Title

STOCAL-II: Computer Assisted Learning System for Stochastic Dyanmic Analysis of Structures Part I - Theory and Development

Authors

Wung, Chih-Dao
Der Kiureghian, Armen

Publication Date

1989-04-01

REPORT NO.
UCB/SEMM-89/10

STRUCTURAL ENGINEERING,
MECHANICS AND MATERIALS

STOCAL-II: COMPUTER-ASSISTED LEARNING SYSTEM FOR STOCHASTIC DYNAMIC ANALYSIS OF STRUCTURES

PART I -- THEORY AND DEVELOPMENT

By
CHIH-DAO WUNG
and
ARMEN DER KIUREGHIAN

APRIL 1989

DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF CALIFORNIA AT BERKELEY
BERKELEY, CALIFORNIA

STOCAL–II
COMPUTER ASSISTED LEARNING SYSTEM FOR STOCHASTIC DYNAMIC ANALYSIS OF STRUCTURES

PART I
THEORY AND DEVELOPMENT

By

Chih-Dao Wung

and

Armen Der Kiureghian

Report No. UCB/SEMM-89/10
Structural Engineering, Mechanics and Materials
Department of Civil Engineering
University of California, Berkeley

April 1989

ABSTRACT

A new formulation of problems in random vibration analysis of linear systems is presented. The formulation facilitates the solution of problems through a process of synthesis. This solution approach is implemented in the instructional software STOCAL–II, which is designed to be used in teaching graduate courses on random vibrations.

In solving random vibration problems with STOCAL–II, the student issues a sequence of commands that carry out the numerical computations required for each basic element of the solution algorithm. The student must have a good understanding of the fundamental concepts in order to choose the proper sequence of commands and the corresponding parameters. The software assists the learning by: (a) freeing the student from tedious computations that are required for the solution but are not necessary for a fundamental understanding of the basic concepts, (b) providing a transparent computing environment where explicit specification of operations is necessary and all intermediate results are made available upon request, (c) facilitating the solution of exercise problems of practical significance that could not be solved by hand, (d) providing the student with an efficient means for parametric study and experimentation, and (e) providing an interactive computing environment with facility for immediate plotting of the results.

STOCAL–II is developed as an extension to CAL. Thus, in addition to nondeterministic analysis, STOCAL-II can perform basic matrix operations and static and dynamic structural analysis. A 2D graphics capability is also implemented.

Commands and algorithms incorporated in STOCAL-II provide the means for random vibration analysis of linear MDOF systems subjected to stationary or nonstationary excitations specified by their power spectral characterizations, or earthquake

excitations specified by their response spectra. Commands for computing the response statistics of engineering interest, such as crossing rates, distributions of local and extreme peaks, and the characteristics of the envelope process are also provided. Additional commands allow the generation of sample functions or the estimation of temporal or ensemble autocorrelation or power spectral density from given sample functions.

A simple example, consisting of a two-story primary structure and an attached, three-node secondary subsystem, is used to illustrate the application and capabilities of STOCAL-II.

ACKNOWLEDGMENTS

The work presented in this report was supported in part by the College of Engineering Dean's Office, and by the University Council on Educational Development.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Introduction

Increasing concern for the safety and economy of structures, and the awareness that most dynamic load environments are random in nature, have made the theory and methods of random vibrations essential for the analysis and design of important structures. Random dynamic loads arise from diverse sources, including seismic ground motions, jet noise, atmospheric turbulence, wind pressure, ocean waves and road unevenness. Because of the high degree of uncertainty of such loads, a deterministic treatment conducted in the usual engineering practice does not produce reliable results. In particular, the practice of introducing a blanket safety factor to account for all uncertainties is not appropriate, since it does not provide a consistent basis for the evaluation and analysis of the underlying uncertainties. Structures designed by such "deterministic" approaches may not have the proper level of safety.

Recent progress in the theory and methods of random vibration analysis, and vast improvements in the speed and power of computers, have facilitated the use of non-deterministic in practice. However, there are very few trained engineers familiar with the subject, and even fewer general-purpose computer codes for their use. As a result, applications of random vibration methods in practice remain limited and rare.

Random vibration as a subject has been taught in mechanical, aerospace, and structural engineering graduate programs of many U.S. universities since 1950s. However, few students interested in the applied aspects of these fields study it. One reason is the abstract nature of the underlying concepts. The student often finds it difficult to visualize a problem or solution, particularly if the subject is taught in a traditional chalk-and-blackboard manner. Another reason is that the subject is both analytically and computationally demanding. Even simple problems may require lengthy and tedious solutions. As a result, many students are discouraged from taking the subject, and others perceive it as too "theoretical" or "abstract" to be useful to their future careers.

It is the premise of this thesis that the effectiveness of teaching the subject of random vibrations can be greatly enhanced through the use of an instructional software operable on a personal computer. Such a software should achieve two main goals: (a) free the student from the routine computations or derivations that are required but are not central to a fundamental understanding of the random vibration theory, and (b) provide a convenient tool for graphic visualization and parametric study in order to help the student in gaining experience and insight. From a pedagogic viewpoint, the software should not act as a "black box", but should require active involvement and decision by the student in solving problems. It is essential that such a software be operable on personal computers for their wide availability and convenience.

The development of such an instructional software, called STOCAL–II (Computer Assisted Learning of STOchastic methods), is the subject of this thesis. This software is an extension of the deterministic structural analysis code CAL developed by Wilson (1979, 1986). It represents a vast improvement over an earlier such program, STOCAL, which was developed at Berkeley by Button et al. (1981). STOCAL–II is useful as an instructional code in a class environment, as well as in an office environment where engineers may use the system not only for self learning, but also to solve practical random vibration problems.

1.2 The Teaching of Random Vibration

The pioneer in teaching random vibrations in the United States has been S. H. Crandall of M.I.T. His first edited book on the subject in 1958 (Crandall et al. 1958) and his second co-authored text in 1963 (Crandall and Mark 1963) greatly influenced the development of this field. The comprehensive text by Y. K. Lin in 1967 (Lin 1967) provided the next turning point. This text presented the first thorough, indepth and systematic treatment of the subject. In spite of vast developments in the field, the text by Lin remains a major source as a textbook and an indispensable reference. After 1960's, the teaching of random vibrations became widespread in graduate programs of U.S. universities. Many new texts have been published on the subject in recent years, including Clough and Penzien (1975), Newland (1984), Bolotin (1984), Nigam (1983), Elishakoff (1983), Piszczek and Niziol (1986), Madsen et al. (1986), and Yang (1986).

Solutions to random vibration problems are usually characterized by long derivations and repeated, tedious calculations. It is often impossible for the student to carry out the required calculations by hand, let alone conducting parametric studies or sensitivity analysis which are so essential in gaining experience and insight. Furthermore, the basic concepts and results of random processes are best explained and understood through the aid of graphical means. However, preparing graphs by hand is virtually impossible for many of the solutions to random vibration problems. From an instructor's viewpoint, one major difficulty in teaching random vibrations is in finding example problems (for demonstration or assignment purposes), which are meaningful from an applied engineering standpoint and yet are manageable in terms of the required effort to solve. In most cases, the instructor has no choice but to use highly idealized and contrived examples which have limited relevance to real, engineering problems. Many of the end-of-chapter problems in the above texts are of this type. For example, problems involving structures with more than one degree of freedom, or problems dealing with nonstationary response are hard to find in these texts. Because of this, the student does not have an opportunity to analyze a realistic random vibration problem on his own, and leaves the course with a lack of self confidence and often the unfortunate perception that the topic of random vibrations is "too theoretical" and not relevant to practical engineering problems.

The obvious way to resolve the above problems is to provide the student with a computer code that carries out the necessary computations and prepares graphs. However, the few available computer codes that have capabilities for random vibration analysis all function as black boxes and do not provide an opportunity for learning and experimentation. Hence the objective of the present thesis is to develop an interactive computational environment for learning the basic concepts of random processes and, in particular, random vibration of structures. The underlying philosophy in developing this software has been to provide a tool which helps the student in computations, but which requires active involvement and decision by the student in every step of solving a problem. This software allows the student to experiment with a problem by, for example, conducting parametric studies or sensitivity analysis. Furthermore, the interactive nature and the on-line graphics capability provide convenient means to review and visualize the results of each analysis. The software allows the instructor to assign exercise problems that are relevant to realistic engineering problems, without worrying about the amount of required computations. It also provides an excellent tool for classroom demonstration of basic concepts and illustrative examples.

Experience at the University of California, Berkeley, has shown that the most effective manner for using this software for instructional purposes is a two-pronged approach where the student is assigned both analytical and numerical exercise problems.

The analytical exercises are small, can be done by hand and are designed to illustrate the basic mathematical concepts and approaches. The numerical exercises, carried out with the help of the instructional software, are designed to illustrate applications of the same concepts to practical engineering problems. Furthermore, frequent numerical examples and analysis with the software give the student a sense of confidence and an interest in exploring possibilities.

1.3 Computer Assisted Instruction

The idea of using computers as a teaching tool, or Computer Assisted Instruction (CAI), has been around for two or three decades. However, personal computers, and their wide availability on campuses, have made CAI a truly effective method. Today, most students entering the universities are well versed in using personal computers, and many possess their own before they graduate. These conditions call for drastic change in the methods of education, and provide opportunities and challenges to the teaching profession ahead.

Software for CAI has been developed for almost every field of education. These include entire systems (e.g., IBM's LOGO; PLATO IV of the University of Illinois) as well as software for specialized subjects. In each field, the approach to CAI can be different, depending on the special needs and requirements of the field. For example, courses in the social sciences emphasize data retrieval techniques, whereas courses in the applied sciences emphasize data computations. Nevertheless, a set of basis rules are common to all CAI systems and software. These are: (1) the system must require active involvement and decision by the student; (2) the system must provide means for easy experimentation by the student; and (3) the system must provide an efficient interactive environment with quick response time and attractive graphics.

The topic of random vibrations involves concepts and techniques from the fields of statistics, probability and structural dynamics. Although there are many CAI systems for statistics and probability, e.g., S (Becker and Chamers 1984), BLSS (Abrahams and Rizzardi 1988), SAS (SAS Inc.), and SPSS (SPSS Inc.), and several for structural dynamic analysis, e.g., ANSWERS (Saul et al. 1972), GIFTS (Kamel and McCabe 1978), CAL (Wilson 1979), FEAP (Taylor 1987), there is very little available that can be used for teaching random vibrations. To the writer's knowledge, the only software that is available is the program STOCAL developed by Button, Der Kiureghian and Wilson at University of California, Berkeley, in 1981 (see Button et al. 1981). This software, which was a precursor of the present development, has limited capabilities (only stationary analysis; restricted choice of response statistics; no graphics) and acts more like a black box than an interactive instructional program. Therefore, for the present study, an entirely new software, denoted STOCAL–II, was developed from scratch. Thus, STOCAL–II should not be regarded as a new version of STOCAL, but an entirely different software. The Roman II extension in the name of this software is used to emphasize this difference.

STOCAL–II is based on and is an extension of the program CAL developed by Wilson (1979, 1986) for structural analysis. STOCAL–II uses the database management system of CAL and has a similar command syntax. Furthermore, CAL commands are available in STOCAL–II for deterministic structural analysis, such as formation of stiffness matrix, eigenvalue analysis, and various matrix manipulations.

STOCAL–II is intended for teaching and research purposes. As a result, it emphasizes the generality of applications and the interactive involvement of the user rather than the efficiency and productivity of computation. It is designed to operate in an interactive mode, although batch operation is also allowed. Generally speaking, the system has been developed based on the philosophy that the solution to a problem can be decomposed into a sequence of logical steps, each step involving a certain amount of routine computations. The student solves the problem in steps by issuing a command for each logical step. The commands are designed such that the student will have to understand the underlying concepts of the theory in order to choose the proper sequence of commands and the parameters for each command. Furthermore, the one-line graphics command, which can be issued at any point in the sequence of commands, provides an excellent means for visualizing intermediate as well as final results of the solution.

1.4 Problems and Methods in Random Vibrations

In the past two decades, the topic of random vibrations has vastly grown in its methods and applications (e.g., reviews by Vanmarcke 1979, Crandall and Zhu 1983, Spanos and Lutes 1986 and Lin et al. 1986). Any instructional software on this subject necessarily will have to limit its scope. The following subsections briefly describe the classes of problems and methods that are included in STOCAL–II. Important classes of problems and methods that are not included in STOCAL–II are briefly outlined in section 1.4.5. It is hoped that these methods will be incorporated in the continuing development of STOCAL–II.

1.4.1 Analysis of Time Series Data

Fourier Transform Analysis

The Fourier transform is one of the most basic tools used in random vibration analysis. For example, the autocorrelation and power spectral density (PSD) functions of a stationary process are a Fourier transform pair (Lin 1967). Also, Fourier transform analysis is often used in data processing for analyzing random signals. In STOCAL–II, the Fourier and inverse Fourier transformations for either a piecewise linear function or a set of discrete data are implemented. For discrete data, the generalized fast Fourier transform algorithm (Cooley and Tukey 1965, and Dahlquist and Bjorck 1974) is used when possible.

Statistical Estimation

A common problem in random vibrations is the analysis of time series data in order to estimate the temporal or ensemble statistics. Commands for such elementary analysis, namely temporal and ensemble estimation of the mean, the autocorrelation function, and the PSD function are included in STOCAL-II. For a more sophisticated analysis of time series data, one of the standard statistical programs mentioned in section 1.3 can be used.

Generation of Sample Functions

Another problems of common interest in random vibration analysis is the generation of sample functions with a specified ensemble statistical characteristics. Such sample functions may be used, for example, in Monte Carlo simulation studies, or in verifying the results of analytical solutions. Methods for artificial generation of sample functions have been used in structural engineering for many years (e.g., see Housner and Jennings 1964, Ruiz and Penzien 1968, Shinozuka and Jan 1972, Samaras et al. 1985, and Yamazaki and Shinozuka 1988). In STOCAL-II, commands for generation of sample functions for both stationary and nonstationary processes are included that make use of well known techniques employing the PSD function.

1.4.2 Random Vibration of Linear Discrete Systems

The bulk of the available literature on random vibrations deals with predicting the response of discrete linear systems to stochastic excitations. This is because the linear model yields satisfactory results for a large class of practical problems, and also because the linear problem lends itself well to analytical solutions. In particular, the Gaussian process is closed under linear transformations, which is convenient since most available results for the statistics of random processes are restricted to the Gaussian process. STOCAL–II presently is restricted to the analysis of linear, discrete systems.

The Normal Mode Approach

The normal mode approach is commonly used in the analysis of linear discrete systems. In this approach, the response of a linear, multi-degree-of-freedom (MDOF) system, which is governed by a set of coupled second-order linear differential equations, is decomposed into a set of modal responses that are represented by uncoupled equations. Usually, a classical damping matrix (Caughey 1960) is assumed to achieve the decoupling. A generic linear response quantity, then, is expressed as the superposition of the modal responses with the associated participation factors as coefficients (see section 2.2.2). STOCAL–II employs this approach. In general, the generic information needed to compute the statistics of the response are the natural frequencies, the modal damping ratios, and the modal effective participation factors of the system for the response quantity of interest, as well as a complete statistical specification of the input excitation.

More generally, an n-degree-of-freedom system (represented by a system of n second-order linear differential equations) can be represented by a set of $2n$ first-order equations (Foss 1956). A general modal approach employing complex eigenvalues and eigenvectors can be used to decouple these equations. This approach is employed when a classical damping matrix cannot be assumed (Igusa et al. 1984). This approach is also employed when a state-vector time-domain formulation is used (Bryson and Ho 1975, DebChaudhury and Gasparini 1982). This formulation is particularly useful when the input is a delta correlated or a filtered, delta correlated process, in which case the solution of the first-order equations constitutes a Markov process. This approach is not implemented in the current version of STOCAL–II.

Stationary Response

The stationary response of a linear system is of interest because some input excitations, such as those induced by ocean waves, jet noise or road unevenness, are appropriately described by stationary models. Furthermore, even though some excitations such as an earthquake ground motion are distinctly nonstationary, a stationary or "quasi-stationary" model may still be used because of its analytical simplicity and also because it may produce an acceptable approximation under certain conditions.

The stationary analysis of linear systems was well developed by statisticians and communication engineers (Rice 1944, 1945) before the theory of random vibrations was applied to structural engineering problems in the 1950s (Liepmann 1952, Miles 1954, Goodman et al. 1955). In structural engineering, a white-noise or banded white-noise model was first used to describe random dynamic loadings (Bycrot 1960, Crandall and Mark 1963). Since then, other models have been proposed to better represent the random loading function in different applications. For example, the Kanai-Tajimi model (Kanai 1957, Tajimi 1960), also known as a filtered white noise process, is widely accepted in modeling earthquake excitations. For each of these models, the response statistics of a linear system are obtained by direct integration or by numerical means.

In general, the statistics of the stationary response that are of most interest are the auto and cross-correlation functions, the auto and cross-PSD functions, and the spectral moments. Other statistics of practical interest, such as the distribution of local peaks or extreme response (Longuet-Higgins 1952, Vanmarcke 1975), are computed in terms of these basic quantities. STOCAL–II is able to compute these statistics for the stationary response of a linear MDOF system, which is subjected to a white noise, a filtered white noise, or a banded linear-noise excitation, or general stationary input described by a piecewise-linear-PSD function.

Nonstationary Response

Nonstationary analysis is required when a structural system is subjected to a nonstationary or a transient stationary excitation. Early works on this subject employed a direct time domain approach. However, in order to simplify the analytical expressions, the input excitations considered were limited to two special classes of nonstationary processes: shot-noise processes (Caughey and Stump 1961, Lin 1963b), and uniformly modulated processes with an exponentially decaying harmonic correlation function (Bucciarelli and Kuo 1970, Hasselman 1972, Sakata and Kimura 1979). Because of the complexity of the solutions, therefore, investigators turned their attention to the frequency domain approach.

The frequency domain approach did not gain popularity until Priestley (1965) proposed the evolutionary spectrum to describe a general class of nonstationary processes. In addition to the evolutionary spectrum, three other spectra have been proposed for describing nonstationary processes: the instantaneous spectrum (Page 1952), the generalized spectrum (Bendat and Piersol 1962, Lin 1967, Barnoski and Maurer 1969), and the physical spectrum (Mark 1970). Among these, only Priestley's definition provides a convenient input-output relation for dynamic systems, and hence it enjoys the wide popularity among the researchers in the field (e.g., Hammond 1968, Shinozuka 1970, Corotis and Vanmarcke 1975, Lin and Yong 1987). The general formulation for the response to an evolutionary input was given by Hammond (1968). Due to the computational complexity involved in the evaluation of the evolutionary PSD, available studies involving this model are limited to cases where the input is a uniformly modulated process (i.e., the modulating function is only a function of time). In particular, Roberts (1971) obtained approximate expressions for the response to a non-white input modulated by a periodic intensity function; Corotis and Marshall (1977) obtained the exact evolutionary PSD and approximate variance functions of the response of a single-degree-of-freedom (SDOF) oscillator under a uniformly modulated input with an exponential modulating function; and To (1982, 1984a, 1986) obtained the evolutionary PSD and covariance functions of the response of an MDOF system (or their derivatives) under a uniformly modulated input with an exponential time modulating function.

Several recent studies have turned to the state-vector, time-domain approach introduced by Wang and Uhlenbeck (1945). This approach is numerically simple when the input is a shot-noise process (Masri 1978, Gasparini 1979) or a filtered shot-noise process (Gasparini and DebChaudhury 1980, DebChaudhury and Gasparini 1982). Under these inputs, the response is a Markov process and several powerful methods based on the Fokker-Planck equation can be used to obtain the response statistics.

In STOCAL–II, the frequency-domain approach is used where the input excitation is defined in terms of Priestley's evolutionary spectrum with the modulating function specified as a general piecewise linear function of time and frequency (see sections 2.4 and 3.3). Thus, STOCAL–II is capable of analyzing rather general classes of nonstationary problems. The output response quantities in STOCAL–II include the evolutionary PSD and the variance/covariance functions of specified responses and their time derivatives. The derivative responses are of interest because they are needed for evaluating the crossing statistics.

1.4.3 Statistics of Response Process

The final objective in random vibration analysis is to assess the reliability of the structural system against the applied stochastic loading. Two types of failures are usually considered in such analysis. One is first-excursion failures, where the structure is assumed to have failed when a critical response exceeds a specified threshold for the first time. The second is fatigue-type failure, where the structure fails under repeated applications of load cycles due to material or strength deterioration. The response statistics that are necessary in order to compute probabilities for these failure events include, in addition to mean and autocorrelation functions, the rates of threshold crossings, distributions of local and extreme peaks, and the characteristics of the envelope process. Analytical solutions for most of these statistics are available only for Gaussian processes. STOCAL–II includes commands for computing these statistics for stationary and nonstationary Gaussian processes.

For first-passage failure, the distribution of the time to the first crossing (the first-passage time) is of interest. If the level crossings of the process are assumed to constitute Poisson events, the distribution of the first passage time becomes exponential (Rice 1944, 1945). A more accurate result for a narrow-band process was obtained by Vanmarcke (1975) by considering the qualified level crossings of the envelope process. This distribution is included in STOCAL–II. More elaborate approximations, primarily based on the Markov model (Lutes and Tzuang 1983, Madsen and Krenk 1984, Toro and Cornell 1986, Nielsen and Sorensen 1988), are also available which are not included in STOCAL–II.

The mechanism for the fatigue-type failure is more complicated. The fatigue-type failure is assumed to occur when the accumulation of fractional damages from repeated response cycles reaches a specified threshold (Kinra and Marshall 1979). Thus, the distribution, or at least the mean and variance, of the accumulated damage in a given time period are of interest. These are estimated in terms of the distribution of local peaks in the random stress history (Crandall and Mark 1963). The distribution of local peaks for a Gaussian process has been decided by Rice (1944, 1945), Longuet-Higgins (1952), Cartwright and Longuet-Higgins (1956).

1.4.4 The Response Spectrum Method

The response spectrum method is a widely used technique for determining the maximum response of linear structures to earthquake excitations. In this method, the ground motion is specified in terms of the response spectrum, which is a plot of the mean maximum response of a linear oscillator to the specified base motion, expressed as a function of the oscillator frequency and damping ratio. The maximum response in each mode of the structure is determined in terms of the response spectrum amplitude at the modal frequency. The maximum response of the structure is obtained approximately by combing the maximum modal responses.

Several rules are available for combining the modal responses. Two traditional methods are the absolute sum (Biot 1943), which always yield conservative results, and the square-root-of-sum-of-squares (SRSS) method (Goodman et al. 1955), which is based on the assumption that the modal responses are statistically independent. More recently, several combination rules have been developed based on probabilistic methods that include the effect of modal correlation (e.g., Rosenblueth and Elorduy 1969, Singh and Chu 1976, Der Kiureghian 1981). In particular, the method developed by Der Kiureghian (1981) provides a basis for complete random vibration analysis in terms of the response spectrum. This method is implemented in STOCAL–II. In its original form, the method by Der Kiureghian employed a white-noise approximation of the input excitation in computing the modal correlations. In STOCAL–II, the method is extended to allow computation of the modal correlations based on various non-white models of the input excitation.

1.4.5 Beyond STOCAL–II

The scope of STOCAL–II is necessarily limited. In this section, problem areas in random vibrations which are not incorporated in STOCAL–II are reviewed. Some of these problems may be implemented in the future development of the software; others are not included because of their limited use in structural engineering. It is believed that the present version of STOCAL–II is appropriate and entirely adequate for use in a first graduate course on random vibrations.

Continuous Systems

Most real-world structures are continuous-mass systems. Thus, a continuous model used to describe such systems is more realistic. An in-depth description of random vibration analysis of continuous systems is given by Lin (1967), Nigam (1983), and Bolotin (1984). However, a continuous model is not convenient for use on digital computers. As a result, in application, a continuous system is usually described by a discrete model by using finite difference or finite element method. Although, STOCAL–II is not designed to solve for continuous systems, it can analyze any discrete model of such systems provided the mass, damping and stiffness matrices are given. Furthermore, STOCAL–II may be used in analyzing continuous systems by the normal mode approach (Lin 1963a, Su and Ahmadi 1988), provided the modal characteristics of the system are given as input. On the other hand, STOCAL–II can not be used in analyzing continuous systems by the influence function approach (see Nigam 1983, section 9.2.3).

Nonlinear Systems

As mentioned earlier, STOCAL–II presently is restricted to linear problems. However, nonlinear problems are becoming of increasing importance, particularly, in studies aimed at determining the safety of structures. It is envisioned that at least some methods from nonlinear random vibrations will be incorporated in the future development of STOCAL–II. Towards that end, this section presents a brief overview of problems and methods in nonlinear random vibrations. More extensive reviews are given by Roberts (1981, 1984), Crandall and Zhu (1983), Spanos and Lutes (1986) and To (1984b, 1987) among others. Nonlinearity in a structure may arise from different sources such as material properties, the nature of damping, large deformations, or nonlinear coupling between the terms in the equation of motion. Mathematical models for nonlinear dynamic systems are represented by nonlinear differential equations. An analytical solution of such an equation is usually difficult, or it is restricted to very specific cases. Although there is no unified method for the solution of general nonlinear problems, five methods are popularly used to obtain approximate solutions. These are the Monte Carlo simulation method, the equivalent linearization method, the Gaussian closure method, the perturbation method, and methods using the Fokker-Planck equation. These are briefly described in the following paragraphs.

The Monte Carlo simulation method usually consists of three steps: (1) generate a large number of sample excitations, (2) compute the corresponding response functions by deterministic nonlinear methods, and (3) process the response functions to obtain the desired statistics. The method is simple and general, and it can be applied to any problem, which has a deterministic solution. This method, however, involves tremendous amount of computations and is often prohibitive even with the most powerful computers today. The Monte Carlo simulation method is usually employed to examine the accuracy of approximate methods in bench mark studies (Yar and Hammond 1986, Iyengar 1988).

In the statistical or equivalent linearization technique, the solution of a nonlinear system is approximated by the solution of an equivalent linear system, where the equivalent system is often obtained by minimizing some measures of the difference between the nonlinear and equivalent linear equations. The method was introduced by Krylov and Bogoliubov in 1931 and was extended to nonlinear random vibration analysis by Booton (1954) and by Caughey (1959). The features and advantages of the method were widely reviewed by Roberts (1981), by Spanos (1981) and by Lin et al. (1986). The method is popularly used for dealing with hysteretic systems (Wen 1980, 1986, Baber 1984, Wu 1987) and can be easily extended to nonlinear MDOF systems (Casciati and Faravelli 1988). Bruckner and Lin (1987) recently generalized the equivalent linearization method to the study of a nonlinear system subject to random parametric excitations.

The Gaussian closure techniques (Kraichan 1962) are used when the excitation and the response of a nonlinear system can be approximated as jointly Gaussian. In this method, a closure technique converts the nonlinear system into an associated linear system with Gaussian excitation (Iyengar and Dash 1978). The response of this linear system is then assumed to be an approximation to the response of the original nonlinear system. Wu (1987) recently showed that the Gaussian closure technique and the equivalent linearization method often lead to identical or similar results, depending on the specific equivalent linearization technique used. Similar closure techniques for non-Gaussian problems have been used by Crandall (1980) and Noori et al. (1987).

The perturbation method (Erdelyi 1956) is often applied to problems where Ωthe amount of nonlinearity in the system is controlled by a small parameter. In this method, the solution is expanded in a power series with respect to this small parameter and then solved according to the first few terms in the series. Crandall (1963b) has applied the classical perturbation theory to determine the approximate response statistics for nonlinear systems under random excitations. A recent application of this method to wave loading was presented by Lipsett (1986).

Methods using the Fokker-Planck equation (Fokker 1914, Planck 1917, Kolmogorov 1931) are most powerful when used for determining the probability structure of a Markovian response process. The response of a linear structural system is a Markov process when the excitation is a delta correlated process. This method is also applicable to nonlinear problems. However, exact solutions are available only to a special class of nonlinear problems (Dimentberg 1982, Yonk and Lin 1987). In order to obtain approximate solutions of the Fokker-Planck equation, a number of techniques have been proposed. These include the stochastic averaging method (see an overreview by Zhu 1988), the iterative method (Caughey 1971, Orabi and Ahmadi 1987b), the series expansion method (Yar and Hammond 1986, Orabi and Ahmadi 1987a), the Galerkin method (Wen 1975, 1976), moment closure methods (Sun and Hsu 1987), and the detailed balance method (Langley 1988).

Parametric Excitation and Uncertain Systems

The parametric excitation means the excitation of a vibratory system due to the time variation of its inertia, damping, or stiffness parameters. When some of these parameters are random in nature, the system is said to have random parametric excitation. The study of the random parametric excitation can be traced back to 1959 by Samuels and Eringen. A review of parametric excitations is given by Ibrahim and Roberts (1978, 1981), and more recently in a thorough monograph by Ibrahim (1985). Most early investigations (even for deterministic analysis) are directed toward delineating the boundaries between stable and unstable response without attempting to characterize the response in any further detail. However, in recent studies of this problem, attempts have been made to obtain the response statistics of linear or nonlinear systems (Benaroya and Rehak 1987, Yong and Lin 1987).

Non-Gaussian Excitation

Most random processes studied in random vibrations are Gaussian processes. This is because large classes of processes resulting from natural phenomena are indeed Gaussian, or can be closely approximated as Gaussian processes, and also because the probabilistic analysis of a Gaussian process is much simple. In some situations, however, non-Gaussian models are needed to study problems, where non-Gaussian characteristics are evident. For example, Ochi (1986) has shown that wave heights recorded in a sea of finite water depth have a skewed non-Gaussian nature. Thus, non-Gaussian models and analysis methods have gained more attention in recent years. Grigorius (1984) has obtained an approximate solution for the crossing rates of a class of non-Gaussian processes; he has also (1986) obtained the response to quadratic Gaussian excitations. Lutes and Hu (1986) have obtained the non-Gaussian response characterized by its first fourth moments, and Yamazaki and Shinozuka (1988) have simulated sample fields of multi-dimensional non-Gaussian functions.

1.5 Objectives and Organization of the Study

The primary purposes of this study are (a) to present a synthesis of solutions to various random vibration problems as stated in Section 1.4 and then (b) to implement the solutions in STOCAL–II such that STOCAL–II unified with CAL can be used as an efficient teaching and research tool for random vibration analysis.

The needs of developing an instructional software are clearly stated in Sections 1.2 and 1.3. The problems and methods, developed in this study and implemented in STOCAL–II, are clearly described in Section 1.4 where the limitations of this study are also stated in details. In general, the response of a linear elastic MDOF system subject to a uni-source stochastic loading is of interest; on the other hand, the nonlinear response, the non-classical damping, and the multi-support input are not considered.

This study is divided into six chapters. All of the analytical results needed for STOCAL–II implementation are derived in Chapter 2. The results required lengthy numerical computations are put in Chapter 3. The development and description of overall features in STOCAL-II are stated in Chapter 4. In Chapter 5, several examples are used to illustrate how to perform the nondeterministic analysis of a linear MDOF structure by using STOCAL-II. Finally, an overview of the work is summarized in the last chapter. Additional details including the STOCAL–II Command Summary are also presented in appendices. Detailed descriptions of the individual STOCAL–II commands can be found in the companion report (Wung and Der Kiureghian 1989).

CHAPTER 2

FORMULATION AND SYNTHESIS OF PROBLEMS IN RANDOM VIBRATIONS

2.1 Introduction

The objectives in developing an instructional software are very different from the objectives in developing a software for conventional engineering applications. Whereas for conventional applications the user-friendliness and efficiency of a software are top priorities, in an instructional software the direct involvement of the user in solving a problem, the transparency of the solution algorithm (versus a black box approach), and the ability to obtain intermediate results and carry out parametric studies are prime qualities. A careful planning is necessary not only in the development of the instructional software, but also in the formulation of problems and solutions that are implemented in the software.

An instructional software should not perform as a black box. It should require the active involvement of the user in solving each problem. In STOCAL-II as in CAL, the fundamental principle has been to decompose a problem into its most basic (or generic) elements and provide simple commands that carry the necessary computations for each basic element. The user solves a problem through a synthesis process by issuing a sequence of commands that solve and combine the required basic elements. Thus, it is necessary that the user understand the basic elements of the problem and the solution algorithm in order to provide the correct ingredients and command sequence.

For example, in order to obtain the response of an MDOF system to a uniformly modulated input, the user has to understand that the basic elements are the modal effective participation factors and the modal cross-correlation functions for the specific input. The basic elements for computing the modal effective participation factors are the mode shapes, the response transfer matrix, and the influence vector of the input loading. These matrices are computed by using elemental commands. The basic ingredients for computing the modal cross-correlations are the modal frequencies and damping ratios and the PSD and modulating function of the input excitation. These are computed by a separate command. A further command is then used to superimpose the modal responses which are scaled by the corresponding effective participation factors. Through this process the student learns, for example, that the modal responses are generic to all different responses of the structure. That is, if a different response is to be computed, the student only needs to compute the new set of modal effective participation factors and then only issue the modal superposition command. No recomputation of the modal cross-correlations (which is the most time-consuming part of the solution) is necessary. Throughout the solution process, the student is able to examine the intermediate results, e.g., the modal effective participation factors for a selected response quantity, the cross-correlation between two selected modes.

The objective of this chapter is to formulate selected problems in random vibrations in a format that clearly identifies the most basic elements and allows their solutions through a process of synthesis. Although no major new results are derived, existing solutions are presented in a new and unified format that facilitates their implementation in the instructional software. As indicated in the introduction, the scope of the study and the software is limited to the analysis of linear, MDOF systems through the normal mode approach. However, extensive capabilities for both stationary and nonstationary analysis are provided. The formulation starts in Section 2.2 from the dynamic analysis of a SDOF oscillator and progresses to the analysis of MDOF systems using the normal mode approach. Both time-domain and frequency-domain analysis are discussed. For stationary inputs, several standard PSD's are introduced, which are incorporated in the library of STOCAL-II. These include PSD's for white noise and filtered white noise processes and a banded, linear PSD. In addition, the user may specify a general PSD as a piecewise linear function. For nonstationary analysis, an evolutionary process is provided which is specified through a PSD function and a modulating function of time or time and frequency. These are presented in Section 2.3. Section 2.4 includes the analysis of the response to stochastic input, where it is shown that the generic element for random vibration analysis of MDOF systems is the cross-correlation or cross-PSD function of the $m1$-th and $m2$-th derivatives of two modal responses. Section 2.5 summarizes formulas for well known statistics of random processes that are of engineering interest, including crossing rates and distributions of local and extreme peaks. The final section of this chapter describes the response spectrum method for random vibrations developed by Der Kiureghian (1981).

Throughout this chapter, indented paragraphs are used to give short descriptions of CAL or STOCAL-II commands that are relevant to the topic under discussion. More detailed descriptions are given in the companion report (Wung and Der Kiureghian 1989).

2.2 Dynamic Response of Linear Systems

2.2.1 The Single-Degree-of-Freedom Oscillator

In this section, the response of a SDOF linear oscillator is reviewed. Characterization of the oscillator in terms of unit impulse and frequency response functions is described, and expressions for the response to general loading are given.

<u>The Equation of Motion</u>

The equation of motion of the oscillator, when it is excited by an externally applied dynamic load, $p(t)$, is (Fig. 2.1a)

$$m\,\ddot{s}(t) \;+\; c\,\dot{s}(t) \;+\; k\,s(t) \;=\; p(t) \tag{2.1}$$

where m, c and k are the mass, damping and stiffness, respectively, and $s(t)$ denotes the displacement. If the oscillator is excited by base motion (Fig. 2.1b), the equation of motion is

$$m\,\ddot{s}(t) \;+\; c\,\dot{s}(t) \;+\; k\,s(t) \;=\; -\,m\,\ddot{x}_g(t) \tag{2.2}$$

in which $\ddot{x}_g(t)$ represents the acceleration of the base and $s(t)$ now denotes the relative displacement with respect to the moving base. Note that Eqs. 2.1 and 2.2 are identical if $-m\ddot{x}_g(t)$, an inertia force, is treated as $p(t)$, an external force.

Introducing the notations

$$\omega_o = \sqrt{\frac{k}{m}} \tag{2.3}$$

$$\varsigma_o = \frac{c}{2\sqrt{mk}} \tag{2.4}$$

both Eqs. 2.1 and 2.2 can be written in the general form

$$\ddot{s}(t) \;+\; 2\varsigma_o\omega_o\dot{s}(t) \;+\; \omega_o^2\,s(t) \;=\; f(t) \tag{2.5}$$

where $f(t) = p(t)/m$ for the external loading and $f(t) = -\ddot{x}_g(t)$ for the base motion. The parameters ω_o and ς_o respectively represent the natural frequency and damping ratio of the oscillator. In structural engineering applications usually ς_o is greater than zero, but much smaller than unity.

In the following sections, the general form of the equation of motion in Eq. 2.5 is considered. Solutions for the particular cases are obtained by specifying the proper form of $f(t)$.

Unit Impulse and Frequency Response Functions

The unit impulse response function, $h(t)$, is defined as the response of the oscillator, starting from at rest conditions, to a unit impulse loading applied at t=0; i.e., to $f(t) = \delta(t)$, where $\delta(t)$ is the Dirac delta function defined by

$$\delta(t) \;=\; \begin{cases} \infty & t=0 \\ 0 & t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)dt \;=\; 1 \tag{2.6}$$

A system is said to be stable if its unit impulse response function approaches zero as time approaches infinity. The oscillator under consideration is stable for all positive values of ς_o. For $0 < \varsigma_o < 1$, the solution of Eq. 2.5 for the unit impulse loading yields (Lin 1967)

$$h(t) \;=\; U(t)\;\frac{1}{\omega_d}\;e^{-\varsigma_o\omega_o t} sin(\omega_d t) \tag{2.7}$$

where $\omega_d = \omega_o\sqrt{1-\varsigma_o^2}$ is the damped frequency and $U(t)$ is the Heaviside's step function

$$U(t) \;=\; \begin{cases} 1 & t>0 \\ 0 & t \leq 0 \end{cases} \tag{2.8}$$

The frequency response function, $H(\omega)$, of a stable system is defined as the amplitude of the steady-state response to a harmonic input, $f(t) = e^{i\omega t}$. For the SDOF oscillator defined by Eq. 2.5, provided $0 < \varsigma_o$,

$$H(\omega) \;=\; \frac{1}{\omega_o^2 - \omega^2 + 2i\varsigma_o\omega_o\omega} \;=\; \frac{-1}{(\omega - z_1)(\omega - z_2)} \tag{2.9}$$

where

$$z_{1,2} \;=\; \pm\omega_d + i\varsigma_o\omega_o \tag{2.10}$$

It can be shown (Lin 1967, p114) that the unit impulse and frequency response functions are Fourier transform pairs, i.e.,

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt \tag{2.11}$$

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i\omega t}d\omega \tag{2.12}$$

This also implies that the frequency response function exists only if the system is stable.

<u>Response to General Loading</u>

The response of the SDOF linear oscillator to a general loading can be determined by the use of either the unit impulse or the frequency response function. For a general loading function, $f(t)$, applying the principle of superposition for linear systems, one has

$$s(t) = s(0)g(t) + \dot{s}(0)h(t) + \int_0^t f(\tau)h(t-\tau)d\tau \tag{2.13}$$

in which $s(0)$ and $\dot{s}(0)$ are the initial displacement and velocity of the oscillator, respectively, and $g(t)$ is the response of the oscillator to an initial unit displacement. For the oscillator defined by Eq. 2.5 with $0 < \varsigma_o < 1$,

$$g(t) = U(t)\ e^{-\varsigma_o\omega_o t}[cos(\omega_d t) + \frac{\varsigma_o\omega_o}{\omega_d}sin(\omega_d t)] \tag{2.14}$$

If the steady-state response of the oscillator is of interest, the loading can be regarded to have begun in the infinite past. The steady-state solution, then, is

$$s_{steady}(t) = \int_{-\infty}^{t} f(\tau)h(t-\tau)d\tau \tag{2.15}$$

The solution to a general loading $f(t)$ employing the frequency response function, $H(\omega)$, requires the Fourier transform of the loading

$$\bar{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}d\tau \tag{2.16}$$

Provide $\bar{f}(\omega)$ exists, the solution for the oscillator defined by Eq. 2.5 with $0 < \varsigma_o < 1$ is given by

$$s(t) = e^{-\varsigma_o\omega_o t}[Acos(\omega_d t) + Bsin(\omega_d t)] + \int_{-\infty}^{\infty} H(\omega)\bar{f}(\omega)e^{i\omega t}d\omega \tag{2.17}$$

where A and B are constants dependent on both the initial conditions of the oscillator and the Fourier transform of the loading, which are

$$s(0) = A + \int_{-\infty}^{\infty} H(\omega)\bar{f}(\omega)d\omega \tag{2.18}$$

$$\dot{s}(0) = -\varsigma_o\omega_o A + \omega_d B + i\int_{-\infty}^{\infty} \omega H(\omega)\bar{f}(\omega)d\omega \tag{2.19}$$

The frequency-domain solution is particularly useful in determining the steady-state response of the oscillator, which is given by

$$s_{steady}(t) = \int_{-\infty}^{\infty} H(\omega)\bar{f}(\omega)e^{i\omega t}d\omega \tag{2.20}$$

Note that for any given loading Eqs. 2.15 and 2.20 should yield identical results.

2.2.2 The Multi-Degree-of-Freedom System

In this section, the response of an n-degree-of-freedom linear system is reviewed. The static condensation method is used to reduce the equations of motion when the formulation involves unwanted degrees of freedom. The modal decomposition method is used to decouple the equations of motion, and the response to general loading is given by modal superposition.

The Equations of Motion

The equations of motion, when the system is excited by an externally applied load vector $\mathbf{F}(t)$, are

$$\mathbf{M}\,\ddot{\mathbf{X}}(t) + \mathbf{C}\,\dot{\mathbf{X}}(t) + \mathbf{K}\,\mathbf{X}(t) = \mathbf{F}(t) \tag{2.21}$$

where $\mathbf{X}(t)$ denotes the nodal displacement vector with n components and $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the n x n mass, damping and stiffness matrices, respectively. Here, we consider the special case where the load vector is expressed in the form,

$$\mathbf{F}(t) = \mathbf{P}\,f(t) \tag{2.22}$$

where $\mathbf{P}$ is a constant vector of nodal load coefficients and $f(t)$ is a time function. When $\mathbf{F}(t)$ involves more than one time function, the principle of superposition is applied to the responses to the individual time functions to obtain the total response.

The above formulation is also applicable to the case of system response to base motion. In that case, $\mathbf{X}(t)$ denotes displacements relative to the base, $f(t)$ equals the base acceleration, $\ddot{x}_g(t)$, and $\mathbf{P} = -\mathbf{MR}$, where $\mathbf{R}$ is the influence vector relating the base motion to the degrees of freedom of the system. This last vector equals the displacement vector of the system for a unit static displacement of the base in the direction of the base motion.

> *The CAL commands TRUSS, SLOPE, FRAME and FRAME3 form local stiffness matrices for truss or frame members of a structure. The CAL command ADDK is used to construct the global stiffness matrix* $\mathbf{K}$ *from these local stiffness matrices. The diagonal mass matrix* $\mathbf{M}$ *and influence vector* $\mathbf{P}$ *are specified by use of the CAL command LOAD.*

Static Condensation

The equations of motion may be formulated incorporating both translational and rotational degrees of freedom. However, the dynamic responses at rotational degrees of freedom are usually dependent on the responses at translational degrees of freedom when the elementary lumped-mass approach is used and no mass moment of inertia or external loading is present at the rotational degrees of freedom. In such cases, the static condensation method is needed to eliminate the rotational degrees of freedom from the equations of motion.

