Construction of the unitary free fermion Segal conformal field theory

by

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A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

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Spring 2014

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Abstract

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This thesis is primarily concerned with the construction and analysis of free fermion Segal CFTs in arbitrary genus, with emphasis placed on analytic aspects of the construction. In particular, our Segal CFTs take values in the "category" of Hilbert spaces and trace class maps. The main results are:

- A detailed construction of the free fermion Segal CFT, in arbitrary genus, using Hilbert spaces and trace class operators (Chapter 4),
- An explicit identification of the operators assigned to three-punctured spheres with the free fermion vertex operator algebra (Chapter 5.1),
- Preliminary construction of (quotients of) tensor products of $SU(k)_\ell$ WZW vertex modules via descent from fermions (Chapter 5.2).

As a technical tool, we will develop a Riemann surface generalization of the Cauchy transform for planar domains (Appendix A).

Contents

Acknowledgments

I woud like to thank Vaughan Jones for his support and guidance during my time at Berkeley. I'm grateful for his constant encouragement, fantastic ideas, limitless patience, good advice, and incredible generosity with time, energy and support.

I would like to thank Antony Wassermann for freely sharing his invaluable expertise and ideas.

I would like to thank André Henriques, Yasu Kawahigashi, Roberto Longo, Peter Teichner, and Feng Xu for many enlightening conversations, and for generously hosting visits. I would also like to thank Terry Gannon for patiently answering many questions.

I am extremely grateful for the support and friendship of the subfactor community. In particular, I would like to thank Dietmar Bisch, Arnaud Brothier, Pinhas Grossman, Mike Hartglass, Scott Morrison, Dave Penneys, Emily Peters, and Noah Snyder.

Thank you to all of the wonderful Berkeley mathematics colleagues I've had the pleasure of sharing these past years with, including Kelty Allen, Adam Boocher, Morgan Brown, Michael Daub, Taryn Flock, Jeff Galkowski, Ian Herbert, Ralph Morrison, Ben Orlin, Doug Rizzolo, Ivan Ventura, and Harold Williams.

I would like to thank the Pomona College mathematics community, especially Stephan Garcia, for their continued involvement in my mathematics education years after my departure from Claremont.

Finally, I would like to thank my parents, Barbara and Marvin, and my brother Ben for their love and support, in this endeavor and all of the others.

Chapter 1

Introduction

At present, there is no completely satisfactory rigorous mathematical foundation for conformal field theory. Such a foundation would have to be sufficiently rigid to imply the standard structural results for conformal field theories (*axiomatic CFT*) while simultaneously not being so strict as to prevent the construction of important models (*constructive CFT*).

This thesis is primarily concerned with constructive conformal field theory. The conformal field theories that we will be interested in are $(1+1)$-dimensional rational chiral theories. Roughly speaking, a chiral theory is one in which the fields depend holomorphically on the space-time parameter. The state space of a full conformal field theory should decompose

$$\bigoplus \mathcal{H}_\lambda \otimes \overline{\mathcal{H}_\lambda}$$

where the $\mathcal{H}_\lambda$ (resp. $\overline{\mathcal{H}_\lambda}$) are the *sectors* of an underlying chiral (resp. antichiral) theory. The theory is called rational if there are finitely many sectors, and each has finite dimension (in an appropriate sense).

It would be impossible to give a full accounting of the history of mathematical chiral conformal field theory in this space, but the interested reader might begin with [FRS10]. We will mainly be interested in three definitions of a chiral conformal field theory: conformal nets (*functional analytic*), vertex operator algebras (*algebraic*), and Segal CFT (*geometric*). While these axiomatizations should be equivalent modulo technical hypotheses, there are no general theorems of this nature. Axiomatic and constructive theorems must be proved separately for each case, and often some theorems can be proven in one framework but not others.

In an unfinished manuscript from the 1980s (later appearing as [Seg04]), Graeme Segal proposed a definition for conformal field theory which is essentially a functor from the category of 1-dimensional complex cobordisms to the category of Hilbert spaces and trace class maps. The theory of Segal CFT has developed slowly, especially with regard to constructive CFT. One of the major goals of this thesis is to carefully construct the Segal CFT for the free fermion model using only bounded operators on Hilbert spaces.

After constructing the free fermion model, the main tool that we will use to analyze it is the connection between Segal CFT and *vertex operator algebras.* The notion of vertex operator algebra [Bor86, FLM88] axiomatizes the structure of a chiral half of the conformally invariant quantum fields of a system. This is closely related to the term *chiral algebra* in the physics literature. Vertex operator algebras provide a combinatorial framework for working with the operator product expansions of fields, which corresponds to the local data of a Segal CFT as punctures get closer together. More precisely, the operators assigned to three punctured spheres by the Segal CFT are given by vertex operators, up to a dampening factor. The result is a geometric extension of a vertex operator algebra, closely related to *geometric vertex operator algebras* [Hua97].

A third notion of conformal field theory is that of a *conformal net* (or *algebraic CFT*), which axiomatizes the algebras of local observables of a chiral conformal field theory. This approach to quantum field theory was developed in [HK64, DHR69a, DHR69b], and adapted to the case of 2-dimensional conformal field theory in [FRS89, FRS92]. This thesis will not deal directly with conformal nets, but was largely motivated by the fundamental paper [Was98] in constructive algebraic CFT. In particular, this work is intended to lay the foundation for the construction of a Segal CFT corresponding to the representation theory of the loop group $LSU(k)$ via descent from fermions.

The main results of this thesis are the following:

- A detailed construction of the free fermion Segal CFT, in arbitrary genus, using Hilbert spaces and trace class operators (Chapter 4),

- An explicit identification of the operators assigned to three-punctured spheres with the free fermion vertex operator algebra (Chapter 5.1),

- Preliminary construction of tensor products of $SU(k)_\ell$ WZW vertex modules via descent from fermions (Chapter 5.2).

While there have been previous descriptions of Segal CFTs (e.g. [Seg04, Hua98, BK01, Pos03]), Chapter 4 gives, to our knowledge, the first detailed construction of a Segal CFT in arbitrary genus based on Hilbert spaces and trace class operators. Regarding the results of Chapter 5, we are very grateful to Antony Wassermann for explaining the connection between Segal CFT and vertex operator algebras, in particular the statement of the crucial Theorem 5.6.

A major feature of our construction of the free fermion Segal CFT is that it emphasizes analytic properties. Our methods are explicit and constructive, and are intended to facilitate computation. In future work we will use the analytic properties of the Segal CFT to construct the corresponding conformal net. While the free fermion conformal net is hardly a new object, this is a proof of concept for the philosophy that Segal CFTs with sufficiently nice analytic properties can be used to construct conformal nets. We also hope to use these properties to extend our theory to cobordisms with corners, as required in [ST04].

This thesis is organized as follows. In Chapter 2, we review several preliminary notions, including super Hilbert spaces, complex manifolds, and spin structures. We also review the machinery of fermionic second quantization, which lies at the heart of our constructions.

In Chapter 3, we carefully define the notion of chiral Segal CFT. The emphasis on construction and calculation influences the nature of these definitions. The definitions are phrased as concretely as possible, and in particular we will require that the boundaries of our cobordisms are parameterized.

In Chapter 4, we give two constructions of free fermion Segal CFTs. The first (preliminary) construction is a non-unitary theory with a single sector. We then use this to construct a unitary free fermion Segal CFT which is defined on cobordisms with a choice of spin structure. The proof that these constructions give Segal CFTs requires two results from other chapters: the main result of Appendix A to verify that the theory is non-trivial and local, and the main result of Section 5.1 to verify holomorphic dependence of the fields.

In Chapter 5, we investigate two applications of the construction from Chapter 4. In Section 5.1, we recover the free fermion vertex operator algebra from the Segal CFT. In Section 5.2, we obtain several consequences of the fact that the vertex operator algebra for the WZW model $SU(k)_\ell$ sits inside the VOA for the free fermion. We define a notion of the fusion of two representations of $LSU(k)_\ell$ embedded in fermionic Fock space, and show that it always produces a quotient of the standard fusion $\bigoplus N_{\lambda\mu}^{\nu}\mathcal{H}_\nu$. As a corollary, we positively answer a conjecture of Vaughan Jones regarding the internal fusion of isotypical components. We also describe how one can extract the $SU(2)_\ell$ fusion rules from this construction.

In Appendix A, we develop a technical tool for our construction, the Cauchy transform for Riemann surfaces. Our development parallels the description of the Cauchy transform for planar domains in [Bel92]. A key theorem is that the Cauchy transform gives a bounded operator which differs from the Hardy space projection by a Hilbert-Schmidt operator. This allows us to replace the Hardy space projection with the Cauchy transform in certain key computations, and then exploit locality properties of the Cauchy transform (Theorem A.15 and preceding remarks). We would like to thank Antony Wassermann for explaining how singular integral operators could be used to prove Theorem A.1(1) in the planar case, which was the motivation for Appendix A.

We conclude with several directions for future research.

- We would like to construct the Segal CFT for $SU(k)_\ell$ WZW models as a subtheory of the free fermion model. We briefly indicate our approach to this problem in Remark 5.19.

- From Section 5.1, we can think of the fields of the free fermion Segal CFT as cylinders with a disk removed. Following a suggestion of André Henriques, we wish to obtain the operators of the corresponding conformal net from these cylinders, in the limit where intervals on the top and bottom circles of the cylinder pinch together. We should also point out very interesting ongoing work which relates conformal nets and Segal CFTs in [BDH13] and subsequent papers.

- The tools used to construct the free fermion Segal CFT have natural extensions to complex manifolds with corners on the boundary. We would like to investigate whether these tools can be used to construct a local conformal field theory on these manifolds, as described in [ST04]. We expect that the locality properties of the Cauchy transform discussed in Appendix A will play an important role.

- We would like to say more about the categorical structures of the Segal CFTs that we construct. In particular, we would like to investigate how easily one can recover facts related to the modularity of the representation categories of the $SU(k)_\ell$ models ([Was98, Xu00, KLM01] for conformal nets, [FZ92, Zhu96] for VOAs, or [MS89, BK01] for modular functors) using the techniques of Section 5.2.

Chapter 2

Preliminaries

2.1 Vector spaces, Hilbert spaces and operators

We begin by fixing notation and basic notions. If V and W are complex vector spaces, we will write $\mathcal{L}(V,W)$ for the vector space of linear maps from V to W. If H and K are (pre-)Hilbert spaces, then we will write $\mathcal{B}(H,K)$ for the space of bounded linear maps from H to K. If $V = W$ or $H = K$, we will write $\mathcal{L}(V)$ or $\mathrm{End}(V)$ for the linear endomorphisms of V, and $\mathcal{B}(H)$ for bounded linear operators on H. We will write $\mathcal{U}(H)$ and $\mathcal{P}(H)$ for the unitary operators and projections on H, respectively.

For $p \geq 1$, define $\mathcal{I}^p(H,K) = \{x \in \mathcal{B}(H,K) : \mathrm{tr}((x^*x)^{p/2}) < \infty\}$. When $H = K$, $\mathcal{I}^p(H,K)$ is a $*$-ideal of $\mathcal{B}(H)$ called the *Schatten p-ideal.* More generally, $\mathcal{I}^p(H,K)$ is a $\mathcal{B}(K) - \mathcal{B}(H)$ bimodule. The vector space $\mathcal{I}^p(H,K)$ is complete with respect to the norm

$$\|x\|_{\mathcal{I}^p} = \mathrm{tr}((x^*x)^{p/2})^{1/p}.$$

We will be interested in the cases $p = 1$ and $p = 2$, in which case the elements of $\mathcal{I}^p(H,K)$ are called *trace class* and *Hilbert-Schmidt* operators, respectively. Since $\mathcal{I}^p(H,K) \subset \mathcal{I}^{p'}(H,K)$ when $p < p'$, all trace class operators are Hilbert-Schmidt. Conversely, the product of Hilbert-Schmidt operators is trace class. The Hilbert-Schmidt operators are a Hilbert space, with inner product

$$\langle x, y\rangle = \mathrm{tr}(y^*x).$$

Our inner products will always be complex linear in the first slot.

There is a natural isomorphism of Hilbert spaces $H^* \otimes K \cong \mathcal{I}^2(H,K)$. Note that whenever we write tensor products of Hilbert spaces, we are referring to the Hilbert space tensor product. The notation H^* will always refer to the continuous dual, and we will denote by $\xi \mapsto \xi^*$ the conjugate linear isomorphism $H \to H^*$.

If V is a complex vector space, denote by $\overline{V}$ the vector space with the conjugate complex structure. If $\xi \in V$, we will write $\overline{\xi} \in \overline{V}$ for the image of ξ under the identity map $V_{\mathbb{R}} \to \overline{V}_{\mathbb{R}}$.

Remark 2.1. If $V = \mathbb{C}^k$, the above notation is confusing as the symbol $\overline{\xi}$ is already associated with componentwise complex conjugation. In this case, we will use the notation

$\iota : \mathbb{C}^k \to \overline{\mathbb{C}^k}$ for the conjugate linear identity map and reserve the conjugate notation $\overline{\xi}$ for the conjugate linear (non-identity) map $\mathbb{C}^k \to \mathbb{C}^k$.

We will identify $\mathcal{B}(H^*, K^*)$ and $\overline{\mathcal{B}(H,K)}$ by setting $\overline{x}\xi^* = (x\xi)^*$ for $x \in \mathcal{B}(H,K)$.

2.2 Super Hilbert spaces

The theory of the free fermion is $\mathbb{Z}/2$-graded, meaning that all Hilbert spaces H will carry a $\mathbb{Z}/2$-grading

$$H = H^0 \oplus H^1.$$

We will call such an H a *super Hilbert space*. We call elements of H^0 and H^1 the *even and odd homogeneous elements*, respectively. The tensor product of super Hilbert spaces $H \otimes K$ is again a super Hilbert space, with

$$(H \otimes K)^0 = (H^0 \otimes K^0) \oplus (H^1 \otimes K^1), \quad (H \otimes K)^1 = (H^0 \otimes K^1) \oplus (H^1 \otimes K^0).$$

The braiding

$$H \otimes K \overset{\beta_{H,K}}{\to} K \otimes H$$

is given on homogeneous elements ξ, η by

$$\beta_{H,K}(\xi \otimes \eta) = (-1)^{p(\xi)p(\eta)} \eta \otimes \xi, \tag{2.1}$$

where $p(\xi), p(\eta) \in \{0,1\}$ are the parities. When we write expressions involving the parity, they should be interpreted as holding for homogeneous elements, and extended linearly otherwise.

The braiding of super Hilbert spaces is symmetric. That is, given a permutation σ of $\{1, \ldots, n\}$, the isomorphism

$$H_1 \otimes \cdots \otimes H_n \to H_{\sigma(1)} \otimes \cdots \otimes H_{\sigma(n)} \tag{2.2}$$

induced by the braidings is independent of the manner in which transpositions are applied.

We will now spell out some of the consequences of (2.1). For a more detailed treatment, see [DM99, Ch. 1].

If H and K are super Hilbert spaces, then $\mathcal{B}(H,K)$ is a super vector space with $\mathcal{B}(H,K)^0$ the grading preserving maps and $\mathcal{B}(H,K)^1$ the grading reversing maps. A particularly important even element of $\mathcal{B}(H)$ is the grading operator d_H which acts by $\mathbf{1}$ on H^0 and $-\mathbf{1}$ on H^1.

Given a finite family $\{H_i\}_{i \in I}$ of super Hilbert spaces, we define the unordered tensor product to be the collection of all ordered tensor products of the H_i, along with the family of isomorphisms (2.2).

A map of unordered tensor products

$$x : \bigotimes_{i\in I} H_i \to \bigotimes_{j\in J} K_j$$

is a natural family of bounded linear maps between ordered tensor products

$$x^\nu_\mu : \bigotimes_{k=1}^{|I|} H_{\mu(k)} \to \bigotimes_{k=1}^{|J|} K_{\nu(k)}$$

where μ and ν are bijections $\mu : \{1, \ldots, |I|\} \to I$ and $\nu : \{1, \ldots, |J|\} \to J$. The naturality of the family of maps means that the diagrams

$$\begin{array}{ccc} \bigotimes_{k=1}^{|I|} H_{\mu(k)} & \xrightarrow{x^\nu_\mu} & \bigotimes_{k=1}^{|J|} K_{\nu(k)} \\ \big\downarrow \beta & & \big\downarrow \beta \\ \bigotimes_{k=1}^{|I|} H_{\mu'(k)} & \xrightarrow{x^{\nu'}_{\mu'}} & \bigotimes_{k=1}^{|J|} K_{\nu'(k)} \end{array} \tag{2.3}$$

commute. Given a bounded linear map between ordered tensor products, there is a unique map of the corresponding unordered tensor products that includes that map. More generally, we can define an unbounded linear map between unordered tensor products in exactly the same way, where (2.3) implicitly asserts that the domains of x^ν_μ and $x^{\nu'}_{\mu'}$ are related by the braiding.

If we have a pair of maps of unordered tensor products

$$\bigotimes_{i\in I} H_i \xrightarrow{x} \bigotimes_{j\in J} K_j, \quad \bigotimes_{j\in J} K_j \xrightarrow{y} \bigotimes_{s\in S} L_s,$$

then the composition yx is unambiguously defined by any composition of ordered maps $y^\rho_\nu x^\nu_\mu$.

Given even maps of unordered tensor products x and x', we can form their tensor product $x\hat{\otimes}x'$ by taking the unique family of maps containing all of the $x^\nu_\mu \otimes (x')^{\nu'}_{\mu'}$. Note that $x\hat{\otimes}x' = x'\hat{\otimes}x$ as maps of unordered tensor products.

We wish to understand the evaluation map $\mathrm{ev} : H^* \otimes H \to \mathbb{C}$, as well as the related notions of operator composition and partial trace. Since this map is unbounded when H is infinite-dimensional, we will temporarily assume $\dim H < \infty$. We will deduce the sign rules for evaluation in this special case, and then impose the same rules in general.

The evaluation map $H^* \otimes H \to \mathbb{C}$ is given by

$$\mathrm{ev}_H(\xi^* \otimes \eta) = \xi^*(\eta) = \langle \eta, \xi \rangle,$$

and is evidently even. More generally, if $\{H_i\}_{i\in I}$ is a finite collection of finite-dimensional Hilbert spaces and $H_{i_0} = H^*_{i_1}$, then we have a graded partial evaluation $\mathrm{ev}_{i_0 i_1} = \mathrm{ev}_{H_{i_1}} \hat{\otimes} \mathbf{1}$.

We now wish to define an even isomorphism $\alpha_{H,K} : (H \otimes K)^* \to K^* \otimes H^*$ of unordered tensor products. Such an isomorphism is determined by requiring that

$$(\mathrm{ev}_H \hat{\otimes} \mathrm{ev}_K)(\alpha_{H,K} \hat{\otimes} \mathbf{1}) = \mathrm{ev}_{H \otimes K} .$$

One can compute that the correct isomorphism is given, with respect to the ordering above, by

$$\alpha_{H,K}((\xi \otimes \eta)^*) = \eta^* \otimes \xi^*.$$

Remaining in the finite-dimensional case, we wish to define an even isomorphism of unordered tensor products

$$\iota_{H,K} : H^* \otimes K \to \mathcal{B}(H, K)$$

in such a way that evaluation of maps is given by

$$\mathcal{B}(H, K) \otimes H \overset{\iota_{H,K}^{-1} \hat{\otimes} \mathbf{1}}{\longrightarrow} H^* \otimes K \otimes H \overset{\mathbf{1} \hat{\otimes} \mathrm{ev}_H}{\longrightarrow} K$$

when the tensor factors are ordered as above. The correct definition is

$$\iota_{H,K}(\xi^* \otimes \eta) = \psi \mapsto (-1)^{p(\xi)p(\eta)} \langle \psi, \xi \rangle \eta.$$

The composition

$$(\iota_{H,L} \hat{\otimes} \iota_{K,M})(\alpha_{H,K} \hat{\otimes} \mathbf{1}) \iota^{-1}_{H \otimes K, L \otimes M}$$

gives an isomorphism

$$\mathcal{B}(H \otimes K, L \otimes M) \cong \mathcal{B}(H, L) \otimes \mathcal{B}(K, M). \tag{2.4}$$

If $x \in \mathcal{B}(H, L)$ and $y \in \mathcal{B}(K, M)$, then we will write $x \otimes y$ for the element of $\mathcal{B}(H \otimes K, L \otimes M)$ that acts by $(x \otimes y)(\xi \otimes \eta) = x\xi \otimes y\eta$. On the other hand, we will write $x \hat{\otimes} y$ for the corresponding element of $\mathcal{B}(H, L) \otimes \mathcal{B}(K, M)$, and identify this with its image under the isomorphism (2.4).

Under this identification, one has

$$x \hat{\otimes} y = x d_H^{p(y)} \otimes y.$$

This definition of $x \hat{\otimes} y$ coincides with the one given earlier for even x and y. If H, K, L and M are unordered tensor products, and x and y are maps of unordered tensor products, then $x \hat{\otimes} y$ and $y \hat{\otimes} x$ induce maps of unordered tensor products $H \otimes K \to L \otimes M$. As maps of unordered tensor products, we have $x \hat{\otimes} y = (-1)^{p(x)p(y)} y \hat{\otimes} x$ and $(x_1 \hat{\otimes} y_1)(x_2 \hat{\otimes} y_2) = (-1)^{p(y_1)p(x_2)} x_1 x_2 \hat{\otimes} y_1 y_2$.

The isomorphism $\iota_{H,K}$ has the following bimodularity property.

Proposition 2.2. *If $\xi \in H^* \otimes K$, $x \in \mathcal{B}(H)$ and $y \in \mathcal{B}(K)$, then*

$$\iota_{H,K}((\overline{x} \hat{\otimes} \mathbf{1})\xi) = d_K^{p(x)} \iota_{H,K}(\xi) x^*, \quad \iota_{H,K}((\mathbf{1} \hat{\otimes} y)\xi) = y \iota_{H,K}(\xi).$$

An operator of particular interest to us is the supertrace τ^s, given by

$$\tau^s = \mathcal{B}(H) \xrightarrow{\iota^{-1}} H^* \otimes H \xrightarrow{\mathrm{ev}_H} \mathbb{C}.$$

It is not hard to check that

$$\tau^s(z) = \tau(d_H z)$$

where τ is the ordinary trace.

More generally, we can define the partial supertrace

$$\tau_L^s : \mathcal{B}(H \otimes L, K \otimes L) \to \mathcal{B}(H, K).$$

by

$$\tau_s = \iota_{H,K} \circ \mathrm{ev}_L \circ \iota^{-1}_{H\otimes L, K\otimes L}. \tag{2.5}$$

Proposition 2.3. *Let $z \in \mathcal{B}(H \otimes L, K \otimes L)$ be a map of unordered tensor products.*

1. $\tau_L^s(z) = \tau_L(z(\mathbf{1} \otimes d_L))$ *where the ordinary partial trace τ_L is computed with respect to the given ordering.*
2. *If $x \in \mathcal{B}(H)$ and $y \in \mathcal{B}(K)$, then $\tau_L^s(z)x = \tau_L^s(z(x\hat{\otimes}\mathbf{1}))$ and $y\tau_L^s(z) = \tau_L^s((y\hat{\otimes}1)z)$.*
3. *If $w \in \mathcal{B}(L)$, then $\tau_L^s((1\hat{\otimes}w)z) = (-1)^{p(z)}\tau_L^s(z(1\hat{\otimes}w))$.*

Remark 2.4. In the case of the (non-partial) supertrace, statement (3) gives $\tau^s(zw) = (-1)^{p(z)}\tau^s(wz)$ instead of the more familiar $\tau^s(zw) = (-1)^{p(z)p(w)}\tau^s(wz)$. However, since τ^s is even the two properties are equivalent.

Proposition 2.5. *Let $z_1 \in \mathcal{B}(H, K \otimes L)$ and $z_2 \in \mathcal{B}(L \otimes M, N)$ be maps of unordered tensor products. Then $\tau_L^s(z_2\hat{\otimes}z_1) = (z_2\hat{\otimes}\mathbf{1}_K)(z_1\hat{\otimes}\mathbf{1}_M)$.*

So far, we have only discussed the partial supertrace for finite-dimensional Hilbert spaces. However, its definition in equation (2.5) makes sense as an unbounded operator. Its domain includes trace class operators, as well as $z_2\hat{\otimes}z_1$ for all $z_1 \in \mathcal{B}(H, L)$ and $z_2 \in \mathcal{B}(L, K)$. All of the sign rules of this chapter extend to the infinite dimensional case by taking limits of compressions by finite-rank projections.

