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MULTIGRID PRECONDITIONERS FOR MIXED FINITE ELEMENT METHODS OF VECTOR LAPLACIAN

LONG CHEN, YONGKE WU, LIN ZHONG, AND JIE ZHOU

ABSTRACT. Due to the indefiniteness and poor spectral properties, the discretized linear algebraic system of the vector Laplacian by mixed finite element methods is hard to solve. A block diagonal preconditioner has been developed and shown to be an effective preconditioner by Arnold, Falk, and Winther [Acta Numerica, 15:1–155, 2006]. The purpose of this paper is to propose alternative and effective block diagonal and block triangular preconditioners for solving this saddle point system. A variable V-cycle multigrid method with the standard point-wise Gauss-Seidel smoother is proved to be a good preconditioner for a discrete vector Laplacian operator. This multigrid solver will be further used to build preconditioners for the saddle point systems of the vector Laplacian and the Maxwell equations with divergent free constraint. The major benefit of our approach is that the point-wise Gauss-Seidel smoother is more algebraic and can be easily implemented as a black-box smoother.

1. INTRODUCTION

Discretization of the vector Laplacian in spaces $\mathbf{H}_0(\text{curl})$ and $\mathbf{H}_0(\text{div})$ by mixed finite element methods is well-studied in [1]. The discretized linear algebraic system is ill-conditioned and in the saddle point form which leads to the slow convergence of classical iterative methods as the size of the system becomes large. In [1], a block diagonal preconditioner has been developed and shown to be an effective preconditioner. The purpose of this paper is to present alternative and effective block diagonal and block triangular preconditioners for solving these saddle point systems.

Due to the similarity of the problems arising from spaces $\mathbf{H}_0(\text{curl})$ and $\mathbf{H}_0(\text{div})$, we use the mixed formulation of the vector Laplacian in $\mathbf{H}_0(\text{curl})$ as an example to illustrate our approach. Choosing appropriate finite element spaces $S_h \subset H_0^1$ (a vertex element space) and $\mathbf{U}_h \subset \mathbf{H}_0(\text{curl})$ (an edge element space), the mixed formulation is: Find $\sigma_h \in S_h, \mathbf{u}_h \in \mathbf{U}_h$ such that

$$\begin{cases} -(\sigma_h, \tau_h) + (\mathbf{u}_h, \text{grad } \tau_h) = 0 & \text{for all } \tau_h \in S_h, \\ (\text{grad } \sigma_h, \mathbf{v}_h) + (\text{curl } \mathbf{u}_h, \text{curl } \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) & \text{for all } \mathbf{v}_h \in \mathbf{U}_h. \end{cases}$$

The corresponding matrix formulation is

$$(1) \quad \begin{pmatrix} -M_v & B \\ B^T & C^T M_f C \end{pmatrix} \begin{pmatrix} \sigma_h \\ \mathbf{u}_h \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \end{pmatrix}.$$

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Here M_v and M_f are mass matrices of the vertex element and the face element, respectively, B^T corresponds to a scaled grad operator, and C corresponds to the curl operator.

Based on the stability of (1) in $H_0^1 \times \mathbf{H}_0(\text{curl})$ norm, in [1], a block diagonal preconditioner in the form

$$\begin{pmatrix} (I + G^T M_e G)^{-1} & O \\ O & (I + C^T M_f C)^{-1} \end{pmatrix},$$

with $G = M_e^{-1} B^T$, is proposed and the preconditioned Krylov space method is shown to converge with optimal complexity. To compute the inverse operators in the diagonal, multigrid methods based on additive or multiplicative overlapping Schwarz smoothers [2], multigrid methods based on Hiptimair smoothers [16, 17], or HX auxiliary space preconditioner [19] can be used. In all these methods, to achieve a mesh independent condition number, a special smoother taking care of the large kernel of the curl (or div) differential operators is needed.

In contrast, we shall apply multigrid methods with the standard point-wise Gauss-Seidel (G-S) smoother to the Schur complement of the (1, 1) block

$$(2) \quad A = B^T M_v^{-1} B + C^T M_f C$$

which is a matrix representation of the following identity of the vector Laplacian

$$-\Delta \mathbf{u} = -\text{grad div } \mathbf{u} + \text{curl curl } \mathbf{u}.$$

In (2), the inverse of the mass matrix, i.e., M_v^{-1} is dense. To be practical, the exact Schur complement can be replaced by an approximation

$$\tilde{A} = B^T \tilde{M}_v^{-1} B + C^T M_f C,$$

with $\tilde{M}_v$ an easy-to-invert matrix, e.g., the diagonal or a mass lumping of M_v .

We shall prove that a variable V-cycle multigrid method using the standard point-wise Gauss-Seidel smoother is a good preconditioner for the Schur complement A or its approximation $\tilde{A}$. The major benefit of our approach is that the point-wise Gauss-Seidel smoother is more algebraic and can be easily implemented as a black-box smoother. The block smoothers proposed in [2] for the $\mathbf{H}(\text{curl})$ and $\mathbf{H}(\text{div})$ problems, however, requires more geometric information and solving local problems in small patches.

Although the finite element spaces are nested and A is symmetric positive definite, due to the inverse of the mass matrix, the bilinear forms in the coarse grid are non-inherited from the fine one. To overcome this difficulty, we shall follow the multigrid framework developed by Bramble, Pasciak, and Xu [4]. In this framework, we need only to verify two conditions: (1) Regularity and approximation assumption; (2) Smoothing property. Since A is symmetric and positive definite, the smoothing property of the Gauss-Seidel smoother is well known, see e.g. [5]. To prove the approximation property, we make use of the L^2 -error estimates of mixed finite element methods established in [2] and thus have to assume the full regularity of elliptic equations. Numerically our method works well for the case when the full regularity does not hold. With the approximation and smoothing properties, we show that one V-cycle is an effective preconditioner. As noticed in [5], W-cycle or two V-cycles may not be a valid preconditioner as the corresponding operator may not be positive definite. In other words, the proposed multigrid method for the Schur complement cannot be used as an iterative method but one V-cycle can be used as an effective preconditioner.

The multigrid preconditioner for $\tilde{A}$ will be used to build preconditioners for (1). We propose a block diagonal preconditioner and a block triangular preconditioner:

$$(3) \quad \begin{pmatrix} M_v^{-1} & O \\ O & \tilde{A}^{-1} \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} I & \tilde{M}_v^{-1}B \\ 0 & I \end{pmatrix} \begin{pmatrix} -\tilde{M}_v & 0 \\ B^T & \tilde{A} \end{pmatrix}^{-1}.$$

The action M_v^{-1} can be further approximated by $\tilde{M}_v^{-1}$ and $\tilde{A}^{-1}$ by one V-cycle multigrid. Following the framework of [20], we prove that the preconditioned system using these two preconditioners has a uniformly bounded conditional number by establishing a new stability result of the saddle point system (1) in the $\|\cdot\| \times \|\cdot\|_A$ norm.

As an application we further consider a prototype of Maxwell equations with divergence-free constraint

$$\text{curl curl } \mathbf{u} = \mathbf{f}, \quad \text{div } \mathbf{u} = 0, \quad \text{in } \Omega.$$

A regularized system obtained by the augmented Lagrangian method [14] has the form

$$(4) \quad \begin{pmatrix} A & B^T \\ B & O \end{pmatrix},$$

where A is the vector Laplacian in $\mathbf{H}_0(\text{curl})$. We then construct a block diagonal preconditioner and a block triangular preconditioner

$$(5) \quad \begin{pmatrix} A^{-1} & 0 \\ 0 & M_v^{-1} \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} I & G \\ O & -\tilde{M}_v^{-1}A_p \end{pmatrix} \begin{pmatrix} \tilde{A} & O \\ B & A_p \end{pmatrix}^{-1},$$

and prove that they are uniformly bounded preconditioners for the Maxwell system (4). Our preconditioners are new and different with the solver proposed in [12].

The paper is organized as follows. In Section 2, we introduce the discretization of the mixed formulation of the vector Laplacian, and prove stability results. In Section 3, we consider the multigrid methods for the discrete vector Laplacian and verify the approximation and smoothing properties. In Section 4, we propose the uniform preconditioner for the vector Laplacian and apply to Maxwell equation in the saddle point form. At last, we support our theoretical results with numerical experiments.

2. DISCRETIZATION

In this section, we first recall the function spaces and finite element spaces, and then present discrete formulations of the vector Laplacian problems in both space $\mathbf{H}_0(\text{curl})$ and space $\mathbf{H}_0(\text{div})$. We shall define a new norm using the Schur complement and prove corresponding Poincaré inequalities and inverse inequalities.

We assume that Ω is a bounded and convex polyhedron in $\mathbb{R}^3$ with a simple topology (homomorphism to a ball), and it is triangulated into a mesh $\mathcal{T}_h$ with size h . We assume that the mesh $\mathcal{T}_h$ belongs to a shape regular and quasi-uniform family.

2.1. Function Spaces and Finite Element Spaces. We use $L^2(\Omega)$ to denote the space of all square integrable scalar or vector functions on Ω and $(\cdot, \cdot)$ for both the scalar and vector L^2 -inner product. Given a differential operator $\mathcal{D} = \text{grad}, \text{curl}, \text{or div}$, we introduce the Sobolev space $H(\mathcal{D}, \Omega) = \{v \in L^2(\Omega), \mathcal{D}v \in L^2(\Omega)\}$. For $\mathcal{D} = \text{grad}$, $H(\text{grad}, \Omega)$ is the standard $H^1(\Omega)$. For simplicity, we will suppress the domain Ω in the notation. Let $\mathbf{n}$ be the unit outwards normal vector of $\partial\Omega$. We further introduce the following Sobolev

spaces on domain Ω with homogenous traces:

$$\begin{aligned} H_0^1 &= \{u \in H^1 : u = 0 \text{ on } \partial\Omega\}, \\ \mathbf{H}_0(\text{curl}) &= \{\mathbf{u} \in \mathbf{H}(\text{curl}) : \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Omega\}, \\ \mathbf{H}_0(\text{div}) &= \{\mathbf{u} \in \mathbf{H}(\text{div}) : \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \\ \text{and } L_0^2 &= \{u \in L^2 : \int_{\Omega} u \, dx = 0\}. \end{aligned}$$

Then, let us recall the following finite element spaces:

- $S_h \subset H_0^1$ is the well-known Lagrange elements, i.e., continuous and piecewise polynomials,
- $\mathbf{U}_h \subset \mathbf{H}_0(\text{curl})$ is the edge element space [23, 24],
- $\mathbf{V}_h \subset \mathbf{H}_0(\text{div})$ is the face element space [25, 23, 7, 24, 6, 8],
- $W_h \subset L_0^2$ is discontinuous and piecewise polynomial space.

To discretize the vector Laplacian problem posed in $\mathbf{H}_0(\text{div})$ or $\mathbf{H}_0(\text{curl})$, we start from the following de Rham complex

$$0 \longrightarrow H_0^1 \xrightarrow{\text{grad}} \mathbf{H}_0(\text{curl}) \xrightarrow{\text{curl}} \mathbf{H}_0(\text{div}) \xrightarrow{\text{div}} L_0^2 \longrightarrow 0.$$

We choose appropriate degrees and types of finite element spaces such that the discrete de Rham complex holds

$$(6) \quad 0 \longrightarrow S_h \xrightarrow{\text{grad}_h} \mathbf{U}_h \xrightarrow{\text{curl}_h} \mathbf{V}_h \xrightarrow{\text{div}_h} W_h \longrightarrow 0.$$

Important examples are: S_h is the linear Lagrange element; $\mathbf{U}_h$ is the lowest order Nedelec edge element; $\mathbf{V}_h$ is the lowest order Raviart-Thomas element, and W_h is the piecewise constant.

We now define weak differential operators and introduce the following exact sequence in the reversed ordering:

$$(7) \quad 0 \longleftarrow S_h \xleftarrow{\text{div}_h} \mathbf{U}_h \xleftarrow{\text{curl}_h} \mathbf{V}_h \xleftarrow{\text{grad}_h} W_h \longleftarrow 0.$$

The weak divergence $\text{div}_h : \mathbf{U}_h \rightarrow S_h$ is defined as the adjoint of $-\text{grad}$ operator in the L^2 -inner product, i.e., $\text{div}_h \mathbf{w}_h \in S_h$, s.t.,

$$(8) \quad (\text{div}_h \mathbf{w}_h, v_h) := -(\mathbf{w}_h, \text{grad } v_h) \quad \text{for all } v_h \in S_h.$$

Weak curl operator curl_h and weak grad operator grad_h are defined similarly. For a given $\mathbf{w}_h \in \mathbf{V}_h$, define $\text{curl}_h \mathbf{w}_h \in \mathbf{U}_h$ as

$$(9) \quad (\text{curl}_h \mathbf{w}_h, \mathbf{v}_h) := (\mathbf{w}_h, \text{curl } \mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in \mathbf{U}_h.$$

For a given $w_h \in W_h$, define $\text{grad}_h w_h \in \mathbf{V}_h$ as

$$(10) \quad (\text{grad}_h w_h, \mathbf{v}_h) := -(w_h, \text{div } \mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in \mathbf{V}_h.$$

In the limiting case when $h \rightarrow 0$, these weak differential operators becomes the so-called co-differential operators, c.f. [1], and will be denoted by $\mathcal{D}^w$.