Assume the translational and rotational degrees of freedom be segregated such that Eq. 2.21 without considering the damping matrix is written in the partitioned form

$$\begin{pmatrix} \mathbf{M}_{tt} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \ddot{\mathbf{X}}_t \\ \ddot{\mathbf{X}}_\theta \end{pmatrix} + \begin{pmatrix} \mathbf{K}_{tt} & \mathbf{K}_{t\theta} \\ \mathbf{K}_{\theta t} & \mathbf{K}_{\theta\theta} \end{pmatrix} \begin{pmatrix} \mathbf{X}_t \\ \mathbf{X}_\theta \end{pmatrix} = \begin{pmatrix} \mathbf{F}_t \\ \mathbf{0} \end{pmatrix} \tag{2.23}$$

where the subscript t denotes the translational degrees of freedom and the subscript θ denotes the rotational degrees of freedom. From the second submatrix equation of Eq. 2.23,

$$\mathbf{X}_\theta = -\mathbf{K}_{\theta\theta}^{-1}\mathbf{K}_{\theta t}\mathbf{X}_t \tag{2.24}$$

Substituting this result into the first submatrix equation leads to

$$\mathbf{M}_{tt}\ddot{\mathbf{X}}_t + \mathbf{K}_r\mathbf{X}_t = \mathbf{F}_t \tag{2.25}$$

where

$$\mathbf{K}_r = \mathbf{K}_{tt} - \mathbf{K}_{t\theta}\mathbf{K}_{\theta\theta}^{-1}\mathbf{K}_{\theta t} \tag{2.26}$$

Note that Eq. 2.25 and Eq. 2.21 are identical with the exception of the damping effect if $\mathbf{M}_{tt}$, $\mathbf{K}_r$, $\mathbf{F}_t$ and $\mathbf{X}_t$ in Eq. 2.25 are treated as $\mathbf{M}$, $\mathbf{K}$, $\mathbf{F}$ and $\mathbf{X}$ in Eq. 2.21, respectively. The effect of damping is included in terms of modal damping ratios, as described in the following section.

> *The CAL command REDUCE performs static condensation to eliminate the unwanted degrees of freedom from the total stiffness matrix* $\mathbf{K}$.

In the following sections, the general form of the equations of motion in Eq. 2.21 is considered.

Modal Decomposition

In the absence of damping and the external force, Eq. 2.21 reduces to

$$\mathbf{M}\,\ddot{\mathbf{X}}(t) + \mathbf{K}\,\mathbf{X}(t) = \mathbf{0} \tag{2.27}$$

The mode shapes, $\mathbf{\Phi}_i$, and natural frequencies, ω_i, $i = 1, 2, ..., n$, of the system are the solutions of the preceding equation in the form

$$\mathbf{X}(t) = \mathbf{\Phi}_i \; sin(\omega_i t + \theta) \tag{2.28}$$

in which θ denotes an arbitrary phase angle. By substituting this relation in Eq. 2.27, the following eigenproblem is obtained:

$$[\; \mathbf{K} \; - \; \omega_i^2 \; \mathbf{M} \;] \; \mathbf{\Phi}_i \; = \; \mathbf{0} \qquad i = 1, 2, ..., n \tag{2.29}$$

The n solutions $\mathbf{\Phi}_i$ are orthogonal with respect to the positive-definite matrices $\mathbf{K}$ and $\mathbf{M}$. In particular, by proper normalization of the mode shapes,

$$\mathbf{\Phi}_i^T \; \mathbf{M} \; \mathbf{\Phi}_j \; = \; \delta_{ij} \tag{2.30}$$

$$\mathbf{\Phi}_i^T \; \mathbf{K} \; \mathbf{\Phi}_j \; = \; \omega_i^2 \; \delta_{ij} \tag{2.31}$$

in which δ_{ij} is the Kronecker delta

$$\delta_{ij} \; = \; \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{2.32}$$

The system is said to have proportional (or classical) damping when the undamped mode shapes are also orthogonal with respect to the damping matrix $\mathbf{C}$. In that case, modal damping ratios ς_i are defined by

$$\mathbf{\Phi}_i^T \; \mathbf{C} \; \mathbf{\Phi}_j \; = \; 2 \; \omega_i \; \varsigma_i \; \delta_{ij} \tag{2.33}$$

Throughout this study proportional damping will be assumed.

The CAL command EIGEN solves for the matrix of mode shapes $\mathbf{\Phi} = [\mathbf{\Phi}_1, \mathbf{\Phi}_2, ... \mathbf{\Phi}_n]$ and the vector of squared undamped frequencies $[\omega_1^2, \omega_2^2, ..., \omega_n^2]$ when the stiffness matrix **K** *and the diagonal mass matrix* **M** *are provided.*

The orthogonality relations in Eqs. 2.30-33 are used to decompose the equations of motion. Introducing the transformation

$$\mathbf{X} \;=\; \mathbf{\Phi}\;\mathbf{Y} \tag{2.34}$$

where $\mathbf{Y}^T = [y_1, y_2, ..., y_n]$ denotes the modal coordinates, use of the orthogonality relations leads to the n uncoupled equations of motion,

$$\ddot{y}_i(t) + 2\zeta_i\omega_i\dot{y}_i(t) + \omega_i^2 y_i(t) = \mathbf{\Phi}_i^T\mathbf{P}f(t) = p_i f(t) \qquad i = 1, ..., n \tag{2.35}$$

in which $p_i = \mathbf{\Phi}_i^T\mathbf{P}$ denote the modal participation factors. For base motion, these factors are given by

$$p_i \;=\; -\mathbf{\Phi}_i^T\mathbf{MR} \tag{2.36}$$

> *The command MPF computes the vector of modal participation factors* $\mathbf{p} = [p_1, ..., p_n]^T$ *when the vector of nodal load coefficients* $\mathbf{P}$ *(or* $-\mathbf{MR}$*) and the matrix of mode shapes* $\mathbf{\Phi}$ *are provided.*

From Eq. 2.35, it is clear that the response in each mode of the system is proportional to the response of a SDOF oscillator of frequency ω_i and damping ζ_i to the forcing function $f(t)$, with the participation factor being the proportionality constant.

Modal Superposition

The solution for the nodal displacements of the MDOF system is given by the superposition rule

$$\mathbf{X} \;=\; \mathbf{\Phi Y} \;=\; \sum_{i=1}^{n} \mathbf{\Phi}_i y_i(t) \tag{2.37}$$

A generic response of the system (e.g., displacement, stress or internal force at a point) can be expressed as a linear function of the nodal displacements,

$$z(t) \;=\; \mathbf{Q}^T\;\mathbf{X}(t) \tag{2.38}$$

where $\mathbf{Q}^T=[q_1, q_2, ..., q_n]$ is a vector of constants. For instance, for the relative displacement between the first and second degrees of freedom $\mathbf{Q}^T = [1, -1, 0, ..., 0]$. More generally, when a stress or internal force is of interest, $\mathbf{Q}$ involves the stiffness properties of the relevant structural member.

Using Eqs. 2.35-38, the generic response is given by

$$z(t) = \mathbf{Q}^T \mathbf{\Phi} \mathbf{Y} = \sum_{i=1}^{n} \mathbf{Q}^T \mathbf{\Phi}_i y_i(t) = \sum_{i=1}^{n} \mathbf{Q}^T \mathbf{\Phi}_i p_i s_i(t) = \sum_{i=1}^{n} a_i s_i(t) \tag{2.39}$$

where

$$a_i = \mathbf{Q}^T \mathbf{\Phi}_i p_i \tag{2.40}$$

is the modal effective participation factor (MEPF) for mode i and $s_i(t)$ is the response of a SDOF oscillator (Eq. 2.5) of properties ω_i and ζ_i to loading $f(t)$, hereafter denoted the i-th modal response. From Eq. 2.13 the i-th modal response is expressed by

$$s_i(t) = s_i(0) g_i(t) + \dot{s}_i(0) h_i(t) + \int_0^t f(\tau) h_i(t-\tau) d\tau \tag{2.41}$$

where $s_i(0)$ and $\dot{s}_i(0)$ are the initial displacement and velocity and $h_i(t)$ and $g_i(t)$ are identical to $h(t)$ and $g(t)$ in Eqs. 2.7 and 2.14, respectively, with ω_o and ζ_o replaced by ω_i and ζ_i. In addition, the frequency response function of mode i, denoted $H_i(\omega)$, is identical to Eq. 2.9 with ω_o and ζ_o replaced by ω_i and ζ_i, respectively.

Eq. 2.39 indicates that any generic response of the system can be expressed as a linear superposition of the modal responses with the associated MEPF's as coefficients. In this superposition, the MEPF's a_i incorporate the influence of the mode shapes and the response transfer vector $\mathbf{Q}$, whereas the modal responses $s_i(t)$ incorporate the influences of the natural frequencies, damping ratios and the loading function $f(t)$. It is important to note that while the modal participation factors p_i are affected by a scaling of the mode shapes (e.g., multiplication by a -1), the MEPF's

a_i remain unaffected. Therefore, the magnitude and algebraic sign of each MEPF is an invariant characteristic of the system and the response quantity of interest. This is not true for the modal participation factors p_i. Also note that $s_i(t)$ are independent of the response transfer vector $\mathbf{Q}$ and, hence, need to be solved only once for all response quantities of interest.

> *The command EPF computes the vector of MEPF's* $\mathbf{A} = [a_1, ..., a_n]^T$ *when the response transfer vector* $\mathbf{Q}$*, the mode shape matrix* $\mathbf{\Phi}$ *and the vector of modal participation factors* $\mathbf{P}$ *are provided.*

One should also note that certain response quantities cannot be expressed as linear combinations of modal responses. An example is the principal stress at a point. Such response quantities cannot be obtained by modal superposition. Instead, one needs to compute intermediate response quantities (e.g., components of stress at the point) and then combine them nonlinearly to obtain the response quantity of interest. The analysis of such responses is beyond the scope of this study.

2.3 Models of Stochastic Excitation

For application in random vibrations, a random process $x(t)$ is most conveniently described in terms of its moment functions. In particular, the first two moment functions are of most interest, which are the mean function

$$\mu_x(t) = E[x(t)] \tag{2.42}$$

and the autocorrelation function

$$\phi_{xx}(t_1,t_2) = E[x(t_1)x(t_2)] \tag{2.43}$$

Furthermore, the autocovariance function is defined and related to the above by

$$\begin{aligned} \kappa_{xx}(t_1,t_2) &= E\{[x(t_1)-\mu_x(t_1)][x(t_2)-\mu_x(t_2)]\} \\ &= \phi_{xx}(t_1,t_2)-\mu_x(t_1)\mu_x(t_2) \end{aligned} \tag{2.44}$$

For $t_1 = t_2 = t$, $\phi_{xx}(t,t) = E[x^2(t)]$ is the mean-square function and $\kappa_{xx}(t,t) = \sigma_x^2(t)$ is the variance function. For a process with zero mean, $\phi_{xx}(t_1,t_2) = \kappa_{xx}(t_1,t_2)$.

For two processes $x(t)$ and $y(t)$, joint moment functions analogous to the above are the cross-correlation function

$$\phi_{xy}(t_1,t_2) = E[x(t_1)y(t_2)] \tag{2.45}$$

and the cross-covariance function

$$\begin{aligned} \kappa_{xy}(t_1,t_2) &= E\{[x(t_1) - \mu_x(t_1)][y(t_2) - \mu_y(t_2)]\} \\ &= \phi_{xy}(t_1,t_2) - \mu_x(t_1)\mu_y(t_2) \end{aligned} \tag{2.46}$$

In this study a random process is usually considered to have a zero mean. This simplifies derivations without a loss of generality, since a general process may be seen as the superposition of a zero-mean process and its mean function. Furthermore, the process is usually considered to be Gaussian. This is because processes encountered in civil engineering practice are often Gaussian or assumed to be Gaussian, and also because most results available in the theory of random processes are limited to Gaussian processes. It is important to note that a Gaussian process is completely defined by its second-moment functions described above. Furthermore, a Gaussian process remains Gaussian under all linear transformations. Thus, the response of a linear system to a Gaussian input is also Gaussian.

In the following analysis, random processes are categorized into two classes: the class of stationary processes and the class of nonstationary processes.

2.3.1 Stationary Processes

A random process is stationary if its probability characterization remains unchanged under an arbitrary shift of the parameter origin (Lin 1967). In most situations, only the mean and autocorrelation functions are available. Thus, a weakly stationary process is defined if only the mean and autocorrelation functions of the process are known to be invariant with respect to the shift in the parameter origin. In that case,

$$\mu_x(t) = \mu_x \tag{2.47}$$

and

$$\phi_{xx}(t_1, t_2) = R_{xx}(t_1 - t_2) = R_{xx}(\tau) \tag{2.48}$$

where $\tau = t_1 - t_2$.

The Power Spectral Density (PSD) function, $\Phi_{xx}(\omega)$, of a stationary process is defined as the Fourier transform of the autocorrelation function:

$$\Phi_{xx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xx}(\tau)e^{-i\omega\tau}d\tau \tag{2.49}$$

This relation and its inverse

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} \Phi_{xx}(\omega)e^{i\omega\tau}d\omega \tag{2.50}$$

are known as Wiener-Khintchine relations (Wiener 1930, Khintchine 1934). It can be shown (Lin 1967, p57) that the PSD function is related to the incomplete Fourier transformation of the process

$$\bar{x}(\omega, T) = \frac{1}{2\pi}\int_{-T}^{T} x(t)e^{-i\omega t}dt \tag{2.51}$$

through

$$\Phi_{xx}(\omega) = \lim_{T\to\infty}\frac{\pi}{T}E[\bar{x}(\omega,T)\bar{x}^*(\omega,T)] \tag{2.52}$$

where $\bar{x}^*(\omega, T)$ denotes the complex conjugate of $\bar{x}(\omega, T)$.

Relations analogous to Eqs. 2.49 and 2.50 for two processes $x(t)$ and $y(t)$ are

$$\Phi_{xy}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xy}(\tau)e^{-\iota\omega\tau}d\tau \tag{2.53}$$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} \Phi_{xy}(\omega) e^{\iota\omega\tau} d\omega \tag{2.54}$$

where $\Phi_{xy}(\omega)$ is denoted cross-PSD function.

> *The commands FTP and FTD compute Fourier transforms using direct integration and discrete Fourier transform methods, respectively. These commands can be used to compute the PSD function for a given auto- or cross-correlation function. The commands IFTP and IFTD compute the inverse Fourier transforms using the respective methods. These commands can be used to compute the auto- or cross-correlation function for a given PSD.*

Models of Stationary Input in STOCAL–II

Four types of stationary input models are included in STOCAL–II. These are specified by their PSD functions as follows:

White Noise (WN): This model (Fig. 2.2a) is defined by

$$\Phi_{ff}(\omega) = \Phi_o \tag{2.55}$$

where Φ_o is a positive constant.

Banded Linear Noise (BLN): This model (Fig. 2.2b) is defined by

$$\Phi_{ff}(\omega) = \begin{cases} \dfrac{\Phi_2 - \Phi_1}{\Omega_2 - \Omega_1}|\omega| + \dfrac{\Phi_1\Omega_2 - \Phi_2\Omega_1}{\Omega_2 - \Omega_1} & \Omega_1 \le |\omega| \le \Omega_2 \\ 0 & \text{elsewhere} \end{cases} \tag{2.56}$$

where $\Omega_1 \ge 0$ and Φ_1 and Φ_2 are such that the PSD is nonnegative over the specified symmetric intervals.

Piecewise linear Noise (PLN): This model is formed by patching together several banded-linear PSD's as shown in Fig. 2.2c. Any arbitrary PSD can be represented by this model through a proper discretization.

Filtered White Noise (FWN): This model (Fig. 2.2d) is defined by

$$\Phi_{ff}(\omega) = \frac{\omega_g^4 + 4\varsigma_g^2\omega_g^2\omega^2}{(\omega_g^2 - \omega^2)^2 + 4\varsigma_g^2\omega_g^2\omega^2}\Phi_o \tag{2.57}$$

where ω_g and ς_g are the natural frequency and damping ratio of the filter, respectively, and Φ_o is a positive constant.

The above stationary process models are accessed in STOCAL–II by providing an indicator number (I=1 for WN, I=2 for BLN, I=3 for PLN, and I=4 for FWN) and appropriate parameters in the command line.

Generation of Sample Functions

When the PSD function $\Phi(\omega)$ of a stationary Gaussian process is specified, sample functions of the process can be generated by using the formula (Clough and Penzien 1975),

$$x(t) = 2\sum_{i=1}^{m}\sqrt{\Phi(\omega_i)\Delta\omega}\ cos(\omega_i t + \Theta_i) \tag{2.58}$$

where Θ_i are random phase angles with uniform distribution in $[0,\ 2\pi]$ and $\Phi(\omega_i)\Delta\omega$ represents the total energy concentrated in frequency band $\Delta\omega$ around frequency ω_i. For large m, this procedure generates samples of a Gaussian process by virtue of the central limit theorem.

The sample functions of a zero-mean stationary Gaussian process having a given autocorrelation function $R_{xx}(\tau)$ can also be generated in the form of a vector $\mathbf{X}$ given by

$$\mathbf{X} = \mathbf{L}\ \mathbf{u} \tag{2.59}$$

where $\mathbf{u}$ is a standard uncorrelated Gaussian vector and $\mathbf{L}$ is the Cholesky decomposition,

$$\mathbf{R} = \mathbf{L}\ \mathbf{L}^T \tag{2.60}$$

where **R** is a positive definite, symmetric Toeplitz matrix,

$$\mathbf{R} = \begin{pmatrix} r_o & r_1 & \cdots & r_{n-1} \\ r_1 & r_o & \cdots & r_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n-1} & r_{n-2} & \cdots & r_o \end{pmatrix} \tag{2.61}$$

having the elements $r_k = R_{xx}(k\Delta\tau)$, $k = 0, 1, ..., n-1$, where n is the number of equidistant sample points. This procedure is applicable for generating short samples of the process, since the accumulation of roundoff errors often hinders the decomposition of a large correlation matrix.

The command GSGP generates sample functions of a zero-mean stationary and asymptotically Gaussian process with a specified PSD function according to Eq. 2.58; the command GSGPT generates sample functions of a zero-mean stationary Gaussian process with a specified autocorrelation function according to Eqs. 2.59-61.

2.3.2 Nonstationary Processes

A zero-mean nonstationary Gaussian process is completely characterized by its autocorrelation function $\phi_{xx}(t_1, t_2)$ defined in the time domain. However, for the modeling and evaluation of random processes and for random vibration analysis of linear systems, a characterization in the frequency domain is more desirable. For this reason, several approaches for spectral characterization of nonstationary processes have been proposed. These include the instantaneous spectrum (Page 1952), the physical spectrum (Mark 1970), the evolutionary spectrum (Priestley 1965), and the generalized spectrum (Bendat and Piersol 1962).

The first three spectra provide mixed time-and-frequency domain characterizations of the nonstationary process. At any given time, these spectra represent the decomposition of a measure of the mean square of the process in the frequency domain. Nonstationarity is achieved by changing the spectral shape and content with time. For random vibration analysis, the instantaneous and physical spectra do not provide convenient input/output relations. For this reason, they are not included in the present version of STOCAL–II. Priestley's evolutionary spectrum, on the other hand, provides a convenient input/output relation for linear systems and is generally favored for random vibration analysis. This spectrum is included in STOCAL–II as described in the following section.

The generalized PSD (Bendat and Piersol 1962, Lin 1967) is defined as the bivariate Fourier transform of the autocorrelation function, $\phi_{xx}(t_1, t_2)$, and is represented in a two-dimensional frequency domain. Although this spectrum provides a convenient input/output relation for linear systems (see Lin 1967), its use in random vibrations has been limited due to the difficulty in its interpretation and evaluation from recorded data. For this reason, this model is also not included in the present development of STOCAL–II.

The Evolutionary Spectrum

A random process in general can be represented in terms of a Fourier-Stieltjes integral (Priestley 1965)

$$x(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dS(\omega) \tag{2.62}$$

where $dS(\omega)$ is an increment process which may or may not be differentiable. The autocorrelation function of the process with this definition is

$$\phi_{xx}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\omega_1 t_1 - \omega_2 t_2)} E[dS(\omega_1) dS^*(\omega_2)] \tag{2.63}$$

where the superposed asterisk denotes the complex conjugate.

Consider the special case where $dS_x(\omega)$ is not differentiable but is an orthogonal-increment process satisfying the relation

$$E[dS(\omega_1) dS^*(\omega_2)] = \Phi(\omega_1)\delta(\omega_1 - \omega_2) d\omega_1 d\omega_2 \tag{2.64}$$

where $\Phi(\omega)$ is a real, even and positive function of frequency. In that case Eq. 2.63 reduces to

$$\phi_{xx}(t_1,t_2) \;=\; \int_{-\infty}^{\infty} \Phi(\omega)e^{i\omega(t_1-t_2)}d\omega \;=\; R_{xx}(t_1-t_2) \tag{2.65}$$

which is the definition of a stationary process with PSD $\Phi(\omega)$. On the other hand, when $dS(\omega)$ is differentiable, one may write

$$E[dS(\omega_1)dS^*(\omega_2)] \;=\; \Phi(\omega_1,\omega_2)d\omega_1 d\omega_2 \tag{2.66}$$

In that case, Eq. 2.63 reduces to

$$\phi_{xx}(t_1,t_2) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\omega_1,\omega_2)\; e^{i(\omega_1 t_1-\omega_2 t_2)}\; d\omega_1 d\omega_2 \tag{2.67}$$

This relation is a bivariate inverse Fourier transform and, therefore, $\Phi(\omega_1,\omega_2)$ denotes the generalized PSD of a nonstationary process as defined by Bendat and Piersol (1962). As mentioned earlier, this definition is not of practical interest.

Priestley (1965) defined an evolutionary process by employing the relation in Eq. 2.64 for the increment process, but introducing a time and frequency modulating function $A(\omega,t)$ in the Stieltjes integral representation, i.e.,

$$x(t) \;=\; \int_{-\infty}^{\infty} A(\omega,t)\; e^{i\omega t}\; dS(\omega) \tag{2.68}$$

Using Eq. 2.64, the autocorrelation function with this definition is

$$\phi_{xx}(t_1,t_2) \;=\; \int_{-\infty}^{\infty} A(\omega,t_1)A^*(\omega,t_2)\Phi(\omega)e^{i\omega(t_1-t_2)}d\omega \tag{2.69}$$

which for $t_1=t_2=t$ yields

$$\begin{aligned} E[x^2(t)] \;&=\; \int_{-\infty}^{\infty} |A(\omega,t)|^2\Phi(\omega)d\omega \\ &=\; \int_{-\infty}^{\infty} \Phi_{xx}(\omega,t)d\omega \end{aligned} \tag{2.70}$$

where

$$\Phi_{xx}(\omega,t) \;=\; |A(\omega,t)|^2\Phi(\omega) \tag{2.71}$$

This defines the evolutionary PSD function of the process. From Eq. 2.70, it is clear that $\Phi_{xx}(\omega,t)$ represents the spectral decomposition of the mean square at time instant t. Also note that this definition reduces to a stationary process if $A(\omega,t)$ is taken to be only a function of ω. This definition of the spectrum of a nonstationary process enjoys broad acceptance in random vibrations and is mathematically convenient.

A special class of nonstationary processes with evolutionary spectrum is defined by specifying the modulating function $A(\omega,t)$ as a function of time only. This class, denoted uniformly modulated processes, is characterized by

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} A(t)\; e^{i\omega t}\; dS(\omega) \\ &= A(t)\int_{-\infty}^{\infty} e^{i\omega t}\; dS(\omega) \\ &= A(t)\; y(t) \end{aligned} \tag{2.72}$$

where $y(t) = \int_{-\infty}^{\infty} e^{i\omega t} dS(\omega)$ is a stationary process. Thus, a uniformly modulated process is obtained by time-modulation of a stationary process. It is important to note that the spectral decomposition of such a process remains unchanged with time, except for a scaling of the spectral amplitudes.

Models of Nonstationary Input in STOCAL–II

In STOCAL–II, an evolutionary process is defined by specifying the modulating function $A(\omega,t)$ as a piecewise linear function of time and frequency. Although the function in general can be complex, in the present development it is restricted to being real. The frequency function $\Phi(\omega)$ can be specified as a piecewise linear function of frequency.

A uniformly modulated process is defined in STOCAL–II by specifying the modulating function $A(t)$ as a piecewise linear function of time. The PSD $\Phi(\omega)$ of the associated stationary process is specified as indicated above.

Generation of Sample Functions

Sample functions of an asymptotically Gaussian process with a specified evolutionary PSD are generated using the following equation:

$$x(t) \;=\; 2\sum_{i=1}^{m} A(\omega_i, t)\sqrt{\Phi(\omega_i)\Delta\omega}\;\; cos(\omega_i t + \Theta_i) \tag{2.73}$$

where Θ_i are independent random phase angles with uniform distribution in $[0,\ 2\pi]$, $A(\omega, t)$ is the specified modulating function, $\Phi(\omega)$ is the time-invariant part of the evolutionary PSD, and $\Delta\omega$ is the frequency increment.

The command GEGP generates sample functions of an evolutionary process with a specified evolutionary PSD. Samples of a uniformly modulated process can be generated by using either the command GEGP or the commands GSGP and TSSF. In the latter case, sample functions of the stationary process are generated first and then multiplied by the time modulating function.

2.4 Response to Stochastic Excitation

A random dynamic loading, $f(t)$, can always be separated into a mean loading, $\mu_f(t)$, and a zero-mean random fluctuating component, $\tilde{f}(t)$, i.e.,

$$f(t) \;=\; \mu_f(t) + \tilde{f}(t) \tag{2.74}$$

Substituting Eq. 2.74 into Eq. 2.13, the response of the linear oscillator can be written as

$$s(t) = s(0)g(t) + \dot{s}(0)h(t) + \int_0^t \mu_f(\tau)h(t-\tau)d\tau + \int_0^t \tilde{f}(\tau)h(t-\tau)d\tau \tag{2.75}$$

where the function $g(t)$ and $h(t)$ are as defined in Section 2.2.1. Assuming deterministic initial conditions, the first three terms on the right hand side can be evaluated by conventional deterministic dynamic analysis methods. Only the fourth term is nondeterministic. Evaluation of the statistical characteristics of this latter term for the SDOF oscillator and MDOF systems is the subject of study for the remainder of this chapter. Thus, unless specified otherwise, in the remainder of this chapter the input excitation will be considered to be a zero-mean process and the system initial conditions will be assumed to be zero. For non-zero initial conditions or non-zero mean excitation, the deterministic solution and the superposition rule in Eq. 2.75 can be used to obtain the complete response. For the convenience of notation, the superposed tilde on the zero-mean random excitation $\tilde{f}(t)$ will be dropped henceforth.

In random vibration analysis of linear systems, usually a second-moment characterization of the response is of interest. For a zero-mean excitation, this characterization is either in terms of the auto- or cross-correlation function or the auto- or cross-PSD function of the response. In addition, many results for stationary processes are given in terms of spectral moments. These quantities are formulated in the following sections for stationary and nonstationary responses.

2.4.1 Stationary Response

The response of a stable system to a stationary excitation $f(t)$ approaches stationarity after the effect of initial conditions is diminished. Mathematically, this is achieved by assuming that the excitation started in the infinite past. Thus, the stationary response of mode i of an MDOF system is given by

$$s_i(t) = \int_{-\infty}^{t} f(t)h_i(t-\tau)d\tau \tag{2.76}$$

where $h_i(t)$ is the impulse response function of the mode i as defined in Section 2.2.2.

The generic term in the solution of the auto- or cross-correlation function of the response of an MDOF system is the modal cross-correlation function given by

$$\begin{aligned}\phi_{ij}(t_1,t_2) &= E\Big[\int_{-\infty}^{t_1} f(\tau_1)h_i(t_1-\tau_1)d\tau_1 \int_{-\infty}^{t_2} f(\tau_2)h_j(t_2-\tau_2)d\tau_2\Big] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ff}(\tau_1-\tau_2)h_i(t_1-\tau_1)h_j(t_2-\tau_2)d\tau_1 d\tau_2\end{aligned} \tag{2.77}$$

in which the upper limits of the integrals are replaced by ∞ by observing that $h(u)=0$ for $u<0$. Expressing the autocorrelation function of the excitation in terms of the PSD and changing the orders of integration, one obtains

$$\begin{aligned}\phi_{ij}(t_1,t_2) = & \int_{-\infty}^{\infty} \Phi_{ff}(\omega)\int_{-\infty}^{\infty} h_i(t_1-\tau_1)\; e^{-i\omega(t_1-\tau_1)}\; d\tau_1 \\ & \int_{-\infty}^{\infty} h_j(t_2-\tau_2)\; e^{i\omega(t_2-\tau_2)}\; d\tau_2\; e^{i\omega(t_1-t_2)}\; d\omega \\ = & \int_{-\infty}^{\infty} \Phi_{ff}(\omega)H_i(\omega)H_j^*(\omega)e^{i\omega(t_1-t_2)}\; d\omega \\ = & \; R_{ij}(t_1-t_2)\end{aligned} \tag{2.78}$$

where $H_i(\omega)=\int_{-\infty}^{\infty} h_i(\tau)e^{-i\omega\tau}d\tau$ is the frequency response function of mode i, and $R_{ij}(.)$ is used to denote the stationary form of the cross-correlation function. This relation defines the modal cross-PSD function

$$\Phi_{ij}(\omega) \;=\; H_i(\omega)H_j^*(\omega)\Phi_{ff}(\omega) \tag{2.79}$$

Eqs. 2.78 and 2.79 are the generic modal solutions needed for computing displacement or displacement-related (e.g., internal forces and stresses) responses of the MDOF system. More generally, one is also interested in the time derivatives of the response. The generic modal solutions needed then are the cross-correlation or cross-PSD function between the m_1-th derivative of mode i and the m_2-th derivative of mode j. Assuming the derivative processes exist, the required functions are given by

$$\begin{aligned}R_{ij}^{(m_1,m_2)}(\tau) &= (-1)^{m_2}\frac{\partial^{m_1+m_2}R_{ij}(\tau)}{\partial\tau^{m_1+m_2}} \\ &= \int_{-\infty}^{\infty} (i\omega)^{m_1}(-i\omega)^{m_2}H_i(\omega)H_j^*(\omega)\Phi_{ff}(\omega)e^{i\omega\tau}d\omega\end{aligned} \tag{2.80}$$

and

$$\Phi_{ij}^{(m_1,m_2)}(\omega) \;=\; (i\omega)^{m_1}(-i\omega)^{m_2}H_i(\omega)H_j^*(\omega)\Phi_{ff}(\omega) \tag{2.81}$$

As described in Section 2.5, most response statistics of interest are expressed in terms of the spectral moments of the PSD. The generic term needed for such evaluation is the cross-modal spectral moments defined by (Der Kiureghian 1980)

$$\lambda_{m,ij} = 2 \; Re \int_0^\infty \omega^m H_i(\omega) H_j^*(\omega) \Phi_{ff}(\omega) d\omega \qquad m = 0, 1, 2, .. \tag{2.82}$$

where Re denotes the real part.

The terms $\Phi_{ij}^{(m_1,m_2)}(\omega)$, $R_{ij}^{(m_1,m_2)}(\tau)$ and $\lambda_{m,ij}$ are all the generic solutions needed for stationary random vibration analysis using the modal approach. Detailed solutions of these functions for the four types of input excitations defined in Section 2.3.1 are described in Chapter 3. In the following section modal superposition rules for computing the total response in terms of these generic terms are presented.

Modal Superposition

Consider two response quantities z_1 and z_2 having the MEPF's, a_{1i} and a_{2i}, for $i = 1, 2, ..., n$, where n is the number of degrees of freedom of the MDOF system. The cross-correlation and cross-PSD functions for the m_1-th and m_2-th derivatives of z_1 and z_2 are obtained by modal superposition from

$$R_{z_1 z_2}^{(m_1,m_2)}(\tau) = \sum_i^n \sum_j^n a_{1i} \; a_{2j} \; R_{ij}^{(m_1,m_2)}(\tau) \tag{2.83}$$

and

$$\Phi_{z_1 z_2}^{(m_1,m_2)}(\omega) = \sum_i^n \sum_j^n a_{1i} \; a_{2j} \; \Phi_{ij}^{(m_1,m_2)}(\omega) \tag{2.84}$$

Obviously, the autocorrelation and auto-PSD functions are obtained if z_1 and z_2 denote the same response quantity. Note that these expressions fully account for the cross-correlation between modal responses.

The spectral moments for a response quantity $z(t)$ are obtained from the superposition rule

$$\lambda_m = \sum_i^n \sum_j^n a_i \; a_j \; \lambda_{m,ij} \tag{2.85}$$

where a_i denote the MEPF's.

The commands SCF and SPSD compute the auto/cross-correlation and auto/cross-PSD functions of the m_1-th and m_2-th derivatives of two response quantities z_1 and z_2 when the modal frequencies, damping ratios, and MEPF's of the MDOF system and the PSD function of the stationary input are provided. The command SRSM computes the spectral moments of a response quantity z when the same modal information is provided.

2.4.2 Nonstationary Response

Using an evolutionary representation of the excitation, the response in mode i of the MDOF system is given by

$$\begin{aligned} s_i(t) &= \int_0^t f(\tau) h_i(t-\tau)\ d\tau \\ &= \int_0^t \left[\int_{-\infty}^{\infty} A_f(\omega,\tau) e^{i\omega\tau} dS_f(\omega)\right]\ h_i(t-\tau)\ d\tau \end{aligned} \tag{2.86}$$

in which $A_f(\omega,t)$ represents the modulating function and $dS_f(\omega)$ denotes the orthogonal-increment process of the input excitation (see Section 2.3.2). After exchanging the order of integration, one obtains

$$s_i(t) = \int_{-\infty}^{\infty} M_i(\omega,t) e^{i\omega t} dS_f(\omega) \tag{2.87}$$

where

$$M_i(\omega,t) = \int_0^t A_f(\omega,\tau)\ h_i(t-\tau)\ e^{-i\omega(t-\tau)}\ d\tau \tag{2.88}$$

Thus, the response is also an evolutionary process with the modulating function $M_i(\omega,t)$. Using the orthogonal property of $dS_f(\omega)$, the generic modal cross-correlation function is given by

$$\phi_{ij}(t_1,t_2) = \int_{-\infty}^{\infty} M_i(\omega,t_1)\ M_j^*(\omega,t_2)\ \Phi_{ff}(\omega)\ e^{i\omega(t_1-t_2)}\ d\omega \tag{2.89}$$

where the superposed asterisk denotes the complex conjugate. Comparing Eqs. 2.69 and 2.89, it is clear that the generic modal evolutionary cross-PSD function is

$$\Phi_{ij}(\omega,t) \;=\; M_i(\omega,t)\; M_j^*(\omega,t)\; \Phi_{ff}(\omega) \tag{2.90}$$

Eqs. 2.89 and 2.90 are the generic modal solutions needed for computing displacement or displacement-related nonstationary responses of the MDOF system. More generally, one is also interested in the time derivatives of the response. The generic modal solutions needed then are the cross-correlation or evolutionary cross-PSD function between the m_1-th derivative of mode i and the m_2-th derivative of mode j. Assuming the derivative processes exist, the required functions are given by

$$\begin{aligned}\phi_{ij}^{(m_1,m_2)}(t_1,t_2) &= \frac{\partial^{m_1}\partial^{m_2} E[s_i(t_1)s_j(t_2)]}{\partial t_1^{m_2}\partial t_2^{m_2}} \\ &= \int_{-\infty}^{\infty} M_i^{(m_1)}(\omega,t_1)M_j^{(m_2)*}(\omega,t_2)\Phi_{ff}(\omega)e^{i\omega(t_1-t_2)}d\omega\end{aligned} \tag{2.91}$$

and

$$\Phi_{ij}^{(m_1,m_2)}(\omega,t) \;=\; M_i^{(m_1)}(\omega,t)M_j^{(m_2)*}(\omega,t)\Phi_{ff}(\omega) \tag{2.92}$$

where

$$M_i^{(m_1)}(\omega,t) \;=\; e^{-i\omega t}\,\frac{\partial^{m_1}}{\partial t^{m_1}}\left[M_i(\omega,t)e^{i\omega t}\right] \tag{2.93}$$

The terms $\Phi_{ij}^{(m_1,m_2)}(\omega,t)$ and $\phi_{ij}^{(m_1,m_2)}(t_1,t_2)$ are both the generic solutions needed for nonstationary random vibration analysis using the modal approach. Detailed solutions of these functions for the particular forms of $A_f(\omega,t)$ and $\Phi_{ff}(\omega)$ defined in Section 2.3.2 are described in Chapter 3. In the following section modal superposition rules for computing the total response in terms of these generic terms are presented.

Modal Superposition

Consider two response quantities z_1 and z_2 having the MEPF's, a_{1i} and a_{2i}, for $i = 1, 2, ..., n$, where n is the number of degrees of freedom of the MDOF system. The cross-correlation and evolutionary cross-PSD functions for the m_1-th and m_2-th derivatives of z_1 and z_2 are obtained by modal superposition from

$$\phi_{z_1 z_2}^{(m_1,m_2)}(t_1,t_2) = \sum_i^n \sum_j^n a_{1i} \; a_{2j} \; \phi_{ij}^{(m_1,m_2)}(t_1,t_2) \tag{2.94}$$

$$\Phi_{z_1 z_2}^{(m_1,m_2)}(\omega,t) = \sum_i^n \sum_j^n a_{1i} \; a_{2j} \; \Phi_{ij}^{(m_1,m_2)}(\omega,t) \tag{2.95}$$

Obviously, the autocorrelation and evolutionary auto-PSD functions are obtained if z_1 and z_2 denote the same response quantity. Note that these expressions fully account for the cross-correlation between modal responses.

The commands ECF and EPSD compute the auto/cross-correlation and evolutionary auto/cross-PSD functions of the m_1-th and m_2-th time derivatives of two response quantities z_1 and z_2 when the modal frequencies, damping ratios, and MEPF's of the MDOF system and the time-frequency modulating and time-invariant PSD functions of the evolutionary input are provided.

When the input is a uniformly modulated process (see Eq. 2.72), the expressions from Eq. 2.86 to Eq. 2.95 can be used used with $A_f(\omega,t)$ replaced by $A_f(t)$.

The commands TCF and TPSD compute the auto/cross-correlation and evolutionary auto/cross-PSD functions of the m_1-th and m_2-th time derivatives of two response quantities z_1 and z_2 when the modal natural frequencies, damping ratios, and MEPF's of the MDOF system and the time modulating and time-invariant PSD functions of the uniformly modulated input are provided.

2.5 Response Statistics of Engineering Interest

The ultimate purpose in using random vibration theory in structural engineering is to assess the reliability of structures. Two kinds of failure events are of interest in such applications: first-excursion failures, which occur at the first crossing of the dynamic response above or below a specified threshold, and fatigue-type failures, which occur due to damage accumulated from repeated response cycles. Detailed review of models used for describing these events is beyond the scope of this study. The emphasis instead is on formulating those statistical measures of the response that are necessary in assessing the reliability against these failure events, and the coding of these formulations in STOCAL–II. The statistics considered include crossing rates and distributions of local and extreme peaks for a general response, and the spectral moments, measures of bandwidth and the characteristics of the envelope process for a stationary response. Since these are well known results in the theory of random processes, derivations of the formulas are not presented.

2.5.1 Spectral Moments

The statistics of a stationary process $x(t)$ are well represented by its first few spectral moments, defined by

$$\lambda_m = \int_0^\infty \omega^m \Phi_{xx}(\omega) d\omega \qquad m = 0, 1, 2, ... \tag{2.96}$$

In particular, for a zero-mean process,

$$\lambda_o = E[x^2(t)] = \sigma_x^2 \tag{2.97}$$

$$\lambda_2 = E[\dot{x}^2(t)] = \sigma_{\dot{x}}^2 \tag{2.98}$$

$$\lambda_4 = E[\ddot{x}^2(t)] = \sigma_{\ddot{x}}^2 \tag{2.99}$$

which are the mean squares (or variances) of the process and its first two derivatives. Furthermore, $\lambda_2 = -E[x(t)\ddot{x}(t)]$ is the negative cross-correlation of the process and its second derivative. The first spectral moment, λ_1, is the cross-correlation of the process and the time derivative of its Hilbert transform (Grigorius 1981), i.e.,

$$\lambda_1 = E[x(t)\dot{\hat{x}}(t)] \tag{2.100}$$

where

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} x(\tau)\frac{1}{t-\tau}d\tau \tag{2.101}$$

The latter spectral moment is of interest in characterizing the envelope process and the extreme peak.

> *The command SM computes the spectral moments of a stationary process when the PSD is provided.*

For a zero-mean evolutionary, nonstationary process $x(t)$, the mean squares of the process and its time derivatives are obtained as the area underneath the corresponding evolutionary PSD functions. In terms of the autocorrelation functions defined in Eq. 2.94, these are

$$\phi_{xx}(t,t) = E[x^2(t)] = \sigma_x^2(t) \tag{2.102}$$

$$\phi_{xx}^{(1,1)}(t,t) = E[\dot{x}^2(t)] = \sigma_{\dot{x}}^2(t) \tag{2.103}$$

$$\phi_{xx}^{(2,2)}(t,t) = E[\ddot{x}^2(t)] = \sigma_{\ddot{x}}^2(t) \tag{2.104}$$

Note that spectral moments defined according to Eq. 2.96 with the evolutionary PSD could not produce the mean squares of the derivative processes due to the time dependent modulating function. In addition to the above, the following cross-correlation coefficient functions are also of interest,

$$\rho_{x\dot{x}}(t) = \frac{\phi_{xx}^{(0,1)}(t,t)}{\sigma_x(t)\sigma_{\dot{x}}(t)} \tag{2.105}$$

$$\rho_{x\ddot{x}}(t) = \frac{\phi_{xx}^{(0,2)}(t,t)}{\sigma_x(t)\sigma_{\ddot{x}}(t)} \tag{2.106}$$

$$\rho_{\dot{x}\ddot{x}}(t) = \frac{\phi_{xx}^{(1,2)}(t,t)}{\sigma_{\dot{x}}(t)\sigma_{\ddot{x}}(t)} \tag{2.107}$$

The command EMS computes the variances and cross-correlation coefficients of an evolutionary process. The command TMS computes the same results when a uniformly modulated process is provided.

The command ERMS computes the variances and cross-correlation coefficients of the response and response derivatives of an MDOF system subjected to an evolutionary input when the modal frequencies, damping ratios, and MEPF's of the system and the modulating and PSD functions of the evolutionary input are provided. The commands TRMS computes the same results when the input is a uniformly modulated process.

2.5.2 Measures of Bandwidth

The shape of the PSD function of a process, in particular its bandwidth, has a distinct influence on the statistical character of the process. Various measures of bandwidth (also known as regularity factors) have been defined for stationary processes. These include

$$\alpha = \frac{\lambda_2}{\sqrt{\lambda_0\lambda_4}} \qquad 0 < \alpha < 1 \tag{2.108}$$

which was defined by Cartwright and Longuet-Higgins (1956), and

$$\delta = \sqrt{1 - \frac{\lambda_1^2}{\lambda_0\lambda_2}} \qquad 0 < \delta < 1 \tag{2.109}$$

which was defined by Vanmarcke (1972). A large α or small δ denotes a narrow-band process.

2.5.3 Crossing Rates

Let $x(t)$ be a mean-square differentiable process. According to Rice (1944, 1945), the mean rate of upcrossing of $x(t)$ above a threshold a at time t is given by

$$\nu_x(a^+,t) = \int_0^\infty \dot{x}\ f_{x\dot{x}}(a,\dot{x};t)d\dot{x} \tag{2.110}$$

where $f_{x\dot{x}}(x,\dot{x};t)$ denotes the joint PDF of $x(t)$ and $\dot{x}(t)$ at time t.