2.3 Fermionic Fock space

Given a complex Hilbert space H, the algebra $\mathrm{CAR}(H)$ is the universal unital C^*-algebra with generators $a(f)$ for $f \in H$ which are linear in f and satisfy the canonical anticommutation relations

$$a(f)a(g) + a(g)a(f) = 0,$$
$$a(f)a(g)^* + a(g)^*a(f) = \langle f, g\rangle\mathbf{1}.$$

There is an irreducible, faithful representation of $\mathrm{CAR}(H)$ on the Hilbert space

$$\Lambda H = \bigoplus_{k=0}^{\infty} \Lambda^k H$$

densely defined by $a(f)\zeta = f \wedge \zeta$. These operators are bounded, and $\|a(f)\| = \|f\|$. The exterior Hilbert space ΛH is naturally a super Hilbert space, with $\mathbb{Z}/2$-grading inherited from the number grading. That is,

$$(\Lambda H)^i = \bigoplus_{k=0}^{\infty} \Lambda^{2k+i} H.$$

The subspace $\Lambda^0 H$ is spanned by a distinguished unit vector Ω which satisfies $a(f)^*\Omega = 0$ for all $f \in H$.

There is a family of irreducible, faithful representations of $\mathrm{CAR}(H)$ indexed by projections $p \in \mathcal{P}(H)$. Let $H_p = pH \oplus ((1-p)H)^*$. We have a representation $\pi_p : \mathrm{CAR}(H) \to \mathcal{B}(\Lambda H_p)$ given by

$$\pi_p(a(f)) = a(pf) + a(((1-p)f)^*)^*.$$

We call ΛH_p *fermionic Fock space*, and denote it by $\mathcal{F}_{H,p}$, or simply $\mathcal{F}_p$ or $\mathcal{F}$ when the decorations are clear from context. Note that $\pi_p(a(f))$ is an odd operator on $\mathcal{F}_{H,p}$.

One can reconstruct the representation $(\mathcal{F}_p, \pi_p)$ via the GNS construction with respect to the vacuum expectation $\omega_p = \langle \,\cdot\, \Omega_p, \Omega_p \rangle$, where Ω_p is the distinguished unit vector in $\Lambda^0 H_p$. This state is determined by the canonical anticommutation relations, as well as the p-vacuum equations

$$\begin{aligned} \pi_p(a(f))^*\Omega_p &= 0 && \text{for } f \in pH, \\ \pi_p(a(f))\Omega_p &= 0 && \text{for } f \in (1-p)H, \end{aligned}$$

which determine Ω_p up to a phase.

A crucial property of the Fock space construction is its functoriality.

Proposition 2.6. *As super Hilbert spaces, we have natural isomorphisms*

$$\mathcal{F}_{H\oplus K, p\oplus q} \cong \mathcal{F}_{H,p} \otimes \mathcal{F}_{K,q}.$$

The induced action of $\mathrm{CAR}(H \oplus K)$ *on* $\mathcal{F}_{H,p} \otimes \mathcal{F}_{K,q}$ *is*

$$a(h+k) \mapsto \pi_p(a(h)) \hat{\otimes} \mathbf{1} + \mathbf{1} \hat{\otimes} \pi_q(a(k)). \tag{2.6}$$

Remark 2.7. The naturality of the isomorphisms from Proposition 2.6 make $\mathcal{F}_{H\oplus K,p\oplus q}$ a model for the unordered tensor product. That is, maps to and from $\mathcal{F}_{H\oplus K,p\oplus q}$ are equivalent to maps to and from the unordered tensor product $\mathcal{F}_{H,p} \otimes \mathcal{F}_{K,q}$. As a result, we will not distinguish between $\mathcal{F}_{H\oplus K,p\oplus q}$ and the unordered tensor product $\mathcal{F}_{H,p} \otimes \mathcal{F}_{K,q}$. We will also freely identify $\pi_{p\oplus q}$ and the representation given in equation (2.6).

Since $H_{1-p} = H_p^*$, we have a natural unitary $\Phi : \mathcal{F}_{1-p} \to \mathcal{F}_p^*$ given by

$$\Phi(\xi_1^* \wedge \cdots \wedge \xi_n^*) = (\xi_n \wedge \cdots \wedge \xi_1)^*$$

for $\xi_i \in H_p$.

Proposition 2.8. *For all $f \in H$ we have*

$$\Phi\pi_{1-p}(a(f))\Phi^* = -\overline{\pi_p(a((2p-\mathbf{1})f))^*}d_{\mathcal{F}_p^*}$$

and

$$\Phi\pi_{1-p}(a(f))^*\Phi^* = \overline{\pi_p(a((2p-\mathbf{1})f))}d_{\mathcal{F}_p^*}.$$

Proof. The two identities are clearly equivalent for every fixed $f \in H$. We prove the first for $f \in (1-p)H$ and the second for $f \in pH$.

If $f \in (1-p)H$, then the first identity reads

$$\Phi\pi_{1-p}(a(f))\Phi^* = \overline{\pi_p(a(f))^*}d_{\mathcal{F}_p^*}.$$

Applying the left-hand side to $\omega^* \in (\Lambda^n H_p)^*$ yields $(\omega \wedge f^*)^*$, and applying the right-hand side yields $(-1)^n(f^* \wedge \omega)^*$. The proof of the second identity when $f \in pH$ is similar. □

The natural question of when π_p and π_q are unitarily equivalent is answered by the following theorem.

Theorem 2.9 (Segal equivalence criterion)**.** *The following are equivalent:*

(i) π_p and π_q are unitarily equivalent representations of $\mathrm{CAR}(H)$*.*

(ii) There exists a unit vector $\tilde{\Omega}_q \in \Lambda H_p$, which will be unique up to phase, satisfying the q-vacuum equations.

(iii) $p - q$ is a Hilbert-Schmidt operator on H.

This theorem is also called the Shale-Stinespring criterion, and there are many proofs in the literature. A simple version of the argument may be found in the textbook [Tha92, Thm. 10.7]. A more concise version of the constructive proof that (iii) implies (ii) and (i) is in [Was98, Sec. 3], and an abstract proof using von Neumann algebra techniques is given in [dlHJ95, Thm. 8.23]

If $u \in \mathcal{U}(H)$, the *Bogoliubov automorphism* α_u of $\mathrm{CAR}(H)$ is characterized by $\alpha_u(a(f)) = a(uf)$. We say that an automorphism α of a C^*-algebra A is *implemented* in a representation $\pi : A \to \mathcal{B}(H_\pi)$ if there is a unitary $U \in \mathcal{U}(H_\pi)$ such that $\mathrm{Ad}\, U \circ \pi = \pi \circ \alpha$. The set of unitaries U implementing α_u is a principal homogeneous space for $\mathcal{U}(\pi(A)')$, and in particular if π is irreducible then an implementing unitary U will be unique up to phase.

Corollary 2.10. *The Bogoliubov automorphism α_u is implemented in π_p if and only if $[u,p]$ is Hilbert-Schmidt. If α_u is implemented by U, then Ω is an eigenvector for U if and only if $[u,p]=0$.*

Definition 2.11. Define the *restricted general linear group*

$$GL_{res}(H,p) = \{x \in GL(H) : [x,p] \text{ is Hilbert-Schmidt}\}.$$

and the *restricted unitary group*

$$\mathcal{U}_{res}(H,p) = GL_{res}(H,p) \cap \mathcal{U}(H).$$

We give $\mathcal{U}_{res}$ the topology generated by the strong operator topology and the pseudometric $\|[u-v,p]\|_{\mathcal{I}^2}$. With this topology, $\mathcal{U}_{res}$ is a topological group.

There is a natural projective representation of $\mathcal{U}_{res}(H,p)$ on $\mathcal{F}_{H,p}$ called the *basic representation*, which we will write $u \mapsto U$. The basic representation is characterized by

$$U\pi_p(a(f))U^* = \pi_p(a(uf))$$

for all $f \in H$. The basic representation restricts to an honest representation on the subgroup of unitary operators u commuting with p. On this subgroup, a lift to $\mathcal{U}(\mathcal{F}_{H,p})$ is given by choosing U so that $U\Omega = \Omega$.

Lemma 2.12. *The basic representation is strongly continuous.*

A proof of this lemma is given in [Was98, Sec. 3].

We will denote by $d_{\mathcal{F}_p}$ the $\mathbb{Z}/2$-grading operator which implements $\alpha_{-\mathbf{1}}$ and fixes Ω. This grading coincides with the one defined earlier on $\mathcal{F}_p$ coming from the number grading . We will simply write d for the grading operator when the Fock space that it acts on is clear.

Proposition 2.13. *The vectors $\tilde{\Omega}_q$ from Theorem 2.9 are homogeneous. The parity of $\tilde{\Omega}_q$ is the parity of $\dim(pH \cap (1-q)H) + \dim((1-p)H \cap qH)$.*

Proof. The homogeneity of $\tilde{\Omega}_q$ follows immediately from the fact that $d\tilde{\Omega}_q$ again satisfies the q-vacuum equations, and thus $\tilde{\Omega}_q$ is an eigenvector for the grading operator. The parity can be read off from an explicit formula for $\tilde{\Omega}_q$ (see e.g. [Was98, Sec. 3] or [Tha92, Thm 10.6]). □

Proposition 2.14. *If U implements the Bogoliubov automorphism α_u in $\mathcal{F}_{H,p}$, then U is homogeneous. The parity of U is the same as the parity of $\tilde{\Omega}_q$, where $q = upu^*$.*

Suppose H and K are Hilbert spaces, $p \in \mathcal{P}(H)$, $r \in \mathcal{P}(K)$, and $q \in \mathcal{P}(H \oplus K)$. If $((1-p \oplus r) - q$ is Hilbert-Schmidt, then using Theorem 2.9 and Proposition 2.8 we obtain a one-dimensional space of vacuum vectors in $\mathcal{F}_{H,p}^* \otimes \mathcal{F}_{K,r}$, or equivalently a one-dimensional space space of Hilbert-Schmidt maps $F_{H,p} \to \mathcal{F}_{K,r}$. The vacuum vectors satisfy the q-vacuum equations, and these induce intertwining relations between the corresponding Hilbert-Schmidt maps and the representations of CAR(H) and CAR(K). The precise relations that these maps satisfy are given by the following lemma.

Lemma 2.15 (q-commutation relations)**.** *Let H and K be Hilbert spaces, $p \in \mathcal{P}(H)$, $r \in \mathcal{P}(K)$, and $q \in \mathcal{P}(H \oplus K)$. Then the following are equivalent.*

1. *$((1-p) \oplus r) - q$ is Hilbert-Schmidt.*

2. *There exists a (necessarily unique up to scalar) $\mathbb{Z}/2$-homogeneous Hilbert-Schmidt map $T_q : \mathcal{F}_{H,p} \to \mathcal{F}_{K,r}$ satisfying*

$$\pi_p(a(g))^* T_q = (-1)^{p(T_q)} T_q \pi(a(f))^* \tag{2.7}$$

for all $(f,g) \in \mathrm{Im}(q)$, and

$$\pi_p(a(g)) T_q = -(-1)^{p(T_q)} T_q \pi(a(f)) \tag{2.8}$$

for all $(f,g) \in \mathrm{Im}(q)^\perp$.

Proof. First assume condition (1) holds. Observe that

$$q' = ((1-2p) \oplus \mathbf{1}) q ((1-2p) \oplus \mathbf{1})$$

also satisfies condition (1). Thus by Theorem 2.9 there exists a non-zero $\tilde{\Omega}_{q'} \in \mathcal{F}_{H \oplus K, (1-p) \oplus r}$ such that

$$\pi_{(1-p)\oplus r}(a(f,g))^* \tilde{\Omega}_{q'} = 0$$

for all $(f,g) \in \mathrm{Im}(q')$ and

$$\pi_{(1-p)\oplus r}(a(f,g)) \tilde{\Omega}_{q'} = 0$$

for all $(f,g) \in \mathrm{Im}(q')^\perp$.

Let $\hat{\Omega}_{q'}$ be the image of $\tilde{\Omega}_{q'}$ under the isomorphism

$$\mathcal{F}_{H \oplus K, (1-p)\oplus r} \to \mathcal{F}_{H, 1-p} \otimes \mathcal{F}_{K,r}$$

from Proposition 2.6. By the same proposition, we see that for $(f,g) \in \mathrm{Im}(q')$ we have

$$(\pi_{1-p}(a(f))^* \hat{\otimes} \mathbf{1} + \mathbf{1} \hat{\otimes} \pi_r(a(g))^*) \hat{\Omega}_{q'} = 0.$$

By Proposition 2.8, we have

$$(\overline{\pi_p(a((2p-1)f))} d \hat{\otimes} \mathbf{1} + \mathbf{1} \hat{\otimes} \pi_r(a(g))^*)(\Phi \otimes \mathbf{1}) \hat{\Omega}_{q'} = 0.$$

Let $T_q \in \mathcal{B}(\mathcal{F}_{H,p}, \mathcal{F}_{K,r})$ be the image of $(\Phi \otimes \mathbf{1})\hat{\Omega}_{q'}$ under the natural isomorphism $\iota_{\mathcal{F}_{H,p}, \mathcal{F}_{K,r}}$. By Proposition 2.13, T_q is homogeneous. Applying Proposition 2.2 gives

$$\pi_r(a(g))^* T_q = -d T_q d \pi_p(a((2p-1)f))^* = (-1)^{p(T_q)} T_q \pi_p(a((1-2p)f))^*.$$

By construction, $(f,g) \in \mathrm{Im}(q')$ if and only if $((1-2p)f, g) \in \mathrm{Im}(q)$, and so T_q satisfies (2.7).

Similarly, if $(f,g) \in \operatorname{Im}(q')^{\perp}$, then

$$(-\overline{\pi_p(a((2p-1)f))^*}d \otimes \mathbf{1} + \mathbf{1}\hat{\otimes}\pi_r(a(g)))(\Phi \otimes \mathbf{1})\hat{\Omega}_{q'} = 0$$

which yields the commutation relation

$$\pi_r(a(g))T_q = -(-1)^{p(T_q)}T_q\pi_p(a((1-2p)f)).$$

Since $(f,g) \in \operatorname{Im}(q')^{\perp}$, the condition (2.8) will hold when $(f,g) \in \operatorname{Im}(q)^{\perp}$. This completes the proof that (1) implies (2).

In fact, every step in the preceding proof was an equivalence, and so we are done. □

Definition 2.16. The commutation relations (2.7) and (2.8) are called the *q-commutation relations* or the *qH-commutation relations.*

The conclusion of Lemma 2.15 can be summarized by saying there is at most a one-dimensional space of Hilbert-Schmidt maps satisfying the q-commutation relations, and this space is one-dimensional precisely when $((1-p)\oplus r) - q$ is Hilbert-Schmidt.

We will find the following observations useful.

Proposition 2.17. *If $u \in \mathcal{U}_{res}(H,p)$, $v \in \mathcal{U}_{res}(K,r)$, and $T_q \in \mathcal{B}(\mathcal{F}_{H,p}, \mathcal{F}_{K,r})$ satisfies the q-commutation relations, then $d_{\mathcal{F}_{K,r}}^{p(U)+p(V)}UT_qV^*$ satisfies the $(u\oplus v)qH$-commutation relations.*

Proposition 2.18. *If c is an idempotent operator on H with $c-p$ Hilbert-Schmidt and q is the range projection of c, then $q-p$ is Hilbert-Schmidt. In particular, if $x \in GL_{res}(H,p)$ and q is the range projection of xpx^{-1}, then $q-p$ is Hilbert-Schmidt.*

Proof. Since $c-p$ is Hilbert-Schmidt, so is $(c-p)-(c-p)^* = c-c^*$. We compute

$$\begin{aligned} q-p &= cq-p \\ &= (c-c^*)q + (qc-p)^* \\ &= (c-c^*)q + (c-p)^* \end{aligned}$$

which is evidently Hilbert-Schmidt. □

2.4 Representations of $\mathrm{Diff}_+(S^1)$

We will use fermionic Fock space primarily in the case where $H = L^2(S^1,\mathbb{C}^k)$ and p_+H is the Hardy space $H^2(S^1,\mathbb{C}^k)$. That is

$$H^2(S^1,\mathbb{C}^k) = \operatorname{cl}(\operatorname{span}\{z^n\xi : n \geq 0, \xi \in \mathbb{C}^k\}).$$

Let $LGL(k)$ be the loop group of smooth maps from S^1 to $GL(k)$, and let $\mathrm{Diff}_+(S^1)$ be the group of orientation-preserving diffeomorphisms of the circle. There is a natural action

$$t : LGL(k) \rtimes \mathrm{Diff}_+(S^1) \to \mathcal{B}(L^2(S^1,\mathbb{C}^k)),$$

given by

$$(t(x,\phi)f)(z) = x(z)f(\phi^{-1}(z)).$$

We define the subgroup $L_\omega GL(k) \rtimes \mathrm{Diff}_{+,\omega}(S^1)$ to consist of elements that extend to holomorphic maps in a neighborhood of S^1.

The given representation of $\mathrm{Diff}_+(S^1)$ is not unitary. We can define a unitary representation $u_R : \mathrm{Diff}_+(S^1) \to \mathcal{U}(H)$ by

$$(u_R(\phi)f)(z) = \left|(\phi^{-1})'(z)\right|^{\frac{1}{2}} f(\phi^{-1}(z)).$$

This representation comes from the embedding of $\mathrm{Diff}_+(S^1)$ into $LGL(k) \rtimes \mathrm{Diff}_+(S^1)$ via

$$\phi \mapsto (\left|(\phi^{-1})'\right|^{\frac{1}{2}}, \phi).$$

Remark 2.19. It is not immediately obvious, but this also gives an embedding of $\mathrm{Diff}_{+,\omega}(S^1)$ into $L_\omega GL(k) \rtimes \mathrm{Diff}_{+,\omega}(S^1)$. Since $\phi^{-1} \in \mathrm{Diff}_+(S^1)$,

$$\left|(\phi^{-1})'(z)\right| = \frac{z(\phi^{-1})'(z)}{\phi^{-1}(z)}.$$

Thus if f is a holomorphic extension of ϕ to a neighborhood of S^1, we can define

$$\sqrt{\frac{z(f^{-1})'(z)}{f^{-1}(z)}}$$

by analytic continuation, and this function gives a holomorphic extension of $|(\phi^{-1})'|^{\frac{1}{2}}$.

The notation u_R is to indicate that these give automorphisms of the Ramond spin structure over the circle (see Sections 2.6 and 3.3). To identify diffeomorphisms with automorphisms of the Neveu-Schwarz spin structure, we will need to pass to a double cover.

Definition 2.20. Define the n-fold cover

$$\mathrm{Diff}_+^{(n)}(S^1) = \{(\eta,\phi) \in LGL(1) \rtimes \mathrm{Diff}_+(S^1)) : \eta^n = (\phi^{-1})'\}.$$

The Neveu-Schwarz representation

$$u_{NS} : \mathrm{Diff}_+^{(2)}(S^1) \to \mathcal{U}(L^2(S^1, \mathbb{C}^k))$$

is given by $u_{NS}(\eta,\phi)f = t(\eta,\phi)f$.

Proposition 2.21. *The representation t of $LGL(k) \rtimes \mathrm{Diff}_+(S^1)$ takes values in $GL_{res}(H,p)$. In particular, u_R and u_{NS} take values in $\mathcal{U}_{res}(H,p)$.*

Proof. This follows from Propositions 6.3.1 and 6.8.2 of [PS86]. $\square$

We give $\mathrm{Diff}_+^{(n)}(S^1)$ the topology induced by the metric

$$d((\eta_1,\phi_1),(\eta_2,\phi_2)) = \|\phi_1-\phi_2\|_\infty + \|\eta_1-\eta_2\|_{C^2},$$

where

$$\|\eta\|_{C^2} = \|\eta\|_\infty + \|\eta'\|_\infty + \|\eta''\|_\infty .$$

While this metric is not particularly natural, it is easy to compute with and we will only be interested in the following properties.

Proposition 2.22. *The representations u_R and u_{NS} are continuous as maps into the restricted unitary group $\mathcal{U}_{res}(H,p_+)$, and the holomorphic subgroup $\mathrm{Diff}_{+,\omega}^{(n)}(S^1)$ is dense in $\mathrm{Diff}_+^{(n)}(S^1)$.*

Proof. To approximate $(\eta,\phi)\in\mathrm{Diff}_+^{(n)}(S^1)$ by elements of $\mathrm{Diff}_{+,\omega}^{(n)}(S^1)$ one can approximate $-i\log(z^{-1}\phi(z))$ by the partial sums of its real Fourier series.

It is easy to check that if $(\eta_n,\phi_n)\to(\eta,\phi)$ in $\mathrm{Diff}_+^{(n)}(S^1)$, then $u_{NS}(\eta_n,\phi_n)$ and $u_R(\phi_n)$ converge strongly to $u_{NS}(\eta,\phi)$ and $u_R(\phi)$, respectively.

The fact that $\|[u_R(\phi_n)-u_R(\phi),p_+]\|_{\mathcal{I}^2}\to 0$ can be read off from the explicit formula for the matrix entries of $[t(\phi),p_+]$ computed in [Seg81, Prop. 5.3].

Observe that $u_{NS}(\eta_n,\phi_n) = M_{\psi_n}u_R(\phi_n)$, where M_{ψ_n} is multiplication by

$$\psi_n := \frac{\eta_n}{|(\phi_n^{-1})'|^{1/2}}.$$

The choice of metric on $\mathrm{Diff}_+^{(2)}(S^1)$ ensures that $\psi_n\to\psi$ and $\psi_n'\to\psi'$ uniformly, so $M_{\psi_n}\to M_\psi$ in $\mathcal{U}_{res}$ by [PS86, Prop. 6.3.1]. Since $\mathcal{U}_{res}$ is a topological group,

$$u_{NS}(\eta_n,\phi_n)\to M_\psi u_R(\phi) = u_{NS}(\eta,\phi).$$

□

Composing with the basic representation gives strongly continuous projective representations U_R (and U_{NS}) of $\mathrm{Diff}_+(S^1)$ (and $\mathrm{Diff}_+^{(2)}(S^1)$) on $\mathcal{F}_{H,p_+}$. The images of U_{NS} and U_R consist of homogeneous operators by Proposition 2.14, and these operators must be even by the connectedness of $\mathrm{Diff}_+^{(2)}(S^1)$ and $\mathrm{Diff}_+(S^1)$.

Let $\mathrm{Rot}(S^1)$ be the subgroup of $\mathrm{Diff}_+(S^1)$ consisting of rotations. If $r_\theta\in\mathrm{Rot}(S^1)$ is clockwise rotation by θ, then $[r_\theta,p_+]=0$ so U_R gives an honest, strongly continuous representation of $\mathrm{Rot}(S^1)$ on $\mathcal{F}_{H,p}$. By Stone's theorem, we can write

$$U_R(r_\theta) = e^{i\theta L_0^R}$$

with L_0^R an unbounded self-adjoint operator on $\mathcal{F}_{H,p}$. A priori we must have $\sigma(L_0)\subseteq\mathbb{Z}$, and so L_0^R is diagonalizable. Explicit calculation (Proposition 2.28) shows that L_0^R is positive and has finite-dimensional eigenspaces.

Definition 2.23. We say that a strongly continuous representation $U : \mathrm{Rot}(S^1) \to \mathcal{U}(\mathcal{H})$ has *positive energy* if $U(r_\theta) = e^{i\theta L_0}$ with L_0 a positive operator having finite-dimensional eigenspaces. We call L_0 the *generator* of the representation.

We will also require a notion of projective positive energy representation. The subgroup $\mathrm{Rot}^{(n)}(S^1) \subset \mathrm{Diff}_+^{(n)}(S^1)$ is given by

$$\mathrm{Rot}^{(n)}(S^1) = \{(\xi, r_\theta) \in S^1 \times \mathrm{Rot}(S^1) : \xi^n = e^{i\theta}\}.$$

There is an isomorphism $s : \mathrm{Rot}^{(n)}(S^1) \to \mathrm{Rot}(S^1)$ which implements the n-fold cover of S^1 by itself and sends (ξ, r_θ) to $z \mapsto \overline{\xi} z$.

Definition 2.24. We say that a strongly continuous unitary representation $U : \mathrm{Rot}^{(n)}(S^1) \to \mathcal{U}(\mathcal{H})$ has *positive energy* if $U \circ s^{-1}$ has positive energy and $U(\xi, \mathbf{1}) \in \mathbb{C}\mathbf{1}$.

We say that a projective representation $\tilde{U}$ of $\mathrm{Rot}(S^1)$ has *positive energy* if it is of the form $\tilde{U}(r_\theta) = [U(e^{2\pi i/n}, r_\theta)]$ for a positive energy representation U of $\mathrm{Rot}^{(n)}(S^1)$, where $x \mapsto [x]$ is the natural projection of $GL(\mathcal{H})$ onto $PGL(\mathcal{H})$.

We define the *generator* L_0 of U to be the generator of $U \circ s^{-1}$, and the generator of $\tilde{U}$ to be $\frac{1}{n} L_0$.

A strongly continuous representation of $\mathrm{Diff}_+(S^1)$ or $\mathrm{Diff}_+^{(n)}(S^1)$ is said to have *positive energy* if the restriction to the rotation subgroup has positive energy.