The exactness of (7) can be easily verified by the definition and the exactness of (6). Note that the inverse of mass matrices will be involved when computing the weak differential operators and thus they are global operators.

We introduce the null space of differential operators:

$$Z_h^c = \mathbf{U}_h \cap \ker(\text{curl}), \quad \text{and } Z_h^d = \mathbf{V}_h \cap \ker(\text{div}),$$

and the null space of weak differential operators

$$K_h^c = \mathbf{U}_h \cap \ker(\text{div}_h), \quad \text{and } K_h^d = \mathbf{V}_h \cap \ker(\text{curl}_h).$$

Similar notation Z^c, Z^d, K^c, K^d will be used for the null spaces in the continuous level when the subscript h is skipped.

According to the exact sequence (6), we have the discrete Hodge decompositions [1]:

$$\begin{aligned}\mathbf{U}_h &= Z_h^c \oplus^\perp K_h^c = \text{grad } S_h \oplus^\perp \text{curl}_h \mathbf{V}_h, \\ \mathbf{V}_h &= Z_h^d \oplus^\perp K_h^d = \text{curl } \mathbf{U}_h \oplus^\perp \text{grad}_h W_h.\end{aligned}$$

The notation $\oplus^\perp$ stands for the L^2 orthogonal decomposition. These discrete version of Hodge decompositions play an important role in the analysis.

We update the exact sequences as:

$$(11) \quad 0 \longrightarrow S_h \xrightarrow{\text{grad}} Z_h^c \oplus K_h^c \xrightarrow{\text{curl}} Z_h^d \oplus K_h^d \xrightarrow{\text{div}} W_h \longrightarrow 0,$$

and

$$(12) \quad 0 \longleftarrow S_h \xleftarrow{\text{div}_h} Z_h^c \oplus K_h^c \xleftarrow{\text{curl}_h} Z_h^d \oplus K_h^d \xleftarrow{\text{grad}_h} W_h \longleftarrow 0.$$

The space in the end of the arrow is the range of the operator above and in the beginning is the real domain. The precise characterization of the null space Z_h or K_h can be found by tracing back of the corresponding operators.

2.2. Discrete Formulations of Vector Laplacian. On the continuous level, the mixed formulation of the vector Laplacian in space $\mathbf{H}_0(\text{curl})$ is: Find $\sigma \in H_0^1, \mathbf{u} \in \mathbf{H}_0(\text{curl})$ such that

$$(13) \quad \begin{cases} -(\sigma, \tau) + (\mathbf{u}, \text{grad } \tau) = 0 & \text{for all } \tau \in H_0^1, \\ (\text{grad } \sigma, \mathbf{v}) + (\text{curl } \mathbf{u}, \text{curl } \mathbf{v}) = (\mathbf{f}, \mathbf{v}) & \text{for all } \mathbf{v} \in \mathbf{H}_0(\text{curl}). \end{cases}$$

The problem (13) on the discrete level is: Find $\sigma_h \in S_h, \mathbf{u}_h \in \mathbf{U}_h$ such that

$$(14) \quad \begin{cases} -(\sigma_h, \tau_h) + (\mathbf{u}_h, \text{grad } \tau_h) = 0 & \text{for all } \tau_h \in S_h, \\ (\text{grad } \sigma_h, \mathbf{v}_h) + (\text{curl } \mathbf{u}_h, \text{curl } \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) & \text{for all } \mathbf{v}_h \in \mathbf{U}_h. \end{cases}$$

Note that the first equation of (14) can be interpreted as $\sigma_h = -\text{div}_h \mathbf{u}_h$ and in the second equation of (14) the term $(\text{grad } \sigma_h, \mathbf{v}_h) = -(\sigma_h, \text{div}_h \mathbf{v}_h)$. After eliminating σ_h from the first equation, we can write the discrete vector Laplacian for edge elements as

$$(15) \quad -\Delta_h^c \mathbf{u}_h := \text{curl}_h \text{curl } \mathbf{u}_h - \text{grad } \text{div}_h \mathbf{u}_h,$$

which is a discretization of the identity

$$-\Delta \mathbf{u} = \text{curl } \text{curl } \mathbf{u} - \text{grad } \text{div } \mathbf{u}.$$

Choosing appropriate bases for the finite element spaces, we can represent the spaces S_h and $\mathbf{V}_h$ by $\mathbb{R}^{\dim S_h}$ and $\mathbb{R}^{\dim \mathbf{V}_h}$ respectively. In the following, we shall use the same notation for the vector representation of a function if no ambiguity arises. Then we have the corresponding operator and matrix formulations as: $\mathcal{L}_h^c : S_h \times \mathbf{U}_h \rightarrow S_h' \times \mathbf{U}_h'$

$$(16) \quad \mathcal{L}_h^c \begin{pmatrix} \sigma_h \\ \mathbf{u}_h \end{pmatrix} := \begin{pmatrix} -M_v & B \\ B^T & C^T M_f C \end{pmatrix} \begin{pmatrix} \sigma_h \\ \mathbf{u}_h \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \end{pmatrix}.$$

Here M_v, M_e and M_f are mass matrices of the vertex element, edge element and the face element, respectively, $B^T = M_e G$ corresponds to a scaling of the grad operator G , and C to the curl operator. We follow the convention of Stokes equations to reserve B for the (negative) divergence operator. Note that to form the corresponding matrices of weak derivative operators, the inverse of mass matrices will be involved. The Schur complement

$$(17) \quad A_h^c = B^T M_v^{-1} B + C^T M_f C$$

is the matrix representation of discrete vector Laplacian (15). The system (16) can be reduced to the Schur complement equation

$$(18) \quad A_h^c \mathbf{u}_h = \mathbf{f}.$$

Similarly, the mixed formulation of the vector Laplacian in space $\mathbf{H}_0(\text{div})$ is: Find $\boldsymbol{\sigma} \in \mathbf{H}_0(\text{curl})$, $\mathbf{u} \in \mathbf{H}_0(\text{div})$ such that

$$(19) \quad \begin{cases} -(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathbf{u}, \text{curl } \boldsymbol{\tau}) = 0 & \text{for all } \boldsymbol{\tau} \in \mathbf{H}_0(\text{curl}), \\ (\text{curl } \boldsymbol{\sigma}, \mathbf{v}) + (\text{div } \mathbf{u}, \text{div } \mathbf{v}) = (\mathbf{f}, \mathbf{v}) & \text{for all } \mathbf{v} \in \mathbf{H}_0(\text{div}). \end{cases}$$

The corresponding discrete mixed formulation is: Find $\boldsymbol{\sigma}_h \in \mathbf{U}_h$, $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$(20) \quad \begin{cases} -(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\mathbf{u}_h, \text{curl } \boldsymbol{\tau}_h) = 0 & \text{for all } \boldsymbol{\tau}_h \in \mathbf{U}_h, \\ (\text{curl } \boldsymbol{\sigma}_h, \mathbf{v}_h) + (\text{div } \mathbf{u}_h, \text{div } \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) & \text{for all } \mathbf{v}_h \in \mathbf{V}_h. \end{cases}$$

Eliminating $\boldsymbol{\sigma}_h$ from the first equation of (20), we have the discrete vector Laplacian for face elements as

$$(21) \quad -\Delta_h^d \mathbf{u}_h := \text{curl curl}_h \mathbf{u}_h - \text{grad}_h \text{div}_h \mathbf{u}_h,$$

and the operator and matrix formulations are: $\mathcal{L}_h^d : \mathbf{U}_h \times \mathbf{V}_h \rightarrow \mathbf{U}_h' \times \mathbf{V}_h'$

$$(22) \quad \mathcal{L}_h^d \begin{pmatrix} \boldsymbol{\sigma}_h \\ \mathbf{u}_h \end{pmatrix} := \begin{pmatrix} -M_e & C^T \\ C & B^T M_t B \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_h \\ \mathbf{u}_h \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \end{pmatrix},$$

where M_t denotes the mass matrix of the discontinuous element. The Schur complement $A_h^d = C M_e^{-1} C^T + B^T M_t B$ is the matrix representation of discrete vector Laplacian (21). Similarly, the reduced equation of (22) is

$$(23) \quad A_h^d \mathbf{u}_h = \mathbf{f}.$$

We shall consider multigrid methods for solving (18) and (23) and use them to construct efficient preconditioners for the corresponding saddle point systems (16) and (22), respectively.

2.3. Discrete Poincaré Inequality and Inverse Inequality. In this subsection, we define the norms associated with the discrete vector Laplacian, and prove discrete Poincaré and inverse inequalities.

Definition 2.1. For $\mathbf{u}_h \in \mathbf{U}_h$, define $\|\mathbf{u}_h\|_{A_h^c}^2 = a_h^c(\mathbf{u}_h, \mathbf{u}_h)$, where the bilinear form $a_h^c(\cdot, \cdot)$ is defined as

$$a_h^c(\mathbf{u}_h, \mathbf{v}_h) := (\text{curl } \mathbf{u}_h, \text{curl } \mathbf{v}_h) + (\text{div}_h \mathbf{u}_h, \text{div}_h \mathbf{v}_h).$$

Similarly, for $\mathbf{u}_h \in \mathbf{V}_h$, define $\|\mathbf{u}_h\|_{A_h^d}^2 = a_h^d(\mathbf{u}_h, \mathbf{u}_h)$, where the bilinear form $a_h^d(\cdot, \cdot)$ is defined as

$$a_h^d(\mathbf{u}_h, \mathbf{v}_h) := (\text{curl}_h \mathbf{u}_h, \text{curl}_h \mathbf{v}_h) + (\text{div } \mathbf{u}_h, \text{div } \mathbf{v}_h).$$

Lemma 2.2 (Discrete Poincaré Inequality). *We have the following discrete Poincaré inequalities:*

$$(24) \quad \|\mathbf{u}_h\| \lesssim \|\mathbf{u}_h\|_{A_h^c} \quad \text{for all } \mathbf{u}_h \in \mathbf{U}_h;$$

$$(25) \quad \|\mathbf{u}_h\| \lesssim \|\mathbf{u}_h\|_{A_h^d} \quad \text{for all } \mathbf{u}_h \in \mathbf{V}_h.$$

Proof. We prove the first inequality (24) and refer to [11] for a proof of (25). From the discrete Hodge decomposition, we have for $\mathbf{u}_h \in \mathbf{U}_h$, there exist $\rho \in S_h$ and $\phi \in Z_h^d$ such that

$$(26) \quad \mathbf{u}_h = \text{grad } \rho + \text{curl}_h \phi.$$

Applying $-\text{div}_h$ to (26), we have $-\text{div}_h \mathbf{u}_h = -\text{div}_h \text{grad } \rho$, thus

$$\|\text{grad } \rho\|^2 = (-\text{div}_h \mathbf{u}_h, \rho) \leq \|\text{div}_h \mathbf{u}_h\| \|\rho\| \lesssim \|\text{div}_h \mathbf{u}_h\| \|\text{grad } \rho\|,$$

which leads to

$$(27) \quad \|\text{grad } \rho\| \lesssim \|\text{div}_h \mathbf{u}_h\|.$$

To control the other part, we first prove a discrete Poincaré inequality in the form

$$(28) \quad \|\phi\| \lesssim \|\text{curl}_h \phi\| \quad \text{for all } \phi \in Z_h^d.$$

By the exactness of the complex (12), there exists $\mathbf{v} \in K_h^c$ such that $\phi = \text{curl } \mathbf{v}$. We recall another Poincaré inequality [22, 18]

$$\|\mathbf{v}\| \lesssim \|\text{curl } \mathbf{v}\| \quad \text{for all } \mathbf{v} \in K_h^c = \mathbf{U}_h \cap \ker(\text{curl})^\perp.$$

Then we have

$$\|\phi\|^2 = (\phi, \text{curl } \mathbf{v}) = (\text{curl}_h \phi, \mathbf{v}) \leq \|\text{curl}_h \phi\| \|\mathbf{v}\| \lesssim \|\text{curl}_h \phi\| \|\text{curl } \mathbf{v}\| = \|\text{curl}_h \phi\| \|\phi\|.$$

Canceling one $\|\phi\|$, we obtain the desired inequality (28).

