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Publication Date
1971-06-01

Submitted to
Physical Review Letters

P. Langacker

UCRL-20608 Rev.
Preprint

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June 4, 1971

AEC Contract No. W-7405-eng-48

UCRL-20608Rev.

TOTAL CROSS SECTION FOR ELECTRON-POSITRON ANNIHILATION AND THE HADRONIC CONTRIBUTION TO THE MUON MAGNETIC MOMENT*

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June 4, 1971

ABSTRACT

A class of inequalities are derived for the integrated cross section for e^+e^- annihilation into hadrons through one photon. They are applied to estimate a lower bound on the hadronic contribution to the muon magnetic moment and also to test some predictions on the asymptotic behavior of the annihilation cross section.

Electron-positron colliding beam experiments have produced a number of interesting data on the annihilation cross section in the vector meson region[1] and also some preliminary data in the few GeV region.[2,3] The purpose of this short note is to investigate relations between the hadronic cross section and the pion form factor. The presently available data for the pion form factor does not lead us to a definite quantitative conclusion, but π-e scattering at Serpukhov, Los Alamos, and NAL in the near future may provide the parameters needed to draw conclusions about the hadronic cross section for electron-positron annihilation into hadrons.

Let us derive an inequality first pointed out by Drell and Zachariasen.[4] With the usual definition of the pion form factor, we obtain an explicit decomposition of the absorptive part as $[F(0) = 1]$

$$e\,\mathrm{Im}\,F(t) = \frac{(2\pi)^4(2p_0)^{\frac{1}{2}}}{t-4\mu^2}\sum_n \delta^{(4)}(p+p'-p_n)\langle p|j(0)|n\rangle \times \langle n|\underset{\sim}{p}\cdot\underset{\sim}{J}(0)|0\rangle\theta(t-4\mu^2)\,, \qquad (1)$$

where μ is the pion mass, $t = -(p+p')^2 = -q^2$, $\underset{\sim}{p}$ is the pion momentum in the $\pi^+\pi^-$ center-of-mass frame, $\underset{\sim}{p}\cdot\underset{\sim}{J}(0)$ is a three-dimensional scalar product of $\underset{\sim}{p}$ and the electromagnetic current, and $j(0)$ is the pion source function. The Schwartz inequality in the summation over intermediate states n leads us to

$$e^2|\mathrm{Im}\,F(t)|^2 \le \pi\left(\frac{t}{t-4\mu^2}\right)^{\frac{1}{2}} \sigma^{\pi\pi}(t)\,\rho(t)\,, \qquad (2)$$

where $\sigma^{\pi\pi}(t)$ is the p-wave $\pi\pi$ scattering cross section at c.m. energy $t^{1/2}$. The isovector spectral function $\rho(t)$, defined by

$$(\delta_\mu{}^\nu - q_\mu q^\nu/q^2)\rho(t) = (2\pi)^3\sum_n \delta^{(4)}(p+p'-p_n)\langle 0|J_\mu(0)|n\rangle\langle n|J^\nu(0)|0\rangle, \qquad (3)$$

is related to the total cross section for $e^+e^- \to$ hadrons through one isovector photon by

$$\rho(t) = t^2\,\sigma^{e\bar{e}}(t)/4\pi^2\alpha\,. \qquad (4)$$

We use the Schwartz inequality once again, this time in the integral over t to obtain[5]

$$\left|\frac{F(q_1{}^2)-F(q_2{}^2)}{q_1{}^2-q_2{}^2}\right|^2 \le \left|\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt\,\frac{|\mathrm{Im}\,F(t)|}{(t+q_1{}^2)(t+q_2{}^2)}\right|^2$$

$$\le \frac{1}{16\pi^4\alpha^2}\left|\int_{4\mu^2}^{\infty} dt\,\sigma^{e\bar{e}}(t)f(t)\right| \times \left|\int_{4\mu^2}^{\infty} dt\,\frac{t^{5/2}\,\sigma^{\pi\pi}(t)}{f(t)(t+q_1{}^2)^2(t+q_2{}^2)^2(t-4\mu^2)^{\frac{1}{2}}}\right| \qquad (5)$$

where $q_1{}^2$ and $q_2{}^2$ are two arbitrary points in the space-like region

$(q_1^2, q_2^2 > 0)$ and $f(t)$ may be an arbitrary non-negative function of t. The second integral on the right-hand side converges if $\{f(t)\}^{-1} = o(t^2)$ as $t \to +\infty$, because of the unitarity bound on the p-wave $\pi\pi$ cross section $[\sigma^{\pi\pi}(t) \leqq 48\pi(t-4\mu^2)^{-1}]$. However, for large values of q_1^2 and q_2^2, the contribution from the large t region $(t \gg m_\rho^2)$ is important relative to that from the low t region. Since at present we do not have much knowledge of the p-wave $\pi\pi$ scattering above $t \simeq 1\ \text{GeV}^2$, Eq. (5) is probably more useful for small values of q_1^2 and q_2^2; for instance, in the case of $q_1^2 = q_2^2 \to 0$:

$$\left(\frac{r_\pi^{\,2}}{6}\right)^2 \leqq \frac{1}{16\pi^4\alpha^2} \left| \int_{4\mu^2}^{\infty} dt\ \sigma^{e\bar{e}}(t) f(t) \right| \times \left| \int_{4\mu^2}^{\infty} dt\ \frac{\sigma^{\pi\pi}(t)}{f(t)t^{3/2}(t-4\mu^2)^{1/2}} \right|, \tag{6}$$

where r_π stands for the electromagnetic charge radius of the pion.

