_0^{t_1} |\Phi(s,0)||\psi(s,0)|(|\varepsilon(s,0)| + |\nu(s)|)ds \\
&\leq |K_1(\xi(t_1, 0))|\gamma\psi_M \int_0^{t_1} \left(\mathrm{e}^{-\omega s}|\varepsilon(0,0)| + \int_0^{t_1} (1+\rho)\|\nu\|_\infty ds\right) \\
&\leq |K_1(\xi(t_j, j-1))|\gamma\psi_M \left(\int_0^{\infty} \mathrm{e}^{-\omega s}|\varepsilon(0,0)|ds + t_1(1+\rho)\|\nu\|_\infty\right) \\
&\leq |K_1(\xi(t_j, j-1))|\gamma\psi_M \left(\frac{1}{\omega}|\varepsilon(0,0)| + t_1(1+\rho)\|\nu\|_\infty\right) \\
&\leq |K_1(\xi(t_j, j-1))|\gamma\psi_M \left(\frac{1}{\omega} + t_1(1+\rho)\right)\|\nu\|_\infty
\end{aligned}$$

where the second line follows from Lemmas 6.18 and 6.19, the fourth line uses the fact that $\omega > 0$, and the last line follows from the fact that $|\varepsilon(0,0)| = |\nu(0)| \leq \|\nu\|_\infty$. Since $t_1 \in (0, t' + T]$, we have

$$\begin{aligned}
|\tilde{\theta}(t_1, 1)| &\leq |K_1(\xi(t_j, j-1))|\gamma\psi_M \left(\frac{1}{\omega} + (t' + T)(1+\rho)\right)\|\nu\|_\infty \\
&\leq \frac{\gamma\psi_M}{1 - \sigma(\gamma, \alpha)} \left(\frac{1}{\omega} + (t' + T)(1+\rho)\right)\|\nu\|_\infty = \kappa_1'(\|\nu\|_\infty)
\end{aligned} \tag{6.59}$$

with κ_1' as in (6.57), where the last inequality follows from the definition of $K_1 = -(\Phi - I)^{-1}$ in (6.8), and we use the equivalence between the dynamics of the state component Φ in (6.54) and the state transition matrix Ω from Lemma 6.9 to upper bound $|(\Phi - I)^{-1}|$ using (6.26).

By construction, the hybrid system $\mathcal{H}'_\nu$ jumps at most once, when $k = 0$. Since k is reset to 1 at the first jump, ξ flows for all hybrid time after the first jump. We have from Lemma 6.17 that, for all $t \in I^1$,

$$\tilde{\theta}(t, 1) = \Phi(t, 1)\tilde{\theta}(t_1, 1) - \gamma \int_{t_1}^{t} \Phi(s, 1)\psi(s, 1)^\top(\varepsilon(s, 1) + \nu(s))ds.$$

Since $\Phi(t_1, 1) = I$ and the dynamics during flows of Φ in (6.54) are equivalent to the dynamics of the state transition matrix Ω for (6.22a), it follows from (6.23) that $|\Phi(t, 1)| \leq 1$ for all $t \in I^1$. Using similar arguments as in (6.58), we obtain

$$\begin{aligned} |\tilde{\theta}(t, 1)| &\leq |\tilde{\theta}(t_1, 1)| + \gamma\psi_M \left(\frac{1}{\omega}\|\nu\|_\infty + \int_0^t \left(|\nu(s)| + \rho\|\nu\|_\infty\right) ds\right) \\ &\leq \kappa'_1(\|\nu\|_\infty) + \kappa'_2(\|\nu\|_\infty) + \int_0^t \kappa'_3(|\nu(s)|, \|\nu\|_\infty)ds \end{aligned} \tag{6.60}$$

for all $t \in I^1$, with $\kappa'_1, \kappa'_2, \kappa'_3$ as in (6.57), where the last line follows from substituting (6.59).

We unify the bounds in (6.58) and (6.60) to obtain that, for all $(t, j) \in \operatorname{dom}\xi$,

$$\begin{aligned} |\tilde{\theta}(t, j)| \leq q(j)|\tilde{\theta}(0, 0)| &+ (1 - q(j))\kappa'_1(\|\nu\|_\infty) \\ &+ \kappa'_2(\|\nu\|_\infty) + \int_0^t \kappa'_3(|\nu(s)|, \|\nu\|_\infty)ds \end{aligned}$$

where $q(j) := 1$ if $j = 0$ and $q(j) := 0$ if $j > 0$. Hence, (6.56) holds. □

6.5 Numerical Example

Consider the following nonlinear system similar to that in [1], with state $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, input $t \mapsto u(t) \in \mathbb{R}$, and dynamics

$$\begin{aligned}
\dot{x}_1 &= x_2 + \theta_1 x_1 \\
\dot{x}_2 &= x_3 + \theta_2 x_1 \\
\dot{x}_3 &= \theta_3 x_1^3 + \theta_4 x_2 + \theta_5 x_3 + (2 + \sin(x_1))u(t)
\end{aligned} \tag{6.61}$$

where $\theta = (\theta_1, \theta_2, \cdots, \theta_5) \in \mathbb{R}^5$ is a vector of unknown parameters, and the input $t \mapsto u(t)$ is designed using the control algorithm in [33] so that the state component x_1 converges asymptotically to a reference signal $t \mapsto x_r(t)$. To estimate θ using our proposed algorithm, we express the dynamics of each solution pair (x, u) to (6.61) in the form of (6.1) with

$$\begin{aligned}
f(x(t), u(t)) &= \begin{bmatrix} x_2(t) \\ x_3(t) \\ (2 + \sin(x_1(t)))u(t) \end{bmatrix} \\
\phi(t) &= \begin{bmatrix} x_1(t) & 0 & 0 & 0 & 0 \\ 0 & x_1(t) & 0 & 0 & 0 \\ 0 & 0 & x_1(t)^3 & x_2(t) & x_3(t) \end{bmatrix}
\end{aligned} \tag{6.62}$$

for all $t \in \mathrm{dom}(x, u)$.

We simulate our estimation algorithm $\mathcal{H}$ in (6.10) with f and ϕ in (6.62). To illustrate the robustness properties of $\mathcal{H}$, we also simulate the system with additive noise $t \mapsto \nu(t) = 0.5\sin(10t)[1\ 1\ 1]^\top$ in the measurements of x. The plant has parameters $\theta = (-1, -2, 1, 2, 3)$, reference signal $x_r(t) = 1$ for all $t \geq 0$, and initial condition $x(0) = (0, 0, 0)$. The estimator has design parameters $\gamma = 1$, $\lambda = 0.1$, and $\alpha = 0.2$, and initial conditions $\hat{\theta}(0, 0) = m(0, 0) = (0, 0, 0, 0, 0)$, $\psi(0, 0) = 0$, $\eta(0, 0) = -x(0, 0)$, $\Phi(0, 0) = I$, and $Q(0, 0) = 0$. With these parameters and initial conditions, it can be shown that ϕ in (6.62) satisfies Assumption 6.3 with $\phi_M = 6.64$, and satisfies Assumption 6.4 with $t' = 0$ sec, $T = 2.93$ sec, and $\mu = 0.2$. Hence, the conditions of Theorem 6.6 are satisfied. Furthermore, since f in (6.62) satisfies Assumption 6.13 with $L_x = 1$, the conditions of Theorem 6.21 are satisfied. When no noise is present, the parameter estimate $\hat{\theta}$ converges in finite-time to θ in accordance with Theorems 6.6, as shown in blue in Figure 6.1.[22] When noise is present, the estimation error remains bounded in accordance with Theorem 6.21, as shown in green in Figure 6.1.

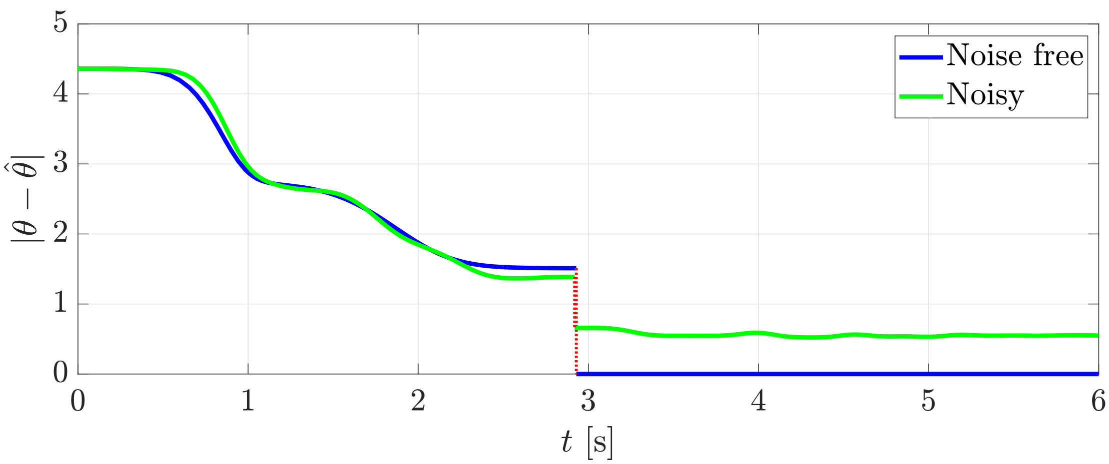

Figure 6.1: The projection onto t of the norm of the parameter estimation error for $\mathcal{H}$.

[22]Code at `https://github.com/HybridSystemsLab/HybridFT_NonlinearSystem`

Chapter 7

Parameter Estimation for Hybrid Systems with Approximately Known Jump Times

In this chapter, we propose an algorithm for estimating unknown parameters of hybrid dynamical systems whose jump times are known only approximately. The algorithm solves an optimization problem to estimate the jump times of the hybrid system.

7.1 Problem Statement

Consider a hybrid plant, denoted by $\mathcal{H}_P$, with dynamics

$$\mathcal{H}_P : \begin{cases} \dot{x} = F_P(x, \theta) & x \in C_P \\ x^+ = G_P(x, \theta) & x \in D_P \end{cases} \tag{7.1}$$

where $x \in \mathbb{R}^n$ is the state vector, F_P is the flow map, G_P is the jump map, $C_P \subset \mathbb{R}^n$ is the flow set, $D_P \subset \mathbb{R}^n$ is the jump set, $\theta \in \Theta$ is a vector of unknown constant parameters that are contained in a known compact set $\Theta \subset \mathbb{R}^p$, and $n, p \in \mathbb{N}$.

If jumps in x are detected instantaneously, then a hybrid estimator can be designed to estimate θ for certain classes of hybrid plants – see Chapters 4 and 5 and [48, 27, 28]. We denote such an estimation algorithm as $\mathcal{H}_E$ with state $z \in \mathbb{R}^m$, input $x \in \mathbb{R}^n$,[23] and dynamics

$$\mathcal{H}_E : \begin{cases} \dot{z} = F_E(x, z) & \text{when } \mathcal{H}_P \text{ flows} \\ z^+ = G_E(x, z) & \text{when } \mathcal{H}_P \text{ jumps} \end{cases} \tag{7.2}$$

where F_E is the flow map, G_E is the jump map, and $m \in \mathbb{N}$. The state z of $\mathcal{H}_E$ is partitioned as $z := (\hat{\theta}, \chi)$, where $\hat{\theta} \in \mathbb{R}^p$ is an estimate of the unknown parameter vector θ in (7.1) and $\chi \in \mathbb{R}^{m-p}$ collects any auxiliary state variables. We denote the interconnection of $\mathcal{H}_P$ and $\mathcal{H}_E$ as $\overline{\mathcal{H}}$, with state $\xi := (x, z)$ and dynamics

$$\overline{\mathcal{H}} : \begin{cases} \left.\begin{array}{l} \dot{x} = F_P(x, \theta) \\ \dot{z} = F_E(x, z) \end{array}\right\} =: \overline{F}(x, z, \theta), \quad (x, z) \in \overline{C} \\ \\ \left.\begin{array}{l} x^+ = G_P(x, \theta) \\ z^+ = G_E(x, z) \end{array}\right\} =: \overline{G}(x, z, \theta), \quad (x, z) \in \overline{D} \end{cases} \tag{7.3}$$

where $\overline{C} := C_P \times \mathbb{R}^m$ and $\overline{D} := D_P \times \mathbb{R}^m$.

The flow map F_E and jump map G_E of $\mathcal{H}_E$ are designed so that, under sufficient excitation conditions, each complete solution (x, z) to $\overline{\mathcal{H}}$ converges to the set

$$\mathcal{A} := \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^m : \hat{\theta} = \theta\}. \tag{7.4}$$

In practice, exact synchronization between a plant and an estimator is difficult to achieve due to delays in sensing, signal transmission, and computation. Moreover, if detection of jumps in the plant state is delayed, resetting z based on G_E may result in divergence of the parameter estimate – see the example in Section 7.5.

[23] See [45] for details on hybrid systems with inputs.

This motivates the design of a new estimation algorithm that accounts for such delays. We denote this algorithm as $\hat{\mathcal{H}}_E$, with dynamics

$$\hat{\mathcal{H}}_E : \begin{cases} \dot{z} = F_E(x,z) & \text{otherwise} \\ z^+ = \hat{G}_E(x,z) & \text{when a jump in } x \text{ is detected} \end{cases} \tag{7.5}$$

where F_E is given in (7.2) and $\hat{G}_E$ is to be designed to solve the following problem.

Problem 7.1: Design the jump map $\hat{G}_E$ of $\hat{\mathcal{H}}_E$ in (7.5) so that, under delays in detection of jumps in the plant state, the parameter estimate $\hat{\theta}$ converges to the unknown parameter vector θ in (7.1), except possibly on the delay intervals.

7.2 Problem Solution

7.2.1 Assumptions

To enable our design of $\hat{G}_E$, we make the following assumption on the data of $\mathcal{H}_P$ and $\mathcal{H}_E$.

Assumption 7.2: Given the hybrid systems $\mathcal{H}_P$ in (7.1) and $\mathcal{H}_E$ in (7.2),

1. $\operatorname{dom} F_P \supset \mathbb{R}^n \times \Theta$;

2. $\operatorname{dom} F_E = \mathbb{R}^n \times \mathbb{R}^m$;

3. $\operatorname{dom} G_E = \mathbb{R}^n \times \mathbb{R}^m$.

We impose the following local Lipschitz continuity condition (that is uniform in θ).

Assumption 7.3: For each $(x, z, \theta) \in \mathbb{R}^n \times \mathbb{R}^m \times \Theta$, there exist $\delta_x, \delta_z > 0$ and $L \geq 0$ such that

$$|\overline{F}(x_1, z_1, \theta) - \overline{F}(x_2, z_2, \theta)| \leq L(|x_1 - x_2| + |z_1 - z_2|)$$

for all $x_1, x_2 \in x + \delta_x\mathbb{B}$ and all $z_1, z_2 \in z + \delta_z\mathbb{B}$, with $\overline{F}$ as in (7.3).

Furthermore, we make the following assumption on solutions to $\mathcal{H}_P$.

Assumption 7.4: Given the hybrid systems $\mathcal{H}_P$ in (7.1), there exist $\mathcal{I}, \varepsilon > 0$, $\Delta \in [0, \mathcal{I})$, and a closed set $\mathcal{P} \subset \mathbb{R}^n$ such that the following conditions hold.

1. Each maximal solution x to $\mathcal{H}_P$ from $x(0,0) \in \mathcal{P}$ satisfies[24]

$$t_j - t_{j-1} \geq \mathcal{I}$$

   for all $j \in \{1, 2, \cdots, \sup_j \operatorname{dom} x\}$, where $\{t_j\}_{j=0}^{\sup_j \operatorname{dom} x}$ is the sequence defining $\operatorname{dom} x$ as in Section 2.1.

2. Each solution x to $\mathcal{H}_P$ from $x(0,0) \in \mathcal{P}$ satisfies

$$|x(t,j) - x(t,j+1)| \geq \varepsilon$$

   for all $(t,j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$.

3. Following each jump in x, there is a delay of $\delta \in [0, \Delta]$ seconds before the jump can be detected.

Item 1 of Assumption 7.4 means that each maximal solution x to $\mathcal{H}_P$ from $x(0,0) \in \mathcal{P}$ has a dwell time of at least $\mathcal{I}$ seconds. Item 2 imposes a lower bound on the change in x at jumps. Item 3 means that each jump in x can be detected in at most Δ seconds following the jump. Since $\Delta < \mathcal{I}$, items 1 and 2 ensure that each time x jumps, it does not jump again in the (hybrid) time required for the estimation algorithm $\hat{\mathcal{H}}_E$ to detect the jump. In Section 7.5 we show that, for the bouncing ball system [22, Example 1.1] with coefficient of restitution equal to one, each solution with initial conditions in a closed set $\mathcal{P}$ satisfies Assumption 7.4.

[24] If x is continuous, we define $t_1 := \sup_t \operatorname{dom} x$.

Remark 7.5: According to item 3 of Assumption 7.4, the delay δ in detecting a jump in x depends on the hybrid time at which the jump occurred. In other words, the detection delay is not necessarily the same for each jump in x. Such a property holds for Zeno solutions as well; however, the dwell time condition imposed by item 1 of Assumption 7.4 prohibits solutions with intervals of flow that have vanishing length as hybrid time evolves.

7.2.2 Hybrid Model

Inspired by [2], we model the interconnection of $\mathcal{H}_P$ and $\hat{\mathcal{H}}_E$ as[25]

$$\hat{\mathcal{H}}: \begin{cases} \left.\begin{aligned} \dot{x} &= F_P(x,\theta) \\ \dot{z} &= F_E(x,z) \\ \dot{\tau}_\delta &= -\min\{\tau_\delta + 1, 1\} \end{aligned}\right\} \xi \in \hat{C} \\ \left.\begin{aligned} x^+ &= G_P(x,\theta) \\ z^+ &= z \\ \tau_\delta^+ &\in [0,\Delta] \end{aligned}\right\} \xi \in \hat{D}_{-1} \\ \left.\begin{aligned} x^+ &= x \\ z^+ &= \hat{G}_E(x,z) \\ \tau_\delta^+ &= -1 \end{aligned}\right\} \xi \in \hat{D}_0 \end{cases} \tag{7.6}$$

with state $\xi := (x, z, \tau_\delta) \in \mathbb{R}^n \times \mathbb{R}^m \times \big(\{-1\} \cup [0,\Delta]\big)$, flow set

$$\hat{C} := \overline{C} \times \big(\{-1\} \cup [0,\Delta]\big),$$

and jump set $\hat{D} := \hat{D}_{-1} \cup \hat{D}_0$, where

$$\hat{D}_{-1} := \overline{D} \times \{-1\}, \qquad \hat{D}_0 := \big(\overline{C} \cup \overline{D}\big) \times \{0\}$$

[25] Note that the jump map in (7.6) encodes a sequential execution of jumps. That is, it does not allow solutions to jump due to the state component x reaching D_P during a delay interval. This modeling decision is justified since Assumption 7.4 imposes that x flows for the duration of each delay interval.

with $\overline{C}$ and $\overline{D}$ given below (7.3).