Remark 2.25. If U is a representation of $\mathrm{Rot}^{(n)}(S^1)$, then tensoring by a character of S^1 gives a new representation in the same projective class. Thus if $\tilde{U}$ is a projective positive energy representation, a lift to $\mathrm{Rot}^{(n)}$ is only well-defined up to the choice of a character. Since the value of n is also not well-defined, the generator $L_0^{\tilde{U}}$ is only well-defined up to adding an element of $\mathbb{Q}\mathbf{1}$. This is a vaguer notion of generator than one requires in general, but it will suffice for our purposes.

For every choice of (projective) generator L_0, we will have $\sigma(L_0) \subseteq q + \mathbb{Z}_{\geq 0}$ for some $q \in \mathbb{Q}_{\geq 0}$. Since L_0 has discrete spectrum, it is diagonalizable and we can write

$$\mathcal{H} = \bigoplus_{h \in \sigma(L_0)} \mathcal{H}(h).$$

where $\mathcal{H}(h)$ is finite-dimensional and $L_0|_{\mathcal{H}(h)} = h\mathbf{1}$.

The representation U_{NS} of $\mathrm{Diff}^{(2)}(S^1)$ does not have positive energy. In particular, one can check that $U_{NS}(-1, \mathbf{1}) = d_{\mathcal{F}_{H,p_+}}$, which is not a multiple of the identity. We will now define the notation of a positive energy spin representation to capture this behavior.

Definition 2.26. Let $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ be a super Hilbert space with projections p^0 and p^1 onto the even and odd subspaces, respectively. We say that a strongly continuous unitary representation $U : \mathrm{Rot}^{(2n)}(S^1) \to \mathcal{U}(\mathcal{H})$ is a *positive energy spin representation* if $[U(\xi, r_\theta), p^i] = 0$ for all (ξ, r_θ), the compressions

$$(\xi, r_\theta) \mapsto p^i U(\xi, r_\theta) p^i$$

are positive energy representations of $\mathrm{Rot}^{(2n)}(S^1)$, and

$$p^0 U(e^{\pi i k/n}, \mathbf{1}) p^0 = \alpha p^0 \iff p^1 U(e^{\pi i k/n}, \mathbf{1}) p^1 = (-1)^k \alpha p^1$$

for all $1 \leq k \leq 2n$. We define the generator L_0 of the representation to be $L_0^0 \oplus L_0^1$, where L_0^i is the generator of the compression of U by p^i.

If $\tilde{U}$ is projective representation of $\mathrm{Rot}^{(2)}(S^1)$, we say that it is a *projective positive energy spin representation* if there is a positive energy spin representation $U : \mathrm{Rot}^{(2n)} \to \mathcal{U}(\mathcal{H})$ such that $U(\xi^n, r_\theta) = [\tilde{U}(\xi, r_\theta)]$ for all $(\xi, r_\theta) \in \mathrm{Rot}^{(2n)}(S^1)$. The generator of $\tilde{U}$ is $\frac{1}{n} L_0^U$.

If U is a representation of $\mathrm{Diff}^{(2n)}(S^1)$, we say that it is a *positive energy spin representation* if the restriction to $\mathrm{Rot}^{(2n)}(S^1)$ has positive energy.

Remark 2.27. If L_0 is the generator of a positive energy spin representation of $\mathrm{Rot}^{(2)}(S^1)$, then $\sigma(L_0) \subseteq \frac{1}{2}\mathbb{Z}_{\geq 0}$. If the representation is projective, then $\sigma(L_0) \subset q + \frac{1}{2}\mathbb{Z}_{\geq 0}$ for some $q \in \mathbb{Q}_{\geq 0}$.

It is not difficult to calculate explicitly the action of the generators of U_R and U_{NS}.

Proposition 2.28. *Let $v_1, \ldots, v_p$, $w_1, \ldots, w_q \in \mathbb{C}^k$, and let $n_1, \ldots, n_p, m_1, \ldots, m_q \in \mathbb{Z}$. If*

$$\xi = \pi_{p_+}(a(z^{n_p} v_p)^* \cdots a(z^{n_1} v_1) a(z^{m_1} w_1) \cdots a(z^{m_q} w_q)^*)\Omega,$$

then

$$L_0^R \xi = \left(\sum_{i=1}^{p} -n_i + \sum_{i=1}^{q} m_i \right) \xi$$

and

$$L_0^{NS} \xi = \left(\sum_{i=1}^{p} -(n_i + \frac{1}{2}) + \sum_{i=1}^{q} (m_i + \frac{1}{2}) \right) \xi.$$

We also have

$$[L_0^{NS}, \pi_{p_+}(a(z^n v))] = (n + \frac{1}{2}) a(z^n v), \quad [L_0^R, \pi_{p_+}(a(z^n v))] = n \pi_{p_+}(a(z^n v))$$

and

$$[L_0^{NS}, \pi_{p_+}(a(z^n v))^*] = -(n + \frac{1}{2}) a(z^n v), \quad [L_0^R, \pi_{p_+}(a(z^n v))] = -n \pi_{p_+}(a(z^n v)).$$

2.5 Complex surfaces and complex vector bundles

Complex manifolds

In this section we will assume that the reader is familiar with the basics of real smooth surfaces and real smooth vector bundles, and recall the corresponding complex notions. A more detailed treatment can be found in [AS60] or [For81].

A surface Σ is a 2-dimensional real manifold. A surface is called *closed* if it is compact and has empty boundary, and a surface is called *open* if it has empty boundary and every connected component is non-compact. All surfaces that we consider will be smooth and either open, or compact with non-empty boundary.

A *holomorphic atlas* is given by an open covering $\{U_i\}$ of Σ, along with diffeomorphisms $\phi_i : U_i \to \mathbb{D}$ or $\phi_i : U_i \to \mathbb{D} \cap \mathrm{cl}(\mathbb{H})$ (where $\mathbb{D}$ is the open unit disk in $\mathbb{C}$ and $\mathbb{H}$ is the open upper half plane) such that the transition maps

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \mathbb{C}$$

are holomorphic. Here, and throughout, if $S \subset \mathbb{C}$ we will say that a function $f : S \to \mathbb{C}$ is holomorphic if it extends to a holomorphic function defined on an open set $U \supseteq S$.

If Σ and Σ' are surfaces equipped with holomorphic atlases (U_i, ϕ_i) and (V_j, ϕ'_j), respectively, a function $f : \Sigma \to \Sigma'$ is called *holomorphic* if $\phi'_j \circ f \circ \phi_i^{-1} : \phi_i(U_i \cap f^{-1}(V_j)) \to \mathbb{C}$ is holomorphic for all i and j where the domain is non-empty. We will denote by $\mathcal{O}(\Sigma)$ the algebra of holomorphic functions valued in $\mathbb{C}$.

A diffeomorphism $f : \Sigma \to \Sigma'$ is called *biholomorphic* if f (and thus f^{-1}) is holomorphic. A *local coordinate* on Σ is a biholomorphic map from an open subset of Σ onto a subset of the complex plane. A function $f : \Sigma \to \Sigma'$ is holomorphic if and only if $\psi \circ f \circ \phi^{-1}$ is holomorphic for all local coordinates (U, ϕ) and (V, ψ) on Σ and Σ', respectively, for which this expression makes sense.

Two holomorphic atlases (U_i, ϕ_i) and (U'_i, ϕ'_i) are called *equivalent* if the identity map $(\Sigma, U, \phi) \to (\Sigma, U', \phi')$ is holomorphic, or equivalently if (Σ, U, ϕ) and (Σ, U', ϕ') have the same holomorphic functions. A *complex structure* on Σ is a choice of equivalence class of holomorphic atlas, and a surface with a complex structure is called a *Riemann surface.*

Given a Riemann surface Σ with a holomorphic atlas (U_i, ϕ_i), define the *conjugate surface* $\overline{\Sigma}$ to be the same smooth manifold with holomorphic atlas $\{U_i, \overline{\phi_i}\}$.

Complex and holomorphic vector bundles

If Σ is a surface, a *complex vector bundle of rank n* $\pi : V \to \Sigma$ is a smooth vector bundle modeled on $\mathbb{C}^n$ with complex linear transition functions. That is, the fibers are complex vector spaces, and there exists an open covering U_i of Σ and a family of diffeomorphisms $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$ such that

$$\pi|_{U_i} = \mathrm{proj}_{U_i} \circ \phi_i,$$

as well as transition functions $t_{ij} : U_i \cap U_j \to \mathrm{GL}(\mathbb{C}^n)$ such that

$$\phi_i \circ \phi_j^{-1} : U_i \cap U_j \times \mathbb{C}^n \to U_i \cap U_j \times \mathbb{C}^n$$

is given by $(\phi_i \circ \phi_j^{-1})(z, \xi) = (z, t_{ij}(z)\xi)$.

If Σ is a Riemann surface, then V is called a *holomorphic vector bundle* if there exists a family of holomorphic trivializations ϕ_i such that the t_{ij} are holomorphic maps (equivalently, if the projection π is a holomorphic map). If V is holomorphic, a *holomorphic section* of V is a section f such that

$$\mathrm{proj}_{\mathbb{C}^n} \circ \phi_i \circ f : U_i \to \mathbb{C}^n$$

is holomorphic for some (equivalently, every) choice of holomorphic local trivialization ϕ_i. Recall that we have defined holomorphic functions in the plane in such a way that $\mathrm{proj}_{\mathbb{C}^n} \circ \phi_i \circ f$ is required to extend holomorphically past the boundary in any local holomorphic coordinate on U_i. We will denote by $\mathcal{O}(V)$ the vector space of holomorphic sections of V.

If V is a holomorphic vector bundle, then the dual bundle V^* is again holomorphic. In fact, if we identify $\mathbb{C}^k$ and $(\mathbb{C}^k)^*$ by the complex linear isomorphism $\kappa : \mathbb{C}^n \to (\mathbb{C}^n)^*$ obtained from the standard basis of $\mathbb{C}^k$, then the trivializations ϕ_i induce transition functions

$$t_{ij}^d(z) = (t_{ij}(z)^t)^{-1}.$$

If $f \in \mathcal{O}(\Sigma \times \mathbb{C}^k)$, we will write $\kappa(f)$ for the corresponding section of $\Sigma \times (\mathbb{C}^k)^*$.

If V is a complex vector bundle over a Riemann surface Σ, then define the *conjugate bundle* $\overline{V}$ to be the bundle over $\overline{\Sigma}$ where the complex structure has been reversed on each fiber. That is, if V_p is a fiber of V then $\overline{V_p}$ is the corresponding fiber of $\overline{V}$. If $f \in \mathcal{O}(V)$, we write $\overline{f}$ for the section of $\overline{V}$ obtained by composing with the identity map on real bundles $V_{\mathbb{R}} = \overline{V}_{\mathbb{R}}$. If V is holomorphic, then $\overline{V}$ is as well.

Remark 2.29. We should emphasize some potentially confusing notation. Recall from Section 2.1 that we denote the conjugate Hilbert space of $\mathbb{C}^k$ by $\iota(\mathbb{C}^k)$ instead of $\overline{\mathbb{C}^k}$, and write $\xi \mapsto \iota(\xi)$ instead of $\xi \mapsto \overline{\xi}$ for the conjugate-linear isomorphism. We must make similar adjustments for trivial bundles $\Sigma \times \mathbb{C}^k$. If $f \in \mathcal{O}(\Sigma \times \mathbb{C}^k)$, we will write $\iota(f)$ for the corresponding section of $\overline{\Sigma} \times \iota(\mathbb{C}^k)$.

If Σ and Σ' are surfaces, a morphism of real vector bundles from $\pi : V \to \Sigma$ to $\pi' : V' \to \Sigma'$ is a pair of smooth maps $f : \Sigma \to \Sigma'$ and $g : V \to V'$ such that g is real linear on the fibers and

$$\begin{array}{ccc} V & \xrightarrow{g} & V' \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi'}\downarrow \\ \Sigma & \xrightarrow{f} & \Sigma' \end{array}$$

commutes. An automorphism of a vector bundle is often, but not always, required to have $f = \mathrm{id}$.

If V and V' are complex bundles, then a *(complex linear) morphism* is a morphism (f, g) such that g is complex linear on fibers. A *conjugate linear morphism* is one in which g is conjugate linear on fibers.

If V and V' are holomorphic, then a complex linear morphism (f, g) is called holomorphic if g (and consequently f) are holomorphic maps. Holomorphic vector bundles are called *(holomorphically) isomorphic* if there exists an invertible holomorphic morphism between them.

A vector bundle over a surface Σ is called *trivial* if the restriction to each connected component Σ' is of the form $\Sigma' \times V$ for some complex vector space V. Trivial bundles are holomorphic.

A holomorphic vector bundle over a Riemann surface Σ is called *(holomorphically) trivializable* if it is holomorphically isomorphic to a trivial bundle. Equivalently, a holomorphic vector bundle of rank n is trivializable if there exists a family of holomorphic sections $F_1, \ldots, F_n$ for which $\{F_1(p), \ldots, F_n(p)\}$ is a linearly independent set for all $p \in \Sigma$. In particular, a holomorphic line bundle is trivializable if and only if it has a non-vanishing holomorphic section.

We will frequently use the following standard result [For81, Thm. 30.4].

Theorem 2.30. *Every holomorphic vector bundle over an open Riemann surface is holomorphically trivializable.*

In particular, open Riemann surfaces are Stein manifolds.

The holomorphic cotangent bundle

The complex structure on a Riemann surface Σ induces an almost complex structure J. That is, J is a smooth family of endomorphisms J_p of the tangent spaces $T_p\Sigma$ such that $J_p^2 = -\mathbf{1}$ for all $p \in \Sigma$. In any local holomorphic coordinate $z = x + iy$, one has

$$J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \quad J\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}.$$

Set $T\Sigma_{\mathbb{C}} = T\Sigma \otimes_{\mathbb{R}} \mathbb{C}$, and extend each J_p to a complex linear operator $J_p \otimes \mathbf{1}$ on $T_p\Sigma_{\mathbb{C}}$. Let $T_p^{(1,0)}\Sigma$ be the i-eigenspace for J_p acting on $T_p\Sigma_{\mathbb{C}}$, and let $T_p^{(0,1)}\Sigma$ be the $(-i)$-eigenspace. With respect to a local holomorphic coordinate $z : U \to \mathbb{C}$, we have basis elements

$$\begin{aligned} \frac{\partial}{\partial z} &:= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \in T^{(1,0)}U, \\ \frac{\partial}{\partial \bar{z}} &:= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \in T^{(0,1)}U. \end{aligned}$$

We give both spaces $T_p\Sigma^{(1,0)}$ and $T_p\Sigma^{(0,1)}$ complex structures coming from the restriction of J_p. On $T_p\Sigma^{(0,1)}$, this coincides with the complex structure inherited from $T_p\Sigma_{\mathbb{C}}$, but on $T_p\Sigma^{(0,1)}$ the complex structure has been reversed. Hence the conjugate linear involution on $T_p\Sigma_{\mathbb{C}}$ given by complex conjugation on the factor of $\mathbb{C}$ restricts to a natural complex linear isomorphism of complex bundles $T\Sigma^{(1,0)} \to T\Sigma^{(0,1)}$. These bundles are holomorphic, as is the isomorphism. In a local holomorphic coordinate, this isomorphism exchanges $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$.

Define the *holomorphic cotangent bundle* (or *canonical bundle*) K_Σ by

$$K_\Sigma = (T^{(1,0)}\Sigma)^*.$$

If (z, U) is a local coordinate, a trivialization of K_U is given by the section $dz = dx + idy$. We also have a trivialization of $(T^{(0,1)}U)^*$ given by $d\overline{z} = dx - idy$. If $u \in C^\infty(\Sigma)$, define a holomorphic section $\partial u \in \mathcal{O}(K_\Sigma)$ so that in every local holomorphic coordinate we have

$$\partial u = \frac{\partial u}{\partial z} dz.$$

Similarly define $\overline{\partial} u \in C^\infty(T^{(0,1)}\Sigma^*)$ by

$$\overline{\partial} u = \frac{\partial u}{\partial \overline{z}} d\overline{z}.$$

The differential operators ∂ and $\overline{\partial}$ are called the *Dolbeault operators*, and are related to the de Rahm differential by $d = \partial + \overline{\partial}$.

As $T\Sigma$ only depends on the smooth structure, we have an equality $T\overline{\Sigma} = T\Sigma$. The complex structure on $\overline{\Sigma}$ corresponds to the almost complex structure $-J$, and thus we have equalities of real vector bundles

$$(T^{(1,0)}\overline{\Sigma})_{\mathbb{R}} = (T^{(0,1)}\Sigma)_{\mathbb{R}}, \quad (T^{(0,1)}\Sigma)_{\mathbb{R}} = (T^{(1,0)}\overline{\Sigma})_{\mathbb{R}}.$$

However, the complex structures on $T^{(1,0)}\overline{\Sigma}$ and $T^{(0,1)}\overline{\Sigma}$ are given by $-J$, so these equalities induce conjugate linear isomorphisms of complex vector spaces. That is,

$$T^{(1,0)}\overline{\Sigma} = \overline{T^{(0,1)}\Sigma}, \quad T^{(0,1)}\overline{\Sigma} = \overline{T^{(1,0)}\Sigma}.$$

Combining with the holomorphic isomorphism $T\Sigma^{(1,0)} \to T\Sigma^{(0,1)}$ from above yields a holomorphic isomorphism of the conjugate bundles $\overline{T^{(1,0)}\Sigma} \to T^{(1,0)}\overline{\Sigma}$.

By dualizing, our construction has produced a holomorphic isomorphism

$$c : \overline{K_\Sigma} \to K_{\overline{\Sigma}}.$$

If f is a section of $\overline{K_\Sigma}$ then we will write $c_* f$ for the corresponding section of $K_{\overline{\Sigma}}$.

In local holomorphic coordinates, we have

$$c_* \overline{f(z) dz} = \overline{f(z)} d\overline{z},$$

where on the right-hand side we have used $\overline{(T^{(0,1)}\Sigma)^*} = K_{\overline{\Sigma}}$ to think of $d\overline{z}$ as a section of $K_{\overline{\Sigma}}$.

2.6 Spin structures

A *spin structure* on Σ is a holomorphic line bundle L along with an isomorphism $\phi : L \otimes L \to K_\Sigma$ that acts identically on the base space. In light of Theorem 2.30, on an open Riemann surface the bundle L does not contain any information and the data of the spin structure is encoded in the isomorphism ϕ. Despite this, we will often refer to L as a spin structure. It follows from Theorem 2.30 that every open Riemann surface can be given a spin structure.

Remark 2.31. The definition of a spin structure that we have given is particular to Riemann surfaces. The definition of a spin structure on a Riemannian manifold is a bundle of irreducible modules for the bundle of Clifford algebras associated to the tangent bundle, or equivalently an equivariant lifting of the orthonormal frame bundle to a principal bundle for the spin group. These notions of spin structure are treated in depth in [LM89, Ch. 2], and the equivalence of these definitions with our definition for a Riemann surface is established in [Ati71, Sec. 3].

If L and L' are spin structures on Σ and Σ', then an isomorphism of spin structures $L \to L'$ is a holomorphic isomorphism of bundles given by $g : L \to L'$ and $f : \Sigma \to \Sigma'$ such that the diagram

$$\begin{array}{ccc} L \otimes L & \xrightarrow{g \otimes g} & L' \otimes L' \\ \phi \downarrow & & \downarrow \phi' \\ K_\Sigma & \xleftarrow{f^*} & K_{\Sigma'} \end{array}$$

commutes. Here, f^* is the pullback induced by the map f.

Now consider the case when $\tilde{\Sigma}$ is an open Riemann surface. Since every holomorphic line bundle on $\tilde{\Sigma}$ is trivializable, we may take $L = \tilde{\Sigma} \times \mathbb{C}$ above. A spin structure then becomes a choice of trivialization of $K_{\tilde{\Sigma}}$, or equivalently a choice of non-vanishing holomorphic 1-form.

If ϕ_1 and ϕ_2 are two spin structures with line bundle $L = \tilde{\Sigma} \times \mathbb{C}$, then a base space preserving isomorphism $(\mathrm{id}, g) : (L, \phi_1) \to (L, \phi_2)$ is given by fiberwise multiplication by a non-vanishing holomorphic function $h \in \mathcal{O}(\tilde{\Sigma})$. If ϕ_1 and ϕ_2 correspond to 1-forms ω_1 and ω_2, then $\phi_1 \cong \phi_2$ if and only if there exists an $h \in \mathcal{O}(\tilde{\Sigma})^\times$ such that $\omega_1 = h^2 \omega_2$. Hence spin structures on an open Riemann surface $\tilde{\Sigma}$, up to base space preserving isomorphism, give a principal homogeneous space for the group $\mathcal{O}(\tilde{\Sigma})^\times / (\mathcal{O}(\tilde{\Sigma})^\times)^2$.

Proposition 2.32. *If (L, ϕ) is a spin structure over Σ and f is a conformal automorphism of Σ, then there exists an automorphism of (L, ϕ) that acts on the base space by f if and only if f' has a holomorphic square root.*

Proposition 2.33. *Up to isomorphism, there is a unique spin structure on the open disk $\mathbb{D}$.*

Proof. If we assume isomorphisms act identically on the base space, it suffices to show that every $h \in \mathcal{O}(\mathbb{D})^\times$ has a holomorphic square root. Observe that if k is a primitive of h'/h with $e^{k(0)} = h(0)$, then $h = e^k$ and in particular $(e^{k/2})^2 = h$. If f is a holomorphic automorphism of $\mathbb{D}$ then f' has a holomorphic square root, and so there is an automorphism of any spin structure acting on the base space by f. Thus spin structures up to base space preserving isomorphism are the same as spin structures up to isomorphism, and we are done. $\square$

Proposition 2.34. *Up to isomorphism, there are exactly two spin structures on the annulus $\mathcal{A} = \{r < |z| < R\}$. They are represented by 1-forms dz and $z^{-1}dz$.*

Proof. If f is a holomorphic automorphism of $\mathcal{A}$, then f is a rotation and f' has a holomorphic square root. Thus it suffices to consider spin structures up to base space preserving isomorphism.

Given $h \in \mathcal{O}(\mathcal{A})^\times$, we can take a $k \in \mathcal{O}(\mathcal{A} \setminus (-R,-r))^\times$ with $k^2 = h$. Let $k_+ : (-R,R) \to \mathbb{C}$ be the continuous function obtained by extending k counterclockiwse around $\mathcal{A} \setminus (-R,-r)$, and let k_- be the analogous function obtained by extending k clockwise. Since $k_-^2 = k_+^2 = h\mid_{(-R,-r)}$ and h is non-vanishing, we must have $k_- = k_+$ or $k_- = -k_+$.

In the former case, k extends to a holomorphic function on $\mathcal{A}$ with $k^2 = h$. In the latter case, h cannot have a holomorphic square root. However, $\sqrt{z}k$ is a holomorphic function on $\mathcal{A}$, where we have chosen a branch of $\sqrt{\cdot}$ with branch cut $(-\infty, 0]$. Hence $zh = (\sqrt{z}k)^2 \in (\mathcal{O}(\mathcal{A})^\times)^2$. Thus $(\mathcal{O}(\mathcal{A})^\times)/(\mathcal{O}(\mathcal{A})^\times)^2$ is cyclic of order 2, and the cosets are represented by the functions $\{\mathbf{1}, z^{-1}\}$. □

We call the spin structure induced by dz the Neveu-Schwarz (or NS or vacuum) spin structure, and the spin structure induced by $z^{-1}dz$ the Ramond (or R) spin structure. We will denote these spin structures by $(\mathcal{A}, NS)$ and $(\mathcal{A}, R)$. Note that only $(\mathcal{A}, NS)$ extends to a spin structure on the disk.

If (L, ϕ) is a spin structure over Σ, then the conjugate bundle $\overline{L}$ can be made a spin structure over $\overline{\Sigma}$ in a natural way. Recall that we have a holomorphic isomorphism

$$c : \overline{K_\Sigma} \to K_{\overline{\Sigma}} = \overline{(T^{(0,1)}\Sigma)^*},$$

and that in local coordinates $c_* \overline{f(z)dz} = \overline{f(z)d\overline{z}}$.

The isomorphism $\phi : L \otimes L \to K_\Sigma$ induces a holomorphic isomorphism $\widetilde{\phi} : \overline{L} \otimes \overline{L} \to \overline{K_\Sigma}$, and composing with c yields a holomorphic isomorphism

$$\overline{\phi} := c \circ \tilde{\phi} : \overline{L} \otimes \overline{L} \to K_{\overline{\Sigma}}$$

making $(\overline{L}, \overline{\phi})$ into a spin structure.

We will also be interested in the Ramond and Neveu-Schwarz spin structures over the unit circle S^1. We will think of these objects as being germs of $(\mathcal{A}_\epsilon, NS)$ and $(\mathcal{A}_\epsilon, R)$, where

$$\mathcal{A}_\epsilon = \{z \in \mathbb{C} : (1+\epsilon)^{-1} < |z| < 1 + \epsilon\}.$$

As we remarked in Section 2.4, the orientation-preserving automorphism groups of these germs are $\mathrm{Diff}^{(2)}_{+,\omega}(S^1)$ and $\mathrm{Diff}_{+,\omega}(S^1)$, respectively. The following proposition makes that statement precise.