When $x(t)$ is a zero-mean stationary Gaussian process, the mean upcrossing rate becomes

$$\nu_x(a^+) = \frac{1}{2\pi}\frac{\sigma_{\dot{x}}}{\sigma_x} exp\left(-\frac{a^2}{2\sigma_x^2}\right) \tag{2.111}$$

which is invariant of time. The mean upcrossing rate of level zero, commonly known as the apparent frequency is

$$\nu_o = \nu_x(0^+) = \frac{1}{2\pi}\frac{\sigma_{\dot{x}}}{\sigma_x} = \frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}} \tag{2.112}$$

When $x(t)$ is a zero-mean nonstationary Gaussian process, the mean upcrossing rate of level a becomes

$$\nu(a^+,t) = \frac{\sqrt{1-\rho_{x\dot{x}}^2(t)}}{\sqrt{2\pi}}\frac{\sigma_{\dot{x}}(t)}{\sigma_x(t)}\ exp\left(-\frac{a^2}{2\sigma_x^2(t)}\right)\ [\psi(r)+r\Phi(r)] \tag{2.113}$$

where $\psi(.)$ and $\Phi(.)$ denote the standard normal probability distribution function (PDF) and cumulative distribution function (CDF), respectively, and

$$r = \frac{\rho_{x\dot{x}}(t)}{\sqrt{1-\rho_{x\dot{x}}^2(t)}}\frac{a}{\sigma_x(t)} \tag{2.114}$$

The zero upcrossing rate is

$$\nu(0^+,t) = \frac{\sqrt{1-\rho_{x\dot{x}}^2(t)}}{2\pi}\frac{\sigma_{\dot{x}}(t)}{\sigma_x(t)} \tag{2.115}$$

Note that the above crossing rates in general are functions of time.

Similar results can be derived for threshold down-crossing rates by use of symmetry principles.

2.5.4 Distribution of Local Peaks

A local peak of a differentiable process $x(t)$ occurs whenever the derivative process $\dot{x}(t)$ crosses down the zero level. The mean rate of local peaks, therefore, is given by

$$\nu_{\dot{x}}(0^-,t) \; = \; \int_{-\infty}^{0} |\ddot{x}| \; f_{\dot{x}\ddot{x}}(0,\ddot{x};t)d\ddot{x} \tag{2.116}$$

where $f_{\dot{x}\ddot{x}}(\dot{x},\ddot{x};t)$ is the joint PDF of $\dot{x}(t)$ and $\ddot{x}(t)$ at same time. The mean rate of local peaks above level a is given by (Rice 1944, 1945)

$$\mu(a,t) \; = \; \int_{a}^{\infty} dx \int_{-\infty}^{0} |\ddot{x}| \; f_{x\dot{x}\ddot{x}}(x,0,\ddot{x};t)d\ddot{x} \tag{2.117}$$

where $f_{x,\dot{x},\ddot{x}}(x,\dot{x},\ddot{x};t)$ is the joint PDF of $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ at time t. From this result, the PDF of peaks at level a is given by (Huston and Skopinski 1956)

$$\begin{aligned} f_p(a;t) \; &= \; \frac{\partial}{\partial a}\Big[1 - \frac{\mu(a,t)}{\nu_{\dot{x}}(0^-,t)}\Big] \\ &= \; \frac{1}{\nu_{\dot{x}}(0^-,t)} \int_{-\infty}^{0} |\ddot{x}| \; f(a,0,\ddot{x};t)d\ddot{x} \end{aligned} \tag{2.118}$$

For a zero-mean stationary Gaussian process, the PDF of local peaks is independent of time and is given by (Cartwright and Longuet-Higgins 1956)

$$f_p(a) = \frac{\sqrt{1-\alpha^2}}{\sqrt{2\pi}\sigma_x} exp\Big[-\frac{1}{2}\frac{a^2}{\sigma_x^2(1-\alpha^2)}\Big] + \frac{\alpha a}{\sigma_x^2} exp\Big(-\frac{a^2}{2\sigma_x^2}\Big)\Phi\Big(\frac{\alpha a}{\sqrt{1-\alpha^2}\sigma_x}\Big) \tag{2.119}$$

where α is the bandwidth factor in Eq. 2.108. The CDF is

$$F_p(a) \; = \; \Phi\Big(\frac{a}{\sqrt{1-\alpha^2}\sigma_x}\Big) - \alpha \; exp\Big(-\frac{1}{2}\frac{a^2}{\sigma_x^2}\Big) \; \Phi\Big(\frac{\alpha a}{\sqrt{1-\alpha^2}\sigma_x}\Big) \tag{2.120}$$

For a zero-mean nonstationary Gaussian process, the use of Eq. 2.118 leads to

$$\begin{aligned} f_p(a;t) \; &= \frac{\sqrt{\rho_0}}{[1-\rho_{x\dot{x}}^2(t)]\sqrt{1-\rho_{\dot{x}\ddot{x}}^2(t)}\sigma_x(t)} \; [\psi(\bar{r}) + \bar{r}\Phi(\bar{r})] \\ & exp\Big\{-\frac{1}{2\rho_0}\big(\frac{a}{\sigma_x(t)}\big)^2\Big[\; 1 - \rho_{\dot{x}\ddot{x}}^2(t) - \frac{[\rho_{x\ddot{x}}(t) - \rho_{x\dot{x}}(t)\rho_{\dot{x}\ddot{x}}(t)]^2}{[1-\rho_{x\dot{x}}^2(t)]^2} \;\Big]\Big\} \end{aligned} \tag{2.121}$$

where

$$\bar{r} = \frac{\rho_{x\ddot{x}}(t) - \rho_{x\dot{x}}(t)\rho_{\dot{x}\ddot{x}}(t)}{\sqrt{\rho_0[1-\rho^2_{x\dot{x}}(t)]}} \frac{a}{\sigma_x(t)} \tag{2.122}$$

and $\rho_0 = 1 + 2\rho_{x\dot{x}}(t)\rho_{x\ddot{x}}(t)\rho_{\dot{x}\ddot{x}}(t) - \rho^2_{x\dot{x}}(t) - \rho^2_{x\ddot{x}}(t) - \rho^2_{\dot{x}\ddot{x}}(t)$.

2.5.5 Envelope of Narrow-Band Stationary Process

The envelope of a narrow-band process $x(t)$ is defined as a pair of smoothly varying processes $E(t)$ such that $|E(t)| \geq |x(t)|$ for all t and $|E(t)| = |x(t)|$ at, or very nearly at, the peaks of $x(t)$. One definition of the envelope for a narrow-band stationary process is (Rice 1944, 1945),

$$x(t) = E(t)\ cos[\omega_m t + \Theta(t)] \tag{2.123}$$

where ω_m represents a midband frequency and $\Theta(t)$ denotes a random phase process. The midband frequency is selected such that $E(t)$ and $\Theta(t)$ are much more slowly varying with time than $x(t)$. Another definition of the envelope by Cramer and Leadbetter (1967) is

$$E^2(t) = x^2(t) + \hat{x}^2(t) \tag{2.124}$$

where $\hat{x}(t)$ is the Hilbert transform in Eq. 2.101. For a stationary Gaussian process both above definitions lead to the following joint PDF of $E(t)$ and its derivative $\dot{E}(t)$,

$$f_{E\dot{E}}(a,v) = \frac{a}{\sqrt{2\pi}\sigma_x^2\sigma_{\dot{x}}\delta} exp\Big[-\frac{1}{2}\big(\frac{a^2}{\sigma_x^2} + \frac{v^2}{\delta^2\sigma_{\dot{x}}^2}\big)\Big] \tag{2.125}$$

where δ is the bandwidth factor in Eq. 2.109. From this equation, $E(t)$ and $\dot{E}(t)$ are found to be statistically independent with Rayleigh and normal marginal distributions,

$$f_E(a) = \frac{a}{\sigma_x^2} exp\big(-\frac{a^2}{2\sigma_x^2}\big) \tag{2.126}$$

$$f_{\dot{E}}(v) = \frac{1}{\sqrt{2\pi}} exp\big(-\frac{v^2}{2\delta^2\sigma_{\dot{x}}^2}\big) \tag{2.127}$$

The means and standard deviations of $E(t)$ and $\dot{E}(t)$ are

$$\mu_E = \sqrt{\frac{\pi}{2}}\,\sigma_x \tag{2.128}$$

$$\sigma_E = \sqrt{2-\frac{\pi}{2}}\,\sigma_x \tag{2.129}$$

and

$$\mu_{\dot{E}} = 0 \tag{2.130}$$

$$\sigma_{\dot{E}} = \delta\sigma_x \tag{2.131}$$

Using Eqs. 2.110 and 2.111, the mean upcrossing rate of the envelope process is

$$\nu_E(a^+) = \sqrt{2\pi}\delta\frac{a}{\sigma_x}\nu_x(a^+) \tag{2.132}$$

2.5.6 Distribution of Extreme Peak

The one-sided extreme peak of a process $x(t)$ over a duration $(0,\ \tau)$ is defined by

$$x_\tau = \max_{0<t\leq\tau} x(t) \tag{2.133}$$

For a zero-mean process, the two-sided extreme peak defined by

$$x_\tau = \max_{0<t\leq\tau} |x(t)| \tag{2.134}$$

is also of interest.

When the crossings of the process above level a are assumed to be statistically independent and to constitute Poisson events, the cumulative distribution of the one-side extreme peak is given by

$$F_{x_\tau}(a) = exp\Big[-\int_0^\tau \nu_x(a^+,t)dt\Big] \tag{2.135}$$

Where $x(0) \leq a$ is assumed. For the two-side extreme peak of a zero-mean process, the CDF is given by

$$F_{x_\tau}(a) = exp\Big[-\int_0^\tau [\nu_x(a^+,t) + \nu_x(-a^-,t)]dt\Big] \tag{2.136}$$

The assumption of statistical independence between upcrossings is not reasonable for a narrow-band process. In fact, for every upcrossing of the envelope process $E(t)$, one may expect a clumping of the upcrossings of the narrow-band process at the same level (see Fig. 2.3). Vanmarcke (1975) has derived the following expression for the expected clump size associated with a qualified upcrossing of the envelope (i.e., an upcrossing of the envelope which includes at least one upcrossing of the process) of a stationary process:

$$E[\text{clump size}] = \frac{1}{1 - exp(-\sqrt{2\pi}\delta\frac{a}{\sigma_x})} \tag{2.137}$$

Considering such qualified crossing as independent events, he has obtained the following CDF for the extreme peak x_τ of a zero-mean stationary Gaussian process,

$$F_{x_\tau}(a) = (1 - e^{-\frac{r^2}{2}})\ exp\Big(-\nu_e\tau\ \frac{1 - e^{-\sqrt{\frac{\pi}{2}}\delta_e r}}{e^{\frac{r^2}{2}} - 1}\Big) \qquad r \geq 0 \tag{2.138}$$

where $r = \frac{a}{\sigma_x}$. This expression denotes the distribution of the one-sided extreme peak when $\delta_e = (2\delta)^{1.2}$ and $\nu_e = \nu_o$, and it denotes the distribution of the two-side extreme peak when $\delta_e = (\delta)^{1.2}$ and $\nu_e = 2\nu_o$, where δ and ν_o denote the bandwidth factor in Eq. 2.109 and the mean zero-crossing rate in Eq. 2.112, respectively. The mean and standard deviation of the extreme peak may be obtained from

$$\mu_{x_\tau} = p\ \sigma_x \tag{2.139}$$

$$\sigma_{x_\tau} = q\ \sigma_x \tag{2.140}$$

where p and q are peak factors, approximately given by (Der Kiureghian 1980)

$$p = \sqrt{2ln(\nu\tau)} + \frac{0.5772}{\sqrt{2ln(\nu\tau)}} \tag{2.141}$$

$$q = \frac{1.2}{\sqrt{2ln(\nu\tau)}} - \frac{5.4}{1.3 + [2ln(\nu\tau)]^{3.2}} \tag{2.142}$$

where

$$\nu\tau \;=\; \begin{cases} max(2.1, 2\delta_e\nu_e\tau) & 0.0 \leq \delta_e < 0.063 \\ (1.63\delta_e^{0.375} - 0.38)\nu_e\tau & 0.063 \leq \delta_e \leq 0.64 \\ \nu_e\tau & 0.64 < \delta_e \end{cases} \tag{2.143}$$

The above relations, which are extended from their original version to account for both the one-sided and two-sided extreme peaks, are appropriated for earthquake engineering applications where $\nu_e\tau < 1000$. For larger values of $\nu_e\tau$, which may occur in wind or ocean engineering problems, the following expressions due to Davenport (1964)

$$p \;=\; \sqrt{2ln(\nu_e\tau)} + \frac{0.5772}{\sqrt{2ln(\nu_e\tau)}} \tag{2.144}$$

$$q \;=\; \frac{\pi}{\sqrt{6}}\frac{1}{\sqrt{2ln(\nu_e\tau)}} \tag{2.145}$$

For a nonstationary process, the current version of STOCAL–II uses Eqs. 2.135 and 2.136 to computes the probability distribution of the extreme peak.

The command SSGP computes the statistics of a stationary Gaussian process, including the root mean squares of the process, its envelope and their derivatives, the apparent frequency, the regularity factors, mean upcrossing rates, the mean clump size, and CDF's and PDF's of the local and extreme peaks and the envelope process when the first few spectral moments are provided. Furthermore, the commands LPKD and EXTD compute the PDF and CDF of the local and extreme peaks, respectively, the results of which can be plotted.

The command NCR computes the mean upcrossing rates at specified thresholds when the matrix containing the variances and cross-correlation coefficients of a zero-mean nonstationary Gaussian process and its derivatives is provided. The commands NDLP and NDEP respectively compute the the PDF of local peaks and the CDF of the extreme peak at specified levels when the same variance matrix in the preceding command is provided.

2.6 The Response Spectrum Method

In earthquake engineering, the input ground motion is usually specified in terms of the response spectrum. By definition, the response spectrum of a ground motion process is a plot of the mean maximum response of an oscillator to the specified base motion, expressed as a function of the oscillator frequency and damping ratio. Der Kiureghian (1980) has shown that this specification of the input can be used to carry out complete random vibration analysis under two assumptions: (a) the input excitation is wide band and the dominant modes of vibration are within the significant range of input frequencies; and (b) the strong-motion phase of the ground motion is at least several times longer than the fundamental period of the structure. Employing the modal combination rule for the spectral moments of a stationary response, Eq. 2.85, Der Kiureghian (1981) developed the following relation for the m-th spectral moment of the MDOF response

$$\lambda_m = \sum_i^n \sum_j^n a_i a_j \rho_{m,ij} w_{m,i} w_{m,j} \frac{1}{p_i p_j} D(\omega_i, \varsigma_i) D(\omega_j, \varsigma_j), \quad m = 0, 1, 2 \tag{2.146}$$

in which

$$\rho_{m,ij} = \frac{\lambda_{m,ij}}{\sqrt{\lambda_{m,ii}\lambda_{m,jj}}} \tag{2.147}$$

$$w_{m,i} = \sqrt{\frac{\lambda_{m,ii}}{\lambda_{0,ii}}} \tag{2.148}$$

In the above relation $D(\omega_i, \varsigma_i)$ is the ordinate of the (displacement) response spectrum for mode i, and p_i is a peak factor for mode i given by Eq. 2.141 in terms of the spectral moments $\lambda_{m,ii}$ $(m = 0,1,2)$ of the response in mode i and the duration of the excitation. The latter factor is used to define the response spectrum ordinate in terms of the modal mean square response.

The above formulation is exact if the input process is stationary and the exact modal spectral moments are employed to compute $\rho_{m,ij}$, $w_{m,i}$ and p_i. In practice, the input is not stationary and the modal spectral moments are not available. In that case, Der Kiureghian (1981) has shown that sufficiently accurate results can be obtained by approximating $\rho_{m,ij}$, $w_{m,i}$ and p_i by their values for simple stationary models, such as the WN model. This approximation works well because these terms are expressed in terms of the ratios of spectral moments rather than their absolute values. More accurate approximations are obtained by using the FWN model (Igusa and Der Kiureghian 1985). In STOCAL–II, the terms can also be approximated by using the BLN or PLN model. In that case, $\rho_{m,ij}$, $w_{m,i}$ and p_i are obtained from Eqs. 2.147, 2,148 and 2.141, based on $\lambda_{m,ij}$ and $\lambda_{m,ii}$ obtained from Eq. 2.82.

With the spectral moments for $m = 0,1,2$ determined, various statistical measures of the response can be obtained as described in Section 2.5.1. In particular, the mean maximum response is obtained by the product $p\sqrt{\lambda_o}$, where p is the peak factor of the total response computed from Eq. 2.141. Noting that the ratios p/p_i appearing in the expression for the product $p\sqrt{\lambda_o}$ are near unity, Der Kiureghian (1981) suggested a simpler modal combination rule for the mean maximum response. This rule, now known as the CQC rule (Wilson et al. 1981), is

$$E[|x_r|] \simeq \left[\sum_i^n \sum_j^n a_i a_j \rho_{0,ij} D(\omega_i, \varsigma_i) D(\omega_j, \varsigma_j)\right]^{1/2} \tag{2.149}$$

where

$$\rho_{0,ij} = \frac{8\sqrt{\varsigma_i\varsigma_j}(\varsigma_i + r\varsigma_j)r^{3/2}}{(1-r^2)^2 + 4\varsigma_i\varsigma_j r(1+r^2) + 4(\varsigma_i^2+\varsigma_j^2)r^2} \tag{2.150}$$

where $r = \omega_j/\omega_i$. One should note that the above modal combination rule does not include the effect of the peak factors.

The command RSM computes the first few spectral moments of the response of an MDOF system when the modal frequencies, damping ratios, MEPF's and the input response spectrum are specified. These spectral moments may be used with the command SSGP to compute the response statistics. The command CQC computes the mean maximum response according to Eq. 2.149.

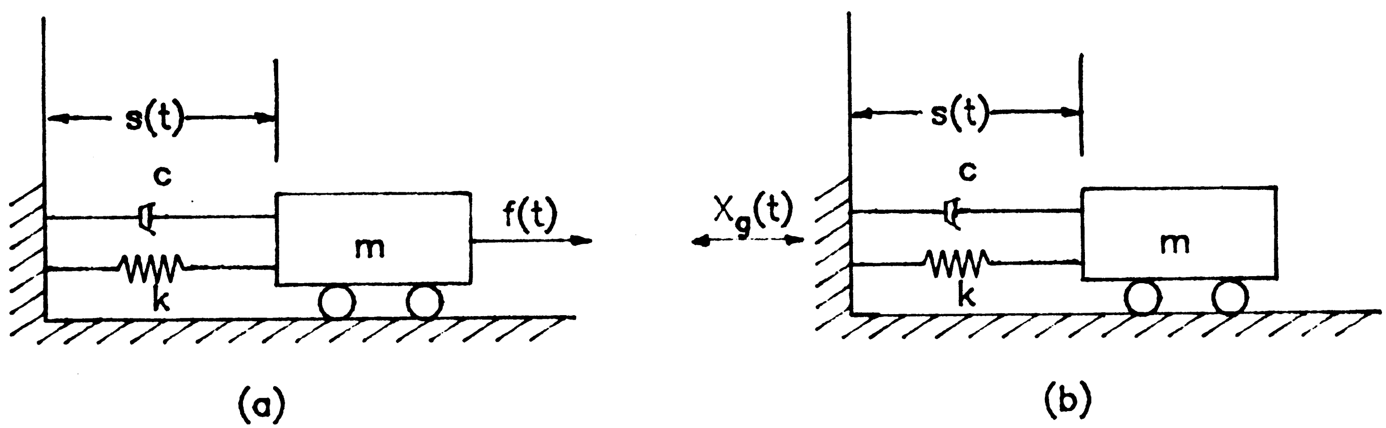


Figure 2.1 SDOF System: (a) to Applied Loading, (b) to Base Motion

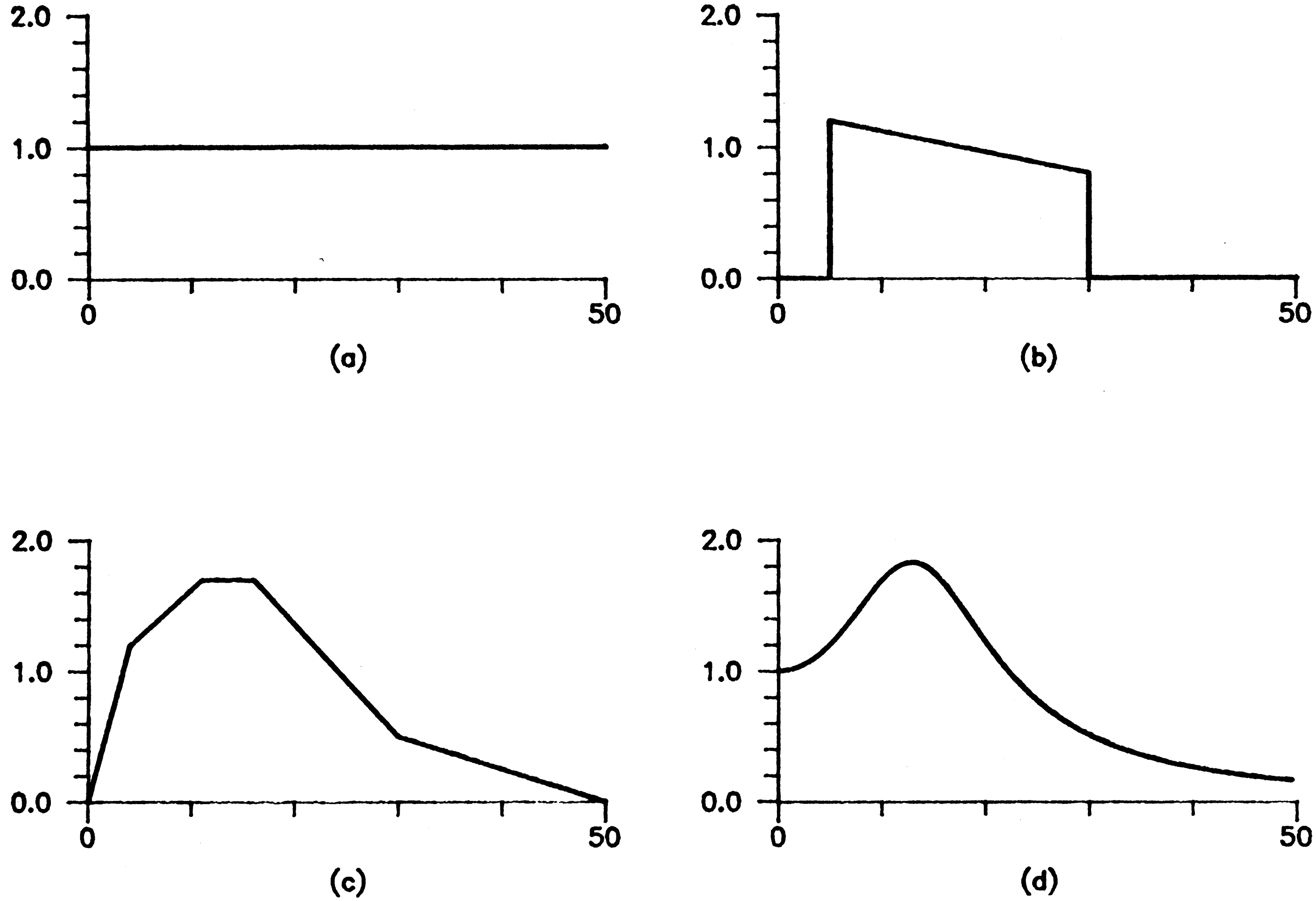


Figure 2.2 PSD Models for Stationary Processes: (a) WN (b) BLN (c) PLN (d) FWN

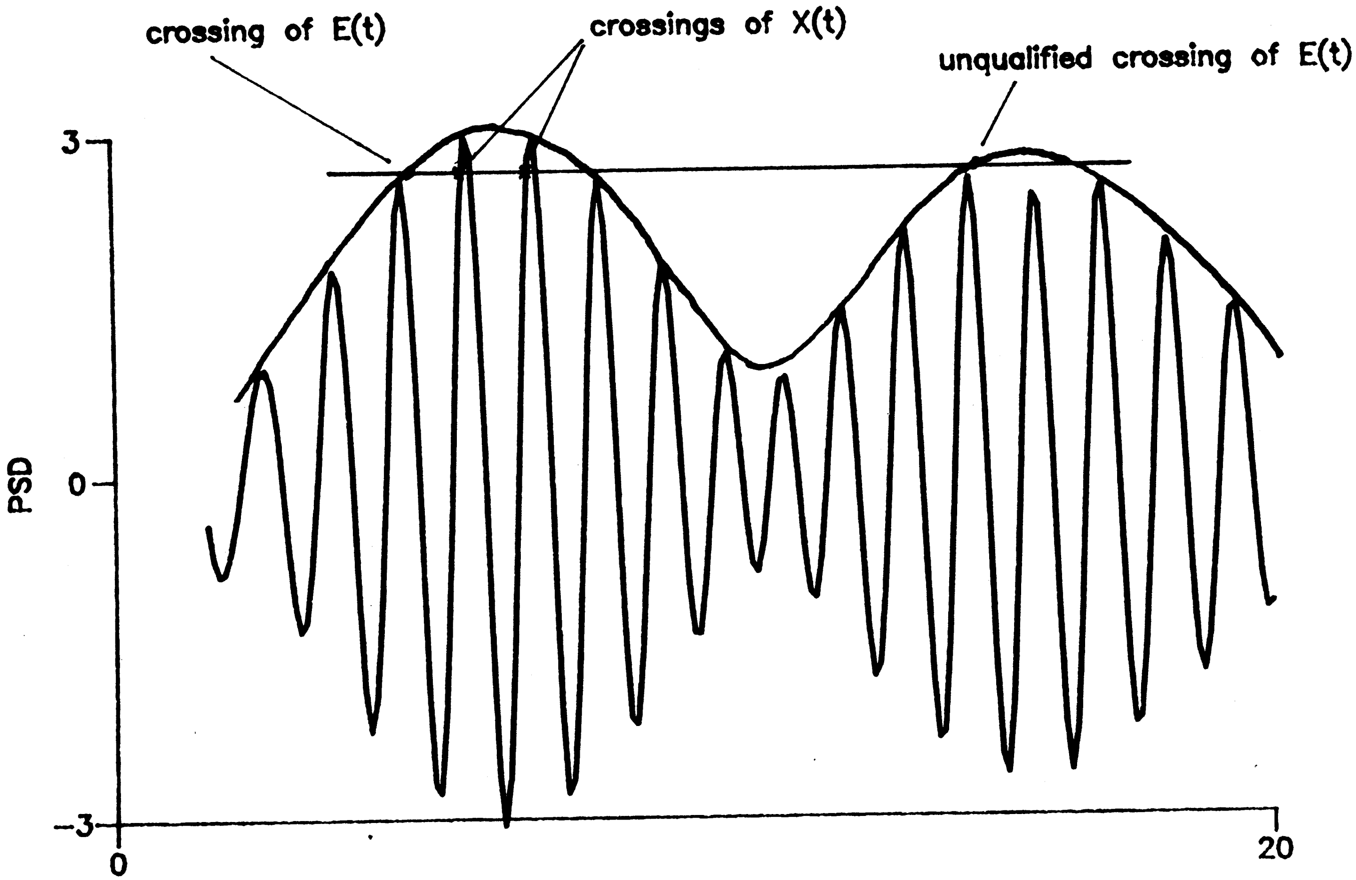


Figure 2.3 Crossings of a Process X(t) and its Envelope E(t)

CHAPTER 3

NUMERICAL COMPUTATION

3.1 Introduction

This chapter describes the formulas and numerical algorithms used for computing the general results derived in Chapter 2 and their implementation in STOCAL–II. The results include the generic solutions for the cross-modal responses, including the modal cross-PSD and cross-correlation functions for stationary and nonstationary responses, and the cross-modal spectral moments for stationary responses. Moreover, the numerical algorithms for the Fourier and inverse Fourier transforms are described in Section 3.4. Consistent with the objectives of STOCAL–II, the formulas and algorithms presented in this chapter stress the generality rather than efficiency of computations.

3.2 Stationary Response

In this section, the expressions for numerical computation of the generic cross-modal terms $\Phi_{ij}^{(m_1,m_2)}(\omega)$, $\phi_{ij}^{(m_1,m_2)}(\tau)$ and $\lambda_{m,ij}$ are presented. Stationary inputs under consideration are the WN, the BLN and the FWN models described in Section 2.3.1. The solution for an input with an arbitrary, piecewise linear PSD is obtained by patching together the solutions for BLN inputs.

3.2.1 Modal Cross-PSD Functions

From Eq. 2.81, the $(m_1 + m_2)$-th order modal cross-PSD function is

$$\Phi_{ij}^{(m_1,m_2)}(\omega) = (\mathrm{i}\omega)^{m_1}(-\mathrm{i}\omega)^{m_2} H_i(\omega) H_j^*(\omega) \Phi_{ff}(\omega) \tag{3.1}$$

where $H_i(\omega)$ is the frequency response function of mode i given in Eq. 2.9 with ω_o and ζ_o replaced by ω_i and ζ_i, respectively, and the asterisk denotes the complex conjugate. For a given input PSD, this function is computed term by term for all specified frequency values. For $i \neq j$, the modal cross-PSD function is complex valued with an even real part and an odd imaginary part; for $i = j$, the imaginary part is zero and the real part is a positive and even function of ω.

3.2.2 Modal Cross-Correlation Functions

The $(m_1 + m_2)$-th order modal cross-correlation function is given in terms of the corresponding cross-PSD function by the inverse Fourier transform

$$R_{ij}^{(m_1,m_2)}(\tau) = \int_{-\infty}^{\infty} \Phi_{ij}^{(m_1,m_2)}(\omega)\ e^{i\omega\tau}\ d\omega \tag{3.2a}$$

When appropriate, this integral is computed by using the residue theorem. It can be shown (Henrici 1974, p256) that

$$\begin{aligned} \int_{-\infty}^{\infty} \Phi_{ij}^{(m_1,m_2)}(\omega) e^{i\omega\tau} d\omega = & \ 2\pi i \sum_{Im z>0} res[\Phi_{ij}^{(m_1,m_2)}(z)] e^{iz\tau} \qquad \tau \geq 0 \\ = & -2\pi i \sum_{Im z<0} res[\Phi_{ij}^{(m_1,m_2)}(z)] e^{iz\tau} \qquad \tau < 0 \end{aligned} \tag{3.2b}$$

where the sum involves the residues of $\Phi_{ij}^{(m_1,m_2)}(z)e^{iz\tau}$ at all poles in the upper half-plane for $\tau \geq 0$, or in the lower half-plane for $\tau < 0$, and $\Phi_{ij}^{(m_1,m_2)}(z)e^{iz\tau}$ is a rational function with no poles on the real axis and a zero of order ≥ 1 at infinity. Other integration methods are employed when the residue theorem is not applicable.

Response to White Noise

The solution for the response to a WN input (Eq. 2.55) is obtained by using Eqs. 3.2. For $m_1 + m_2 < 3$

$$\begin{aligned} R_{ij}^{(m_1,m_2)}(\tau) = & \ 2\pi\Phi_0 (i)^{m_1+1}(-i)^{m_2}[R_1(z_1) + R_2(z_2)] \qquad \tau \geq 0 \\ = & \ 2\pi\Phi_0 (i)^{m_1}(-i)^{m_2+1}[R_3(z_3) + R_4(z_4)] \qquad \tau < 0 \end{aligned} \tag{3.3a}$$

where z_k, $k = 1,2,3,4$, are the poles in the complex plane given by

$$\begin{aligned} z_{1,2} = & \ (\pm\sqrt{1-\varsigma_i^2} + i\varsigma_i)\omega_i \\ z_{3,4} = & \ (\pm\sqrt{1-\varsigma_j^2} - i\varsigma_j)\omega_j \end{aligned} \tag{3.3b}$$

The residue at pole z_1 is given by

$$R_1(z_1) = \frac{z_1^{m_1+m_2}\ e^{iz_1\tau}}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} \tag{3.3c}$$

Similar expressions hold for the residues at poles z_2, z_3 and z_4.

Note that $R_{ij}^{(m_1,m_2)}(\tau)$ does not exist for $m_1 + m_2 \geq 3$.

Response to Banded Linear Noise

The solution for the response to a BLN input (Eq. 2.56) is obtained by using the method of partial fractions (Kreyszig 1972, p158) and direct integration:

$$\begin{aligned} R_{ij}^{(m_1,m_2)}(\tau) = {} & (\mathrm{i})^{m_1}(-\mathrm{i})^{m_2}\Phi_0 \sum_{k=1}^{4}\left[\int_{\Omega_1}^{\Omega_2} c_k \frac{a\omega^{m+1} + b\omega^m}{\omega - z_k} e^{\mathrm{i}\omega\tau}\,d\omega \right. \\ & \left. + \int_{-\Omega_2}^{-\Omega_1} c_k \frac{-a\omega^{m+1} + b\omega^m}{\omega - z_k} e^{\mathrm{i}\omega\tau}\,d\omega\right] \\ = {} & \mathrm{i}^{m_1}(-\mathrm{i})^{m_2}\Phi_0 \sum_{k=1}^{4} c_k\big[aJ(z_k, m+1, \Omega_1, \Omega_2, \tau) \\ & + bJ(z_k, m, \Omega_1, \Omega_2, \tau) - aJ(z_k, m+1, -\Omega_2, -\Omega_1, \tau) \\ & + bJ(z_k, m, -\Omega_2, -\Omega_1, \tau)\big] \end{aligned} \tag{3.4a}$$

where $m = m_1 + m_2$, z_k, $k = 1,2,3,4$, are as in Eq. 3.3b, $J[.]$ is the compound exponential integral described in Appendix A, and c_k are as follows:

$$\begin{aligned} c_1 &= \frac{1}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} \\ c_2 &= \frac{1}{(z_2 - z_1)(z_2 - z_3)(z_2 - z_4)} \\ c_3 &= \frac{1}{(z_3 - z_1)(z_3 - z_2)(z_3 - z_4)} \\ c_4 &= \frac{1}{(z_4 - z_1)(z_4 - z_2)(z_4 - z_3)} \end{aligned} \tag{3.4b}$$

Response to Filtered White Noise

The solution for the response to a FWN input (Eq. 2.57) is obtained by using Eqs. 3.2. For $m_1 + m_2 < 5$,

$$R_{ij}^{(m_1,m_2)}(\tau) = 2\pi\Phi_0(\mathrm{i})^{m_1+1}(-\mathrm{i})^{m_2}\mathrm{R} \tag{3.5a}$$

where

$$
\begin{aligned}
R &= \sum_{k=1,2,5,6} R_k(z_k) && \varsigma_g \neq \varsigma_i \text{ or } \omega_g \neq \omega_i,\ \tau \geq 0 \\
&= \sum_{k=1,2} \bar{R}_k(z_k) && \varsigma_g = \varsigma_i \text{ and } \omega_g = \omega_i,\ \tau \geq 0 \\
&= -\sum_{k=3,4,7,8} R_k(z_k) && \varsigma_g \neq \varsigma_j \text{ or } \omega_g \neq \omega_j,\ \tau < 0 \\
&= -\sum_{k=3,4} \bar{R}_k(z_k) && \varsigma_g = \varsigma_i \text{ and } \omega_g = \omega_j,\ \tau < 0
\end{aligned}
\tag{3.5b}
$$

where z_k, $k = 1,2,3,4$, are as in Eq. 3.3b and

$$
\begin{aligned}
z_{5,6} &= (\pm\sqrt{1-\varsigma_g^2} + \mathrm{i}\varsigma_g)\omega_g \\
z_{7,8} &= (\pm\sqrt{1-\varsigma_g^2} - \mathrm{i}\varsigma_g)\omega_g
\end{aligned}
\tag{3.5c}
$$

An order 1 residue occurs at pole z_1 when $\varsigma_g \neq \varsigma_i$ or $\omega_g \neq \omega_i$, which is given by

$$
R_1(z_1) = \frac{(\omega_g^4 + 4\varsigma_g^2\omega_g^2 z_1^2) z_1^{m_1+m_2} e^{\mathrm{i}z_1\tau}}{\Pi_{k=2}^{8}(z_1 - z_k)}
\tag{3.5d}
$$

Similar expressions hold for the first order residues at the remaining poles. An order 2 residue, denoted by a superposed bar, occurs at pole z_1 when $\varsigma_g = \varsigma_i$ and $\omega_g = \omega_i$, and is given by

$$
\begin{aligned}
\bar{R}_1(z_1) = & \frac{z_1^{m_1+m_2} e^{\mathrm{i}z_1\tau}}{(z_1 - z_2)^2 \Pi_{k=3,4,7,8}(z_1 - z_k)} \Big[8\varsigma_g^2\omega_g^2 z_1 + (\omega_g^4 + 4\varsigma_g^2\omega_g^2 z_1^2) \\
& (\frac{m_1 + m_2}{z_1} + \mathrm{i}\tau - \frac{2}{z_1 - z_2} - \sum_{k=3,4,7,8} \frac{1}{z_1 - z_k}) \Big]
\end{aligned}
\tag{3.5e}
$$

The residue $\bar{R}_2(z_2)$ is similar to the above with the positions of z_1 and z_2 switched. Also, $\bar{R}_3(z_3)$ is given by

$$
\begin{aligned}
\bar{R}_3(z_3) = & \frac{z_3^{m_1+m_2} e^{\mathrm{i}z_3\tau}}{(z_3 - z_4)^2 \Pi_{k=1,2,5,6}(z_3 - z_k)} \Big[8\varsigma_g^2\omega_g^2 z_3 + (\omega_g^4 + 4\varsigma_g^2\omega_g^2 z_3^2) \\
& (\frac{m_1 + m_2}{z_3} + \mathrm{i}\tau - \frac{2}{z_3 - z_4} - \sum_{k=1,2,5,6} \frac{1}{z_3 - z_k}) \Big]
\end{aligned}
\tag{3.5f}
$$

and $\bar{R}_4(z_4)$ is similar with the positions of z_3 and z_4 switched.

Note that $R_{ij}^{(m_1,m_2)}(\tau)$ does not exist for $m_1 + m_2 \geq 5$.

3.2.3 Modal Cross-Spectral Moments

The modal cross-spectral moments are defined by

$$\lambda_{m,ij} \;=\; 2\ Re \int_0^\infty \omega^m\ \Phi_{ij}(\omega)\ d\omega \qquad m = 0,1,2,... \tag{3.6a}$$

When m is even, these moments can be computed as values of the modal cross-correlation function, Eqs. 3.2, for $\tau = 0$. However, here, a computationally more efficient formulation is presented, which is applicable to even as well as odd values of m from the residue theorem (Henrici 1974, p259):

$$\int_0^\infty \omega^\alpha \omega^m \Phi_{ij}(\omega) d\omega = \frac{2\pi i}{1 - e^{2\pi i \alpha}} \sum res\ [z^\alpha z^m \Phi_{ij}(z)] \tag{3.6b}$$

where z^α, $0 < \alpha < 1$, is computed with $arg(z) \in (0, 2\pi)$, the sum involves the residues at all nonzero poles of $z^m \Phi_{ij}(z)$, and $z^m \Phi_{ij}(z)$ is a rational function with no poles on the positive real axis, a pole of order ≤ 1 at 0, and a zero of order ≥ 2 at infinity. Applying L'Hopital's rule to Eq. 3.6b when α approaches zero, gives

$$\int_0^\infty \omega^m \Phi_{ij}(\omega) d\omega = -\sum res\ [ln(z) z^m \Phi_{ij}(z)] \tag{3.6c}$$

where $ln(z)$ is the logarithmic function defined in Appendix A. This approach is used to compute the spectral moments for WN and FWN inputs.

<u>Response to White Noise</u>

The solution for the response to a WN input is obtained by using Eqs. 3.6. For $m \leq 2$,

$$\lambda_{m,ij} \;=\; Re \left[-\sum_{k=1}^{4} R_k(z_k) \right] \tag{3.7a}$$

where z_k are as in Eq. 3.3b and the residue for pole z_1 is

$$R_1(z_1) = \frac{ln(z_1)(z_1)^m}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} \tag{3.7b}$$

Similar expressions hold for the residues at the remaining poles.

Note that $\lambda_{m,ij}$ does not exist when $m > 2$.

Response to Banded Linear Noise

The solution for the response to a BLN input is by using partial fractions and direct integration:

$$\begin{aligned}\lambda_{m,ij} &= 2\Phi_0 Re\Big[\sum_{k=1}^{4}\int_{\Omega_1}^{\Omega_2} c_k \frac{a\omega^{m+1}+b\omega^m}{\omega - z_k}d\omega\Big] \\ &= 2\Phi_0 Re\Big[\sum_{k=1}^{4} c_k[a\ \ I(z_k, m+1, \Omega_1, \Omega_2) + b\ \ I(z_k, m, \Omega_1, \Omega_2)]\Big]\end{aligned} \tag{3.8}$$

where $I[.]$ is the compound logarithmic integral described in Appendix A. Other terms in the above equation are defined in Eq. 3.4b.

Response to Filtered White Noise

The solution for the response to a FWN input is obtained by using Eqs. 3.6. For $m \leq 5$,

$$\begin{aligned}\lambda_{m,ij} &= -\ Re\Big[\sum_{k=1}^{8} R_k(z_k)\Big] && \text{case 1} \\ &= -\ Re\Big[\sum_{k=1}^{4} \bar{R}_k(z_k)\Big] && \text{case 2} \\ &= -\ Re\Big[\sum_{k=1,2} \bar{R}_k(z_k) + \sum_{k=3,4,7,8} R_k(z_k)\Big] && \text{case 3} \\ &= -\ Re\Big[\sum_{k=3,4} \bar{R}_k(z_k) + \sum_{k=1,2,5,6} R_k(z_k)\Big] && \text{case 4}\end{aligned} \tag{3.9a}$$

where case 1: $\varsigma_g \neq \varsigma_i$ or $\omega_g \neq \omega_i$, and $\varsigma_g \neq \varsigma_j$ or $\omega_g \neq \omega_j$, case 2: $\varsigma_g = \varsigma_i$ and $\omega_g = \omega_i$, and $\varsigma_g = \varsigma_j$ and $\omega_g = \omega_j$, case 3: $\varsigma_g = \varsigma_i$ and $\omega_g = \omega_i$, and $\varsigma_g \neq \varsigma_j$ or $\omega_g \neq \omega_j$, and case 4: $\varsigma_g \neq \varsigma_i$ or $\omega_g \neq \omega_i$, and $\varsigma_g = \varsigma_j$ and $\omega_g = \omega_j$. Also, z_k are as in Eqs. 3.3b and 3.5c. The residue of order 1 at pole z_1 is given by

$$R_1(z_1) = \frac{ln(z_1)z_1^m(\omega_g^4 + 4\varsigma_g^2\omega_g^2 z_1^2)}{\Pi_{k=2}^{8}(z_1 - z_k)} \tag{3.9c}$$

Similar expressions hold for the residues of order 1 at the remaining poles. The residue of order 2 at pole z_1 is given by

$$\bar{R}_1(z_1) = \frac{z_1^{m-1}}{(z_1 - z_2)^2 \Pi_{k=3,4,7,8}(z_1 - z_k)} \Big[8\varsigma_g^2 \omega_g^2 z_1^2 ln(z_1) + [1 + m\ ln(z_1)](\omega_g^4 + 4\varsigma_g^2 \omega_g^2 z_1^2) - z_1 ln(z_1)(\omega_g^4 + 4\varsigma_g^2 \omega_g^2 z_1^2)\big(\frac{2}{z_1 - z_2} + \sum_{k=3,4,7,8} \frac{1}{z_1 - z_k}\big)\Big] \tag{3.9d}$$

and $\bar{R}_2(z_2)$ is similar with the positions of z_1 and z_2 switched. Also, the residue $\bar{R}_3(z_3)$ is given by

$$\bar{R}_3(z_3) = \frac{z_3^{m-1}}{(z_3 - z_4)^2 \Pi_{k=1,2,5,6}(z_3 - z_k)} \Big[8\varsigma_g^2 \omega_g^2 z_3^2 ln(z_3) + (1 + m\ ln(z_3))(\omega_g^4 + 4\varsigma_g^2 \omega_g^2 z_3^2) - z_3 ln(z_3)(\omega_g^4 + 4\varsigma_g^2 \omega_g^2 z_3^2)\big(\frac{2}{z_3 - z_4} + \sum_{k=1,2,5,6} \frac{1}{z_3 - z_k}\big)\Big] \tag{3.9e}$$

and $\bar{R}_4(z_4)$ is similar with the positions of z_3 and z_4 switched.