Compared to $\overline{\mathcal{H}}$ in (7.3), $\hat{\mathcal{H}}$ contains a new state component $\tau_\delta \in [0, \Delta]$ that models the delay between the jumps of the plant and the jumps of the estimator. When $\tau_\delta = -1$ and x does not jump, $\hat{\mathcal{H}}$ flows with x and z flowing according to F_P and F_E, respectively, and τ_δ remains equal to -1. When the plant state x jumps, τ_δ is set to a value in $[0, \Delta]$ thus starting a delay period. $\hat{\mathcal{H}}$ then flows and the timer τ_δ decreases until it reaches 0, at which point a delay interval of length smaller than or equal to Δ has elapsed. Once τ_δ reaches zero, the estimator state is reset based on $\hat{G}_E$, and τ_δ is reset back to -1.

We design the jump map $\hat{G}_E$ of $\hat{\mathcal{H}}$ so that the dynamics of the $\hat{\theta}$ component of solutions to $\hat{\mathcal{H}}$ are equivalent to the dynamics of the $\hat{\theta}$ component of solutions to $\overline{\mathcal{H}}$, except perhaps on the delay intervals. The algorithm we propose requires sampling of solutions to $\hat{\mathcal{H}}$, which we describe in the following section.

7.2.3 Sampling of Solutions to $\hat{\mathcal{H}}$

Let $\xi = (x, z, \tau_\delta)$ be a solution to $\hat{\mathcal{H}}$ in (7.6). During flows, we sample x and z at hybrid time instants $\{(\tilde{t}_k, \tilde{j}_k)\}_{k=0}^{S(t)} \in \operatorname{dom}\xi$, where $t \mapsto S(t) \in \mathbb{N} \setminus \{0\}$ indicates a time-dependent number of samples that is to be designed, and $(\tilde{t}_0, \tilde{j}_0) := (0, 0)$. We also record the time t at which each sample is taken. We store the samples of x, z, and t in time-varying matrices X, Z, and T, respectively, defined as[26]

$$\begin{aligned}
\mathrm{X}(t,j) &:= \begin{bmatrix} x_1(t,j) & x_2(t,j) & \cdots & x_N(t)(t,j) \end{bmatrix} \in \mathbb{R}^{n \times N(t)} \\
\mathrm{Z}(t,j) &:= \begin{bmatrix} z_1(t,j) & z_2(t,j) & \cdots & z_N(t)(t,j) \end{bmatrix} \in \mathbb{R}^{m \times N(t)} \\
\mathrm{T}(t,j) &:= \begin{bmatrix} \tau_1(t,j) & \tau_2(t,j) & \cdots & \tau_N(t)(t,j) \end{bmatrix} \in \mathbb{R}^{1 \times N(t)}
\end{aligned} \tag{7.7}$$

[26]We denote the columns of T as τ_i, rather than t_i, in order to avoid confusion with the sequence of times $\{t_j\}_{j=0}^{\sup_j \operatorname{dom}\xi}$ that define $\operatorname{dom}\xi$ as in Section 2.1.

for all $(t, j) \in \operatorname{dom} \xi$, where $t \mapsto N(t) \in \mathbb{N} \setminus \{0\}$ indicates that the number of columns of X, Z, and T is time-dependent. The matrices X, Z, and T are initialized as $\mathrm{X}(0,0) = x(0,0)$, $\mathrm{Z} = z(0,0)$, and $\mathrm{T}(0,0) = 0$, respectively, with $N(0) = 1$. Each time a sample of x (resp. z, t) is stored, we append a new column to the right of the last column of X (resp. Z, T), thereby increasing the value of $N(t)$ by one, and store the sample in the new column. Samples are stored whenever the current value of t or x differs sufficiently from the value stored in the last column of T or X, respectively. In particular, when

$$|t - \tau_{N(t)}(t, j)| \geq \alpha_t \tag{7.8a}$$

or

$$|x(t, j) - x_{N(t)}(t, j)| \geq \alpha_x \tag{7.8b}$$

where $\alpha_t \in (0, (\mathcal{I} - \Delta)/3]$ and $\alpha_x \in (0, \varepsilon/2]$ are design parameters, with $\mathcal{I}, \varepsilon > 0$ from Assumption 7.4. For each $(t, j) \in \operatorname{dom} \xi$ and each $\ell \in \{1, 2, \cdots, N(t) - 1\}$, we discard column ℓ of X, Z, and T, when $\tau_{\ell+1}(t, j) \leq t - \mathcal{I}$. Each time a column of X, Z, and T is discarded, the value of $N(t)$ is decreased by one. Thus, the elements of X, Z, and T are piecewise constant right-continuous signals, with values changing only at the sample times.

Omitting the arguments of X, Z, T, and N for readability, suppose that a jump in x is detected at hybrid time $(t^*, j^*) \in \operatorname{dom} \xi$. Since each solution ξ to $\hat{\mathcal{H}}$ in (7.6) jumps each time the plant state x jumps, it follows that the jump in x occurred at hybrid time $(t^* - \delta, j^* - 1)$, where δ is unknown and satisfies $\delta \in [0, \Delta]$, with $\Delta \in [0, \mathcal{I})$ from Assumption 7.4. Note that, due to Assumption 7.4, (7.8a) guarantees that, at hybrid time $(t^*, j^*) \in \operatorname{dom} \xi$, there are at least three samples of x (resp. z, t) stored in X (resp. Z, T) from the interval of flow prior to the jump at hybrid time $(t^* - \delta, j^* - 1)$. Furthermore, (7.8b) guarantees that a sample of x (resp. z, t) is stored in X (resp. Z, T) immediately after the jump at hybrid time $(t^* - \delta, j^* - 1)$. Finally, since column x_ℓ (resp. z_ℓ, τ_ℓ) of X (resp. Z, T) is discarded when $\tau_{\ell+1} \leq t^* - \mathcal{I}$ for each $\ell \in \{1, 2, \cdots, N-1\}$, it follows that there is only one jump among the samples of x in X. In the next section, we describe a method of determining the index of the jump in the samples of x.

7.2.4 Jump Index Determination

We wish to find the index of the last sample of x that was stored before the jump. We explain our approach in words before formally defining the algorithm.

Pick $\ell \in \{1, 2, \cdots, N-1\}$ and $\vartheta \in \Theta$, with $\Theta \subset \mathbb{R}^p$ from (7.1), where ℓ is a candidate for the index of the last sample of x that was stored before the jump, and ϑ is a candidate for the unknown parameter vector θ. Using the flow map F_P in (7.1), we compute two solutions, denoted by $t \mapsto \hat{x}_1(t)$ with initial condition x_1, and $t \mapsto \hat{x}_{\ell+1}(t)$, with initial condition $x_{\ell+1}$. In particular, we solve

$$\dot{\hat{x}}_1 = F_P(\hat{x}_1, \vartheta), \qquad \hat{x}_1(\tau_1) = x_1 \tag{7.9}$$

for all $t \in [\tau_1,\ \tau_\ell]$, and

$$\dot{\hat{x}}_{\ell+1} = F_P(\hat{x}_{\ell+1}, \vartheta), \qquad \hat{x}_{\ell+1}(\tau_{\ell+1}) = x_{\ell+1} \tag{7.10}$$

for all $t \in [\tau_{\ell+1},\ \tau_N]$, where x_i and τ_i, $i \in \{1, 2, \cdots, N\}$, are the columns of X and T, respectively, in (7.7). Note that, due to item 1 of Assumption 7.2, the systems in (7.9) and (7.10) are well defined for all $\hat{x}_1, \hat{x}_{\ell+1} \in \mathbb{R}^n$ and all $\vartheta \in \Theta$. Finally, we compare the solutions $\hat{x}_1$ and $\hat{x}_{\ell+1}$ against the samples stored in X by computing

$$\alpha(\ell, \vartheta) := \sum_{i=1}^{\ell} |x_i - \hat{x}_1(\tau_i)|^2 + \sum_{i=\ell+1}^{N} |x_i - \hat{x}_{\ell+1}(\tau_i)|^2. \tag{7.11}$$

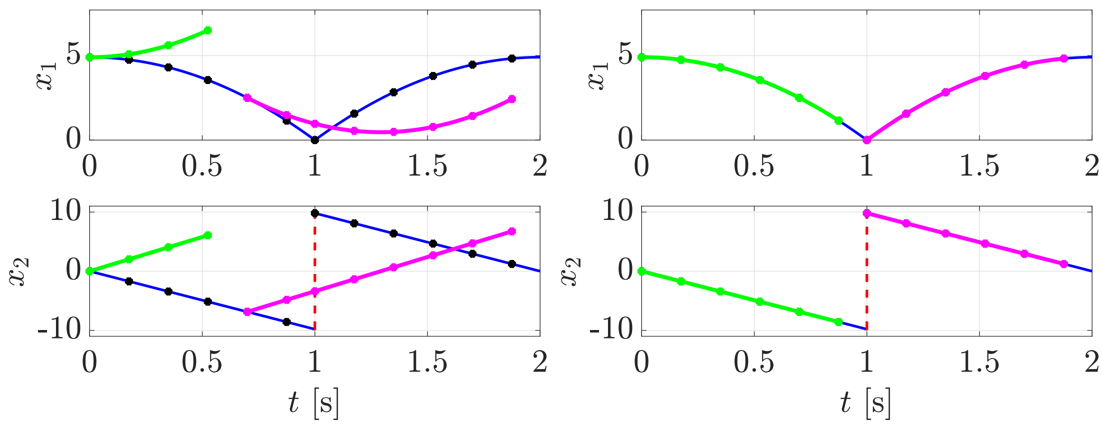

Figure 7.1: The projection onto t of a hybrid arc $x = (x_1, x_2)$, shown in blue, with a jump at $t = 1$ second. The stored samples of x are shown in black. The trajectories $\hat{x}_1$ and $\hat{x}_{\ell+1}$ that minimize α in (7.11) are shown in green and magenta, respectively, for $\ell = 4$ on the left and $\ell = 6$ on the right. The right plot, in which ℓ corresponds to the index of the last sample of x from before the jump, results in the smallest value for α.

Let $\vartheta \in \Theta$ be such that the value of $\alpha(\ell, \vartheta)$ is minimized. If the jump in x occurred between the samples x_ℓ and $x_{\ell+1}$, then the value of $\alpha(\ell, \vartheta)$ will be small. On the other hand, if the jump did not occur between x_ℓ and $x_{\ell+1}$, then $\alpha(\ell, \vartheta)$ may be large. Example trajectories $\hat{x}_1$ and $\hat{x}_{\ell+1}$ that minimize α for an example hybrid arc $x = (x_1, x_2)$ are shown in Figure 7.1 for two different values of ℓ.

Remark 7.6: Note that, for each $\ell \in \{1, 2, \cdots, N\}$, multiple distinct values of $\vartheta \in \Theta$ may yield the same minimum value of $\alpha(\ell, \vartheta)$. In particular, if the flow map F_P in (7.1) does not depend on components of θ, then α can be minimized for any values of the corresponding components of ϑ. An example of such a case is described in Section 7.5, where F_P does not depend on the second component of θ.

Formally, we determine the index of the last sample of x that was stored before the jump by solving the following optimization problem

$$\begin{aligned}
\text{minimize} \quad & \alpha(\ell, \vartheta) \\
\text{subject to} \quad & \ell \in \{1, 2, \cdots, N-1\}, \quad \vartheta \in \Theta, \\
& \frac{d}{dt}\hat{x}_1 = F_P(\hat{x}_1, \vartheta), \quad \hat{x}_1(\tau_1) = x_1, \\
& \frac{d}{dt}\hat{x}_{\ell+1} = F_P(\hat{x}_{\ell+1}, \vartheta), \quad \hat{x}_{\ell+1}(\tau_{\ell+1}) = x_{\ell+1}.
\end{aligned} \tag{7.12}$$

7.2.5 Design of $\hat{G}_E$

Let $\ell \in \{1, 2, \cdots, N-1\}$ be the result of (7.12). Since, by (7.8b) and item 2 of Assumption 7.4, a sample of x is stored in X immediately after each jump, it follows that the jump in x occurred at hybrid time $(\tau_{\ell+1}, j^* - 1)$. We design the jump map $\hat{G}_E$ in (7.5) to reset the estimator state, z, using the data stored in Z, based on the knowledge that a jump in x occurred at hybrid time $(\tau_{\ell+1}, j^* - 1)$. To do so, we first compute solutions to the system

$$\begin{aligned}
\dot{\hat{x}} &= F_P(\hat{x}, \vartheta), \qquad \hat{x}(\tau_\ell) = x_\ell \\
\dot{\mu} &= F_E(\hat{x}, \mu), \qquad \mu(\tau_\ell) = z_\ell
\end{aligned} \tag{7.13}$$

for all $t \in [\tau_\ell, \tau_{\ell+1}]$, where $\vartheta \in \Theta$ is constant and the result of (7.12). Note that, due to items 1 and 2 of Assumption 7.2, the system in (7.13) is well defined for all $(\hat{x}, \mu) \in \mathbb{R}^n \times \mathbb{R}^m$ and all $\vartheta \in \Theta$ and, due to Assumption 7.3, a solution to (7.13) from a given initial condition is unique. Then, we reset column $z_{\ell+1}$ of Z in (7.7) using the value of jump map G_E in (7.2) evaluated on $\hat{x}(\tau_{\ell+1})$ and $\mu(\tau_{\ell+1})$, which are obtained from computing the solution to (7.13). That is,

$$z_{\ell+1} = G_E(\hat{x}(\tau_{\ell+1}), \mu(\tau_{\ell+1})) \tag{7.14}$$

which is well defined due to item 3 of Assumption 7.2. Next, we forward propagate the system in (7.13) from time $\tau_{\ell+1}$ up to the current time t^* by solving

$$\begin{aligned}
\dot{\hat{x}} &= F_P(\hat{x}, \vartheta), \qquad \hat{x}(\tau_{\ell+1}) = x_{\ell+1} \\
\dot{\mu} &= F_E(\hat{x}, \mu), \qquad \mu(\tau_{\ell+1}) = z_{\ell+1}
\end{aligned} \tag{7.15}$$

for all $t \in [\tau_{\ell+1}, t^*]$. Given the solution $t \mapsto (x(t), \mu(t))$ to (7.15), we reset columns $\ell+2$ through N of Z in (7.7) as $z_i = \mu(\tau_i)$ for all $i \in \{\ell+2, \ell+3, \cdots, N\}$. Finally, we reset the estimator state z as

$$z^+ = \mu(t^*). \tag{7.16}$$

Given a solution ξ to $\hat{\mathcal{H}}$ and matrices X, Z, and T as in (7.7), we implement (7.12)–(7.16) using the following algorithm.

Algorithm 1 An algorithm for computing $\hat{G}_E$ in (7.5)

Require: $\xi(t^*, j^*) \in D_0$
- Create an empty vector $Q \in \mathbb{R}^{N-1}$
- **for** $\ell = 1$ to $N-1$ **do**
  - Solve for $\vartheta \in \Theta$ that minimizes $\alpha(\ell, \vartheta)$ in (7.11)
  - Store $\alpha(\ell, \vartheta)$ in row ℓ of Q
- **end for**
- Find $\min Q$ and set ℓ as the corresponding row index
- Compute the solution $(\hat{x}, \mu)$ to (7.13) for all $t \in [\tau_\ell, \tau_{\ell+1}]$
- Set $z_{\ell+1} = G_E(\hat{x}(\tau_{\ell+1}), \mu(\tau_{\ell+1})$ as in (7.14)
- Compute the solution $(\hat{x}, \mu)$ to (7.15) for all $t \in [\tau_{\ell+1}, t^*]$
- **for** $i = \ell+2$ to N **do**
  - Set $z_i = \mu(\tau_i)$
- **end for**
- Set $z^+ = \mu(t^*)$

7.3 Stability Analysis

To analyze the stability properties induced by our proposed algorithm, we rewrite the dynamics of $\hat{\mathcal{H}}$ in (7.6) to incorporate our design for the jump map $\hat{G}_E$. To do so, we augment the state vector of $\hat{\mathcal{H}}$ with a new component, μ, that evolves based on the dynamics of μ in (7.13)–(7.15). The resulting hybrid system, denoted by $\mathcal{H}$, has dynamics

$$\mathcal{H}: \begin{cases} \left.\begin{aligned} \dot{x} &= F_P(x,\theta) \\ \dot{z} &= F_E(x,z) \\ \dot{\mu} &= F_E(x,\mu) \\ \dot{\tau}_\delta &= -\min\{\tau_\delta + 1, 1\} \end{aligned}\right\} =: F(\xi,\theta), \quad \xi \in C \\ \left.\begin{aligned} x^+ &= G_P(x,\theta) \\ z^+ &= z \\ \mu^+ &= G_E(x,\mu) \\ \tau_\delta^+ &\in [0,\Delta] \end{aligned}\right\} =: G_{-1}(\xi,\theta), \qquad \xi \in D_{-1} \\ \left.\begin{aligned} x^+ &= x \\ z^+ &= \mu \\ \mu^+ &= \mu \\ \tau_\delta^+ &= -1 \end{aligned}\right\} =: G_0(\xi,\theta), \qquad \xi \in D_0 \end{cases} \tag{7.17}$$

with state $\xi := (x, z, \mu, \tau_\delta) \in \mathcal{X} := \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \big(\{-1\} \cup [0,\Delta]\big)$, flow set

$$C := \overline{C} \times \mathbb{R}^m \times \big(\{-1\} \cup [0,\Delta]\big)$$

and jump set $D := D_{-1} \cup D_0$, where

$$D_{-1} := \overline{D} \times \mathbb{R}^m \times \{-1\}, \qquad D_0 := \big(\overline{C} \cup \overline{D}\big) \times \mathbb{R}^m \times \{0\}$$

with $\overline{C}$ and $\overline{D}$ given below (7.3).

Compared to $\hat{\mathcal{H}}$ in (7.6), $\mathcal{H}$ contains a new state component $\mu \in \mathbb{R}^m$. When $\tau_\delta = -1$ and x does not jump, μ flows per the flow map F_E, as in (7.13). When the plant state x jumps, μ is reset to the value of $G_E(x,\mu)$, and then continues flowing per F_E until τ_δ reaches zero, as in (7.14)–(7.15). At the end of the delay interval, the estimator state z is reset to value of μ, as in (7.16). Note that the dynamics of μ in (7.17) are equivalent to the dynamics of z in (7.3), except that μ has an extra trivial jump each time $\tau_\delta = 0$.

To establish the stability properties induced by $\mathcal{H}$ in (7.17), we first make the following assumption regarding the parameter estimation error for $\overline{\mathcal{H}}$ in (7.3).

Assumption 7.7: Let $\mathcal{P} \subset \mathbb{R}^n$ come from Assumption 7.4. For each compact set $K \subset \mathbb{R}^m$, there exists $\beta \in \mathcal{KL}$ such that the parameter estimation error $(t, j) \mapsto \tilde{\theta}(t, j) := \theta - \hat{\theta}(t, j)$ for each solution (x, z) to the hybrid system $\overline{\mathcal{H}}$ in (7.3) from $(x(0, 0), z(0, 0)) \in \mathcal{P} \times K$ satisfies

$$|\tilde{\theta}(t, j)| \leq \beta(|\tilde{\theta}(0, 0)|, t + j) \quad \forall (t, j) \in \operatorname{dom}(x, z).$$

In words, Assumption 7.7 states that $\overline{\mathcal{H}}$ ensures *semiglobal $\mathcal{KL}$ pre-asymptotic stability* of the set $\mathcal{A}$ in (7.4) – see Definition 2.11. Several recent works proposed hybrid parameter estimation algorithms that, under sufficient excitation conditions, satisfy this assumption. In particular, [28, 48, 60] propose algorithms for hybrid linear regression, and Chapters 4 and 5 propose algorithms for estimating unknown parameters in hybrid dynamical systems.