Proposition 2.35. *Let* $(\eta, \phi) \in \mathrm{Diff}^{(2)}_{+,\omega}(S^1)$.

- *The map* $S^1 \times \mathbb{C} \to S^1 \times \mathbb{C}$ *given by*

$$(z, \alpha) \mapsto (\phi^{-1}(z), \eta^{-1}(z)\alpha)$$

extends to a holomorphic isomorphism of spin structures

$$(f^{-1}, g) : (\mathcal{A}_\epsilon, NS) \to (\mathcal{A}_\epsilon, NS)|_V$$

for some $\epsilon > 0$ *and some neighborhood* V *of* S^1.

- *The map $S^1 \times \mathbb{C} \to S^1 \times \mathbb{C}$ given by*

$$(z, \alpha) \mapsto (\phi^{-1}(z), \left|(\phi^{-1})'(z)\right|^{-1/2} \alpha)$$

extends to a holomorphic isomorphism of spin structures

$$(f^{-1}, g) : (\mathcal{A}_\epsilon, R) \to (\mathcal{A}_\epsilon, R)|_V$$

for some $\epsilon > 0$ and some neighborhood V of S^1.

Proof. We first consider the Neveu-Schwarz case. By hypothesis, ϕ and η extend to a pair of holomorphic functions f and h on some $\mathcal{A}_\epsilon$. On $f(\mathcal{A}_\epsilon) \cap \mathcal{A}_\epsilon$, we have $h^2 = (f^{-1})'$. We now choose ϵ' sufficiently small that

$$\mathcal{A}_{\epsilon'} \subseteq f(\mathcal{A}_\epsilon) \cap \mathcal{A}_\epsilon,$$

and

$$V := f^{-1}(\mathcal{A}_{\epsilon'}) \subseteq \mathcal{A}_\epsilon.$$

Define an isomorphism of bundles

$$g : \mathcal{A}_{\epsilon'} \times \mathbb{C} \to V \times \mathbb{C}$$

by

$$g(z, \alpha) = (f^{-1}(z), h(z)^{-1}\alpha).$$

We now check that that (f^{-1}, g) gives the desired isomorphism of spin structures. That is, we check that the diagram

$$\begin{array}{ccc} \mathcal{A}_{\epsilon'} \times \mathbb{C}^{\otimes 2} & \xrightarrow{g^{\otimes 2}} & V \times \mathbb{C}^{\otimes 2} \\ {\scriptstyle \phi_{NS}}\downarrow & & {\scriptstyle \phi_{NS}}\downarrow \\ K_{\mathcal{A}_{\epsilon'}} & \xleftarrow{(f^{-1})^*} & K_V \end{array}$$

commutes.

If k is a smooth function on $\mathcal{A}_{\epsilon'}$, the pushforward $(g^{\otimes 2})_* k$ is given by

$$((g^{\otimes 2})_* k)(z) = h(f(z))^{-2} k(f(z)) = f'(z) k(f(z)).$$

The pullback $(f^{-1})^* : K_{V_j} \to K_{\mathcal{A}_{\epsilon'}}$ is given on 1-forms by

$$(f^{-1})^* k(z) dz = k(f^{-1}(z))(f^{-1})'(z) dz.$$

We now compute

$$\begin{aligned} (f^{-1})^*(\phi_{NS})_*(g^{\otimes 2})_* k &= (f^{-1})^*(\phi_{NS})_*(f'(k \circ f)) \\ &= (f^{-1})^* f'(k \circ f) dz \\ &= (f' \circ f^{-1}) k df^{-1} \\ &= f dz \\ &= (\phi_{NS})_* f. \end{aligned}$$

A similar construction can be done in the Ramond case, except replacing h by

$$h_R(z) = \sqrt{\frac{z(f^{-1})'(z)}{f^{-1}(z)}},$$

where the square root is interpreted as in Remark 2.19.

Now set $g(z, \alpha) = (f^{-1}(z), h_R(z)^{-1}\alpha)$. We have

$$((g^{\otimes 2})_* k)(z) = h_R(f(z))^{-2} k(f(z))$$

To show that (f^{-1}, g) is an isomorphism of spin structures, we compute

$$\begin{aligned}
(f^{-1})^*(\phi_R)_*(g^{\otimes 2})_* k &= (f^{-1})^*(\phi_R)_* \frac{1}{(h_R \circ f)^2}(k \circ f) \\
&= (f^{-1})^* \frac{1}{z(h_R \circ f)^2}(k \circ f) dz \\
&= \frac{1}{f^{-1} h_R^2} k df^{-1} \\
&= z^{-1} k dz \\
&= (\phi_R)_* k
\end{aligned}$$

Hence the diagram

$$\begin{array}{ccc}
\mathcal{A}_{\epsilon'} \times \mathbb{C}^{\otimes 2} & \xrightarrow{g^{\otimes 2}} & V_j \times \mathbb{C}^{\otimes 2} \\
\phi_R \downarrow & & \phi_R \downarrow \\
K_{\mathcal{A}_{\epsilon'}} & \xleftarrow{(f^{-1})^*} & K_{V_j}
\end{array}$$

commutes, and (f^{-1}, g) is an isomorphism of spin structures. □

Chapter 3

Segal CFT

In Segal's manuscript [Seg04], he proposes axiomatizations for what would today be called 2-dimensional *(full) conformal field theories* and *chiral conformal field theories* (under the names "conformal field theories" and "weakly conformal field theories," respectively). Informally, a full Segal CFT is a projective monoidal functor from the category of 1-complex dimensional cobordisms to the category of Hilbert spaces and trace class maps. That is, one has a Hilbert space $\mathcal{H}$ and for every Riemann surface Σ with parameterized boundary a one-dimensional space of trace class maps

$$\bigotimes_{\partial_{in}\Sigma} \mathcal{H} \to \bigotimes_{\partial_{out}\Sigma} \mathcal{H}.$$

The locality of the theory is expressed by the functorality of this assignment; that is, the requirement that gluing of Riemann surfaces corresponds to composition of maps.

While not a theorem in general, it is a standard assumption that the Hilbert space of a full conformal field theory decomposes

$$\mathcal{H} = \bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda \otimes \overline{H_\lambda}$$

where the H_λ encode the portion of the theory that varies holomorphically with Σ. The elements of Λ index the *sectors* of the theory.

The structure of the sectors of a conformal theory is encoded by the notion of a chiral conformal field theory. Under Segal's axioms, the data of a chiral conformal field theory is: a finite set Λ, corresponding to sectors, a Hilbert space $\mathcal{H}_\lambda$ for every $\lambda \in \Lambda$, and for every complex 1-dimensional cobordism with parameterized boundary components labeled by elements of Λ a finite-dimensional vector space of trace-class maps

$$\bigotimes_{j \in \partial_{in}\Sigma} \mathcal{H}_{\lambda_j} \to \bigotimes_{j \in \partial_{out}\Sigma} \mathcal{H}_{\lambda_j}.$$

These finite-dimensional vector spaces are called the spaces of *conformal blocks*, and they are required to satisfy a gluing axiom similar to the one satisfied by the operators assigned to surfaces by a full CFT. We also require that the dependence of the conformal blocks upon the surface be holomorphic. Alternatively, we call the theory *antichiral* if the dependence is antiholomorphic.

From a physical perspective, conformal blocks consist of maps that satisfy the chiral Ward identities, and they are the building blocks with which one should be able to construct the correlators of the corresponding full CFT. An exposition on conformal blocks and gauge symmetry is given in [Gaw00]. There has been extensive work by Fjelstad, Fuchs, Runkel and Schweigert to construct correlators from conformal blocks [FRS02, FRS04a, FRS04b, FRS05, FFRS06], although their setting is not the same as the one we are working in.

In this chapter we will carefully lay out the axioms that a chiral Segal CFT will satisfy, and in Chapter 4 we will construct examples of chiral conformal field theories corresponding to the free (complex) fermion.

3.1 Rigged Riemann surfaces

We will consider Riemann surfaces that are compact with no closed components. Each connected component of the boundary will be equipped with a label $\lambda \in \Lambda$, as well as a parameterization by the standard circle S^1. We will also equip our Riemann surfaces with holomorphic vector bundles V, and assume that the parameterizations come from a trivialization of $V|_\Gamma$. For convenience, we will assume that our Riemann surfaces come with germs of collars of complex structure, and that our bundles and trivializations extend holomorphically to these collars (although this assumption is not necessary due to conformal welding). We will call a Riemann surface with these properties and with this additional data a *rigged Riemann surface*.

We now describe the precise nature of the data of a rigged Riemann surface. Let Σ be a compact Riemann surface with holomorphic vector bundle V, and let $\Gamma = \partial\Sigma$. Assume that we have partitioned Γ into two closed subsets, Γ^0 and Γ^1 (called the *incoming* and *outgoing* boundary, respectively). Let $\pi_0(\Gamma)$ denote the set of connected components of Γ, and similarly for $\pi_0(\Gamma^i)$.

Let $\tilde{\Sigma}$ be an open Riemann surface, and assume that Σ is holomorphically embedded in $\tilde{\Sigma}$. Let $\tilde{V}$ be a holomorphic extension of V to $\tilde{\Sigma}$.

Assume further that for every $j \in \pi_0(\Gamma)$ we have chosen neighborhoods $U_j \supset j$ in $\tilde{\Sigma}$, as well as holomorphic isomorphisms

$$g_j : \mathcal{A}_\epsilon \times \mathbb{C}^{k(j)} \to V|_{U_j},$$

where

$$\mathcal{A}_\epsilon = \{z \in \mathbb{C} : (1+\epsilon)^{-1} < |z| < 1+\epsilon\}.$$

Suppose further that

$$g_j((\mathbb{D}^c \cap \mathcal{A}_\epsilon) \times \mathbb{C}^{k(j)}) = \tilde{V}|_{\Sigma \cap U_j}, \quad j \in \pi_0(\Gamma^0)$$

and

$$g_j((\mathrm{cl}(\mathbb{D}) \cap \mathcal{A}_\epsilon) \times \mathbb{C}^{k(j)}) = \tilde{V}|_{\Sigma \cap U_j}, \quad j \in \pi_0(\Gamma^1).$$

By restricting g_j, we obtain trivializations

$$\beta_j : S^1 \times \mathbb{C}^{k(j)} \to V|_j.$$

Definition 3.1. Let Σ be a compact Riemann surface with boundary Γ partitioned as above, and let V be a holomorphic vector bundle over Σ. *Analytic boundary data for* (Σ, V) is a family of smooth bundle isomorphisms $\beta_j : S^1 \times \mathbb{C}^{k(j)} \to V|_j$, indexed by $j \in \pi_0(\Gamma)$, that can be obtained by the above procedure.

In particular, *analytic boundary data for* Σ is analytic boundary data for the trivial bundle over Σ that acts identically on the fibers.

Remark 3.2. The complex structure for Σ gives it, and consequently Γ, an orientation. If we give S^1 the standard orientation, then $\beta_j \mid_{S^1}$ is orientation-preserving if and only if j is an outgoing boundary component.

Definition 3.3. Fix a finite set Λ, and let $\mathcal{E}_\Lambda$ be the class of tuples $(\Sigma, V, \beta, \alpha)$, where

- Σ is a compact Riemann surface with no closed components and boundary $\Gamma = \partial\Sigma$ partitioned $\Gamma = \Gamma^0 \cup \Gamma^1$ as above,
- V is a holomorphic vector bundle over Σ,
- $\beta = (\beta_j)_{j \in \pi_0(\Gamma)}$ is analytic boundary data for (Σ, V),
- $\alpha : \pi_0(\Gamma) \to \Lambda$.

We call an element $X \in \mathcal{E}$ *a rigged Riemann surface.* If we want to emphasize the underlying Riemann surface, we will call X a *rigging of* Σ. We denote by $k : \pi_0(\Gamma) \to \mathbb{Z}_{>0}$ the boundary rank function $k(j) = \mathrm{rank}(V|_j)$.

We will suppress the dependence of $\mathcal{E}$ upon Λ whenever possible.

Involutions

Fix $X = (\Sigma, V, \beta, \alpha) \in \mathcal{E}$. Since V is a holomorphic vector bundle over Σ, the conjugate bundle $\overline{V}$ is a holomorphic vector bundle over the conjugate surface $\overline{\Sigma}$. Analytic boundary data for $(\overline{\Sigma}, \overline{V})$ is given by $\overline{\beta}$, where

$$\overline{\beta}_j(z, \xi) = \overline{\beta_j(\overline{z}, \overline{\xi})}.$$

Note that $\overline{\xi}$ refers to the element of $\mathbb{C}^k$ given by entry-wise complex conjugation, while $\overline{\beta_j(\cdot)}$ refers to the conjugate linear identity map $V \to \overline{V}$ (see Remark 2.1).

Frequently, our set of labels Λ will come with an involution $\lambda \mapsto \overline{\lambda}$. In this case, define $\overline{\alpha}(j) = \overline{\alpha(j)}$, and otherwise set $\overline{\alpha}(\lambda) = \lambda$.

Definition 3.4 (Conjugate)**.** If $X = (\Sigma, V, \beta, \alpha) \in \mathcal{E}$, the *conjugate* $\overline{X}$ is given by $\overline{X} = (\overline{\Sigma}, \overline{V}, \overline{\beta}, \overline{\alpha})$.

For $j \in \pi_0(\Gamma)$, let $\Sigma^{t,j}$ be the same Riemann surface as Σ, but with the partition on the boundary changed so that j has switched between incoming and outgoing. Define analytic boundary data for $(\Sigma^{t,j}, V)$ by

$$\beta^{t,j}_{j'}(z,\xi) = \begin{cases} \beta_{j'}(z,\xi), & j' \neq j, \\ \beta_{j'}(z^{-1},\xi), & j' = j. \end{cases}$$

Define

$$\alpha^{t,j}(j') = \begin{cases} \alpha(j'), & j' \neq j, \\ \overline{\alpha(j')}, & j' = j. \end{cases}$$

Definition 3.5 (Transpose)**.** If $X = (\Sigma, V, \beta, \alpha) \in \mathcal{E}$ and $j \in \pi_0(\Gamma)$, the *transpose at j* $X^{t,j}$ is given by $X^{t,j} = (\Sigma^{t,j}, V, \beta^{t,j}, \alpha^{t,j})$. The transpose X^t is given by taking the transpose at every $j \in \pi_0(\Gamma)$.

Definition 3.6 (Adjoint)**.** If $X = (\Sigma, V, \beta, \alpha) \in \mathcal{E}$, the *adjoint* X^* is given by composing the conjugate and transpose. We will write β^* for the resulting analytic boundary data, so that $X^* = (\Sigma^*, \overline{V}, \beta^*, \alpha)$.

The dual bundle V^* is again holomorphic over Σ. Let

$$\widetilde{\beta}_j : S^1 \times (\mathbb{C}^k)^* \to V^*|_j$$

be the isomorphism induced by β_j. Analytic boundary data for (Σ, V^*) is given by

$$\beta^d_j(z,\xi) = \widetilde{\beta}_j(z, \kappa(\xi)),$$

where $\kappa : \mathbb{C}^k \to (\mathbb{C}^k)^*$ is the natural complex linear isomorphism coming from the choice of standard basis.

Definition 3.7 (Dual)**.** If $X = (\Sigma, V, \beta, \alpha) \in \mathcal{E}$, the *dual* X^d is given by $X^d = (\Sigma, V^*, \beta^d, \alpha)$.

Proposition 3.8. *If X is a rigged Riemann surface, then so are X^*, $X^{t,j}$, X^d and X^*.*

Sewing, gluing, and cutting

Once again fix $(\Sigma, V, \beta, \alpha) \in \mathcal{E}$, and fix choices of incoming and outgoing boundary components $j^0 \in \pi_0(\Gamma^0)$ and $j^1 \in \pi_0(\Gamma^1)$. If $k(j^0) = k(j^1)$, one can obtain a new manifold $\check{\Sigma}$ and new vector bundle $\check{V}$ by identifying $V|_{j^0}$ and $V|_{j^1}$ via $\beta_{j^1} \circ \beta_{j^0}^{-1}$. The sewn surface $\check{\Sigma}$ naturally has the structure of a Riemann surface and the bundle $\check{V}$ is holomorphic. The holomorphic sections of $\check{V}$ are precisely the $f \in \mathcal{O}(V)$ that extend continuously to both j^i and satisfy

$$f \circ \beta_{j^0} = f \circ \beta_{j^1}.$$

By restriction., the labeling α and analytic boundary data β for (Σ, V) give a labeling $\check{\alpha}$ and analytic boundary data $\check{\beta}$ for $(\check{\Sigma}, \check{V})$.

Definition 3.9 (Sewing and gluing)**.** If $X \in \mathcal{E}_\Lambda$, $j^0 \in \pi_0(\Gamma^0)$, $j^1 \in \pi_0(\Gamma^1)$, and $k(j^0) = k(j^1)$, then we define the sewn rigged surface $\check{X}$ by $\check{X} = (\check{\Sigma}, \check{V}, \check{\beta}, \check{\alpha})$. We let $\mathcal{E}_{\Lambda,*}$ be the class of (X, j^0, j^1) for which $\check{X} \in \mathcal{E}_\Lambda$, and in this case we say that X is *sewable at* $j^0 j^1$. If j^0 and j^1 lie on distinct connected components of Σ, we will sometimes refer to this operation as *gluing*.

As with $\mathcal{E}_\Lambda$, we will write $\mathcal{E}_{\Lambda,*}$ as $\mathcal{E}_*$ whenever Λ is understood from context. We can write the sewing as $\check{X}^{j^0 j^1}$ if the boundary components to be sewn are unclear, and $(X, j^0, j^1) \mapsto \check{X}^{j^0 j^1}$ gives a map $E_* \to E$. The notions of conjugate, transpose, dual and adjoint rigged surfaces extend to $\mathcal{E}_*$, with the caveat that we must exchange the roles of j^0 and j^1 for transpose and adjoint.

Definition 3.10 (Cutting)**.** If $X = (\Sigma, V, \beta, \alpha) \in \mathcal{E}$ and $g : \mathcal{A}_\epsilon \times \mathbb{C}^k \to V$ is a holomorphic embedding, then the Riemann surface $\hat{\Sigma}$ obtained by cutting Σ along $g(S^1)$ naturally has analytic boundary data given by β and $g|_{S^1 \times \mathbb{C}^k}$. The partition of incoming and outgoing boundary components is determined by g. We define $\hat{X}_\lambda$, called X *cut along* g, to be the corresponding element of $\mathcal{E}$ where the labeling has been extended to have value λ at the new boundary components.

We can also naturally think of $\hat{X}_\lambda$ as an element of $\mathcal{E}_*$.

Reparameterization

Let (Σ, V, β) be a Riemann surface with holomorphic vector bundle and analytic boundary data, and let

$$(x, \phi) \in \left(L_\omega GL(k(j)) \rtimes \mathrm{Diff}_{+,\omega}(S^1)\right)^{\pi_0(\Gamma)}.$$

Then (x, ϕ) acts on β by

$$((x, \phi) \cdot \beta)_j(z, \xi) = \beta_j(\phi_j^{-1}(z), x_j(z)^{-1}\xi)$$

and $(x, \phi) \cdot \beta$ again gives analytic boundary data for (Σ, V).

Definition 3.11. If $X = (\Sigma, V, \beta, \alpha) \in \mathcal{E}$ and

$$(x, \phi) \in \left(L_\omega GL(k(j)) \rtimes \mathrm{Diff}_{+,\omega}(S^1)\right)^{\pi_0(\Gamma)},$$

then define the *reparameterization of* X *by* (x, ϕ) by

$$(x, \phi) \cdot X = (\Sigma, V, (x, \phi) \cdot \beta, \alpha).$$

We define $\phi \cdot X := (\mathbf{1}, \phi) \cdot X$.

Tensor product, direct sum, and disjoint union

Suppose that for $m \in \{1,2\}$ we have $X_m = (\Sigma, V_m, \beta_m, \alpha) \in \mathcal{E}$, and suppose that $\beta_1|_{S^1} = \beta_2|_{S^1}$. Define trivializations

$$(\beta_1 \oplus \beta_2) : S^1 \times (\mathbb{C}^{k_1(j)+k_2(j)}) \to (V_1 \oplus V_2)|_j$$

and

$$(\beta_1 \otimes \beta_2) : S^1 \times (\mathbb{C}^{k_1(j)k_2(j)}) \to (V_1 \otimes V_2)|_j$$

in the obvious way.

Definition 3.12 (Direct sum and tensor product)**.** If X_1 and X_2 are as above, we define the *direct sum* and *tensor product* of rigged surfaces by

$$X_1 \oplus X_2 = (\Sigma, V_1 \oplus V_2, \beta_1 \oplus \beta_2, \alpha)$$

and

$$X_1 \otimes X_2 = (\Sigma, V_1 \otimes V_2, \beta_1 \otimes \beta_2, \alpha).$$

Up to natural isomorphisms, we have the expected distributivity, commutativity and associativity of these operations.

If $X = (\Sigma, V, \beta, \alpha) \in \mathcal{E}$ and K_Σ is the canonical bundle, then there is a natural way to interpret $K_\Sigma \otimes X$. This is done by extending (Σ, K_Σ) to a rigged surface with labeling α, and boundary parameterization $(\beta|_{S^1})_*$ induced by β.

Definition 3.13 (Disjoint union)**.** If $X_1, X_2 \in \mathcal{E}$, we define the disjoint union $X_1 \sqcup X_2$ to be the rigged surface $(\Sigma_1 \sqcup \Sigma_2, V_1 \sqcup V_2, \beta_1 \sqcup \beta_2, \alpha_2 \sqcup \alpha_2)$.

Important families of rigged Riemann surfaces

If $\lambda \in \Lambda$ and $k \in \mathbb{Z}_{>0}$, define $\mathrm{cl}(\mathbb{D})_{k,\lambda} \in \mathcal{E}$ by

$$\mathrm{cl}(\mathbb{D})_{k,\lambda} = (\mathrm{cl}(\mathbb{D}), \mathrm{cl}(\mathbb{D}) \times \mathbb{C}^k, \mathrm{id}, \alpha_\lambda)$$

where $\alpha_\lambda(S^1) = \lambda$.

For $q \in \mathbb{D} \setminus \{0\}$, we denote by ${}_{(\lambda,q)}\mathcal{A}^k_\mu$ the element of $\mathcal{E}$ with surface

$${}_q\mathcal{A} := \mathrm{cl}(\mathbb{D}) \setminus |q|\,\mathbb{D},$$

incoming boundary $|q|\,S^1$, outgoing boundary S^1, vector bundle ${}_q\mathcal{A} \times \mathbb{C}^k$, labeling $\alpha(|q|\,S^1) = \lambda$, $\alpha(S^1) = \mu$, and parameterizations

$$\beta_j(z) = \begin{cases} qz & j = |q|\,S^1 \\ z & j = S^1 \end{cases} .$$

We define in the same way $\mathcal{A}^k_{(\lambda,q),\mu}$, except all boundary is outgoing and we set $\beta'_{|q|S^1}(z) = qz^{-1}$. Similarly, ${}_{(\lambda,q),\mu}\mathcal{A}^k$ is given by the same data as ${}_{(\lambda,q)}\mathcal{A}^k_{\mu}$, except all boundary is incoming and $\beta''_{S^1}(z) = z^{-1}$.

Now suppose $\lambda_1, \lambda_2, \mu \in \Lambda$ and $(q_1, w_1, q_2, w_2) \in M_{\mathcal{P}}$, where

$$M_{\mathcal{P}} = \{(q_1, w_1, q_2, w_2) \in \mathbb{D}^4 : |q_i| > 0, |w_i| + |q_i| < 1, |w_1 - w_2| > |q_1| + |q_2|\}$$

is the moduli space of pairs of disjoint disks removed from the unit disk, with marked points on the inside boundary.

Given the above data, we have an element $\mathcal{P}((\lambda_1, q_1, w_1), (\lambda_2, q_2, w_2); \mu; k) \in \mathcal{E}$. The surface for this element is

$$\mathcal{P}_{(q_1,w_1),(q_2,w_2)} := \mathrm{cl}(\mathbb{D}) \setminus ((w_1 + |q_1| \, \mathbb{D}) \cup (w_2 + |q_2| \, \mathbb{D})) \, .$$

with outgoing boundary S^1, and the internal circles as incoming boundary. The vector bundle for this element is $\mathcal{P}_{(q_1,w_1),(q_2,w_2)} \times \mathbb{C}^k$, and the labeling is $\alpha(S^1) = \mu$, $\alpha(w_i + |q_i| \, S^1) = \lambda_i$. The boundary data is given by

$$\beta_j(z) = \begin{cases} z, & j = S^1, \\ w_i + q_i z, & j = w_i + q_i S^2. \end{cases}$$

3.2 Definition of Segal CFT

We now give the definition of a $\mathbb{Z}/2$-graded chiral Segal CFT; the ungraded version of the definition can be obtained by taking all of the Hilbert spaces to be even.