Applying curl to the Hodge decomposition (26) and using the inequality (28), we have $\text{curl } \mathbf{u}_h = \text{curl } \text{curl}_h \phi$, thus

$$\|\text{curl}_h \phi\|^2 = (\text{curl } \mathbf{u}_h, \phi) \leq \|\text{curl } \mathbf{u}_h\| \|\phi\| \lesssim \|\text{curl } \mathbf{u}_h\| \|\text{curl}_h \phi\|,$$

which leads to the inequality

$$(29) \quad \|\text{curl}_h \phi\| \lesssim \|\text{curl } \mathbf{u}_h\|.$$

Combine inequalities (27) and (29), we have proved that

$$\|\mathbf{u}_h\| \leq \|\text{grad } \rho\| + \|\text{curl}_h \phi\| \lesssim \|\text{div}_h \mathbf{u}_h\| + \|\text{curl } \mathbf{u}_h\| \lesssim \|\mathbf{u}_h\|_{A_h^c}.$$

□

Remark 2.3. The result and the proof can be easily generalized to mixed discretization of Hodge Laplacian in discrete differential forms [1]. We keep the concrete form in $\mathbf{H}(\text{curl})$ and $\mathbf{H}(\text{div})$ conforming finite element spaces for the easy access of these results. □

It is easy to prove the following inverse inequalities:

$$\begin{aligned} \|\mathbf{u}_h\|_{A_h^c} &\lesssim h^{-1} \|\mathbf{u}_h\| \quad \text{for all } \mathbf{u}_h \in \mathbf{U}_h; \\ \|\mathbf{u}_h\|_{A_h^d} &\lesssim h^{-1} \|\mathbf{u}_h\| \quad \text{for all } \mathbf{u}_h \in \mathbf{V}_h. \end{aligned}$$

3. MULTIGRID METHODS FOR DISCRETE VECTOR LAPLACIAN

In this section, we describe a variable V-cycle multigrid algorithm to solve the Schur complement equations (18) and (23), and prove that it is a good preconditioner.

3.1. Problem Setting. Let us assume that nested tetrahedral partitions of Ω are given as

$$\mathcal{T}_1 \subset \cdots \subset \mathcal{T}_J = \mathcal{T}_h,$$

and the corresponding H_0^1 , $\mathbf{H}_0(\text{curl})$ and $\mathbf{H}_0(\text{div})$ finite element spaces are

$$S_1 \subset \cdots \subset S_J = S_h, \quad \mathbf{U}_1 \subset \cdots \subset \mathbf{U}_J = \mathbf{U}_h, \quad \mathbf{V}_1 \subset \cdots \subset \mathbf{V}_J = \mathbf{V}_h.$$

For a technical reason, we assume that the edge element space and the face element space contain the full linear polynomial which rules out only the lowest order case. When no ambiguity can arise, we replace subscripts h by the level index k for $k = 1, 2, \dots, J$.

The discretization (13) of the mixed formulation of the vector Laplacian in space $\mathbf{H}_0(\text{curl})$ based on $\mathcal{T}_k$, for $k = 1, 2, \dots, J$, can be written as

$$(30) \quad \begin{pmatrix} -M_{v,k} & B_k \\ B_k^T & C_k^T M_{f,k} C_k \end{pmatrix} \begin{pmatrix} \sigma_k \\ \mathbf{u}_k \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f}_k \end{pmatrix}.$$

Eliminating σ_k from (30), we get the reduced Schur complement equation

$$(31) \quad A_k^c \mathbf{u}_k = (B_k^T M_{v,k}^{-1} B_k + C_k^T M_{f,k} C_k) \mathbf{u}_k = \mathbf{f}_k.$$

The discretization (19) of the mixed formulation of vector Laplacian in space $\mathbf{H}_0(\text{div})$ on $\mathcal{T}_k$, for $k = 1, 2, \dots, J$, can be written as

$$(32) \quad \begin{pmatrix} -M_{e,k} & C_k^T \\ C_k & B_k^T M_{t,k} B_k \end{pmatrix} \begin{pmatrix} \sigma_k \\ \mathbf{u}_k \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f}_k \end{pmatrix},$$

and the reduced Schur complement equation is

$$(33) \quad A_k^d \mathbf{u}_k = (B_k^T M_{t,k} B_k + C_k M_{e,k}^{-1} C_k^T) \mathbf{u}_k = \mathbf{f}_k.$$

We are interested in preconditioning the Schur complement equations (31) and (33) in the finest level, i.e., $k = J$.

Notice that, for $k < J$, A_k^c and A_k^d are defined by the discretization of the vector Laplacian on the triangulation $\mathcal{T}_k$, but not by the Galerkin projection of A_J^c or A_J^d since the inverse of a mass matrix is involved. In other words, A_k^c and A_k^d are non-inherited from A_J^c or A_J^d for $k < J$.

When necessary, the notation without the superscript c and d is used to unify the discussion. The notation $\mathcal{V}_k$ is used to represent both $\mathbf{U}_k$ and $\mathbf{V}_k$ spaces.

3.2. A Variable V-cycle Multigrid Method. We introduce some operators first. Let R_k denote a smoothing operator on level k , which is assumed to be symmetric and convergent. Let I^k denote the prolongation operator from level $k-1$ to level k , which is the natural inclusion since finite element spaces are nested. The transpose $Q_{k-1} = (I^k)^T$ then represents the restriction from level k to level $k-1$. The Galerkin projection P_{k-1} , which is from level k to level $k-1$, is defined as: for any given $\mathbf{u}_k \in \mathcal{V}_k$, $P_{k-1} \mathbf{u}_k \in \mathcal{V}_{k-1}$ satisfies

$$a_{k-1}(P_{k-1} \mathbf{u}_k, \mathbf{v}_{k-1}) = a_k(\mathbf{u}_k, I^k \mathbf{v}_{k-1}) = a_k(\mathbf{u}_k, \mathbf{v}_{k-1}) \quad \text{for all } \mathbf{v}_{k-1} \in \mathcal{V}_{k-1}.$$

The variable V-cycle multigrid algorithm is as following.

Algorithm 2. Multigrid Algorithm: $\mathbf{u}_k^{MG} = MG_k(\mathbf{f}_k; \mathbf{u}_k^0, m_k)$

Set $MG_1 = A_1^{-1}$.

For $k \geq 2$, assume that MG_{k-1} has been defined. Define $MG_k(\mathbf{f}_k; \mathbf{u}_k^0, m_k)$ as follows:

- Pre-smoothing: Define $\mathbf{u}_k^l$ for $l = 1, 2, \dots, m_k$ by

$$\mathbf{u}_k^l = \mathbf{u}_k^{l-1} + R_k(\mathbf{f}_k - A_k \mathbf{u}_k^{l-1}).$$

- Coarse-grid correction: Define $\mathbf{u}_k^{m_k+1} = \mathbf{u}_k^{m_k} + I^k \mathbf{e}_{k-1}$, where

$$\mathbf{e}_{k-1} = MG_{k-1}(Q_{k-1}(\mathbf{f}_k - A_k \mathbf{u}_k^{m_k}); 0, m_{k-1}).$$

- Post-smoothing: Define $\mathbf{u}_k^l$ for $l = m_k + 2, \dots, 2m_k + 1$ by

$$\mathbf{u}_k^l = \mathbf{u}_k^{l-1} + R_k(\mathbf{f}_k - A_k \mathbf{u}_k^{l-1}).$$

$$\text{Define } \mathbf{u}_k^{MG} = \mathbf{u}_k^{2m_k+1}.$$

In this algorithm, m_k is a positive integer which may vary from level to level, and determines the number of smoothing iterations on the k -th level, see [4, 5].

3.3. Multigrid Analysis Framework. We employ the multigrid analysis framework developed in [4]. Denoted by λ_k the largest eigenvalue of A_k . For the multigrid algorithm to be a good preconditioner to A_k , we need to verify the following assumptions:

(A.1): “Regularity and approximation assumption”: For some $0 < \alpha \leq 1$,

$$|a_k((I - P_{k-1})\mathbf{u}_k, \mathbf{u}_k)| \leq C_A \left(\frac{\|A_k \mathbf{u}_k\|^2}{\lambda_k} \right)^\alpha a_k(\mathbf{u}_k, \mathbf{u}_k)^{1-\alpha} \quad \text{for all } \mathbf{u}_k \in \mathcal{V}_k,$$

holds with constant C_A independent of k ;

(A.2): “Smoothing property”:

$$\frac{\|\mathbf{u}_k\|^2}{\lambda_k} \leq C_R(R_k \mathbf{u}_k, \mathbf{u}_k) \quad \text{for all } \mathbf{u}_k \in \mathcal{V}_k,$$

holds with constant C_R independent of k .

Following the standard arguments, we can show that the largest eigenvalue of A_k , λ_k , satisfies $\lambda_k \approx h_k^{-2}$ for $k = 1, 2, \dots, J$.

3.4. Smoothing Property. The symmetric Gauss-Seidel (SGS) or a properly weighted Jacobi iteration both satisfy the smoothing property (A.2), a proof of which can be found in [5]. For completeness we present a short proof below.

Recall that Gauss-Seidel iteration can be understood as a successive subspace correction method applied to the basis decomposition $\mathcal{V}_k = \sum_{i=1}^{N_k} \mathcal{V}_{k,i}$ with exact local solvers [26]. For $\mathbf{u} \in \mathcal{V}_k$, let $\mathbf{u} = \sum_{i=1}^{N_k} \mathbf{u}_i$ be the basis decomposition. By the X-Z identity [27, 10] for the multiplicative method, we have

$$(R_{\text{SGS}}^{-1} \mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|_{A_k}^2 + \sum_{i=0}^N \|P_i \sum_{j>i} \mathbf{u}_j\|_{A_k}^2,$$

where P_i is the A_k orthogonal projection to $\mathcal{V}_{k,i}$. For an index i , we denote by $n(i)$ the set of indices such that the corresponding basis function has overlapping support with basis function at i . We then estimate the second term as

$$\sum_{i=0}^N \|P_i \sum_{j>i} \mathbf{u}_j\|_{A_k}^2 \leq \sum_{i=0}^N \sum_{j \in n(i)} \|\mathbf{u}_j\|_{A_k}^2 \lesssim \lambda_k \sum_{i=0}^N \|\mathbf{u}_i\|^2 \lesssim \lambda_k \|\mathbf{u}\|^2.$$

Here we use the sparsity of A_k such that the repetition in the summation, i.e, the number of indices in $n(i)$, is uniformly bounded above by a constant. The last step is from the

stability of the basis decomposition in L^2 -norm which holds for all finite element spaces under consideration.

We have thus proved that $(R_{\text{SGS}}^{-1}\mathbf{u}, \mathbf{u}) \lesssim \lambda_k \|\mathbf{u}\|^2$ which is equivalent to the smoothing property by a simple change of variable. Similar proof can be adapted to the weighted Jacobi smoother.

3.5. Regularity Results. In this subsection, we will present some regularity results for Maxwell equation. Recall that, we assume Ω is a bounded and convex polyhedron throughout of this paper.

Lemma 3.1 (Theorem 3.7 and 3.9 in [15]). *The space $\mathbf{H}(\text{div}; \Omega) \cap \mathbf{H}_0(\text{curl}; \Omega)$ and $\mathbf{H}_0(\text{div}; \Omega) \cap \mathbf{H}(\text{curl}; \Omega)$ are continuously imbedded into $\mathbf{H}^1(\Omega)$ and*

$$\|\phi\|_1 \lesssim \|\text{curl } \phi\| + \|\text{div } \phi\|.$$

for all functions $\phi \in \mathbf{H}(\text{div}; \Omega) \cap \mathbf{H}_0(\text{curl}; \Omega)$ or $\mathbf{H}_0(\text{div}; \Omega) \cap \mathbf{H}(\text{curl}; \Omega)$.

In the sequel, we are going to develop an H^2 regularity result of Maxwell equation.