We now establish our main stability result.

Theorem 7.8: Given the hybrid system $\mathcal{H}$ in (7.17) and $K \subset \mathbb{R}^m$ compact, suppose that Assumption 7.7 holds, and let $\mathcal{P} \subset \mathbb{R}^n$ and $\beta \in \mathcal{KL}$ come from that assumption. Then, the parameter estimation error $(t, j) \mapsto \tilde{\theta}(t, j) := \theta - \hat{\theta}(t, j)$ for each solution ξ to $\mathcal{H}$ from $\xi(0, 0) \in \mathcal{X}_0 := \{\xi \in \mathcal{X} : x \in \mathcal{P}, z \in K, \mu = z, \tau_\delta = -1\}$ satisfies

$$|\tilde{\theta}(t, j)| \leq \beta(|\tilde{\theta}(0, 0)|, t + \eta(j)) \tag{7.18}$$

for all $(t, j) \in \operatorname{dom} \xi$ such that $\tau_\delta(t, j) = -1$, where

$$\eta(j) := j - \left\lfloor \frac{j}{2} \right\rfloor. \tag{7.19}$$

Theorem 7.8 provides an upper bound on the norm of the parameter estimation error for $\mathcal{H}$, except possibly on the delay intervals, namely, for $(t,j)'s$ such that $\tau_\delta(t,j) = -1$.

Remark 7.9: Note that Assumptions 7.2, 7.3, and 7.4 are not included in Theorem 7.8. These assumptions are used to justify the hybrid model $\hat{\mathcal{H}}$ in (7.6) and to design the jump map $\hat{G}_E$ in Sections 7.2.3–7.2.5. We then rewrite $\hat{\mathcal{H}}$ in view of $\hat{G}_E$ to obtain the hybrid system $\mathcal{H}$ in (7.17). However, given $\mathcal{H}$, we need only Assumption 7.7 to prove Theorem 7.8.

7.3.1 Proof of Theorem 7.8

To prove Theorem 7.8, we require the following result that employs a notion of j-reparamaterization of a hybrid arc – see Definition 2.8.

Lemma 7.10: Let $\xi = (x, z, \mu, \tau_\delta)$ be a solution to $\mathcal{H}$ in (7.17) with $\tau_\delta(0,0) = -1$, and let x^r and μ^r be j-reparameterizations of x and μ, respectively, given by

$$x^r(t, \eta(j)) := x(t,j), \qquad \mu^r(t, \eta(j)) := \mu(t,j), \tag{7.20}$$

for all $(t,j) \in \operatorname{dom}\xi$, with η as in (7.19). Then, $\xi^r := (x^r, \mu^r)$ is a solution to the hybrid system $\overline{\mathcal{H}}$ in (7.3).

Remark 7.11: In words, the j-reparameterization of the solution components x and μ in Lemma 7.10 removes the even-numbered jumps (excluding zero) from $\operatorname{dom}\xi$. Since, for each solution ξ to $\mathcal{H}$ with $\tau_\delta(0,0) = -1$, the even-numbered jumps in $\operatorname{dom}\xi$ correspond to jumps from the set D_0, this reparameterization removes the trivial jumps of x and μ.

Proof of Lemma 7.10: Pick a solution $\xi = (x, z, \mu, \tau_\delta)$ to $\mathcal{H}$ with $\tau_\delta(0,0) = -1$. We show that $\xi^r = (x^r, \mu^r)$ below (7.20) satisfies the dynamics of $\overline{\mathcal{H}}$ – see Definition 2.1. Since ξ is a solution to $\mathcal{H}$, we have

$$\xi(0,0) \in \mathrm{cl}(C) \cup D \implies (x(0,0), \mu(0,0)) \in \mathrm{cl}(\overline{C}) \cup \overline{D}.$$

Then, since $x^r(0,0) = x(0,0)$ and $\mu^r(0,0) = \mu(0,0)$, it follows that $\xi^r(0,0) \in \mathrm{cl}(\overline{C}) \cup \overline{D}$.

Let $j \in \mathbb{N}$. If the interior of $I^j := \{t : (t,j) \in \operatorname{dom}\xi\}$ is nonempty, then $\xi(t,j) \in C$ for all $t \in \operatorname{int} I^j$, which, from (7.17), implies that

$$\begin{aligned}\xi^r(t,\eta(j)) &= (x^r(t,\eta(j)), \mu^r(t,\eta(j))) \\ &= (x(t,j), \mu(t,j)) \in C_P \times \mathbb{R}^m \subset \overline{C}\end{aligned}$$

for all $t \in \operatorname{int} I^j$. Furthermore, it follows from (7.17) that

$$\begin{aligned}\dot{\xi}^r(t,\eta(j)) &= (\dot{x}^r(t,\eta(j)), \dot{\mu}^r(t,\eta(j))) \\ &= (\dot{x}(t,j), \dot{\mu}(t,j)) \\ &= \big(F_P(x(t,j),\theta), F_E(x(t,j), z(t,j))\big) \\ &= \big(F_P\big(x^r(t,\eta(j)),\theta\big), F_E\big(x^r(t,\eta(j)), z^r(t,\eta(j))\big)\big) \\ &= \overline{F}\big(x^r(t,\eta(j)), z^r(t,\eta(j)), \theta\big)\end{aligned}$$

for almost all $t \in I^j$, with $\overline{F}$ as in (7.3).

Next, let $(t,j) \in \operatorname{dom}\xi$ be such that $(t,j+1) \in \operatorname{dom}\xi$. If j is odd, then, since $\tau_\delta(0,0) = -1$, it follows that $\xi(t,j) \in D_0$, which implies that

$$\begin{aligned}\xi^r(t,\eta(j)) &= (x^r(t,\eta(j)), \mu^r(t,\eta(j))) \\ &= (x(t,j), \mu(t,j)) \in (C_P \cup D_P) \times \mathbb{R}^m \subset \overline{C} \cup \overline{D}.\end{aligned}$$

Moreover, it follows from (7.17) that

$$\begin{aligned}\xi^r(t,\eta(j+1)) &= (x^r(t,\eta(j+1)),\mu^r(t,\eta(j+1)))\\ &= (x(t,j+1),\mu(t,j+1))\\ &= (x(t,j),\mu(t,j))\\ &= (x^r(t,\eta(j)),\mu^r(t,\eta(j)))\\ &= \xi^r(t,\eta(j))\end{aligned}$$

which trivially satisfies the dynamics of $\overline{\mathcal{H}}$ since, from (7.19), $\eta(j+1)=\eta(j)$ when j is odd. On the other hand, if j is even, then $\xi(t,j)\in D_{-1}$, which implies that

$$\begin{aligned}\xi^r(t,\eta(j)) &= (x^r(t,\eta(j)),\mu^r(t,\eta(j)))\\ &= (x(t,j),\mu(t,j)) \in D_P\times\mathbb{R}^m \subset \overline{D}.\end{aligned}$$

Moreover, it follows from (7.17) that

$$\begin{aligned}\xi^r(t,\eta(j+1)) &= (x^r(t,\eta(j+1)),\mu^r(t,\eta(j+1)))\\ &= (x(t,j+1),\mu(t,j+1))\\ &= \big(G_P(x(t,j),\theta),G_E(x(t,j),\mu(t,j))\big)\\ &= \big(G_P\big(x^r(t,\eta(j)),\theta\big),G_E\big(x^r(t,\eta(j)),\mu^r(t,\eta(j))\big)\big)\\ &= \overline{G}\big(x^r(t,\eta(j)),z^r(t,\eta(j)),\theta\big)\end{aligned}$$

with $\overline{G}$ as in (7.3). Therefore $\xi^r=(x^r,\mu^r)$ is a solution to $\overline{\mathcal{H}}$. □

We now have the ingredients to prove Theorem 7.8.

Proof of Theorem 7.8: Pick a solution $\xi=(x,z,\mu,\tau_\delta)$ to $\mathcal{H}$ from $\mathcal{X}_0$ and let x^r and μ^r be j-reparameterizations of x and μ, respectively, as in (7.20). We partition μ and μ^r as $\mu:=(\hat{\vartheta},\omega)$ and $\mu^r:=(\hat{\vartheta}^r,\omega^r)$, respectively, where $\hat{\vartheta},\hat{\vartheta}^r\in\mathbb{R}^p$ are estimates of θ, and $\omega,\omega^r\in\mathbb{R}^{m-p}$. From Lemma 7.10, it follows that $\xi^r=(x^r,\mu^r)$

is a solution to $\overline{\mathcal{H}}$. Thus, from Assumption 7.7, we have that the $\hat{\vartheta}^r$ component of ξ^r satisfies

$$|\tilde{\vartheta}^r(t,j)| \leq \beta(|\tilde{\vartheta}^r(0,0)|, t+j)$$

for all $(t,j) \in \operatorname{dom} \xi^r$, where $\tilde{\vartheta}^r := \theta - \hat{\vartheta}^r$. Then, from the definition of μ^r in (7.20), it follows that

$$|\tilde{\vartheta}(t,j)| = |\tilde{\vartheta}^r(t,\eta(j))| \leq \beta(|\tilde{\vartheta}(0,0)|, t+\eta(j)) \tag{7.21}$$

for all $(t,j) \in \operatorname{dom} \xi$, where $\tilde{\vartheta} := \theta - \hat{\vartheta}$.

To complete the proof, let $(t,j) \in \operatorname{dom} \xi$ be such that $\tau_\delta(t,j) = -1$. From the flow map F in (7.17), it follows that $\tau_\delta(s,j) = -1$ for all $s \in I^j$. Next, since $\xi(0,0) \in \mathcal{X}_0$, it follows from the jump map G_0 in (7.17) that $z(t_j,j) = \mu(t_j,j)$ – that is, the solution components z and μ are equal at the beginning of the I^j interval of flow. From the equivalence between the dynamics of z and μ during flows, we conclude that $z(s,j) = \mu(s,j)$ for all $s \in I^j$, which, from the partitioning of z and μ, implies that $\hat{\theta}(s,j) = \hat{\vartheta}(s,j)$ for all $s \in I^j$. Since this result does not depend on the particular choice of $(t,j) \in \operatorname{dom} \xi$ such that $\tau_\delta(t,j) = -1$, we conclude from (7.21) that

$$|\tilde{\theta}(t,j)| = |\tilde{\vartheta}(t,j)| \leq \beta(|\tilde{\theta}(0,0)|, t+\eta(j))$$

for all $(t,j) \in \operatorname{dom} \xi$ such that $\tau_\delta(t,j) = -1$, where $\tilde{\theta} = \theta - \hat{\theta}$. Hence, (7.18) holds. □

7.4 Robustness Analysis

In this section, we study the robustness properties induced by $\mathcal{H}$ with respect to vanishing state perturbations. We first impose the following well-posedness assumption on $\overline{\mathcal{H}}$.

Assumption 7.12: The hybrid system $\overline{\mathcal{H}}$ in (7.3) satisfies the *hybrid basic conditions* in Definition 2.2.

We impose the following assumption on solutions to $\overline{\mathcal{H}}$.

Assumption 7.13: All maximal solutions to the hybrid system $\overline{\mathcal{H}}$ in (7.3) are complete.

We now introduce the notion of a *ρ-perturbation* of a hybrid system [22, Definition 6.27]. Namely, given the hybrid system $\mathcal{H}$ in (7.17) and a function $\rho : \mathcal{X} \to \mathbb{R}_{\geq 0}$, the ρ-perturbation of $\mathcal{H}$, denoted by $\mathcal{H}_\rho$, is the hybrid system

$$\mathcal{H}_\rho : \begin{cases} \dot{\xi} = F_\rho(\xi, \theta) & \xi \in C_\rho \\ \xi^+ = G_\rho(\xi, \theta) & \xi \in D_\rho \end{cases} \tag{7.22}$$

where

$$\begin{aligned} C_\rho &:= \{\xi \in \mathcal{X} \ : \ (\xi + \rho(\xi)\mathbb{B}) \cap C \neq \varnothing \}, \\ F_\rho(\xi, \theta) &:= \overline{\text{con}} F((\xi + \rho(\xi)\mathbb{B}) \cap C, \theta) + \rho(\xi)\mathbb{B} \quad \forall \xi \in \mathcal{X}, \\ D_\rho &:= \{\xi \in \mathcal{X} \ : \ (\xi + \rho(\xi)\mathbb{B}) \cap D \neq \varnothing \}, \\ G_\rho(\xi, \theta) &:= \big\{ v \in \mathcal{X} \ : \ v \in g + \rho(g)\mathbb{B}, \\ & \qquad g \in G((\xi + \rho(\xi)\mathbb{B}) \cap D, \theta) + \rho(\xi)\mathbb{B} \big\} \quad \forall \xi \in \mathcal{X}. \end{aligned}$$

and

$$G(\xi, \theta) := \begin{cases} G_{-1}(\xi, \theta) & \text{if } \ \xi \in D_{-1} \\ G_0(\xi, \theta) & \text{if } \ \xi \in D_0. \end{cases}$$

We establish the following robustness result, which relies on the notion of (τ, ε)-closeness of trajectories in Definition 2.7.

Theorem 7.14: Given the hybrid system $\overline{\mathcal{H}}$ in (7.3), suppose that Assumptions 7.12 and 7.13 hold. Then, for every solution ξ_ρ to the hybrid system $\mathcal{H}_\rho$ in (7.22), there exists a solution ξ to the hybrid system $\mathcal{H}$ in (7.17) such that ξ_ρ and ξ are (τ, ε)-close.

To prove Theorem 7.14, we require the following results.

Lemma 7.15: If Assumption 7.12 holds for $\overline{\mathcal{H}}$, then $\mathcal{H}$ satisfies the hybrid basic conditions.

Proof. This proof is in Appendix F.1. □

Lemma 7.16: If Assumption 7.13 holds for $\overline{\mathcal{H}}$, then all maximal solutions to $\mathcal{H}$ are complete.

Proof. This proof is in Appendix F.2. □

We recall [22, Proposition 6.34].

Proposition 7.17: Given a hybrid system $\mathcal{H}$ as in (2.1), let $\mathcal{H}$ be well posed in the sense of [22, Definition 6.29]. Suppose that $\mathcal{H}$ is pre-forward complete (see Definition 2.6) from a compact set $K \subset \mathbb{R}^n$ and $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. Then, for every $\varepsilon > 0$ and $\tau \geq 0$, there exists $\delta > 0$ with the following property: for every solution x_δ to $\mathcal{H}_{\delta\rho}$ with $x_\delta(0,0) \in K + \delta\mathbb{B}$, there exists a solution x to $\mathcal{H}$ with $x(0,0) \in K$ such that x_δ and x are (τ, ε)-close.

We now have the ingredients to prove Theorem 7.14.

Proof of Theorem 7.14: We rely on Proposition 7.17. Indeed, well-posedness of $\mathcal{H}$ follows from Assumption 7.12, Lemma 7.15, and Theorem 2.3. Pre-forward completeness of $\mathcal{H}$ follows from Assumption 7.13 and Lemma 7.16. The (τ, ε)-closeness result follows from Proposition 7.17. □

7.5 Numerical Example

Consider the problem of estimating the acceleration due to gravity and the restitution coefficient for a bouncing ball. The ball has state $x = (x_1, x_2) \in \mathbb{R}^2$, where x_1 is the height above the ground and x_2 is the vertical velocity. The bouncing ball system has dynamics [22, Example 1.1]

$$\mathcal{H}_P : \begin{cases} \dot{x} = \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} = F_P(x, \theta) & x \in C_P \\ x^+ = \begin{bmatrix} 0 \\ -\lambda x_2 \end{bmatrix} = G_P(x, \theta) & x \in D_P \end{cases} \tag{7.23}$$

with flow set $C_P := \{x \in \mathbb{R}^2 : x_1 \geq 0\}$, jump set $D_P := \{x \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0\}$, and $\theta := (\gamma, \lambda) \in \Theta := [0, 10] \times [0, 1]$, where γ is the acceleration due to gravity and λ is the restitution coefficient.

For the estimation algorithm $\mathcal{H}_E$, we employ the algorithm proposed in Chapter 4. This algorithm has state $z := (\hat{\theta}, \psi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \times \mathbb{R}^2$, input $x \in \mathbb{R}^2$, and data

$$\begin{aligned} F_E(x, z) &= \begin{bmatrix} \gamma_c \psi^\top (y - \psi\hat{\theta}) \\ -\lambda_c \psi + \phi_c(x) \\ -\lambda_c (x + \eta) - f_c(x) \end{bmatrix} \\ G_E(x, z) &= \begin{bmatrix} \hat{\theta} + \frac{\psi^{+\top}}{\gamma_d + |\psi^+|^2}(y^+ - \psi^+\hat{\theta}) \\ (1 - \lambda_d)\psi + \phi_d(x) \\ (1 - \lambda_d)(x + \eta) - g_d(x) \end{bmatrix} \end{aligned} \tag{7.24}$$

where $\gamma_c, \lambda_c, \gamma_d > 0$, $\lambda_d \in (0, 2)$ are design parameters, $y := x + \eta$, and

$$f_c(x) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \qquad \phi_c(x) = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

$$g_d(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \phi_d(x) = \begin{bmatrix} 0 & 0 \\ 0 & -x_2 \end{bmatrix}.$$

The interconnection of $\mathcal{H}_P$ and $\mathcal{H}_E$ is simulated with $\theta = (9.81, 1)$, $\gamma_c = 1.5$, $\lambda_c = 0.01$, $\gamma_d = 0.5$, and $\lambda_d = 1.99$ from the initial conditions $x(0,0) = (4.91, 0)$, $\hat{\theta}(0,0) = (0,0)$, $\psi(0,0) = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, and $\eta(0,0) = -x(0,0)$. It can be shown numerically that the trajectory of the plant state x, shown in Figure 7.2, is sufficiently exciting to ensure exponential convergence of $\hat{\theta}$ to θ for $\mathcal{H}_E$ (see Theorem 4.7). Hence,

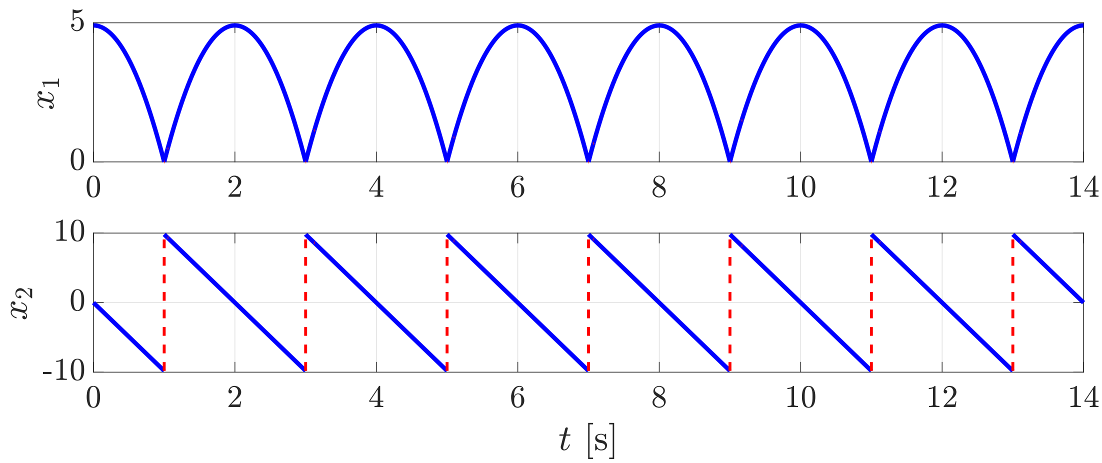

Figure 7.2: The projection onto t of the bouncing ball state x.

if jumps in x are detected instantaneously, the parameter estimate $\hat{\theta}$ converges exponentially to θ as shown in blue in Figure 1.3. However, when the detection of each jump is delayed by up to 0.2 seconds, the parameter estimation error fails to converge to zero, as shown in green in Figure 1.3.