Definition 3.14. A rational, chiral $\mathbb{Z}/2$-graded Segal CFT consists of the following data:

(1) a finite set of labels Λ,

(2) a super Hilbert space $\mathcal{H}^k_\lambda$ for every $\lambda \in \Lambda$ and $k \in \mathbb{Z}_{>0}$,

(3) a strongly continuous projective positive energy representation $U_{k,\lambda}$ of $\mathrm{Rot}(S^1)$ by even operators on each $\mathcal{H}^k_\lambda$,

(4) for every $X = (\Sigma, V, \beta, \alpha) \in \mathcal{E}$, a finite-dimensional vector subspace of homogeneous maps of unordered tensor products $E(X) \subset \mathcal{I}^1 \left(\mathcal{H}_{\Gamma^0}, \mathcal{H}_{\Gamma^1}\right)$, where

$$\mathcal{H}_{\Gamma^i} = \bigotimes_{j \in \pi_0(\Gamma^i)} \mathcal{H}^{k(j)}_{\alpha(j)}.$$

This data must have the following properties.

(a) *(Conformal invariance)* If $(\Sigma, V, \beta, \alpha) \in \mathcal{E}$ and $\phi : (\Sigma, V) \to (\Sigma', V')$ is a holomorphic isomorphism of bundles, then

$$E(\Sigma, V, \beta, \alpha) = E(\Sigma', V', \phi \circ \beta, \alpha \circ \phi^{-1}).$$

(b) *(Non-degeneracy)* For every $k \in \mathbb{Z}_{>0}$ and $\lambda \in \Lambda$, we have

$$\dim E({}_{(\lambda,q)}\mathcal{A}^k_\lambda) > 0,$$

and non-zero elements of this space are injective with dense image.

(c) *(Vacuum sector)* There is a label $0 \in \Lambda$ such that for all $k \in \mathbb{Z}_{>0}$ we have

$$\dim(E(\mathrm{cl}(\mathbb{D})_{k,\lambda})) = \delta_{0,\lambda}.$$

The representation $U_{k,0}$ of $\mathrm{Rot}(S^1)$ is required to be an honest representation. We fix a distinguished unit vector $\Omega^k \in \mathcal{H}^k_0$ in the image of elements of $E(\mathrm{cl}(\mathbb{D})_{k,0})$, and require that the generator $L^{k,0}_0$ of $U_{k,0}$ annihilate Ω^k.

(d) *(Monoidal)* If $X_1, X_2 \in \mathcal{E}$ then $E(X_1 \sqcup X_2) = E(X_1) \hat{\otimes} E(X_2)$.

(e) *(Chirality)* For every $\lambda_1, \lambda_2, \mu \in \Lambda$ and $k \in \mathbb{Z}_{>0}$, the subbundle of the trivial bundle $M_{\mathcal{P}} \times \mathcal{I}^2(\mathcal{H}^k_{\lambda_1} \otimes \mathcal{H}^k_{\lambda_2}, \mathcal{H}^k_\mu)$ given by restricting to $E(\mathcal{P}((\lambda_1, q_1, w_1), (\lambda_2, q_2, w_2); \mu; k))$ on fibers is a holomorphic vector bundle over $M_{\mathcal{P}}$.

(f) *(Locality)* Suppose that β is analytic boundary data for (Σ, V), $j^0 \in \pi_0(\Gamma^0)$, $j^1 \in \pi_0(\Gamma^1)$, and that we have a partial labeling $\alpha : \pi_0(\Gamma) \setminus \{j^0, j^1\} \to \Lambda$. For every $\lambda \in \Lambda$, we can extend α to a labeling $\alpha_\lambda : \pi_0(\Gamma) \to \Lambda$ by setting $\alpha_\lambda(j^0) = \alpha_\lambda(j^1) = \lambda$. Let $X_\lambda = (\Sigma, V, \beta, \alpha_\lambda)$. If $(X_\lambda, j^0, j^1) \in \mathcal{E}_*$, then we require that

$$\tau^s_{j^0 j^1} : \bigoplus_{\lambda \in \Lambda} E(X_\lambda) \to E(\check{X})$$

and that this map is an isomorphism.

(g) *(Rotation covariance)* Let $X \in \mathcal{E}$ and $\phi \in (\mathrm{Rot}(S^1))^{\pi_0(\Gamma)}$. If $X' = \phi \cdot X$, then we have

$$E(X') = \left(\bigotimes_{j \in \pi_0(\Gamma^1)} U_{k(j),\alpha(j)}(\phi_j) \right) E(X) \left(\bigotimes_{j \in \pi_0(\Gamma^0)} U_{k(j),\alpha(j)}(\phi_j)^* \right).$$

Additionally, the theory is called *unitary* if it comes with extensions of the representations $U_{k,\lambda}$ to projective unitary representation $U_{k,\lambda} : \mathrm{Diff}_+(S^1) \to \mathcal{PU}(\mathcal{H}^k_\lambda)$, and has the following properties.

(h) *(Diffeomorphism covariance)* Condition (g) holds for all $\phi \in \mathrm{Diff}_{+,\omega}(S^1)^{\pi_0(\Gamma)}$.

(i) *(Adjoint preservation)* For every $X \in \mathcal{E}$, we have $E((X^d)^*) = E(X)^*$.

We analogously define an antichiral Segal CFT by altering Definition 3.14(e) to require the vector bundle be holomorphic over $\overline{M_{\mathcal{P}}}$.

Remark 3.15. Since every holomorphic vector bundle on an open Riemann surface is trivializable, conformal invariance tells us that it is sufficient to define the theory for trivial bundles $V = \Sigma \times \mathbb{C}^k$. However, in future work we will consider extensions of this definition that allow closed surfaces, and in that case the vector bundles will play an important role.

One can also consider versions of Definition 3.14 that only allow a certain class of holomorphic vector bundles V, or require that these vector bundles have additional structure (e.g. require that V is a principal G-bundle for a gauge group G).

3.3 Rigged spin Riemann surfaces

A spin Segal CFT will assign vector spaces to rigged Riemann surfaces that come equipped with a spin structure. Like our vector bundles, our spin structures will be holomorphic and come with analytic boundary trivializations.

Let Σ be a compact Riemann surface with boundary Γ, and let L be a spin structure over Σ. Suppose that we have a holomorphic embedding $\Sigma \subset \tilde{\Sigma}$ for some open Riemann surface $\tilde{\Sigma}$, and we have an extension of L to a spin structure $\tilde{L}$ over $\tilde{\Sigma}$. Suppose that for every $j \in \pi_0(\Gamma)$ we have a neighborhood $j \subset U_j$ in $\tilde{\Sigma}$, and a trivialization of the spin structure $L|_{U_j}$,

$$(f_j, g_j) : (\mathcal{A}_\epsilon, \sigma(j)) \to \tilde{L}|U_j$$

Here $\sigma \in \{NS, R\}^{\pi_0(\Gamma)}$.

Assume that g_j satisfies

$$\begin{cases} g_j\left((\mathcal{A}_\epsilon \times \mathbb{C})|_{\mathcal{A}_\epsilon \cap \mathrm{cl}(\mathbb{D})}\right) = L|_{U_j \cap \Sigma}, & j \text{ outgoing}, \\ g_j\left((\mathcal{A}_\epsilon \times \mathbb{C})|_{\mathcal{A}_\epsilon \cap \mathbb{D}^c}\right) = L|_{U_j \cap \Sigma}, & j \text{ incoming}. \end{cases}$$

Note that for each j there is a unique value of σ for which such a g_j exists, and this choice is independent of the extensions $\tilde{\Sigma}$ and $\tilde{L}$; to indicate this value of σ, we will say that "*L is σ near j.*"

By restriction, we get a smooth bundle isomorphism

$$\gamma_j := g_j|_{S^1 \times \mathbb{C}} : S^1 \times \mathbb{C} \to L|_j.$$

Definition 3.16. If L is a spin structure over a compact Riemann surface Σ, then *analytic boundary data for* (Σ, L) is a family of smooth isomorphisms $\gamma_j : S^1 \times \mathbb{C} \to L\,|_j$ indexed by $j \in \pi_0(\Gamma)$ that can be obtained by the above procedure.

If V is a holomorphic vector bundle over Σ, then *analytic boundary data for* (Σ, V, L) is analytic boundary data $(\beta_j)_{j \in \pi_0(\Gamma)}$ for (Σ, V) and analytic boundary data $(\gamma_j)_{j \in \pi_0(\Gamma)}$ for (Σ, L) such that $\beta_j|_{S^1} = \gamma_j|_{S^1}$.

Definition 3.17 (Rigged spin Riemann surface)**.** Fix a finite set Λ^0, and let $\Lambda = \Lambda^0 \times \{NS, R\}$. Let $\mathcal{E}_\Lambda^{spin}$ be the collection of tuples $X = (X_0, L, \gamma)$ where

- $X_0 = (\Sigma, V, \beta, \alpha) \in \mathcal{E}_\Lambda$,

- L is a spin structure over Σ,
- (β, γ) gives analytic boundary data for (Σ, V, L),
- $\alpha(j) \in \Lambda^0 \times \{\sigma\}$ if L is σ near j.

We call X a *rigged spin Riemann surface.*

We define $\mathcal{E}^{spin}_{\Lambda,*}$ to be the collection of tuples (X, j^0, j^1) where $X \in \mathcal{E}^{spin}$, $j^i \in \pi_0(\Gamma^i)$, and $(X_0, j^0, j^1) \in \mathcal{E}_{\Lambda,*}$.

We will suppress the dependence of $\mathcal{E}^{spin}_{\Lambda}$ and $\mathcal{E}^{spin}_{\Lambda,*}$ on Λ whenever possible.

Tensor product, direct sum, and disjoint union

Definition 3.18 (Inherited operations)**.** For $m \in \{1, 2\}$, let $X_m = (X_{0,m}, L_m, \gamma_m) \in \mathcal{E}^{spin}$. Define the *disjoint union* by

$$X_1 \sqcup X_2 = (X_{0,1} \sqcup X_{0,2}, L_1 \sqcup L_2, \gamma_1 \sqcup \gamma_2).$$

If $L_1 = L_2 = L$ and $\gamma_1 = \gamma_2 = \gamma$, then we define the *direct sum* and *tensor product* by

$$X_1 \oplus X_2 = (X_{0,1} \oplus X_{0,2}, L, \gamma), \quad X_1 \otimes X_2 = (X_{0,1} \otimes X_{0,2}, L, \gamma)$$

whenever those operations are defined on the $X_{0,m}$.

Sewing, gluing, and cutting

To define the sewing of a rigged spin Riemann surface, we need the following observation.

Proposition 3.19. *If $(X_0, L, \gamma) \in \mathcal{E}^{spin}$, the sewn spin structure $\check{L}$ is naturally a spin structure over $\check{\Sigma}$.*

Definition 3.20 (Sewing)**.** For $X = (X_0, L, \gamma, j^0, j^1) \in \mathcal{E}^{spin}_*$, define the *sewn* rigged spin surface $\check{X}$ by $\check{X} = (\check{X}_0, \check{L}, \check{\gamma})$.

We define gluing and cutting of rigged spin surfaces the same way they were defined for ordinary rigged surfaces.

Reparameterization

To reparameterize $L|_j$, we need an analytic automorphism of the spin structure of type $\sigma(j)$ over S^1. If $\sigma = NS$, this is given by $(\eta, \phi) \in \mathrm{Diff}^{(2)}_{+,\omega}(S^1)$. We have an action of (η, ϕ) on the analytic boundary data $\gamma_j : (S^1 \times \mathbb{C}) \to L|_j$ inherited from the action of $L_\omega GL(k) \rtimes \mathrm{Diff}_{+,\omega}$ defined in Section 3.1. It is given by

$$((\eta, \phi) \cdot \gamma_j)(z, \alpha) = \gamma_j(\phi^{-1}(z), \eta(z)^{-1}\alpha).$$

On the other hand, if L is Ramond near j then an automorphism of $L|_j$ is simply given by $\phi \in \mathrm{Diff}_{+,\omega}(S^1)$. The action of ϕ on γ is

$$(\phi \cdot \gamma)(z, \xi) = \gamma_j \left(\phi^{-1}(z), \left|(\phi^{-1})'(z)\right|^{-\frac{1}{2}} \xi\right).$$

If $(\eta, \phi) \in \mathrm{Diff}^{(2)}_{+,\omega}(S^1)^{\pi_0(\Gamma)}$, then we let (η, ϕ) act on γ by

$$((\eta, \phi) \cdot \gamma)_j = \begin{cases} (\eta_j, \phi_j) \cdot \gamma_j & L \text{ is } NS \text{ near } j \\ \phi_j \cdot \gamma_j & L \text{ is } R \text{ near } j. \end{cases}$$

Definition 3.21 (Reparameterization)**.** If $X = (X_0, L, \gamma) \in \mathcal{E}^{spin}$ and

$$(\eta, \phi) \in \left(\mathrm{Diff}^{(2)}_{+,\omega}(S^1)\right)^{\pi_0(\Gamma)},$$

then define the *reparameterization of X by (η, ϕ)* by

$$(\eta, \phi) \cdot X = (\phi \cdot X_0, L, (\eta, \phi) \cdot \gamma).$$

Note that the action of $(\eta, \phi) \in \mathrm{Diff}^{(2)}(S^1)$ on Ramond boundary components only depends on ϕ.

Proposition 3.22. *If $X \in \mathcal{E}^{spin}$ and $(\eta, \phi) \in \left(\mathrm{Diff}^{(2)}_{+,\omega}(S^1)\right)^{\pi_0(\Gamma)}$, then $(\eta, \phi) \cdot X \in \mathcal{E}^{spin}$.*

Proof. We only need to check that $(\eta, \phi) \cdot \gamma$ gives analytic boundary data for L.

Fix an embedding $\Sigma \subset \tilde{\Sigma}$, a spin structure $\tilde{L}$ on $\tilde{\Sigma}$ extending L, neighborhoods U_j of j in $\tilde{\Sigma}$ and isomorphisms $(f_j, g_j) : (\mathcal{A}_\epsilon, \sigma(j)) \to \tilde{L}|_{U_j}$ such that $\gamma_j = g_j|_{S^1 \times \mathbb{C}}$.

By Proposition 2.35, we may decrease ϵ so that when L is of type NS near j,

$$(z, \alpha) \mapsto (\phi_j^{-1}(z), \eta_j^{-1}(z)\alpha)$$

extends to an isomorphism $h_j : (\mathcal{A}_\epsilon, NS) \to (\mathcal{A}_\epsilon, NS)|_{V_j}$ for some open neighborhood V_j of S^1. Similarly, if L is of type R near j, then

$$(z, \alpha) \mapsto (\phi_j^{-1}(z), \left|(\phi^{-1})'(z)\right|^{-1/2} \alpha)$$

extends to an isomorphism of spin structures $h_j : (\mathcal{A}_\epsilon, R) \to (\mathcal{A}_\epsilon, R)|_{V_j}$ for some open neighborhood V_j of S^1.

After suitably restricting domains, $g_j \circ h_j$ is a holomorphic trivialization of L in a neighborhood of j, and $(g_j \circ h_j)|_{S^1 \times \mathbb{C}} = ((\eta, \phi) \cdot \gamma)_j$. ∎

Involutions

We want notions of conjugate, transpose, and adjoint for rigged spin Riemann surfaces $X = (X_0, L, \gamma)$, and to do this we must explain how these operations will act on the boundary data γ. That is, we will have

$$\overline{X} = (\overline{X_0}, \overline{L}, \overline{\gamma}),\ X^{t,j} = (X_0^{t,j}, L, \gamma^{t,j}),\ X^* = (X_0^*, \overline{L}, \gamma^*),$$

where $\overline{\gamma}, \gamma^{t,j}$, and γ^* are to be defined.

Just as in the non-spin case, we define

$$\overline{\gamma}_j(z, \alpha) = \overline{\gamma_j(\overline{z}, \overline{\alpha})}.$$

We define $\gamma^{t,j}$ by

$$\gamma^{t,j}_{j'}(z,\alpha) = \begin{cases} \gamma_{j'}(z,\alpha), & j' \neq j, \\ \gamma_{j'}(z^{-1}, -iz\alpha), & j' = j \text{ and } \sigma(j) = NS, \\ \gamma_{j'}(z^{-1}, -i\alpha), & j' = j \text{ and } \sigma(j) = R, \end{cases}$$

and γ^t by taking transpose at every $j \in \pi_0(\Gamma)$. Let $\gamma^* = \overline{\gamma^t}$

Definition 3.23 (Conjugate, transpose, adjoint, and dual)**.** We define the conjugate, transpose and adjoint of $X \in \mathcal{E}^{spin}$ by

$$\overline{X} = (\overline{X_0}, \overline{L}, \overline{\gamma}),\ X^{t,j} = (X_0^{t,j}, L, \gamma^{t,j}),\ X^* = (X_0^*, \overline{L}, \gamma^*), X^d = (X_0^d, L, \gamma).$$

Remark 3.24. Note that $\overline{\gamma^t} \neq \overline{\gamma}^t$. They differ by the spin involution $(z, \alpha) \mapsto (z, -\alpha)$. We have defined γ^* as $\overline{\gamma^t}$, which is given by the formula

$$\overline{\gamma^t}(z, \alpha) = \overline{\gamma(z, \overline{iz\alpha})}.$$

If $F \in \mathcal{O}(L)$, we will write the pullback of F by γ as $F \circ \gamma$ to avoid the potentially confusing standard notation $\gamma^* F$. We have

$$\overline{F} \circ \gamma^* = -iM_{\overline{z}}^{NS} \overline{F \circ \gamma},$$

where $M_{\overline{z}}^{NS}$ acts by multiplication by $\overline{z}$ on circles with the Neveu-Schwarz spin structure, and by $\mathbf{1}$ on the complement.

Proposition 3.25. *If $X \in \mathcal{E}^{spin}$, then $\overline{X} \in \mathcal{E}^{spin}$.*

Proof. To show that $\overline{X} \in \mathcal{E}^{spin}$, we need to check that $\overline{\gamma}$ gives analytic boundary data for $\overline{L}$. We will show that the isomorphism $S^1 \times \mathbb{C} \to S^1 \times \overline{\mathbb{C}}$ given by

$$(z, \alpha) \mapsto (\overline{z}, \iota(\overline{\alpha}))$$

extends to an isomorphism of spin structures

$$(f, g) : (\mathcal{A}_\epsilon, \sigma) \to \overline{(\mathcal{A}_\epsilon, \sigma)}$$

for $\sigma \in \{NS, R\}$. In both cases, the holomorphic isomorphism of bundles (f, g) is given by $f(z) = \overline{z}$ and

$$g(z, \alpha) = (\overline{z}, \iota(\overline{\alpha})) = (\overline{z}, \alpha\iota(1)).$$

We must check that the diagrams

$$\begin{array}{ccc} \mathcal{A}_\epsilon \times \mathbb{C}^{\otimes 2} & \xrightarrow{g^{\otimes 2}} & \overline{\mathcal{A}_\epsilon} \times \overline{\mathbb{C}}^{\otimes 2} \\ \phi_\sigma \downarrow & & \overline{\phi_\sigma} \downarrow \\ K_{\mathcal{A}_\epsilon} & \xleftarrow{f^*} & K_{\overline{\mathcal{A}_\epsilon}} \end{array}$$

commute.

If k is a smooth function on $\mathcal{A}_\epsilon$, then

$$(g_*^{\otimes 2} k)(z) = \iota(\overline{k(\overline{z})}) = k(\overline{z})\iota(1).$$

Recall that $\overline{\phi_\sigma}$ is defined by the composition $c \circ \widetilde{\phi}_\sigma$, where

$$\widetilde{\phi}_\sigma : \overline{\mathcal{A}_\epsilon \times \mathbb{C}^{\otimes 2}} \to \overline{K_{\mathcal{A}_\epsilon}}$$

is the isomorphism obtained naturally from ϕ_σ, and $c : \overline{K_{\mathcal{A}_\epsilon}} \to K_{\overline{\mathcal{A}_\epsilon}}$ is the isomorphism discussed in Section 2.5. Recall that if $\overline{f(z)dz}$ is a section of $\overline{K_{\mathcal{A}_\epsilon}}$, then

$$c_* \overline{f(z)dz} = \overline{f(z)d\overline{z}},$$

where we have used the equality $K_{\overline{\mathcal{A}_\epsilon}} = \overline{(T^{0,1}\mathcal{A}_\epsilon)^*}$ on the right-hand side..

Thus

$$(\overline{\phi_{NS}})_* \iota(f(z)) = \overline{f(z)d\overline{z}}$$

and

$$(\overline{\phi_R})_* \iota(f(z)) = \overline{f(z)z^{-1}d\overline{z}}.$$

Hence

$$((f^*(\overline{\phi_{NS}})_* g_*^{\otimes 2} k = kdz = (\phi_{NS})_* k,$$

and

$$((f^*(\overline{\phi_R})_* g_*^{\otimes 2} k = kdz = (\phi_R)_* k.$$

We have shown that (f, g) gives an isomorphism of spin structures, and thus that $\overline{\gamma}$ gives analytic boundary data. □

Proposition 3.26. *If $X \in \mathcal{E}^{spin}$, then $X^{t,j}, X^* \in \mathcal{E}^{spin}$.*

Proof. We proceed in similar fashion to the proof of Proposition 3.25. We need to show that $\gamma^{t,j}$ gives analytic boundary data by showing that the isomorphism $S^1 \times \mathbb{C} \to S^1 \times \overline{\mathbb{C}}$ given by

$$(z, \alpha) \mapsto (z^{-1}, iz\alpha)$$

extends to an isomorphism of spin structures

$$(f, g) : (\mathcal{A}_\epsilon, NS) \to \overline{(\mathcal{A}_\epsilon, NS)},$$

and that

$$(z, \alpha) \mapsto (z^{-1}, i\alpha)$$

extends to an isomorphism of spin structures

$$(f, g) : (\mathcal{A}_\epsilon, R) \to \overline{(\mathcal{A}_\epsilon, R)}.$$

The proof is nearly identical to the proof of Proposition 2.35, making the necessary changes to account for z^{-1} being orientation reversing.

Combining with Proposition 3.25 shows $X^* \in \mathcal{E}^{spin}$ as well. □

Important families of rigged spin Riemann surfaces

For $\lambda \in \Lambda^0 \times \{NS\}$ and $k \in \mathbb{Z}_{>0}$, the standard spin disk $\mathrm{cl}(\mathbb{D})_{k,\lambda} \in \mathcal{E}^{spin}$ is given by

$$\mathrm{cl}(\mathbb{D})_{k,\lambda} = (\mathrm{cl}(\mathbb{D})_{k,\lambda}, (\mathrm{cl}(\mathbb{D}), NS), \mathrm{id}).$$

If $q \in \mathbb{D} \setminus \{0\}$ and $q^{1/2}$ is a square root of q, we define the annular elements ${}_{(\lambda, q^{1/2})}\mathcal{A}^k_\lambda$ of $\mathcal{E}^{spin}$ by

$${}_{(\lambda, q^{1/2})}\mathcal{A}^k_\lambda = ({}_{(\lambda,q)}\mathcal{A}^k_\lambda, \sigma(\lambda), \gamma_{q^{1/2}})$$

where $\gamma_{q^{1/2}, S^1}(z, \xi) = (z, \xi)$ and

$$\gamma_{q^{1/2}, |q|S^1}(z, \xi) = \begin{cases} (qz, q^{-1/2}\xi), & \sigma(\lambda) = NS, \\ (qz, \xi), & \sigma(\lambda) = R. \end{cases}$$

We can similarly extend the standard pairs of pants $\mathcal{P}((\lambda_1, q_1, w_1), (\lambda_2, q_2, w_2); \mu; k) \in \mathcal{E}$ to elements

$$\mathcal{P}((\lambda_1, q_1^{1/2}, w_1), (\lambda_2, q_2^{1/2}, w_2); \mu; k) \in \mathcal{E}^{spin}$$

by specifying a spin structure and boundary trivializations. While there are several spin structures on these surfaces, we will choose in this case the one induced by the 1-form dz, which is NS near each boundary component. The moduli space of standard spin pairs of pants is

$$M^{spin}_{\mathcal{P}} = \{((q_1^{1/2}, w_1, q_2^{1/2}, w_2) \in \mathbb{D}^4 : (q_1, w_1, q_2, w_2) \in M_{\mathcal{P}}\},$$

which is a 4-fold cover of $M_{\mathcal{P}}$.