Lemma 3.2. *For functions $\psi \in \mathbf{H}(\text{div}; \Omega) \cap \mathbf{H}_0(\text{curl}; \Omega)$ or $\mathbf{H}_0(\text{div}; \Omega) \cap \mathbf{H}(\text{curl}; \Omega)$, satisfying $\text{curl } \psi \in \mathbf{H}^1(\Omega)$ and $\text{div } \psi \in \mathbf{H}^1(\Omega)$. Then $\psi \in H^2(\Omega)$ and*

$$\|\psi\|_2 \lesssim \|\text{curl } \psi\|_1 + \|\text{div } \psi\|_1.$$

Proof. Let $\tilde{\psi}$ be the zero extension of ψ from Ω to $\mathbb{R}^3$ and $\mathcal{F}\tilde{\psi}$ denote the Fourier transform of $\tilde{\psi}$ defined as usual by

$$\mathcal{F}\tilde{\psi} = \int_{\mathbb{R}^3} e^{-2i\pi(x,\mu)} \tilde{\psi} dx, \quad (x, \mu) = \sum_{i=1}^3 x_i \mu_i.$$

By carefully calculation, we can prove that

$$\left\| \mathcal{F} \frac{\partial^2 \tilde{\psi}_l}{\partial x_i \partial x_j} \right\| \lesssim \|\text{curl } \psi\|_1 + \|\text{div } \psi\|_1.$$

The desired result follows by the properties of Fourier transform. $\square$

Then we have the following H^2 regularity of Maxwell equation.

Lemma 3.3. *For any $\psi \in K^c$, define $\zeta \in K^c$ to be the solution of*

$$(34) \quad (\text{curl } \zeta, \text{curl } \theta) = (\psi, \theta) \quad \text{for all } \theta \in K^c.$$

Then $\text{curl } \zeta \in \mathbf{H}^2(\Omega)$ and

$$(35) \quad \|\text{curl } \zeta\|_1 \lesssim \|\psi\|,$$

$$(36) \quad \|\text{curl } \zeta\|_2 \lesssim \|\text{curl } \psi\|.$$

Proof. Indeed $\text{curl } \zeta \in \mathbf{H}_0(\text{div}; \Omega)$ with $\text{div } \text{curl } \zeta = 0$ and (34) implies $\text{curl } {}^w \text{curl } \zeta = \psi$ holds in L^2 . The desired H^1 regularity (35) of $\text{curl } \zeta$ then follows from Lemma 3.1.

For any $w \in \mathbf{H}_0(\text{div})$, let $\theta = \text{curl } {}^w w$. Then equation (34) implies

$$(\text{curl } \text{curl } {}^w \text{curl } \zeta, w) = (\text{curl } \psi, w) \quad \text{for all } w \in \mathbf{H}_0(\text{div}).$$

Thus, we have

$$\text{curl } \text{curl } {}^w \text{curl } \zeta = \text{curl } \psi \text{ in } L^2, \text{ and } \text{div } {}^w \text{curl } {}^w \text{curl } \zeta = 0.$$

Again by Lemma 3.1, it holds

$$\|\text{curl } {}^w \text{curl } \zeta\|_1 \lesssim \|\text{curl } \psi\|.$$

The desired result (36) is then obtained by Lemma 3.2. $\square$

3.6. Error Estimate of Several Projection Operators. We define several projection operators to the null space $K_h^{\mathcal{D}}$. Given $u \in H(\mathcal{D})$, find $P_h^{\mathcal{D}}u \in K_h^{\mathcal{D}}$ such that

$$(37) \quad (\mathcal{D}P_h^{\mathcal{D}}u, \mathcal{D}v_h) = (\mathcal{D}u, \mathcal{D}v_h), \quad \text{for all } v_h \in K_h^{\mathcal{D}}.$$

Equation (37) determines $P_h^{\mathcal{D}}u$ uniquely since $(\mathcal{D}\cdot, \mathcal{D}\cdot)$ is an inner product on the subspace $K_h^{\mathcal{D}}$ which can be proved using the Poincaré inequality (Lemma 2.2). For $\mathcal{D} = \text{grad}$, we understand K_h^{grad} as S_h .

Lemma 3.4 (Theorem 2.4 in Monk [21]). *Suppose that $\text{curl } \mathbf{u} \in \mathbf{H}^k$ and let $\mathbf{u}_h = P_h^c \mathbf{u}$ to $\mathbf{U}_h$ which contains polynomial of degree less than or equal to k . Then we have the error estimate*

$$\|\text{curl}(\mathbf{u} - \mathbf{u}_h)\| \lesssim h^r \|\text{curl } \mathbf{u}\|_r, \quad \text{for } 1 \leq r \leq k.$$

We are also interested in the estimate of projections between two consecutive finite element spaces. Following the convention of multigrid community, for any $2 < k \leq J$, let $\mathcal{T}_H = \mathcal{T}_{k-1}$ and $\mathcal{T}_h = \mathcal{T}_k$. Notice that the ratio $H/h \leq C$.

The following error estimates are obtained in [2].

Lemma 3.5. *Given $\mathbf{u}_h \in K_h^c$, let $\mathbf{u}_H = P_H^c \mathbf{u}_h$. Then*

$$\begin{aligned} \|\mathbf{u}_h - \mathbf{u}_H\| &\lesssim H \|\text{curl } \mathbf{u}_h\|, \\ \|\text{curl}(\mathbf{u}_h - \mathbf{u}_H)\| &\lesssim H \|\text{curl}_h \text{curl } \mathbf{u}_h\|. \end{aligned}$$

Lemma 3.6. *Give $\mathbf{v}_h \in K_h^d$, let $\mathbf{v}_H = P_H^d \mathbf{v}_h$. Then*

$$\begin{aligned} \|\mathbf{v}_h - \mathbf{v}_H\| &\lesssim H \|\text{div } \mathbf{v}_h\|, \\ \|\text{div}(\mathbf{v}_h - \mathbf{v}_H)\| &\lesssim H \|\text{grad}_h \text{div } \mathbf{v}_h\|. \end{aligned}$$

We now introduce a projection to K^c . Let $Q_K^c : \mathbf{L}^2 \rightarrow K^c$ be the L^2 -projection to K^c . Notice that for $\mathbf{u} \in \mathbf{L}^2$, $Q_K^c \mathbf{u} = \mathbf{u} - \nabla p$ where $p \in H_0^1$ is determined by the Poisson equation $(\nabla p, \nabla q) = (\mathbf{u}, \nabla q)$ for all $q \in H_0^1$. Therefore $\text{curl } Q_K^c \mathbf{u} = \text{curl } \mathbf{u}$. Similarly we define $Q_h^c : \mathbf{L}^2 \rightarrow K_h^c$ as $Q_h^c \mathbf{u} = \mathbf{u} - \nabla p$ where $p \in S_h$ is determined by the Poisson equation $(\nabla p, \nabla q) = (\mathbf{u}, \nabla q)$ for all $q \in S_h$. We have the error estimate, c.f. [2, 28].

Lemma 3.7. *For $\mathbf{u}_h \in K_h^c$, we have*

$$(38) \quad \|Q_K^c \mathbf{u}_h - \mathbf{u}_h\| \lesssim h \|\text{curl } \mathbf{u}_h\|.$$

And for $\mathbf{u}_H \in K_H^c$

$$(39) \quad \|Q_h^c \mathbf{u}_H - \mathbf{u}_H\| \lesssim H \|\text{curl } \mathbf{u}_H\|.$$

In the estimate (38)-(39), we lift a function in a coarse space to a fine space while in Lemma 3.5, we estimate the projection. The L^2 -projection $Q_h^c : K_H^c \rightarrow K_h^c$ can be thought of as a prolongation of non-nested spaces K_H^c and K_h^c .

3.7. Approximation Property of Edge Element Spaces. Let $\mathbf{u}_h \in \mathbf{U}_h$ be the solution of equation

$$(40) \quad a_h^c(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}_h, \mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in \mathbf{U}_h,$$

and $\mathbf{u}_H \in \mathbf{U}_H \subset \mathbf{U}_h$ be the solution of equation

$$(41) \quad a_H^c(\mathbf{u}_H, \mathbf{v}_H) = (\mathbf{f}_h, \mathbf{v}_H) \quad \text{for all } \mathbf{v}_H \in \mathbf{U}_H.$$

We have the Hodge decomposition

$$(42) \quad \mathbf{u}_h = \text{grad } \phi_h \oplus^\perp \mathbf{u}_{0,h}, \quad \text{with unique } \phi_h \in S_h, \mathbf{u}_{0,h} \in K_h^c,$$

$$(43) \quad \mathbf{u}_H = \text{grad } \phi_H \oplus^\perp (\mathbf{u}_{0,H} + \mathbf{e}_H), \quad \text{with unique } \phi_H \in S_H, \mathbf{u}_{0,H} \text{ and } \mathbf{e}_H \in K_H^c,$$

$$(44) \quad \mathbf{f}_h = \text{grad } g_h \oplus^\perp \text{curl}_h \mathbf{q}_h, \quad \text{with unique } g_h \in S_h, \mathbf{q}_h \in Z_h^d,$$

where $\mathbf{u}_{0,H} = P_H^c \mathbf{u}_{0,h}$. Then by Lemma 3.5, we immediately get the following estimate.

Lemma 3.8. *Let $\mathbf{u}_{0,h}$ and $\mathbf{u}_{0,H}$ be defined as in equations (42) and (43). It holds*

$$\|\mathbf{u}_{0,h} - \mathbf{u}_{0,H}\| \lesssim H \|\mathbf{u}_h\|_{A_h^c}.$$

Now we turn to the estimate of $\mathbf{e}_H$ being given in equation (43).

Lemma 3.9. *Let $\mathbf{e}_H \in K_H^c$ be defined as in equation (43). It holds*

$$\|\mathbf{e}_H\|_{A_h^c} \lesssim H \|A_h^c \mathbf{u}_h\|.$$

Proof. By equations (40) and (41), we have

$$(\text{curl } \mathbf{u}_{0,h}, \text{curl } \mathbf{v}_h) = (\mathbf{q}_h, \text{curl } \mathbf{v}_h), \quad \text{for all } \mathbf{v}_h \in K_h^c$$

$$(\text{curl } (\mathbf{u}_{0,H} + \mathbf{e}_H), \text{curl } \mathbf{v}_H) = (\text{grad } g_h, \mathbf{v}_H) + (\mathbf{q}_h, \text{curl } \mathbf{v}_H), \quad \text{for all } \mathbf{v}_H \in K_H^c,$$

where g_h and $\mathbf{q}_h$ are defined in equation (44). Then

$$(45) \quad (\text{curl } \mathbf{e}_H, \text{curl } \mathbf{v}_H) = (\text{grad } g_h, \mathbf{v}_H) \quad \text{for all } \mathbf{v}_H \in K_H^c.$$

Let $\mathbf{e}_h = Q_h^c \mathbf{e}_H$, then $\text{div}_h \mathbf{e}_h = 0$ and by Lemma 3.5 $\|\mathbf{e}_h - \mathbf{e}_H\| \lesssim H \|\text{curl } \mathbf{e}_H\|$. Thus it holds

$$(\text{curl } \mathbf{e}_H, \text{curl } \mathbf{e}_H) = (\text{grad } g_h, \mathbf{e}_H) = (\text{grad } g_h, \mathbf{e}_H - \mathbf{e}_h) \lesssim H \|\text{grad } g_h\| \|\text{curl } \mathbf{e}_H\|,$$

which implies

$$\|\text{curl } \mathbf{e}_H\| \lesssim H \|\text{grad } g_h\| \leq H \|\mathbf{f}_h\| = H \|A_h^c \mathbf{u}_h\|.$$

Using the fact that $\text{div}_h \mathbf{e}_h = 0$, the inverse inequality and the above inequality, we immediately get

$$\|\text{div}_h \mathbf{e}_H\| = \|\text{div}_h (\mathbf{e}_H - \mathbf{e}_h)\| \lesssim h^{-1} \|\mathbf{e}_h - \mathbf{e}_H\| \lesssim \frac{H}{h} \|\text{curl } \mathbf{e}_H\| \lesssim H \|A_h^c \mathbf{u}_h\|.$$

The desired result then follows. $\square$

We now explore the relation between ϕ_h, ϕ_H , and g_h defined in equations (42)-(44).