To estimate θ in the presence of delays in jump detection, we employ our proposed algorithm $\hat{\mathcal{H}}$ in (7.6), where the jump map $\hat{G}_E$ of $\hat{\mathcal{H}}$ is computed using Algorithm 1. It can be shown that the maps F_P in (7.23) and F_E, G_E in (7.24) satisfy Assumptions 7.2 and 7.3. Furthermore, with $\theta = (9.81, 1)$, for each maximal solution x to (7.23) from $x(0,0) \in [4.91, \infty) \times [0, \infty)$, the initial interval of flow in $\operatorname{dom} x$ has length of at least 1 second, and x flows for at least 2 seconds after each jump. Since the detection of jumps in x is delayed by at most 0.2 seconds, it follows that Assumption 7.4 holds with $\mathcal{P} = [4.91, \infty) \times [0, \infty)$, $\mathcal{I} = 1$, and

$\Delta = 0.2$. Moreover, since, for each compact set $K \subset \mathbb{R}^m$, the $\hat{\theta}$ component of each solution to $\overline{\mathcal{H}}$ from the set $\mathcal{P} \times K$ converges exponentially to θ, it follows that Assumption 7.7 holds. Hence, the conditions of Theorem 7.8 are satisfied. The parameter estimation error for $\hat{\mathcal{H}}$, shown in green in Figure 7.3, converges to zero and is equal to the error for $\mathcal{H}_E$ shown in blue (with no delays in jump detection), except possibly on the delay intervals, in accordance with Theorem 7.8.

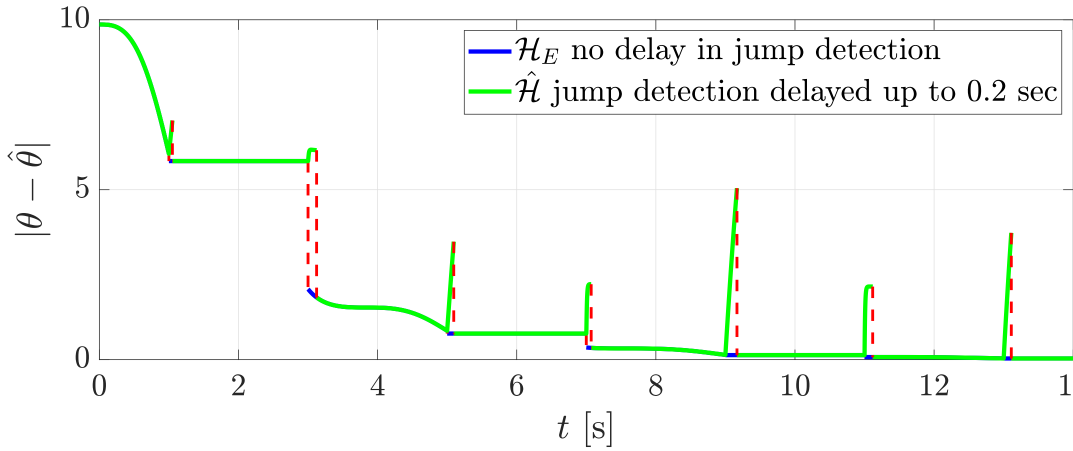

Figure 7.3: The projection onto t of the parameter estimation error for $\mathcal{H}_E$ with no delay in jump detection (blue) and for $\hat{\mathcal{H}}$ with a delay of up to 0.2 seconds (green). The estimation error for $\hat{\mathcal{H}}$ converges to zero, except possibly on the delay intervals.

Chapter 8

Conclusion and Future Work

In this dissertation, we proposed algorithms for estimating unknown parameters in hybrid dynamical systems. In this chapter, we present a summary of the major contributions and describe several potential future research directions.

8.1 Summary

In Chapter 3, we established key properties of a class of time-varying hybrid systems. For (3.1), we established global exponential stability of a closed set under the hybrid persistence of excitation condition in Assumption 3.2. Then, for (3.8), we established input-to-state stability with respect to a class of hybrid disturbances in the state dynamics. These results are used in Chapters 4 and 5 to prove the properties induced by our proposed hybrid parameter estimation algorithms.

In Chapter 4, we developed a hybrid algorithm for estimating unknown parameters in hybrid systems of the form in (4.1). The algorithm flows and jumps in accordance with the plant, under the assumption that jumps in the plant state are detected instantaneously. The parameter estimate is updated continuously during flows and discretely at jumps, using dynamics inspired by the continuous-time and discrete-time gradient descent algorithms in Section 2.2. We show that our proposed algorithm ensures exponential convergence of the parameter estimation error to zero under the hybrid persistence of excitation condition in (4.13). Furthermore, we show that the algorithm is ISS with respect to bounded hybrid noise on the measurements of the plant state.

In Chapter 5, we developed a hybrid algorithm for estimating unknown parameters in hybrid systems of the form in (5.1). Inspired by the continuous-time ICL algorithm in Section 2.3, our proposed hybrid ICL algorithm integrates the input and output signals of the plant during flows, and sums these signals at jumps, over a hybrid time window of specified length. The resulting values are selectively stored using the algorithms in Section 5.6, and the stored data is used alongside current measurements to adapt the parameter estimate. We show that our proposed algorithm ensures exponential convergence of the parameter estimation error to zero when the stored data satisfies the hybrid richness condition in (5.25). Furthermore, we show that the algorithm is ISS with respect to bounded hybrid noise on the measurements of the plant state.

In Chapter 6, we developed a hybrid algorithm for estimating unknown parameters in continuous-time systems of the form in (6.1). We show that our proposed algorithm ensures finite-time convergence of the parameter estimate to the true value when the regressor is exciting over only a finite time interval. As a result, our algorithm can also be applied to estimate unknown parameter of hybrid systems when the regressor is sufficiently exciting over a single interval of flow. We show that our proposed algorithm is ISS with respect to bounded hybrid noise on the measurements of the plant state if the regressor satisfies a persistence of excitation condition. Furthermore, we show that, if the regressor is exciting over only a finite time interval, the parameter estimation error is bounded by a function of the integral of the measurement noise.

In Chapter 7, we developed a hybrid algorithm for estimating unknown parameters in hybrid systems of the form in (7.1), whose jump times are known only approximately. We propose a method of modifying an estimation algorithm that is designed to jump coincident with jumps in the plant state, so that its jumps can be delayed with respect to jumps in the plant state. By sampling and storing the input and output signals of the plant, we solve an optimization problem that estimates the jump times of the plant. We show that the estimation algorithm that results from our proposed modification preserves the stability properties induced by the unmodified version, except possibly during the delays in detection of jumps in the plant state. Furthermore, we show that our proposed algorithm is robust to vanishing state perturbations.

8.2 Future Directions

The following research directions arise from the results in this dissertation.

- **Estimation of parameters for hybrid systems with unknown jump times:** The algorithms in this dissertation are derived assuming that the jump times of the plant are known, either exactly or approximately. As a result, we do not use information about the flow and jump sets of the plant to adapt the parameter estimate. By extending the algorithms in this dissertation to incorporate information about the flow and jump sets of the plant, it may be possible to relax the assumption that the jump times of the plant state are known.

- **Nonlinear dependence on unknown parameters:** This dissertation considers parameter estimation for classes of hybrid systems whose dynamics depend linearly on the unknown parameters. However, this dependence may be nonlinear in practice, which motivates developing algorithms for estimating unknown parameters for hybrid systems whose dynamics depend nonlinearly on the unknown parameters. The development of hybrid algorithms employing the state augmentation approach in [17] or the multi-observer approach in [13, 61] are promising directions for this research.

- **General robustness to perturbations:** In this dissertation, we establish robustness results for each of the proposed algorithms. Namely, in Chapters 4, 5, and 6, we study input-to-state stability with respect to hybrid noise in the measurements of the plant state, and, in Chapter 7, we study robustness with respect to vanishing state perturbations. These results motivate studying the robustness properties induced by our proposed algorithms for wider varieties of disturbances and uncertainties. The analysis tools in [22, 45] are promising approaches for this research.

- **Simultaneous state and parameter estimation for hybrid systems:** The algorithms in this dissertation are derived assuming that the full plant state is measurable. However, full state measurement is not always feasible in practice, which motivates extending the results in this dissertation to develop algorithms for simultaneous state and parameter estimation for hybrid systems with only output measurements. The development of hybrid extensions of the algorithms in [13, 17, 34, 58, 59, 62] are promising directions for this research.

- **Estimation of slowly time-varying parameters for hybrid systems:** The algorithms in this dissertation are derived assuming that unknown parameters are constant. However, such parameters may be slowly time-varying it practice, which motivates developing algorithms for estimating slowly time-varying unknown parameters in hybrid dynamical systems. The development of hybrid extensions of the algorithms in [32, 34, 58, 59] are promising directions for this research.

- **Experimental validation using real-world testbeds:** This dissertation includes simulation results for several practical systems, including a bouncing ball system in Chapters 5 and 7, and a spacecraft system in Chapter 4. However, experimental validation of our results using real-world testbeds would further highlight the practicality of the proposed estimation algorithms. The juggling apparatus in [56] and the boost converter system in [26, 55] are promising applications for this work.

Appendix A

Proofs of Stability Results for Chapter 3

A.1 Proof of Theorem 3.3

Consider the following Lyapunov function

$$V_0(\xi) := \frac{1}{2}\vartheta^\top \vartheta = \frac{1}{2}|\xi|_{\mathcal{A}}^2 \quad \forall \xi \in C_0 \cup D_0. \tag{A.1}$$

By (3.1), we have that, for all $\xi \in C_0$,

$$\langle \nabla V_0(\xi), F_0(\xi) \rangle = -\vartheta^\top A(\tau, k)\vartheta \leq 0 \tag{A.2}$$

and, for all $\xi \in D_0$,

$$V_0(G_0(\xi)) - V_0(\xi) \leq -\frac{1}{2}\vartheta^\top B(\tau, k)\vartheta \leq 0 \tag{A.3}$$

where the inequality follows from the fact that, by Assumption 3.1, $B(t, j) \geq 0$ and $|B(t, j)| \leq 1$ for all $(t, j) \in \Upsilon(E)$. Hence, V_0 is nonincreasing during flows and jumps.

To prove pre-exponential stability of $\mathcal{A}$ for $\mathcal{H}_0$, we follow the approach in [48] and show that, for each solution ξ to $\mathcal{H}_0$ and each $(t', j'), (t^*, j^*) \in \operatorname{dom} \xi$ satisfying (3.4), with Δ from Assumption 3.2, the following inequality holds:

$$V_0(\xi(t^*, j^*)) \leq (1 - \sigma) V_0(\xi(t', j')), \tag{A.4}$$

with σ as in (3.7). To show (A.4), we define, for the given solution ξ to $\mathcal{H}_0$ and for each $(t', j'), (t^*, j^*) \in \operatorname{dom} \xi$ satisfying $t^* + j^* \geq t' + j'$,

$$\begin{aligned} \widetilde{V} &:= V_0(\xi(t^*, j^*)) - V_0(\xi(t', j')) \\ &= \sum_{j=j'}^{j^*} V_F(\max\{t', t_j\}, \min\{t^*, t_{j+1}\}, j) + \sum_{j=j'}^{j^*-1} V_G(t_{j+1}, j) \end{aligned} \tag{A.5}$$

where for each $(t, j), (t_o, j) \in \operatorname{dom} \xi$ satisfying $t \geq t_o$,

$$V_F(t_o, t, j) := V_0(\xi(t, j)) - V_0(\xi(t_o, j)) \tag{A.6}$$

and for each $(t, j) \in \operatorname{dom} \xi$ such that $(t, j+1) \in \operatorname{dom} \xi$,

$$V_G(t, j) := V_0(\xi(t, j+1)) - V_0(\xi(t, j)). \tag{A.7}$$

To complete the proof, we rely on the following lemmas.

Lemma A.1: For each solution $\xi = (\vartheta, \tau, k)$ to the hybrid system $\mathcal{H}_0$ in (3.1) *and each $(t', j'), (t, j) \in \operatorname{dom} \xi$ satisfying $t + j \geq t' + j'$, the x component of ξ satisfies*

$$\begin{aligned} \vartheta(t, j) = \vartheta(t', j') &- \sum_{i=j'}^{j} \int_{\max\{t', t_i\}}^{\min\{t, t_{i+1}\}} A(s, i)\vartheta(s, i) ds \\ &- \sum_{i=j'}^{j-1} B(t_{i+1}, i)\vartheta(t_{i+1}, i). \end{aligned} \tag{A.8}$$

Proof. This proof is given in Appendix A.2. □

Lemma A.2: Given the hybrid system $\mathcal{H}_0$ *in* (3.1)*, suppose that Assumption 3.1 holds and let* $a_M > 0$ *come from that assumption. Then, for each solution* $\xi = (\vartheta, \tau, k)$ *to* $\mathcal{H}_0$*, each* $(t', j'), (t^*, j^*) \in \operatorname{dom} \xi$ *satisfying* $t^* + j^* \geq t' + j'$*, and each* $\rho > 0$*, the function* V_F *in* (A.6) *satisfies*

$$\begin{aligned}
&\sum_{j=j'}^{j^*} V_F(\max\{t', t_j\}, \min\{t^*, t_{j+1}\}, j) \\
&\leq -\frac{\rho}{1+\rho}\vartheta(t', j')^\top \left(\sum_{j=j'}^{j^*} \int_{\max\{t', t_j\}}^{\min\{t^*, t_{j+1}\}} A(s, j) ds \right) \vartheta(t', j') \\
&\quad - \rho a_M (a_M + 2)(t^* - t' + 1)^2 (j^* - j' + 1)^2 \widetilde{V}
\end{aligned}$$

with $\widetilde{V}$ *defined in* (A.5).

Proof. This proof is given in Appendix A.3. □

Lemma A.3: Given the hybrid system $\mathcal{H}_0$ *in* (3.1)*, suppose that Assumption 3.1 holds and let* $a_M > 0$ *come from that assumption. Then, for each solution* $\xi = (\vartheta, \tau, k)$ *to* $\mathcal{H}_0$*, each* $(t', j'), (t^*, j^*) \in \operatorname{dom} \xi$ *satisfying* $t^* + j^* \geq t' + j'$*, and each* $\rho > 0$*, the function* V_G *in* (A.7) *satisfies*

$$\begin{aligned}
\sum_{j=j'}^{j^*-1} V_G(t_{j+1}, j) \leq &-\frac{1}{2}\frac{\rho}{1+\rho}\vartheta(t', j')^\top \left(\sum_{j=j'}^{j^*-1} B(t_{j+1}, j) \right) \vartheta(t', j') \\
&- \frac{\rho}{2}(a_M + 2)(t^* - t' + 1)(j^* - j' + 1)^2 \widetilde{V}
\end{aligned}$$

with $\widetilde{V}$ *defined in* (A.5).

Proof. This proof is given in Appendix A.4. □

Combining the results in Lemma A.2 and Lemma A.3, we obtain the following upper bound on $\widetilde{V}$ in (A.5),

$$\begin{aligned}\widetilde{V} \leq &-\frac{\rho}{1+\rho}\\ &\times \vartheta(t',j')^\top \left(\sum_{j=j'}^{j^*}\int_{\max\{t',t_j\}}^{\min\{t^*,t_{j+1}\}} A(s,j)ds + \frac{1}{2}\sum_{j=j'}^{j^*-1} B(t_{j+1},j)\right)\vartheta(t',j')\\ &-\rho(a_M+2)(t^*-t'+1)(j^*-j'+1)^2(a_M(t^*-t'+1)+1/2)\widetilde{V}\end{aligned}$$

From Assumption 3.2 and the definition of V_0 in (A.1), it follows that

$$\begin{aligned}\widetilde{V} \leq &-\frac{2\rho\mu}{1+\rho}V_0(\xi(t',j'))\\ &-\rho(a_M+2)(t^*-t'+1)(j^*-j'+1)^2(a_M(t^*-t'+1)+1/2)\widetilde{V}\end{aligned}$$

Since, by (A.2) and (A.3), $\widetilde{V}$ is nonpositive, the second term on the right-hand side of the expression above is nonnegative. Using the fact that, by Assumption 3.2, $t^*-t' < \Delta+1$ and $j^*-j' < \Delta+1$, we upper bound the expression above by

$$\begin{aligned}\widetilde{V} \leq &-\frac{2\rho\mu}{1+\rho}V_0(\xi(t',j'))\\ &-\rho(a_M+2)(\Delta+2)^3(a_M(\Delta+2)+1/2)\widetilde{V}.\end{aligned}$$

Using the definition of $\widetilde{V}$ in (A.5), we have

$$V_0(\xi(t^*,j^*)) \leq (1-\sigma)V_0(\xi(t',j')). \tag{A.9}$$

with

$$\sigma = \frac{2\rho\mu}{(1+\rho)\left(1+\rho(a_M+2)(\Delta+2)^3(a_M(\Delta+2)+1/2)\right)}$$

where $\rho > 0$ is chosen so that $\sigma \in (0,1)$. By choosing ρ as

$$\rho := 1\Big/\sqrt{(a_M+2)(\Delta+2)^3(a_M(\Delta+2)+1/2)}$$

we conclude that (A.4) holds with σ in (3.7). From the fact that

$$\mu \leq a_M(t^* - t') + \frac{1}{2}(j^* - j') \leq (\Delta + 1)(a_M + 1/2)$$

it follows that σ in (3.7) satisfies $\sigma \in (0, 1)$.

Since (A.9) holds for each $(t', j'), (t^*, j^*) \in \operatorname{dom} \xi$ satisfying (3.4) and, by (A.2) and (A.3), V_0 is nonincreasing during flows and jumps, it follows that, for all $(t, j) \in \operatorname{dom} \xi$,

$$\begin{aligned} V_0(\xi(t,j)) &\leq (1-\sigma)^{\left\lfloor \frac{t+j}{\Delta+1} \right\rfloor} V_0(\xi(0,0)) \\ &\leq (1-\sigma)^{\left(\frac{t+j}{\Delta+1} - 1\right)} V_0(\xi(0,0)) \\ &\leq \frac{1}{1-\sigma}(1-\sigma)^{\frac{t+j}{\Delta+1}} V_0(\xi(0,0)). \end{aligned}$$

Using the definition of V_0 in (A.1), we conclude that (3.6) holds with κ, λ in (3.7). □

A.2 Proof of Lemma A.1

Pick a maximal solution $\xi = (\vartheta, \tau, k)$ to $\mathcal{H}_0$ and hybrid time instants $(t', j'), (t, j) \in \operatorname{dom} \xi$ satisfying $t + j \geq t' + j'$. To prove the result, we proceed by induction on $i \in \{j', j' + 1, \cdots, j\}$. First, let $i = j'$. We consider the following two cases.