Let us first apply this relation to the muon magnetic moment. The hadronic contribution to $(g-2)/2$ is given to the order of α^2 by[6]

$$\Delta a_\mu(\text{hadronic}) = \frac{1}{4\pi^3} \int_{4\mu^2}^{\infty} dt\ \sigma^{e\bar{e}}(t)\ G(t)\ , \tag{7}$$

where

$$G(t) = \int_0^1 dz\ \frac{z^2(1-z)}{z^2 + (1-z)t/m_\mu^2}\ . \tag{8}$$

Let us choose $f(t) = G(t)$ in (6) (which is positive definite in the integral region). We split the integral involving $\sigma^{\pi\pi}(t)$ into three regions; (a) $280\ \text{MeV} < \sqrt{t} < 500\ \text{MeV}$, (b) $500\ \text{MeV} < \sqrt{t} < 1000\ \text{MeV}$, and (c) $1000\ \text{MeV} < \sqrt{t}$. The region (b), where the ρ-meson is located, has been studied by applying the Chew-Low extrapolation to the reaction $\pi^- p \to \pi^- \pi^0 p$. We use the results of Baton et al. for this region, reinforcing them with a few other measurements.[7] There is no evidence for anomalous enhancement in the p-wave near threshold, so we interpolate between the threshold and 500 MeV with a phase shift $\propto (t - 4\mu^2)^{3/2}$ incorporating the p-wave threshold behavior. We have little knowledge of region (c) except that there are no conspicuous resonances up to $\sqrt{t} \simeq 2$ GeV. We therefore extrapolate $\sigma^{\pi\pi}(t)$ as a constant, using its value at the upper edge of the region (b) to higher energies until the unitarity bound is reached, and replace $\sigma^{\pi\pi}(t)$ by the unitarity bound above this point. This is very likely to overestimate the contribution from region (c) by a large amount so that it underestimates the lower bound. The lower bound on Δa_μ (hadronic) thus obtained is[8]

$$\Delta a_\mu(\text{hadronic}) > 2.4\, r_\pi^{\,4} \times 10^{-7}\ , \tag{9}$$

where r_π is the pion charge radius in fermi. There are a few data for r_π, all of which were extrapolated from $\pi^\pm$ electroproduction. Two typical values are: (i) $r_\pi = (0.86 \pm 0.14)$ fermi.[9] This was obtained through a dispersion relation of Zagury.[10] The value is considerably larger than the prediction of ρ-dominance (0.63 fermi). The other is a very new result, (ii) $r_\pi = 0.60$ fermi (dipole fit) or 0.68 fermi (simple pole fit).[11] This analysis is based on Berends' dispersion relation incorporating the assumption of Born diagram dominance.[12] The form factor is less accurate for small q^2 than for large q^2. If the value (i) were true, we would have a lower bound

$$\Delta a_\mu(\text{hadronic}) > 13.0\ {}^{+11.0}_{-\ 7.6} \times 10^{-8}\ , \tag{10}$$

which should be compared with the ρ-dominance value calculated from Orday data,[6,13] $\Delta a_\mu(\text{hadronic}) = (5.4 \pm 0.3) \times 10^{-8}$. If one recalls our very conservative estimates in deriving the lower bound, (10) indicates that the hadronic contribution to Δa_μ might be much larger than the ρ-dominance value. Our lower bound is toward improving a small discrepancy between theory and experiment.[14] If the value (ii) were true, one could not derive an interesting lower bound from (6). But, as is mentioned in the Ref. 11, the new data of the pion form factor are less accurate at low momentum transfers than at high momentum transfers. The slope of the form factor between the highest two points ($q^2 = 0.795$ and $1.188\ \mathrm{GeV}^2$) is steeper by about 30% than that of a simple ρ-pole fit. Therefore, there will be a chance to obtain a meaningful bound from these data points using the inequality (5) when $\pi\pi$ scattering data become available up to the few GeV region.