Lemma 3.10. *Let $\phi_h \in S_h$ and $\phi_H \in S_H$ be defined as in equations (42) and (43). It holds*

$$\|\text{grad } \phi_h - \text{grad } \phi_H\| \lesssim H \|\mathbf{u}_h\|_{A_h^c}.$$

Proof. For equation (40), test with $\mathbf{v}_h \in \text{grad } S_h$ to get

$$(\text{div}_h \text{grad } \phi_h, \text{div}_h \mathbf{v}_h) = (\text{grad } g_h, \mathbf{v}_h) = -(\mathbf{q}_h, \text{div}_h \mathbf{v}_h),$$

which implies $-\text{div}_h \text{grad } \phi_h = -\text{div}_h \mathbf{u}_h = g_h$, i.e.,

$$(46) \quad -\Delta_h \phi_h = g_h.$$

From equation (46), we can see that ϕ_h is the Galerkin projection of ϕ to S_h , where $\phi \in H_0^1(\Omega)$ satisfies the Poisson equation:

$$-\Delta \phi = g_h.$$

Therefore by the standard error estimate of finite element methods, we have

$$\|\nabla\phi - \nabla\phi_h\| \lesssim H\|g_h\|.$$

For equation (41), choose $\mathbf{v}_H = \text{grad } \psi_H \in \text{grad } S_H$, we have

$$(\text{div}_H \text{grad } \phi_H, \text{div}_H \mathbf{v}_H) = (\text{grad } g_h, \text{grad } \psi_H) = (\text{grad } P_H^g g_h, \text{grad } \psi_H),$$

which implies $-\text{div}_H \text{grad } \phi_H = P_H^g g_h$, i.e.,

$$(47) \quad -\Delta_H \phi_H = P_H^g g_h.$$

From equation (47), we can see that ϕ_H is the Galerkin projection of $\tilde{\phi}$ to S_H , where $\tilde{\phi} \in H_0^1(\Omega)$ satisfies the Poisson equation:

$$-\Delta \tilde{\phi} = P_H^g g_h.$$

The H^1 -projection P_H^g is not stable in L^2 -norm. Applied to functions in S_h , however, we can recover one as follows

$$\|(I - P_H^g)g_h\| \lesssim H\|\text{grad } (I - P_H^g)g_h\| \lesssim H\|\text{grad } g_h\| \lesssim H/h\|g_h\| \lesssim \|g_h\|.$$

In the last step, we use the fact that the ratio of the mesh size between consecutive levels is bounded, i.e., $H/h \leq C$.

We then have

$$\|\text{grad } (\tilde{\phi} - \phi_H)\| \lesssim H\|P_H^g g_h\| \leq H\|g_h\| + H\|(I - P_H^g)g_h\| \lesssim H\|g_h\|.$$

And by the triangle inequality and the stability of the projection operator P_H^g

$$\begin{aligned} \|\text{grad } (\phi_h - \phi_H)\| &\leq \|\text{grad } (\phi_h - \phi)\| + \|\text{grad } (\phi_H - \tilde{\phi})\| + \|\text{grad } (\phi - \tilde{\phi})\| \\ &\lesssim H\|g_h\| + \|g_h - P_H^g g_h\|_{-1}. \end{aligned}$$

Using the error estimate of negative norms and the inverse inequality, we have

$$\|g_h - P_H^g g_h\|_{-1} \lesssim H^2\|g_h\|_1 \lesssim H\|g_h\|.$$

Here we use H^{-1} norm estimate for S_H having degree greater than or equal to 2. Noticing that $g_h = \text{div}_h \mathbf{u}_h$, we thus get

$$(48) \quad \|\text{grad } (\phi_h - \phi_H)\| \lesssim H\|\text{div}_h \mathbf{u}_h\| \lesssim H\|\mathbf{u}_h\|_{A_h^c}.$$

□

As a summary of the above results, we have the following approximation result.

Theorem 3.11. *Condition (A.1) holds with $\alpha = \frac{1}{2}$, i.e. for any $\mathbf{u}_k \in \mathbf{U}_k$, there hold*

$$(49) \quad a_k^c((I - P_{k-1})\mathbf{u}_k, \mathbf{u}_k) \lesssim \left(\frac{\|A_k^c \mathbf{u}_k\|^2}{\lambda_k} \right)^{\frac{1}{2}} a_k^c(\mathbf{u}_k, \mathbf{u}_k)^{\frac{1}{2}}.$$

Proof. We use h to denote k and H to denote $k - 1$. Let $\mathbf{u}_h$, $\mathbf{u}_H$, and $\mathbf{f}_h$ as in equations (40)-(41) which have Hodge decompositions, c.f. (42)-(44). Let $\delta_1 = \mathbf{u}_{0,h} - \mathbf{u}_{0,H}$, $\delta_2 = \text{grad } \phi_h - \text{grad } \phi_H$, by Lemmas 3.8, 3.9 and 3.10, it holds

$$\begin{aligned} a_h^c((I - P_H)\mathbf{u}_h, \mathbf{u}_h) &= a_h^c(\delta_1, \mathbf{u}_h) + a_h^c(\delta_2, \mathbf{u}_h) + a_h^c(\mathbf{e}_H, \mathbf{u}_h) \\ &\leq \|\delta_1\| \|A_h^c \mathbf{u}_h\| + \|\delta_2\| \|A_h^c \mathbf{u}_h\| + \|\mathbf{e}_H\|_{A_h^c} \|\mathbf{u}_h\|_{A_h^c} \\ &\lesssim H\|\mathbf{u}_h\|_{A_h^c} \|A_h^c \mathbf{u}_h\|. \end{aligned}$$

□

3.8. Approximation Property of Face Element Spaces. Let $\mathbf{u}_h \in \mathbf{V}_h$ be the solution of equation

$$(50) \quad a_h^d(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}_h, \mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in \mathbf{V}_h,$$

and $\mathbf{u}_H \in \mathbf{V}_H \subset \mathbf{V}_h$ be the solution of equation

$$(51) \quad a_H^d(\mathbf{u}_H, \mathbf{v}_H) = (\mathbf{f}_h, \mathbf{v}_H) \quad \text{for all } \mathbf{v}_H \in \mathbf{V}_H.$$

We can easily see that $\mathbf{f}_h = A_h^d \mathbf{u}_h$.

By the Hodge decomposition, we have

$$(52) \quad \mathbf{u}_h = \text{curl } \phi_h \oplus \mathbf{u}_{0,h}, \quad \text{with unique } \phi_h \in K_h^c, \mathbf{u}_{0,h} \in K_h^d,$$

$$(53) \quad \mathbf{u}_H = \text{curl } \phi_H \oplus (\mathbf{u}_{0,H} + \mathbf{e}_H) \quad \text{with unique } \phi_H \in K_H^c, \mathbf{u}_{0,H}, \mathbf{e}_H \in K_H^d,$$

$$(54) \quad \mathbf{f}_h = \text{curl } \mathbf{g}_h \oplus \text{grad}_h q_h \quad \text{with unique } \mathbf{g}_h \in K_h^c, q_h \in W_h,$$

where $\mathbf{u}_{0,H} = P_H^d \mathbf{u}_{0,h}$. By Lemma 3.6, we immediately have the following result.

Lemma 3.12. *Let $\mathbf{u}_{0,h} \in \text{grad}_h W_h$ and $\mathbf{u}_{0,H} \in \text{grad}_H W_H$ be defined as in equations (52) and (53). It holds*

$$\|\mathbf{u}_{0,h} - \mathbf{u}_{0,H}\| \lesssim H \|\text{div } \mathbf{u}_{0,h}\|.$$

The estimate of $\mathbf{e}_H \in K_H^d$ defined in equation (53) can be proved analog to Lemma 3.9 and thus skipped.

Lemma 3.13. *Assume that $\mathbf{e}_H \in \text{grad}_H W_H$ be defined as in equation (53). Then it holds*

$$\|\mathbf{e}_H\|_{A_H^d} \lesssim \|A_h^d \mathbf{u}_h\|.$$

We now explore the relation between ϕ_h , ϕ_H , and $\mathbf{g}_h$ defined in equations (52)-(54).

Lemma 3.14. *Assume that $\psi_h \in K_h^c$. Let $\zeta_h \in K_h^c$ be the solution of equation*

$$(\text{curl } \zeta_h, \text{curl } \tau_h) = (\psi_h, \tau_h) \quad \text{for all } \tau_h \in K_h^c,$$

and let $\zeta \in K^c$ be the solution of equation

$$(\text{curl } \zeta, \text{curl } \tau) = (Q_K^c \psi_h, \tau) \quad \text{for all } \tau \in K^c.$$

Then, it holds

$$\|\text{curl } (\zeta - \zeta_h)\| \lesssim h \|\psi_h\|.$$

Proof. Let $\tilde{\zeta}_h = P_h^c \zeta$. By Lemma 3.4, we have

$$\|\text{curl } (\zeta - \tilde{\zeta}_h)\| \lesssim h \|\text{curl } \zeta\|_1 \lesssim h \|Q_K^c \psi_h\| \lesssim h \|\psi_h\|.$$

But $\zeta_h \neq \tilde{\zeta}_h$. Indeed by the definition of ζ_h and $\tilde{\zeta}_h$, we have

$$(\text{curl } (\zeta_h - \tilde{\zeta}_h), \text{curl } \tau_h) = (\psi_h - Q_K^c \psi_h, \tau_h) \quad \text{for all } \tau_h \in K_h.$$

Thus, with $\delta_h = \zeta_h - \tilde{\zeta}_h$, we have

$$\begin{aligned} \|\text{curl } (\zeta_h - \tilde{\zeta}_h)\|^2 &= (\text{curl } (\zeta_h - \tilde{\zeta}_h), \text{curl } \delta_h) = (Q_K^c \psi_h - \psi_h, \delta_h) \\ &= (Q_K^c \psi_h - \psi_h, \delta_h - Q_K^c \delta_h) \lesssim h \|\psi_h\| \|\text{curl } \delta_h\|. \end{aligned}$$

The desired result follows by canceling one $\|\text{curl } \delta_h\|$ and the triangle inequality. $\square$

We are in the position to estimate ϕ_h and ϕ_H .

Lemma 3.15. *Let $\phi_h \in U_h$ and $\phi_H \in U_H$ be defined as in equations (52) and (53). It holds*

$$\|\text{curl } \phi_h - \text{curl } \phi_H\| \lesssim H \|\mathbf{u}_h\|_{A_h^d}.$$

Proof. Chose the test function $\mathbf{v}_h = \text{curl } \mathbf{w}_h$ with $\mathbf{w}_h \in \mathbf{U}_h$ in equation (50) to simplify the left hand side of (50) as

$$(\text{curl}_h \mathbf{u}_h, \text{curl}_h \text{curl } \mathbf{w}_h) = (\text{curl}_h \text{curl } \phi_h, \text{curl}_h \text{curl } \mathbf{w}_h) = (\text{curl } \phi_h, \text{curl } \text{curl}_h \text{curl } \mathbf{w}_h),$$

and the right hand side becomes

$$(\mathbf{f}_h, \text{curl } \mathbf{w}_h) = (\text{curl } \mathbf{g}_h, \text{curl } \mathbf{w}_h) = (\mathbf{g}_h, \text{curl}_h \text{curl } \mathbf{w}_h)$$

Denoted by $\boldsymbol{\tau}_h = \text{curl}_h \text{curl } \mathbf{w}_h \in K_h^c$. We get

$$(\text{curl } \phi_h, \text{curl } \boldsymbol{\tau}_h) = (\mathbf{g}_h, \boldsymbol{\tau}_h) \quad \text{for all } \boldsymbol{\tau}_h \in K_h^c.$$

Let $\phi \in K^c$ satisfy the Maxwell equation:

$$(\text{curl } \phi, \text{curl } \boldsymbol{\tau}) = (Q_K^c \mathbf{g}_h, \boldsymbol{\tau}) \quad \text{for all } \boldsymbol{\tau} \in K^c.$$

By Lemma 3.14, we have

$$\|\text{curl}(\phi - \phi_h)\| \lesssim h \|\mathbf{g}_h\|.$$

When moving to the coarse space, the left hand side of equation (51) can be still simplified to $(\text{curl } \phi_H, \text{curl } \boldsymbol{\tau}_H)$. But the right hand side becomes

$$(\mathbf{f}_h, \text{curl } \mathbf{w}_H) = (\text{curl } \mathbf{g}_h, \text{curl } \mathbf{w}_H) \neq (\mathbf{g}_h, \text{curl}_H \text{curl } \mathbf{w}_H).$$

We need to project $\mathbf{g}_h$ to the coarse space and arrives at the equation

$$(\text{curl } \phi_H, \text{curl } \boldsymbol{\tau}_H) = (P_H^c \mathbf{g}_h, \boldsymbol{\tau}_H) \quad \text{for all } \boldsymbol{\tau}_H \in K_H^c.$$

Let $\tilde{\phi} \in K^c$ satisfy the Maxwell equation:

$$(\text{curl } \tilde{\phi}, \text{curl } \boldsymbol{\tau}) = (Q_K^c P_H^c \mathbf{g}_h, \boldsymbol{\tau}) \quad \text{for all } \boldsymbol{\tau} \in K^c.$$

By Lemma 3.14, we have

$$\|\text{curl}(\tilde{\phi} - \phi_H)\| \lesssim H \|Q_K^c P_H^c \mathbf{g}_h\| \leq H \|P_H^c \mathbf{g}_h\| \lesssim H \|\mathbf{g}_h\|.$$

By the triangle inequality, it remains to estimate $\|\text{curl}(\phi - \tilde{\phi})\|$. We first write out the error equation for $\phi - \tilde{\phi}$

$$(\text{curl}(\phi - \tilde{\phi}), \text{curl } \boldsymbol{\psi}) = (Q_K^c (\mathbf{g}_h - P_H^c \mathbf{g}_h), \boldsymbol{\psi}), \quad \text{for all } \boldsymbol{\psi} \in K^c.$$

We then apply the standard duality argument. Let $\boldsymbol{\zeta} \in K^c$ satisfies

$$(\text{curl } \boldsymbol{\zeta}, \text{curl } \boldsymbol{\tau}) = (\boldsymbol{\psi}, \boldsymbol{\tau}) \quad \text{for all } \boldsymbol{\tau} \in K^c$$

Then

$$\begin{aligned} (Q_K^c (\mathbf{g}_h - P_H^c \mathbf{g}_h), \boldsymbol{\psi}) &= (\text{curl } \boldsymbol{\zeta}, \text{curl } Q_K^c (\mathbf{g}_h - P_H^c \mathbf{g}_h)) \\ &= (\text{curl } \boldsymbol{\zeta}, \text{curl } (\mathbf{g}_h - P_H^c \mathbf{g}_h)) \\ &= (\text{curl } (\boldsymbol{\zeta} - P_H^c \boldsymbol{\zeta}), \text{curl } (\mathbf{g}_h - P_H^c \mathbf{g}_h)) \\ &\lesssim H^2 \|\text{curl } \boldsymbol{\zeta}\|_2 \|\text{curl } \mathbf{g}_h\| \lesssim H \|\text{curl } \boldsymbol{\psi}\| \|\mathbf{g}_h\|, \end{aligned}$$

which implies

$$\|\text{curl}(\phi - \tilde{\phi})\| \lesssim H \|\mathbf{g}_h\|.$$

The estimate of $\|\text{curl } \phi_h - \text{curl } \phi_H\|$ then follows from the triangle inequality. $\square$

As a summary the the above results, we have the following theorem.