1. If ξ is flowing at hybrid time (t', j'), then it follows from (3.1) that, for all $(r, j') \in \operatorname{dom} \xi$ satisfying $r \geq t'$,

$$\begin{aligned} \vartheta(r, j') &= \vartheta(t', j') - \int_{t'}^{r} A(s, j')\vartheta(s, j')ds \\ &= \vartheta(t', j') - \sum_{i=j'}^{j'} \int_{\max\{t', t'_j\}}^{\min\{r, t_{j'+1}\}} A(s, i)\vartheta(s, i)ds \\ &\quad - \sum_{i=j'}^{j'-1} B(t_{i+1}, i)\vartheta(t_{i+1}, i) \end{aligned}$$

where the second equality above follows from the fact that the third term on the right-hand side is equal to zero. Hence, (A.8) holds.

2. If ξ jumps at hybrid time (t', j'), then it follows from (3.1) that, at hybrid time $(t', j'+1) \in \operatorname{dom}\xi$,

$$\begin{aligned}\vartheta(t', j'+1) &= \vartheta(t', j') - B(t', j')\vartheta(t', j') \\ &= \vartheta(t', j') - \sum_{i=j'}^{j'+1} \int_{\max\{t', t'_j\}}^{\min\{t', t_{j'+1}\}} A(s, i)\vartheta(s, i)ds \\ &\quad - \sum_{i=j'}^{j'} B(t_{i+1}, i)\vartheta(t_{i+1}, i)\end{aligned}$$

where the second equality above follows from the fact that the integral on the right-hand side is equal to zero. Hence, (A.8) holds.

Now, for the given hybrid time instant $(t, j) \in \operatorname{dom}\xi$, we consider the following two cases.

1. If the interior of $I^j := \{t : (t, j) \in \operatorname{dom}\xi\}$ is nonempty, then ξ is flowing at time (t_j, j), where $\{t_j\}_{j=0}^{\sup_j \operatorname{dom}\xi}$ is the sequence defining $\operatorname{dom}\xi$ as in Section 2.1. Then, assume that (A.8) holds for all $i \in \{j', j'+1, \cdots, j\}$. That is,

$$\begin{aligned}\vartheta(t_j, j) = \vartheta(t', j') &- \sum_{i=j'}^{j} \int_{\max\{t', t_i\}}^{\min\{t_j, t_{i+1}\}} A(s, i)\vartheta(s, i)ds \\ &- \sum_{i=j'}^{j-1} B(t_{i+1}, i)\vartheta(t_{i+1}, i)\end{aligned} \tag{A.10}$$

It follows from (3.1) that, at time $(t, j) \in \operatorname{dom}\xi$,

$$\begin{aligned}\vartheta(t, j) &= \vartheta(t_j, j) - \int_{t_j}^{t} A(s, j)\vartheta(s, j)ds \\ &= \vartheta(t_j, j) - \sum_{i=j}^{j} \int_{\max\{t_j, t_i\}}^{\min\{t, t_{i+1}\}} A(s, i)\vartheta(s, i)ds\end{aligned} \tag{A.11}$$

Subsituting (A.10) into (A.11), we obtain

$$\vartheta(t,j) = \vartheta(t',j') - \sum_{i=j'}^{j} \int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}} A(s,i)\vartheta(s,i)ds \\ - \sum_{i=j'}^{j-1} B(t_{i+1},i)\vartheta(t_{i+1},i)$$

Hence, (A.8) holds.

2. If there exists $(t, j-1) \in \operatorname{dom}\xi$ such that $(t,j) \in \operatorname{dom}\xi$, then ξ jumps at hybrid time $(t, j-1)$. Then, assume that (A.8) holds for all $i \in \{j', j'+1, \cdots, j-1\}$. That is,

$$\vartheta(t,j-1) = \vartheta(t',j') - \sum_{i=j'}^{j-1} \int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}} A(s,i)\vartheta(s,i)ds \\ - \sum_{i=j'}^{j-2} B(t_{i+1},i)\vartheta(t_{i+1},i). \tag{A.12}$$

It follows from (3.1) that, at time $(t,j) \in \operatorname{dom}\xi$,

$$\begin{aligned}\vartheta(t,j) &= \vartheta(t,j-1) - B(t,j-1)\vartheta(t,j-1) \\ &= x(t,j-1) - \sum_{i=j-1}^{j} \int_{\max\{t,t_i\}}^{\min\{t,t_{i+1}\}} A(s,i)\vartheta(s,i)ds \\ &\quad - \sum_{i=j-1}^{j-1} B(t_{i+1},i)\vartheta(t_{i+1},i)\end{aligned} \tag{A.13}$$

where the second equality follows from the fact that the integral on the right-hand side is equal to zero. Subsituting (A.12) into (A.13), we obtain

$$\vartheta(t,j) = \vartheta(t',j') - \sum_{i=j'}^{j} \int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}} A(s,i)\vartheta(s,i)ds \\ - \sum_{i=j'}^{j-1} B(t_{i+1},i)\vartheta(t_{i+1},i).$$

Hence, (A.8) holds.

From the analysis above, we conclude by induction that (A.8) holds for all $(t', j'), (t, j) \in \operatorname{dom}\xi$ satisfying $t \geq t'$ and $j \geq j'$. $\square$

A.3 Proof of Lemma A.2

We begin by bounding the evolution of V_F in (A.6) over a single interval of flow. Since, by Assumption 3.1, A is positive semidefinite, let $A^{1/2}$ denote the unique positive semidefinite square root of A. Then, for each solution ξ to $\mathcal{H}_0$ and each $(t_o, j), (t, j) \in \operatorname{dom}\xi$ satisfying $t \geq t_o$, we have from (A.2) that

$$\begin{aligned} V_F(t_o, t, j) &= V_0(t, j) - V_0(t_o, j) \\ &= -\int_{t_o}^{t} \vartheta(s, j)^\top A^{\frac{1}{2}}(s, j) A^{\frac{1}{2}}(s, j) \vartheta(s, j) ds \\ &= -\int_{t_o}^{t} \left| A^{\frac{1}{2}}(s, j) \vartheta(s, j) \right|^2 ds. \end{aligned}$$

For the given $(t_o, j), (t, j) \in \operatorname{dom}\xi$, we have from Lemma A.1 that, for each $(t', j') \in \operatorname{dom}\xi$ satisfying $t_o + j \geq t' + j'$,

$$\begin{aligned} V_F(t_o, t, j) = -\int_{t_o}^{t} \Bigg| & A^{\frac{1}{2}}(s, j) \vartheta(t', j') - A^{\frac{1}{2}}(s, j) \\ & \times \left(\sum_{i=j'}^{j} \int_{\max\{t', t_i\}}^{\min\{s, t_{i+1}\}} A(v, i) \vartheta(v, i) dv \right. \\ & \left. + \sum_{i=j'}^{j-1} B(t_{i+1}, i) \vartheta(t_{i+1}, i) \right) \Bigg|^2 ds. \end{aligned}$$

Using the fact that, for any $a, b \in \mathbb{R}$ and any $\rho > 0$,

$$|a - b|^2 \geq \frac{\rho}{1 + \rho} |a|^2 - \rho |b|^2$$

we obtain

$$
\begin{aligned}
V_F(t_o,t,j) \leq &-\frac{\rho}{1+\rho}\int_{t_o}^{t}\left|A^{\frac{1}{2}}(s,j)\vartheta(t',j')\right|^2 ds \\
&+\rho\int_{t_o}^{t}\left|A^{\frac{1}{2}}(s,j)\right|^2\Bigg(\Bigg|\sum_{i=j'}^{j}\int_{\max\{t',t_i\}}^{\min\{s,t_{i+1}\}}A(v,i)\vartheta(v,i)dv \\
&+\sum_{i=j'}^{j-1}B(t_{i+1},i)\vartheta(t_{i+1},i)\Bigg|^2\Bigg)ds
\end{aligned}
$$

Using the triangle inequality, the Cauchy–Schwarz inequality, and the boundedness of A according to Assumption 3.1, we obtain

$$
\begin{aligned}
V_F(t_o,t,j) \leq &-\frac{\rho}{1+\rho}\int_{t_o}^{t}\left|A^{\frac{1}{2}}(s,j)\vartheta(t',j')\right|^2 ds \\
&+\rho a_M\int_{t_o}^{t}\Bigg((j-j'+1)\sum_{i=j'}^{j}\Bigg|\int_{\max\{t',t_i\}}^{\min\{s,t_{i+1}\}}A(v,i)\vartheta(v,i)dv\Bigg|^2 \\
&+(j-j')\sum_{i=j'}^{j-1}\left|B(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2\Bigg)ds
\end{aligned}
$$

Using the Cauchy–Schwarz inequality again, we obtain

$$
\begin{aligned}
V_F(t_o,t,j) \leq &-\frac{\rho}{1+\rho}\int_{t_o}^{t}\left|A^{\frac{1}{2}}(s,j)\vartheta(t',j')\right|^2 ds \\
&+\rho a_M\int_{t_o}^{t}\Bigg((j-j'+1)\sum_{i=j'}^{j}(\min\{s,t_{i+1}\}-\max\{t',t_i\}) \\
&\times\int_{\max\{t',t_i\}}^{\min\{s,t_{i+1}\}}\left|A(v,i)\vartheta(v,i)\right|^2 dv \\
&+(j-j')\sum_{i=j'}^{j-1}\left|B(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2\Bigg)ds
\end{aligned}
$$

Using the fact that $t \geq s \geq t_o$,

$$
0 \leq \sum_{i=j'}^{j}\int_{\max\{t',t_i\}}^{\min\{s,t_{j+1}\}}\left|A^{1/2}(v,j)\vartheta(v,j)\right|^2 dv \leq \sum_{i=j'}^{j}\int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}}\left|A^{1/2}(v,j)\vartheta(v,j)\right|^2 dv.
$$

Hence,

$$
\begin{aligned}
V_F(t_o,t,j) \leq & -\frac{\rho}{1+\rho}\int_{t_o}^{t}\left|A^{\frac{1}{2}}(s,j)\vartheta(t',j')\right|^2 ds \\
& + \rho a_M(t-t_o) \\
& \times \Bigg((j-j'+1)\sum_{i=j'}^{j}(\min\{t,t_{i+1}\}-\max\{t',t_i\}) \\
& \quad \times \int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}}\left|A(v,i)\vartheta(v,i)\right|^2 dv \\
& + (j-j')\sum_{i=j'}^{j-1}\left|B(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2\Bigg).
\end{aligned}
$$

Since $\min\{t,t_{i+1}\}-\max\{t',t_i\} \leq t-t'$ for all $i \in \{j',j'+1,\cdots,j\}$,

$$
\begin{aligned}
V_F(t_o,t,j) \leq & -\frac{\rho}{1+\rho}\int_{t_o}^{t}\left|A^{\frac{1}{2}}(s,j)\vartheta(t',j')\right|^2 ds \\
& + \rho a_M(t-t')(j-j'+1) \\
& \times \Bigg((t-t')\sum_{i=j'}^{j}\int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}}\left|A(v,i)\vartheta(v,i)\right|^2 dv \\
& + \sum_{i=j'}^{j-1}\left|B(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2\Bigg)
\end{aligned}
$$

which is upper bounded by

$$
\begin{aligned}
V_F(t_o,t,j) \leq & -\frac{\rho}{1+\rho}\int_{t_o}^{t}\left|A^{\frac{1}{2}}(s,j)\vartheta(t',j')\right|^2 ds \\
& + \rho a_M(t-t'+1)^2(j-j'+1) \\
& \times \Bigg(\sum_{i=j'}^{j}\int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}}\left|A(v,i)\vartheta(v,i)\right|^2 dv \\
& + \sum_{i=j'}^{j-1}\left|B(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2\Bigg).
\end{aligned}
$$

Using the decomposition of A as $A^{1/2}A^{1/2}$ and B as $B^{1/2}B^{1/2}$,

$$\begin{aligned}V_F(t_o,t,j) \leq &- \frac{\rho}{1+\rho}\int_{t_o}^{t}\left|A^{\frac{1}{2}}(s,j)\vartheta(t',j')\right|^2 ds \\ &+ \rho a_M(t-t'+1)^2(j-j'+1) \\ &\times \left(a_M \sum_{i=j'}^{j}\int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}}\left|A^{\frac{1}{2}}(v,i)\vartheta(v,i)\right|^2 dv \right.\\ &\left. + 2\frac{1}{2}\sum_{i=j'}^{j-1}\left|B^{\frac{1}{2}}(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2\right)\end{aligned}$$

which is upper bounded by

$$\begin{aligned}V_F(t_o,t,j) \leq &- \frac{\rho}{1+\rho}\int_{t_o}^{t}\left|A^{\frac{1}{2}}(s,j)\vartheta(t',j')\right|^2 ds \\ &+ \rho a_M(a_M+2)(t-t'+1)^2(j-j'+1) \\ &\times \left(\sum_{i=j'}^{j}\int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}}\left|A^{\frac{1}{2}}(v,i)\vartheta(v,i)\right|^2 dv \right.\\ &\left. + \frac{1}{2}\sum_{i=j'}^{j-1}\left|B^{\frac{1}{2}}(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2\right).\end{aligned}$$

For the given $(t',j'),(t,j) \in \operatorname{dom}\xi$, it follows from (A.2), (A.3), and the definition of $\widetilde{V}$ in (A.5) that, for each $(t^*,j^*) \in \operatorname{dom}\xi$ satisfying $t^*+j^* \geq t+j$,

$$\begin{aligned}-\widetilde{V} &\geq \sum_{i=j'}^{j^*}\int_{\max\{t',t_i\}}^{\min\{t^*,t_{i+1}\}}\left|A^{\frac{1}{2}}(s,i)\vartheta(s,i)\right|^2 ds + \frac{1}{2}\sum_{i=j'}^{j^*-1}\left|B^{\frac{1}{2}}(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2 \\ &\geq \sum_{i=j'}^{j}\int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}}\left|A^{\frac{1}{2}}(s,i)\vartheta(s,i)\right|^2 ds + \frac{1}{2}\sum_{i=j'}^{j-1}\left|B^{\frac{1}{2}}(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}V_F(t_o,t,j) \leq &- \frac{\rho}{1+\rho}\int_{t_o}^{t}\left|A^{\frac{1}{2}}(s,j)\vartheta(t',j')\right|^2 ds \\ &- \rho a_M(a_M+2)(t-t'+1)^2(j-j'+1)\widetilde{V}.\end{aligned}$$

Since, by (A.2) and (A.3), $\widetilde{V}$ is nonpositive, the second term on the right-hand side of the expression above is nonnegative. Using the fact that $t^* \geq t$ and $j^* \geq j$, we upper bound the expression above by

$$\begin{aligned} V_F(t_o, t, j) \leq &- \frac{\rho}{1+\rho} \int_{t_o}^{t} \left| A^{\frac{1}{2}}(s, j) \vartheta(t', j') \right|^2 ds \\ &- \rho a_M (a_M + 2)(t^* - t' + 1)^2 (j^* - j' + 1) \widetilde{V}. \end{aligned}$$

Since this inequality holds for each $(t', j'), (t_o, j), (t, j), (t^*, j^*) \in \operatorname{dom} \xi$ satisfying $t^* \geq t \geq t_o \geq t'$ and $j^* \geq j \geq j'$, we upper bound the evolution of V_F between (t', j') and (t^*, j^*) as

$$\begin{aligned} &\sum_{j=j'}^{j^*} V_F(\max\{t', t_j\}, \min\{t^*, t_{j+1}\}, j) \\ &\leq -\frac{\rho}{1+\rho} \sum_{j=j'}^{j^*} \int_{t_o}^{t} \left| A^{\frac{1}{2}}(s, j) \vartheta(t', j') \right|^2 ds \\ &\quad - \sum_{j=j'}^{j^*} \rho a_M (a_M + 2)(t^* - t' + 1)^2 (j^* - j' + 1) \widetilde{V} \end{aligned}$$

which we rewrite as

$$\begin{aligned} &\sum_{j=j'}^{j^*} V_F(\max\{t', t_j\}, \min\{t^*, t_{j+1}\}, j) \\ &\leq -\frac{\rho}{1+\rho} \vartheta(t', j')^\top \left(\sum_{j=j'}^{j^*} \int_{\max\{t', t_j\}}^{\min\{t^*, t_{j+1}\}} A(s, j) ds \right) \vartheta(t', j') \\ &\quad - \rho a_M (a_M + 2)(t^* - t' + 1)^2 (j^* - j' + 1)^2 \widetilde{V}. \end{aligned}$$

□

A.4 Proof of Lemma A.3

We begin by bounding the evolution of V_G in (A.7) over a single interval of flow. Since, by Assumption 3.1, B is positive semidefinite, let $B^{1/2}$ denote the unique positive semidefinite square root of B. Then, for each solution ξ to $\mathcal{H}_0$ and each $(t, j) \in \operatorname{dom} \xi$ such that $(t, j+1) \in \operatorname{dom} \xi$, we have from (A.3) that

$$\begin{aligned} V_G(t,j) &= V_0(t,j+1) - V_0(t,j) \\ &\leq -\frac{1}{2}\vartheta(t,j)^\top B^{\frac{1}{2}}(t,j) B^{\frac{1}{2}}(t,j)\vartheta(t,j) \\ &\leq -\frac{1}{2}\left|B^{\frac{1}{2}}(t,j)\vartheta(t,j)\right|^2. \end{aligned}$$

For the given $(t, j) \in \operatorname{dom} \xi$, we have from Lemma A.1 that, for each $(t', j') \in \operatorname{dom} \xi$ satisfying $t + j \geq t' + j'$,

$$\begin{aligned} V_G(t,j) \leq -\frac{1}{2}\Bigg| B^{\frac{1}{2}}(t,j)\vartheta(t',j') - B^{\frac{1}{2}}(t,j) \\ \times \left(\sum_{i=j'}^{j} \int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}} A(v,i)\vartheta(v,i)dv \right. \\ \left. + \sum_{i=j'}^{j-1} B(t_{i+1},i)\vartheta(t_{i+1},i) \right) \Bigg|^2. \end{aligned}$$

Next, using the fact that for any $a, b \in \mathbb{R}$ and any $\rho > 0$,

$$|a-b|^2 \geq \frac{\rho}{1+\rho}|a|^2 - \rho|b|^2$$

we obtain

$$\begin{aligned} V_G(t,j) \leq -\frac{1}{2}\frac{\rho}{1+\rho}\left|B^{\frac{1}{2}}(t,j)\vartheta(t',j')\right|^2 + \frac{\rho}{2}\left|B^{\frac{1}{2}}(t,j)\right|^2 \\ \times \left(\left| \sum_{i=j'}^{j} \int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}} A(v,i)\vartheta(v,i)dv \right. \right. \\ \left. \left. + \sum_{i=j'}^{j-1} B(t_{i+1},i)\vartheta(t_{i+1},i) \right|^2 \right) \end{aligned}$$