As a second application we consider the possibility of testing some predictions on the asymptotic behavior of $\sigma^{e\bar{e}}(t)$ for large t. For instance, let us choose $f(t) = 1$ in (b). The hadron annihilation cross section (sum of those through isovector and isoscalar photons) has been measured at $\sqrt{t} = 644$ MeV and 886 MeV and at three points in between.[1] We extrapolate $\sigma^{e\bar{e}}$ at 644 MeV smoothly down to the threshold and evaluate $\int dt\, \sigma^{e\bar{e}}$ between the threshold and $\sqrt{t} = 886$ MeV (because of the threshold behavior of $\sigma^{e\bar{e}}$, an error introduced here is not so large). The integral involving $\sigma^{\pi\pi}(t)$ is evaluated in the same way as for the muon magnetic moment. Then we obtain a lower bound on the integrated cross section above $\sqrt{t} = 886$ MeV as

$$\int_{(886\ \mathrm{MeV})^2}^{\infty} dt\, \sigma^{e\bar{e}}(t) \geq 5.6\, r_\pi^{\,4} \times 10^{-3} - \int_{4\mu^2}^{(886\ \mathrm{MeV})^2} dt\, \sigma^{e\bar{e}}(t)$$

$$= (5.6\, r_\pi^{\,4} - 1.0) \times 10^{-3}\ . \qquad (11)$$

If r_π turns out to be as large as 0.86 fermi,[8] then (11) would set a

lower bound

$$\int_{(886\ \mathrm{MeV})^2}^{\infty} dt\, \sigma^{e\bar{e}}(t) \geq 2.1 \times 10^{-3}\ . \qquad (12)$$

It is always a difficult problem to find at what energy cross sections become asymptotic. If we assume that $\sigma^{e\bar{e}}(t)$ becomes asymptotic above $\sqrt{t} \sim 1$ GeV and that $\sigma^{e\bar{e}}(t)$ at 886 MeV < t < 1 GeV is equal to the value reached by a smooth extrapolation of a Breit-Wigner shape, the left-hand side is equal to $\simeq 0.10 \times 10^{-3}$ for an asymptotic behavior of $\sigma^{e\bar{e}}(t) = o(t^{-2})$ as $t \to \infty$ and about half of this for an asymptotic behavior of $\sigma^{e\bar{e}}(t) = o(t^{-3})$.[15] Therefore an asymptotic behavior like t^{-2} or t^{-3} would be ruled out. If the new data[11] are taken instead of the old one,[9] we can still obtain a meaningful result from the two highest data points. Choosing $q_1^{\,2} = 0.795\ \mathrm{GeV}^2$ and $q_2^{\,2} = 1.188\ \mathrm{GeV}^2$ ($F(q_1^{\,2}) = 0.400$ and $F(q_2^{\,2}) = 0.276$) with $f(t) = t$ in (5), we obtain

$$\int_{4\mu^2}^{\infty} dt\, t\, \sigma^{e\bar{e}}(t) \geq 1.6 \times (\rho\text{-dominance value}),$$

or

$$\int_{(886\ \mathrm{MeV})^2}^{\infty} dt\, t\, \sigma^{e\bar{e}}(t) \geq 0.33 \times 10^{-3}\ \mathrm{GeV}^2\ . \qquad (13)$$

The same argument as above rules out an asymptotic behavior like t^{-3}. Because of the weight function $f(t)$ suppressing the unknown high-energy part of $\sigma^{\pi\pi}(t)$, we have succeeded in deriving a nontrivial result even from the new data.[11] As has been seen in the preceding discussions, if the slope of the pion form factor turns out to be consistent with or smaller than the ρ-dominance value, our whole discussion will collapse.

In principle, we can derive a similar formula for the nucleon charge radius. Although this quantity is known much more accurately, one would have the disadvantage of having to evaluate the large unphysical region between $4\mu^2$ and $4m_N^2$ of the $N\bar{N}$ amplitude.

We are very grateful to Professor F. M. Pipkin of the Harvard University for a very helpful communication concerning the pion form factor experiment.

FOOTNOTES AND REFERENCES

* This work was supported in part by the U. S. Atomic Energy Commission.

+ National Science Foundation Graduate Fellow.

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2. V. Alles-Borelli, Proceedings of the Fifteenth International Conference on High Energy Physics (Kiev, USSR, 1970), Vol. 1, p.308.

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5. A once-subtracted dispersion relation is assumed for $F(t)$. If the ρ-meson dominance were exact, (5) would become equality.

6. See, for instance, S. J. Brodsky and S. D. Drell, Annual Review of Nuclear Science (1970), Vol. 20, p. 147. This contains further references.

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8. In the narrow width approximation to $\sigma^{\pi\pi}(t)$, the right-hand side of (9) is $2.7\, r_\pi^4 \times 10^{-7}$. The nonresonant p-wave background below the ρ region enhances the integral of $\sigma^{\pi\pi}(t)$ to push down the lower bound on Δa_μ.

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14. $\Delta a_\mu^{theor} - \Delta a_\mu^{exp} = -(27 \pm 34) \times 10^{-8}$. See Ref. 6.

15. According to Gatto,[3] $\sigma^{e\bar{e}} = o(t^{-3})$ for a finite field algebra and $\sigma^{e\bar{e}} = o(t^{-2})$ for a divergent field algebra.

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