3.4 Definition of spin Segal CFT

Definition 3.27. A *chiral spin Segal CFT* is given by

(1) a finite set of labels $\Lambda = \Lambda^0 \times \{NS, R\}$,

(2) a super Hilbert space $\mathcal{H}_\lambda^k$ for every $\lambda \in \Lambda$ and $k \in \mathbb{Z}_{>0}$,

(3) a strongly continuous projective positive energy spin representation $U_{k,\lambda}$ of $\mathrm{Diff}_+^{(2)}(S^1)$ by even operators on each $\mathcal{H}_\lambda^k$ with $\lambda \in \Lambda^0 \times \{NS\}$,

(4) a strongly continuous projective positive energy representation $U_{k,\lambda}$ of $\mathrm{Diff}_+(S^1)$ by even operators on each $\mathcal{H}_\lambda^k$ with $\lambda \in \Lambda^0 \times \{R\}$,

(5) for every $X \in \mathcal{E}^{spin}$, a finite-dimensional vector subspace of homogeneous maps of unordered tensor products $E(X) \subset \mathcal{I}^1\left(\mathcal{H}_{\Gamma^0}, \mathcal{H}_{\Gamma^1}\right)$.

This data must have the following properties.

(a) *(Conformal invariance)* If $(\Sigma, V, \beta, \alpha, L, \gamma) \in \mathcal{E}^{spin}$, $\phi : (\Sigma, V) \to (\Sigma', V')$ is a holomorphic isomorphism of bundles, and $\psi : (\Sigma, L) \to (\Sigma', L')$ is a holomorphic isomorphism of spin structures that agrees with ϕ on Σ, then

$$E(\Sigma, V, \beta, \alpha, L, \gamma) = E(\Sigma', V', \phi \circ \beta, \alpha \circ \phi^{-1}, L', \psi \circ \gamma).$$

(b) *(Non-degeneracy)* For every $k \in \mathbb{Z}_{>0}$ and $\lambda \in \Lambda$, we have

$$\dim E({}_{(\lambda,q)}\mathcal{A}_\lambda^k) > 0,$$

and non-zero elements of this space are injective with dense image.

(c) *(Vacuum sector)* There is a label $0 \in \Lambda^0 \times \{NS\}$ such that for all $k \in \mathbb{Z}_{>0}$ we have

$$\dim(E(\mathrm{cl}(\mathbb{D})_{k,\lambda})) = \delta_{0,\lambda}.$$

The representation $U_{k,0}$ of $\mathrm{Rot}^{(2)}(S^1)$ is required to be an honest representation. We fix a distinguished unit vector $\Omega^k \in \mathcal{H}_0^k$ in the image of elements of $E(\mathrm{cl}(\mathbb{D})_{k,0})$, and require that the generator $L_0^{k,0}$ of $U_{k,0}$ annihilate Ω^k.

(d) *(Monoidal)* $E(X_1 \sqcup X_2) = E(X_1) \hat{\otimes} E(X_2)$.

(e) *(Chirality)* For every $\lambda_1, \lambda_2, \mu \in \Lambda$ and $k \in \mathbb{Z}_{>0}$, the subbundle of the trivial bundle $M_{\mathcal{P}}^{spin} \times \mathcal{I}^2(\mathcal{H}_{\lambda_1}^k \otimes \mathcal{H}_{\lambda_2}^k, \mathcal{H}_\mu^k)$ given by restricting to $E(\mathcal{P}((\lambda_1, q_1^{1/2}, w_1), (\lambda_2, q_2^{1/2}, w_2); \mu; k))$ on fibers is a holomorphic vector bundle over $M_{\mathcal{P}}^{spin}$.

(f) *(Locality)* Suppose that (β, γ) is analytic boundary data for (Σ, V, L), $j^0 \in \pi_0(\Gamma^0)$, $j^1 \in \pi_0(\Gamma^1)$, and that we have a partial labeling $\alpha : \pi_0(\Gamma) \setminus \{j^0, j^1\} \to \Lambda$. Suppose also that L is σ near j^0 and j^1. For every $\lambda \in \Lambda^0 \times \{\sigma\}$, we can extend α to a labeling $\alpha_\lambda : \pi_0(\Gamma) \to \Lambda$ by setting $\alpha_\lambda(j^0) = \alpha_\lambda(j^1) = \lambda$. Let $X_\lambda = (\Sigma, V, \beta, \alpha_\lambda, L, \gamma)$. If $(X_\lambda, V, \beta, \alpha_\lambda, j^0, j^1, L, \gamma) \in \mathcal{E}^{spin}_*$, then we require that

$$\tau^s_{j^0 j^1} : \bigoplus_{\lambda \in \Lambda^0 \times \{\sigma\}} E(X_\lambda) \to E(\check{X})$$

and that this map is an isomorphism.

(g) *(Spin diffeomorphism covariance)* Let $X \in \mathcal{E}^{spin}$ and $(\eta, \phi) \in (\mathrm{Diff}^{(2)}_{+,\omega}(S^1))^{\pi_0(\Gamma)}$. Set $X' = (\eta, \phi) \cdot X$. Then we have

$$E(X') = \left(\bigotimes_{j \in \pi_0(\Gamma^1)} U_{k(j), \alpha(j)}(\eta_j, \phi_j) \right) E(X) \left(\bigotimes_{j \in \pi_0(\Gamma^0)} U_{k(j), \alpha(j)}(\eta_j, \phi_j)^* \right).$$

If $\lambda \in \Lambda^0 \times \{R\}$, we have implicitly extended $U_{k,\lambda}$ to a representation of $\mathrm{Diff}^{(2)}_+(S^1)$ by $U_{k,\lambda}(\eta, \phi) = U_{k,\lambda}(\phi)$.

(h) *(Adjoint preservation)* For every $X \in \mathcal{E}^{spin}$, we have $E((X^d)^*) = E(X)^*$.

As in the non-spin case, we can define an antichiral spin Segal CFT by altering Definition 3.27(e) to require the vector bundle be holomorphic over $\overline{M^{spin}_{\mathcal{P}}}$.

Chapter 4

Construction of the free fermion Segal CFTs

4.1 Overview

In this chapter, we will construct two versions of the Segal CFT corresponding to the free fermion: first a non-unitary version, and then a unitary spin version. The main tool will be the machinery of fermionic second quantization. Our construction roughly follows the outlines in [Seg04, Ch. 8] and [Pos03], and gives an accounting of the technical details required.

These theories will have the minimal number of labels: one for the non-unitary theory, and just $\{NS, R\}$ for the unitary spin theory. All of the spaces of conformal blocks will be one-dimensional, so the result is a projective monoidal functor from a category of 1-dimensional complex cobordisms to the category of Hilbert spaces and trace-class operators. While these "categories" fail to be true categories in certain respects (e.g. lacking identity morphisms, composition not being strictly associative), we will freely use categorical language for descriptions throughout this section and the reader is cautioned that such statements are to be understood modulo technicalities. As a result, we will avoid categorical language when stating theorems or definitions.

The construction will proceed by factoring the desired functor through H_{pol}, the "category" of polarized Hilbert spaces. An object in H_{pol} is a pair (H, p) where p is a projection on H. A morphism $(H, p_1) \to (K, p_2)$ is a projection q on $H \oplus K$ such that $q - ((1-p_1) \oplus p_2)$ is Hilbert-Schmidt. If q_2 is a morphism $(K, p_2) \to (L, p_3)$, the composition $q_2 \star q_1$ is the projection

$$(q_2 \star q_1)(H \oplus L) = \{(h, l) : \exists k \in K, (h, k) \in q_1(H \oplus K), (k, l) \in q_2(K \oplus L)\}.$$

If the q_i are projections onto the graphs of linear operators, then this composition coincides with composition of operators.

The reader is cautioned that H_{pol} fails to be a category, as the composition of morphisms need not be a morphism. However, a sufficient condition that this composition be a morphism in H_{pol} is given by the following lemma.

Lemma 4.1. *If $(H, p_1) \xrightarrow{q_1} (K, p_2)$ and $(K, p_2) \xrightarrow{q_2} (L, p_3)$ are morphisms in H_{pol}, then $q_2 \star q_1$ is again a morphism in H_{pol} if the following conditions are satisfied*

- *the image of $(q_2 \star q_1)(H \oplus L)$ under the projection onto H is dense, and similarly for the image of $q_2(K \oplus L)$ under the projection onto K,*
- $(\mathbf{1} - q_1) \star (\mathbf{1} - q_2) = (p_H - p_L)(\mathbf{1} - q_2 \star q_1)(p_H - p_L)$.

We call the two conditions of Lemma 4.1 the *non-degeneracy condition* and the *compatibility condition*, respectively. We outline a proof of Lemma 4.1 in Remark 4.9.

The role of the non-degeneracy and compatibility conditions can be understood most easily in the context of graphs of unbounded operators. If the q_i are graphs of unbounded operators x_i, then the non-degeneracy condition says that x_1, x_2, and $x_2 x_1$ are densely defined, and the compatibility condition says that $(x_2 x_1)^* = x_1^* x_2^*$. The unitary operator $(p_H - p_L)$ is the familiar term required to relate the orthogonal complement of the graph of an operator with the graph of the adjoint.

We now outline the construction of the non-unitary free fermion; the necessary adjustments needed to construct the unitary spin theory are discussed in Section 4.3. We construct our functor as a composition

$$\mathbb{C}\text{-bord} \xrightarrow{H^2} H_{pol} \xrightarrow{\mathcal{F}} (\text{Hilb}, \mathcal{I}^1),$$

where H^2 takes a surface to its Hardy space and $\mathcal{F}$ is fermionic second quantization. We will now discuss these functors in more detail.

First, we fix some notation. Let H^k be the Hilbert space $L^2(S^1, \mathbb{C}^k)$, with standard polarization given by the projection p_+ onto $H^2(S^1, \mathbb{C}^k)$. We denote $\mathbf{1} - p_+$ by p_-. If $X = (\Sigma, V, \beta, \alpha) \in \mathcal{E}$, the pre-quantized boundary Hilbert space is defined by

$$H_\Gamma = \bigoplus_{j \in \pi_0(\Gamma^0)} H^{k(j)} \oplus \bigoplus_{j \in \pi_0(\Gamma^1)} H^{k(j)}.$$

This space has a natural polarization, given by

$$p_\Gamma = \bigoplus_{j \in \pi_0(\Gamma^0)} p_- \oplus \bigoplus_{j \in \pi_0(\Gamma^1)} p_+.$$

We denote by H_{Γ^i} the subspace of H_Γ consisting of summands indexed by $j \in \pi_0(\Gamma^i)$, and set

$$p_{\Gamma^i} = \bigoplus_{j \in \pi_0(\Gamma^i)} p_+,$$

so that $p_\Gamma = (\mathbf{1} - p_{\Gamma^0}) \oplus p_{\Gamma^1}$.

Definition 4.2. For $X \in \mathcal{E}$, define the Hardy space $H^2(X)$ to be the closed subspace of H_Γ

$$H^2(X) = \mathrm{cl}(\{\beta^* F : F \in \mathcal{O}(V)\}).$$

We denote by q_X the projection of H_Γ onto $H^2(X)$.

Here, $\beta^* F$ is the pullback of F from a section of $V|_\Gamma$ to a section of the trivial bundle over copies of the standard circle S^1. In particular, $\beta^* F$ should not be confused with the boundary parameterization β^*.

In Theorem A.1 we will show that $p_\Gamma - q_X$ is Hilbert-Schmidt and so q_X gives a morphism from $(H_{\Gamma^0}, p_{\Gamma^0})$ to $(H_{\Gamma^1}, p_{\Gamma^1})$ in H_{pol}.

The functor $\mathcal{F}$ from H_{pol} to $(\mathrm{Hilb}, \mathcal{I}^1)$ is projective, and was described in Section 2.3. To a pair (H, p) in H_{pol} we assign the Fock space $\mathcal{F}_{H,p}$, and to a morphism q we assign the one-dimensional space of maps satisfying the $H^2(X)$-commutation relations of Lemma 2.15. Since composition isn't always defined in H_{pol}, we must be careful when discussing the functoriality of $\mathcal{F}$. It will turn out that the morphisms in H_{pol} that arise from the functor H^2 will satisfy the hypothesis of Lemma 4.1, and we will prove that $\mathcal{F}$ intertwines composition in H_{pol} and composition in Hilb when that hypothesis is satisfied.

4.2 The non-unitary free fermion

We begin by giving the data of the (non-unitary) free fermion model.

1. The set of labels is $\Lambda = \{0\}$, and will be omitted hereafter.
2. For $k \in \mathbb{Z}_{>0}$, the corresponding super Hilbert space is $\mathcal{F}_{H^k, p_+}$.
3. The strongly continuous positive energy representations U_k of $\mathrm{Rot}(S^1)$ are obtained by restricting the Ramond representation of $\mathrm{Diff}_+(S^1)$ (see Section 2.4).
4. To $X \in \mathcal{E}$ we assign the vector space of Hilbert-Schmidt maps satisfying the q_X-commutation relations.

By Theorem A.1, the spaces $E(X)$ are one-dimensional.

Remark 4.3. We will frequently write $\mathcal{F}_{\Gamma^i}$ for $\mathcal{F}_{H_{\Gamma^i}, p_{\Gamma^i}}$. The space $E(X)$ consists of maps

$$\mathcal{F}_{\Gamma^0} \to \mathcal{F}_{\Gamma^1},$$

while the axioms of a Segal CFT require a vector space of maps between unordered tensor products

$$\bigotimes_{j \in \pi_0(\Gamma^0)} \mathcal{F}_{H^{k(j)}, p_+} \to \bigotimes_{j \in \pi_0(\Gamma^1)} \mathcal{F}_{H^{k(j)}, p_+}.$$

However, we can naturally identify these spaces of maps as described in Remark 2.7.

Theorem 4.4. *The preceding data gives an antichiral Segal CFT, called the non-unitary free fermion.*

In this section, we will prove that the non-unitary free fermion satisfies every requirement of Definition 3.14 except antichirality. While it is possible to establish this fact using the tools of this section, we will instead give a proof by explicitly computing $E(X)$ for the relevant surfaces. This will be done in Section 5.1.

To establish properties of the spaces $E(X)$, we will want to know how $H^2(X)$ changes when operations are performed on X. The following proposition follows immediately from the definitions.

Proposition 4.5. *Let $X = (\Sigma, V, \beta, \alpha) \in \mathcal{E}$, and $(x, \phi) \in \mathrm{Diff}_{+,\omega}(S^1)^{\pi_0(\Gamma)}$. Let $\gamma_j = (\beta_j)|_{S^1}$ and extend γ_j to an isomorphism $S^1 \times \mathbb{C} \to j \times \mathbb{C}$. We write $H^2(\Sigma, \gamma)$ for $H^2(\tilde{X})$, where $\tilde{X} = (\Sigma, \Sigma \times \mathbb{C}, \gamma, \alpha)$.*

- $H^2(X^{t,j}) = \{(f_{j'}) \in H_\Gamma : (f \circ z^{-\delta_{jj'}}) \in H^2(X)\}$.
- $H^2(\overline{X}) = \{(f_j) \in H_\Gamma : (\overline{f_j \circ z^{-1}}) \in H^2(X)\}$.
- $H^2(X^*) = \overline{H^2(X)}$.
- $H^2((x, \phi) \cdot X) = \left(\bigoplus_{j \in \pi_0(\Gamma)} t(x_j, \phi_j)\right) H^2(X)$.
- $H^2((x \cdot X)^d) = (x^{-1})^t H^2(X^d)$
- *If V is a line bundle $H^2(X) = (\beta^* F) H^2(\Sigma, \gamma)$, where F is any non-vanishing element of $\mathcal{O}(V)$.*
- *If $Y \in \mathcal{E}$, $H^2(X \sqcup Y) = H^2(X) \oplus H^2(Y)$.*
- *If $\phi : (\Sigma, V) \to (\Sigma', V')$ is a holomorphic isomorphism of bundles, then $H^2(X) = H^2(\Sigma', V', \phi \circ \beta)$.*

Let $X_1, X_2 \in \mathcal{E}$ with $X_i = (\Sigma, V_i, \beta_i, \alpha)$.

- $H^2(X_1 \oplus X_2) = \{f_1 \oplus f_2 : f_i \in H^2(X_i)\}$.
- $H^2(X_1 \otimes X_2) = \{f_1 \otimes f_2 : f_i \in H^2(X_i)\}$.

It is clear from Proposition 4.5 that our model satisfies Definition 3.14(a), conformal invariance. The uniqueness of the vacuum sector, Definition 3.14(c), is clear. The line in $\mathcal{F}_{H^k, p_+}$ corresponding to $E(\mathbb{D})$ is the vacuum line of the Fock space, and it is fixed by $\mathrm{Rot}(S^1)$ since rotation was canonically quantized.

Rotation covariance, Definition 3.14(g), follows immediately from Propositions 4.5 and 2.17.

We now turn our attention to proving that operators $T \in E(X)$ are trace class, as well as establishing the monoidal property and locality.

Suppose that $(X, j^0, j^1) \in \mathcal{E}_*$. It is almost obvious that elements of $H^2(\check{X})$ correspond one-to-one with elements of $H^2(X)$ that agree on j^0 and j^1. That is, that

$$H^2(\check{X}) = \{(f_j)_{j \in \pi_0(\check{\Gamma})} : \exists f_{j^0} = f_{j^1} \in H^{k(j^0)} \text{ with } (f_j)_{j \in \pi_0(\Gamma)} \in H^2(X)\}.$$

Since our Hardy space was defined as *the closure* of a space of boundary values of holomorphic functions, there is still a technical point to address. This is done in Lemma A.16.

Less obviously, we have a similar relation between the orthogonal complements $H^2(\check{X})^\perp$ and $H^2(X)^\perp$. By Corollary A.2, we have

$$H^2(\check{X})^\perp = \{(f_j)_{j \in \pi_0(\check{\Gamma})} : \exists f_{j^0} = -f_{j^1} \in H^{k(j^0)} \text{ with } (f_j)_{j \in \pi_0(\Gamma)} \in H^2(X)^\perp\}.$$

The inclusion "$\supseteq$" follows immediately from the relationship between $H^2(X)$ and $H^2(\check{X})$, but the reverse inclusion is a non-trivial consequence of the geometric structure underlying the spaces $H^2(X)$. This establishes the compatibility condition of Lemma 4.1.

As we start to compute with $H^2(X)$ and $E(X)$, we will rely on the representations of $\mathrm{CAR}(H^k)$ on $\mathcal{F}_{H^k,p_+}$ and $\mathcal{F}_{\Gamma^i}$. Since $\mathrm{CAR}(H^k)$ will be acting almost exclusively on these spaces, whenever possible we will omit the notation "π_{H^k,p_+}" and simply write the action of $a(f) \in \mathrm{CAR}(H^k)$ on Fock spaces as $a(f)$.

We begin with a warmup.

Proposition 4.6. *The non-unitary free fermion has the monoidal property.*

Proof. Let $X_1, X_2 \in \mathcal{E}$, and suppose $T_i \in E(X_i)$. We must show that $T_1 \hat{\otimes} T_2$ satisfies the commutation relations for $H^2(X_1 \sqcup X_2) = H^2(X_1) \oplus H^2(X_2)$.

Suppose that $(f^0, f^1) \in H^2(X_1) \subseteq H_{\Gamma^0_{X_1}} \oplus H_{\Gamma^1_{X_1}}$, and that $(g^0, g^1) \in H^2(X_2) \subseteq H_{\Gamma^0_{X_2}} \oplus H_{\Gamma^1_{X_2}}$. By Proposition 2.6, $a(f^0, g^0)$ acts on $\mathcal{F}_{\Gamma^0_{X_1}} \otimes \mathcal{F}_{\Gamma^0_{X_2}}$ by $a(f^0) \hat{\otimes} \mathbf{1} + \mathbf{1} \hat{\otimes} a(g^0)$. We have

$$\left(a(f^1)^* \hat{\otimes} \mathbf{1} + \mathbf{1} \hat{\otimes} a(g^1)^*\right)(T_1 \hat{\otimes} T_2) = (-1)^{p(z_1)+p(z_2)}(T_1 \hat{\otimes} T_2)\left(a(f^0)^* \hat{\otimes} \mathbf{1} + \mathbf{1} \hat{\otimes} a(g^0)^*\right).$$

The computation for $H^2(X_1 \sqcup X_2)^\perp$ is similar. □

We now check that the non-unitary free fermion satisfies non-degeneracy. We actually prove a stronger result.

Lemma 4.7. *Let $X \in \mathcal{E}$ and suppose that Σ is connected and $T \in E(X) \setminus \{0\}$. Then T is injective if $\Gamma^1 \neq \emptyset$, and T has dense image if $\Gamma^0 \neq \emptyset$.*

Proof. Suppose first that $\Gamma^1 \neq \emptyset$, and assume without loss of generality that $\Gamma^0 \neq \emptyset$. By the main theorem of [Sch78], $p_{\Gamma^0} H^2(X)$ is dense in H_{Γ^0}. The commutation relations for $H^2(X)$ now imply that the kernel of T is invariant under $a(f)^*$ for $f \in p_{\Gamma^0} H^2(X)$, and thus for all $f \in H_{\Gamma^0}$. By Theorem A.1, $p_{\Gamma^0} H^2(X)^\perp$ is dense in H_{Γ^0} as well, so $\ker(T)$ is invariant under $\mathrm{CAR}(H_{\Gamma^0})$. But $\mathrm{CAR}(H_{\Gamma^0})$ acts irreducibly on $\mathcal{F}_{\Gamma^0}$, so either T is injective or $T = 0$. The same proof shows that either the image of T is dense or $T = 0$ if $\Gamma^0 \neq \emptyset$. □

Lemma 4.8. *Suppose that $X \in \mathcal{E}_*$, and that either*

(i) the elements of $E(X)$ are trace class, or

(ii) the components j^0 and j^1 lie in distinct connected components of X.

Then $\tau^s_{j^0 j^1}(E(X)) \subseteq E(\check{X})$.

Proof. Observe that either condition (i) or condition (ii) ensures that $E(X)$ lies in the domain of $\tau^s_{j^0 j^1}$. Let $T \in E(X)$, and we will show that $\tau^s_{j^0 j^1}(T)$ satisfies that commutation relations for $H^2(\check{X})$.

Fix $f = (f^0, f^1) \in H^2(\check{X}) \subseteq H_{\Gamma^0_{\check{X}}} \oplus H_{\Gamma^1_{\check{X}}}$. By Lemma A.16, there is some $g = (g^0, g^1) \in H^2(X)$ with $g_j = f_j$ when $j \notin \{j^0, j^1\}$, and $g_{j^0} = g_{j^1}$. Using the partial supertrace properties from Proposition 2.3, we have

$$\begin{aligned} a(f^1)^* \tau^s_{j^0 j^1}(T) &= \tau^s_{j^0 j^1}\left((a(f^1)^* \hat{\otimes} \mathbf{1}_{j^1})T\right) \\ &= \tau^s_{j^0 j^1}\left(\left(a(g^1)^* - \mathbf{1}_{\pi_0(\Gamma^1)\setminus j^1} \hat{\otimes} a(g_{j^1})^*\right) T\right) \\ &= (-1)^{p(T)} \tau^s_{j^0 j^1}\left(T\left(a(g^0)^* - \mathbf{1}_{\pi_0(\Gamma^0)\setminus j^0} \hat{\otimes} a(g_{j^1})^*\right)\right) \\ &= (-1)^{p(T)} \tau^s_{j^0 j^1}\left(T\left(a(f^0)^* \hat{\otimes} \mathbf{1}_{j^0}\right)\right) \\ &= (-1)^{p(T)} \tau^s_{j^0 j^1}(T) a(f^0)^*. \end{aligned}$$

The same proof establishes the corresponding relations for $(f^0, f^1) \in H^2(\check{X})^\perp$, where now we must rely on Corollary A.2 instead of Lemma A.16. □

Remark 4.9. We can now outline a proof of Lemma 4.1. Given the subspaces $W_1 = q_1(H \oplus K)$ and $W_2 = q_2(K \oplus L)$, we apply Lemma 2.15 to get non-zero operators satisfying the W_1 and W_2 commutation relations. The proof of Lemma 4.7 shows that these operators are injective (using the non-degeneracy condition), and the proof of Lemma 4.8 shows that the composition satisfies the commutation relations for $(q_2 \star q_1)(H \oplus L)$ (using the compatibility condition). Since these maps were injective, the composition is non-zero and by Lemma 2.15 $(q_2 \star q_1)(H \oplus L)$ must be a morphism, as required.

Lemma 4.10. *For all $X \in \mathcal{E}$, the elements of $E(X)$ are trace class. If $X \in \mathcal{E}_*$, then $\tau^s_{j^0 j^1} : E(X) \to E(\check{X})$ is an isomorphism.*

Proof. First we show that $\tau^s_{j^0 j^1} : E(X) \to E(\check{X})$ is an isomorphism under the assumption that either elements of $E(X)$ are trace class, or j^0 and j^1 lie in separate connected components. We will then use this to show that the elements of $E(X)$ are always trace class, which will complete the proof of the lemma. Since $E(X)$ and $E(\check{X})$ are one-dimensional, it suffices to show that $\tau^s_{j^0 j^1}$ acts injectively.