Using the triangle inequality, the Cauchy–Schwarz inequality, and the boundedness of B according to Assumption 3.1, we obtain

$$\begin{aligned} V_G(t,j) \leq & -\frac{1}{2}\frac{\rho}{1+\rho}\left|B^{\frac{1}{2}}(t,j)\vartheta(t',j')\right|^2 \\ & + \frac{\rho}{2}\Bigg((j-j'+1)\sum_{i=j'}^{j}\left|\int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}} A(v,i)\vartheta(v,i)dv\right|^2 \\ & + (j-j')\sum_{i=j'}^{j-1}\left|B(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2\Bigg) \end{aligned}$$

Using the Cauchy–Schwarz inequality again, we obtain

$$\begin{aligned} V_G(t,j) \leq & -\frac{1}{2}\frac{\rho}{1+\rho}\left|B^{\frac{1}{2}}(t,j)\vartheta(t',j')\right|^2 \\ & + \frac{\rho}{2}\Bigg((j-j'+1)\sum_{i=j'}^{j}(\min\{t,t_{j+1}\}-\max\{t',t_j\}) \\ & \times \int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}} \left|A(v,i)\vartheta(v,i)\right|^2 dv \\ & + (j-j')\sum_{i=j'}^{j-1}\left|B(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2\Bigg) \end{aligned}$$

Since $\min\{t,t_{i+1}\}-\max\{t',t_i\} \leq t-t'$ for all $i \in \{j', j'+1, \cdots, j\}$,

$$\begin{aligned} V_G(t,j) \leq & -\frac{1}{2}\frac{\rho}{1+\rho}\left|B^{\frac{1}{2}}(t,j)\vartheta(t',j')\right|^2 \\ & + \frac{\rho}{2}\Bigg((j-j'+1)(t-t')\sum_{i=j'}^{j}\int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}} \left|A(v,i)\vartheta(v,i)\right|^2 dv \\ & + (j-j')\sum_{i=j'}^{j-1}\left|B(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2\Bigg) \end{aligned}$$

which is upper bounded by

$$
\begin{aligned}
V_G(t,j) \leq & -\frac{1}{2}\frac{\rho}{1+\rho}\left|B^{\frac{1}{2}}(t,j)\vartheta(t',j')\right|^2 \\
& + \frac{\rho}{2}(t-t'+1)(j-j'+1) \\
& \times \Bigg(\sum_{i=j'}^{j}\int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}} \left|A(v,i)\vartheta(v,i)\right|^2 dv \\
& + \sum_{i=j'}^{j-1}\left|B(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2\Bigg).
\end{aligned}
$$

Using the decomposition of A as $A^{1/2}A^{1/2}$ and B as $B^{1/2}B^{1/2}$,

$$
\begin{aligned}
V_G(t,j) \leq & -\frac{1}{2}\frac{\rho}{1+\rho}\left|B^{\frac{1}{2}}(t,j)\vartheta(t',j')\right|^2 \\
& + \frac{\rho}{2}(t-t'+1)(j-j'+1) \\
& \times \Bigg(a_M\sum_{i=j'}^{j}\int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}} \left|A^{\frac{1}{2}}(v,i)\vartheta(v,i)\right|^2 dv \\
& + 2\frac{1}{2}\sum_{i=j'}^{j-1}\left|B^{\frac{1}{2}}(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2\Bigg)
\end{aligned}
$$

which is upper bounded by

$$
\begin{aligned}
V_G(t,j) \leq & -\frac{1}{2}\frac{\rho}{1+\rho}\left|B^{\frac{1}{2}}(t,j)\vartheta(t',j')\right|^2 \\
& + \frac{\rho}{2}(a_M+2)(t-t'+1)(j-j'+1) \\
& \times \Bigg(\sum_{i=j'}^{j}\int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}} \left|A^{\frac{1}{2}}(v,i)\vartheta(v,i)\right|^2 dv \\
& + \frac{1}{2}\sum_{i=j'}^{j-1}\left|B^{\frac{1}{2}}(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2\Bigg).
\end{aligned}
$$

For the given $(t',j'),(t,j)\in\operatorname{dom}\xi$, it follows from (A.2), (A.3), and the defintion of $\widetilde{V}$ in (A.5) that, for each $(t^*,j^*)\in\operatorname{dom}\xi$ satisfying $t^*+j^*\geq t+j$,

$$\begin{aligned}-\widetilde{V} &\geq \sum_{i=j'}^{j^*}\int_{\max\{t',t_i\}}^{\min\{t^*,t_{i+1}\}}\left|A^{\frac{1}{2}}(s,i)\vartheta(s,i)\right|^2 ds+\frac{1}{2}\sum_{i=j'}^{j^*-1}\left|B^{\frac{1}{2}}(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2\\ &\geq \sum_{i=j'}^{j}\int_{\max\{t',t_i\}}^{\min\{t,t_{i+1}\}}\left|A^{\frac{1}{2}}(s,i)\vartheta(s,i)\right|^2 ds+\frac{1}{2}\sum_{i=j'}^{j-1}\left|B^{\frac{1}{2}}(t_{i+1},i)\vartheta(t_{i+1},i)\right|^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}V_G(t,j)\leq &-\frac{1}{2}\frac{\rho}{1+\rho}\left|B^{\frac{1}{2}}(t,j)\vartheta(t',j')\right|^2\\ &-\frac{\rho}{2}(a_M+2)(t-t'+1)(j-j'+1)\widetilde{V}.\end{aligned}$$

Since, by (A.2) and (A.3), $\widetilde{V}$ is nonpositive, the second term on the right-hand side of the expression above is nonnegative. Using the fact that $t^*\geq t$ and $j^*\geq j$, we upper bound the expression above by

$$\begin{aligned}V_G(t,j)\leq &-\frac{1}{2}\frac{\rho}{1+\rho}\left|B^{\frac{1}{2}}(t,j)\vartheta(t',j')\right|^2\\ &-\frac{\rho}{2}(a_M+2)(t^*-t'+1)(j^*-j'+1)\widetilde{V}.\end{aligned}$$

Since this inequality holds for each $(t',j'),(t,j),(t^*,j^*)\in\operatorname{dom}\xi$ satisfying $(t,j+1)\in\operatorname{dom}\xi$, $t^*\geq t\geq t'$, and $j^*\geq j\geq j'$, we upper bound the evolution of V_G between (t',j') and (t^*,j^*) as

$$\begin{aligned}\sum_{j=j'}^{j^*-1}V_G(t_{j+1},j)\leq &-\frac{1}{2}\frac{\rho}{1+\rho}\sum_{j=j'}^{j^*-1}\left|B^{\frac{1}{2}}(t_{j+1},j)\vartheta(t',j')\right|^2\\ &-\sum_{j=j'}^{j^*-1}\frac{\rho}{2}(a_M+2)(t^*-t'+1)(j^*-j'+1)\widetilde{V}\end{aligned}$$

which is upper bounded by

$$\begin{aligned}\sum_{j=j'}^{j^*-1}V_G(t_{j+1},j)\leq &-\frac{1}{2}\frac{\rho}{1+\rho}\sum_{j=j'}^{j^*-1}\vartheta(t',j')^\top\left(\sum_{j=j'}^{j^*}B(t_{j+1},j)\right)\vartheta(t',j')\\ &-\frac{\rho}{2}(a_M+2)(t^*-t'+1)(j^*-j'+1)^2\widetilde{V}.\end{aligned}$$

□

Appendix B

Proofs of Robustness Results for Chapter 3

B.1 Proof of Lemma 3.4

To prove Lemma 3.4, we first recall the following result from [24].

Lemma B.1: Given $B \in \mathbb{R}^{p\times p}$, if $|B| < 1$, then $I - B$ is invertible.

Proof of Lemma 3.4: Let $U : E \to \mathbb{R}^{p\times p}$ be such that $U(0,0)$ is invertible and, for each $j \in \mathbb{N}$ and almost all $t \in I^j$,

$$\frac{d}{dt}U(t,j) = -A(t,j)U(t,j) \tag{B.1}$$

and, for all $(t,j) \in \Upsilon(E)$, with Υ as in (2.2),

$$U(t,j+1) = U(t,j) - B(t,j)U(t,j). \tag{B.2}$$

Then, for all $(t,j),(t',j') \in E$, we define

$$\Phi(t,j,t',j') := U(t,j)U(t',j')^{-1}, \tag{B.3}$$

where, in view of Lemma B.1, $U(t,j)$ is invertible for all $(t,j) \in E$ since $U(0,0)$ is invertible and, by Assumption 3.1, $|B(t,j)| < 1$ for all $(t,j) \in \Upsilon(E)$.

By the equivalence between the dynamics of U and the ϑ component of ξ in (3.1), we have that, for each solution ξ to $\mathcal{H}_0$ and each $(t,j), (t',j') \in \operatorname{dom}\xi$ satisfying $t + j \geq t' + j'$,[27]

$$\vartheta(t,j) = \Phi(t,j,t',j')\vartheta(t',j'). \tag{B.4}$$

Hence, Φ is the state transition matrix for ϑ. Note that Φ is not necessarily smooth at jumps.

Next, we define $(t,j) \mapsto P(t,j)$ as

$$P(t,j) := P_c(t,j) + P_d(t,j) + q_m I \tag{B.5}$$

for all $(t,j) \in E$, with

$$\begin{aligned} P_c(t,j) &:= \sum_{i=j}^{J} \int_{\max\{t,t_i\}}^{t_{i+1}} \Phi(s,i,t,j)^\top Q(s,i)\Phi(s,i,t,j)ds \\ P_d(t,j) &:= \sum_{i=j}^{J} \Phi(t_{i+1},i,t,j)^\top Q(t_{i+1},i)\Phi(t_{i+1},i,t,j), \end{aligned}$$

where $t_{J+1} := T$, with $J := \sup_j E$ and $T := \sup_t E$. Note that the term $q_m I$ in (B.5) was chosen for simplicity – any positive definite matrix would suffice.

We first show that (3.10) holds. Since, for all $(t,j) \in E$, $P_c(t,j) \geq 0$ and $P_d(t,j) \geq 0$, a lower bound on P in (B.5) is $P(t,j) \geq q_m I$ for all $(t,j) \in E$. Next, we develop an upper bound on P. Since, by Assumptions 3.1 and 3.2, the conditions of Theorem 3.3 are satisfied, it follows from (3.6) and from the

[27]Since each solution ξ to $\mathcal{H}_0$ inherits the hybrid time domain E, it follows that $\operatorname{dom}\xi = E$, and thus $\Phi(t,j,t',j')$ is well defined for all $(t,j), (t',j') \in \operatorname{dom}\xi$,

equivalence between $|\xi|_{\mathcal{A}}$ and $|\vartheta|$ that, for each solution $\xi = (\vartheta, \tau, k)$ to $\mathcal{H}_0$ and each $(s,i),(t,j) \in \operatorname{dom}\xi$ satisfying $s+i \geq t+j$,

$$|\vartheta(s,i)| \leq \kappa_0 \mathrm{e}^{-\lambda_0(s+i-t-j)}|\vartheta(t,j)|$$

with κ_0 and λ_0 from Theorem 3.3. By substituting (B.4) into the expression above, we have that, for each $(s,i),(t,j) \in \operatorname{dom}\xi$ satisfying $s+i \geq t+j$,

$$|\Phi(s,i,t,j)\vartheta(t,j)| \leq \kappa_0 \mathrm{e}^{-\lambda_0(s+i-t-j)}|\vartheta(t,j)|$$

which, if $|\vartheta(t,j)| \neq 0$, implies that

$$|\Phi(s,i,t,j)\vartheta(t,j)|/|\vartheta(t,j)| \leq \kappa_0 \mathrm{e}^{-\lambda_0(s+i-t-j)}.$$

Since this inequality holds for any $\vartheta(t,j) \in \mathbb{R}^p \setminus \{0\}$, it follows from the equivalence between $\operatorname{dom}\xi$ and E that, for each $(s,i),(t,j) \in E$ satisfying $s+i \geq t+j$,

$$|\Phi(s,i,t,j)| = \sup_{r \in \mathbb{R}^p \setminus \{0\}} \frac{|\Phi(s,i,t,j)r|}{|r|} \leq \kappa_0 \mathrm{e}^{-\lambda_0(s+i-t-j)}.$$

Then, from the definitions of P_c and P_d below (B.5),

$$P_c(t,j) \leq q_M \int_0^\infty \kappa_0^2 \mathrm{e}^{-2\lambda_0 s} ds I = \frac{q_M \kappa_0^2}{2\lambda_0} I$$

and

$$P_d(t,j) \leq q_M \sum_{i=0}^{\infty} \kappa_0^2 \mathrm{e}^{-2\lambda_0 i} I = \frac{q_M \kappa_0^2 \mathrm{e}^{2\lambda_0}}{\mathrm{e}^{2\lambda_0} - 1} I.$$

From the bounds above and the definition of P in (B.5), we conclude that (3.10) holds with p_m, p_M in (3.11).

Next, we show that (3.12) holds. We differentiate P during flows and use that, for each $(s,i) \in E$ and each $j \in \mathbb{N}$ and for almost all $t \in I^j$,

$$\frac{d}{dt}\Phi(s,i,t,j) = \Phi(s,i,t,j)A(t,j).$$

This property follows from (B.1) and from the definition of Φ in (B.3). For readability, we define

$$\Pi(s,i,t,j) := \Phi(s,i,t,j)^\top Q(s,i)\Phi(s,i,t,j). \tag{B.6}$$

Using the Leibniz integral rule, we obtain that, for each $j \in \mathbb{N}$ and for almost all $t \in I^j$,

$$\begin{aligned}\frac{d}{dt}P_c(t,j) &= A(t,j)^\top\left(\sum_{i=j}^{J}\int_{\max\{t,t_i\}}^{t_{i+1}}\Pi(s,i,t,j)ds\right) \\ &\quad + \left(\sum_{i=j}^{J}\int_{\max\{t,t_i\}}^{t_{i+1}}\Pi(s,i,t,j)ds\right)A(t,j) - Q(t,j) \\ &= A(t,j)^\top P_c(t,j) + P_c(t,j)A(t,j) - Q(t,j)\end{aligned} \tag{B.7}$$

and

$$\begin{aligned}\frac{d}{dt}P_d(t,j) &= A(t,j)^\top\left(\sum_{i=j}^{J}\Pi(t_{i+1},i,t,j)\right) \\ &\quad + \left(\sum_{i=j}^{J}\Pi(t_{i+1},i,t,j)\right)A(t,j) \\ &= A(t,j)^\top P_d(t,j) + P_d(t,j)A(t,j).\end{aligned} \tag{B.8}$$

Combining the expressions in (B.7) and (B.8), and using the definition of P in (B.5), we have that, for each $j \in \mathbb{N}$ and for almost all $t \in I^j$,

$$\begin{aligned}\frac{d}{dt}P(t,j) &= A(t,j)^\top P(t,j) + P(t,j)A(t,j) \\ &\quad - Q(t,j) - 2q_m A(t,j) \\ &\leq A(t,j)^\top P(t,j) + P(t,j)A(t,j) - Q(t,j).\end{aligned}$$

The inequality follows from the fact that, by Assumption 3.1, $A(t,j) \geq 0$ for all $(t,j) \in E$. Hence, (3.12) holds.

To conclude the proof, we show that (3.13) holds. We use the property that, for each $(t,j),(s,i) \in \Upsilon(E)$,

$$\Phi(s,i,t,j+1)(I - B(t,j)) = \Phi(s,i,t,j).$$

This property follows from (B.2) and from the definition of Φ in (B.3). Then, for each $(t,j) \in \Upsilon(E)$,

$$\begin{aligned}&(I - B(t,j))^\top P_c(t,j+1)(I - B(t,j)) \\ &\quad = \sum_{i=j+1}^{J} \int_{\max\{t,t_i\}}^{t_{i+1}} \Pi(s,i,t,j)ds.\end{aligned}$$

Since the value of ordinary time t is the same immediately before and after each jump, it follows that, for each $(t,j) \in \Upsilon(E)$, $\max\{t,t_j\} = t = t_{j+1}$. Hence, we rewrite the expression above as

$$\begin{aligned}&(I - B(t,j))^\top P_c(t,j+1)(I - B(t,j)) \\ &\quad = \sum_{i=j}^{J} \int_{\max\{t,t_i\}}^{t_{i+1}} \Pi(s,i,t,j)ds = P_c(t,j).\end{aligned} \tag{B.9}$$

Focusing now on P_d, for each $(t,j) \in \Upsilon(E)$,

$$\begin{aligned}&(I - B(t,j))^\top P_d(t,j+1)(I - B(t,j)) - P_d(t,j) \\ &\quad = -\Pi(t_{j+1},j,t,j).\end{aligned}$$

From (B.6) and the fact that $t = t_{j+1}$ at each jump,

$$\begin{aligned}&(I - B(t,j))^\top P_d(t,j+1)(I - B(t,j)) - P_d(t,j) \\ &\quad = -\Phi^\top(t,j,t,j)Q(t,j)\Phi(t,j,t,j) = -Q(t,j).\end{aligned} \tag{B.10}$$

Using the definition of P in (B.5), it follows from (B.9) and (B.10) that, for each $(t,j) \in \Upsilon(E)$,

$$\begin{aligned}&(I - B(t,j))^\top P(t,j+1)(I - B(t,j)) - P(t,j)\\&\quad = -Q(t,j) - q_m B(t,j)(2I - B(t,j)) \leq -Q(t,j),\end{aligned}$$

where the inequality holds since, by Assumption 3.1, $B(t,j) \geq 0$ and $|B(t,j)| < 1$ for all $(t,j) \in \Upsilon(E)$. □

B.2 Proof of Theorem 3.5

Since, by Assumptions 3.1 and 3.2, the conditions of Lemma 3.4 are satisfied, it follows that, given $q_M \geq q_m > 0$ and a symmetric matrix function $Q : E \to \mathbb{R}^{p\times p}$ satisfying (3.9), there exists a symmetric matrix function $P : E \to \mathbb{R}^{p\times p}$ satisfying (3.10)–(3.13). Given such P, consider the Lyapunov function

$$V(\xi) := \vartheta^\top P(\tau,k)\vartheta \quad \forall \xi \in C \cup D.$$

From (3.10) and from the equivalence between $|\vartheta|$ and $|\xi|_{\mathcal{A}}$, we have that

$$p_m|\xi|_{\mathcal{A}}^2 \leq V(\xi) \leq p_M|\xi|_{\mathcal{A}}^2 \quad \forall \xi \in C \cup D, \tag{B.11}$$

with p_m, p_M as in (3.11). We first study the change in V during flows. Omitting the (τ,k) arguments for readability, we have from (3.12) that, for all $\xi \in C$,

$$\langle \nabla V(\xi), F(\xi)\rangle \leq -\vartheta^\top Q\vartheta + 2\vartheta^\top P\nu_c.$$

We use that for any $\varrho > 0$, $2\vartheta^\top P\nu_c \leq \varrho\vartheta^\top P\vartheta + \varrho^{-1}\nu_c^\top P\nu_c$. Choosing $\varrho = q_m/(2p_M)$,

$$\langle \nabla V(\xi), F(\xi)\rangle \leq -\frac{q_m}{2p_M}V(\xi) + \frac{2p_M^2}{q_m}|\nu_c(\tau,k)|^2.$$

Let $\zeta \in (0,1)$. By adding and subtracting $\zeta\frac{q_m}{2p_M}V(\xi)$ to the right-hand side of the expression above, we conclude

$$\begin{aligned}\langle \nabla V(\xi), F(\xi)\rangle &\leq -\frac{q_m}{2p_M}(1-\zeta)V(\xi) \\ \forall \xi \in C:\ V(\xi) &\geq \frac{4p_M^3}{q_m^2\zeta}|\nu_c(\tau,k)|^2.\end{aligned} \tag{B.12}$$