Theorem 3.16. *Condition (A.1) holds with $\alpha = \frac{1}{2}$, i.e. for any $\mathbf{u}_k \in \mathbf{V}_k$, there hold*

$$(55) \quad a_k^d((I - P_{k-1})\mathbf{u}_k, \mathbf{u}_k) \lesssim \left(\frac{\|A_k^d \mathbf{u}_k\|^2}{\lambda_k} \right)^{\frac{1}{2}} a_k^d(\mathbf{u}_k, \mathbf{u}_k)^{\frac{1}{2}}.$$

3.9. Results. According to the multigrid framework in [4], we conclude that the variable V-cycle multigrid algorithm is a good preconditioner for the Schur complement equations (18) and (23). We summarize the result in the following theorem.

Theorem 3.17. *Let V_k denote the operator of one V-cycle of MG_k in Algorithm 2 with homogenous data, i.e., $\mathbf{f}_k = 0$. Assume the smoothing steps m_k satisfy*

$$\beta_0 m_k \leq m_{k-1} \leq \beta_1 m_k.$$

Here we assume that β_0 and β_1 are constants which are greater than one and independent of k . Then the condition number of $V_J A_J$ is $\mathcal{O}(1)$.

Remark 3.18. As noticed in [5], W-cycle or two V-cycles may not be a valid preconditioner as the corresponding operator may not be positive definite. In other words, the proposed multigrid method for the Schur complement cannot be used as an iterative method but one V-cycle can be used as an effective preconditioner. $\square$

4. UNIFORM PRECONDITIONER

In this section, we will show that the multigrid solver for the Schur complement equations can be used to build efficient preconditioners for the mixed formulations of vector Laplacian (16) and (22). We also apply the multigrid preconditioner of the vector Laplacian to the Maxwell equation discretized as a saddle point system. We prove that the preconditioned systems have condition numbers independent of mesh parameter h .

4.1. Block Diagonal Preconditioner. It is easy to see that the inverses of the symmetric positive definite matrices M_v , M_e , A_h^c and A_h^d exist, which implies the existence of the operators $(\mathcal{L}_h^c)^{-1}$, $(\mathcal{L}_h^d)^{-1}$, and the block diagonal preconditioners defined as following.

Definition 4.1. *We define the operator $\mathcal{P}_h^c : S'_h \times U'_h \rightarrow S_h \times U_h$ with the matrix representation*

$$(56) \quad \mathcal{P}_h^c = \begin{pmatrix} M_v^{-1} & 0 \\ 0 & (A_h^c)^{-1} \end{pmatrix},$$

and the operator $\mathcal{P}_h^d : U'_h \times V'_h \rightarrow U_h \times V_h$ with the matrix representation

$$(57) \quad \mathcal{P}_h^d = \begin{pmatrix} M_e^{-1} & 0 \\ 0 & (A_h^d)^{-1} \end{pmatrix}.$$

Follow the framework in [20], it suffices to prove the boundedness of operators $\mathcal{L}_h^c$ and $\mathcal{L}_h^d$ and their inverse in appropriate norms. In the sequel, to unify the notation, we use M for the mass matrix and A the vector Laplacian. The inverse of the mass matrix can be thought of as the matrix representation of the Riesz representation induced by the L^2 -inner product and the inverse of A is the Riesz representation of the A -inner product. The preconditioners $\mathcal{P}_h^c$ and $\mathcal{P}_h^d$ are Riesz representation of $L^2 \times A$ -inner product. Let $\langle \cdot, \cdot \rangle$ be the duality pair in $\mathcal{V}_h$. We clarify the norm notations using M and A as follows:

- $\| \cdot \|_M$: $\|\sigma_h\|_M^2 = \langle M\sigma_h, \sigma_h \rangle$;
- $\| \cdot \|_A$: $\|u_h\|_A^2 = \langle A_h u_h, u_h \rangle$;
- $\| \cdot \|_{M^{-1}}$: $\|g_h\|_{M^{-1}}^2 = \langle M^{-1}g_h, g_h \rangle$;
- $\| \cdot \|_{A^{-1}}$: $\|f_h\|_{A^{-1}}^2 = \langle A_h^{-1}f_h, f_h \rangle$.

The following lemma gives a bound of the Schur complement $BA^{-1}B^T$ similar to the corresponding result of the Stokes equation.

Lemma 4.2. *We have the inequality*

$$(58) \quad \langle B(A_h^c)^{-1}B^T\phi_h, \phi_h \rangle \leq \langle M_v\phi_h, \phi_h \rangle \quad \text{for all } \phi_h \in S_h,$$

Proof. Let $\mathbf{v}_h = (A_h^c)^{-1}B^T\phi_h$. Then

$$\langle B(A_h^c)^{-1}B^T\phi_h, \phi_h \rangle = \langle (A_h^c)^{-1}B^T\phi_h, B^T\phi_h \rangle = \langle A_h^c\mathbf{v}_h, \mathbf{v}_h \rangle = \|\mathbf{v}_h\|_A^2.$$

Now we identify $\mathbf{v}_h \in \mathbf{V}'_h$ by the Riesz map in the A -inner product, and then we have

$$\begin{aligned} \|\mathbf{v}_h\|_A &= \sup_{\mathbf{u}_h \in \mathbf{V}_h} \frac{\langle \mathbf{v}_h, \mathbf{u}_h \rangle_A}{\|\mathbf{u}_h\|_A} = \sup_{\mathbf{u}_h \in \mathbf{V}_h} \frac{\langle B^T\phi_h, \mathbf{u}_h \rangle}{\|\mathbf{u}_h\|_A} = \sup_{\mathbf{u}_h \in \mathbf{V}_h} \frac{\langle \phi_h, B\mathbf{u}_h \rangle}{\|\mathbf{u}_h\|_A} \\ &\leq \sup_{\mathbf{u}_h \in \mathbf{V}_h} \frac{\|\phi_h\|_M \|B\mathbf{u}_h\|_{M^{-1}}}{\|\mathbf{u}_h\|_A} \leq \|\phi_h\|_M. \end{aligned}$$

In the last step, we have used the identity (17) which implies $\|B\mathbf{u}_h\|_{M^{-1}} \leq \|\mathbf{u}_h\|_A$. The desired result (58) then follows easily. $\square$

We present a stability result of the mixed formulation of the vector Laplacian which is different with that established in [1].

Theorem 4.3. *The operators $\mathcal{L}_h^c, \mathcal{L}_h^d$ and there inverse are both bounded operators:*

$$\|\mathcal{L}_h^c\|_{L(S_h \times \mathbf{U}_h, S'_h \times \mathbf{U}'_h)}, \|\mathcal{L}_h^d\|_{L(\mathbf{U}_h \times \mathbf{V}_h, \mathbf{U}'_h \times \mathbf{V}'_h)},$$

are bounded and independent of h from $(\|\cdot\|_{M^{-1}}, \|\cdot\|_{A^{-1}}) \rightarrow (\|\cdot\|_M, \|\cdot\|_A)$, and

$$\|(\mathcal{L}_h^c)^{-1}\|_{L(S'_h \times \mathbf{U}'_h, S_h \times \mathbf{U}_h)}, \|(\mathcal{L}_h^d)^{-1}\|_{L(\mathbf{U}'_h \times \mathbf{V}'_h, \mathbf{U}_h \times \mathbf{V}_h)}$$

are bounded and independent of h from $(\|\cdot\|_M, \|\cdot\|_A) \rightarrow (\|\cdot\|_{M^{-1}}, \|\cdot\|_{A^{-1}})$.

Proof. We prove the $\mathbf{H}_0(\text{curl})$ case below. The proof of the $\mathbf{H}_0(\text{div})$ case is similar.

Let $(\sigma_h, \mathbf{u}_h) \in S_h \times \mathbf{U}_h$ and $(g_h, \mathbf{f}_h) \in S'_h \times \mathbf{U}'_h$ be given by the relation with

$$(59) \quad \mathcal{L}_h^c \begin{pmatrix} \sigma_h \\ \mathbf{u}_h \end{pmatrix} = \begin{pmatrix} -M_v & B \\ B^T & C^T M_f C \end{pmatrix} \begin{pmatrix} \sigma_h \\ \mathbf{u}_h \end{pmatrix} = \begin{pmatrix} g_h \\ \mathbf{f}_h \end{pmatrix}.$$

To prove $\|\mathcal{L}_h^c\|_{L(S_h \times \mathbf{U}_h, S'_h \times \mathbf{U}'_h)} \lesssim 1$, it is sufficient to prove

$$(60) \quad \|g_h\|_{M^{-1}} + \|\mathbf{f}_h\|_{A^{-1}} \lesssim \|\sigma_h\|_M + \|\mathbf{u}_h\|_A.$$

From (59), we have $g_h = -M_v\sigma_h + B\mathbf{u}_h$ and $\mathbf{f}_h = A_h^c\mathbf{u}_h - B^T M_v^{-1}g_h$. The norm of g_h is easy to bound as follows

$$\|g_h\|_{M^{-1}}^2 \leq 2\|M_v\sigma_h\|_{M^{-1}}^2 + 2\|B\mathbf{u}_h\|_{M^{-1}}^2 \leq 2\|\sigma_h\|_M^2 + 2\|\mathbf{u}_h\|_A^2.$$

To bound the norm of $\mathbf{f}_h$, we first have

$$\|\mathbf{f}_h\|_{A^{-1}}^2 \leq 2\|B^T M_v^{-1}g_h\|_{A^{-1}}^2 + 2\|A_h^c\mathbf{u}_h\|_{A^{-1}}^2 \leq 2\|B^T M_v^{-1}g_h\|_{A^{-1}}^2 + 2\|\mathbf{u}_h\|_A^2.$$

Let $\phi_h = M_v^{-1}g_h$, by Lemma 4.2, we have

$$\|B^T M_v^{-1}g_h\|_{A^{-1}}^2 = \|B^T\phi_h\|_{A^{-1}}^2 = \langle B(A_h^c)^{-1}B^T\phi_h, \phi_h \rangle \leq \|\phi_h\|_M^2 = \|g_h\|_{M^{-1}}^2.$$

Thus we get

$$\|\mathbf{f}_h\|_{A^{-1}}^2 \leq 2\|g_h\|_{M^{-1}}^2 + 2\|\mathbf{u}_h\|_A^2 \leq 4\|\sigma_h\|_M^2 + 6\|\mathbf{u}_h\|_A^2.$$

Then the desired inequality (60) follows from the bound of $\|g_h\|_{M^{-1}}$ and $\|\mathbf{f}_h\|_{A^{-1}}$.