Suppose $T \in E(\check{X})$ and $\tau^s_{j^0 j^1}(T) = 0$. Since $X \in \mathcal{E}_*$, at least one of j^0 and j^1 lies on a connected component of Σ that has a free boundary component (i.e. a boundary component that is not j^0 or j^1). Assume without loss of generality that this is j^1, and the proof will be similar in the other case.

Fix $f \in L^2(S^1, \mathbb{C}^{k(j^0)})$. By [Sch78], there exists a sequence $f^{(n)} \in H^2(X)$ with $f^{(n)}_{j^0} \to 0$ and $f^{(n)}_{j^1} \to -f$. Let $(f^{(n),0}, f^{(n),1})$ be the decomposition of $f^{(n)}$ with respect to $H_\Gamma = H_{\Gamma^0} \oplus H_{\Gamma^1}$, and let $g^{(n)} \in H_{\check{\Gamma}}$ be the restriction of $f^{(n)}$. We then have

$$\begin{aligned}
0 &= a(g^{(n),1})^* \tau^s_{j^0 j^1}(T) \\
&= \tau^s_{j^0 j^1}\left(\left(a(g^{(n),1})^* \hat{\otimes} \mathbf{1}_{j^1}\right) T\right) \\
&= \tau^s_{j^0 j^1}\left(\left(a(f^{(n),1})^* - \mathbf{1} \hat{\otimes} a(f^{(n)}_{j^1})^*\right) T\right) \\
&= (-1)^{p(T)} \tau^s_{j^0 j^1}\left(T\left(a(f^{(n),0})^*\right)\right) - \tau^s_{j^0 j^1}\left(\left(\mathbf{1} \hat{\otimes} a(f^{(n)}_{j^1})^*\right) T\right) \\
&= (-1)^{p(T)} \tau^s_{j^0 j^1}\left(T\left(\mathbf{1} \otimes a(f^{(n)}_{j^0})^*\right)\right) - \tau^s_{j^0 j^1}\left(\left(\mathbf{1} \hat{\otimes} a(f^{(n)}_{j^1})^*\right) T\right).
\end{aligned}$$

This last term converges to $\tau^s_{j^0 j^1}\left(\left(\mathbf{1} \hat{\otimes} a(f)^*\right) T\right)$ as $n \to \infty$, and thus

$$\tau^s_{j^0 j^1}\left(\left(\mathbf{1} \hat{\otimes} a(f)^*\right) T\right) = 0.$$

Using the formula for $H^2(X)^\perp$ from Theorem A.1, we can apply the same argument using a sequence $f^{(n)} \in H^2(X)^\perp$. We can also iterate the argument to get that

$$\tau^s_{j^0 j^1}\left(\left(\mathbf{1} \hat{\otimes} x\right) T\right) = 0 \tag{4.1}$$

whenever x is a word in $a(f)^*$ and $a(g)$. But such words span a dense subspace of $\mathcal{B}(H^{k(j^1)})$, so equation (4.1) holds for arbitrary x. But this means $T = 0$, as desired.

We can now show that the elements of $E(X)$ are always trace class. If Σ has no outgoing boundary components, then the operators have rank one. Otherwise, using the analytic boundary data we may cut off a standard annulus from each outgoing boundary component. By the above result, gluing each of these annuli back on gives an isomorphism to $E(X)$. By Proposition 2.5, this isomorphism corresponds to composition of operators. Since the operators being composed are Hilbert-Schmidt, the composition is trace class. □

We have shown that the non-unitary free fermion satisfies all of the requirements of a Segal CFT, except for antichirality. We will establish this fact and complete the proof of Theorem 4.4 in Section 5.1.

4.3 The unitary free fermion

The most obvious way that the Segal CFT constructed in Section 4.2 fails to be unitary is that the operators implementing boundary reparameterization are not unitary, and in fact generally not even bounded. In this section, we will refer to the non-unitary theory as E^{nu}, and construct a related unitary antichiral spin CFT, E^s.

The data of the unitary free fermion is the following

1. The set of labels is $\{0\} \times \{NS, R\}$, which we will think of as $\{NS, R\}$.
2. For $k \in \mathbb{Z}_{>0}$, the Ramond and Neveu-Schwarz super Hilbert spaces are both $\mathcal{F}_{H^k,p_+}$.
3. The positive energy spin representation $U_{k,NS} : \mathrm{Diff}^{(2)}_{+,\omega}(S^1) \to \mathcal{PU}(\mathcal{F}_{H^k,p_+})$ is U_{NS} (see Section 2.4).
4. The positive energy representation $U_{k,R} : \mathrm{Diff}_{+,\omega}(S^1) \to \mathcal{PU}(\mathcal{F}_{H^k,p_+})$ is U_R.
5. If $X = (\Sigma, V, \beta, \alpha, L, \gamma) \in \mathcal{E}^{spin}$, then $E^s(X)$ is given by

$$E^s(X) = E^{nu}(\Sigma, V \otimes L, \beta \otimes \gamma, \alpha).$$

Conformal invariance, uniqueness of the vacuum, the monoidal property, and locality are inherited from the non-unitary theory. Non-degeneracy and rotation invariance of the vacuum are proven in exactly the same way. Diffeomorphism covariance follows from Proposition 2.17, along with the following observation.

Proposition 4.11.

$$H^2((\eta, \phi) \cdot X) = \left(\bigoplus_{j \in \pi_0(\Gamma)} u_{\alpha(j)}(\eta_j, \phi_j) \right) H^2(X).$$

As in the non-unitary case, we will check antichirality by explicit computation in Section 5.1. All that remains to check is adjoint preservation.

Theorem 4.12. *The unitary free fermion model preserves adjoints.*

Proof. From the definitions of the $H^2(X)$ vacuum equations, it is clear that T satisfies the $H^2(X)$ vacuum equations if and only if T^* satisfies the $M_{\pm\mathbf{1}} H^2(X)^\perp$ vacuum equations, where $M_{\pm\mathbf{1}}$ acts by $\mathbf{1}$ on outgoing boundary components, and $-\mathbf{1}$ on incoming boundary components. Thus we must show that if $X = (X_0, L, \gamma) \in \mathcal{E}^{spin}$, then

$$M_{\pm\mathbf{1}} H^2(\Sigma^*, \overline{V^*} \otimes \overline{L}, (\beta^d)^* \otimes \gamma^*) = H^2(\Sigma, V \otimes L, \beta \otimes \gamma)^\perp. \tag{4.2}$$

Note that β^* refers to the adjoint of trivializations of bundles, while γ^* refers to the adjoint of trivializations of spin structures. From the definition of γ^* and Remark 3.24 we have that if $F \in \mathcal{O}(L)$ and $F \circ \gamma = f \in H^2(L, \gamma)$, then

$$\overline{F} \circ \gamma^* = M^{NS}_{\overline{z}} \overline{if},$$

where $M^{NS}_{\overline{z}}$ is multiplication by $\overline{z}$ on boundary components of type NS. Fix a non-vanishing section $F \in \mathcal{O}(L)$.

By conformal invariance, we may assume that V is a trivial bundle. By Proposition 4.5, equation (4.2) is invariant under fiberwise reparameterizations of β. Thus we may assume that β acts trivially on fibers, and since equation (4.2) is invariant under direct sums in (V, β) we may assume that $V = \Sigma \times \mathbb{C}$.

Applying Proposition 4.5 and Theorem A.1 under these assumptions, equation (4.2) is equivalent to

$$\overline{M_z^{NS} i f H^2(\Sigma, \gamma|_{S^1})} = \overline{f^{-1} r M_z H^2(\Sigma, \gamma|_{S^1})},$$

where we are free to pick r as the boundary value of any non-vanishing section of K_Σ. Here, the boundary value is taken with respect to the boundary parameterization of K_Σ induced by $\gamma|_{S^1}$. Such a section is furnished by $i\phi_L(F \otimes F)$, which gives $r = iM_{\overline{z}}^R f^2$. The above identity now holds. $\square$

Chapter 5

Applications

5.1 Vertex operator algebras

Segal's definition given in Chapter 3 is not the only axiomatization of a chiral conformal field theory. In [Bor86], Richard Borcherds introduced the concept of a vertex algebra, en route to proving the monstrous moonshine conjectures of Conway and Norton. A modified version of the definition appears in [FLM88], under the name vertex operator algebra (VOA). Vertex operator algebras are a mathematical formulation of the physical notion of chiral algebras, and they describe conformally invariant chiral quantum fields.

Vertex operator algebras are an algebraic notion, and we will replace $\mathrm{Diff}_+(S^1)$ by its (complexified, centrally extended) Lie algebra Vir.

Definition 5.1. The *Virasoro algebra* Vir is the Lie algebra spanned by $\{L_n\}_{n\in\mathbb{Z}}$ and a central element c, with commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}c.$$

A representation of Vir on a pre-Hilbert space V is called *unitary* if $\langle L_n\xi, \eta\rangle = \langle \xi, L_{-n}\eta\rangle$ for all $\xi, \eta \in V$.

For more information on the Virasoro algebra, the reader is encouraged to consult [KR87].

Our definition of a vertex operator algebra is a combination of the ones given in [FZ92] and [Was11]. It is not minimal, and includes some requirements of a unitary vertex operator algebra. Another valuable resource on vertex operator algebras is [Kac98].

Definition 5.2. A *vertex operator (super)algebra* is given by the following data

- A $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded super pre-Hilbert space V, and an even unit vector $\Omega \in V$,
- a state-field correspondence $Y : V \to \mathrm{End}(V)[[z^{\pm 1}]]$, which we write $Y(\xi, w) = \sum_{m\in\mathbb{Z}} \xi_m z^{-m-1}$ with $\xi_m \in \mathrm{End}(V)$,

- a unitary representation of Vir on V by even operators.

The data is required to satisfy the following axioms.

- Ω spans $V(0)$, and $Y(\Omega, w) = \mathbf{1}$ (i.e. $\Omega_m = \delta_{m+1,0}\mathbf{1}$).
- $\xi_m\Omega = 0$ for $m \geq 0$ and $\xi_{-1}\Omega = \xi$.
- $[L_{-1}, Y(\xi, w)] = \frac{d}{dw}Y(\xi, w)$, where the derivative is interpreted formally.
- The grading on V is given by L_0, and $Y(L_{-2}\Omega, w) = \sum_{m\in\mathbb{Z}} L_m w^{-m-2}$.
- The eigenspaces of L_0 are finite-dimensional.
- If $L_0\xi = \alpha\xi$, then $[L_0, \xi_m] = (\alpha - m - 1)\xi_m$.
- $\langle Y(\xi_1, w_1)Y(\xi_2, w_2)\eta_1, \eta_2\rangle$ is a polynomial in $w_i^{\pm 1}$ and $(w_1 - w_2)^{-1}$ for all $\xi_i, \eta_i \in V$, and the order of the pole at $w_1 - w_2$ is bounded independently of η_1 and η_2.
- $\langle Y(\xi_1, w_1)Y(\xi_2, w_2)\eta_1\eta_2\rangle = (-1)^{p(\xi_1)p(\xi_2)}\langle Y(\xi_2, w_2)Y(\xi_1, w_1)\eta_1\eta_2\rangle$ as a rational function of w_1 and w_2.

We will not work with vertex operator algebras in general; we will be interested in the VOA corresponding to the free fermion model (and later, the WZW models for $SU(N)_\ell$ - see Section 5.2). We will describe these VOAs by explicitly giving the fields $Y(\xi, w)$ corresponding to certain generating vectors ξ. We can then deduce fields corresponding to other vectors from the following theorem.

Theorem 5.3 (Borcherds product formula)**.**

$$(\eta_n\xi)_m = \sum_{j\geq 0}(-1)^j\binom{n}{j}\left(\eta_{n-j}\xi_{m+j} - (-1)^{p(\eta)p(\xi)+n}\xi_{m+n-j}\eta_j\right).$$

The following relation will also be useful.

Theorem 5.4 (Borcherds commutator formula)**.**

$$[\xi_m, Y(\eta, w)] = \sum_{j=0}^{\infty} w^{m-j}\binom{m}{j}Y(\xi_j\eta, w).$$

The free fermion vertex operator superalgebra

The state space for the free fermion VOA is the subspace of finite energy vectors in $\mathcal{F}_{H,p_+}$, where $H = L^2(S^1, \mathbb{C}^k)$ and p_+ is the projection onto the Hardy space. The finite energy vectors are those that lie in the algebraic direct of eigenspaces for L_0^{NS},

$$\mathcal{F}^0_{H,p_+} = \sum_{d \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \mathcal{F}_{H,p_+}(d).$$

We will generally refer to this space simply as $\mathcal{F}^0$.

The generating fields are, for $v \in \mathbb{C}^k$,

$$Y(a(v)\Omega, w) = \sum_{m \in \mathbb{Z}} a(z^{-m-1}v)w^{-m-1}$$

and

$$Y(a(z^{-1}v)^*\Omega, w) = \sum_{m \in \mathbb{Z}} a(z^m v)^* w^{-m-1}.$$

The representation of the Virasoro algebra corresponds to the representation U_{NS}, and in particular $L_0 = L_0^{NS}$. The existence of the free fermion VOA is covered in more detail in most references on VOAs, e.g. [Kac98, Ch. 5] or [Was11].

We may now use the Borcherds product formula to give explicit formulas for higher energy vectors. For example, if $\xi = a(v)a(z^{-1}w)^*\Omega$, then

$$\begin{aligned}\xi_m &= \sum_{j \geq 0} a(z^j v)a(z^{j+m}w)^* - \sum_{j \geq 0} a(z^{m-j-1}w)^* a(z^{-j-1}v) \\ &= \sum_{j \geq 0} a(z^{j-m} v)a(z^j w)^* - \sum_{j \geq 0} a(z^{-j-1}w)^* a(z^{-j-1-m}v).\end{aligned} \tag{5.1}$$

While ξ_m is not bounded as an operator on $\mathcal{F}$, it gives as well-defined endomorphism of $\mathcal{F}^0$ as all but finitely many terms vanish when ξ_m is applied to a finite energy vector.

Connection between free fermion Segal CFT and VOA

In Chapter 4, we constructed two examples of Segal CFTs. The maps assigned to Riemann surfaces were characterized by commutation relations derived from the Hardy space of the surface. In this section, we will make the theory more concrete by explicitly describing the action of the important families of surfaces described in Sections 3.1 and Section 3.3. In particular, we will identify the operators assigned to the standard spin pairs of pants with the operators of the free fermion vertex operator algebra.

For ease of computation, we will assume that all vector bundles are trivial. By Theorem 2.30, we do not lose any generality by this assumption. To simplify notation, we will continue to regard $k \in \mathbb{Z}_{>0}$ as fixed and write $\mathcal{F}$ for the Hilbert space $\mathcal{F}_{L^2(S^1,\mathbb{C}^k),p_+}$.

Recall from Section 3.1 that we defined $\mathrm{cl}(\mathbb{D})_{0,k} \in \mathcal{E}$ by

$$\mathrm{cl}(\mathbb{D})_{0,k} = (\mathrm{cl}(\mathbb{D}), \mathrm{cl}(\mathbb{D}) \times \mathbb{C}^k, \mathrm{id}, \alpha_0),$$

where 0 is the vacuum label. By construction, we have

$$E^{nu}(\mathrm{cl}(\mathbb{D})_{0,k}) = \mathbb{C}\Omega.$$

In the spin case, $\mathrm{cl}(\mathbb{D})_{0,k}$ is given the standard Neveu-Schwarz spin structure, and we again have

$$E^s(\mathrm{cl}(\mathbb{D})_{0,k}) = \mathbb{C}\Omega.$$

The next simplest surface is the standard annulus. As with the disk, we will simplify the notation of Sections 3.1 and 3.3.

For $q \in \mathrm{cl}(\mathbb{D}) \setminus \{0\}$, let

$$\mathcal{A}_q = \{z \in \mathbb{C} : |q| \leq z \leq 1\}.$$

We will think of $\mathcal{A}_q$ as an element of $\mathcal{E}$ by making the circle of radius $|q|$ incoming, the circle of radius 1 outgoing, giving it the trivial $\mathbb{C}^k$-bundle, and parameterizing as in Section 3.1.

We define $\mathcal{A}^{NS}_{q,q^{1/2}}, \mathcal{A}^R_q \in \mathcal{E}^{spin}$ by giving $\mathcal{A}_q$ the standard NS or R spin structure, and trivializing the boundary as described in Section 3.3.

We have

$$H^2(\mathcal{A}_q) = H^2(\mathcal{A}^R_q) = \mathrm{cl}(\mathrm{span}\{(q^n z^n v, z^n v) : n \in \mathbb{Z}, v \in \mathbb{C}^k\}),$$

and

$$H^2(\mathcal{A}^{NS}_{q,q^{1/2}}) = \mathrm{cl}(\mathrm{span}\{(q^{n+\frac{1}{2}} z^n v, z^n v) : n \in \mathbb{Z}, v \in \mathbb{C}^k\}).$$

The orthogonal complements are given by

$$H^2(\mathcal{A}_q)^\perp = \mathrm{cl}(\mathrm{span}\{(-\overline{q}^{-n} z^n v, z^n v) : n \in \mathbb{Z}, v \in \mathbb{C}^k\}),$$

and

$$H^2(\mathcal{A}^{NS}_{q,q^{1/2}})^\perp = \mathrm{cl}(\mathrm{span}\{(-\overline{q}^{-n-\frac{1}{2}} z^n v, z^n v) : n \in \mathbb{Z}, v \in \mathbb{C}^k\}).$$

Proposition 5.5.

- $E^{nu}(\mathcal{A}_q) = E^s(\mathcal{A}^R_q) = \mathbb{C}\overline{q}^{L_0^R}$
- $E^s(\mathcal{A}^{NS}_{q,q^{1/2}}) = \mathbb{C}\overline{q}^{L_0^{NS}}$

Proof. It suffices to show that $\overline{q}^{L_0^R}$ and $\overline{q}^{L_0^{NS}}$ are Hilbert-Schmidt and satisfy the commutation relations for $H^2(\mathcal{A}_q)$ and $H^2(\mathcal{A}^{NS}_{q,q^{1/2}})$, and this is easily checked. □

To simplify notation, we will write $\mathcal{P}_{w,q_1,q_2}$ for the element

$$\mathcal{P}((0,q_1,w),(0,q_2,0);0;k) \in \mathcal{E}$$

defined in Section 3.1. The underlying surface of $\mathcal{P}_{w,q_2,q_2}$ is the pair of pants

$$\{z \in \mathbb{C} : |q_2| \leq |z| \leq 1, |z-w| \geq |q_1|\},$$

and we only define $\mathcal{P}_{w,q_2,q_2}$ when this surface is non-degenerate.

We define $\mathcal{P}^{NS}_{w,q_1^{1/2},q_2^{1/2}} \in \mathcal{E}^{spin}$ by giving it the standard Neveu-Schwarz spin structure inherited from the unit disk, and parameterizations as described in Section 3.3.

The statement of the following theorem was told to us by Antony Wassermann.

Theorem 5.6. *Let $T \in E^s(\mathcal{P}^{NS}_{w,q_1^{1/2},q_2^{1/2}})$, and $\xi \in \mathcal{F}^0$. If the free fermion vertex operator is given by*

$$Y(\xi,w) = \sum_{n\in\mathbb{Z}} \xi_n w^{-n-1},$$

then $\xi_n \overline{q_2}^{L_0^{NS}}$ is a bounded operator and

$$T(\xi \otimes \cdot) = \alpha \sum_{n\in\mathbb{Z}} (\overline{q_1}^{L_0^{NS}}\xi)_n \overline{q_2}^{L_0^{NS}} \overline{w}^{-n-1}$$

for some $\alpha \in \mathbb{C}$, with the sum converging uniformly in norm on compact subsets of $\{w : |q_2| < w < 1\}$.

Proof. It suffices to prove the theorem for

$$\xi = a(z^{n_p}v_p)^* \cdots a(z^{n_1}v_1)a(z^{m_1}v_1') \cdots a(z^{m_q}v_q')\Omega,$$

where $v_i, v_i' \in \mathbb{C}^k$, $n_i \in \mathbb{Z}_{<0}$ and $m_i \in \mathbb{Z}_{\geq 0}$. We proceed by induction on $p+q$.

When $\xi = \Omega$, $Y(\xi,w) = \mathbf{1}$ and

$$T(\Omega \otimes \cdot) \in \mathbb{C}\overline{q_2}^{L_0^NS}$$

by locality. Hence $T(\Omega \otimes \Omega) \in \mathbb{C}\Omega$, and since T is homogeneous, it is even. We normalize T so that $T(\Omega \otimes \Omega) = \Omega$, and establish the theorem with $\alpha = 1$.

Now assume that the conclusion holds for ξ, and we will show that it holds for $a(z^n v)^*\xi$ and $a(z^m v)\xi$. From the holomorphic function $(z-w)^n v \in \mathcal{O}(\mathcal{P}_{w,q_1,q_2})$, we have

$$(q_1^{n+\frac{1}{2}} z^n v, q_2^{\frac{1}{2}}(q_2 z - w)^n v, (z-w)^n v) \in H^2(\mathcal{P}^{NS}_{w,q_1,q_2}).$$

Thus T satisfies the commutation relation

$$T(a(q_1^{n+\frac{1}{2}} z^n v)^* \hat{\otimes} \mathbf{1}) = a((z-w)^n v)^* T - T(\mathbf{1} \hat{\otimes} a(q_2^{1/2}(q_2 z - w)^n v)^*).$$

We can expand $(z-w)^n$ and $(q_2z-w)^n$ in power series converging uniformly in norm on compact subsets of $|q_2|<|w|<1$, and by the inductive hypothesis we can do the same with $T(\xi\otimes\cdot)$, so we have

$$\begin{aligned}
T(\overline{q_1}^{-L_0^{NS}}a(z^nv)^*\xi\otimes\cdot) &= T(a(q_1^{n+1/2}z^nv)^*\overline{q_1}^{-L_0^{NS}}\xi\otimes\cdot)\\
&= a((z-w)^nv)^*T({q_1}^{-L_0^{NS}}\xi\otimes\cdot)-\\
&\quad -T((-1)^{p(\xi)}{q_1}^{-L_0^{NS}}\xi\otimes a(q_2^{\frac{1}{2}}(q_2z-w)^nv)^*\cdot)\\
&= \sum_{j\ge 0}(-1)^j\binom{n}{j}a(z^{n-j}v)^*T(\overline{q_1}^{-L_0^{NS}}\xi\otimes\cdot)\overline{w}^j-\\
&\quad -(-1)^{p(\xi)}\sum_{j\ge 0}(-1)^j\binom{n}{j}(-1)^nT({q_1}^{-L_0^{NS}}\xi\otimes a(z^jv)^*\cdot)\overline{q_2}^{j+\frac{1}{2}}\overline{w}^{n-j}\\
&= \sum_{j\ge 0}\sum_{m\in\mathbb{Z}}(-1)^j\binom{n}{j}a(z^{n-j}v)^*\xi_m\overline{q_2}^{L_0^{NS}}\overline{w}^{j-m-1}-\\
&\quad -(-1)^{p(\xi)}\sum_{j\ge 0}\sum_{m\in\mathbb{Z}}(-1)^j\binom{n}{j}(-1)^n\xi_m a(z^jv)^*\overline{q_2}^{L_0^{NS}}\overline{w}^{n-j-m-1}\\
&= \sum_{m\in\mathbb{Z}}\tilde{\xi}_m\overline{q_2}^{L_0^{NS}}\overline{w}^{-m-1},
\end{aligned}$$

where

$$\tilde{\xi}_m=\sum_{j\ge 0}(-1)^j\binom{n}{j}\left(a(z^{n-j}v)^*\xi_{m+j}-(-1)^{p(\xi)+n}\xi_{m+n-j}a(z^jv)^*\right).$$

Note that $\tilde{\xi}_m\overline{q_2}^{L_0^{NS}}$ is bounded, $\sum_{m\in\mathbb{Z}}\tilde{\xi}_m q_2^{L_0^{NS}}\overline{w}^{-m-1}$ converges uniformly on compact subsets of $|q_2|<|w|<1$, and $\tilde{\xi}_m\in\operatorname{End}(\mathcal{F}^0)$. In fact,

$$\tilde{\xi}_m=(a(z^nv)^*\xi)_m$$

by Theorem 5.3, which completes the first case of the induction.