Next, we study the change in V at jumps. For readability, we omit the (τ,k) arguments and denote $P(\tau,k+1)$ as P^+. Then, we have from (3.13) that, for all $\xi \in D$,

$$V(G(\xi)) - V(\xi) \leq -\vartheta^\top Q\vartheta + 2|\vartheta^\top P^+\nu_d| + \nu_d^\top P^+\nu_d.$$

We use that for any $\varrho > 0$, $2|\vartheta^\top P^+\nu_d| \leq \varrho\vartheta^\top P^+\vartheta + \varrho^{-1}\nu_d^\top P^+\nu_d$. Choosing $\varrho = q_m/(2p_M)$,

$$V(G(\xi)) - V(\xi) \leq -\frac{q_m}{2p_M}V(\xi) + \left(\frac{2p_M^2}{q_m} + p_M\right)|\nu_d(\tau,k)|^2.$$

Let $\zeta \in (0,1)$. By adding and subtracting $\zeta\frac{q_m}{2p_M}V(\xi)$ to the right-hand side of the expression above, we conclude

$$\begin{aligned}V(G(\xi)) - V(\xi) &\leq -\frac{q_m}{2p_M}(1-\zeta)V(\xi), \\ \forall \xi \in D:\ V(\xi) &\geq \frac{2p_M^2}{q_m\zeta}\left(\frac{2p_M}{q_m}+1\right)|\nu_d(\tau,k)|^2.\end{aligned} \tag{B.13}$$

Note that the lower bound on V for which (B.13) holds is more restrictive than the lower bound on V for which (B.12) holds. Using the function ν defined in (3.15), we combine the expressions in (B.12) and (B.13) and obtain

$$\begin{aligned}\langle \nabla V(\xi), F(\xi)\rangle &\leq -\frac{q_m}{2p_M}(1-\zeta)V(\xi) \quad \forall \xi \in C \cap S \\ V(G(\xi)) - V(\xi) &\leq -\frac{q_m}{2p_M}(1-\zeta)V(\xi) \quad \forall \xi \in D \cap S,\end{aligned}$$

where $S := \left\{\xi \in C \cup D : V(\xi) \geq \frac{2p_M^2}{q_m\zeta}\left(\frac{2p_M}{q_m}+1\right)|\nu(\tau,k)|^2\right\}$. Then, for each solution ξ to $\mathcal{H}$, by integration using the bounds above, we have that, for all $(t,j) \in \operatorname{dom}\xi$,

$$V(\xi(t,j)) \leq \exp\left\{\left(-\frac{q_m}{2p_M}(1-\zeta)t + \ln\left(1-\frac{q_m}{2p_M}(1-\zeta)\right)j\right)\right\}$$
$$\times V(\xi(0,0)) + \frac{2p_M^2}{q_m\zeta}\left(\frac{2p_M}{q_m}+1\right)|\nu|_{(t,j)}.$$

Using (B.11), we conclude that (3.14) holds. □

B.3 Proof of Theorem 3.6

Let ξ be a maximal solution to $\mathcal{H}$. First, we upper bound $(t,j) \mapsto |\xi(t,j)|_{\mathcal{A}}$ for all $(t,j) \in \operatorname{dom}\xi$. Since, by Assumptions 3.1 and 3.2, the conditions of Theorem 3.5 are satisfied, it follows from (3.14) that

$$|\xi(t,j)|_{\mathcal{A}} \leq \beta(|\xi(0,0)|_{\mathcal{A}}, 0) + a\rho|\nu(0,0)| =: \xi_M \tag{B.14}$$

for all $(t,j) \in \operatorname{dom}\xi$, where the second inequality follows from (3.16). Next, we define $\delta \mapsto c_1(\delta) \in \mathbb{R}$ as

$$c_1(\delta) := -\frac{1}{b}\ln\left(\frac{\delta/2}{a\rho|\nu(0,0)|}\right) \quad \forall \delta > 0. \tag{B.15}$$

Let $\delta > 0$ be such that there exists $(t',j') \in \operatorname{dom}\xi$ such that $t'+j' \geq c_1(\delta)$. Then, it follows from (3.16) that for all $(t,j) \in \operatorname{dom}\xi$ satisfying $t+j \geq t'+j'$, $|d(t,j)| \leq a\mathrm{e}^{-b(t+j)}|\nu(0,0)| \leq a\mathrm{e}^{-bc_1(\delta)}|\nu(0,0)| = \delta/(2\rho)$. Hence, for all $(t,j) \in \operatorname{dom}\xi$ satisfying $t+j \geq t'+j'$, the supremum norm of $(t,j) \mapsto |d(t,j)|$ from (t',j') to (t,j) is less than or equal to $\delta/(2\rho)$. Thus, from (3.14), for all $(t,j) \in \operatorname{dom}\xi$ satisfying $t+j \geq t'+j'$,

$$|\xi(t,j)|_{\mathcal{A}} \leq \beta(\xi_M, t+j-c_1(\delta)) + \delta/2 \tag{B.16}$$

with ξ_M as in (B.14). Next, we define $\delta \mapsto c_2(\delta) \in \mathbb{R}$ as

$$c_2(\delta) := -\frac{1}{\omega} \ln\left(\frac{\delta/2}{\beta(\xi_M, 0)}\right) \quad \forall \delta > 0. \tag{B.17}$$

Omitting the argument δ of c_1 and c_2 for readability, we have that, for all $(t, j) \in \operatorname{dom} \xi$ satisfying $t + j \geq c_2 + c_1$,

$$\beta(\xi_M, t + j - c_1) \leq \beta(\xi_M, c_2 + c_1 - c_1) = \delta/2. \tag{B.18}$$

By combining (B.16) and (B.18), it follows that, for each $\delta > 0$ and each $(t, j) \in \operatorname{dom} \xi$,

$$t+j \geq \max\{c_1(\delta), c_2(\delta)+c_1(\delta)\} \implies |\xi(t,j)|_{\mathcal{A}} \leq \delta. \tag{B.19}$$

Since c_1 in (B.15) and c_2 in (B.17) are continuous monotonically increasing functions of δ with $\operatorname{rge} c_1 = \operatorname{rge} c_2 = \mathbb{R}$, it follows that, for each $(t, j) \in \operatorname{dom} \xi$, there exists a unique $\delta > 0$ such that $t + j = \max\{c_1(\delta), c_2(\delta) + c_1(\delta)\}$. For such δ, (B.19) holds. Hence, we develop a bound for $(t, j) \mapsto |\xi(t, j)|_{\mathcal{A}}$ by bounding, for each $(t, j) \in \operatorname{dom} \xi$, the corresponding value of δ for which $t + j = \max\{c_1(\delta), c_2(\delta) + c_1(\delta)\}$.

Given $(t, j) \in \operatorname{dom} \xi$ and $\delta > 0$ satisfying $t + j = \max\{c_1(\delta), c_2(\delta) + c_1(\delta)\}$, we consider two cases: $\max\{c_1(\delta), c_2(\delta) + c_1(\delta)\} = c_1(\delta)$ and $\max\{c_1(\delta), c_2(\delta) + c_1(\delta)\} = c_2(\delta) + c_1(\delta)$.

1. If $\max\{c_1(\delta), c_2(\delta) + c_1(\delta)\} = c_1(\delta)$, then $t + j = c_1(\delta)$ which, from (B.15), implies that $\delta = 2a\rho \mathrm{e}^{-b(t+j)}|\nu(0, 0)|$.

2. If $\max\{c_1(\delta), c_2(\delta) + c_1(\delta)\} = c_2(\delta) + c_1(\delta)$, then $t + j = c_2(\delta) + c_1(\delta)$. Since $c_2(\delta) + c_1(\delta) \geq c_1(\delta)$, it follows that $c_2(\delta) \geq 0$. Then, we consider two cases: $c_1(\delta) \leq 0$ and $c_1(\delta) > 0$.

   a. If $c_1(\delta) \leq 0$, then $t + j \leq c_2(\delta)$ which, from (B.17), implies that $\delta \leq 2\mathrm{e}^{-\omega(t+j)}\beta(\xi_M, 0)$. Substituting ξ_M given in (B.14) yields $\delta \leq \max\left\{2\frac{p_M}{p_m}, 2a\rho\sqrt{\frac{p_M}{p_m}}\right\} \mathrm{e}^{-\omega(t+j)}(|\xi(0, 0)|_{\mathcal{A}} + |\nu(0, 0)|)$.

b. If $c_1(\delta) > 0$, we define $\sigma := \min\{\omega, b\}$ and then

$$\begin{aligned} t+j &= c_1(\delta) + c_2(\delta) \\ &\leq -\frac{1}{\sigma}\left(\ln\left(\frac{\delta/2}{\beta(\xi_M,0)}\right) + \ln\left(\frac{\delta/2}{a\rho|\nu(0,0)|}\right)\right) \end{aligned}$$

which implies $\delta \leq \sqrt{4a\rho \mathrm{e}^{-\sigma(t+j)}\beta(\xi_M,0)|\nu(0,0)|)}$. By substituting ξ_M in (B.14) and completing the square yields $\delta \leq \max\left\{\left(\frac{p_M}{p_m}\right)^{3/4}, 2a\sqrt{\rho}\left(\frac{p_M}{p_m}\right)^{1/4}\right\}$ $\times\, \mathrm{e}^{-\frac{\sigma}{2}(t+j)}\left(|\xi(0,0)|_{\mathcal{A}} + |\nu(0,0)|\right)$.

By combining the bounds in the items above, and using that $p_M/p_m > 1$ and $\rho > 1$, it follows from (B.19) that (3.17) holds. $\square$

Appendix C

Proofs of Results for Chapter 4

C.1 Proof of Lemma 4.8

Consider the Lyapunov function

$$V_\varepsilon(\xi) := \frac{1}{2}\varepsilon^\top \varepsilon \quad \forall \xi \in \widetilde{C}_g \cup \widetilde{D}_g, \tag{C.1}$$

with ε as in (4.11). Since θ is constant, we have from (4.15) that $\dot{\varepsilon} = \dot{z} + \dot{\eta} - \dot{\psi}\theta = -\lambda_c \varepsilon$. Thus, for all $\xi \in \widetilde{C}_g$, $\langle \nabla V_\varepsilon(\xi), \widetilde{F}_g(\xi)\rangle = -2\lambda_c V_\varepsilon(\xi) \leq 0$. At jumps, since θ is constant, we have from (4.15) that $\varepsilon^+ = z^+ + \eta^+ - \psi^+\theta = (1-\lambda_d)\varepsilon$. Thus, for all $\xi \in \widetilde{D}_g$, $V_\varepsilon(\widetilde{G}_g(\xi)) - V_\varepsilon(\xi) = -\lambda_d(2-\lambda_d)V_\varepsilon(\xi) \leq 0$, where the inequality holds since $\lambda_d \in (0,2)$. Then, for each solution ξ to $\widetilde{\mathcal{H}}_g$, by integration using the bounds above and the definition of V_ε in (C.1), (4.17) holds. □

C.2 Proof of Lemma 4.9

Consider the Lyapunov function

$$V_\psi(\xi) := \frac{1}{2}\mathrm{tr}(\psi^\top \psi) = \frac{1}{2}|\psi|_{\mathrm{F}}^2 \quad \forall \xi \in \widetilde{C}_g \cup \widetilde{D}_g. \tag{C.2}$$

For all $\xi \in \widetilde{C}_g$, we have from (4.15) that $\langle \nabla V_\psi(\xi), \widetilde{F}_g(\xi)\rangle = -2\lambda_c V_\psi(\xi) + \mathrm{tr}(\psi^\top \phi_c(\tau,k))$. Applying the Cauchy-Schwarz inequality yields $\langle \nabla V_\psi(\xi), \widetilde{F}_g(\xi)\rangle \leq -2\lambda_c V_\psi(\xi) + \sqrt{2V_\psi(\xi)}|\phi_c(\tau,k)|_{\mathrm{F}}$. Hence,

$$\begin{aligned} &\langle \nabla V_\psi(\xi), \widetilde{F}_g(\xi)\rangle \leq 0 \\ &\qquad \forall \xi \in \widetilde{C}_g : V_\psi(\xi) \geq \frac{1}{2\lambda_c^2}|\phi_c(\tau,k)|_{\mathrm{F}}^2. \end{aligned} \tag{C.3}$$

Let us now analyze the variation of V_ψ at jumps. Omitting the (τ,k) arguments for readability, for all $\xi \in \widetilde{D}_g$, we have from (4.15) that $V_\psi(\widetilde{G}_g(\xi)) - V_\psi(\xi) \leq -\bar{\lambda}_d V_\psi(\xi) + |\mathrm{tr}(\psi^\top \phi_d)| + \frac{1}{2}\mathrm{tr}(\phi_d^\top \phi_d)$, where the inequality follows since $\lambda_d \in (0,2)$, and we define $\bar{\lambda}_d := \lambda_d(2-\lambda_d)$ for readability. We apply the Cauchy-Schwarz inequality and use that for any $\varrho > 0$, $\sqrt{\mathrm{tr}(\psi^\top\psi)}\sqrt{\mathrm{tr}(\phi_d^\top\phi_d)} \leq \varrho \mathrm{tr}(\psi^\top\psi) + \varrho^{-1}\mathrm{tr}(\phi_d^\top\phi_d)$. Choosing $\varrho = \bar{\lambda}_d/4$ yields $V_\psi(\widetilde{G}_g(\xi)) - V_\psi(\xi) \leq -\frac{\bar{\lambda}_d}{2}V_\psi(\xi) + \frac{\bar{\lambda}_d+8}{2\bar{\lambda}_d}|\phi_d|_{\mathrm{F}}^2$. Hence,

$$\begin{aligned} &V_\psi(\widetilde{G}_g(\xi)) - V_\psi(\xi) \leq 0 \\ &\qquad \forall \xi \in \widetilde{D}_g : \ V_\psi(\xi) \geq \frac{\bar{\lambda}_d + 8}{\bar{\lambda}_d^2}|\phi_d(\tau,k)|_{\mathrm{F}}^2. \end{aligned} \tag{C.4}$$

Using the bounds in Assumption 4.6, we combine the expressions in (C.3) and (C.4) to obtain

$$\begin{aligned} \langle \nabla V_\psi(\xi), \widetilde{F}_g(\xi)\rangle \leq 0 \quad & \forall \xi \in \widetilde{C}_g \cap S_\psi \\ V_\psi(\widetilde{G}_g(\xi)) - V_\psi(\xi) \leq 0 \quad & \forall \xi \in \widetilde{D}_g \cap S_\psi, \end{aligned}$$

where

$$S_\psi := \left\{ \xi \in \widetilde{C}_g \cup \widetilde{D}_g : V_\psi(\xi) \geq \max\left\{\frac{1}{2\lambda_c^2}, \frac{\bar{\lambda}_d+8}{\bar{\lambda}_d^2}\right\}\phi_M^2 \right\}.$$

Then, for each solution ξ to $\widetilde{\mathcal{H}}_g$ from $\mathcal{X}_0$, by integration using the bounds above, we conclude that, for all $(t,j) \in \mathrm{dom}\,\xi$, $V_\psi(\xi(t,j)) \leq V_\psi(\xi(0,0)) + \max\left\{1/(2\lambda_c^2), (\bar{\lambda}_d+8)/\bar{\lambda}_d^2\right\}\phi_M^2$. Using the definition of V_ψ in (C.2), we obtain $|\psi(t,j)| \leq |\psi(t,j)|_{\mathrm{F}} \leq |\psi(0,0)|_{\mathrm{F}} + \max\left\{1/\lambda_c, \sqrt{2\bar{\lambda}_d + 16}/\bar{\lambda}_d\right\}\phi_M \leq \psi_0 + \max\left\{1/\lambda_c, \sqrt{2\bar{\lambda}_d+16}/\bar{\lambda}_d\right\}\phi_M$ for all $(t,j) \in \mathrm{dom}\,\xi$, where the last inequality follows from the fact that, since $\xi(0,0) \in \mathcal{X}_0$, $|\psi(0,0)|_{\mathrm{F}} \leq \psi_0$. Hence, (4.18) holds. $\square$

C.3 Proof of Lemma 4.12

Consider the Lyapunov function

$$V_\varepsilon(\xi) := \frac{1}{2}\varepsilon^\top \varepsilon \quad \forall \xi \in C_\nu \cup D_\nu, \tag{C.5}$$

with ε as in (4.11). Since θ is constant, we have from (4.11) that $\dot{\varepsilon} = \dot{z} + \dot{\eta} - \dot{\psi}\theta = -\lambda_c \varepsilon + \alpha_c(\tau, k)$, with α_c as in (4.23b). Thus, for all $\xi \in C_\nu$, $\langle \nabla V_\varepsilon(\xi), F_\nu(\xi) \rangle = -2\lambda_c V_\varepsilon(\xi) - \varepsilon^\top \alpha_c(\tau, k) \leq -\lambda_c V_\varepsilon(\xi) + \frac{2}{\lambda_c}|\alpha_c(\tau, k)|^2$. Then, for any $\zeta \in (0, 1)$,

$$\begin{gathered} \langle \nabla V_\varepsilon(\xi), F_\nu(\xi) \rangle \leq -\lambda_c(1-\zeta)V_\varepsilon(\xi) \\ \forall \xi \in C_\nu : \ V_\varepsilon(\xi) \geq \frac{2}{\lambda_c^2 \zeta}|\alpha_c(\tau, k)|^2. \end{gathered} \tag{C.6}$$

Let us now analyze the variation of V_ε at jumps. Omitting the (τ, k) arguments for readability, since θ is constant, we have from (4.11) that $\varepsilon^+ = z^+ + \eta^+ - \psi^+\theta = (1-\lambda_d)\varepsilon + \alpha_d(\tau, k)$, with α_d as in (4.23d). Thus, for all $\xi \in D_\nu$, $V_\varepsilon(G_\nu(\xi)) - V_\varepsilon(\xi) \leq -\bar{\lambda}_d V_\varepsilon(\xi) + |\varepsilon^\top \alpha_d| + \frac{1}{2}\alpha_d^\top \alpha_d \leq -\frac{\bar{\lambda}_d}{2}V_\varepsilon(\xi) + \frac{\bar{\lambda}_d + 8}{2\bar{\lambda}_d}|\alpha_d(\tau, k)|^2$, where we define $\bar{\lambda}_d := \lambda_d(2 - \lambda_d)$ for readability. Then, for any $\zeta \in (0, 1)$,

$$\begin{gathered} V_\varepsilon(G_\nu(\xi)) - V_\varepsilon(\xi) \leq -\frac{\bar{\lambda}_d}{2}(1-\zeta)V_\varepsilon(\xi), \\ \forall \xi \in D_\nu : \ V_\varepsilon(\xi) \geq \frac{\bar{\lambda}_d + 8}{\bar{\lambda}_d^2 \zeta}|\alpha_d(\tau, k)|^2. \end{gathered} \tag{C.7}$$

Using the function d_ε defined in (4.22), we combine the expressions in (C.6) and (C.7) and obtain that

$$\begin{gathered} \langle \nabla V_\varepsilon(\xi), F_\nu(\xi) \rangle \leq -\lambda_c(1-\zeta)V_\varepsilon(\xi) \quad \forall \xi \in C_\nu \cap S_\varepsilon \\ V_\varepsilon(G_\nu(\xi)) - V_\varepsilon(\xi) \leq -\frac{\bar{\lambda}_d}{2}(1-\zeta)V_\varepsilon(\xi) \quad \forall \xi \in D_\nu \cap S_\varepsilon, \end{gathered}$$

where

$$S_\varepsilon := \left\{ \xi \in C_\nu \cup D_\nu : V_\varepsilon(\xi) \geq \max\left\{ \frac{2}{\lambda_c^2 \zeta}, \frac{\bar{\lambda}_d + 8}{\bar{\lambda}_d^2 \zeta} \right\} |d_\varepsilon(\tau, k)|^2 \right\}.$$

Then, for each solution ξ to $\mathcal{H}_\nu$, by integration using the bounds above, and using the definition of V_ε in (C.5), we conclude that (4.24) holds. $\square$

Appendix D

Data Recording Algorithms for Hybrid ICL

In this section only, we make explicit the dependence of $\mathcal{Z}_c$ in (5.21) and $\mathcal{Z}_d$ in (5.23) on the history stacks Z_c and Z_d by writing

$$\mathcal{Z}_c(Z_c(t,j)) := \sum_{\ell=1}^{N_c} Z_{c,\ell}(t,j)^\top Z_{c,\ell}(t,j)$$
$$\mathcal{Z}_d(Z_d(t,j)) := \sum_{\ell=1}^{N_d} Z_{d,\ell}(t,j)^\top Z_{d,\ell}(t,j)$$

for all $(t,j) \in E$, where $Z_{c,\ell}$ (resp. $Z_{d,\ell}$) are the layers of Z_c (resp. Z_d) as in (5.8) (resp. (5.15)). Furthermore, write $\mathcal{Z}$ in (5.62) as

$$\mathcal{Z}(Z_c(t,j), Z_d(t,j)) := \mathcal{Z}_c(Z_c(t,j)) + \mathcal{Z}_d(Z_d(t,j)).$$

for all $(t,j) \in E$.