Note that $\lambda_{m,ij}$ does not exist when $m \geq 5$.

3.3 Nonstationary Response

In this section, expressions for the the numerical computation of the generic cross-modal response of an MDOF system to an evolutionary excitation are developed. In section 3.3.1, a convenient model for the modulation function $A(\omega, t)$ is proposed, which is used to describe the evolutionary, nonstationary input. Based on this model, expressions for computing the generic modal terms, $\Phi_{ij}^{(m_1,m_2)}(\omega, t_1, t_2)$ and $\phi_{ij}^{(m_1,m_2)}(t_1, t_2)$, are presented in sections 3.3.2 and 3.3.3, respectively.

3.3.1 Models of Modulating Function

A nonstationary process with the evolutionary spectrum is described by the PSD function, $\Phi(\omega)$, of the independent-increment process $dS(\omega)$ and the modulating function $A(\omega, t)$, see Eq. 2.71. $\Phi(\omega)$ defines the overall frequency content of the process, whereas the modulating function $A(\omega, t)$ describes its evolutionary behavior in both frequency and time domains. In general, the modulating function $A(\omega, t)$ is assumed to be slowly varying in both ω and t.

Exact, closed form solutions of the response of a linear oscillator or MDOF system to an evolutionary input with a general modulating function are not available, or are extremely lengthy due to the multi-fold integrals that are involved. As a result, only a few simple cases of the modulating function have been considered, mostly as a function $A(t)$ of time alone. Lin (1963) used $A(t)$ as a step or half sine function; Barnoski and Maurer (1969) used a rectangular function; Hasselman (1972) used a stair case function; Roberts (1971) used a periodic function; Saragoni and Hart (1974) used a gamma function; Shinozuka and Sato (1967), Corotis and Marshall (1977), and To (1982, 1984) used an exponential function; and Gasparini (1979) used a piecewise linear time function. The above models all produce nonstationary processes which are evolutionary only in the time domain; i.e., their frequency contents remain fixed in time. This class of evolutionary processes are known as uniformly modulated process.

In order to provide a flexible yet effective means for implementing the evolutionary process model in STOCAL–II, we have assumed $A(\omega, t)$ to be a piecewise linear function in both frequency and time variables:

$$A(\omega, t) = \sum_{l} \sum_{k} A_{lk}(\omega, t) \tag{3.10a}$$

where

$$A_{lk}(\omega, t) = a_{lk}\omega t + b_{lk}\omega + c_{lk}t + d_{lk} \qquad \Omega_l \le \omega \le \Omega_{l+1};\; T_k \le t \le T_{k+1} \tag{3.10b}$$

$$= 0 \qquad \text{elsewhere}$$

where Ω_l, $l = 1, 2, ...$, and T_k, $k = 1, 2, ...$, are the selected frequency and time points (denoted transition points), respectively, and a_{lk}, b_{lk}, c_{lk} and d_{lk} are given by

$$a_{lk} = \frac{A(\Omega_l,T_k) - A(\Omega_l,T_{k+1}) - A(\Omega_{l+1},T_k) + A(\Omega_{l+1},T_{k+1})}{(T_{k+1} - T_k)(\Omega_{l+1} - \Omega_l)}$$

$$b_{lk} = \frac{T_{k+1}[A(\Omega_{l+1},T_k) - A(\Omega_l,T_k)] + T_k[A(\Omega_l,T_{k+1}) - A(\Omega_{l+1},T_{k+1})]}{(T_{k+1} - T_k)(\Omega_{l+1} - \Omega_l)}$$

$$c_{lk} = \frac{\Omega_{l+1}[A(\Omega_l,T_{k+1}) - A(\Omega_l,T_k)] + \Omega_l[A(\Omega_{l+1},T_k) - A(\Omega_{l+1},T_{k+1})]}{(T_{k+1} - T_k)(\Omega_{l+1} - \Omega_l)}$$

$$d_{lk} = \frac{T_{k+1}\Omega_{l+1}A(\Omega_l,T_k) - T_k\Omega_{l+1}A(\Omega_l,T_{k+1}) - T_{k+1}\Omega_l A(\Omega_{l+1},T_k)}{(T_{k+1} - T_k)(\Omega_{l+1} - \Omega_l)} + \frac{T_k\Omega_l A(\Omega_{l+1},T_{k+1})}{(T_{k+1} - T_k)(\Omega_{l+1} - \Omega_l)} \tag{3.10c}$$

In STOCAL–II, $A(\omega,t)$ is specified by providing the values $A(\Omega_l,T_k)$ at the transition points.

When $A(\omega,t) = A(t)$ is only a function of time, i.e., a uniform modulating function, we assume the piecewise linear model

$$A(t) = \sum_k A_k(t) \tag{3.11a}$$

$$\begin{aligned} A_k(t) &= c_k t + d_k \qquad && T_k \le t \le T_{k+1} \\ &= 0 && \text{elsewhere} \end{aligned} \tag{3.11b}$$

in which T_k, $k = 1,2,...$, are the selected transition points and a_k and b_k are given by

$$\begin{aligned} c_k &= \frac{A(T_{k+1}) - A(T_k)}{T_{k+1} - T_k} \\ d_k &= \frac{-T_k A(T_{k+1}) + T_{k+1}A(T_k)}{T_{k+1} - T_k} \end{aligned} \tag{3.11c}$$

In STOCAL–II, $A(t)$ is specified by providing the values $A(T_k)$ at the transition points. Obviously, the uniform modulating function $A(t)$ is a simple case of the more general modulating function $A(\omega,t)$ in Eqs. 3.10 with $l = 2$, $a_{lk} = 0$, $b_{lk} = 0$, $\Omega_1 = 0$ and $\Omega_2 = \infty$. However, this model is included in STOCAL–II because the corresponding computations are much efficient that they would be if $A(t)$ was specified as a special case of $A(\omega, t)$. An illustration of this model for fitting a target modulating function is shown in Fig. 3.1.

It must be pointed out that in general $A(\omega, t)$ can be a complex valued function. However, in the present development we have considered $A(\omega, t)$ only as a real function. The extension to the more general case can be easily done with the present formulation. However, at present, it does not seem to have practical relevance.

Since $A(\omega, t)$ and $A(t)$ are piecewise linear functions, their time derivatives at the transition points along the time axis do not exist. This creates a problem in computing the statistics of the response derivatives at the transition points. This problem is avoided by taking the derivatives of $A(\omega, t)$ or $A(t)$ slightly removed the transition points. This is a reasonable approximation, since the modulating function is slowly varying and its derivative has a small contribution to the response statistics.

The effort needed for computing the autocorrelation function with the evolutionary model is roughly proportional to the number of frequency points, Ω_l, and to the square of the number of time points, T_k. For the uniform modulating function, computing effort is roughly proportional to the square of the number of time points. Thus, to reduce the computation time, one should use as few frequency and time points (especially the latter) as are necessary to adequately describe the target modulating function.

3.3.2 Modal Evolutionary Cross-PSD Functions

In Eq. 2.90, the $(m_1 + m_2)$-th order modal evolutionary cross-PSD function is given by

$$\Phi_{ij}^{(m_1,m_2)}(\omega, t_1, t_2) = M_i^{(m_1)}(\omega, t_1) M_j^{*(m_2)}(\omega, t_2) \Phi_{ff}(\omega) \tag{3.12a}$$

where the asterisk denotes a complex conjugate, and

$$M_i^{(m)}(\omega,t) = e^{-i\omega t}\frac{\partial^m}{\partial t^m}\int_0^t A_f(\omega,\tau)h_i(t-\tau)e^{i\omega\tau}\,d\tau \tag{3.12b}$$

Specific formulas for computing $M_i^{(m)}(\omega,t)$ for the modulating functions described in the previous section are presented below.

Using Eqs. 3.10 and 3.12b,

$$M_i^{(m)}(\omega,t) = \sum_k M_{ik}^{(m)}(\omega,t) \tag{3.13a}$$

where

$$M_{ik}^{(m)}(\omega,t) = e^{-i\omega t}\frac{\partial^m}{\partial t^m}\left\{\int_{T_k}^{min(t,T_{k+1})}[(a_k\omega+c_k)\tau+(b_k\omega+d_k)]\; h_i(t-\tau)e^{i\omega\tau}\,d\tau\right\} \tag{3.13b}$$

Using direct integration,

$$\begin{aligned} M_{ik}^{(m)}(\omega,t) =& \frac{-1}{2\omega_i\sqrt{1-\varsigma_i^2}}\sum_{r=0}^{1}\sum_{p=1}^{2}(-1)^{r+p}e^{iz_{ip}(t_1-T_{k+r})}e^{-i\omega(t_1-T_{k+r})} \\ & (iz_{ip})^m\exists(a_k,b_k,c_k,d_k,z_{ip},\omega,T_{k+r},0) \qquad T_{k+r}\le t \\ =& \frac{-1}{2\omega_i\sqrt{1-\varsigma_i^2}}\sum_{r=0}^{1}\sum_{p=1}^{2}(-1)^{r+p}e^{iz_{ip}(t_1-T_{k+r})}e^{-i\omega(t_1-T_{k+r})} \\ & \exists(a_k,b_k,c_k,d_k,z_{ip},\omega,t,m) \qquad T_{k+r}>t \end{aligned} \tag{3.13c}$$

where z_{i1} and z_{i2} respectively are z_1 and z_2 defined as in Eq. 3.3b, and

$$\begin{aligned}\exists(a,b,c,d,z,\omega,T,m) =& (i)^m\left[\frac{i(a\omega^{m+1}+c\omega^m)}{(\omega-z)^2}+\frac{(aT+b)\omega^{m+1}}{\omega-z}\right. \\ & \left.+\frac{(d-ima+cT)\omega^m - imc\omega^{m-1}}{\omega-z}\right]\end{aligned} \tag{3.13d}$$

The results for the uniform modulating function are obtained from the above expressions by setting $a_k=0$ and $b_k=0$.

3.3.3 Modal Cross-Correlation Functions

Two ways are used to compute the correlation function of the evolutionary response. One is to compute the correlation function as the Fourier transform of the evolutionary PSD function. For example, the cross-correlation function between the m_1-th and m_2-th derivatives of the responses $z_1(t)$ and $z_2(t)$ is obtained from

$$\phi_{z_1 z_2}^{(m_1,m_2)}(t_1,t_2) = \int_{-\infty}^{\infty} \Phi_{z_1 z_2}^{(m_1,m_2)}(\omega,t_1,t_2) e^{i\omega(t_1-t_2)} d\omega \tag{3.14}$$

in which $\Phi_{z_1 z_2}^{(m_1,m_2)}(\omega,t_1,t_2)$ denotes the (m_1+m_2)-th order evolutionary cross-PSD function that is computed in terms of the modal cross-PSD function $\Phi_{ij}^{(m_1,m_2)}(\omega,t_1,t_2)$ by use of the modal superposition rule stated in Eq. 2.95. The STOCAL–II commands for computing Fourier transform is used for this purpose.

In the second method, the modal cross-correlation functions are first computed by use of direct integration, which are then combined to obtain the response correlation function. In this approach, the (m_1+m_2)-th order modal cross-correlation function $\phi_{ij}^{(m_1,m_2)}(t_1,t_2)$ is the generic solution sought. Expressions for computing this term are presented in the remainder of this section.

To compute $\phi_{ij}^{(m_1,m_2)}(t_1,t_2)$, the PSD function $\Phi_{ff}(\omega)$ associated with the independent-increment process of the evolutionary input is assumed to be a piecewise linear function

$$\Phi_{ff}(\omega) = \sum_l \Phi_l(\omega) \tag{3.15a}$$

$$\begin{aligned} \Phi_l(\omega) &= \alpha_l|\omega| + \beta_l \qquad && \Omega_l \le |\omega| \le \Omega_{l+1} \\ &= 0 && \text{elsewhere} \end{aligned} \tag{3.15b}$$

Based on this model, the (m_1+m_2)-th order modal cross-correlation function is given by

$$\phi_{ij}^{(m_1,m_2)}(t_1,t_2) = \sum_l \phi_{ij,l}^{(m_1,m_2)}(t_1,t_2) \tag{3.16a}$$

$$\phi_{ij,l}^{(m_1,m_2)}(t_1,t_2) = \int_{\Omega_l}^{\Omega_{l+1}} (\alpha_l\omega + \beta_l) M_i^{(m_1)}(\omega,t_1) M_j^{(m_2)*}(\omega,t_2) e^{i\omega(t_1-t_2)} d\omega \tag{3.16b}$$

in which $M_i^{(m)}(\omega,t)$ is as in Eq. 3.12b. Based on Eqs. 3.13c for the piecewise linear modulating function, and using direct integration,

$$\begin{aligned}\phi_{ij,l}^{(m_1,m_2)}(t_1,t_2) &= \int_{\Omega_l}^{\Omega_{l+1}} (\alpha_l\omega + \beta_l) M_i^{(m_1)}(\omega,t_1) M_j^{*(m_2)}(\omega,t_2) e^{i\omega(t_1-t_2)} d\omega \\ &= \frac{1}{2\omega_i\sqrt{1-\varsigma_i^2}} \frac{1}{2\omega_j\sqrt{1-\varsigma_j^2}} \sum_{k_1}\sum_{k_2}\sum_{r_1=0}^{1}\sum_{r_2=0}^{1}\sum_{p_1=1}^{2}\sum_{p_2=1}^{2} \\ &\quad (-1)^{r_1+r_2+p_1+p_2} e^{iz_{ip_1}(t_1-T_{k_1+r_1})} e^{-iz_{jp_2}^*(t_2-T_{k_2+r_2})} \\ &\quad (iz_{ip_1})^{\overline{m}_1}(-iz_{ip_2})^{\overline{m}_1} \Pi(\alpha_l,\beta_{l,k_1},b_{k_1},c_{k_1},d_{k_1},a_{k_2},b_{k_2}, \\ &\quad c_{k_2},d_{k_2},z_{ip_1},z_{jp_2}^*,\Omega_l,\Omega_{l+1},\overline{T}_1,\overline{T}_2,\overline{m}_1,\overline{m}_2)\end{aligned} \tag{3.16c}$$

where $\overline{m}_1 = m_1$, $\overline{m}_2 = m_2$, $\overline{T}_1 = t_1$ and $\overline{T}_2 = t_2$ for $T_{k_1+r_1} > t_1$ and $T_{k_2+r_2} > t_2$; $\overline{m}_1 = m_1$, $\overline{m}_2 = 0$, $\overline{T}_1 = t_1$ and $\overline{T}_2 = T_{k_2+r_2}$ for $T_{k_1+r_1} > t_1$ and $T_{k_2+r_2} < t_2$; $\overline{m}_1 = 0$, $\overline{m}_2 = m_2$, $\overline{T}_1 = T_{k_1+r_1}$ and $\overline{T}_2 = t_2$ for $T_{k_1+r_1} < t_1$ and $T_{k_2+r_2} > t_2$; $\overline{m}_1 = 0$, $\overline{m}_2 = 0$, $\overline{T}_1 = T_{k_1+r_1}$ and $\overline{T}_2 = T_{k_2+r_2}$ for $T_{k_1+r_1} < t_1$ and $T_{k_2+r_2} < t_2$; and

$$\begin{aligned}&\Pi(\alpha,\beta,a_1,b_1,c_1,d_1,a_2,b_2,c_2,d_2,Z_1,Z_2,\Omega_1,\Omega_2,T_1,T_2,m_1,m_2,) = \\ &\quad (i)^{m_1}(-i)^{m_2}\Big\{(a_1T_1+b_1)(a_2T_2+b_2)[\alpha J2(Z_1,Z_2,m+3,\Omega_1,\Omega_2,\Delta T) \\ &\quad + \beta J2(Z_1,Z_2,m+2,\Omega_1,\Omega_2,\Delta T)] + [(d_1 - im_1a_1 + c_1T_1)(a_2T_2+b_2) \\ &\quad + (a_1T_1+b_1)(d_2 + im_2a_2 + c_2T_2)][\alpha J2(Z_1,Z_2,m+2,\Omega_1,\Omega_2,\Delta T) \\ &\quad + \beta J2(Z_1,Z_2,m+1,\Omega_1,\Omega_2,\Delta T)] + [i(a_1T_1+b_1)m_2c_2 \\ &\quad + (d_1 - im_1a_1 + c_1T_1)(d_2 + im_2a_2 + c_2T_2) - im_1c_1(a_2T_2+b_2)]\end{aligned}$$

$$[\alpha J2(Z_1, Z_2, m+1, \Omega_1, \Omega_2, \Delta T) + \beta J2(Z_1, Z_2, m, \Omega_1, \Omega_2, \Delta T)]$$

$$- [\mathrm{i}(d_1 - \mathrm{i}m_1 a_1 + c_1 T_1) m_2 c_2 - \mathrm{i}m_1 c_1 (d_2 + \mathrm{i}m_2 a_2 + c_2 T_2)]$$

$$[\alpha J2(Z_1, Z_2, m, \Omega_1, \Omega_2, \Delta T) + \beta J2(Z_1, Z_2, m-1, \Omega_1, \Omega_2, \Delta T)]$$

$$+ m_1 c_1 m_2 c_2 [\alpha J2(Z_1, Z_2, m-1, \Omega_1, \Omega_2, \Delta T)$$

$$+ \beta J2(Z_1, Z_2, m-2, \Omega_1, \Omega_2, \Delta T)] - \mathrm{i}(a_1 T_1 + b_1) a_2$$

$$[\alpha J3(Z_1, Z_2, m+3, \Omega_1, \Omega_2, \Delta T) + \beta J3(Z_1, Z_2, m+2, \Omega_1, \Omega_2, \Delta T)]$$

$$- \mathrm{i}[(a_1 T_1 + b_1) c_2 + (d_1 - \mathrm{i}m_1 a_1 + c_1 T_1) a_2]$$

$$[\alpha J3(Z_1, Z_2, m+2, \Omega_1, \Omega_2, \Delta T) + \beta J3(Z_1, Z_2, m+1, \Omega_1, \Omega_2, \Delta T)]$$

$$- \mathrm{i}[(d_1 - \mathrm{i}m_1 a_1 + c_1 T_1) c_2 - \mathrm{i}m_1 c_1 a_2]$$

$$[\alpha J3(Z_1, Z_2, m+1, \Omega_1, \Omega_2, \Delta T) + \beta J3(Z_1, Z_2, m, \Omega_1, \Omega_2, \Delta T)]$$

$$- m_1 c_1 c_2 [\alpha J3(Z_1, Z_2, m, \Omega_1, \Omega_2, \Delta T)$$

$$+ \beta J3(Z_1, Z_2, m-1, \Omega_1, \Omega_2, \Delta T)] + \mathrm{i}a_1 (a_2 T_2 + b_2)$$

$$[\alpha J3(z_{iq}^*, z_{jp}, m+3, \Omega_1, \Omega_2, \Delta T) + \beta J3(z_{iq}^*, z_{jp}, m+2, \Omega_1, \Omega_2, \Delta T)]$$

$$+ \mathrm{i}[c_1 (a_2 T_2 + b_2) + a_1 (d_2 + \mathrm{i}m_2 a_2 + c_2 T_2)]$$

$$[\alpha J3(z_{iq}^*, z_{jp}, m+2, \Omega_1, \Omega_2, \Delta T) + \beta J3(z_{iq}^*, z_{jp}, m+1, \Omega_1, \Omega_2, \Delta T)]$$

$$+ \mathrm{i}[\mathrm{i}a_1 m_2 c_2 + c_1 (d_2 + \mathrm{i}m_2 a_2 + c_2 T_2)]$$

$$[\alpha J3(z_{iq}^*, z_{jp}, m+1, \Omega_1, \Omega_2, \Delta T) + \beta J3(z_{iq}^*, z_{jp}, m, \Omega_1, \Omega_2, \Delta T)]$$

$$- c_1 m_2 c_2 [\alpha J3(z_{iq}^*, z_{jp}, m, \Omega_1, \Omega_2, \Delta T)$$

$$+ \beta J3(z_{iq}^*, z_{jp}, m-1, \Omega_1, \Omega_2, \Delta T)] + a_1 a_2$$

$$[\alpha J4(Z_1, Z_2, m+3, \Omega_1, \Omega_2, \Delta T) + \beta J4(Z_1, Z_2, m+2, \Omega_1, \Omega_2, \Delta T)]$$

$$+ (a_1 c_2 + c_1 a_2)[\alpha J4(Z_1, Z_2, m+2, \Omega_1, \Omega_2, \Delta T) + \beta$$

$$J4(Z_1, Z_2, m+1, \Omega_1, \Omega_2, \Delta T)] + c_1 c_2 [\alpha J4(Z_1, Z_2, m+1, \Omega_1, \Omega_2, \Delta T)$$

$$+ \beta J4(Z_1, Z_2, m, \Omega_1, \Omega_2, \Delta T)] \Big\} \qquad (3.16d)$$

where $m = m_1 + m_2$, $\Delta T = T_1 - T_2$ and the remaining terms are as follows:

$$
\begin{aligned}
J2(z_1, z_2, m, a, b, \tau) &= \int_a^b \frac{\omega^m e^{i\omega t}}{(\omega - z_1)(\omega - z_2)} d\omega \\
&= \frac{1}{z_1 - z_2}[J(z_1, m, a, b, \tau) - J(z_2, m, a, b, \tau)] \qquad z_1 \neq z_2 \\
&= \frac{a^m}{a - z_1} e^{ia\tau} - \frac{b^m}{b - z_1} e^{ib\tau} \\
&\quad + mJ(z_1, m-1, a, b, \tau) + i\tau J(z_1, m, a, b, \tau) \qquad z_1 = z_2
\end{aligned} \tag{3.16e}
$$

$$
\begin{aligned}
J3(z_1, z_2, m, a, b, \tau) &= \int_a^b \frac{\omega^m e^{i\omega t}}{(\omega - z_1)(\omega - z_2)^2} d\omega \\
&= \frac{1}{(z_1 - z_2)^2}[J(z_1, m, a, b, \tau) - J(z_2, m, a, b, \tau) \\
&\quad - (z_1 - z_2) J2(z_2, z_2, m, a, b, \tau)] \qquad z_1 \neq z_2
\end{aligned} \tag{3.16f}
$$

$$
\begin{aligned}
J4(z_1, z_2, m, a, b, \tau) &= \int_a^b \frac{\omega^m e^{i\omega t}}{(\omega - z_1)^2(\omega - z_2)^2} d\omega \\
&= \frac{1}{(z_1 - z_2)^3}\{-2J(z_1, m, a, b, \tau) + 2J(z_2, m, a, b, \tau) \\
&\quad + (z_1 - z_2)[J2(z_1, z_1, m, a, b, \tau) \\
&\quad + J2(z_2, z_2, m, a, b, \tau)]\} \qquad z_1 \neq z_2
\end{aligned} \tag{3.16g}
$$

in which $J(.)$ is the compound exponential integral defined in Appendix A.

3.4 Fourier and Inverse Fourier Transforms

The Fourier transform and its inverse for a function $f(t)$ are defined, respectively, by

$$
\bar{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \; e^{-i\omega t} dt \tag{3.17}
$$

and

$$
f(t) = \int_{-\infty}^{\infty} \bar{f}(\omega) \; e^{i\omega t} d\omega \tag{3.18}
$$

Three algorithms for computing the Fourier and inverse Fourier transforms are included in STOCAL–II. These are the Fourier transform of a piecewise linear function, the discrete Fourier transform and the fast Fourier transform. Among other tasks, these routines are used to compute the PSD function for a given autocorrelation function, and vice versa.

3.4.1 Fourier Transform of a Piecewise Linear Function

Consider $f(t)$ as a real, piecewise linear function defined by

$$f(t) = \sum_k f_k(t) \tag{3.19a}$$

$$\begin{aligned} f_k(t) &= a_k t + b_k \qquad && t_k \le t \le t_{k+1} \\ &= 0 && \text{elsewhere} \end{aligned} \tag{3.19b}$$

By substituting Eqs. 3.19 into Eq. 3.17, the Fourier transform of $f(t)$ is

$$\begin{aligned} \bar{f}(\omega) &= \bar{f}_R(\omega) + \mathrm{i}\ \bar{f}_I(\omega) \\ &= \sum_k \frac{1}{2\pi} \int_{t_k}^{t_{k+1}} f_k(t)\ e^{-\mathrm{i}\omega t} dt \\ &= \sum_k \Big\{ \frac{a_k}{\omega^2}[cos(\omega t_{k+1}) - cos(\omega t_k)] + \frac{b_k}{\omega}[sin(\omega t_{k+1}) - sin(\omega t_k)] \\ &+ \frac{a_k}{\omega}[t_{k+1} sin(\omega t_{k+1}) - t_k sin(\omega t_k)] \Big\} + \mathrm{i} \Big\{ \\ &\frac{-a_k}{\omega^2}[sin(\omega t_{k+1}) - sin(\omega t_k)] + \frac{b_k}{\omega}[cos(\omega t_{k+1}) - cos(\omega t_k)] \\ &+ \frac{a_k}{\omega}[t_{k+1} cos(\omega t_{k+1}) - t_k cos(\omega t_k)] \Big\} \end{aligned} \tag{3.20}$$

When $f(t)$ is a complex, piecewise linear function

$$f(t) = f^R(t) + \mathrm{i}\ f^I(t) \tag{3.21}$$

with $f^R(t)$ and $f^I(t)$ denoting its real and imaginary parts, respectively, the Fourier transform of $f(t)$ is computed from

$$\bar{f}(\omega) = \bar{f}_R(\omega) + \mathrm{i}\bar{f}_I(\omega) = \bar{f}_R^R(\omega) - \bar{f}_I^I(\omega) + \mathrm{i}\ [\bar{f}_I^R(\omega) + \bar{f}_R^I(\omega)] \tag{3.22}$$

where $\bar{f}_R^R(\omega)$ and $\bar{f}_I^R(\omega)$ are the imaginary and real parts of the Fourier transform of $f^R(t)$ and $\bar{f}_R^I(\omega)$ and $\bar{f}_I^I(\omega)$ are those of $f^I(t)$.

Expressions for the inverse Fourier transform are derived in a similar manner and will not be stated here. The STOCAL–II commands used to compute the Fourier and inverse Fourier transforms of a piecewise linear function are *FTP* and *IFTP*.

3.4.2 Fourier Transform for Discrete Data

Discrete Fourier Transform

When the ordinates of $f(t)$ (real or complex) are provided at equidistance time points beginning at t_o and ending at t_{n-1}, for small Δt, Eq. 3.17 can be rewritten in the form

$$\bar{f}(\omega) = \frac{\Delta t}{2\pi}\sum_{k=0}^{n-1} f(t_k)e^{-i\omega t_k} \tag{3.23}$$

In STOCAL–II, the command *FTD* computes the Fourier transform according to this equation by specifying two end-limits of ω_b and ω_e and the number of frequency points m. In a similar manner, the command *IFTD* computes the inverse Fourier transform of a function specified by its ordinates at equidistance time points.

Fast Fourier Transform

When the number of output frequency points is not specified, the fast Fourier transform algorithm is used to compute the Fourier transform of the discrete function. In this case, by substituting $t_k = t_o + k\Delta t$ and letting $\Delta\omega = \frac{2\pi}{n\Delta t}$, Eq. 3.23 becomes

$$\begin{aligned} \bar{f}(\omega_o + j\Delta\omega) &= \frac{\Delta t}{2\pi}\sum_{k=0}^{n-1} f(t_o + k\Delta t)e^{-i(\omega_o + j\Delta\omega)(t_o + k\Delta t)} \\ &= \frac{\Delta t}{2\pi}e^{-i\omega_o t_o}\sum_{k=0}^{n-1} f(t_o + k\Delta t)e^{-i\omega_o k\Delta t}e^{-ij\Delta\omega k\Delta t} \\ &= \frac{\Delta t}{2\pi}e^{-i\omega_o t_o}\sum_{k=0}^{n-1} a_k e^{-ijk\frac{2\pi}{n}} \\ &= \frac{\Delta t}{2\pi}c_j e^{-i\omega_o t_o} \end{aligned} \tag{3.24a}$$

where

$$a_k = f(t_o + k\Delta t)e^{-i\omega_o k\Delta t} \tag{3.24b}$$

$$c_j = \sum_{k=0}^{n-1} a_k e^{-ijk\frac{2\pi}{n}} \tag{3.24c}$$

Eq. 3.24c is in the form of generalized fast Fourier transform algorithm (Dahlquist and Bjorck 1974, Section 9.3). The routines used in CAL were developed by Dicken and Wilson (1980) and are retained in STOCAL–II. These routines can accommodate any number of data points for a discrete function $f(t)$. However, it is more advantageous to have n as a highly composite number. In particular, it is best to have n as a power of 2.

Expressions for the inverse fast Fourier transform for discrete data are derived in a similar manner and will not be stated here. The STOCAL–II commands *FTD* and *IFTD* use the algorithms in Eqs. 3.24 when the number of output data points is not specified. Otherwise, the algorithm in Eq. 3.23 is used.

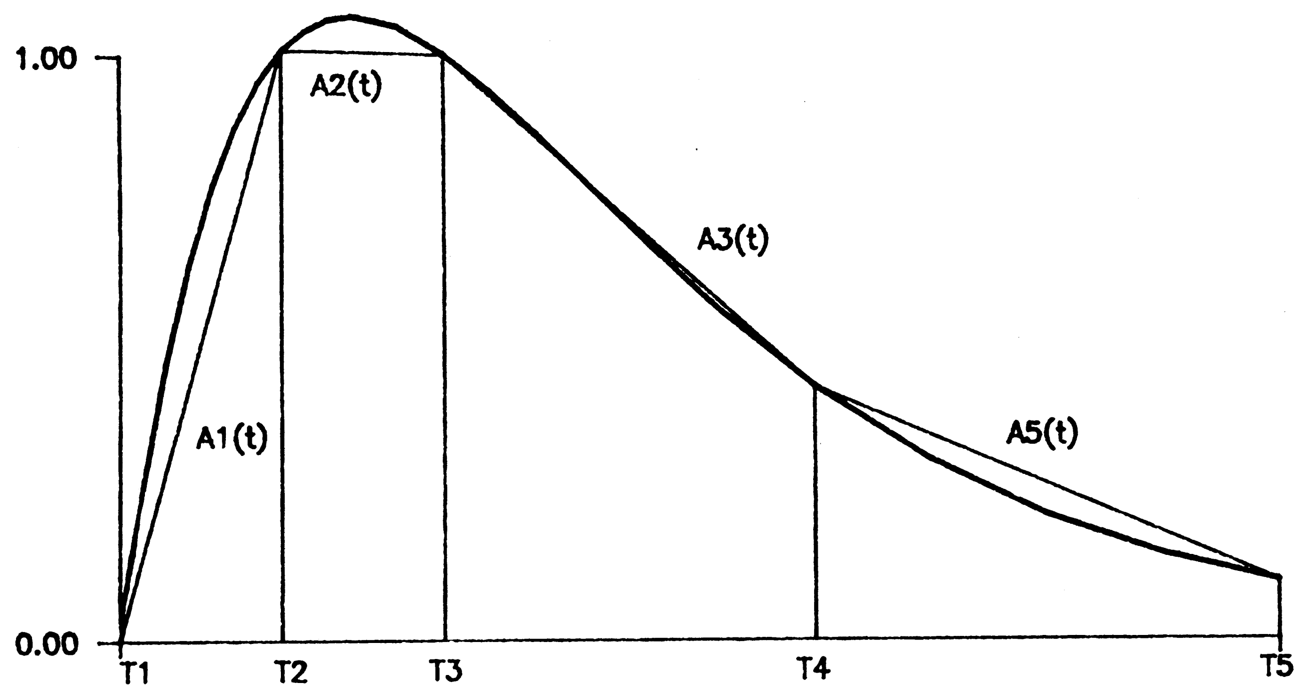


Figure 3.1 The Uniform Modulating Functions

CHAPTER 4

DEVELOPMENT OF STOCAL–II

4.1 Introduction

The development of an instructional program such as STOCAL–II requires a more systematic approach than the development of a program for personal use. The developers, users and maintainers of an instructional software are often distinct entities. Lack of planning will cause difficulties not only in the development and use but also in the future maintenance and expansion of the program. With these concerns in mind, careful consideration and planning were made in the development of STOCAL–II.

An early decision that had to be made was whether the program should be an extension to the program STOCAL by Button et al. (1981), or it should be developed anew. STOCAL is an extension of CAL (Wilson 1979) and performs matrix manipulations, deterministic structural analysis (static and dynamic) and nondeterministic structural analysis. However, the commands for nondeterministic analysis in STOCAL are limited to very simple applications of the stationary random vibration theory and are designed in a black box fashion. Thus, the program has limited instructional capability. Moreover, the solutions in STOCAL are not derived in a general form and cannot be easily extended to nonstationary analysis. Therefore, it was decided to develop STOCAL–II anew and independently of STOCAL. However, the facilities in CAL for matrix manipulations, and for static and dynamic structural analysis are essential for random vibration analysis. Thus, STOCAL–II is developed as an extension to CAL.

STOCAL–II, like CAL, is written in FORTRAN-77. Most application programs in civil engineering have been written in FORTRAN, not only because the language has been in use for a long time, but also because of its proven transferability and portability. Moreover, programs in FORTRAN are efficient and are easy to write for numerical computations. Today, many compilers have the ability to link routines written in different languages into one executable file. Therefore, by proper planning, routines in different languages can be mixed so that functions available in C or other languages can be used in a FORTRAN program.

In 1983, the Civil Engineering Division of the American Society for Engineering Education (ASEE) conducted a survey on the use of microcomputers in civil engineering education (Mcdonough 1985). This survey showed that the use of microcomputers in civil engineering departments was very popular in 1983. Since then, the price of a microcomputer has substantially dropped and the hardware has dramatically improved. Today, the microcomputer has become an affordable educational tool for many college students, and it is gradually becoming a necessity. With these in mind, it was decided to develop STOCAL–II for use on microcomputers. Nevertheless, the program is written in a manner that it can be easily transferred to other machine. Owing to its popularity, the IBM-PC was selected for developing the program.

The next consideration was the necessity of including graphic capabilities, which were not available in CAL. For random vibration analysis graphic features are extremely important as results such as the autocorrelation and PSD functions cannot be easily interpreted without graphic aids. Such aids are particularly essential in an instructional environment. It was decided, therefore, to include a two-dimensional on-line graphics capability in STOCAL–II. Unfortunately, FORTRAN is not a good language for graphics purposes. Therefore, it was decided to use a commercial graphics library available to FORTRAN programmers on IBM-PC. After an evaluation of several options, IBM's Graphics Development Toolkit was selected. This package is hardware independent when the appropriate device driver files are used.

4.2 The Architecture of STOCAL–II

The architecture of STOCAL-II is similar to that of CAL. Some minor modifications, including changes in some basic commands, were made to make the program more user friendly. A detailed description of CAL can be found in the Ph.D. dissertation by Marc I. Hoit (1983). The architecture of STOCAL-II is described in this section.

The structure of STOCAL–II includes four parts: the user interface, command interpreter, analysis modules, and databases (see Fig. 4.1). The user interface refers to the routines that read the input data and produce the output. The command interpreter translates the input data into a form in which the command and its parameters are identified. The routines allow a free-form input. STOCAL–II is a modular system including seven analysis modules. Beside CAL (Wilson 1979) and OTHERS, each module contains a group of commands with similar functions. Each module is able to work independently or in combination with other modules, thus forming a complete program. STOCAL–II includes an internal and an external database. The internal database includes arrays stored in the internal storage while STOCAL–II is in use, whereas the external database includes data files stored on an external memory device.

The detailed description of the user interface, command interpreter, analysis modules and databases is described below.

4.2.1 User Interface

STOCAL-II with the features of CAL allows the user to supply the input data interactively or through batch files. In an interactive mode, the user supplies the data through the keyboard, and the output is shown on the monitor. However, the command *PRINT* sends the output to both the monitor and a specified output file. In the batch mode, the command *SUBMIT* is used to execute a sequence of commands stored in a batch command file. CAL only allows one command file to be accessed within each job, although a different command file may be specified by reinitializing the job. In STOCAL-II, the user is allowed to access more than one command file within one job. The format of the command *SUBMIT* in STOCAL-II is designed to be

SUBMIT Sep **FileName** *N=n*

This command causes the input data to be read from an input file, **FileName**. The program executes the commands in **FileName** starting with the separator Sep until a *RETURN* or a blank statement is encountered. The process is executed *n* times. If **FileName** is not specified, the default name, specified when the program is initialized, is used. The output message in a batch mode is sent to the default output file.

If the available RAM size of the computer when STOCAL–II in use is big enough, STOCAL-II also allows the user to execute a DOS command by using the command *SYS*. With this feature, the user may edit a sequence of commands in a command file without exiting STOCAL–II.

Graphic operations are allowed only in an interactive mode. The graphic output can be directed to the monitor screen and/or to a plotter. The monitor screen is divided into three windows: an input window, and an output window, and a command window (see Fig. 4.2). The input data are supplied through the input window located on the bottom of the screen, the graphic output is shown on the output window located on the upper left of the screen, and the command window on the right side of the screen lists the available commands. The command *PLOT* sends the graphic output to the plotter. A description of each graphic command is shown on the output window by the use of the command *HELP*.

4.2.2 Command Interpreter

Command lines in the input file to STOCAL-II have the syntax

CommandName **Input Matrices** **Output Matrices**+ (**Conditional Matrix**) \
Required Parameters (*Conditional Parameters*) [*Optional Parameters*]

where *CommandName* is the name of the command, **Input Matrices** are the names of a set of previously defined matrices, which are separated by commas or blanks, **Output Matrix** is the name of the generated matrix storing the results of the analysis invoked by the command, **Conditional Matrix** is the name of an additional input matrix which conditionally might be required, *Required Parameters* consist of one or more sets of identifiers and parameters in the form *P=p1,p2,p3* or *I=i1,i2,i3*, and *Conditional Parameters* and *Optional Parameters* are similar, but are either conditionally used or are optional. The only exceptions to the above rule are continuation lines and data lines immediately following the *LOAD* command. Thus, a typical command line may appear as

CommandName **M1 M2 M3– M4+** *P=p1,p2,p3,.... I=i1,i2,i3,...*

For clarity in describing the syntax of each command, a "+" sign is used to indicate a newly created matrix, a "–" sign is used to indicate a modified matrix. Note that the parentheses or brackets indicating the conditional or optional entries need not be used in the actual command line.

As an example, the command *GSGP*, which is used to generate samples of a stationary Gaussian process with a specified PSD function, has the form:

GSGP **MSG+ (MPI)** *I=type [P=p1,p2,...] N=n,nt T=tb,te [M=m RS=rs]*

where *GSGP* is the command name, **MSG+** is a new matrix to store the generated samples, **MPI** is needed only when *I=3*, and *rs* (a random seed) is selected either by the user or by the default value. Some simple mathematical operations producing numerical data are allowed in the parameter list but they are evaluated only in sequential order to compute a real type data. Integer, real, exponential and Hollerith data are allowed in the operation. For example, a particular use of the command *GSGP* may appear as

GSGP **MSG+** *I=1 P=10E3*12/3 N=0,101 T=0,5*

which is the same as

GSGP **MSG+** *I=1 N=0,101 T=0,5 P=4000*

The input command lines are read and interpreted by the command interpreter through a set of routines. The most basic routines in CAL for this purpose are FREE, FREEI, FREER and FREEH. The FREE subroutine can read up to 160 characters from either the input file or the terminal. Any input line can be continued for as many times as desired until the 160-character limit is reached. A backslash character may be used at the end of any input line to indicate that the command is to be continued on the following line. The subroutines, FREEI, FREER and FREEH are then used to read integer, real and alphanumeric data, respectively.

4.2.3 Modules

STOCAL–II is designed as a modular system. Each module in STOCAL–II with a basic library routine can be separately compiled and linked to form an independent executable file. With this formation, the complete STOCAL–II system includes several executable files sharing with each other the data stored in the internal or external data files. This system is easy for testing and debugging. However, it is inconvenient when needed commands reside in different modules. In that case, the user has to exit from one module and execute another module.

Another way to form STOCAL–II is to link all the modules into a single executable file. In this case, the data generated by one command can be immediately used by all other commands. In general, this formulation of STOCAL–II is more desirable, except when the required memory exceeds the available RAM size of the computer. Fortunately, an overlay feature in the modern compiler allows reducing the required resident RAM size of the program. Therefore, the program can be installed in a microcomputer, although a very limited RAM size is available in such a machine. The regular version of STOCAL–II is formed in this way, and, with the use of Microsoft (version 4.0) compiler, the minimum load size of STOCAL–II is around 47000 bytes. For cases where the smaller load size of the program is required, the graphics module may be separated from the main program. In this case, two executable files are formed where the load size of the larger program are around 42000 bytes.

STOCAL–II is divided into seven modules: CAL, CALSR, CALNR, CALGEN, CALFT, CALPLT and OTHERS. CAL developed by Wilson (Wilson 1979) is for the purpose of deterministic (static and dynamic) analysis of structures, and is incorporated into **STOCAL–II** without major modifications. The commands available in CAL is listed Table 4.1 and their detailed descriptions can be found in the Ph. D. dissertation by Marc. I. Hoit (1983). The STOCAL–II commands available in each module are also listed in Table 4.1, and a simple description of each module, with the exception of CAL is given in the following paragraphs:

CALSR – This module includes commands for computing the response of linear systems to stationary excitations. The input excitation can be a white noise, banded linear noise, a filtered white noise or through a piecewise linear PSD function. The computed results include the auto and cross PSD and correlation functions and the spectral moments of the response process and its time derivatives. In addition, by providing the spectral moments of order 0, 1, 2, and 4, various statistics of a stationary Gaussian process are computed. The statistics include the apparent frequency, regularity factors, mean upcrossing rates, CDF and PDF of local peaks, mean, standard deviation, CDF and PDF of the extreme over a specified interval, and statistics of the envelope process.

CALNR – This module includes commands for computing the response of linear systems to nonstationary excitations. The evolutionary model is used for nonstationary analysis. The input is specified in terms of a time or a frequency-and-time modulated PSD function. The computed results include the evolutionary auto and cross PSD and correlation functions of the response and its time derivatives. The upcrossing rates and the distributions of local and extreme peaks can also be obtained.

CALGEN – This module includes commands for generation or estimation of samples of random numbers or random processes. Commands are available for generation of samples of random numbers with prescribed probability distributions, and for generation of sample functions of stationary and nonstationary Gaussian processes. For stationary processes, samples are generated by either specifying the PSD or the autocorrelation function. For nonstationary processes, samples are generated by specifying the modulating and PSD functions in a piecewise linear form. Parallel commands are available for estimating the second-order statistics, i.e., the temporal and ensemble mean, variance and autocorrelation functions, of specified samples.

CALFT – This module includes commands for Fourier and inverse Fourier transformation of piecewise linear functions by an exact integration method, and commands for discrete Fourier and inverse Fourier transformation using DFT and FFT algorithms. These commands can be used to compute the PSD for a given autocorrelation function, or vice versa.

CALPLT – This module includes commands for interactive graphics specifically designed for an IBM-PC (or compatible) computer. The commands can be used interactively to draw 2-D curves on a display and a plotter. This facility is particularly helpful for gaining insight into the stochastic response, which is achieved by readily observing plots of response quantities such as PSD and correlation functions. The graphics facility is also useful in verifying the accuracy of input data.

MISCELLANEOUS – This module includes several commands of general nature which are not included in the above modules. Commands are available for computing the modal participation factors and MEPF's, which are needed in the structural analysis. Commands are also available for writing the data from the internal database to an ASCII file. The command *HELP* can be used to invoke aid on the computer.

See the command summary in Appendix B or the companion report (Wung and Der Kiureghian 1989) for a detailed description of each command in STOCAL–II.