To prove $\|(\mathcal{L}_h^c)^{-1}\|_{L(S'_h \times \mathbf{U}'_h, S_h \times \mathbf{U}_h)} \lesssim 1$, we need to prove

$$(61) \quad \|\sigma_h\|_M + \|\mathbf{u}_h\|_A \lesssim \|g_h\|_{M^{-1}} + \|\mathbf{f}_h\|_{A^{-1}}.$$

From (59), we have $\mathbf{u}_h = (A_h^c)^{-1}(f_h + B^T M_v^{-1} g_h)$. Then

$$\begin{aligned} \|\mathbf{u}_h\|_A^2 &= \|\mathbf{f}_h + B^T M_v^{-1} g_h\|_{A^{-1}}^2 \\ &\leq 2\|\mathbf{f}_h\|_{A^{-1}}^2 + 2\|B^T M_v^{-1} g_h\|_{A^{-1}}^2 \leq 2\|\mathbf{f}_h\|_{A^{-1}}^2 + 2\|g_h\|_{M^{-1}}^2. \end{aligned}$$

We also have $\sigma_h = M_v^{-1}(B\mathbf{u}_h - g_h)$ and thus

$$\|\sigma_h\|_M^2 = \|B\mathbf{u}_h - g_h\|_{M^{-1}}^2 \leq 2\|B\mathbf{u}_h\|_{M^{-1}}^2 + 2\|g_h\|_{M^{-1}}^2 \leq 2\|\mathbf{u}_h\|_A^2 + 2\|g_h\|_{M^{-1}}^2.$$

Combining with the bound for $\|\mathbf{u}_h\|_A$, we obtain the desirable stability (61). $\square$

Remark 4.4. By choosing $\mathbf{v}_h = M_e^{-1} B^T \sigma_h$, we can obtain the stability

$$\|B^T \sigma\|_{M^{-1}} \leq \|\mathbf{f}_h\|_{M^{-1}}.$$

From Theorem 4.3, we can conclude that the proposed preconditioners are uniformly bounded with respect to h .

Theorem 4.5. The $\mathcal{P}_h^c$ and $\mathcal{P}_h^d$ are uniform preconditioners for $\mathcal{L}_h^c$ and $\mathcal{L}_h^d$, respectively, i.e., the corresponding operator norms

$$\begin{aligned} \|\mathcal{P}_h^c \mathcal{L}_h^c\|_{L(S_h \times \mathbf{U}_h, S_h \times \mathbf{U}_h)}, \quad & \|(\mathcal{L}_h^c \mathcal{P}_h^c)^{-1}\|_{L(S_h \times \mathbf{U}_h, S_h \times \mathbf{U}_h)}, \\ \|\mathcal{P}_h^d \mathcal{L}_h^d\|_{L(\mathbf{U}_h \times \mathbf{V}_h, \mathbf{U}_h \times \mathbf{V}_h)}, \quad & \|(\mathcal{P}_h^d \mathcal{L}_h^d)^{-1}\|_{L(\mathbf{U}_h \times \mathbf{V}_h, \mathbf{U}_h \times \mathbf{V}_h)} \end{aligned}$$

are bounded and independent with parameter h .

4.2. Mass Lumping. The inverse of the mass matrices M_v^{-1} and M_e^{-1} are in general dense. To be practical, the exact Schur complement can be replaced by an approximation

$$(62) \quad \tilde{A}_h^c = B^T \tilde{M}_v^{-1} B + C^T M_f C,$$

$$(63) \quad \tilde{A}_h^d = C \tilde{M}_e^{-1} C^T + B^T M_t B,$$

with $\tilde{M}_v$ and $\tilde{M}_e$ easy-to-invert matrices, e.g., diagonal or mass lumping of M_v and M_e , respectively. In this way, we actually change the L^2 -inner product into a discrete L^2 inner product. We then define the adjoint operators with respect to the discrete L^2 -inner product. For example, we define $\widetilde{\text{div}}_h \mathbf{w}_h \in S_h$, s.t.,

$$(64) \quad \langle \widetilde{\text{div}}_h \mathbf{w}_h, v_h \rangle_h := -(\mathbf{w}_h, \text{grad } v_h) \quad \text{for all } v_h \in S_h,$$

where $\langle \cdot, \cdot \rangle_h$ is the discrete L^2 -inner product defined by $\tilde{M}_v$.

The operator and matrix formulations of the vector Laplacian $\tilde{\mathcal{L}}_h^c : S_h \times \mathbf{U}_h \rightarrow S_h' \times \mathbf{U}_h'$

$$(65) \quad \tilde{\mathcal{L}}_h^c \begin{pmatrix} \sigma_h \\ \mathbf{u}_h \end{pmatrix} := \begin{pmatrix} -\tilde{M}_v & B \\ B^T & C^T M_f C \end{pmatrix} \begin{pmatrix} \sigma_h \\ \mathbf{u}_h \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \end{pmatrix}.$$

And $\tilde{\mathcal{L}}_h^d : \mathbf{U}_h \times \mathbf{V}_h \rightarrow \mathbf{U}_h' \times \mathbf{V}_h'$

$$(66) \quad \tilde{\mathcal{L}}_h^d \begin{pmatrix} \sigma_h \\ \mathbf{u}_h \end{pmatrix} := \begin{pmatrix} -\tilde{M}_e & C^T \\ C & B^T M_t B \end{pmatrix} \begin{pmatrix} \sigma_h \\ \mathbf{u}_h \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \end{pmatrix}.$$

The associated diagonal preconditioners are

$$(67) \quad \tilde{\mathcal{P}}_h^c = \begin{pmatrix} \tilde{M}_v^{-1} & 0 \\ 0 & (\tilde{A}_h^c)^{-1} \end{pmatrix}$$

and

$$(68) \quad \tilde{\mathcal{P}}_h^d = \begin{pmatrix} \tilde{M}_e^{-1} & 0 \\ 0 & (\tilde{A}_h^d)^{-1} \end{pmatrix}.$$

It is not hard to see that the modification of the L^2 -inner product will not bring any essential difficulty to the proof of the previous results. We can easily reproduce all the results that we have proved in the previous sections with the help of the following proposition whose proof can be found in [11].

Proposition 4.6. *Assume that the discrete L^2 norm is equivalent to the L^2 norm. Then the norm $\|\cdot\|_{\tilde{A}_h^c}$ is equivalent to $\|\cdot\|_{A_h^c}$, and $\|\cdot\|_{\tilde{A}_h^d}$ is equivalent to $\|\cdot\|_{A_h^d}$ i.e.,*

$$(69) \quad \|\mathbf{u}\|_{\tilde{A}_h^c} \lesssim \|\mathbf{u}\|_{A_h^c} \lesssim \|\mathbf{u}\|_{\tilde{A}_h^c} \quad \text{for all } \mathbf{u} \in \mathbf{U}_h;$$

$$(70) \quad \|\mathbf{u}\|_{\tilde{A}_h^d} \lesssim \|\mathbf{u}\|_{A_h^d} \lesssim \|\mathbf{u}\|_{\tilde{A}_h^d} \quad \text{for all } \mathbf{u} \in \mathbf{V}_h.$$

4.3. Triangular Preconditioner. When a diagonal mass matrix is used, we can make use of the block decomposition

$$(71) \quad \begin{pmatrix} -\tilde{M}_v & B \\ B^T & C^T M_f C \end{pmatrix} \begin{pmatrix} I & \tilde{M}_v^{-1} B \\ 0 & I \end{pmatrix} = \begin{pmatrix} -\tilde{M}_v & 0 \\ B^T & \tilde{A}_h^c \end{pmatrix}$$

to obtain a triangular preconditioner.

Definition 4.7. *We define the operator $\mathcal{G}_h^c : S'_h \times \mathbf{U}'_h \rightarrow S_h \times \mathbf{U}_h$*

$$(72) \quad \mathcal{G}_h^c = \begin{pmatrix} I & \tilde{M}_v^{-1} B \\ 0 & I \end{pmatrix} \begin{pmatrix} -\tilde{M}_v & 0 \\ B^T & \tilde{A}_h^c \end{pmatrix}^{-1},$$

and the operator $\mathcal{G}_h^d : \mathbf{U}'_h \times \mathbf{V}'_h \rightarrow \mathbf{U}_h \times \mathbf{V}_h$

$$(73) \quad \mathcal{G}_h^d = \begin{pmatrix} I & \tilde{M}_e^{-1} C^T \\ 0 & I \end{pmatrix} \begin{pmatrix} -\tilde{M}_e & 0 \\ C^T & \tilde{A}_h^d \end{pmatrix}^{-1}.$$

From the definition, it is trivial to verify that $\mathcal{G}_h^c = \tilde{\mathcal{L}}_h^c{}^{-1}$ and $\mathcal{G}_h^d = \tilde{\mathcal{L}}_h^d{}^{-1}$ and thus conclude that the proposed triangular preconditioners are uniform.

Theorem 4.8. *Assume $\tilde{M}$ is spectrally equivalent to M . Then the $\mathcal{G}_h^c$ and $\mathcal{G}_h^d$ are uniform preconditioners for $\mathcal{L}_h^c$ and $\mathcal{L}_h^d$, respectively, i.e., the corresponding operator norms*

$$\begin{aligned} & \|\mathcal{G}_h^c \mathcal{L}_h^c\|_{L(S_h \times \mathbf{U}_h, S_h \times \mathbf{U}_h)}, \|(\mathcal{L}_h^c \mathcal{G}_h^c)^{-1}\|_{L(S_h \times \mathbf{U}_h, S_h \times \mathbf{U}_h)}, \\ & \|\mathcal{G}_h^d \mathcal{L}_h^d\|_{L(\mathbf{U}_h \times \mathbf{V}_h, \mathbf{U}_h \times \mathbf{V}_h)}, \|(\mathcal{G}_h^d \mathcal{L}_h^d)^{-1}\|_{L(\mathbf{U}_h \times \mathbf{V}_h, \mathbf{U}_h \times \mathbf{V}_h)} \end{aligned}$$

are bounded and independent with parameter h .

In both diagonal and triangular preconditioners, to be practical, we do not compute A^{-1} or $\tilde{A}^{-1}$. Instead we apply one and only one V-cycle multigrid for $\tilde{A}^{-1}$.

4.4. Maxwell Equations with Divergence-Free Constraint. We consider a prototype of Maxwell equations with divergence-free constraint

$$\text{curl curl } \mathbf{u} = \mathbf{f}, \quad \text{div } \mathbf{u} = 0, \quad \text{in } \Omega, \quad \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Omega.$$

The solution $\mathbf{u}$ is approximated in the edge element space $\mathbf{U}_h$. The divergence-free constraint can then be understood in the weak sense, i.e., $\text{div}_h \mathbf{u} = 0$. By introducing a Lagrangian multiplier $p \in S_h$, the matrix form is

$$(74) \quad \begin{pmatrix} C^T M_f C & B^T \\ B & O \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ g \end{pmatrix}.$$

We can apply the augmented Lagrangian method [14], by adding $B^T M_v^{-1} B$ to the first equation, to get an equivalent matrix equation

$$(75) \quad \begin{pmatrix} A & B^T \\ B & O \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f} + B^T M_v^{-1} g \\ g \end{pmatrix}.$$

Now the $(1, 1)$ block $A = C^T M_f C + B^T M_v^{-1} B$ in (75) is a discrete vector Laplacian and the whole system (75) is in Stokes type.

We can thus use the following diagonal preconditioner.

Theorem 4.9. *The following block-diagonal matrix*

$$(76) \quad \begin{pmatrix} A^{-1} & 0 \\ 0 & M_v^{-1} \end{pmatrix}$$

is a uniform preconditioner for the regularized Maxwell operator $\begin{pmatrix} A & B^T \\ B & O \end{pmatrix}$.

Proof. It suffices to prove that the Schur complement $S = B A^{-1} B^T$ is spectral equivalent to M_v . The inequality $(Sp, p) \leq (M_v p, p)$ for all $p \in S_h$ has been proved in Lemma 4.2. To prove the inequality in the other way, it suffices to prove the inf-sup condition: there exists a constant β independent of h such that

$$(77) \quad \inf_{p_h \in S_h} \sup_{v_h \in U_h} \frac{\langle B v_h, p_h \rangle}{\|v_h\|_A \|q_h\|} = \beta > 0.$$

Given $p_h \in S_h$, we solve the Poisson equation $\Delta \phi = p_h$ with homogenous Dirichlet boundary condition and let $\mathbf{v} = \text{grad } \phi$. Then $\mathbf{v} \in \mathbf{H}_0(\text{curl})$ and $\text{div } \mathbf{v} = p_h$ holds in L^2 . We define $\mathbf{v}_h = Q_h \mathbf{v}$ where $Q_h : \mathbf{H}_0(\text{curl}) \rightarrow \mathbf{U}_h$ is the L^2 projection. Then $(\text{div}_h \mathbf{v}_h, q_h) = (\mathbf{v}_h, \text{grad } q_h) = (\mathbf{v}, \text{grad } q_h) = -(\text{div } \mathbf{v}, q_h) = (p_h, q_h)$, i.e., $\text{div}_h \mathbf{v}_h = p_h$. To control the norm of $\text{curl } \mathbf{v}_h$, we denote $\mathbf{v}_0$ as the piecewise constant projection of $\mathbf{v}$. Then

$$\|\text{curl } \mathbf{v}_h\| = \|\text{curl } (\mathbf{v}_h - \mathbf{v}_0)\| \lesssim h^{-1} \|\mathbf{v}_h - \mathbf{v}_0\| \leq \|\mathbf{v}\|_1 \lesssim \|p_h\|.$$

In the last step, we have used the H^2 -regularity result.