Algorithm 2 Data Recording During Flows

Initialize: $\ell = 1$

Require: $(t, j) \in E$ and $(t, j-1) \notin E$

  if $\ell \leq N_c$ **then** ▷ fill empty elements

    if $|\Phi_c(t, j)| > 0$ **then**

      Store $\Phi_c(t, j)$ in layer ℓ of Z_c

      Store $\mathcal{Y}_c(t, j)$ in column ℓ of Y_c

      $\ell = \ell + 1$

    end if

  else ▷ overwrite stored data

    $R_{\text{old}} = \text{rank}(\mathcal{Z}_c(Z_c))$

    $S_{1,\text{old}} = \lambda_{\min}(\mathcal{Z}_c(Z_c))$

    $S_{2,\text{old}} = \lambda_{\min}(\mathcal{Z}(Z_c, Z_d))$

    Create empty vectors $R, S_1, S_2 \in \mathbb{R}^{N_c}$ and set $q = 0$

    for $r = 1$ to N_c **do**

      $Q = Z_c$; Store $\Phi_c(t, j)$ in layer r of Q

      Store $\text{rank}(\mathcal{Z}_c(Q))$ in row r of R

      Store $\lambda_{\min}(\mathcal{Z}_c(Q))$ in row r of S_1

      Store $\lambda_{\min}(\mathcal{Z}(Q, Z_d))$ in row r of S_2

    end for

    if $\max S_1 > S_{1,\text{old}}$ **then**

      Set q as the row index of $\max S_1$

    else if $\max R > R_{\text{old}}$ **then**

      Set q as the row index of $\max R$

    else if $\max S_2 > S_{2,\text{old}}$ **then**

      Set q as the row index of $\max S_2$

    end if

    if $q \geq 1$ **then**

      Store $\Phi_c(t, j)$ in layer q of Z_c

      Store $\mathcal{Y}_c(t, j)$ in column q of Y_c

    end if

  end if

Algorithm 3 Data Recording At Jumps

Initialize: $\ell = 1$

Require: $(t, j), (t, j-1) \in E$

  if $\ell \leq N_d$ **then** ▷ fill empty elements

    if $|\Phi_d(t, j)| > 0$ **then**

      Store $\Phi_d(t, j)$ in layer ℓ of Z_d

      Store $\mathcal{Y}_d(t, j)$ in column ℓ of Y_d

      $\ell = \ell + 1$

    end if

  else ▷ overwrite stored data

    $R_{\text{old}} = \text{rank}(\mathcal{Z}_d(Z_d))$

    $S_{1,\text{old}} = \lambda_{\min}(\mathcal{Z}_d(Z_d))/(\rho_d + \lambda_{\max}(\mathcal{Z}_d(Z_d)))$

    $S_{2,\text{old}} = \lambda_{\min}(\mathcal{Z}(Z_c, Z_d))$

    Create empty vectors $R, S_1, S_2 \in \mathbb{R}^{N_d}$ and set $q = 0$

    for $r = 1$ to N_d **do**

      $Q = Z_d$; Store $\Phi_d(t, j)$ in layer r of Q

      Store $\text{rank}(\mathcal{Z}_d(Q))$ in row r of R

      Store $\lambda_{\min}(\mathcal{Z}_d(Q))/(\rho_d + \lambda_{\max}(\mathcal{Z}_d(Q)))$ in row r of S_1

      Store $\lambda_{\min}(\mathcal{Z}(Z_c, Q))$ in row r of S_2

    end for

    if $\max S_1 > S_{1,\text{old}}$ **then**

      Set q as the row index of $\max S_1$

    else if $\max R > R_{\text{old}}$ **then**

      Set q as the row index of $\max R$

    else if $\max S_2 > S_{2,\text{old}}$ **then**

      Set q as the row index of $\max S_2$

    end if

    if $q \geq 1$ **then**

      Store $\Phi_d(t, j)$ in layer q of Z_d

      Store $\mathcal{Y}_d(t, j)$ in column q of Y_d

    end if

  end if

Appendix E

Proofs of Results for Chapter 6

E.1 Proof of Lemma 6.17

Let ξ be a solution to $\mathcal{H}_\nu$ from $\xi(0,0) \in \mathcal{X}_0^\nu$, where $\mathcal{X}_0^\nu$ is defined above (6.44). We first analyze the evolution of the parameter estimation error $\tilde{\theta} = \theta - \hat{\theta}$ during flows. Since θ is constant, it follows from F_ν in (6.41) that

$$\begin{aligned}
\frac{d}{dt}\tilde{\theta}(t,j) &= -\frac{d}{dt}\hat{\theta}(t,j) \\
&= -\gamma\psi(t,j)^\top(x(t,j) + \nu(t) + \eta(t,j) - \psi(t,j)\hat{\theta}(t,j)) \\
&= -\gamma\psi(t,j)^\top\psi(t,j)\tilde{\theta}(t,j) - \gamma\psi(t,j)^\top(\varepsilon(t,j) + \nu(t))
\end{aligned}$$

for each $j \in \mathbb{N}$ and all $t \in I^j$, with ε as in (6.3). Since $\Phi(0,0) = I$ and, by the jump map G_ν in (6.41), Φ is equal to the identity immediately after each jump, it follows from F_ν in (6.41) that, since Φ is the state transition matrix for (6.5),

$$\tilde{\theta}(t,j) = \Phi(t,j)\tilde{\theta}(t_j,j) - \gamma\int_{t_j}^{t}\Phi(s,j)\psi(s,j)^\top(\varepsilon(s,j) + \nu(s))ds$$

for each $j \in \mathbb{N}$ and all $t \in I^j$. Thus, (6.46) holds.

Next, we analyze the evolution of $\tilde{\theta}$ at jumps. Let $(t,j) \in \operatorname{dom}\xi$ be such that $(t,j+1) \in \operatorname{dom}\xi$. Denoting $\tilde{m} = \theta - m$ and omitting the argument ξ of K_1 for readability, it follows from G_ν in (6.41) that

$$\begin{aligned}
\tilde{\theta}(t,j+1) &= \theta - R(\xi(t,j)) \\
&= K_1(\tilde{\theta}(t,j) - \tilde{m}(t,j)) + \tilde{m}(t,j) \\
&= K_1\left(\Phi(t,j)\tilde{\theta}(t_j,j) - \gamma\int_{t_j}^{t}\Phi(s,j)\psi(s,j)^\top(\varepsilon(s,j)+\nu(s))ds - \tilde{m}(t,j)\right) + \tilde{m}(t,j) \\
&= K_1(\Phi(t,j) - I)\tilde{m}(t_j,j) + \tilde{m}(t_j,j) - K_1\gamma\int_{t_j}^{t}\Phi(s,j)\psi(s,j)^\top(\varepsilon(s,j)+\nu(s))ds \\
&= -\tilde{m}(t_j,j) + \tilde{m}(t_j,j) - K_1\gamma\int_{t_j}^{t}\Phi(s,j)\psi(s,j)^\top(\varepsilon(s,j)+\nu(s))ds \\
&= -K_1\gamma\int_{t_j}^{t}\Phi(s,j)\psi(s,j)^\top(\varepsilon(s,j)+\nu(s))ds
\end{aligned}$$

where the second line follows from the definition of R in (6.7), the third line follows from substituting (6.46), and the fourth line follows from the fact that, since $\xi(0,0) \in \mathcal{X}_0^\nu$, $\tilde{m} = \tilde{\theta}$ at the beginning of each interval of flow in ξ, and the fact that m is constant during flows. Thus, (6.47) holds. □

E.2 Proof of Lemma 6.19

Consider the Lyapunov function

$$V_\varepsilon(\xi) := \frac{1}{2}\varepsilon^\top\varepsilon \quad \forall \xi \in C_\nu \cup D_\nu \tag{E.1}$$

with ε as in (6.3). Since θ is constant, we have from the definition of F_ν below (6.40) that, during flows,

$$\begin{aligned}
\dot{\varepsilon} &= \dot{x} + \dot{\eta} - \dot{\psi}\theta \\
&= -\lambda\varepsilon + f(x,u(\tau)) - f(x+\nu(\tau),u(\tau)) - \lambda\nu(\tau).
\end{aligned}$$

Hence, for all $\xi \in C_\nu$,

$$\begin{aligned}\langle \nabla V_\varepsilon(\xi), F_\nu(\xi)\rangle &= -\lambda\varepsilon^\top\varepsilon + \varepsilon^\top(f(x,u(\tau)) - f(x+\nu(\tau),u(\tau)) - \lambda\nu(\tau)) \\ &\leq -\lambda\varepsilon^\top\varepsilon + |\varepsilon|(L_x+\lambda)|\nu(\tau)| \\ &\leq -2\lambda V_\varepsilon(\xi) + |\varepsilon|\chi(\tau)\end{aligned}$$

where the first inequality follows from Assumption 6.13, and we define $\chi(\tau) := (L_x+\lambda)|\nu(\tau)|$ for readability. We use the fact that, via Young's inequality, for any $\varrho > 0$, $|\varepsilon|\chi(\tau) \leq \varrho|\varepsilon|^2 + \varrho^{-1}\chi(\tau)^2$ for all $\tau \geq 0$. Choosing $\varrho = \lambda/2$,

$$\langle \nabla V_\varepsilon(\xi), F_\nu(\xi)\rangle \leq -\lambda V_\varepsilon(\xi) + \frac{2}{\lambda}\chi(\tau)^2.$$

Let $\zeta \in (0,1)$. We rewrite the expression above as

$$\langle \nabla V_\varepsilon(\xi), F_\nu(\xi)\rangle \leq -\lambda(1-\zeta)V_\varepsilon(\xi) - \lambda\zeta V_\varepsilon(\xi) + \frac{2}{\lambda}\chi(\tau)^2.$$

Hence,

$$\begin{aligned}\langle \nabla V_\varepsilon(\xi), F_\nu(\xi)\rangle &\leq -\lambda(1-\zeta)V_\varepsilon(\xi) \\ \forall \xi \in C_\nu :\ V_\varepsilon(\xi) &\geq \frac{2(L_x+\lambda)^2}{\lambda^2\zeta}|\nu(\tau)|^2.\end{aligned} \tag{E.2}$$

Let us now analyze the variation of V_ε at jumps. Since θ is constant, we have from the definition of G_ν below (6.40) that, at jumps,

$$\varepsilon^+ = x^+ + \eta^+ - \psi^+\theta = -\nu(\tau)$$

Hence, for all $\xi \in D_\nu$,

$$\begin{aligned}&V_\varepsilon(G_\nu(\xi)) - V_\varepsilon(\xi) \\ &\quad = -\frac{1}{2}\varepsilon^\top\varepsilon + \frac{1}{2}\nu(\tau)^\top\nu(\tau) = -V_\varepsilon(\xi) + \frac{1}{2}|\nu(\tau)|^2\end{aligned}$$

Let $\zeta \in (0, 1)$. We rewrite the expression above as

$$V_\varepsilon(G_\nu(\xi)) - V_\varepsilon(\xi) = -(1-\zeta)V_\varepsilon(\xi) - \zeta V_\varepsilon(\xi) + \frac{1}{2}|\nu(\tau)|^2$$

Hence,

$$\begin{aligned} V_\varepsilon(G_\nu(\xi)) - V_\varepsilon(\xi) \leq -(1-\zeta)V_\varepsilon(\xi), \\ \forall \xi \in D_\nu : \; V_\varepsilon(\xi) \geq \frac{1}{2\zeta}|\nu(\tau)|^2. \end{aligned} \tag{E.3}$$

Combining the expressions in (E.2) and (E.3), we obtain

$$\begin{aligned} \langle \nabla V_\varepsilon(\xi), F_\nu(\xi) \rangle &\leq -\lambda(1-\zeta)V_\varepsilon(\xi) \qquad && \forall \xi \in C_\nu \cap S_\varepsilon \\ V_\varepsilon(G_\nu(\xi)) - V_\varepsilon(\xi) &\leq -(1-\zeta)V_\varepsilon(\xi) \qquad && \forall \xi \in D_\nu \cap S_\varepsilon \end{aligned}$$

where

$$S_\varepsilon := \left\{ \xi \in C_\nu \cup D_\nu : V_\varepsilon(\xi) \geq \max\left\{ \frac{2(L_x+\lambda)^2}{\lambda^2\zeta}, \frac{1}{2\zeta} \right\} |\nu(\tau)|^2 \right\}.$$

Then, for each solution ξ to $\mathcal{H}_\nu$, by integration using the bounds above, we obtain

$$\begin{aligned} V_\varepsilon(\xi(t,j)) \leq \exp\left\{-\lambda(1-\zeta)(t-t') + \ln(\zeta)(j-j')\right\} V_\varepsilon(\xi(t',j')) \\ + \max\left\{ \frac{2(L_x+\lambda)^2}{\lambda^2\zeta}, \frac{1}{2\zeta} \right\} \|\nu\|_\infty^2 \end{aligned}$$

for all $(t, j), (t', j') \in \operatorname{dom} \xi$ satisfying $t + j \geq t' + j'$. Using the definition of V_ε in (E.1), we conclude that (6.49) holds. □

Appendix F

Proofs of Results for Chapter 7

F.1 Proof of Lemma 7.15

That the sets C and D are closed follows from the closesedness of the sets $\overline{C}$ and $\overline{D}$ by Assumption 7.12. The flow map F is outersemicontinuous and locally bounded relative to C since, by Assumption 7.12, $\overline{F}$ is outersemicontinuous and locally bounded relative to $\overline{C}$, and the flow map for τ_δ is globally Lipschitz. Outersemicontinuity and boundedness of G relative to D holds since, by Assumption 7.12, $\overline{G}$ is outersemicontinuous and locally bounded relative to $\overline{D}$. □

F.2 Proof of Lemma 7.16

We rely on Proposition 2.5. Let $T_C(\xi)$ be the tangent cone of the flow set C at $\xi \in \mathcal{X}$. For $\xi \in C \setminus D$,

- If $\tau_\delta = -1$, then $T_C(\xi) \cap F(\xi, \theta) \neq \varnothing$ since Assumption 7.13 holds and $\dot{\tau}_\delta = 0$.
- If $\tau_\delta \in (0, \Delta]$, then $T_C(\xi) \cap F(\xi, \theta) \neq \varnothing$ since Assumption 7.13 holds and $\dot{\tau}_\delta = -1$.

Hence, there exists a nontrivial solution to $\mathcal{H}$ from all points in $C \cup D$.

Next, we show that cases *(b)* and *(c)* in Proposition 2.5 cannot hold, and hence only case *(a)* is true.

- Case *(c)* (solutions jumping outside $C \cup D$) We consider the following two cases:

  1. If $\xi \in D_{-1}$ then we have from (7.17) and Assumption 7.13 that $G_{-1}(\xi, \theta) \in (\overline{C} \cup \overline{D}) \times \mathbb{R}^m \times [0, \Delta] \subset C \cup D$.
  2. If $\xi \in D_0$ then we have from (7.17) and Assumption 7.13 that $G_0(\xi, \theta) \in (\overline{C} \cup \overline{D}) \times \mathbb{R}^m \times \{-1\} \subset C \cup D$.

  Thus solutions to $\mathcal{H}$ cannot terminate by jumping outside $C \cup D$, and case *(c)* of Proposition 2.5 cannot hold.

- *(b)* (finite escape time) We exclude this case by contradiction. Suppose that there exists a maximal solution ξ to $\mathcal{H}$ with $\xi(0,0) \in C \cup D$ that is not complete. Let $(T, J) = \sup \operatorname{dom} \xi$ and note that, since ξ is not complete, $T + J < \infty$. By definition of solutions, if $(T, J) \in \operatorname{dom} \xi$, then $(T, J) \in C \cup D$. We consider the following two cases:

  1. If $(T, J) \in D$, then from the first bullet of this list, ξ may be extended by jumping.
  2. $(T, J) \in C \setminus D$, then since Assumption 7.13 holds and the flow map for τ_δ is globally Lipschitz, it follows that solutions will not escape in finite-time due to evolution via the flow map, and thus ξ can be extended via flow.

  Hence, case *(b)* of Proposition 2.5 cannot hold.

By the arguments above, we conclude that cases *(b)* and *(c)* in Proposition 2.5 cannot hold, and thus only case *(a)* is true. □

Appendix G

Code for Numerical Examples

The numerical examples in this dissertation were simulated in MATLAB using the Hybrid Equations Toolbox version 3.0.0.76 [46]. The Hybrid Equations Toolbox can be downloaded for free from
`https://www.mathworks.com/matlabcentral/fileexchange/41372-hybrid-equations-toolbox`

Code for each of the numerical examples can be found at the following GitHub repositories.

- Figures 1.1 and 4.1: `https://github.com/HybridSystemsLab/HybridGD_Motivation`
- Figure 1.2: `https://github.com/HybridSystemsLab/HybridICL_Motivation`
- Figure 4.2: `https://github.com/HybridSystemsLab/HybridGD_SpacecraftBiasTorque`
- Figures 5.1, 5.2, and 5.3: `https://github.com/HybridSystemsLab/HybridICL_BouncingBall`

- Figure 6.1: `https://github.com/HybridSystemsLab/HybridFT_NonlinearSystem`
- Figures 1.3, 7.2, and 7.3: `https://github.com/HybridSystemsLab/ApproximatelyKnownJumpTimes_BouncingBall`

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