4.2.4 Data Management

The data management in STOCAL–II, as in CAL, employs the method of dynamic storage allocation. In this method, all arrays are stored in a single one dimensional storage named **IA** declared as an integer array, but the storage is used to store all kinds of data. A 4-byte word is used to store an integer-type datum, an 8-byte word is used to store a floating point datum with double precision, and an 1-byte word is used to store a Hollerith-type datum. The location of each integer or real array is calculated and kept track of by the program. The layout of the storage is illustrated in Fig. 4.3. This approach greatly increases the capacity of the program by allowing the data storage to adjust to the requirements of each problem. However, care must be taken in counting the array addresses, since character, integer and real data are stored in the same storage. In practice, it is undesirable for the user to change the maximum storage frequently. Therefore, a sufficiently large array is opened to accommodate most analysis needs, and which satisfies the hardware and compiler limitations. In this manner, the usage of the storage is not efficient for most analysis. Fortunately, this is not a problem in a modern computer, where the technique of memory paging is used. Nevertheless, care must be taken so that no word is split by a page boundary.

Another important feature of **STOCAL–II** is that it allows storing data on an external file, which may be used in a later job or by other programs with the same data management. The FORTRAN subroutine transferring the internal data to an external file is as follows.

```
SUBROUTINE SAVE1
CHARACTER*1 NOP,CH*20
COMMON MTOT,NP,IA(66000)
COMMON /DBSYS/ NUMA,NEXT,IDIR,IP(3)
COMMON /IOLIST/ NTM,NTR,NIN,NOT,NSP,NFL,NT7,NT8
WRITE (NFL) NUMA,NEXT,IP
IC = IDIR
DO 10 J=1,NUMA
   WRITE(NFL) (IA(K),K=IC,IC+9)
   IC = IC + 10
```

```
10      CONTINUE
        WRITE (NFL) (IA(I),I=1,NEXT)
        CLOSE(NFL)
        RETURN
        END
```

where IA, NEXT and IDIR are as shown in Fig. 4.3, NUMA is the number of arrays available in the storage, and IP(3) is a vector containing the word length in bytes for integer, real and character data. Note that the directory information for each array requires the length of 10 integer-type data.

A series of commands are available in STOCAL–II for the user to manipulate the internal and external data bases. These are as follows:

START **F1**

The *START* command initializes the program and its internal database. **F1** is assigned as the default name of the database.

READC [**F1**] [(**M1**+)]

The *READC* command reads all arrays (or only the array **M1**) from the external data file **F1**.COR and puts the arrays (or **M1**) into the internal database. The original names of the arrays are retained. If **F1** is not supplied, the default data file name is used. Also, if **M1** is not supplied all arrays are transferred into the internal database; otherwise, only **M1** is transferred.

LIST [**F1**] or *L* [**F1**]

The *LIST* or *L* command displays a list of the names and sizes of all arrays in the external data file **F1**.COR. When **F1** is not supplied, the arrays in the internal database are listed.

SAVE [**F1**]

The *SAVE* command saves all arrays in the internal database to the external data file **F1**.COR. When **F1** is not supplied, the default name is used.

STOP or *S*

The *STOP* or *S* command terminates the use of the program and returns the control to the computer's operating system. All arrays in the internal database are stored in the external data file with the default name.

QUIT or *Q*

The *QUIT* or *Q* command leaves the program without saving any data.

DELETE **M-** or *D* **M-**

The *DELETE* or *D* command deletes the array **M** in the internal database. The storage, therefore, is compacted.

LOAD **M+** *R*=? *C*=?

The *LOAD* command creates a real matrix **M** with *R* rows and *C* columns. The data must be supplied one row per line, and must immediately follow the LOAD command. The data is separated by commas, or one or more blanks. A line of data may be continued by the use of a backslash at the end of the line. However, the total length of the line may not exceed 160 characters. If the data for a row is greater than 160 characters, the matrix must be loaded by the use of submatrix operations.

PRINT **M** or *P* **M**

The *PRINT* or *P* command displays the contents of the matrix **M** both on the terminal screen and in the default output file.

4.3 Documentation

The documentation of STOCAL-II includes an on-line help and the user's manual.

4.3.1 On-Line Help

The on-line help feature is designed such that the syntax and usage of each command in STOCAL–II can be seen on the terminal screen by using the command *HELP* **command-name** in the STOCAL–II environment. The description of each command is prepared in an external, ASCII file with an extension name **.HLP**. These files are stored under the directory CAL.HLP, where CAL.HLP and the executable file are installed in the same directory. Since the help files, not part of the executable file, are readable, they can be edited or extended by the user to enhance the help feature. The separation of the help files also makes the executable file as compact as possible. When the computer storage is limited, the help files may not be installed. In general, the help file for each command is a condensed version of the command documentation given in the user's manual in Appendix B.

4.3.2 User's Manual

A complete description of each STOCAL–II command is given in the companion report (Wung and Der Kiureghian 1989). The report used as an user's manual includes an introduction, a command summary, the command syntax, free format conventions, and a description of each available command. In general, the command descriptions in the User's Manual are more detailed than in the help files. Each command description includes the syntax of the command, a detailed description of the required parameters and matrices, and, when appropriate, simple mathematical formulae to illustrate the function of the command.

Table 4.1 Commands in each module

CAL		CALSR	CALNR	CALGEN	CALFT	CALPLT	OTHERS
MULT	LOADI	SPSD	TPSD	GSU	FTP	under PLOT	HELP
TMULT	ADDK	SCF	TCF	GSGP	IFTP	AXIS	AMP
ADD	MEMFRC	SRSM	EPSD	GSGPT	FTD	CLEAN	VECTOR
SUB	EIGEN	SM	ECF	GEGP	IFTD	ERASE	MPF
DUP	JACOBI	SMSM	TRMS	TSSF		HIDE	EPF
STODG	SQREL	SMR	ERMS	TFSU		INIT	WRITE
DUPDG	INVEL	RCQC	TMS	TTSU		PLOT	
SCALE	DYNAM	RSM	EMS	ACF		QUIT	
INVERT	NORM	SSGP	NCR	TACF		RDATA	
SOLVE	MAX	LPKD	NDLP	STAT		SET	
STOSM	STEP	EXTD	NDEP	NFD		TEXT	
DUPSM	RITZ			NCFD		UNIT	
TRUSS	DFT					VIEW	
SLOPE	IDFT					WIND	
FRAME	RADIUS					ZOOM	
FRAME3	FSOLVE						

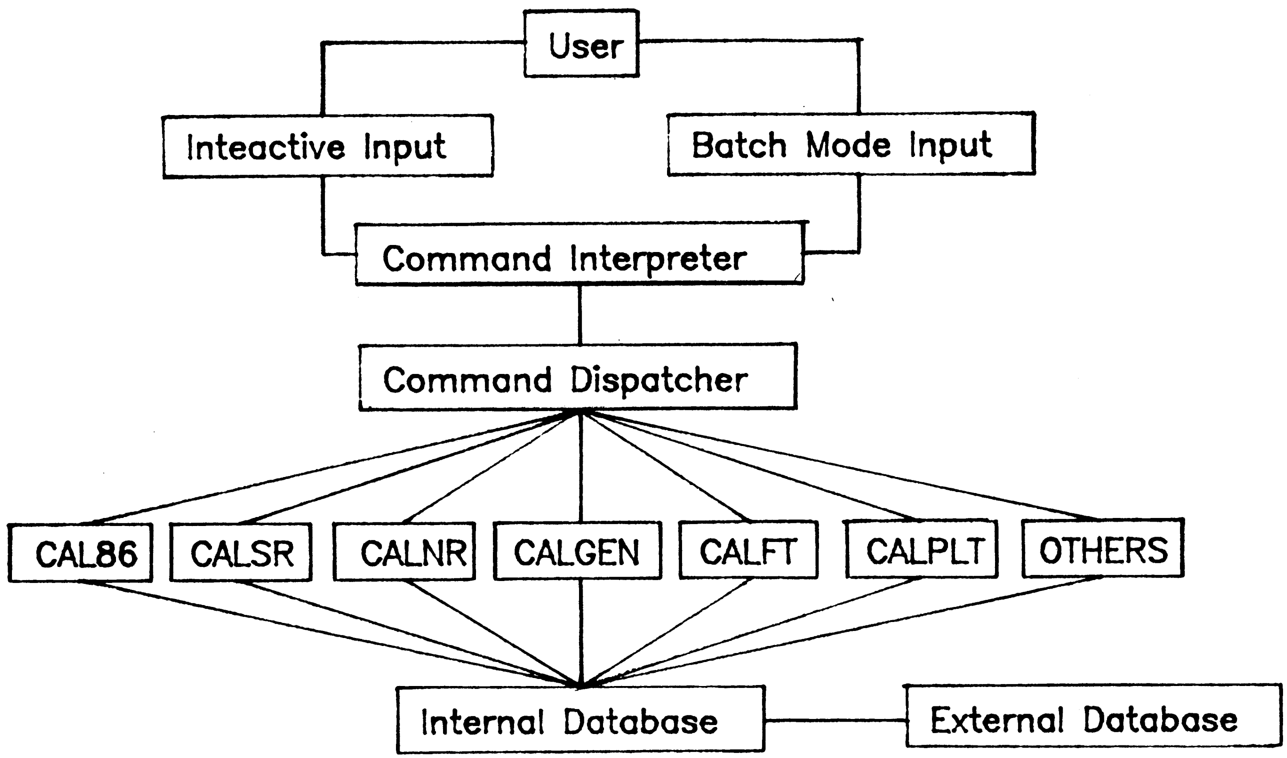


Figure 4.1 Architecture of STOCAL–II

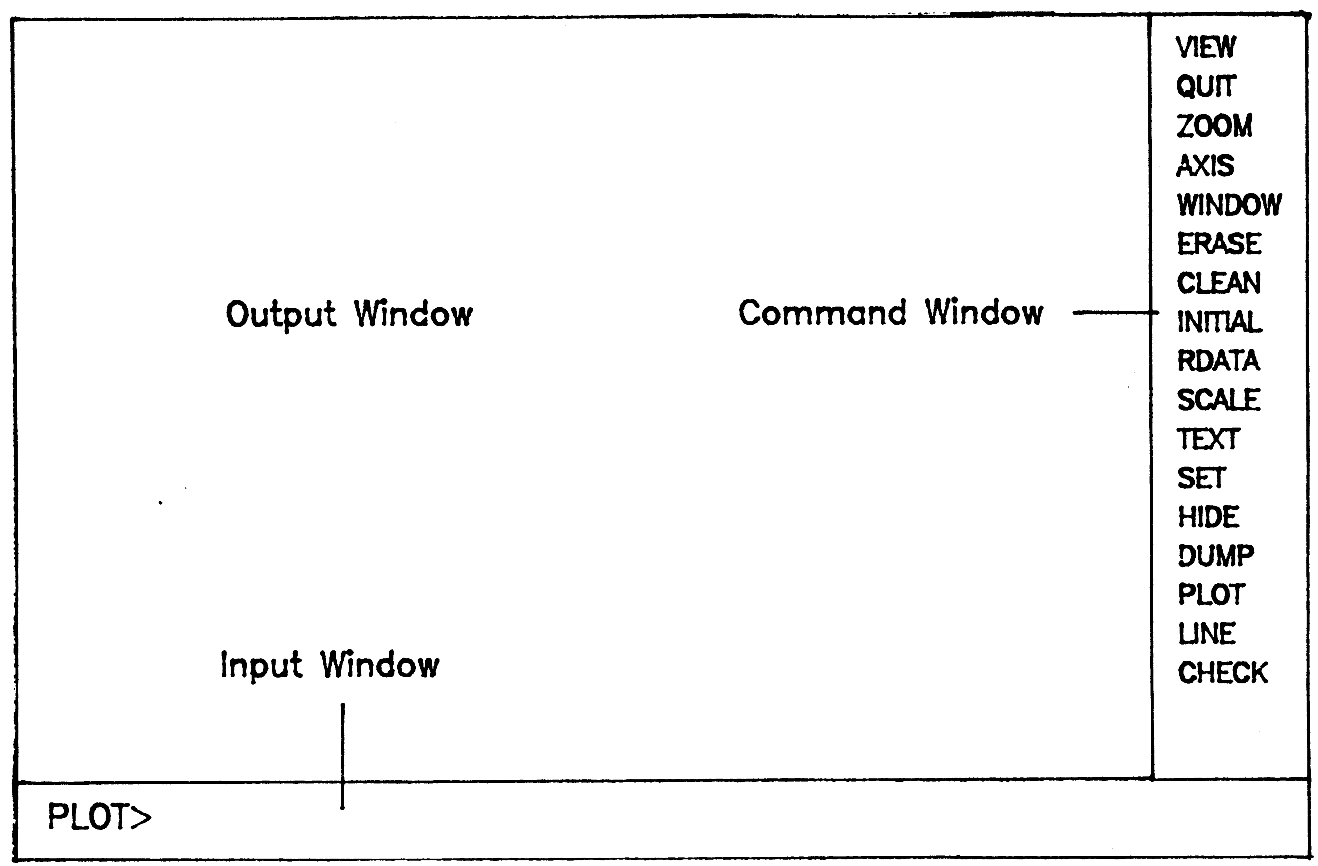


Figure 4.2 Layout of Graphics Screen

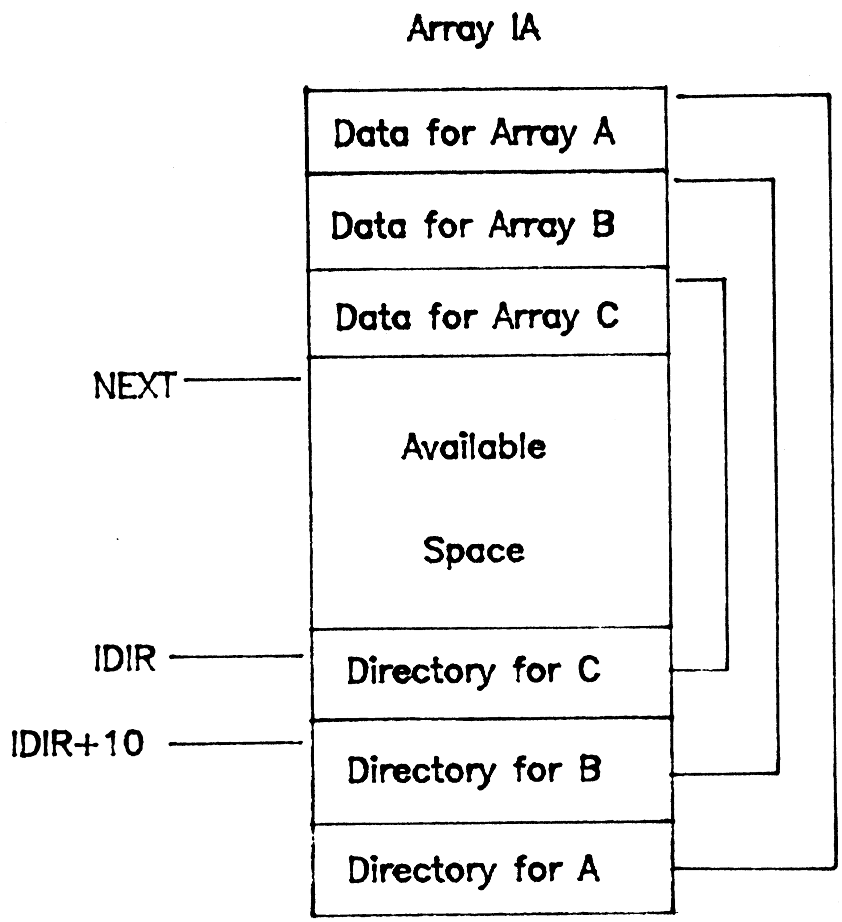


Figure 4.3 Layout of Internal Database

CHAPTER 5

EXAMPLES AND APPLICATIONS

5.1 Introduction

This chapter includes a series of numerical examples which are designed with three goals in mind: (a) To illustrate the use of STOCAL–II in conjunction with CAL; (b) to demonstrate the capabilities of STOCAL–II; and (c) to show how the CAL-STOCAL–II software can be used in teaching random vibrations. The examples are in three parts: Section 5.2 deals with the generation and estimation of random processes; Section 5.3 deals with the modeling and eigenvalue analysis of the structural system; and Section 5.4 deals with the analysis of the structural response to stochastic input. A simple, combined structural system, consisting of a two-story primary structure and an attached secondary subsystem (i.e., a pipe) is considered for this purpose. This system is chosen for its special characteristics, i.e., tuning, interaction, and closely spaced modes, which provide an opportunity to investigate interesting characteristics of the response.

The examples included in this chapter do not cover all the commands of STOCAL–II. However, it is believed that by reviewing these examples, the reader can easily learn the use of the other commands by consulting the User's Manual in Appendix B or the on-line *HELP* command of STOCAL–II.

5.2 Generation and Estimation of Random Processes

The examples in this section illustrate the use of commands *GSGPT* and *GSGP* for generation of a stationary Gaussian process, and the commands *TSSF* and *GEGP* for generation of a nonstationary Gaussian process. Moreover, the commands *ACF* and *TACF* are used to estimate the autocorrelation function of the generated sample, which is then compared with the assumed autocorrelation function of the process.

5.2.1 Stationary Gaussian Process

The CAL command *LOAD* is used to define a triangular autocorrelation function in matrix **CF**:

```
LOAD CF R=2 C=2
0.0 1.0
1.0 0.0
PLOT CF
```

The plot of the triangular autocorrelation function is shown in Fig. 5.1a.

The command *GSGPT* is used to generate 100 sample functions of a zero-mean stationary Gaussian process having the triangular autocorrelation function defined in matrix **CF**. The generated samples, contained in matrix **GV**, are specified at 41 time points over a 4 unit interval of time. **RS** is the seed for the random number generator:

```
GSGPT CF GV N=100,41 T=0,4 RS=0.499
```

The command *VECTOR* is used to generate a vector **GVX** containing 41 equally spaced time points in the interval 0 to 4. This vector is used as the X-coordinate in plotting the first sample function, which is contained in the first row of matrix **GV**:

```
VECTOR GVX T=0,4 N=41
TRAN GV GVT
PLOT GVX GVT
```

The result of the above commands is shown in Fig. 5.1b.

The command *ACF* is used to compute the autocorrelation function of the 100 sample functions in **GV** by ensemble averaging. The computed autocorrelation function is contained in matrix **CF1**:

```
ACF GV CF1 DT=0.1
PLOT CF1
```

The result of the above commands is compared with the known autocorrelation function in Fig. 5.1a.

The command *GPSD* is used to create a matrix **PFWN** to contain the data for a FWN PSD function with the parameters Φ_o=100 in^2/sec^3, ω_g=15.7 rad/sec, and ς_g=0.6 (see Section 2.3.1). The PSD function is specified at 101 equally spaced frequency points in the interval 0 to 50 rad/sec:

```
GPSD PFWN I=4 P=100,15.7,0.6 W=0,50 N=101
PLOT PFWN
```

The plot of the PSD function is shown in Fig. 5.2a.

The command *GSGP* is used to generate one sample function of a zero-mean stationary Gaussian process having the FWN PSD function described above. The sample function, stored in matrix **SFW**, is specified at 401 time points over a 20 unit interval of time. It is generated by superimposing 50 sine functions with random phase angles:

```
GSGP SFW I=4 P=100,15.7,0.6 N=1,401 T=0,20 M=50
TRAN SFW SFWT
VECTOR VT T=0,20 N=401
PLOT VT SFWT
```

The result of the above commands is shown in Fig. 5.2b.

The command *TACF* is used to compute the temporal autocorrelation function of the sample function given in matrix **SFW**. Assuming the process is ergodic, the temporal autocorrelation function asymptotically approaches the true autocorrelation function as the averaging time approaches infinity. An estimate of the PSD function of the process is, therefore, obtained by Fourier transforming the temporal autocorrelation function. Only the first 4 units of time lag of the temporal autocorrelation function are used to obtain the estimated PSD function:

```
TACF SFW SFWC DT=0.05
DUPSM SFWC SFWD L=1,1 R=80 C=2
FTP SFWD SFWP W=0,50 N=51 I=1
PLOT SFWP
```

The result of the above commands is compared with the true PSD function in Fig. 5.2a.

5.2.2 Nonstationary Gaussian Process

The CAL command *LOAD* is used to define a time-modulating function in matrix **MT**. The first column of **MT** stores the time coordinates while the second column stores the ordinates of the function:

```
LOAD MT R=5 C=2
0.0 0.0
2.8 1.0
5.6 1.0
12.0 0.43
20.0 0.1
PLOT MT
```

The plot of the modulating function is shown in Fig. 5.3a.

The command *TSSF* is used to construct one sample function of a uniformly modulated process by the multiplication of the stationary Gaussian process **SFW** constructed in Section 5.2.1 and the time-modulating function defined in matrix **MT**:

```
TSSF SFW MT TFW T=0,20
TRAN TFW TFWT
```

```
PLOT VT TFWT
```

The result of the above commands is shown in Fig. 5.3b.

The CAL command *LOAD* is used to define a time-frequency modulating function in matrix **MWT**. The first row of **MWT** specifies the frequency coordinates, the first column specifies the time coordinates, and the remaining entries specify the ordinates of the function at the corresponding points. The first entry of **MWT** must be −1.1:

```
LOAD MWT R=6 C=5
-1.1 0.0 13.  30.  60.
0.00 0.0 0.0 0.0 0.0
2.80 0.8 1.0 1.2 1.3
5.60 1.2 1.0 0.8 0.7
12.0 0.5 0.4 0.3 0.2
20.0 0.2 0.2 0.0 0.0
```

The plot of the time-frequency modulating function is shown in Fig. 5.4a.

The command *GEGP* is used to generate one sample function of an evolutionary Gaussian process, described through the FWN PSD function and the time-frequency modulating function defined in matrix **MWT**. The generated sample **EFW** is specified at 401 time points over a 20 unit interval of time:

```
GEGP MWT EFW I=4 P=100,15.7,0.6 N=1,401 T=0,20 M=50
TRAN EFW EFWT
PLOT VT EFWT
```

The result of the above commands is shown in Fig. 5.4b.

5.3 Structural Analysis

The examples in this section illustrate the use of commands for constructing the structure stiffness matrix, performing static condensation to eliminate unwanted degrees of freedom, eigenvalue analysis, computing modal participation factors, and computing MEPF's for selected measures of response. Most commands for these analyses were available in CAL and are not new. They are described here for the purpose of completeness of the presentation. It is important to note that for the response analysis in the following section only the natural frequencies and the MEPF's will be necessary.

The configuration and properties of the example structural system are shown in Fig. 5.5a. The system consists of a two-story primary structure and an attached secondary subsystem which is discretized at three nodes. The assumed degrees of freedom and their numbering system is shown in Fig. 5.5b. The structure mass is modeled by lumped nodal masses with negligible rotational inertias. The condensed diagonal mass matrix is represented by the the vector **VM** (in units of $kip * sec^2/in$) with the elements corresponding to the first translational five degrees of freedom:

```
LOAD VM R=1 C=5
0.2 0.1 0.005 0.01 0.005
```

5.3.1 Formation of Structure Stiffness

The CAL command *SLOPE* is used to form the element stiffness matrices for the members of the primary structure and the secondary subsystem. For the primary, the length "L" of each element is 100 *in*, Young's modulus "E" is 30,000 *ksi*, and the moment of inertia "I" is 1,200 in^4. For the secondary members, the length is 50 *in*, Young's modulus is 30,000 *ksi*, and the moment of inertia is 5 in^4. Matrices **KP** and **KS** store the typical element stiffness matrices for the primary and secondary members, respectively:

```
SLOPE KP E=30000 I=1200 L=100
P KP
     COL =           1            2            3            4
     ROW 1    .14400E+07   .72000E+06   .21600E+05  -.21600E+05
```

```
    ROW 2   .72000E+06    .14400E+07   .21600E+05  -.21600E+05
    ROW 3   .21600E+05    .21600E+05   .43200E+03  -.43200E+03
    ROW 4  -.21600E+05   -.21600E+05  -.43200E+03   .43200E+03
SLOPE KS E=30000 I=5 L=50
P KS
    COL =             1             2           3           4
    ROW 1  12000.00000    6000.00000  360.00000  -360.00000
    ROW 2   6000.00000   12000.00000  360.00000  -360.00000
    ROW 3    360.00000     360.00000   14.40000   -14.40000
    ROW 4   -360.00000    -360.00000  -14.40000    14.40000
```

The CAL command *TRUSS* is used to form the element stiffness matrix for the connecting spring members where the section modulus "EA" is 50 kips. The length of the element is specified through the coordinate matrix "XYZ". The spring stiffness matrix is denoted in matrix **KPS**:

```
LOAD XYZ R=2 C=3
0 0 0
40 0 0
TRUSS KPS TTT E=50 A=1 N=1,2
P KPS
    COL =          1        2       3          4       5       6
    ROW 1    1.25000   .00000  .00000   -1.25000  .00000  .00000
    ROW 2     .00000   .00000  .00000     .00000  .00000  .00000
    ROW 3     .00000   .00000  .00000     .00000  .00000  .00000
    ROW 4   -1.25000   .00000  .00000    1.25000  .00000  .00000
    ROW 5     .00000   .00000  .00000     .00000  .00000  .00000
    ROW 6     .00000   .00000  .00000     .00000  .00000  .00000
```

The CAL command *ADDK* is used to assemble the element stiffness matrices into the global structure stiffness matrix **K** in accordance to the identification arrays **ID1** and **ID2**, which provide the correspondence between the local and global degrees of freedom:

```
LOADI ID1 R=4 C=4
6 7 9 10
```

```
0 6 8 9
1 2 4 5
0 1 3 4
LOADI ID2 R=6 C=2
1 2
0 0
0 0
3 5
0 0
0 0
ZERO K R=10 C=10
ADDK K KP ID1 N=1
ADDK K KP ID1 N=2
ADDK K KS ID1 N=3
ADDK K KS ID1 N=4
ADDK K KPS ID2 N=1
ADDK K KPS ID2 N=2
P K
```

COL =	1	2	3	4	5
ROW 1	.86525E+03	-.43200E+03	-.12500E+01	.00000E+00	.00000E+00
ROW 2	-.43200E+03	.43325E+03	.00000E+00	.00000E+00	-.12500E+01
ROW 3	-.12500E+01	.00000E+00	.15650E+02	-.14400E+02	.00000E+00
ROW 4	.00000E+00	.00000E+00	-.14400E+02	.28800E+02	-.14400E+02
ROW 5	.00000E+00	-.12500E+01	.00000E+00	-.14400E+02	.15650E+02
ROW 6	.00000E+00	.21600E+05	.00000E+00	.00000E+00	.00000E+00
ROW 7	-.21600E+05	.21600E+05	.00000E+00	.00000E+00	.00000E+00
ROW 8	.00000E+00	.00000E+00	-.36000E+03	.36000E+03	.00000E+00
ROW 9	.00000E+00	.00000E+00	-.36000E+03	.00000E+00	.36000E+03
ROW 10	.00000E+00	.00000E+00	.00000E+00	-.36000E+03	.36000E+03

COL =	6	7	8	9	10
ROW 1	.00000E+00	-.21600E+05	.00000E+00	.00000E+00	.00000E+00
ROW 2	.21600E+05	.21600E+05	.00000E+00	.00000E+00	.00000E+00
ROW 3	.00000E+00	.00000E+00	-.36000E+03	-.36000E+03	.00000E+00
ROW 4	.00000E+00	.00000E+00	.36000E+03	.00000E+00	-.36000E+03
ROW 5	.00000E+00	.00000E+00	.00000E+00	.36000E+03	.36000E+03
ROW 6	.28800E+07	.72000E+06	.00000E+00	.00000E+00	.00000E+00
ROW 7	.72000E+06	.14400E+07	.00000E+00	.00000E+00	.00000E+00

ROW 8	.00000E+00	.00000E+00	.12000E+05	.60000E+04	.00000E+00
ROW 9	.00000E+00	.00000E+00	.60000E+04	.24000E+05	.60000E+04
ROW 10	.00000E+00	.00000E+00	.00000E+00	.60000E+04	.12000E+05

5.3.2 Static Condensation

The CAL command *REDUCE* is used to perform static condensation on the stiffness matrix **K** to eliminate the rotational degrees of freedom. The retained degrees of freedom are specified through the vector **IRD**. The condensed stiffness matrix is denoted **K1**:

```
LOADI IRD R=1 C=5
1 2 3 4 5
DUP K KT
REDUCE KT K1 IRD
P K1
```

COL =	1	2	3	4	5
ROW 1	494.96429	-154.28571	-1.25000	.00000	.00000
ROW 2	-154.28571	62.96429	.00000	.00000	-1.25000
ROW 3	-1.25000	.00000	3.05000	-3.60000	1.80000
ROW 4	.00000	.00000	-3.60000	7.20000	-3.60000
ROW 5	.00000	-1.25000	1.80000	-3.60000	3.05000

In some versions of CAL, the command *REDUCE* does not exit. In that case, static condensation can be performed by matrix manipulations:

```
DUPSM K K11 L=1,1 R=5 C=5
DUPSM K K12 L=1,6 R=5 C=5
DUPSM K K21 L=6,1 R=5 C=5
DUPSM K K22 L=6,6 R=5 C=5
DUP K22 K22I
INVERT K22I
MULT K22I K21 K1T
MULT K12 K1T KTT
DUP K11 K1
SUB K1 KTT
```

where the CAL commands *DUP*, *DUPSM*, *INVERT*, *MULT*, and *SUB* are used to duplicate a matrix, duplicate a submatrix, invert a matrix, multiply two matrices, and subtract two matrices, respectively.

5.3.3 Eigenvalue Analysis

The CAL command *EIGEN* is used to compute the eigenvalues and eigenvectors of the structure, which are stored in matrix **PHIC** and vector **W2**, respectively. The natural frequencies are computed as the square roots of the eigenvalues, which are stored in vector **W** in units of *rad/sec*. The condensed mode shapes are the same as the eigenvectors **PHIC**:

```
DUP K1 KR
DUP VM W2
EIGEN KR PHIC W2
DUP W2 W
SQREL W
P W
     COL =        1          2          3          4          5
     ROW 1  9.26721   12.09267   15.93769   39.70709   54.60483
P PHIC
     COL =        1          2          3          4          5
     ROW 1   .63574     .66258    -.20878     .01401   -2.02806
     ROW 2  1.93764    2.04796    -.52343     .07826    1.33090
     ROW 3  3.83769   -5.91833   -9.57771   -7.64669     .20257
     ROW 4  5.48333   -5.33247     .35668    6.43198    -.01558
     ROW 5  5.82089   -2.58058   10.03941   -7.65879    -.10482
```

Note that the mode shapes are stored columnwise. Also note that the command *EIGEN* automatically scales the mode shapes by the mass matrix such that the modal masses are equal to one.

The CAL command *RECOVE* is used to obtain the uncondensed mode shapes, which are stored in matrix **PHI**. Matrix **KT** is from Section 5.3.2:

```
RECOVE KT PHI PHIC IRD N=1
P PHI
```

COL =	1	2	3	4	5
ROW 1	.63574	.66258	−.20878	.01401	−2.02806
ROW 2	1.93764	2.04796	−.52343	.07826	1.33090
ROW 3	3.83769	−5.91833	−9.57771	−7.64669	.20257
ROW 4	5.48333	−5.33247	.35668	6.43198	−.01558
ROW 5	5.82089	−2.58058	10.03941	−7.65879	−.10482
ROW 6	−.01103	−.01162	.00314	−.00040	.00299
ROW 7	−.01401	−.01497	.00315	−.00077	−.05188
ROW 8	−.03945	−.00089	−.19995	−.42242	.00501
ROW 9	−.01983	−.03338	−.19617	.00012	.00307
ROW 10	−.00021	−.06587	−.19240	.42266	.00114

The five mode shapes are qualitatively plotted in Fig. 5.6.

The uncondensed mode shapes can also be obtained through matrix manipulation as follows:

```
ZERO K2T R=5 C=5 D=1.0
ZERO K2 R=10 C=5
ZERO MT R=1 C=1 D=-1.0
SCALE K1T MT
STOSM K2 K2T L=1,1
STOSM K2 K1T L=6,1
MULT K2 PHIC PHI
P PHI
```

5.3.4 Modal Participation Factors

The modal participation factors **MPF** are obtained as the product of the modal matrix **PHI** and the vector of nodal load coefficients **P** (see Section 2.2.2). For the base input to be considered, the vector of nodal load coefficients is obtained by multiplying the uncondensed mass matrix (10 x 10) and the influence vector $[1,1,1,1,1,0,0,0,0,0]^T$ (see Section 2.2.2):

```
LOAD P R=1 C=10
0.2 0.1 0.005 0.01 0.005 0 0 0 0 0
MULT P PHI MPF
P MPF
     COL =       1        2         3         4         5
     ROW 1  .42404   .24149   -.08822   -.00158   -.27219
```

5.3.5 Modal Effective Participation Factors

STOCAL–II command *EPF* is used to compute MEPF's for selected responses. For each response quantity, a transfer matrix relating the response to the nodal degrees of freedom must be provided (see Section 2.2.2). For the analysis in Section 5.4, the MEPF's for the following responses are considered:

MEPF's for Nodal Displacements

The response transfer matrix for the five nodal displacement responses at degrees of freedom (DOF) 1 to 5, denoted **QND**, is loaded as follows:

```
LOAD QND R=5 C=10
1 0 0 0 0 0 0 0 0 0
0 1 0 0 0 0 0 0 0 0
0 0 1 0 0 0 0 0 0 0
0 0 0 1 0 0 0 0 0 0
0 0 0 0 1 0 0 0 0 0
```

The corresponding MEPF's are:

```
EPF PHI MPF QND END
P END
     COL =        1         2          3          4          5
     ROW 1   .26958    .82163    1.62733    2.32514    2.46828
     ROW 2   .16001    .49457   -1.42924   -1.28776    -.62319
     ROW 3   .01842    .04618     .84498    -.03147    -.88571
     ROW 4  -.00002   -.00012     .01207    -.01016     .01209
     ROW 5   .55202   -.36226    -.05514     .00424     .02853
```

Note that the effective participation factors for each mode are stored row-wise in matrix **END**.

<u>MEPF's for Member Forces</u>

The response transfer matrix for the member forces is determined by considering an identity matrix **I** of nodal displacements. The CAL command *MEMFRC* is then used to compute the corresponding forces for each member, which are identical to the required response transfer matrix.

For the analysis in Section 5.4, consider the four end forces of column AB of the primary structure. The response transfer matrix **QCF** is computed by

```
ZERO I R=10 C=10 D=1.0
MEMFRC KP I ID1 QCF N=1
P QCF
```

COL =	1	2	3	4	5
ROW 1	.21600E+05	.00000E+00	.00000E+00	.00000E+00	.00000E+00
ROW 2	.21600E+05	.00000E+00	.00000E+00	.00000E+00	.00000E+00
ROW 3	.43200E+03	.00000E+00	.00000E+00	.00000E+00	.00000E+00
ROW 4	−.43200E+03	.00000E+00	.00000E+00	.00000E+00	.00000E+00

COL =	6	7	8	9	10
ROW 1	.14400E+07	.00000E+00	.00000E+00	.00000E+00	.00000E+00
ROW 2	.72000E+06	.00000E+00	.00000E+00	.00000E+00	.00000E+00
ROW 3	−.21600E+05	.00000E+00	.00000E+00	.00000E+00	.00000E+00
ROW 4	.21600E+05	.00000E+00	.00000E+00	.00000E+00	.00000E+00

The MEPF's for the member forces, stored in matrix **EPF**, are computed by using the command *ECF*:

```
EPF PHI MPF QCF ECF
P ECF
```

COL =	1	2	3	4
ROW 1	−911.46838	2455.69293	15.44225	−15.44225
ROW 2	−583.47339	1436.37027	8.52897	−8.52897

```
ROW 3       -.81299    198.51696   1.97704    -1.97704
ROW 4        .42132      -.02826    .00393     -.00393
ROW 5  10752.47629  11338.01953  220.90496  -220.90496
```

MEPF's for Spring Axial Force

The response transfer vector **QSA** for the axial force in spring CF is obtained by the product of the transfer vector of the relative displacement between nodes C and F and the axial stiffness of the spring. The corresponding MEPF's are stored in the column vector **ESA**.

```
LOAD QSA R=1 C=10
0 1.25 0 0 -1.25 0 0 0 0 0
EPF PHI MPF QSA ESA
P ESA
     COL =          1
     ROW 1  -2.05830
     ROW 2   1.39720
     ROW 3   1.16486
     ROW 4   -.01527
     ROW 5   -.48849
```

5.4 Response to Stochastic Excitation

The examples in this section illustrate the use of STOCAL–II in determining the statistics of structural response to stationary and nonstationary excitations. The structure considered is that shown in Fig. 5.5. As indicated before, the only information needed from the previous section for the analysis in this section are the natural frequencies and the MEPF's for the selected responses. In addition, modal damping ratios are required, which are assumed here to be 0.05 for each mode of the combined system. The modal damping ratios are specified in vector **D**, which is defined by the use of the CAL command *LOAD*:

```
LOAD D R=1 C=5
```

```
0.05 0.05 0.05 0.05 0.05
```

The input excitation considered is motion at the base of the primary structure. This input is specified in terms of the base acceleration process, $\ddot{X}_g(t)$. Various forms of the input process are considered in this section.

The examples in this section are in three parts. Part one in Section 5.4.1 deals with the stationary response of the structure to a stationary base input; part two in Section 5.4.2 deals with the nonstationary response of the structure; and part three in Section 5.4.3 deals with the input specified through a response spectrum.

5.4.1 Stationary Responses

For the examples in this section, unless indicated otherwise, the input base acceleration is the FWN process described in Section 5.2.1 having the parameters Φ_o=100 in^2/sec^3, $\omega_g = 15.7$ rad/sec, and $\varsigma_g = 0.6$.

5.4.1.1 Response PSD Functions

The command *SPSD* is used to compute the PSD function of the base shear response. The MEPF's for this response are contained in the third column of matrix **ECF**. The response PSD, stored in matrix **PS4**, is specified at 181 frequency points along the frequency band 0–90 rad/sec:

```
SPSD W D ECF ECF PS4 I=4 P=100,15.7,0.6 W=0,90 N=181 IC=3,3
```

To investigate the effect of the input "filter" (which may be representing the effect of the soil layer at the base of the structure), the PSD for the same response is computed assuming the input is a WN process with a constant PSD at Φ_o=100 in^2/sec^3:

```
SPSD W D ECF ECF PS1 I=1 P=100 W=0,90 N=181 IC=3,3
PLOT PS1
```

```
PLOT PS4
```

The above two response PSD functions are compared in Fig. 5.7.

Next, the command *SPSD* is used to compute the cross-PSD function of the nodal displacement responses at DOF's 1 and 3 for the FWN input. The MEPF's for the two responses are stored in the first and third columns of matrix **END**, respectively. The result, consisting of a real and an imaginary part, is stored in matrix **PD13** for 121 frequency points along the frequency band 0–30 *rad/sec*:

```
SPSD W D END END PD13 I=4 P=100,15.7,0.6 W=0,30 N=121 IC=1,3
PLOT PD13 N=2
```

The result of the above commands is shown in Fig. 5.8a.

Next, the command *SPSD* is used to compute the cross-PSD function of the nodal velocity responses at DOF's 1 and 3 for the FWN input:

```
SPSD W D END END PV13 I=4 P=100,15.7,0.6 W=0,30 N=121 IC=1,3 M=1,1
PLOT PV13 N=2
```

The result of the above commands is shown in Fig. 5.8b.

Similarly, the command *SPSD* is used to compute the cross-PSD function of the nodal acceleration responses at DOF's 1 and 3 for the FWN input:

```
SPSD W D END END PA13 I=4 P=100,15.7,0.6 W=0,30 N=121 IC=1,3 M=2,2
PLOT PA13 N=2
```

The result of the above commands is shown in Fig. 5.8c.

5.4.1.2 Response Correlation Functions

The command *SCF* is used to compute the autocorrelation function of the nodal displacement response at DOF 1 for the FWN input. The MEPF's for this response are contained in the first column of matrix **END**. The autocorrelation function, stored in matrix **C1**, is specified at 201 points along the time-lag band 0–10 *sec*:

```
SCF W D END END C1 I=4 P=100,15.7,0.6 TA=0,10 N=201 IC=1,1
PLOT C1
```

The result of the above commands is shown in Fig. 5.9a.

Next, the command *SCF* is used to compute the autocorrelation function of the nodal displacement response at DOF 3 for the FWN input. The MEPF's for this response are contained in the third column of matrix **END**. The autocorrelation function is stored in matrix **C3**:

```
SCF W D END END C3 I=4 P=100,15.7,0.6 TA=0,10 N=201 IC=3,3
PLOT C3
```

The result of the above commands is shown in Fig. 5.9b.

The command *SCF* is used to compute the cross-correlation function of the nodal displacement responses at DOF's 1 and 3 for the FWN input. The result, stored in matrix **CD13**, is specified at 401 points along the time-lag band from −10 to 10 *sec*:

```
SCF W D END END CD13 I=4 P=100,15.7,0.6 TA=-10,10 N=401 IC=1,3
PLOT CD13
```

The result of the above commands is shown in Fig. 5.10a.

The above cross-correlation function can also be generated approximately by inverse Fourier transformation of the corresponding cross-PSD function stored in matrix **PD13** (see Section 5.4.1.1). This is accomplished by issuing the STOCAL–II command *IFTD* as follows:

```
IFTD PD13 C13A T=-10,10 N=401 I=1
PLOT C13A
```

The exact and approximate cross-correlation functions, from **CD13** and **C13A** respectively, are compared in Fig. 5.10a and are found to be in perfect agreement.

Next, the command *SCF* is used to compute the cross-correlation functions of the nodal velocities and accelerations responses at DOF's 1 and 3:

```
SCF W D END END CV13 I=4 P=100,15.7,0.6 TA=-10,10 N=401 IC=1,3 M=1,1
PLOT CV13
SCF W D END END CA13 I=4 P=100,15.7,0.6 TA=-10,10 N=401 IC=1,3 M=2,2
PLOT CA13
```

The resulting functions, stored in matrices **CV13** and **CA13** are shown in Figs. 5.10b and 5.10c, respectively

5.4.1.3 Response Spectral Moments

The command *SRSM* is used to compute the zeroth, first, second, and fourth spectral moments of the nodal displacement response at DOF 2 for the FWN input. The MEPF's for this response are stored in the second column of matrix **END**. The units for the zeroth, first, second and fourth spectral moments are in^2, in^2/sec, in^2/sec^2, and in^2/sec^4, respectively. The results are stored in vector **SN2**:

```
SRSM W D END END SN2 I=4 P=100,15.7,0.6 IC=2,2

    0th spectral moment l0 =  5.3028
    1st spectral moment l1 =  50.032
    2nd spectral moment l2 =  505.24
    4th spectral moment l4 =  70278.
```

The same is repeated for the nodal displacement at DOF 5:

```
SRSM W D END END SN5 I=4 P=100,15.7,0.6 IC=5,5
```

```
0th spectral moment l0 =   39.019
1st spectral moment l1 =   361.62
2nd spectral moment l2 =   3523.1
4th spectral moment l4 =   .37934E+06
```

An alternative method to compute the spectral moments is to use the command **SM**, which computes the moments of the response PSD by direct integration assuming it to be a piecewise linear function. The following commands compute the PSD and spectral moments for the nodal displacement at DOF 2:

```
SPSD W D END END P2 I=4 P=100,15.7,0.6 W=0,90 N=181 IC=2,2
SM P2 SN2A

    0th spectral moment l0 =   5.2755
    1st spectral moment l1 =   49.868
    2nd spectral moment l2 =   507.35
    4th spectral moment l4 =   70063.
```

Note that the above results are approximate because of the piecewise assumption.