In summary, given $p_h \in S_h$, we have found a $\mathbf{v}_h \in U_h$ such that $\langle B \mathbf{v}_h, p_h \rangle = \|p_h\|^2$ while $\|\mathbf{v}_h\|_A^2 = \|\text{div}_h \mathbf{v}_h\|^2 + \|\text{curl } \mathbf{v}_h\|^2 \lesssim \|p_h\|^2$. Therefore the inf-sup condition (77) has been proved which implies the inequality $\langle Sp, p \rangle \geq \beta^2 \langle M_v p, p \rangle$. $\square$

To design an efficient triangular preconditioner for (75), we explore the commutator

$$(78) \quad AG = \tilde{G} A_p,$$

where $G = M_e^{-1} B^T$ is the matrix representation of the gradient operator $S_h \rightarrow U_h$, $\tilde{G} = B^T M_v^{-1}$ is another scaled gradient operator, and $A_p = BG$ represents the discrete Laplacian operator $S_h \rightarrow S_h$. The identity (78) is a discrete version of the following identity

$$(79) \quad \Delta \text{grad} = \text{grad } \Delta,$$

where the first Δ is the vector Laplacian operator and the second Δ is the scalar Laplacian, and can be verified by noticing that $CG = \text{curl grad} = 0$.

With (78), we have the following block factorization

$$(80) \quad \begin{pmatrix} A & B^T \\ B & O \end{pmatrix} \begin{pmatrix} I & G \\ O & -M_v^{-1} A_p \end{pmatrix} = \begin{pmatrix} A & O \\ B & A_p \end{pmatrix}.$$

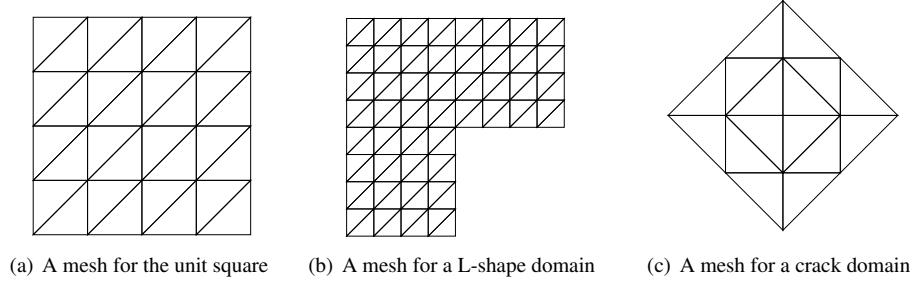


FIGURE 1. Meshes for Example 5.1

TABLE 1. Iteration steps and CPU time of the diagonal and the triangular preconditioners for the vector Laplace equation in $\mathbf{H}_0(\text{curl})$ space: the square domain $(0, 1)^2$.

h	Dof	Iteration (D)	Time	Iteration (T)	Time
1/32	4,225	28	0.20 s	13	0.18s
1/64	16,641	28	0.68 s	14	0.34s
1/128	66,049	27	1.90 s	14	1.30s
1/256	263,169	27	8.80 s	14	6.80s

TABLE 2. Iteration steps and CPU time of the diagonal and the triangular preconditioners for the lowest order discretization of the vector Laplace equation in $\mathbf{H}_0(\text{curl})$ space: the L-shape domain $(-1, 1)^2 \setminus \{[0, 1] \times [-1, 0]\}$.

h	Dof	Iteration (D)	Time	Iteration (T)	Time
1/32	3,201	33	0.24 s	15	0.19s
1/64	12,545	35	0.63 s	16	0.40s
1/128	49,665	39	2.50 s	16	1.90s
1/256	197,633	41	7.20 s	16	5.50s

TABLE 3. Iteration steps and CPU time of the diagonal and the triangular preconditioners for the lowest order discretization of the vector Laplace equation in $\mathbf{H}_0(\text{curl})$ space: the crack domain $\{|x| + |y| < 1\} \setminus \{0 \leq x \leq 1, y = 0\}$.

h	Dof	Iteration (D)	Time	Iteration (T)	Time
1/16	2,145	34	0.13 s	15	0.08 s
1/32	8,385	38	0.54 s	15	0.30 s
1/64	33,153	41	1.60 s	16	1.00 s
1/128	131,841	44	6.70 s	16	3.60 s

When S_h is the linear (P_1) element, M_v^{-1} can be approximated accurately by using the mass lumping of the P_1 element. Therefore we can easily solve (74) by inverting two Laplacian operators: one is a vector Laplacian of the edge element and another is a scalar Laplacian for the P_1 element. In general M_v^{-1} will be replaced by a sparse approximation

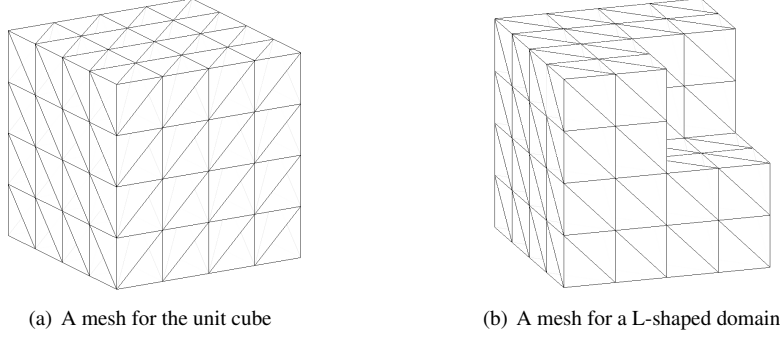


FIGURE 2. Meshes for Example 5.2

TABLE 4. Iteration steps and CPU time of the diagonal and triangular preconditioners for the lowest order discretization of the vector Laplace equation in $\mathbf{H}_0(\text{curl})$ space in three dimensions: the unit cube domain.

h	Dof	Iteration (D)	Time	Iteration (T)	Time
1/4	729	21	0.25 s	12	0.15 s
1/8	4,913	29	0.48 s	16	0.28 s
1/16	35,937	33	3.90 s	18	4.0 s
1/32	274,625	33	40 s	19	27 s

TABLE 5. Iteration steps and CPU time of the diagonal and triangular preconditioners for the lowest order discretization of the vector Laplace equation in $\mathbf{H}_0(\text{curl})$ space in three dimensions: L-shape domain $(-1, 1)^3 \setminus \{(-1, 0) \times (0, 1) \times (0, 1)\}$.

h	Dof	Iteration (D)	Time	Iteration (T)	Time
1/2	665	20	0.03 s	12	0.06 s
1/4	4,401	34	0.54 s	16	0.37 s
1/8	31,841	42	5.50 s	20	3.60 s
1/16	241,857	48	48 s	23	33 s

TABLE 6. Iteration steps and CPU time of the diagonal and triangular preconditioners for the lowest order discretization of Maxwell equations in the saddle point form in three dimensions: the unit cube domain.

h	Dof	Iteration (D)	Time	Iteration (T)	Time
1/4	729	21	0.40 s	12	0.80 s
1/8	4,913	27	1.3 s	16	1.3 s
1/16	35,937	31	4.30 s	18	4.8 s
1/32	274,625	31	40 s	19	39 s

$\tilde{M}_v^{-1}$ and (80) can be used to construct effective block-triangular preconditioners:

$$(81) \quad \begin{pmatrix} I & G \\ O & -\tilde{M}_v^{-1} A_p \end{pmatrix} \begin{pmatrix} \tilde{A} & O \\ B & A_p \end{pmatrix}^{-1}.$$

TABLE 7. Iteration steps and CPU time of the diagonal and triangular preconditioners for the lowest order discretization of Maxwell equations in the saddle point form in three dimensions: L-shape domain $(-1, 1)^3 \setminus \{(-1, 0) \times (0, 1) \times (0, 1)\}$.

h	Dof	Iteration (D)	Time	Iteration (T)	Time
1/2	665	20	0.47 s	10	0.68 s
1/4	4,401	28	0.58 s	14	1.10 s
1/8	31,841	34	5.70 s	17	4.00 s
1/16	241,857	37	40 s	19	38 s

Again in practice, $\tilde{A}^{-1}$ and A_p^{-1} will be replaced by one multigrid V-cycle for the vector Laplacian or scalar Laplacian, respectively.

5. NUMERICAL EXAMPLES

In this section, we will show the efficiency and the robustness of the proposed diagonal and triangular preconditioners. We perform the numerical experiments using the *iFEM* package [9].

Example 5.1 (Two Dimensional Vector Laplacian using Edge Elements). We first consider the mixed system (16) arising from the lowest order discretization of the vector Laplace equation in $\mathbf{H}_0(\text{curl})$ space.

We consider three domains in two dimensions: the unit square $(0, 1)^2$, the L-shape domain $(-1, 1)^2 \setminus \{[0, 1] \times [-1, 0]\}$, and the crack domain $\{|x| + |y| < 1\} \setminus \{0 \leq x \leq 1, y = 0\}$; see Fig. 4.4.

We use the diagonal preconditioner (67) in the MINRES method and the triangular preconditioner (72) in GMRES (with the restart step 20) to solve (16). In these preconditioners, one and only one variable V-cycle is used for approximating $\tilde{A}^{-1}$. In the variable V-cycle, we chose $m_J = 2$ and $m_k = \lceil 1.5^{J-k} m_J \rceil$ for $k = J, \dots, 1$. We stop the Krylov space iteration when the relative residual is less than or equal to 10^{-8} . Iteration steps and CPU time are summarized in Table 1, 2, and 3.

Example 5.2 (Three Dimensional Vector Laplacian using Edge Elements). We then consider the three dimensional case. Still consider the lowest order discretization of the vector Laplace equation in $\mathbf{H}_0(\text{curl})$ space. We use almost the same setting except $m_J = 3$ for which the performance is more robust.

We consider two domains. One is the unit cube $(0, 1)^3$ for which the full regularity assumption holds and another is a L-shape domain $(-1, 1)^3 \setminus \{(-1, 0) \times (0, 1) \times (0, 1)\}$ which violates the full regularity assumption. Iteration steps and CPU time are summarized in Table 4 and 5.

Based on these tables, we present some discussion on our preconditioners.

- (1) Both diagonal and triangular preconditioners perform very well. The triangular one is more robust and efficient.
- (2) The diagonal preconditioner is more sensitive to the elliptic regularity result as the iteration steps are slowly increased, which is more evident in the three dimensional case; see the third column of Table 4 and 5. For general domains, the $\mathbf{H}_0(\text{curl}) \cap \mathbf{H}(\text{div})$ is a strict subspace of $\mathbf{H}^1$ and thus the approximation property may fail. On the other hand, the numerical effectiveness even in the partial regularity cases

is probably due to the fact that the full regularity of elliptic equations always holds in the interior of the domain. Additional smoothing for near boundary region might compensate the loss of full regularity.

- (3) Only the lowest order element is tested while our theory assumes the finite element space should contain the full linear polynomial to ensure the approximation property. This violation may also contribute to the slow increase of the iteration steps. We do not test the second type of edge element due to the complication of the prolongation operators. The lowest order edge element is the most popular edge element. For high order edge elements, we prefer to use the V-cycle for the lowest order element plus additional Gauss-Seidel smoothers in the finest level to construct preconditioners.

Example 5.3 (Three dimensional Maxwell equations with divergent-free constraint). We consider the lowest order discretization of Maxwell equations in the saddle point form (74) and solve the regularized formulation (75). We test the block-diagonal preconditioner (76) and triangular preconditioner (81). We use the same setting as in Example 5.2 and report the iteration steps and corresponding CPU time in Table 6 and 7.

From these results, we conclude our block-diagonal and block-triangular preconditioners works pretty well for the Maxwell equations discretized in the saddle point form. The iteration steps may increase but very slowly. Although the block-triangular preconditioner requires less iteration steps, the computational time is almost the same. This is due to the fact, now for the $(2, 2)$ block, the block-triangular preconditioners requires a V-cycle for the scalar Laplacian while in the block-diagonal preconditioner it is only a diagonal approximation of the mass matrix.

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