A further method to compute the spectral moments is to first use the command *SMSM* to compute the modal spectral moments and correlation coefficients, and then use the command SMR to perform modal superposition. For the 0th, 1st, 2nd and 4th spectral moments of the nodal displacement at DOF 2, this is accomplished by

```
SMSM W D MSM0 RHO0 I=4 P=100,15.7,0.6 M=0
SMR MSM0 RHO0 END END SN20 IC=2,2
    Spectral Moment =   5.3028

SMSM W D MSM1 RHO1 I=4 P=100,15.7,0.6 M=1
SMR MSM1 RHO1 END END SN21 IC=2,2
    Spectral Moment =   50.032

SMSM W D MSM2 RHO2 I=4 P=100,15.7,0.6 M=2
SMR MSM2 RHO2 END END SN22 IC=2,2
    Spectral Moment =   505.24
```

```
SMSM W D MSM4 RHO4 I=4 P=100,15.7,0.6 M=4
SMR MSM4 RHO4 END END SN24 IC=2,2
     Spectral Moment =   70278.
```

5.4.1.4 Statistics of Response Process

The command *SSGP* is used to compute various statistics of the nodal displacement response at DOF 2 when its 0th, 1st, 2nd and 4th spectral moments are provided in matrix SN2. The response is assumed to be a zero-mean stationary Gaussian process with a 10 sec duration. Five equally spaced threshold between 2–10 *in* are considered. The statistics including the crossing rates of the process and its envelope, and the PDF and CDF of the local and extreme peaks of the process:

```
SSGP SN2 T=10 X=2,10 N=5

---- STATISTICS OF ZERO-MEAN STATIONARY GAUSSIAN PROCESS ----

Process X(t):

  Standard deviation of X(t)                  sqrt(l0) = 2.3028
  E [ X(t) d(Hilbert tran X)/dt ]**1/2        sqrt(l1) = 7.0733
  Standard deviation of dX/dt                 sqrt(l2) = 22.478
  Standard deviation of dX2/dt2               sqrt(l4) = 265.10

Cramer-Leadbetter envelope E(t):

  Mean of E(t)                                         = 2.8861
  Standard deviation of E(t)                           = 1.5086
  Standard deviation of dE(t)/dt                       = .59027

Regularity factors (measures of bandwidth):

  delta                         (1 - l1*l1/l0/l2)**1/2 = .25633
  alpha                                    l2*l2/l0/l4 = .82763

Mean zero upcrossing rate (apparent frequency):

  nu(0+)                         (1/2pi)*(l2/l0)**1/2 = 1.5535
```

Threshold crossings:

Threshold Level	Mean Upcrossing Rates X(t)	E(t)	Mean Clump Size (qualified crossings of E)
2.0000	1.06541	.59454	2.33826
4.0000	.34366	.38355	1.48713
6.0000	.05214	.08728	1.23073
8.0000	.00372	.00830	1.12019
10.000	.00012	.00035	1.06543

Probability distribution:

Level	PDF			CDF		
	X	E	Xpeak	X	E	Xpeak
2.0000	.1188	.2220	.2587	.80744	.42837	.31419
4.0000	.3832E-01	.1382	.1669	.95881	.81689	.77879
6.0000	.5814E-02	.3143E-01	.3797E-01	.99541	.97223	.96644
8.0000	.4148E-03	.2990E-02	.3613E-02	.99974	.99802	.99761
10.000	.1392E-04	.1254E-03	.1515E-03	.99999	.99993	.99992

Statistics of max[|X(t)|] for duration T=10:

Peak factors: p = 2.5913

q = .48983

Mean = 5.9671

Standard deviation = 1.1280

Probability distribution:

Level	PDF	CDF
2.0000	.76359E-05	.00000
4.0000	.93909E-01	.03667
6.0000	.33367	.58115
8.0000	.61957E-01	.95590
10.000	.31364E-02	.99829

Next, the command *SSGP* is used to compute the statistics of the nodal displacement response at DOF 5 where the required spectral moments are provided in matrix **SN5**. In this case, the response is assumed to be a stationary Gaussian process with a mean equal to 10 *in*. Such a non-zero mean response is possible, for example, when the displacement due to a static loading is combined with the dynamic displacement response. Four threshold levels 20, 25, 30 and 35 *in* are considered:

```
SSGP SN5 T=10 X=20,35 N=4 MU=5

----- STATISTICS OF STATIONARY GAUSSIAN PROCESS -----

Process X(t):

   Mean of X(t)                                        = 5.0000
   Standard deviation of X(t)               sqrt(l0)   = 6.2465
   E [ X(t) d(Hilbert tran X)/dt ] 1/2      sqrt(l1)   = 19.016
   Standard deviation of dX/dt              sqrt(l2)   = 59.355
   Standard deviation of dX2/dt2            sqrt(l4)   = 615.90

Cramer-Leadbetter envelope E(t):

   Mean of E(t)                                        = 12.829
   Standard deviation of E(t)                          = 4.0923
   Standard deviation of dE(t)/dt                      = 1.3783

Regularity factors (measures of bandwidth):

   delta                        (1 - l1*l1/l0/l2) 1/2  = .22066
   alpha                               l2*l2/l0/l4     = .91573

Mean upcrossing rate at mean level (apparent frequency):

   nu(mean+)                     (1/2pi)*(l2/l0) 1/2   = 1.5123

Threshold crossings:

   Threshold  Mean Upcrossing Rates   Mean Clump Size
    Level        X(t)       E(t)     (qualified crossings)
    20.000     .08462     .11239         1.36046
    25.000     .00899     .01591         1.20507
    30.000     .00050     .00111         1.12271
    35.000     .00001     .00004         1.07550

Probability distribution:
```

Level	PDF			CDF		
	X	E	Xpeak	X	E	Xpeak
20.000	.3574E-02	.1970E-01	.2151E-01	.99183	.94876	.94405
25.000	.3795E-03	.2789E-02	.3046E-02	.99932	.99456	.99406
30.000	.2123E-04	.1951E-03	.2130E-03	.99997	.99970	.99967
35.000	.6261E-06	.6902E-05	.7537E-05	1.00000	.99999	.99999

Statistics of max[X(t)] for duration T=10:

Peak factors: p = 2.4647

q = .51296

Mean = 25.396

Standard deviation = 3.2042

Probability distribution:

Level	PDF	CDF
20.000	.12760	.51492
25.000	.35027E-01	.92657
30.000	.28737E-02	.99542
35.000	.10839E-03	.99986

Note that the command *SSGP* considers the simple maximum of the process (rather than the maximum of its absolute value) when a mean value for the process is specified. To obtain the statistics for the simple maximum of a zero-mean process, MU=0 may be specified.

5.4.1.5 Distribution of Local and Extreme Peaks

The command *LPKD* is used to compute the PDF and CDF of the local peaks of the nodal displacement response at DOF 2. The response is assumed to be a zero-mean stationary Gaussian process with the required spectral moments provided in matrix **SN2**. The result is specified at 121 points between the levels –2 to 4 times the root-mean-square response:

```
LPKD SN2 MLPD R=-2,4 N=121
```

The command *EXTD* is used to compute the PDF and CDF of the extreme peak (defined as $max[|X(t)|]$) of the nodal displacement response at DOF 2 over a 10 sec duration. The result is specified at 101 points between the levels 0 to 5 times the root-mean-square response.

```
EXTD SN2 MEPD T=10 R=0,5 N=101 MU=0
PLOT MLPD N=1
PLOT MEPD N=1
PLOT MLPD MLPD N=1 X=1 Y=3
PLOT MEPD MEPD N=1 X=1 Y=3
```

The distributions of the two local and extreme peaks are compared in Fig. 5.11a-b.

5.4.2 Nonstationary Response

Two types of nonstationary input are considered in STOCAL–II. One is a uniformly modulated process and the other is a fully evolutionary input. The uniformly modulated process is described through a time-invariant PSD function and a modulating function of time. The evolutionary model is described through a time-invariant PSD function and a real-valued modulating function of time and frequency. In this section, two modulating functions of time are used. One is defined in matrix **MT** from Section 5.2.2 and is shown in Fig. 5.3a. The other is the unit step function, which is defined by matrix **MT1** as follows:

```
LOAD MT1 R=1 C=2
0 1
```

For the evolutionary input, the modulating function of time and frequency is defined by matrix **MWT** of Section 5.2.2, which is shown in Fig. 5.4a. The time-invariant PSD is a piecewise linear function defined by matrix **PI** below:

```
LOAD PI R=4 C=2
0.00 90.
13.0 190
```

```
30.0 46.
60.0 0.0
PLOT PI
```

The plot of the time-invariant PSD is shown in Fig. 5.12

In order to observe the evolutionary nature of the response, it is useful to investigate the individual modal responses. For this purpose, a fictitious MEPF matrix **MI** is defined as follows:

```
LOAD MI R=5 C=2
1 0
0 0
0 1
0 0
0 0
```

This matrix is used in the following analysis to study the responses in modes 1 and 3.

5.4.2.1 Response Evolutionary PSD Functions

The command *TPSD* is used to compute the evolutionary PSD of the first mode of the structure. The effective participation factors for this modal response are specified in the first column of **MI**. The input is specified by a WN PSD with $\Phi_o = 100\ in^2/sec^3$ modulated by the unit step function defined in matrix **MT1**. Since only the response in the first mode is required, L=1 is specified. The resulting evolutionary PSD is computed at time instants 1, 2 and 3 *sec* and for 121 frequency points between 0 and 30 *rad/sec*. For comparison, the stationary PSD of the response is also determined by using the command SPSD:

```
TPSD W D MI MI MT1 TP1 I=1 P=100 W=0,30 N=121 T1=1,2,3 L=1
SPSD W D MI MI SP1 I=1 P=100 W=0,30 N=121 IC=1,1 L=1
PLOT TP1 TP1 N=3 X=1,1,1 Y=2,4,6
```

```
PLOT SP1
```

The results of the above commands are shown in Fig. 5.13a.

The above analysis is repeated for the response in mode 3, for which the effective participation factors are specified in the second column of **MI**:

```
TPSD W D MI MI MT1 TP3 I=1 P=100 W=0,30 N=121 T1=1,2,3 IC=2,2 L=3
SPSD W D MI MI SP3 I=1 P=100 W=0,30 N=121 IC=2,2 L=3
PLOT TP3 TP3 N=3 X=1,1,1 Y=2,4,6
PLOT SP3
```

The results of the above commands are shown in Fig. 5.13b.

Next, the command *TPSD* is used to compute the evolutionary cross-PSD of the responses in modes 1 and 3. The resulting evolutionary PSD, consisting of a real and an imaginary part, is obtained for time instants 1, 2 and 3 sec. For comparison, the stationary cross-PSD is also computed:

```
TPSD W D MI MI MT1 TP13 I=1 P=100 W=0,30 N=121 T1=1,2,3 IC=1,2 L=3
SPSD W D MI MI SP13 I=1 P=100 W=0,30 N=121 IC=1,2 L=3
PLOT TP13 TP13 N=3 X=1,1,1 Y=2,4,6
PLOT SP13
PLOT TP13 TP13 N=3 X=1,1,1 Y=3,5,7
PLOT SP13 SP13 N=1 X=1 Y=3
```

The results of the above commands are shown in Fig. 5.14.

Observe in Figs. 5.13 and 5.14 the modal responses initially are relatively wide band processes, but with time the energy concentrates around the modal frequency and after a sufficiently long time the response in each mode approaches the stationary response.

The command *EPSD* is used to compute the evolutionary PSD of the nodal displacement response at DOF 1. The MEPF's for this response are contained in the first column of matrix **END** from Section 5.3.5. The evolutionary input considered is specified by the PSD function defined in matrix **PI** (Fig. 5.12) and the time-frequency modulating function defined in matrix **MWT** (Fig. 5.4a). L=3 is specified because only the contributions from the first three modes are included in the analysis. The resulting evolutionary PSD at time instants 3, 4 and 5 sec is obtained. For comparison, the stationary PSD is also computed:

```
EPSD W D END END MWT EP11 PI I=3 W=0,30 N=121 T1=3,4,5 L=3
SPSD W D END END SP11 PI I=3 W=0,30 N=121 L=3
PLOT EP11 EP11 N=3 X=1,1,1 Y=2,4,6
PLOT SP11
```

The results of the above commands are shown in Fig. 5.15a.

The above analysis is repeated for the nodal displacement at DOF 3, for which the MEPF's are contained in the third column of matrix **END** from Section 5.3.5:

```
EPSD W D END END MWT EP33 PI I=3 W=0,30 N=121 T1=3,4,5 IC=3,3 L=3
SPSD W D END END SP33 PI I=3 W=0,30 N=121 IC=3,3 L=3
PLOT EP33 EP33 N=4 X=1,1,1 Y=2,4,6
PLOT SP33
```

The results of the above commands are shown in Fig. 5.15b.

Next, the command *EPSD* is used to compute the evolutionary cross-PSD of the nodal displacement responses at DOF's 1 and 3. The resulting evolutionary PSD, consisting of a real and an imaginary part, is obtained. For comparison, the stationary cross-PSD is also computed:

```
EPSD W D END END MWT EP13 PI I=3 W=0,30 N=121 T1=3,4,5 IC=1,3 L=3
SPSD W D END END SP1A PI I=3 W=0,30 N=121 IC=1,3 L=3
PLOT EP13 EP13 N=3 X=1,1,1 Y=2,4,6
PLOT SP1A
PLOT EP13 EP13 N=3 X=1,1,1 Y=3,5,7
PLOT SP1A SP1A N=1 X=1 Y=3
```

The results of the above commands are shown in Fig. 5.15b.

5.4.2.2 Response Correlation Functions

The command *TCF* is used to compute the variance function of the first modal displacement response of the structure. The effective participation factors for this modal response are contained in the first column of matrix **MI**. The input is the WN modulated by the unit step function. Since only the response of the first mode is required, L=1 is specified. For comparison, the stationary variance of the response is also computed:

```
TCF W D MI MI MT1 TT1 I=1 P=100 T1=0,5 N=101 L=1
SCF W D MI MI TS1 I=1 P=100 TA=0,1 N=1 L=1
P TS1
     COL =      1        2
     ROW 1  .00000  3.94732
PLOT TT1 TT1 N=1 X=1 Y=3
```

The results of the above commands are shown in Fig. 5.17a.

Next, the command *TCF* is used to compute the covariance function of the first modal displacement and velocity responses of the structure:

```
TCF W D MI MI MT1 TT2 I=1 P=100 T1=0,5 N=101 M=1,0 L=1
PLOT TT2 TT2 N=1 X=1 Y=3
```

The result of the above commands is shown in Fig. 5.17b. Note that the corresponding stationary covariance in this case is zero.

In a similar manner, the command *TCF* is used to compute the variance function of the first modal velocity response:

```
TCF W D MI MI MT1 TT3 I=1 P=100 T1=0,5 N=101 M=1,1 L=1
SCF W D MI MI TS3 I=1 P=100 TA=0,1 N=1 M=1,1 L=1
P TS3
```

```
COL =     1          2
ROW 1  .00000  339.00082
PLOT TT3 TT3 N=1 X=1 Y=3
```

The results of the above commands are shown in Fig. 5.17c.

The results in Fig. 5.17 are in agreement with previously published results by Caughey and Stumpf (1961) and others.

Next, the command *TCF* is used to compute the variance function of the nodal displacement at DOF 2 for a uniformly modulated input having the PSD function defined in matrix **PI** (Fig. 5.12) and the modulating function defined in matrix **MT** (Fig. 5.3a). The MEPF's for this response are contained in the second column of matrix **END** from Section 5.3.5. The variance function is computed at 81 points along time band 0–20 *sec*. Only the contributions from the first three modes are included in the analysis:

```
TCF W D END END MT TC2 PI I=3 T1=0,20 N=81 IC=2,2 L=3
PLOT TC2 TC2 N=1 X=1 Y=3
```

The command *TCF* is repeated to compute the variance function of the nodal displacement at DOF 5:

```
TCF W D END END MT TC5 PI I=3 T1=0,20 N=81 IC=5,5 L=3
PLOT TC5 TC5 N=1 X=1 Y=3
```

The two variance functions are compared in Fig. 5.18.

Next, the command *ECF* is used to compute the variance functions of the nodal displacements at DOF's 2 and 5 for an evolutionary input with a constant PSD and the time-frequency modulating function defined in matrix **MWT** (Fig. 5.4a). The MEPF's for this response are contained in the second and fifth columns of matrix **END** from Section 5.3.5.

```
ECF W D END END MWT EC2 I=1 P=100 T1=0,20 N=81 IC=2,2 L=3
PLOT EC2 EC2 N=1 X=1 Y=3
ECF W D END END MWT EC5 I=1 P=100 T1=0,20 N=81 IC=5,5 L=3
PLOT EC5 EC5 N=1 X=1 Y=3
```

The results the above commands are shown in Fig. 5.19.

5.4.3 Response Spectrum Analysis

In this section the input ground motion is specified through a mean response spectrum. The standard response spectrum suggested in the NRC Regulatory Guide 1.60 (NRC 1976), which is shown in Fig. 5.20, is used for this purpose. For 0.05 modal damping, the spectrum ordinates at modal frequencies of the structure are loaded in vector SPC (in units of *in*) as follows:

```
LOAD SPC R=1 C=5
9.118 6.664 4.746 0.672 0.340
```

This input is used to investigate several modal combination rules as follows:

The command *RCQC* is used to compute the approximate mean of the maximum base shear force in the connecting member AB according to the CQC (Wilson et al. 1981). The MEPF's for this response are obtained from matrix **ECF** (in Section 5.3.5) and store in vector **E1**:

```
DUPSM ECF E1 R=5 C=1 L=1,3
RCQC W D E1 SPC VCQC
      The mean of absolute maximum response = 176.09
```

The above rule accounts for the correlation between modal responses.

To investigate the effect of ignoring modal correlations, the above maximum response is estimated by the SRSS rule through the use of matrix manipulations:

```
ZERO M1 R=5 C=5
```

```
STODG M1 SPC
MULT M1 E1 M2
TRAN M2 M2T
MULT M2T M2 SRSS
SQREL SRSS
P SRSS
  COL =          1
  ROW 1  169.66134
```

Another common modal combination rule is the absolute sum method. The maximum response based on this rule is computed as follows:

```
DUP E1 E1A
ABS E1A
MULT SPC E1A VMA
P VMA
  COL =          1
  ROW 1  282.13280
```

From the above results, it is seen that the effect of modal correlations is around 4 percent and that the absolute sum method grossly overestimates the response.

The command *RSM* is used to compute the spectral moments of the base shear response according to the response spectrum method proposed by Der Kiureghian (1981). The modal combination rule uses a modal correlation coefficient matrix obtained from the assumption of a FWN input. The duration of strong shaking is assumed to be 10 sec. The command *SSGP* is then used to compute the response statistics for a 10 sec duration:

```
RSM W D E1 SPC VMW T=10 I=4 P=100,15.7,0.6
SSGP VMW T=10

---- STATISTICS OF ZERO-MEAN STATIONARY GAUSSIAN PROCESS ----

Process X(t):

   Standard deviation of X(t)                  sqrt(l0) = 67.676
```

E [X(t) d(Hilbert tran X)/dt]**1/2 sqrt(l1) = 252.38

Standard deviation of dX/dt sqrt(l2) = 1260.0

Standard deviation of dX2/dt2 sqrt(l4) = 58265.

Cramer-Leadbetter envelope E(t):

Mean of E(t) = 84.819

Standard deviation of E(t) = 44.337

Standard deviation of dE(t)/dt = 44.995

Regularity factors (measures of bandwidth):

delta (1 - l1*l1/l0/l2)**1/2 = .66485

alpha l2*l2/l0/l4 = .40263

Mean zero upcrossing rate (apparent frequency):

nu(0+) (1/2pi)*(l2/l0)**1/2 = 2.9632

Statistics of max[|X(t)|] for duration T=10:

Peak factors: p = 3.0593

q = .41356

Mean = 207.04

Standard deviation = 27.988

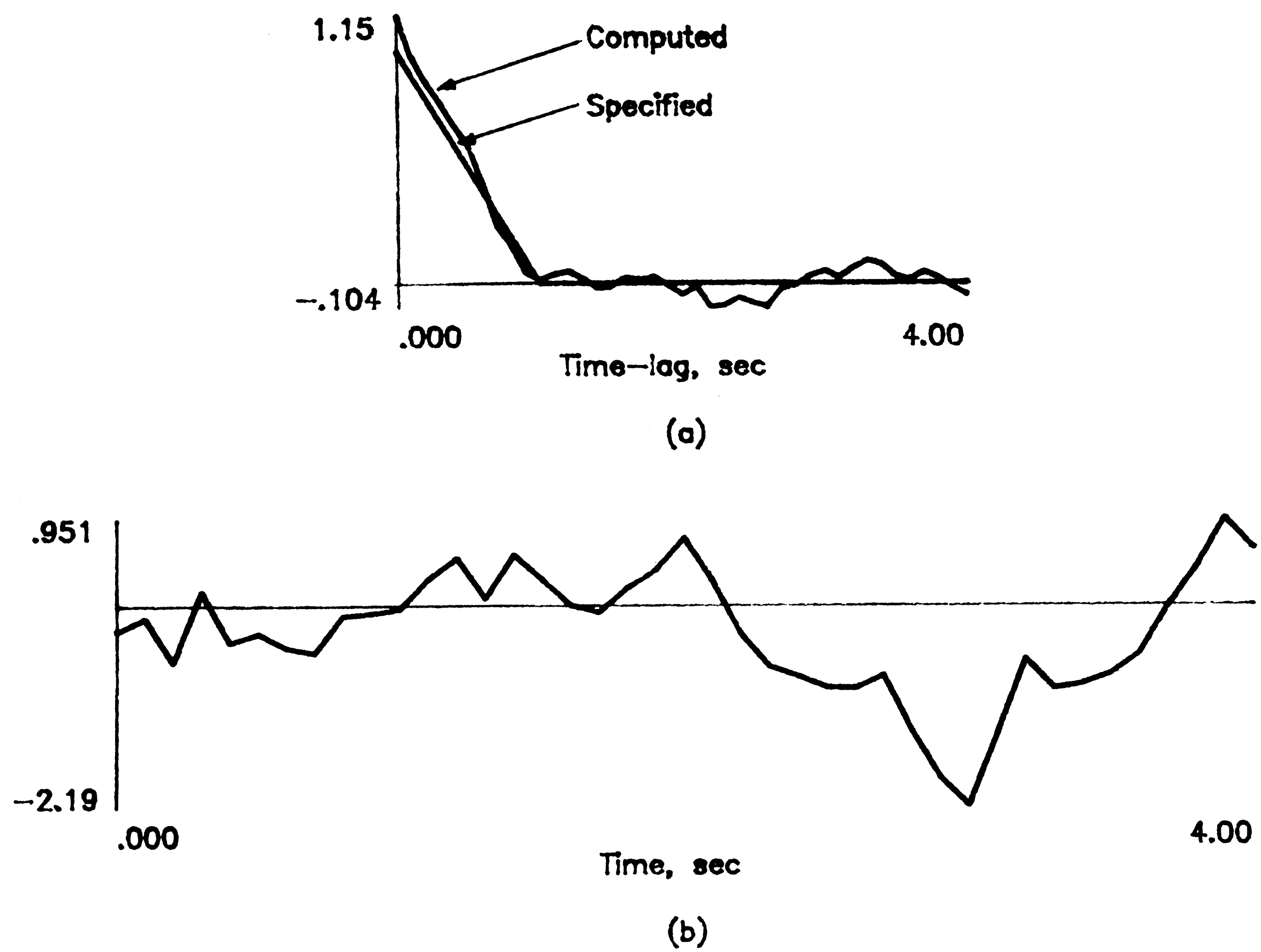
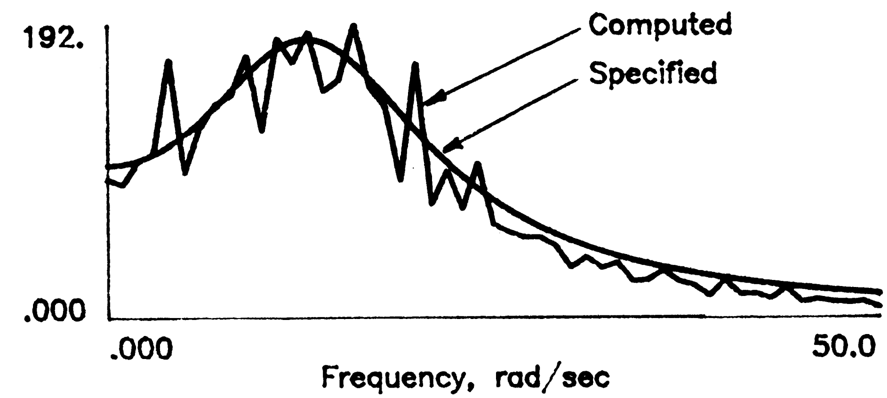
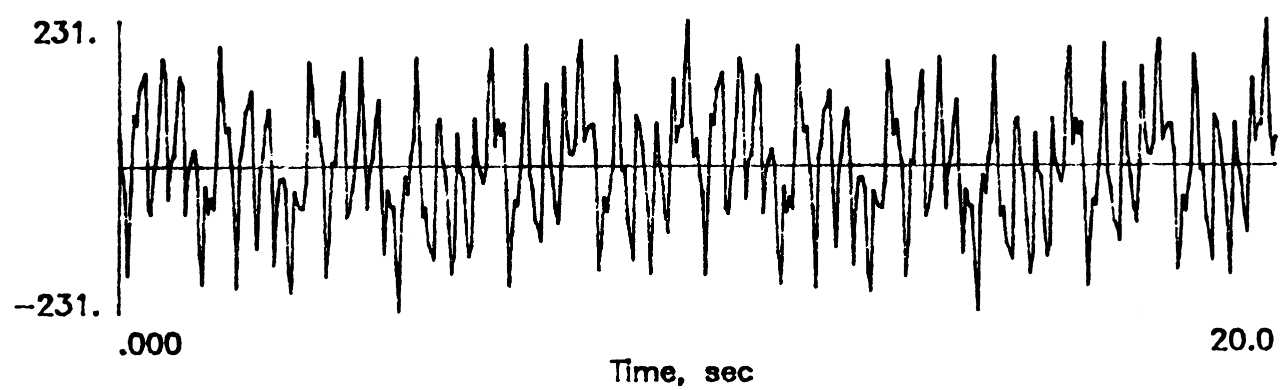


Figure 5.1 Generation of Stationary Gaussian Process: (a) Specified and Computed Autocorrelation Functions, (b) Sample Function



(a)



(b)

Figure 5.2 Generation of Stationary Gaussian Process: (a) Specified and Computed PSD Functions (in^2/sec^3) (b) Sample Function (in/sec^2)

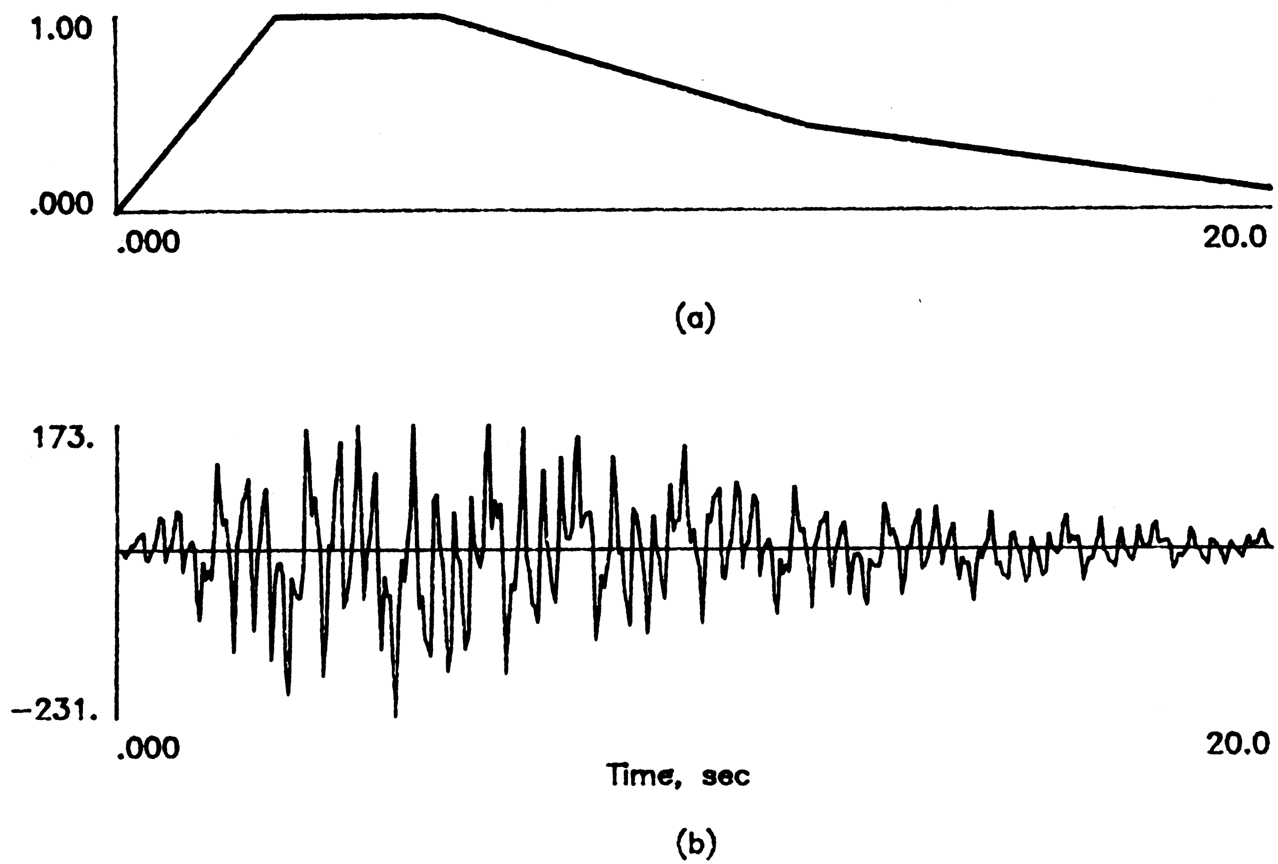
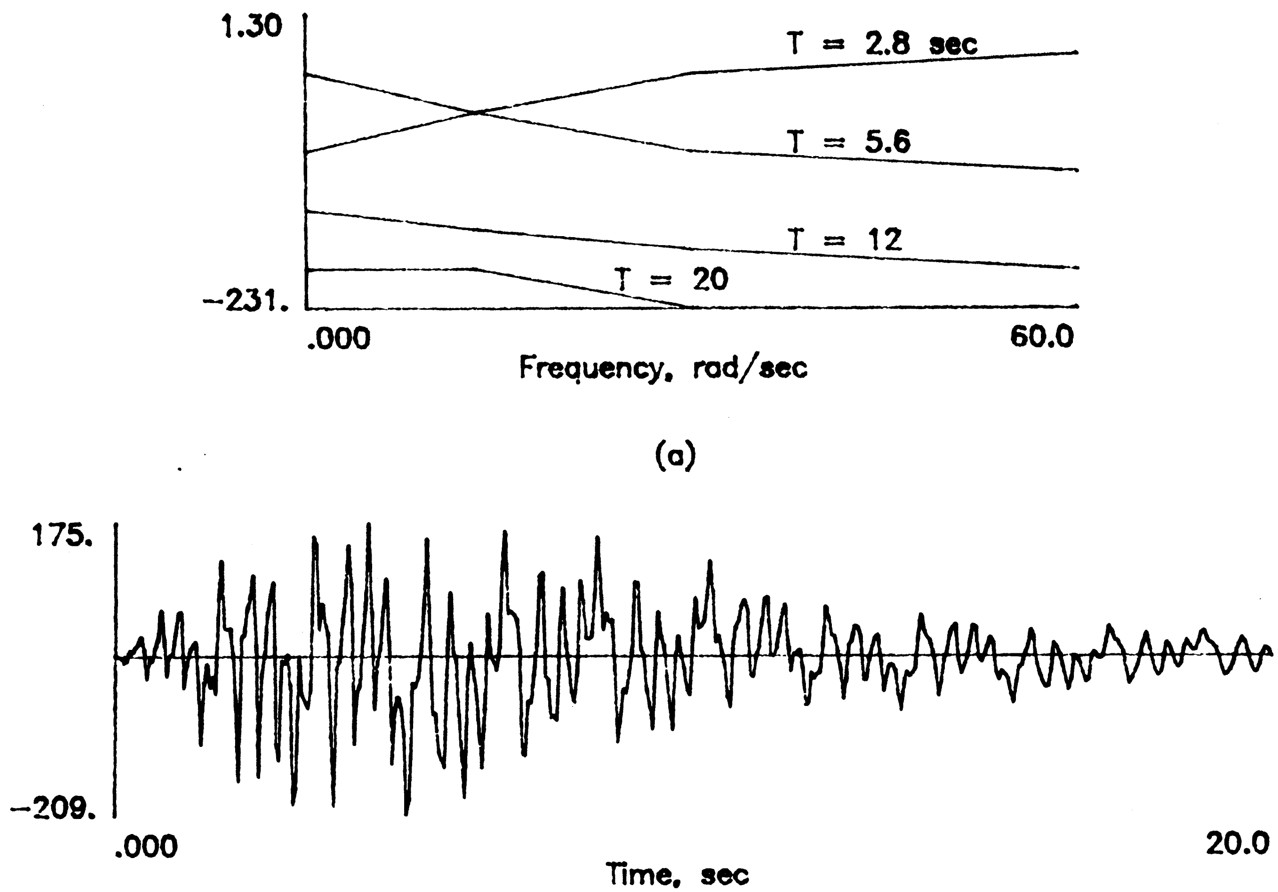


Figure 5.3 Generation of Uniformly Modulated Gaussian Process:
(a) Modulating Function, (b) Sample Function (in/sec^2)



(b)

Figure 5.4 Generation of Evolutionary Gaussian Process: (a) Time-Frequency Modulating Function , (b) Sample Function (in/sec^2)

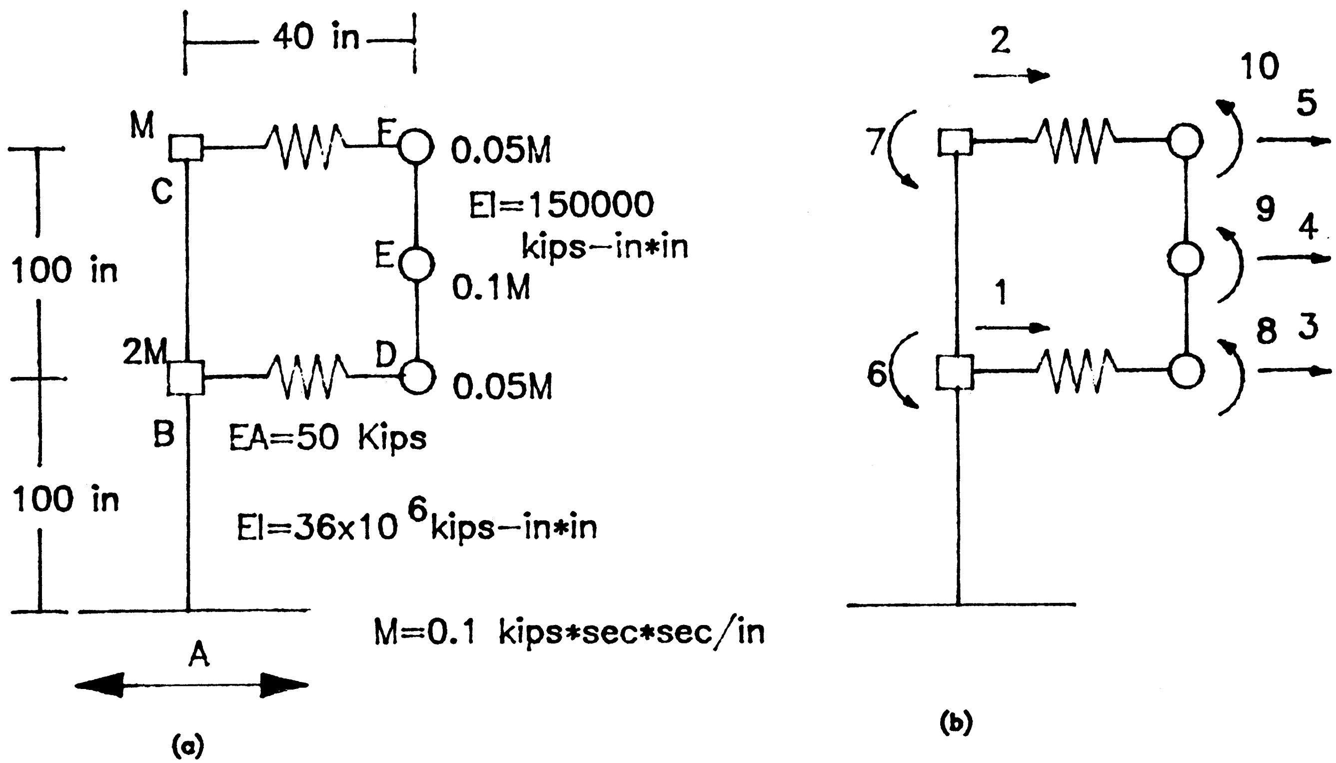


Figure 5.5 Example Primary-Secondary Structure System: (a) Configuration and Properties, (b) Assigned Degrees of Freedom

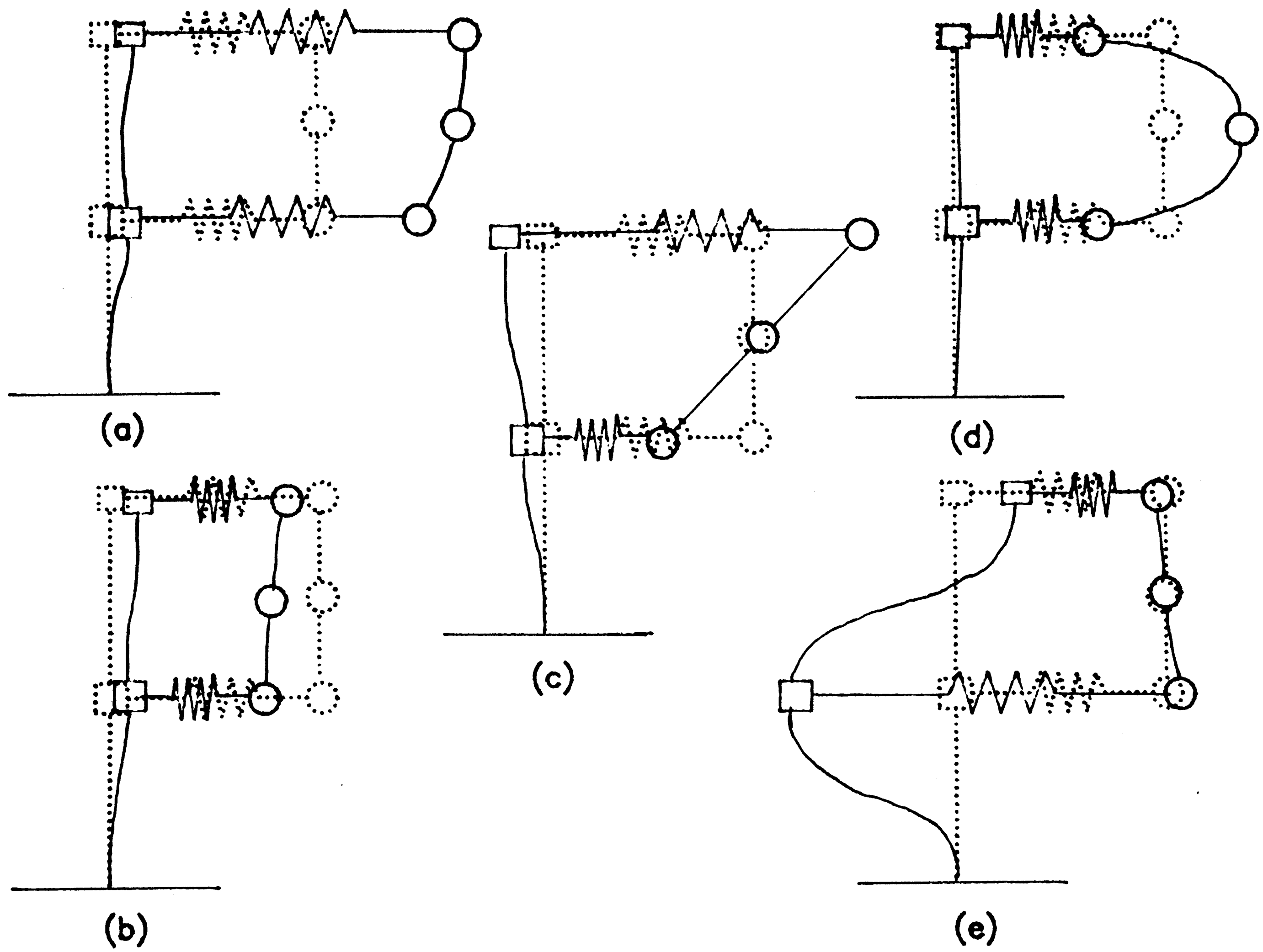


Figure 5.6 Five Mode Shapes of the Example Structure (a–e: mode 1–5)

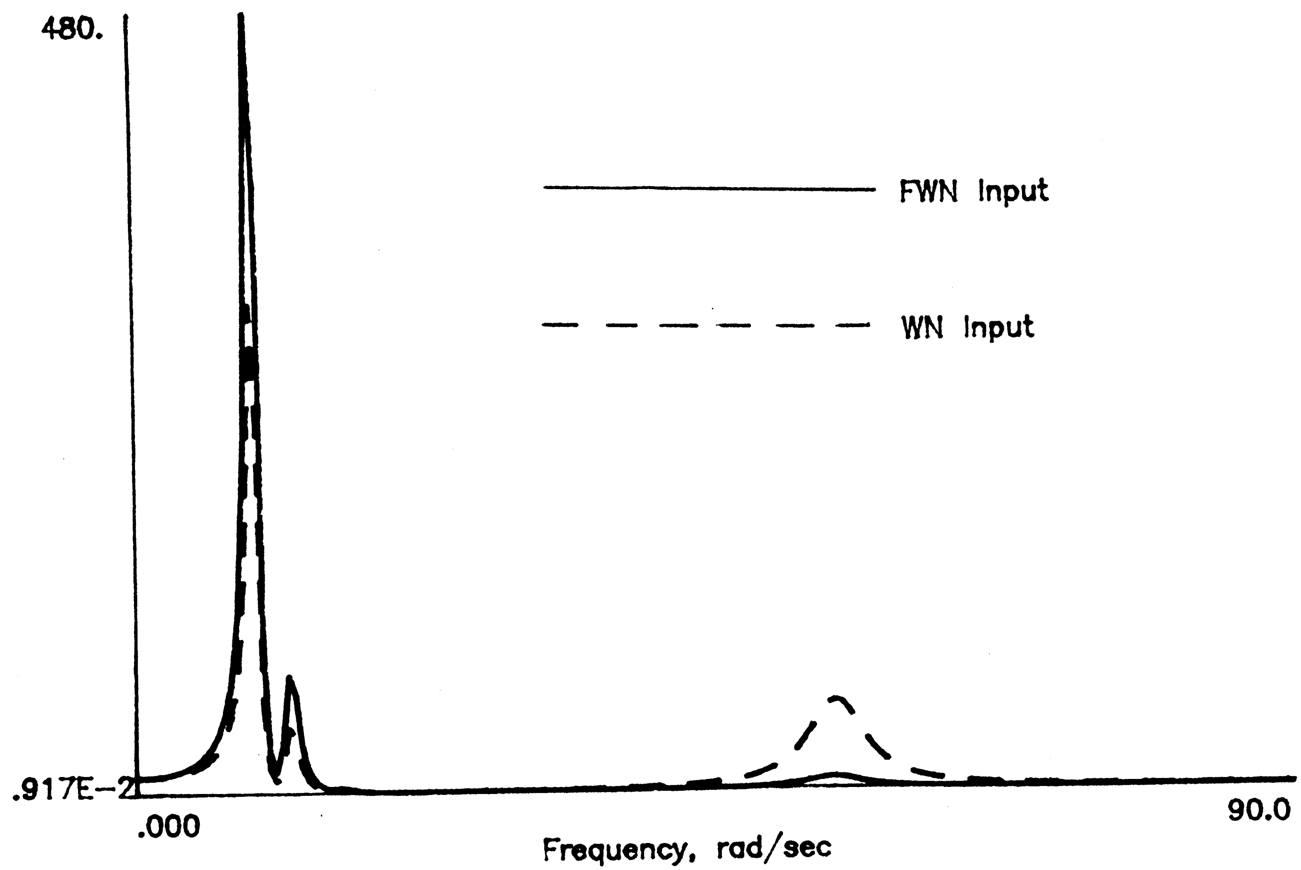


Figure 5.7 PSD's of the Base Shear Response ($kip * sec$)

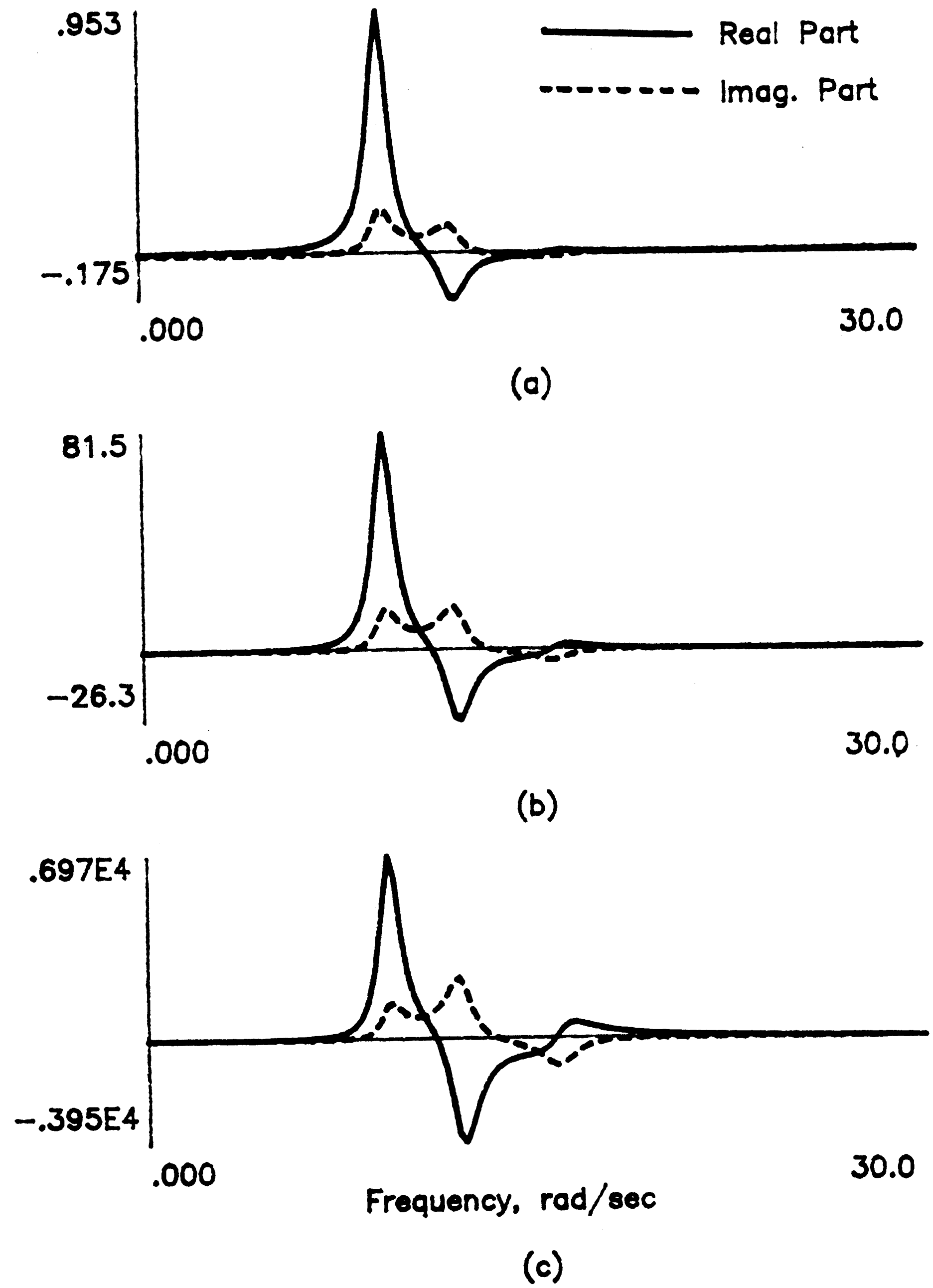


Figure 5.8 Cross-PSD's of Responses at DOF's 1 and 3: (a) Displacements ($in^2 * sec$), (b) Velocities (in^2/sec), (c) Accelerations (in^2/sec^3)

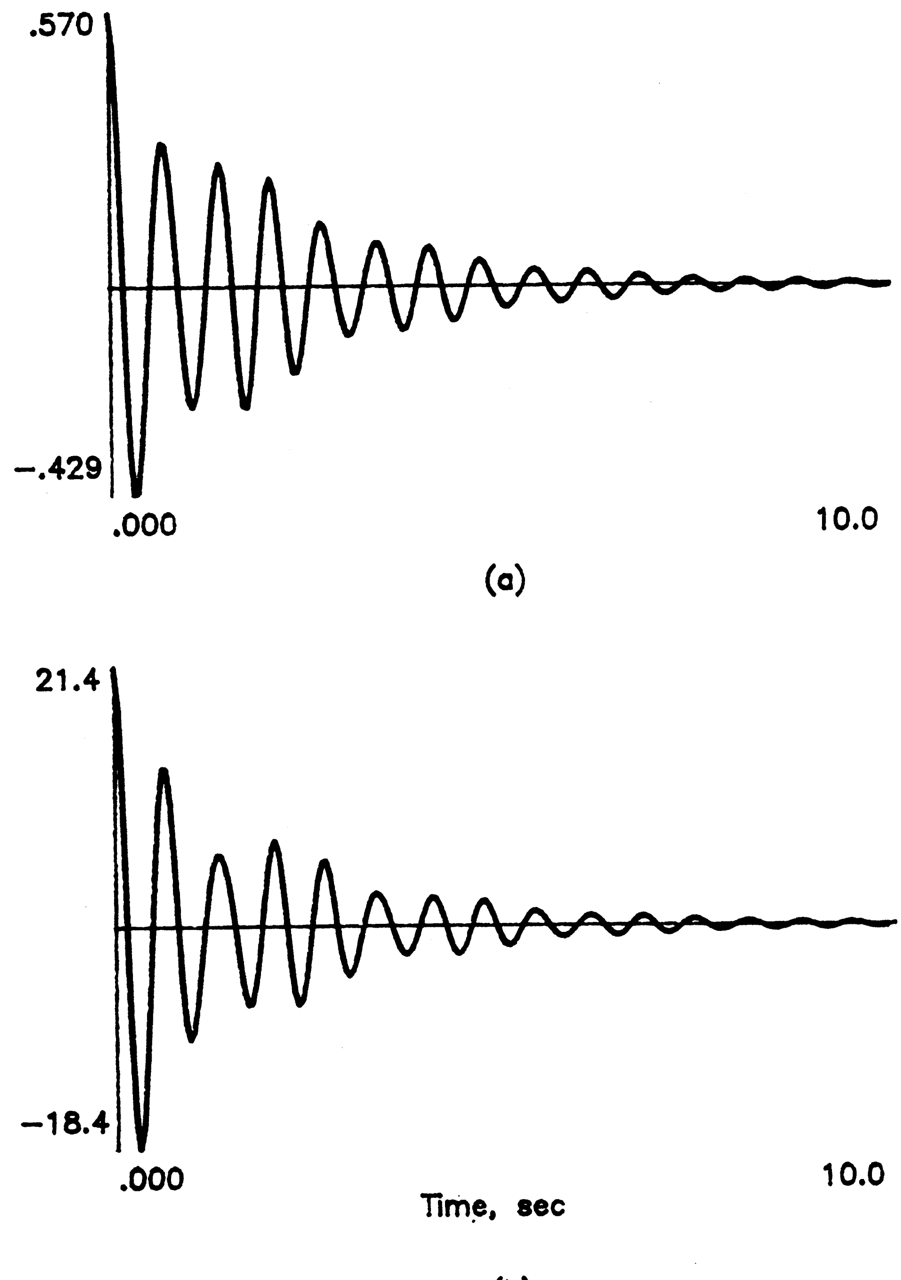


Figure 5.9 Displacement Autocorrelation Functions (in^2)
(a) DOF 1 (b) DOF 3

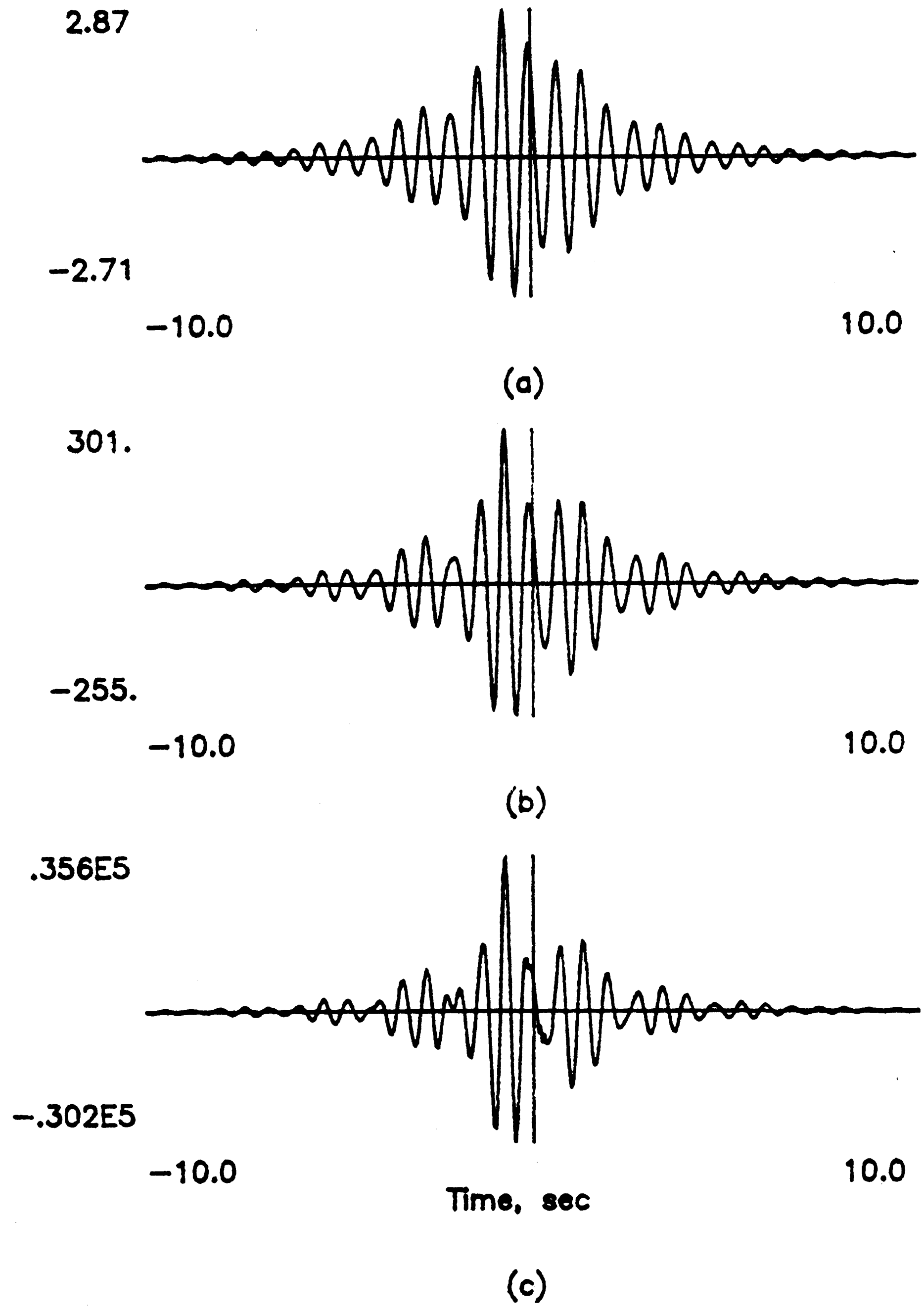
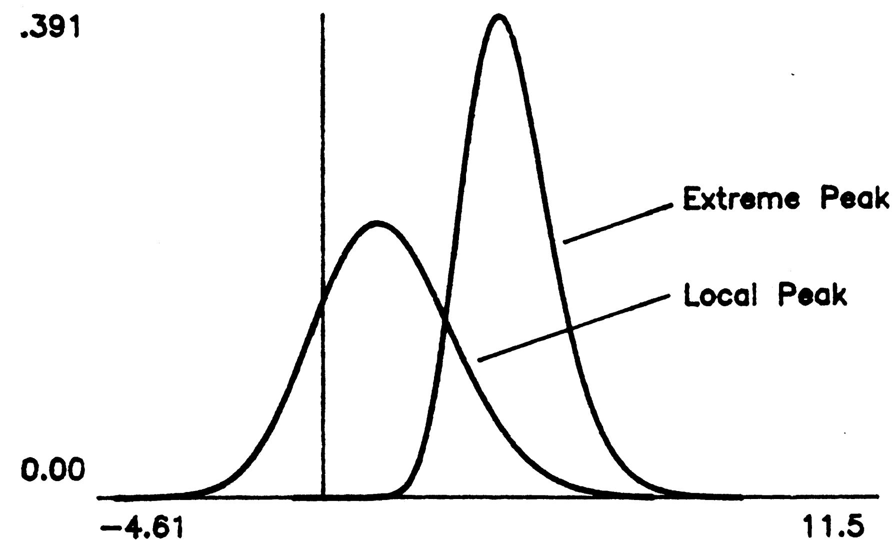
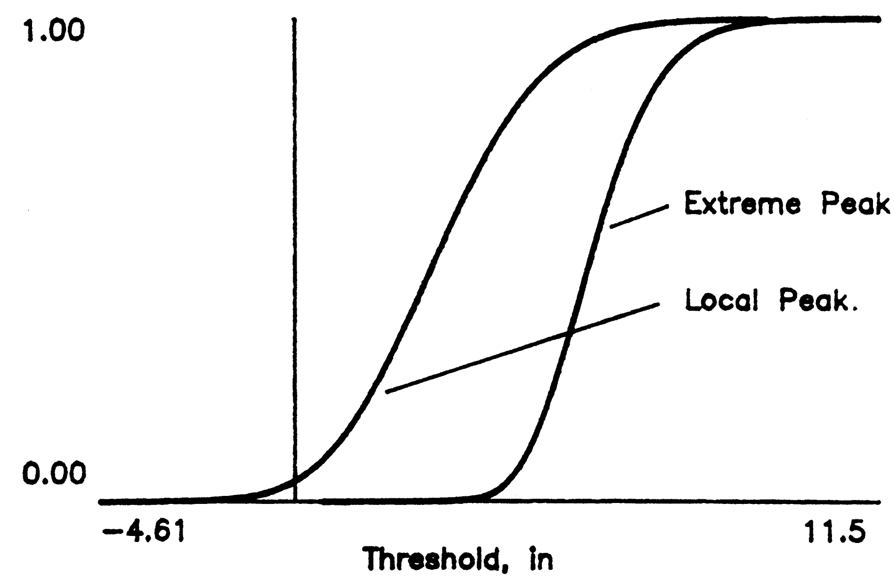


Figure 5.10 Cross-Correlations of Responses at DOF's 1 and 3: (a) Displacements (in^2), (b) Velocities (in^2/sec^2), (c) Accelerations (in^2/sec^4)



(a)



(b)

Figure 5.11 Distribution of Local and Extreme Peaks:
(a) PDF (b) CDF

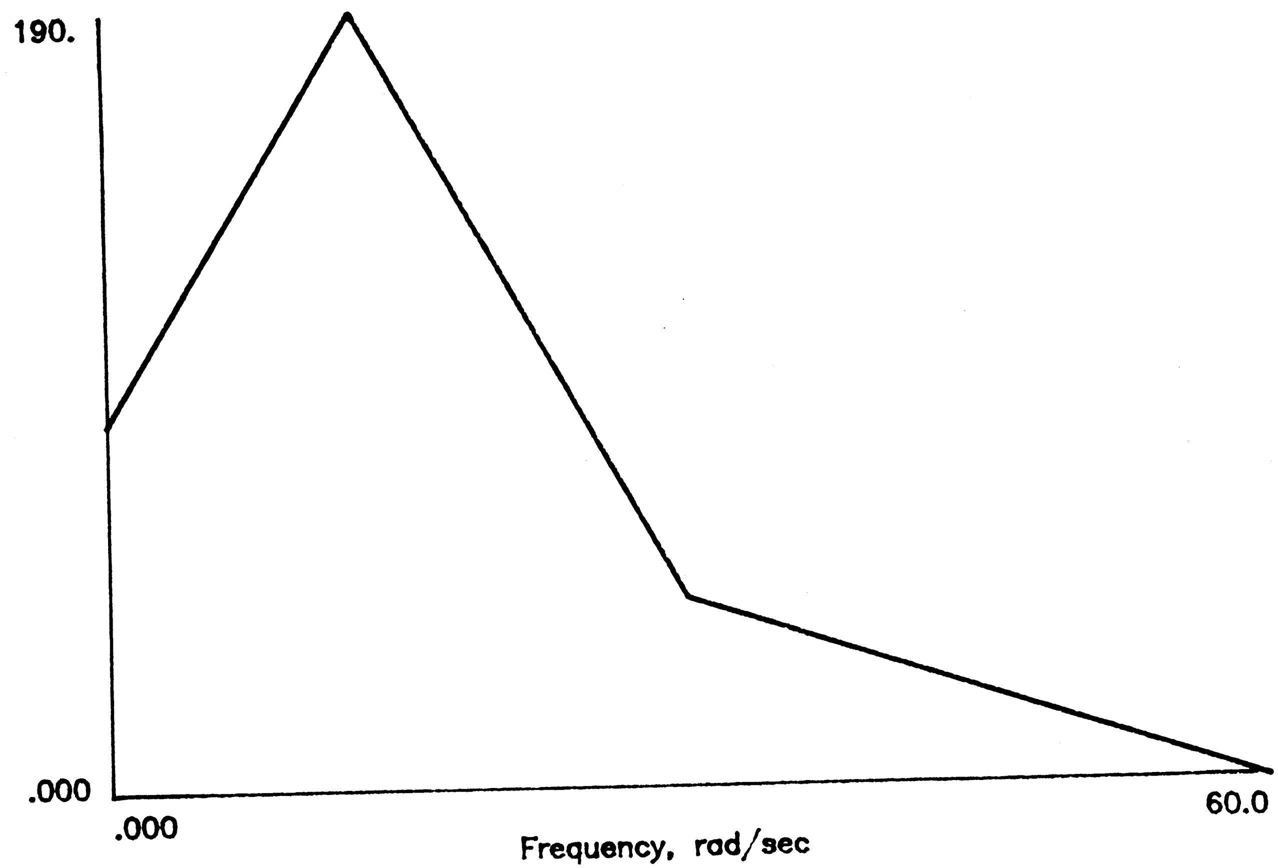


Figure 5.12 Piecewise linear PSD Function (in^2/sec^3)

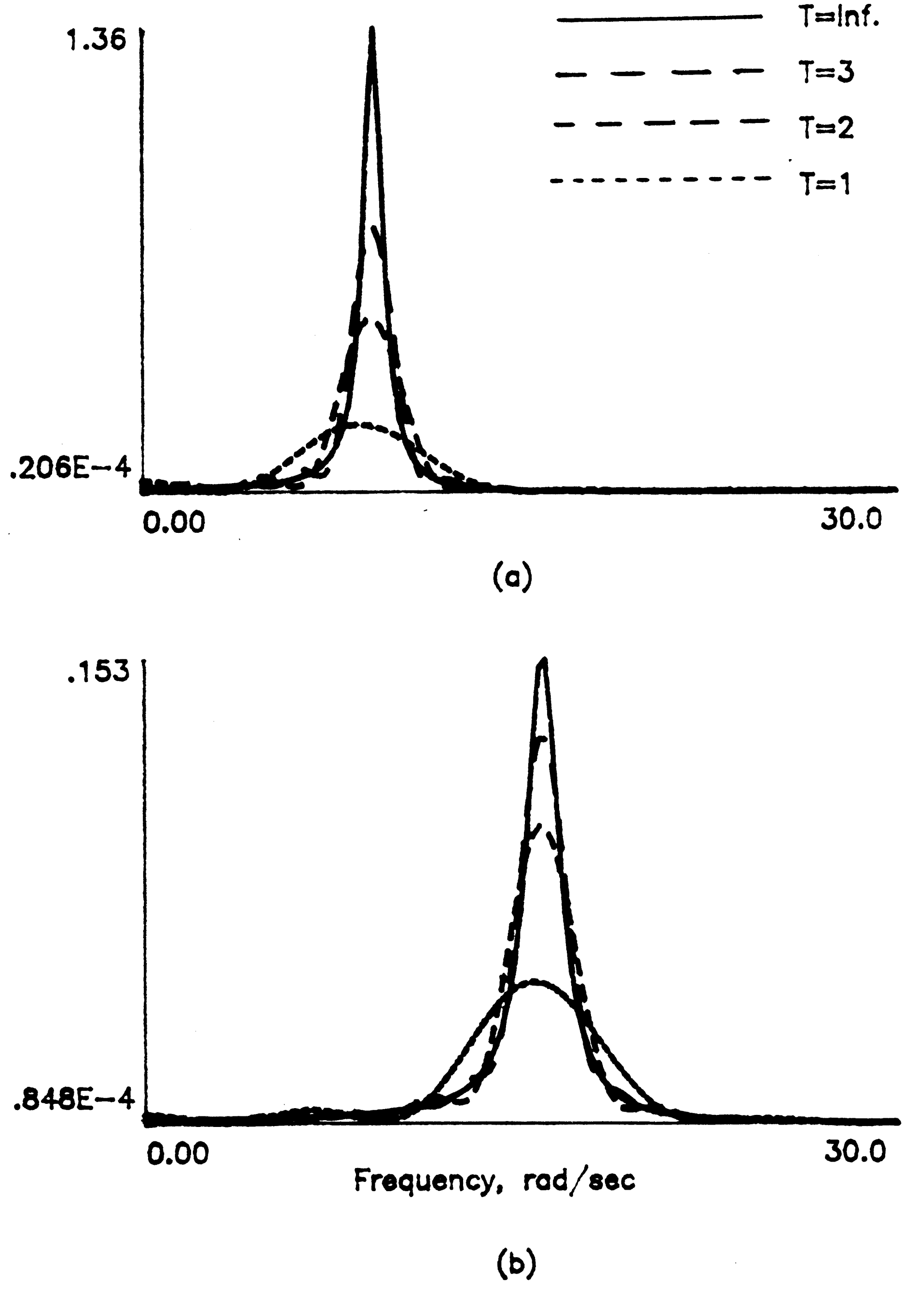


Figure 5.13 Evolutionary Response PSD Functions ($in^2 * sec$): (a) DOF 1 (b) DOF 2

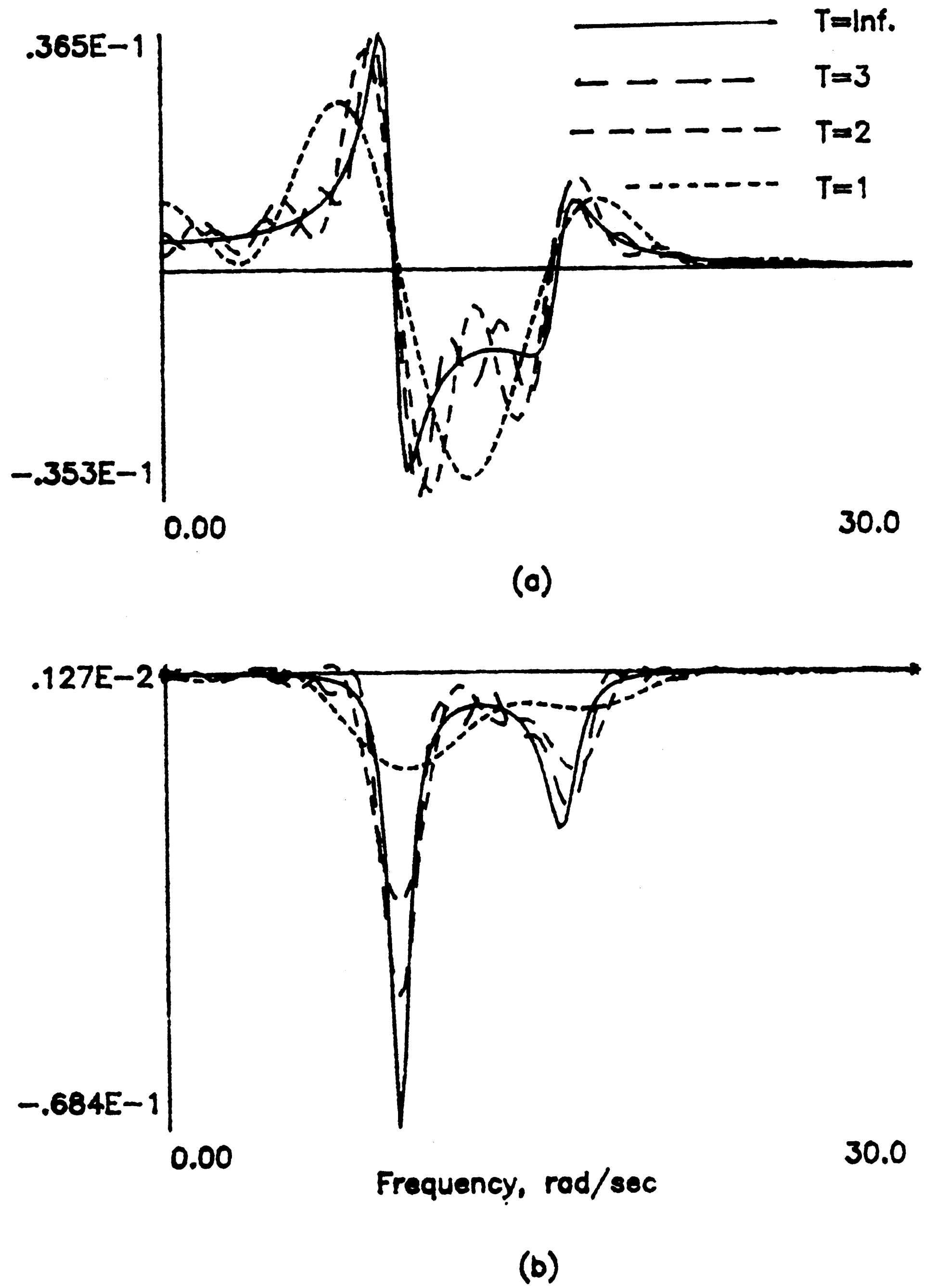


Figure 5.14 Evolutionary Response Cross-PSD Function of Modes 1 and 3 ($in^2 * sec$): (a) Reap Part (b) Imaginary Part

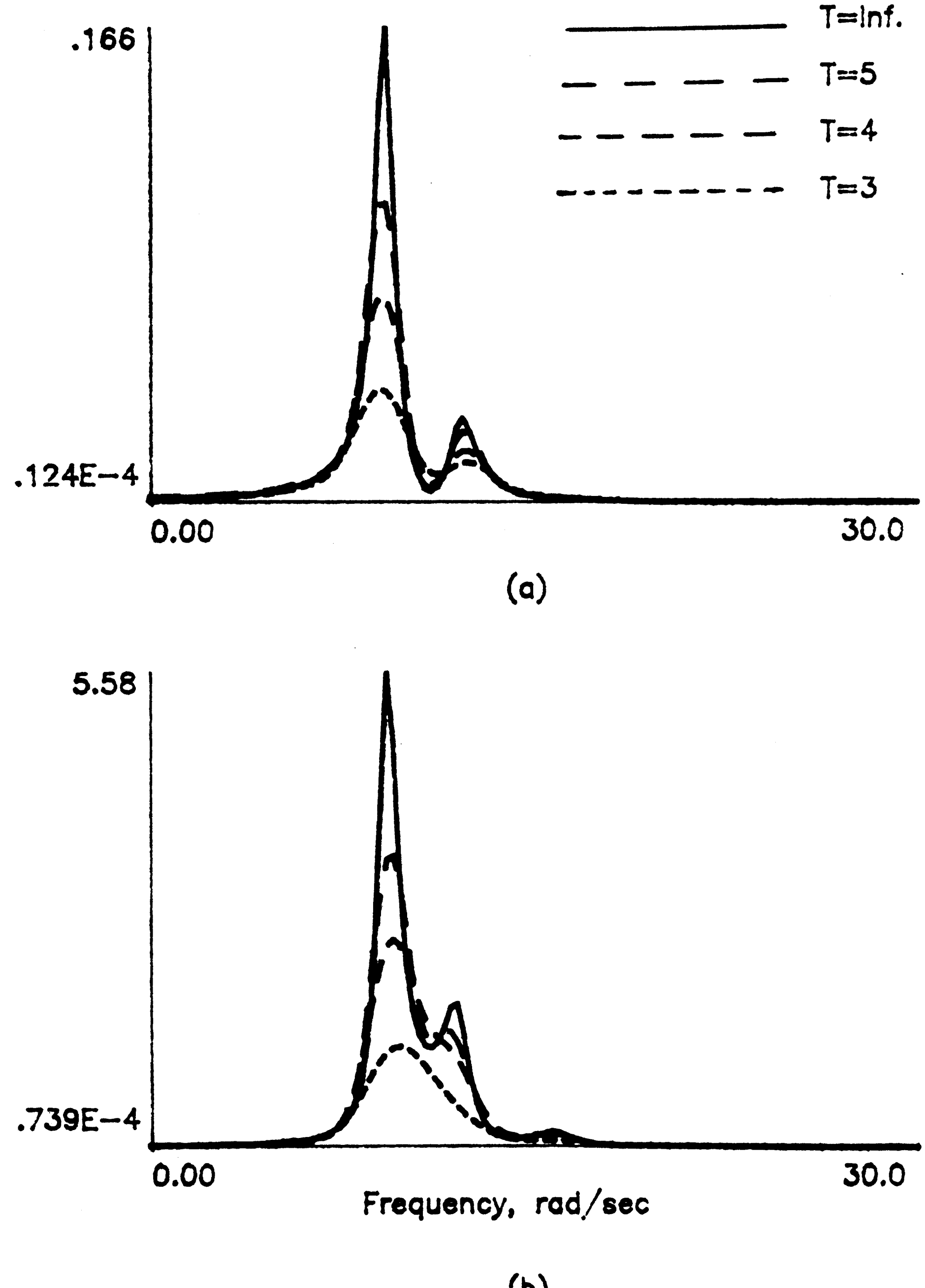


Figure 5.15 Displacement Evolutionary Response PSD Functions ($in^2 * sec$): (a) DOF 1 (b) DOF 2

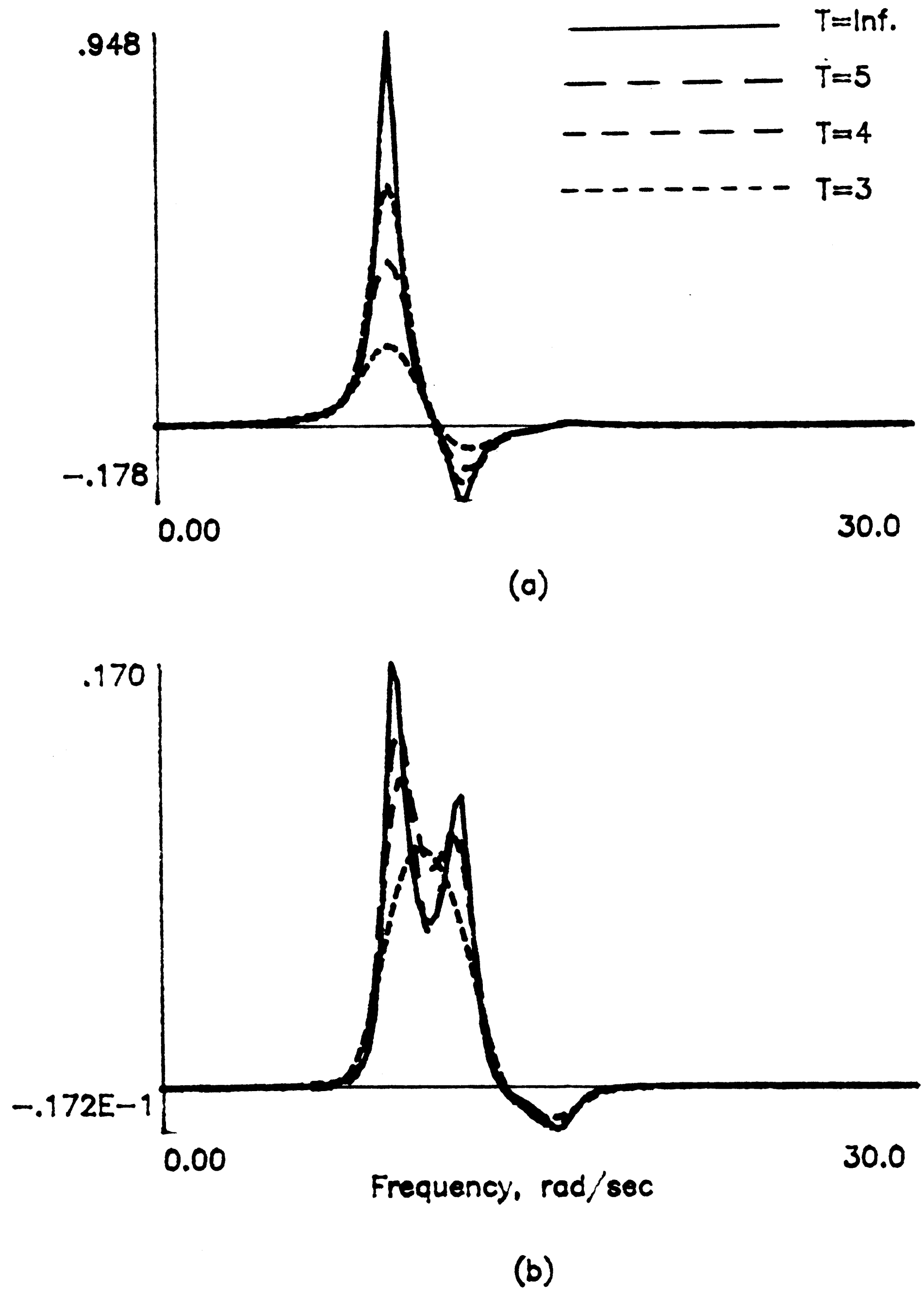


Figure 5.16 Evolutionary Cross-PSD Function of the Displacement Response at DOF's 1 and 3 ($in^2 * sec$): (a) Real Part (b) Imaginary Part

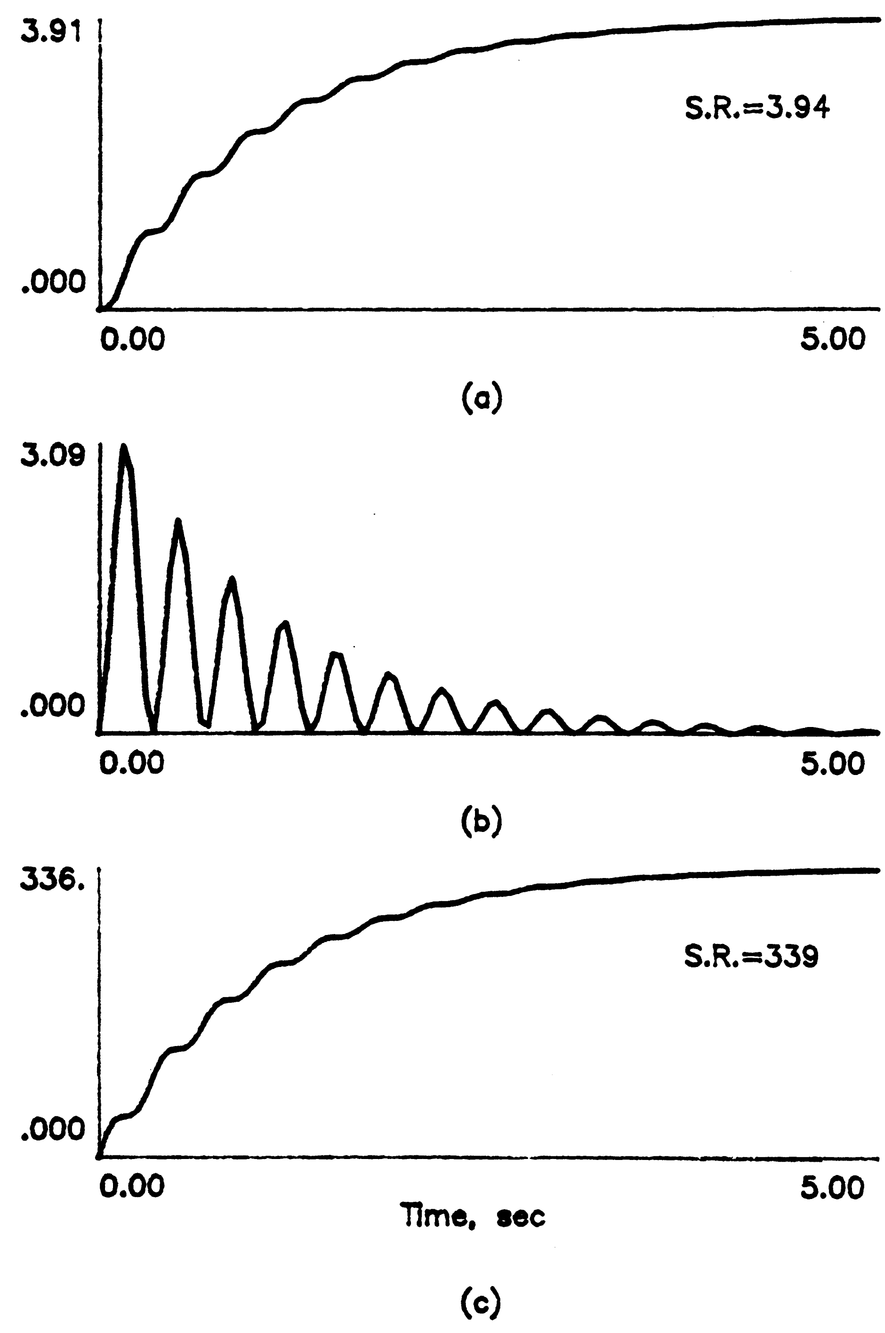


Figure 5.17 Modal Covariance Functions: (a) Displacement /Displacement (in^2), (b) Displacement/Velocity (in^2/sec), (c) Velocity (in^2/sec^2)

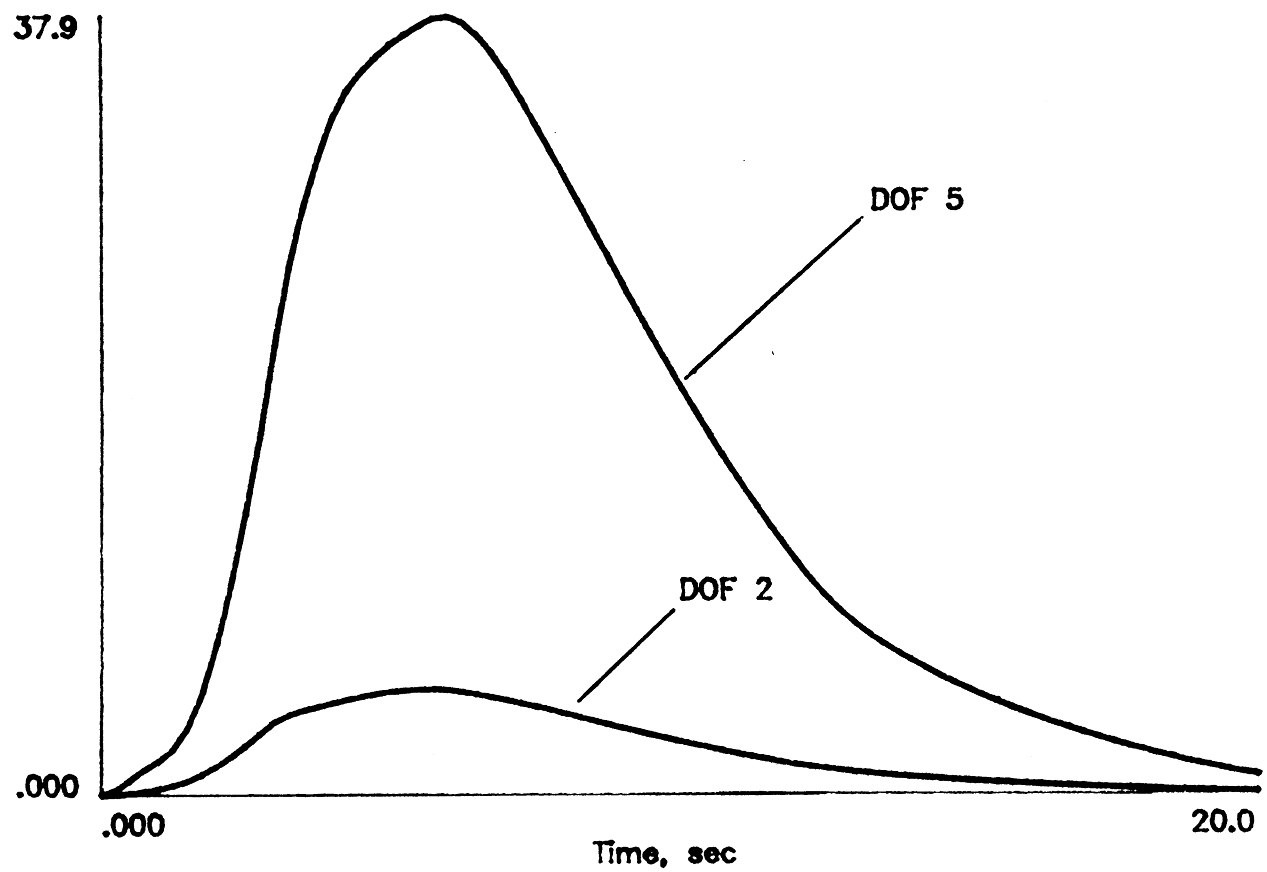


Figure 5.18 Variance Functions of Displacement Responses (in^2)

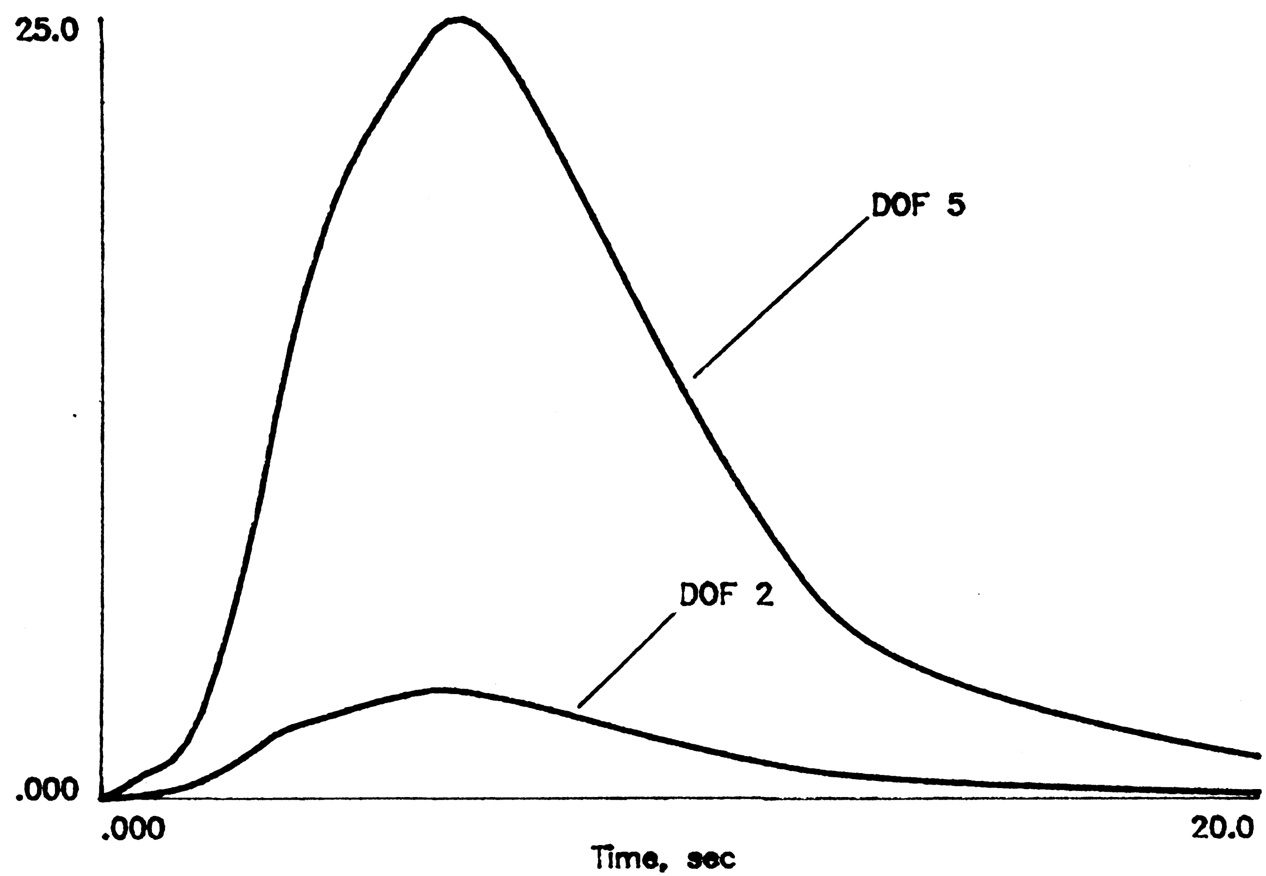


Figure 5.19 Variance Functions of Displacement Responses (in^2)

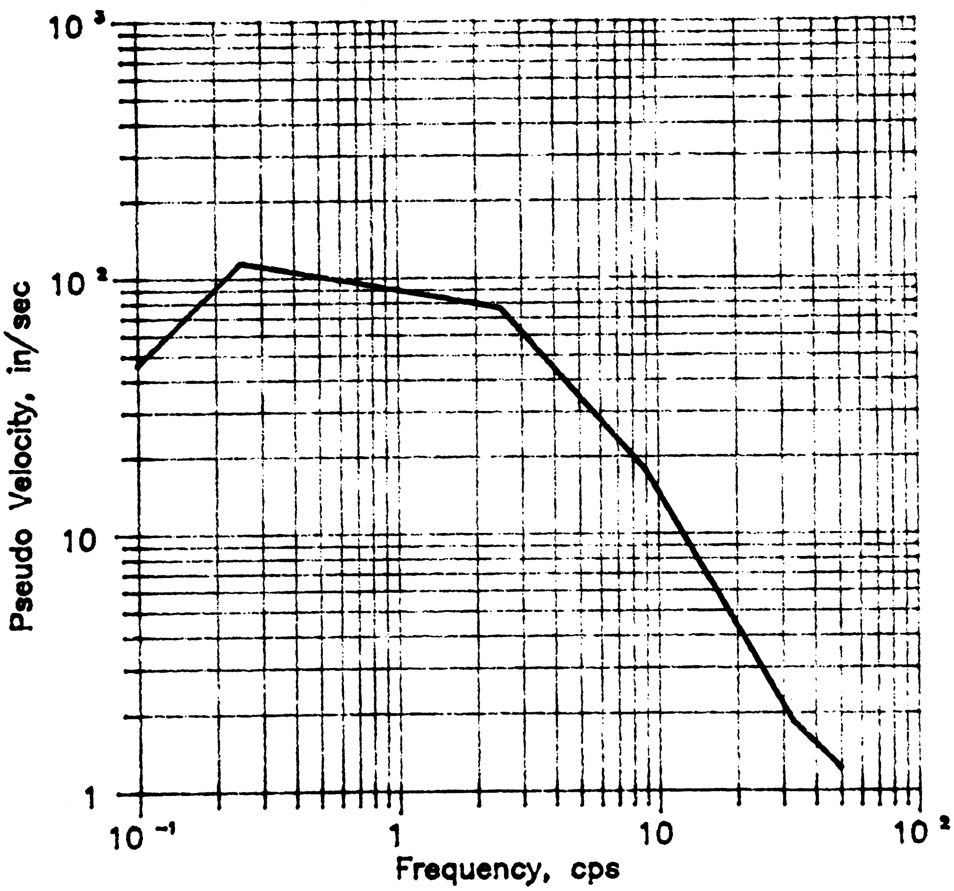


Figure 5.20 NRC Standard Design Response Spectrum of Damping Factor 5%

CHAPTER 6

SUMMARY AND CONCLUSIONS

6.1 Summary

A new formulation of problems in random vibration analysis of linear systems is presented. The formulation facilitates the solution of problems through a process of synthesis. This solution approach is implemented in the instructional software STOCAL–II, which is designed to be used in teaching graduate courses on random vibrations.

In solving random vibration problems with STOCAL–II, the student issues a sequence of commands that carry out the numerical computations required for each basic element of the solution algorithm. The student must have a good understanding of the fundamental concepts in order to choose the proper sequence of commands and the corresponding parameters. The software assists the learning by: (a) freeing the student from tedious computations that are required for the solution but are not necessary for a fundamental understanding of the basic concepts, (b) providing a transparent computing environment where explicit specification of operations is necessary and all intermediate results are made available upon request, (c) facilitating the solution of exercise problems of practical significance that could not be solved by hand, (d) providing the student with an efficient means for parametric study and experimentation, and (e) providing an interactive computing environment with facility for immediate plotting of the results. STOCAL–II is the first instructional software in the area of random vibrations to possess these qualities.

STOCAL–II is developed as an extension to CAL. Only minor modifications to CAL have been made in this development, and all the commands in CAL are available in STOCAL-II. Thus, in addition to nondeterministic analysis, STOCAL-II can perform basic matrix operations and static and dynamic structural analysis. A 2D graphics capability, employing the IBM Graphics ToolKit (1984), is implemented which allows visualization of the results on the monitor screen or hard-copy plotting on a standard pen plotter.

Commands and algorithms incorporated in STOCAL-II provide the means for random vibration analysis of linear MDOF systems subjected to stationary or nonstationary excitations specified by their power spectral characterizations, or earthquake excitations specified by their response spectra. Commands for computing the response statistics of engineering interest, such as crossing rates, distributions of local and extreme peaks, and the characteristics of the envelope process are also provided. Additional commands allow the generation of sample functions or the estimation of temporal or ensemble autocorrelation or power spectral density from given sample functions.

A simple example, consisting of a two-story primary structure and an attached, three-node secondary subsystem, is used to illustrate the application and capabilities of STOCAL-II. The example does not illustrate the use of all STOCAL-II commands. However, after reviewing this illustration and the command descriptions in Appendix B the use of other commands should be straightforward.

6.2 Conclusions

The main contribution of this thesis has been in the new formulation of random vibration problems employing the solution by a process of synthesis, and in developing the instructional software STOCAL–II. It is hoped that these developments will facilitate the teaching and learning of the topic of random vibrations, both in a classroom setting as well as through self-learning or in the preparation of lectures. It is also hoped that the software will be useful in research and application employing the methods of random vibrations.

In developing a software one always has to put limits on the scope. However, STOCAL-II has been developed in a manner that further developments or expansion can be done rather easily. Although the current version has extensive capabilities, there are many areas where it can be further improved or expanded. For example, it would be useful to allow a complex-valued modulating function for the evolutionary input. This would then allow the solution of the response of cascaded systems subjected to nonstationary excitation. Another useful extension would be the capability to compute the frequency response function for general systems, including those with non-classical damping. Finally, it is recalled that presently STOCAL-II is limited to linear systems with deterministic properties. Extensions of the instructional software for the study of nonlinear problems, or problems with uncertain systems and parametric excitation would be significant additions to this work.

APPENDIX A

BASIC INTEGRALS

This appendix summarizes formulas used for computing six integrals that are basic elements in the evaluation of stochastic modal response.

A.1 Logarithmic Integral

The logarithmic integral, $ln(z)$, (Abramowitz and Stegun 1970, p67) is defined by

$$\begin{aligned} ln(z) &= \int_1^z \frac{1}{t} dt \\ &= ln(|z|) + i arg(z) \end{aligned} \qquad (A.1)$$

where $z = x + iy$ is a complex number and $|z|^2 = x^2 + y^2$. Note that $ln(1) = 0$, since $ln(1) = 0 + i2k\pi$ with $k = 0$.

A.2 Exponential Integral

The exponential integral, $E_1(z)$, is defined by

$$\begin{aligned} E_1(z) &= \int_z^\infty \frac{e^{-t}}{t} dt \\ &= -\gamma - ln(z) - \sum_{n=1}^{\infty} \frac{(-z)^n}{n \text{ x } n!} \end{aligned} \qquad (A.2a)$$

where $\gamma = 0.5772156649$, $z = x + iy$ and $|arg(z)| < \pi$. In addition, $E_1(z^*) = E_1^*(z)$, where the asterisk denotes the complex conjugate.

Since an exact evaluation $E_1(z)$ is not available from Eq. A.2a is not possible, approximate are used. For values of z for which

$$\frac{x^2}{25} + \frac{y^2}{36} > 1 \text{ and } x > 0 \qquad (A.2b)$$

or

$$\frac{x^2}{120} + \frac{y^2}{50} > 1 \text{ and } x < 0 \qquad (A.2c)$$

an approximate solution given by Abramowitz and Stegun (1970, p228-p254) is

$$e^z E_1(z) = \frac{0.711093}{z+0.415775} + \frac{0.278518}{z+2.29428} + \frac{0.010389}{z+6.2900} + \epsilon \quad |\epsilon| < 5\text{x}10^{-4} \qquad (A.2d)$$

For other values of z, a truncated Taylor series expansion is used. The number of terms in the series expansion is chosen such that the at least a 4 digit accuracy is achieved.

A.3 Expanded Logarithmic Integral

If the integration range of a logarithmic integral is from z_1 to z_2, then we call it an expanded logarithmic integral:

$$\begin{aligned} ln(z_1, z_2) &= \int_{z_1}^{z_2} \frac{1}{t} dt \\ &= ln(z_2) - ln(z_1) + \mathrm{i}\Delta(z_1, z_2) \end{aligned} \qquad (A.3a)$$

where $\Delta(z_1, z_2)$ is an integration path function and is defined as

$$\begin{aligned} \Delta(z_1, z_2) = &\quad 0 && A < B \\ = &\quad 2\pi && A = B,\ C \neq 0,\ \text{and } y_1 > y_2 \\ = &\ -2\pi && A = B,\ C \neq 0,\ \text{and } y_1 < y_2 \\ = &\quad \pi && A = B,\ C = 0,\ \text{and } y_1 > y_2 \\ = &\ -\ \pi && A = B,\ C = 0,\ \text{and } y_1 < y_2 \end{aligned} \qquad (A.3b)$$

where $z_1 = x_1 + \mathrm{i}y_1$, $z_2 = x_2 + \mathrm{i}y_2$, $C = x_1 y_2 - y_1 x_2$, $A = |C + y_1 - y_2|$ and $B = |C| + |y_1| + |y_2|$. In other words, Δ is equal to $\pm\pi$ when the integration path passes through the origin, it is $\pm 2\pi$ when the path passes over the negative axis, and it is zero elsewhere. When the path passes through the origin, the integration value may not exist. However, using a generalization of the Cauchy principal value (Greenberg 1978, p285) the integral may be defined as $\pm\pi$.

A.4 Expanded Exponential Integral

If the integration range of an exponential integral is from z_1 to z_2, then we call it an expanded exponential integral;

$$\begin{aligned} E_{12}(z_1,z_2) &= \int_{z_1}^{z_2} \frac{e^{-t}}{t} dt \\ &= E_1(z_1) - E_1(z_2) + \mathrm{i}\Delta(z_1,z_2) \end{aligned} \tag{A.4}$$

where the value of Δ depends on the integration path from z_1 to z_2 as described above.

A.5 Compound Logarithmic Integral

The compound logarithmic integral, I(z,m,a,b), for $m \geq 1$ is defined by

$$\begin{aligned} I(z,m,a,b) &= \int_a^b \frac{\omega^m}{\omega - z} d\omega \\ &= \int_a^b \omega^{m-1} d\omega + zI(z,m-1,a,b) \\ &= \frac{1}{m}(b^m - a^m) + zI(z,m-1,a,b) \end{aligned} \tag{A.5a}$$

which is a recursive formula. For $m = 0$,

$$I(z,0,a,b) = ln(a-z, b-z) \tag{A.5b}$$

A.6 Compound Exponential Integral

The compound exponential integral, $J(z,m,a,b,\tau)$, for $m \geq 1$ is defined by

$$\begin{aligned} J(z,m,a,b,\tau) = & \int_a^b \frac{\omega^m e^{\mathrm{i}\omega\tau}}{\omega - z} d\omega \\ = & \int_a^b \omega^{m-1} e^{\mathrm{i}\omega\tau} d\omega + zJ(z,m-1,a,b,\tau) \\ = & \; zJ(z,m-1,a,b,\tau) + \frac{\mathrm{i}}{\tau}[a^{m-1}e^{\mathrm{i}a\tau} - b^{m-1}e^{\mathrm{i}b\tau}] \\ & + \sum_{k=2}^{m} (\frac{\mathrm{i}}{\tau})^k (m-1)(m-2)\cdots(m-k+1) \\ & [a^{m-k}e^{\mathrm{i}a\tau} - b^{m-k}e^{\mathrm{i}b\tau}] \end{aligned} \tag{A.6a}$$

which is a recursive formula. For $m = 0$ and $\tau \neq 0$

$$J(z, 0, a, b, \tau) = e^{\mathrm{i}z\tau} E_{12}(z_1, z_2) \tag{A.6b}$$

where $z_1 = -\mathrm{i}\tau(a - z)$ and $z_2 = -\mathrm{i}\tau(b - z)$. For $\tau = 0$, the compound logarithmic integral is used.

APPENDIX B

STOCAL-II COMMAND SUMMARY

STOCAL-II commands are categorized into the following groups:

(1) Two Dimension Graphics

PLOT draws 2-D curves on the screen and the plotter by supplying the x and y coordinates in a single matrix or two separate matrices. A series of secondary commands are available to draw axes, zoom, etc.

(2) Generation of Samples

GSU generates random numbers between 0 and 1 with uniform distribution.

GSGP generates an ensemble of sample functions for a stationary Gaussian process with a specified PSD function.

GSGPT generates an ensemble of sample functions for a stationary Gaussian process with a specified autocorrelation function.

GEGP generates an ensemble of sample functions for a Gaussian process with an evolutionary PSD function.

TSSF multiplies generated sample functions by a time modulating function.

GPSD discretizes a specified PSD function.

(3) Transformation of Samples

TFSU transforms a uniformly distributed sample to a sample with a specified distribution.

TTSU transforms a sample with a specified distribution to a uniformly distributed sample.

(4) Estimation of Samples

STAT computes the means, standard deviations and skewness coefficients of specified samples of random variables.

NFD constructs the normalized frequency diagram of a given sample.

NCFD constructs the normalized cumulative frequency diagram of a given sample.

ACF computes the ensemble autocorrelation function of a random process from specified sample functions.

TACF computes the temporal autocorrelation function of a random process from a specified sample function.

(5) Fourier Transform

FTP computes the Fourier transform of a piecewise linear function.

IFTP computes the inverse Fourier transform of a piecewise linear function.

FTD computes the Fourier transform for discrete data.

IFTD computes the inverse Fourier transform for discrete data.

(6) Response PSD Functions

SPSD computes the stationary response PSD function.

TPSD computes the evolutionary response PSD function, where the input is specified by a uniformly modulated PSD function.

EPSD computes the evolutionary response PSD function, where the input is specified by an evolutionary PSD function.

(7) Response Correlation Functions

SCF computes the stationary response auto or cross-correlation function.

TCF computes the evolutionary response auto or cross-correlation function, where the input is specified by a uniformly modulated PSD function.

ECF computes the evolutionary response auto or cross-correlation function, where the input is specified by an evolutionary PSD function.

(8) Spectral Moments

SM computes the spectral moments for a specified PSD function.

SRSM computes the spectral moments of a stationary response when the input is specified by a PSD function.

SMSM computes the spectral moments for stationary modal responses when the input is specified by a PSD function.

SMR computes the spectral moments of a stationary response by superposition of modal spectral moments.

RCQC computes the mean of absolute maximum of a response quantity using the CQC response spectrum method.

RSM computes the spectral moments of the response when the input is specified by a mean response spectrum.

(9) Statistics of Stationary Gaussian Process

SSGP computes various statistics of a stationary Gaussian process, including crossing rates, distributions of peaks and the statistics of the envelope process.

LPKD computes the PDF and CDF of the local peaks of a stationary Gaussian process.

EXTD computes the PDF and CDF of the extreme peak of a stationary Gaussian process.

(10) Statistics of Nonstationary Process

TMS computes the variances and cross-correlation coefficients of a uniformly modulated process and its derivatives.

TRMS computes the variances and cross-correlation coefficients of the response and/or its derivatives when the input is specified by a uniformly modulated PSD function.

EMS computes the variances and cross-correlation coefficients of an evolutionary process and its derivatives.

ERMS computes the variances and cross-correlation coefficients of the response and/or its derivatives when the input is specified by an evolutionary PSD function.

NCR computes the mean upcrossing rate of a zero-mean nonstationary Gaussian process above specified thresholds.

NDLP computes the PDF of the local peaks of a zero-mean nonstationary Gaussian process.

NDEP computes the PDF and CDF of the extreme peak of a zero-mean nonstationary Gaussian process.

(11) Miscellaneous

AMP transforms complex numbers expressed by real and imaginary parts into an amplitude and phase angle expression.

MPF computes modal participation factors.

EPF computes modal effective participation factors.

VECTOR constructs a vector containing a sequence of equally spaced ascending numbers.

WRITE writes numerical data onto an external file.

REFERENCES

Abrahams, D. M., and F. Rizzardi (1988), *BLSS - The Berkeley Interactive Statistical System*, Norton Company, New York, N.Y.

Abramowitz, M., and I. A. Stegun (ed. 1970), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York, N.Y.

Baber, T. T. (1984), "Nonzero Mean Random Vibration of Hysteretic systems," *Journal of Engineering Mechanics*, ASCE, 110(7), pp. 1036-1049.

Barnoski, R. L., and J. R. Maurer (1969), "Mean-square Response of Simple Mechanical Systems to Nonstationary Random Excitation," *Journal of Applied Mechanics*, ASME, 36(2), pp. 211-227.

Becker, R. A., and J. M. Chamers (1984), *S: An Interactive Environment for Data Analysis and Graphics*, Belmont, California, Wadsworth Advanced Book Program.

Benaroya, H., and M. Rehak (1987), "Parametric Random Excitation, II: White-noise Parameters," *Journal of Engineering Mechanics*, ASCE, 113(6), pp. 875-884.

Bendat, J. S., and A. G. Piersol (1962, 1985), *Measurement and Analysis of Random Data*, John Wiley and Sons, Inc., New York, N.Y.

Biot, M. A. (1943), "Analytical and Experimental Methods in Engineering Seismology," *Transactions* of the ASCE, 108, Paper No. 2183, pp. 365-385.

Bolotin, V. V. (c1979, 1984) *Random Vibrations of Elastic Systems*, 1979, translated by Leipholz, H.H.E., Martinus Nijhoff Publishers, The Hague, 1984.

Booton, R. C. (1954), "The Analysis of Nonlinear Control Systems with Random Inputs," *Transactions* of the IRE Professional Group on Circuit Theory, 1, pp. 32-34.

Bruckner, A., and Y. K. Lin (1987), "Generalization of the Equivalent Linearization Method for Nonlinear Random Vibration Problems," *International Journal of Non-Linear Mechanics*, 22(3), pp. 227-235.

Bryson, A. E., and Y.-C. Ho (1975), *Applied Optimal Control*, Hemisphere Pub. Corp., New York, N.Y.

Bucciarelli, L. L. Jr., and C. Kuo (1970), "Mean-Square Response of a Second-Order System to Nonstationary Random Excitation," *Journal of Applied Mechanics*, ASME, 37(3), pp. 612-616.

Button, M. R., A. Der Kiureghian, and E. L. Wilson (1981), "STOCAL – User Information Manual", Report No. UCB-SESM/81-2 Department of Civil Engineering, University of California, Berkeley, CA.

Bycroft, G. N. (1960), "White Noise Representation of Earthquakes," *Journal of Engineering Mechanics*, ASCE, 86(2), pp. 1-16.

Cartwright, D. E., and M. S. Longuet-Higgins (1956), "The Statistical Distribution of Maxima of a Random Function," *Proceedings of the Royal Society of London*, Series A, 327, pp. 212-232.

Casciati, F., and L. Faravelli (1988), "Stochastic Equivalent Linearization for 3-D Frames," *Journal of Engineering Mechanics*, ASCE, 114(10), 1760-1771.

Caughey, T. K. (1959), "Response of Nonlinear String to Random Loading," *Journal of Applied Mechanics*, ASME, 26, pp. 341-344.

Caughey, T. K. (1960), "Classical Normal Modes in Damped Linear Dynamic Systems," *Journal of Applied Mechanics*, ASME, 27, pp. 267-271.

Caughey, T. K., and H. F. Stumpf (1961), "Transient Response of a Dynamic System under Random Excitation," *Journal of Applied Mechanics*, ASME, 28, pp. 563-566.

Caughey, T. K. (1971), "Nonlinear Theory of Random Vibrations," *Advances in Applied Mechanics*, ASME, 11, pp. 209-253.

Clough, R. W., and J. Penzien (1975), *Dynamics of Structures*, McGraw-Hill, Inc., New York, p. 634.

Cooley, J. W., and J. W. Tukey (1965), "An Algorithm for the Machine Calculation of Complex Fourier Series," *Mathematics of Computation*, 19(90), pp. 297-301.

Corotis, R. B., and A. M. Vanmarcke (1975), "Time-Dependent Spectral Content of System Response," *Journal of Engineering Mechanics*, ASCE, 101(5), pp. 623-635.

Corotis, R. B., and T. A. Marshall (1977), "Oscillator Response to Modulated Random Excitation," *Journal of Engineering Mechanics*, ASCE, 103(4), pp. 501-513.

Cramer, H., and M. R. Leadbetter (1967), *Stationary and Related Stochastic Processes*, John Wiley and Sons, Inc., New York, N.Y.

Crandall, S. H. (ed. 1958) *Random Vibration*, Vol. 1, MIT Technology Press, Cambridge, Massachusetts.

Crandall, S. H., and W. D. Mark (1963), *Random Vibration in Mechanical Systems*, Academic Press, New York, p. 166.

Crandall, S. H. (1963), "Perturbation Techniques for Random Vibration of Nonlinear System," *Journal of the Acoustical Society of America*, 35(11), pp. 1700-1705.

Crandall, S. H. (1980), "Non-Gaussian Closure for Random Vibration of Non-Linear Oscillator," *International Journal of Non-Linear Mechanics*, 15, pp. 303-313.

Crandall, S. H., and W. Q. Zhu (1983), "Random Vibration: A Survey of Recent

Developments," *Journal of Applied Mechanics*, ASME, 50, pp. 953-962.

Dahlquist, G., and A. Bjorck (1974), *Numerical Methods*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Davenport, A. G. (1964), "Note on the Distribution of the Largest Value of a Random Function with Application to Gust Loading," *Proceedings*, Institution of Civil Engineering, London, England, 28, pp. 187-196.

DebChaudhury, A., and D. A. Gasparini (1982), "Response of MDOF Systems to Vector Random Excitation," *Journal of Engineering Mechanics*, ASCE, 108(2), pp. 367-385.

Der Kiureghian, A. (1980), "Structural Response to Stationary Excitation," *Journal of Engineering Mechanics*, ASCE, 106(6), pp. 1195-1213.

Der Kiureghian, A. (1981), "A Response Spectrum Method for Random Vibration Analysis of MDOF Systems," *Earthquake Engineering and Structural Dynamics*, 9, pp. 419-435.

Dickens, J. M., and E. L. Wilson (1980), "Numerical Methods for Dynamic Substructure Analysis," Report No. EERC-80/20, Earthquake Engineering Research Center, University of California, Berkeley, CA.

Dimentberg, M. F. (1982) "An Exact Solution to a Certain Non-Linear Random Vibration Problems," *International Journal of Non-Linear Mechanics*, 17(4), pp. 231-236.

Elishakoff, I. (1983), *Probabilistic Method in the Theory of Structures*, John Wiley and Sons, Inc., New York, N.Y.

Erdelyi, A. (1956), *Asymptotic Expansions*, Dover Publications, New York, N.Y.

Fokker, A. P. (1914), "Die Mittlere Energie Rotierender Electricher Dipole im Stahlungsfeld," *Annals of Physics*, 43, pp. 810-815.

Foss, K. A. (1956), "Coordinates Which Uncouple the Equations of Motion of Damped Linear Systems," Tech. Rep. 23-30, MIT, Cambridge, Massachusetts.

Gasparini, D. A. (1979), "Response of MDOF Systems to Nonstationary Random Excitation," *Journal of Engineering Mechanics*, ASCE, 105(1), pp. 13-27.

Gasparini, D. A., and A. DebChaudhury (1980), "Dynamic Response to Nonstationary Nonwhite Excitation," *Journal of Engineering Mechanics*, ASCE, 106(6), pp. 1233-1248.

Goodman, L. E., E. Rosenblueth, and N. M. Newmark (1955) "A Seismic Design of a Firmly Founded Elastic Structure," *Transactions* of the ASCE, 120, Paper No. 2762, pp. 782-802.

Greenberg, Michael D. (1978), *Foundations of Applied Mathematics*, Prentice-Hall, Inc., Englewood Cliffs, N.J.

Grigoriu, M. (1981), Discussion of "Structural Response to Stationary Excitation," by Armen Der Kiureghian, December 1980, *Journal of Engineering Mechanics*, ASCE, 107(6), pp. 1255-1257.

Grigoriu, M. (1984), "Crossing of Non-Gaussian Translation Process," *Journal of Engineering Mechanics*, ASCE, 110(4), pp. 610-620.

Grigoriu, M. (1986), "Response of Linear Systems to Quadratic Gaussian Excitations," *Journal of Engineering Mechanics*, ASCE 112(6), pp. 523-535.

Hammond, J. K. (1968), "On the Response of Single and Multidegree of Freedom Systems to Non-Stationary Random Excitations," *Journal of Sound and Vibration*, 7(3), pp. 393-416.

Hasselman, T. (1972), "Linear Response to Nonstationary Random Excitation," *Journal of Engineering Mechanics*, ASCE, 98(3), pp. 519-530.

Henrici, P. (1974), *Applied and Computational Complex Analysis*, Vol. 1, John Wiley and Sons, Inc., New York, N.Y.

Hoit, M. I. (1983), "New Computer Programming Techniques for Structural Engineering," Ph.D. Dissertation in Civil Engineering, University of California, Berkeley, CA.

Huston, W. B., and T. H. Skopinski (1956), "Probability and Frequency Characteristics of Some Flight Buffet Loads," NACA, TN 3733.

Housner, G. W., and P. C. Jennings (1964), "Generation of Artificial Earthquakes," *Journal of Engineering Mechanics*, ASCE, 90(1), pp. 113-150.

Ibrahim, R. A., and J. W. Roberts (1978), "Parameter Vibration, Part V: Stochastic Problems," *Shock and Vibration Digest*, 10(5), pp. 17-38.

Ibrahim, R. A., and J. W. Roberts (1981), "Parameter Vibration, Part VI: Stochastic Problems," *Shock and Vibration Digest*, 13(9), pp. 23-35.

Ibrahim, R. A. (1985), *Parameter Random Vibration*, John Wiley and Sons, Inc., New York, N.Y.

Igusa, T., A. Der Kiureghian, and J. L. Sackman (1984), "Modal Decomposition Method for Stationary Response of Non-Stationary Damped Systems," *Earthquake Engineering and Structural Dynamics*, 12, pp. 121-136.

Igusa, T., and A. Der Kiureghian (1985), "Dynamic Response of Multiply Supported

Secondary Systems," *Journal of Engineering Mechanics*, ASCE, 111(1), pp. 20-41.

Iyengar, R .N., and P. K. Dash (1978), "Study of the Random Vibration System by the Gaussian Closure Technique," *Journal of Applied Mechanics*, ASME, 45, pp. 393-399.

Iyengar, R. N. (1988), "Stochastic Response and Stability of the Duffing Oscillator under Narrowband Excitation," *Journal of Sound and Vibration*, 126(2), pp. 255-263.

Kamel, H., and M. W. McCabe (1978), "GIFTS System Manual," Department of Aerospace and Mechanical Engineering, University of Arizona, Tucson, Arizona.

Kanai, K. (1957), "Seismic-empirical Formula for the Seismic Characteristics of the Ground," *Bull. Earthquake Research Inst.*, Japan, 35, pp. 309-325.

Khintchine, A. (1934), "Korrelations Theorie der Stationaren Stochastischen Prozesse," *Mathematische Annalen*, 109, pp. 604-615.

Kinra, R. K., and P. W. Marshall (1979), "Fatigue Analysis of the Cognac Platform," *Offshore Technology Conference Proceedings*, OTC 3378.

Kolmogorov, A. (1931), "Uber die Analytischen Methoden in Wahrsheinlichkeitsrechnung," *Mathematische Annalen*, 104, pp. 415-458.

Kraichan, R. H. (1962), "The Closure Problem of Turbulence Theory," *Proceedings of Symposia in Applied Mathematics*, 13, pp. 199-225.

Kreyszig, E. (1972), *Advanced Engineering Mathematics*, third edition, John Wiley and Sons, Inc., New York, N.Y.

Krylov, N. M., and N. Bogoliubov (1931), *Introduction to Nonlinear Mechanics*, Kiev. English translation, 1943, Princeton University Press, Princeton, N.J.

Langley, R. S. (1988), "Application of the Principle of Detailed Balance to the Random Vibration of Non-Linear Oscillators," *Journal of Sound and Vibration*, 125(1), pp. 85-92.

Liepmann, H. W. (1952), "On the Application of Statistical Concepts to the Buffeting Problem," *Journal of the Aeronautical Sciences*, 19(12), pp. 793-822.

Lin, Y. K. (1963a), "Nonstationary Response of Continuous Structures to Random Loading," *Journal of the Acoustical Society of America*, 35(2), pp. 222-237.

Lin, Y. K. (1963b), "Application of Nonstationary Shot Noise in the Study of System Response to a Class of Nonstationary Excitations," *Journal of Applied Mechanics*, ASME, 30(4), pp. 555-558.

Lin, Y. K. (1967), *Probabilistic Theory of Structural Dynamics*, McGraw-Hill Book Co., Inc., New York, N.Y.

Lin, Y. K., et al. (1986) "Methods of Stochastic Dynamics, *Structural Safety*, 3, pp. 167-194.

Lin, Y. K., and Y. Yong (1987), "Evolutionary Kanai-Tajimi Earthquake Models," *Journal of Engineering Mechanics*, ASCE, 113(8), pp. 1119-1137.

Lipsett, A. W. (1986), "Nonlinear Structural Response in Random Waves," *Journal of Structural Mechanics*, 112(11), pp. 2416-2429.

Longuet-Higgins, M. S. (1952), "On the Statistical Distribution of the Heights of Sea Waves," *Journal of Marine Research*, 3, pp. 245-266.

Lutes, L. D. and S.-H. Tzuang (1983), "First-Passage Probability for Two-Mode Systems," *Journal of Engineering Mechanics*, ASCE, 109(6), pp. 1358-1374.

Lutes, L. D. and S.-L. J. Hu (1986), "Non-Normal Stochastic Response of Linear Systems," *Journal of Engineering Mechanics*, ASCE, 112(2), pp. 127-141.

Madsen, P. H., and S. Krenk (1982), "Stationary and Transient Response Statistics," *Journal of Engineering Mechanics*, ASCE, 108(4), pp. 622-635.

Madsen, P. H., and S. Krenk (September 1984), "An Integral Equation Method for the First-Passage Problem in Random Vibration," *Journal of Applied Mechanics*, ASME, 51, pp. 674-679.

Madsen, H. O., S. Krenk, and N. C. Lind (1986), *Methods of Structural Safety*, Prentice-Hall, Inc., Englewood Cliffs, N.J.

Mark, W. D. (1970), "Spectral Analysis of the Convolution and Filtering of Non-stationary Stochastic Process" *Journal of Sound and Vibration*, 11(1) pp. 19-63.

Masri, S. F. (1978), "Response of a Multidegree-of-Freedom System to Nonstationary Random Excitation," *Journal of Applied Mechanics*, ASME, 45, pp. 649-656.

Miles, J. W. (1954), "On Structural Fatigue Under Random Loading," *Journal of Aeronautical Sciences*, 21(11), pp. 753-762.

Newland, D. E. (c1975, 1984), *An Introduction to Random Vibrations and Spectral Analysis*, 2nd edition, Longman, Inc., New York.

Nigam, N. C. (1983), *Introduction to Random Vibrations*, MIT Technology Press, Cambridge, Massachusetts.

Nielsen, S. R. J., and J. D. Sorensen (1988), "Probability of Failure in Random Vibration," *Journal of Engineering Mechanics*, ASCE, 114(7), pp. 1218-1230.

Noori, M. N., A. Saffar, and H. Davoodi (1987), "A Comparison between Non-

Gaussian Closure and Statistical Linearization Techniques for Random Vibration of a Nonlinear Oscillator," *Computers and Structures*, 26(6), pp. 925-931.

NRC (1976), U.S. Nuclear Regulatory Commission, Regulatory Guide 1.9.2, Revision 1: "Combining Modal Responses and Spatial Components in Seismic Response Analysis,", Washington, D.C.

Ochi, M. K. (1986), "Non-Gaussian Random Processes in Ocean Engineering," *Probability Engineering Mechanics*, 1(1), pp. 28-39.

Orabi, I. I., and G. Ahmadi (1987a), "Functional Series Expansion Method for Response Analysis of Non-Linear Systems Subjected to Random Excitations," *International Journal of Non-Linear Mechanics*, 22(6), pp. 451-465.

Orabi, I. I., and G. Ahmadi (1987b), "iterative Method for Nonstationary Response of Nonlinear Random Systems," *Journal of Sound and Vibration*, 119(1), pp. 145-157.

Page, C. H. (1952), "Instantaneous Power Spectra," *Journal of Applied Physics*, AIP, 23(1), pp. 103-106.

Piszczek, K., and J. Niziol (1986), *Random Vibration of Mechanical Systems*, translated by Beards, C.F., PWN-Polish Scientific Publishers, Warszawa.

Planck, M. (1917), "Uber einen Satz der Statischen Dynamik und Seine Erweiterung in der Quanten-Theorie," Preuss, Akadamie, Weiss, pp. 324-341.

Priestley, M. B. (1965), "Evolutionary Spectra and Non-stationary Processes," *Journal of the Royal Statistical Society*, Series B, 27, pp. 204-237.

Rice, S. O. (1944, 1945), "Mathematical Analysis of Random Noise," *Bell System Technical Journal*, AT&T, 23, pp. 282-332, 1944, 24, pp. 46-156. 1945.

Roberts, J. B. (1971), "The Covariance Response of Linear Systems to Non-Stationary Random Excitation," *Journal of Sound and Vibration*, 14(3), pp. 385-400.

Roberts, J. B. (1981), "Response of Nonlinear Mechanical Systems to Random Excitation, Part 2: Equivalent Linearization and Other Methods," *Shock and Vibration Digest*, 13(5), pp. 15-29.

Roberts, J. B. (1984), "Techniques for Nonlinear Random Vibration Problems," *Shock and Vibration Digest*, 16(9), pp. 3-14.

Rosenblueth, E., and J. Elorduy (1969), "Response of Linear Systems to Certain Transient Disturbances," *Proceedings*, 4th World Conference Earthquake Engineering, Santiago, Chile.

Ruiz, P., and J. Penzien (1968), "PSEQGN - Artificial Generation of Earthquake Accelegrams," Report No. UCB/EERC 69-3, Earthquake Engineering Research Center, University of California, Berkeley, CA.

Sakata, M., and K. Kimura (1979) "The Use of Moment Equations for Calculating the Mean Square Response of a Linear System to Non-Stationary Random Excitation," *Journal of Sound and Vibration*, 67(3), pp. 383-393.

Samaras, E., M. Shinozuka, and A. Tsurui (1985), "ARMA Representation of Random Processes," *Journal of Engineering Mechanics*, ASCE, 111(3), pp. 449-461.

Samuels, J. C., and A. C. Eringen (1959), "On Stochastic Linear Systems," *Journal of Mathematics and Physics*, 38, pp. 83-103.

SAS Institute, Inc., at Cary, North Carolina, *SAS User's Guide*, 1982 ed.

Saul, W. E., A. H. Peyrot, and P. Jayachandran (1972), "ANSWERS - Structural Analysis Software System for Static and Dynamic Analysis," University of Wisconsin, Madison, WI.

Schrage, L. (1979), "A more Portable Fortran Random Number Generator," *ACM Transactions on Mathematical Software*, 5(2), pp. 132-138.

Shinozuka, M. and Y. Sato (1967), "Simulation of Nonstationary Random Process," *Journal of Engineering Mechanics*, ASCE, 93(1), pp. 11-40.

Shinozuka, M. (1970), "Random Processes with Evolutionary Power," *Journal of Engineering Mechanics*, ASCE, 96(4), pp. 543-545.

Shinozuka, M., and C.-M. Jan (1972) "Digital Simulation of Random Processes and its Applications," *Journal of Sound and Vibration*, 25(1), pp. 111-128.

Singh, M. P., and S. L. Chu (1976), "Stochastic Considerations in Seismic Analysis of Structures," *Earthquake Engineering and Structural Dynamics*, 4, pp. 295-307.

Spanos, P. D. (1981), "Stochastic Linearization in Structural Dynamics," *Applied Mechanics Reviews*, 34(1), pp. 1-8.

Spanos, P. D., and L. D. Lutes (1986), "A Primer of Random Vibration Techniques in Structural Engineering," *Shock and Vibration Digest*, 18(4), pp. 3-9.

SPSS, Inc., at Chicago, Illinois *SPSS User's Guide*, 2nd ed., McGraw-Hill Book, Company, 1986.

Su, Lin, and G. Ahmadi (1988), "Earthquake Response of Linear Continuous Structures by the Method of Evolutionary Spectral," *Engineering Structures*, 10, pp. 47-56.

Sun, J.-Q., and C. S. Hsu (1987), "Cumulant-neglect Closure Method for Nonlinear Systems Under Random Excitations," *Journal of Applied Mechanics*, ASME, 54((3), pp. 649-655.

Tajimi, H. (1960), "A Statistical Method of Determining the Maximum Response of a Building Structure during an Earthquake," *Proceedings*, 2nd World Conference on Earthquake Engineering, Japan, pp. 781-798.

Taylor, R. L. (1987) "Finite Element Tools For Computational Mechanics Instruction," IBM ACIS University Conference, Boston, Massachusetts, June 27-30.

To, C. W. S. (1982) "Non-Stationary Random Responses of a Multi-Degree-of-Freedom System by the Theory of Evolutionary Spectra," *Journal of Sound and Vibration*, 83(2), pp. 273-291.

To, C. W. S. (1984a), "Time-dependent Variance and Covariance of Responses of Structures to Non-stationary Random Excitations'" *Journal of Sound and Vibration*, 89(1), pp. 135-156.

To, C. W. S. (1984b), "The Response of Nonlinear Structures to Random Excitation," *Shock and Vibration Digest*, 16(4), pp. 13-33.

To, C. W. S. (1986), "Response Statistics of Discretized Structures to Non-Stationary Random Excitation," *Journal of Sound and Vibration*, 105(2), pp. 217-231.

To, C. W. S. (1987), "Random vibration of Nonlinear Systems," *Shock and Vibration Digest*, 19(3), pp. 3-9.

Toro, G. R., and C. A. Conell (1986), "Extremes of Gaussian Processes with Bimodal Spectra," *Journal of Engineering Mechanics*, ASCE, 112(5), pp. 465-484.

Vanmarcke, E. H. (1972), "Properties of Spectral Moments with Applications to Random Vibration'" *Journal of Engineering Mechanics*, ASCE, 98(2), pp. 425-445.

Vanmarcke, E. H. (1975), "On the Distribution of First-Passage Time for Normal Stationary Processes," *Journal of Applied Mechanics*, ASME, 42, pp. 215-220.

Vanmarcke, E. H. (1979), "Some Recent Developments in Random Vibration," *Applied Mechanics Reviews*, 32(10), pp. 1197-1202.

Wang, M. C., and G. E. Uhlenbeck (1945), "On the Theory of the Brownian Motion II," *Reviews of Modern Physics*, 17, pp. 323-342.

Wen, Y.-K. (1975), "Approximate Method for Nonlinear Random Vibration," *Journal of Engineering Mechanics*, ASCE, 101(4), pp. 389-401.

Wen, Y. K. (1976), "Method for Random Vibration of Hysteretic System," *Journal of Engineering Mechanics*, ASCE, 102(2), pp. 249-263.

Wen, Y. K. (1980), "Equivalent Linearization for Hysteretic System under Random Loading," *Journal of Applied Mechanics*, ASME, 47, pp. 150-154.

Wen, Y. K. (1986), "Stochastic Response and Damage Analysis of Inelastic Structures," *Probabilistic Engineering Mechanics*, 1(1), pp. 49-57.

Wiener, N. (1930), "Generalized Harmonic Analysis," *Acta Mathematica*, 55, pp. 117-258.

Wilson, E. L. (1979), "CAL - A Computer Analysis Language for Teaching Structural Analysis", *Computers and Structures*, 10, pp. 127-132.

Wilson, E. L., A. Der Kiureghian, and E. P. Bayo (1981), "A Replacement for the SRSS Method in Seismic Analysis," *Earthquake Engineering and Structural Dynamics*, 9(5), pp. 187-194.

Wilson, Edward L. (c1977, 1986), "CAL86 – Computer Assisted Learning of Structural Analysis and The CAL/SAP Development System," Report No. UCB/SEMM-86/05, Department of Civil Engineering, University of California, Berkeley. CA.

Wu, W.-F. (1987), "Comparison of Gaussian Closure Technique and Equivalent Linearization Method," *Probabilistic Engineering Mechanics*, 2(1), pp. 1-8.

Wung, C.-D., and A. Der Kiureghian (1989), "STOCAL–II: Computer-Assisted Learning System for Stochastic Dynamic Analysis of Structures, Part II – User's Manual," Report No. UCB/SEMM-89/11, Department of Civil Engineering, University of California, Berkeley, CA.

Yamazaki, F., and M. Shinozuka (1988), "Digital Generation of Non-Gaussian Stochastic Fields," *Journal of Engineering Mechanics*, ASCE, 117(7), pp. 1183-1197.

Yang, T. Y. (1986), *Random Vibration of Structures*, John Wiley and Sons, Inc., New York.

Yar, M., and J. K. Hammond (1986), "Approximate Eigenfunction Analysis of First-order Non-linear Systems with Application to a Cubic System," *Journal of Sound and Vibration*, 111(3), pp. 457-466.

Yong, Y., and Y. K. Lin (1987), "Exact Stationary-Response Solution for Second-order Nonlinear system under Parametric and External White-Noise Excitations," *Journal of Applied Mechanics*, ASME, 54(2), pp. 414-418.

Zhu, W. Q. (1988), "Stochastic Averaging Methods in Random Vibration," *Applied Mechanics Reviews*, 41(5), pp. 189-199.