We must establish an analogous result for $T(\overline{q_1}^{-L_0^{NS}}a(z^{-n-1}v)\xi\otimes\cdot)$. We have

$$(-\overline{q_1}^{n+\frac{1}{2}}z^{-n-1}v,-\overline{q_2}^{\frac{1}{2}}z^{-1}(\overline{q_2}z^{-1}-\overline{w})^nv,z^{-1}(z^{-1}-\overline{w})^nv)\in H^2(\mathcal{P}^{NS}_{w,q_1,q_2})^\perp.$$

By the same argument as before, we have

$$\begin{aligned}
T(\overline{q_1}^{-L_0^{NS}}a(z^{-n-1}v)\xi\otimes\cdot) &= \sum_{j\ge 0}\sum_{m\in\mathbb{Z}}(-1)^j\binom{n}{j}a(z^{-n-1-j}v)\xi_m\overline{q_2}^{L_0^{NS}}\overline{w}^{j-m-1}-\\
&\quad -(-1)^{p(\xi)}\sum_{j\ge 0}\sum_{m\in\mathbb{Z}}(-1)^j\binom{n}{j}(-1)^n\xi_m a(z^{-j-1}v)\overline{q_2}^{L_0^{NS}}\overline{w}^{n-j-m-1}\\
&= \sum_{m\in\mathbb{Z}}\tilde{\xi}_m\overline{q_2}^{L_0^{NS}}\overline{w}^{-m-1},
\end{aligned}$$

where

$$\tilde{\xi}_m = \sum_{j\geq 0}(-1)^j\binom{n}{j}\left(a(z^{-n-1-j}v)\xi_{m+j} - (-1)^{p(\xi)+n}\xi_{m+n-j}a(z^{-j-1}v)^*\right).$$

We have the same uniform convergence as before, and by Theorem 5.3, $\tilde{\xi}_m = (a(z^{-n-1}v)\xi)_m$. □

We write the conclusion of Theorem 5.6 as $T(\eta\otimes\xi) = Y(\overline{q_1}^{L_0^{NS}}\eta, \overline{w})\overline{q_2}^{L_0^{NS}}\xi$. The following theorem may be established by the same method.

Theorem 5.7. *Let $T \in E^{nu}(\mathcal{P}_{w,q_1,q_2})$, and $\xi \in \mathcal{F}^0$. If the free fermion vertex operator is given by*

$$Y(\xi, w) = \sum_{n\in\mathbb{Z}} \xi_n w^{-n-1},$$

then $\xi_n\overline{q_2}^{L_0^R}$ is a bounded operator and

$$T(\xi\otimes\cdot) = \alpha\sum_{n\in\mathbb{Z}}(\overline{q_1}^{L_0^R}\xi)_n\overline{q_2}^{L_0^R}\overline{w}^{-n-1}$$

for some $\alpha\in\mathbb{C}$, with the sum converging uniformly in norm on compact subsets of $\{w : |q_1| < w < 1\}$.

Corollary 5.8. *The non-unitary free fermion and the unitary free fermion satisfy antichirality.*

This completes the proof that E^s and E^{nu} give antichiral Segal CFTs.

5.2 WZW models

One of our primary motivations for studying the free fermion is to study the $SU(k)$ WZW subtheory. We will briefly introduce the representation theory of the loop group $LSU(k)$ on fermionic Fock space, following [Was98, Sec. 4-9]. The reader is encouraged to consult that reference for proofs of the following statements.

Let $\mathcal{F}_{k\ell}$ be the fermionic Fock space for $H^{k\ell} = L^2(S^1, \mathbb{C}^{k\ell})$. We represent the loop group $LSU(k) = C^\infty(S^1, SU(k))$ on $H^{k\ell}$ by pointwise multiplication by $f(z)\otimes\mathbf{1}$. When $LSU(k)$ is given the appropriate topology, this gives a continuous embedding of $LSU(k)$ into $\mathcal{U}_{res}(H^{k\ell}, p_+)$, and consequently a strongly continuous projective representation π_ℓ of $LSU(k)$ on $\mathcal{F}_{k\ell}$.

Let $L^0\mathfrak{gl}(k\ell)$ be the Lie subalgebra of $C^\infty(S^1, \mathfrak{gl}(k\ell))$ spanned by

$$x_m := z^{-m}x$$

for $x \in \mathfrak{gl}(k\ell)$. We have a representation $\pi_\ell : x_m \mapsto x(m)$ of $L^0\mathfrak{gl}(k\ell)$ on $\mathcal{F}^0_{k\ell}$ given by

$$e_{pq}(m) = \sum_{j>0} a(z^{j-m}v_p)a(z^j v_q)^* - \sum_{j\geq 0} a(z^{-j}v_q)^* a(z^{-j-m}v_p) \tag{5.2}$$

where $1 \leq p, q \leq k\ell$, $\{v_p\}$ is the standard basis for $\mathbb{C}^{k\ell}$, and e_{pq} is an elementary matrix.

Theorem 5.9 ([Was98, Sec. 8])**.** *For $y \in L^0\mathfrak{su}(k\ell)$, $\pi_\ell(y)$ is essentially skew-adjoint and the unitaries* $\exp(\pi_\ell(y))$ *and* $\pi_\ell(\exp(y))$ *agree up to a scalar.*

Since products of terms $\exp(y)$ are dense in $LSU(k\ell)$, if $V \subseteq \mathcal{F}_{k\ell}$ is a subspace invariant under $L^0\mathfrak{sl}(k\ell)$ then $\mathrm{cl}(V)$ is invariant under $LSU(k\ell)$. Similarly if V is invariant under $L^0\mathfrak{sl}(k)$, then $\mathrm{cl}(V)$ is invariant under $LSU(k)$.

Lemma 5.10. *For every $x \in \mathfrak{gl}(k\ell)$, there is a $\xi \in \mathcal{F}^0_{k\ell}$ such that*

$$Y(\xi, w) = \sum_{m\in\mathbb{Z}} \pi(x_m)w^{-m-1}.$$

Proof. It suffices to prove this for $x = e_{pq}$, and in that case comparing equations (5.2) and (5.1) shows that we may take $\xi = a(v_p)a(z^{-1}v_q)^*\Omega$. □

Let $\mathcal{H}_0$ be the irreducible $LSU(k)$ representation generated by Ω, and let $\mathcal{H}^0_0$ be the subspace of finite energy vectors, which carry an irreducible representation of $L^0\mathfrak{sl}(k)$.

Lemma 5.11. *If $V \subseteq \mathcal{F}^0_{k\ell}$ is invariant under $L^0\mathfrak{sl}(k)$, then V is also invariant under ξ_m for every $\xi \in \mathcal{H}^0_0$.*

Proof. It suffices to prove the theorem for $\xi = x_1(m_1)\cdots x_s(m_s)\Omega$. Since $\sum x_i(m)w^{-m-1}$ is a field in the free fermion VOA, the modes of $Y(\xi, w)$ are (infinite) sums of monomials in terms $x(m)$ by Theorem 5.3. □

In particular, $\mathcal{H}^0_0$ is invariant under $Y(\xi, w)$ for every $\xi \in \mathcal{H}^0_0$. We thus have the state-field correspondence for a VOA based on the pre-Hilbert space $\mathcal{H}^0_0$. To completely specify this VOA, we need to describe the operators L_0 and L_{-1}.

The representation π is completely reducible, and we can write

$$\mathcal{F}_{k\ell} = \bigoplus_{\lambda\in\Lambda} \mathcal{H}_\lambda \otimes M_\lambda,$$

with respect to which π decomposes

$$\pi = \bigoplus_{\lambda\in\Lambda} \pi_\lambda \otimes \mathbf{1}.$$

The finite set Λ indexes the irreducible positive energy representations of $LSU(k)$ of level ℓ, and is given by

$$\Lambda = \{(\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k/\mathbb{Z}(1, \ldots, 1) : \lambda_k - \lambda_1 \leq \ell\}.$$

Since $LSU(k)$ acts by even operators, the Hilbert spaces $\mathcal{H}_\lambda$ are even.

By the coset construction, U_{NS} decomposes as projective representations

$$U_{NS}^{\otimes \ell} = \bigoplus_{\lambda \in \Lambda} U_\lambda \otimes V_\lambda,$$

and we can write

$$L_n^{NS} = \bigoplus_{\lambda \in \Lambda} L_n^\lambda \otimes \mathbf{1} + \mathbf{1} \otimes L_n'^\lambda.$$

Since the action of the Möbius group on $\mathcal{F}_{k\ell}$ is canonically quantized, we have $(L_n^0 \otimes \mathbf{1})\Omega = 0$ for $|n| \leq 1$.

Each U_λ and V_λ is a projective positive energy representation of $\mathrm{Diff}_+(S^1)$, and so $\mathcal{H}_\lambda$ has a grading by the spectrum of L_0^λ, called the *energy grading.* If h_λ is the smallest eigenvalue of L_0^λ, then $H_\lambda(h_\lambda)$ is the irreducible representation of $SU(k)$ corresponding to λ (see e.g. [Was98, Sec. 2]).

If ξ is an eigenvector of L_0^λ, then we will write $\mathrm{wt}(\xi)$ for the eigenvalue. We define $\mathrm{wt}(\eta)$ similarly when η is an eigenvector for $L_0'^\lambda$, and $\mathrm{wt}(\xi \otimes \eta) = \mathrm{wt}(\xi) + \mathrm{wt}(\eta)$. Expressions involving $\mathrm{wt}(\xi)$ or $\mathrm{wt}(\eta)$ are to be understood as holding when ξ and η are eigenvectors, and extended linearly otherwise.

Lemma 5.12. *The state-field correspondence $\xi \mapsto Y(\xi, w)$, along with L_0^0 and L_{-1}^0, make $\mathcal{H}_0^0$ into a vertex operator algebra, and this agrees with the standard VOA structure on $\mathcal{H}_0^0$ defined in [Bor86].*

Proof. By Lemma 5.10, we have

$$Y(x(-1)\Omega, w) = \sum_{m \in \mathbb{Z}} x(m) w^{-m-1}, \tag{5.3}$$

which agrees with the standard VOA structure on $\mathcal{H}_0^0$. Since every field can be constructed from linear combinations of Borcherds products of the fields (5.3), $Y(\xi, w)$ agrees with the corresponding field from [Bor86] for all ξ.

Let L_{-1} and L_0 be the operators from the standard VOA structure on $\mathcal{H}_0^0$, and we must show that $L_n = L_n^0$. Since $[L_n^{NS}, x(m)] = [L_n^0, x(m)]$ on $\mathcal{H}_0^0$, we have $[L_{-1}^0, Y(\xi, w)] = \frac{d}{dw} Y(\xi, w)$ for all $\xi \in \mathcal{H}_0^0$. Hence $[L_{-1}^0, \xi_m] = [L_{-1}, \xi_m]$ for all m. But L_{-1} and L_{-1}^0 annihilate Ω, which is cyclic for the ξ_m, so we must have $L_{-1} = L_{-1}^0$. Similarly, L_0 and L_0^0 annihilate Ω and satisfy $[L_0, x(m)] = -mx(m) = [L_0^0, x(m)]$, and so $L_0 = L_0^0$. The other properties of a VOA now follow immediately. □

Definition 5.13. If $\lambda, \mu, \nu \in \Lambda$, we define the vector space of *intertwiners* $I\binom{\nu}{\lambda\mu}$ to be the space of linear maps

$$Y_{\lambda\mu}^\nu : H_\lambda^0 \to \mathcal{L}(\mathcal{H}_\mu^0, \mathcal{H}_\nu^0)[[w^{\pm 1}]] w^{h_\nu - h_\lambda - h_\mu}$$

which satisfy

$$[x(m), Y^{\nu}_{\lambda\mu}(\xi, w)] = \sum_{j=0}^{\infty} w^{m-j} \binom{m}{j} Y^{n}_{\lambda\mu}(x(j)\xi, w) \tag{5.4}$$

and

$$[L_0, Y^{\nu}_{\lambda\mu}(\xi, w)] = w\frac{d}{dw} Y^{\nu}_{\lambda\mu}(\xi, w) + \mathrm{wt}(\xi) Y^{\nu}_{\lambda\mu}(\xi, w) \tag{5.5}$$

for every $\xi \in \mathcal{H}^0_\lambda$, $x \in \mathfrak{sl}(k)$, and $m \in \mathbb{Z}$.

The *fusion rules* are given by $N^{\nu}_{\lambda\mu} = \dim I\binom{\nu}{\lambda\mu}$.

The finiteness of the fusion rules is a fundamental result (see [TK88, Thm 2.3] for $SU(2)$, [TUY89, Thm 4.2.4], or [Was98, Sec. 25]).

Theorem 5.14. $N^{\nu}_{\lambda\mu} \leq \dim \mathrm{Hom}_{SU(k)}(\mathcal{H}_\lambda(h_\lambda) \otimes_{\mathbb{C}} \mathcal{H}_\mu(h_\mu), \mathcal{H}_\nu(h_\nu))$.

We can use the unitary free fermion Segal CFT to define a related notion of fusion for embedded irreducible positive energy representations. The following are the main results of this section.

Theorem 5.15 (Internal fusion)**.** *Suppose $\tilde{\mathcal{H}}_\lambda$ and $\tilde{\mathcal{H}}_\mu$ are irreducible L_0^{NS}-graded $LSU(k)$ subrepresentations of $\mathcal{F}_{k\ell}$ isomorphic to $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$, respectively. Let $T \in E^s(\mathcal{P}^{NS}_{w, q_1^{\frac{1}{2}}, q_2^{\frac{1}{2}}})$. Then $\mathrm{cl}(T(\tilde{\mathcal{H}}_\lambda \otimes \tilde{\mathcal{H}}_\mu))$ is a representation of $LSU(k)$, and*

$$\mathrm{cl}(T(\tilde{\mathcal{H}}_\lambda \otimes \tilde{\mathcal{H}}_\mu)) \cong \bigoplus_{\nu \in \Lambda} \tilde{N}^{\nu}_{\lambda\mu} \mathcal{H}_\nu$$

with $\tilde{N}^{\nu}_{\lambda\mu} \leq N^{\nu}_{\lambda\mu}$.

In the language of vertex operator algebras, T makes its image into a $P(w)$-tensor product [HL94].

As an immediate consequence, we get the following positive answer to a conjecture of Vaughan Jones.

Corollary 5.16 (Fusion of isotypical components)**.** *Let $\mathcal{K}_\lambda = \mathcal{H}_\lambda \otimes M_\lambda$ be the isotypical component of $\mathcal{F}_{k\ell}$ of type λ. Then*

$$\mathrm{cl}(T(\mathcal{K}_\lambda \otimes \mathcal{K}_\mu)) \subseteq \bigoplus_{\nu, N^{\nu}_{\lambda\mu} \neq 0} \mathcal{K}_\nu. \tag{5.6}$$

When $T \neq 0$, we expect the inclusion (5.6) to be an equality.

We will need to develop some machinery to prove Theorem 5.15. For $\eta \in M_\lambda$, let p_η be the projection of $\mathcal{F}_{k\ell}$ onto $\mathcal{H}_\lambda \otimes \{\eta\}$, and let $\tilde{p}_\eta$ be the map $\mathcal{F}_{k\ell} \to \mathcal{H}_\lambda$

$$\tilde{p}_\eta(\xi \otimes \eta') = \frac{\langle \eta', \eta \rangle}{\|\eta\|^2} \xi.$$

Note that $\mathcal{H}_\lambda \otimes \{\eta\}$ is L_0^{NS}-graded if and only if η is an eigenvector for $L_0'^{\lambda}$.

Theorem 5.17 (Construction of intertwiners)**.** *If $\eta_\lambda \in M^0_\lambda$ is an eigenvector of L'^{λ}_0, and similarly with η_μ and η_ν, then*

$$Y^\nu_{\lambda\mu}(\xi, w) = w^{h_\nu - h_\lambda - h_\mu} \sum_{m\in\mathbb{Z}} \tilde{p}_{\eta_\nu}(\xi \otimes \eta_\lambda)_{m-s}\tilde{p}^*_{\eta_\mu} w^{-m-1} \in I\binom{\nu}{\lambda\mu},$$

where

$$s = (\mathrm{wt}(\eta_\nu) - \mathrm{wt}(\eta_\lambda) - \mathrm{wt}(\eta_\mu)) + (h_\nu - h_\lambda - h_\mu).$$

Proof. We first must check that s is an integer when $Y^\nu_{\lambda\mu}$ is non-zero. If ξ_λ is an eigenvector of L^λ_0 with eigenvalue h_λ, then

$$s = \mathrm{wt}(\xi_\nu \otimes \eta_\nu) - \mathrm{wt}(\xi_\lambda \otimes \eta_\lambda) - \mathrm{wt}(\xi_\mu \otimes \eta_\mu)$$

and so $s \in \sigma(L^{NS}_0) \subset \frac{1}{2}\mathbb{Z}$. Since the $x(m)$ are even, $s \in \mathbb{Z}$ if and only if

$$p(\mathcal{H}_\nu \otimes \{\eta_\nu\}) + p(\mathcal{H}_\lambda \otimes \{\eta_\lambda\}) + p(\mathcal{H}_\mu \otimes \{\eta_\mu\}) = 0.$$

But $(\zeta_\lambda \otimes \eta_\lambda)_m$ is even if and only if $\zeta_\lambda \otimes \eta_\lambda$ is, so if $s \in \frac{1}{2} + \mathbb{Z}$ then $\tilde{p}_{\eta_\nu}(\xi \otimes \eta_\lambda)_{m-s}\tilde{p}^*_{\eta_\mu}$ vanishes for all $\xi \in \mathcal{H}^0_\lambda$.

Condition (5.4) follows immediately from Theorem 5.4.

Next, observe that

$$\begin{aligned} L^\nu_0 \tilde{p}_{\eta_\nu} &= \tilde{p}_{\eta_\nu}(L^\nu_0 \otimes \mathbf{1}) \\ &= \tilde{p}_{\eta_\nu}(L^{NS}_0 - \mathrm{wt}(\eta_\nu)\mathbf{1}). \end{aligned}$$

Similarly,

$$\tilde{p}^*_{\eta_\mu} L^\mu_0 = (L^{NS}_0 - \mathrm{wt}(\eta_\mu)\mathbf{1})\tilde{p}^*_{\eta_\mu}.$$

Setting $\tilde{\xi}_m = \tilde{p}_{\eta_\nu}(\xi \otimes \eta_\lambda)_{m-s}\tilde{p}^*_{\eta_\mu}$, we have

$$\begin{aligned} [L_0, \tilde{\xi}_m] &= \tilde{p}_{\eta_\nu}[L^{NS}_0, (\xi \otimes \eta_\lambda)_{m-s}]\tilde{p}^*_{\eta_\mu} + (\mathrm{wt}(\eta_\mu) - \mathrm{wt}(\eta_\nu))\tilde{\xi}_m \\ &= (-m - 1 + s + \mathrm{wt}(\xi \otimes \eta_\lambda) + \mathrm{wt}(\eta_\mu) - \mathrm{wt}(\eta_\nu))\tilde{\xi}_m \\ &= (-m - 1 + h_\nu - h_\lambda - h_\mu + \mathrm{wt}(\xi))\tilde{\xi}_m, \end{aligned}$$

which establishes condition (5.5). □

Remark 5.18. Theorem 5.17 generalizes Wassermann's construction of (dual) vector primary fields as compressions of fermions [Was98, Sec. 25]. It is straightforward to show that for the loop algebra $L^0\mathfrak{sl}(2)$, all intertwining operators can be constructed via Theorem 5.17. We expect that this holds for $L^0\mathfrak{sl}(k)$, but will reserve that question for future work.

Remark 5.19. When combined with Theorem 5.15, we have shown that

$$\dim \operatorname{span}\{w^{-s}\tilde{p}_{\eta_\nu} E(\mathcal{P}^{NS}_{w,q_1^{\frac{1}{2}},q_2^{\frac{1}{2}}})(p^*_{\eta_\lambda} \otimes p^*_{\eta_\mu}) : \eta_\nu, \eta_\lambda, \eta_\mu\} \leq N^\nu_{\lambda\mu},$$

with equality in the case of $SU(2)$. Here, $\eta_\nu, \eta_\lambda, \eta_\mu$ range over eigenvectors of the corresponding L_0' operators and s is as in Theorem 5.15. This suggests an approach to constructing the $SU(k)_\ell$ Segal CFTs, which we will investigate in future work.

We will require the following exercise in linear algebra.

Lemma 5.20. *Let H be a complex vector space, let K and L be Hilbert spaces with $\dim K \geq \dim L$, and let B be an orthonormal basis for L. Let $x : H \to K \otimes L$ be a linear map with $\operatorname{cl}(x(H)) = K \otimes \tilde{L}$. For $\eta \in L$ let $\tilde{p}_\eta : K \otimes L \to K$ be given by*

$$\tilde{p}_\eta : \xi \otimes \eta' \mapsto \frac{\langle \eta', \eta \rangle}{\|\eta\|^2}\xi.$$

Then for every number $n \leq \dim \tilde{L}$, there exists a set $S \subseteq B$ such that $|S| = n$ and

$$\{\tilde{p}_\eta x : \eta \in S\} \subset \mathcal{L}(H, K)$$

is linearly independent.

Proof. Let $S \subseteq B$ be a finite set, and suppose that

$$\sum_{\eta \in S} c_\eta \tilde{p}_\eta x = 0.$$

Fix an orthonormal set $\{\psi_\eta\}_{\eta \in S} \subset K$, and let $\tilde{\eta} \in \tilde{L}$. By hypothesis, there is a sequence $\xi_m \in H$ such that $x\xi_m \to \psi_\eta \otimes \tilde{\eta}$. Then

$$0 = \sum_{\eta \in S} c_\eta \tilde{p}_\eta x \xi_m \to \sum_{\eta \in S} c_\eta \langle \tilde{\eta}, \eta \rangle \psi_\eta.$$

Thus $c_\eta \langle \tilde{\eta}, \eta \rangle = 0$ for all $\eta \in S$ and $\tilde{\eta} \in \tilde{L}$. Now if we can choose $S \subseteq B$ so that $|S| = n$ and $S \cap \tilde{L}^\perp = \emptyset$, we are done. This is always possible, since otherwise we would have $\dim \tilde{L} \leq n - 1$. □

Proof of Theorem 5.15. Write $\tilde{\mathcal{H}}_\lambda = \mathcal{H}_\lambda \otimes \{\eta_\lambda\}$ and $\tilde{\mathcal{H}}_\mu = \mathcal{H}_\mu \otimes \{\eta_\mu\}$, where η_λ and η_μ are eigenvectors for the respective L_0' operators. Assume without loss of generality that $\|\eta_\mu\| = \|\eta_\lambda\| = 1$.

Let $T_w \in E^s(\mathcal{P}^{NS}_{\overline{w},\overline{q_1}^{\frac{1}{2}},\overline{q_2}^{\frac{1}{2}}})$, normalized so that

$$T_w(\xi \otimes \eta) = q_1^{-\operatorname{wt}(\eta_\lambda)} q_2^{-\operatorname{wt}(\eta_\mu)} Y(q_1^{L_0^{NS}}\xi, w) q_2^{L_0^{NS}} \eta.$$

for $\xi, \eta \in \mathcal{F}^0_{k\ell}$. Hence

$$T_w(\tilde{p}^*_{\eta_\lambda} \otimes \tilde{p}^*_{\eta_\mu})(\xi \otimes \eta) = Y(\tilde{p}^*_{\eta_\lambda} q_1^{L_0^\lambda} \xi, w)\tilde{p}^*_{\eta_\mu} q_2^{L_0^\mu} \eta \tag{5.7}$$

for $\xi \in \mathcal{H}_\lambda$ and $\eta \in \mathcal{H}_\mu$. We regard $q_1^{\frac{1}{2}}$ and $q_2^{\frac{1}{2}}$ as fixed, and also fix values of $\log(q_i^{\frac{1}{2}})$ to define the above expressions.

That

$$T_w(\tilde{\mathcal{H}}_\lambda \otimes \tilde{\mathcal{H}}_\mu) = T_w(\tilde{p}^*_{\eta_\lambda} \otimes \tilde{p}^*_{\eta_\mu})(\mathcal{H}_\lambda \otimes \mathcal{H}_\mu)$$

is invariant under $L^0\mathfrak{sl}(k)$ follows from equation (5.7), along with Theorems 5.6 and 5.4. Hence the closure is invariant under $LSU(k)$ by Theorem 5.9.

Fix w, and write

$$\mathrm{cl}(T_w(\tilde{\mathcal{H}}_\lambda \otimes \tilde{\mathcal{H}}_\mu)) = \bigoplus_{\nu \in \Lambda} \mathcal{H}_\nu \otimes \tilde{M}_\nu.$$

We wish to show that $\dim \tilde{M}_\nu \leq N^\nu_{\lambda\mu}$. By Lemma 5.20, if $n \leq \dim \tilde{M}_\nu$, then there exists a set $\{\eta_1, \ldots \eta_n\}$ of eigenvectors of $L_0'^\nu$ such that

$$\{\tilde{p}_{\eta_1} T_w(\tilde{p}^*_{\eta_\lambda} \otimes \tilde{p}^*_{\eta_\mu}), \ldots, \tilde{p}_{\eta_n} T_w(\tilde{p}^*_{\eta_\lambda} \otimes \tilde{p}^*_{\eta_\mu})\} \tag{5.8}$$

is linearly independent. Since $w \mapsto T_w$ is holomorphic, the set (5.8) is also linearly independent as functions of w defined in a small neighborhood of the fixed value. Hence the corresponding formal power series are linearly independent, and by Theorem 5.17 we have $n \leq N^\nu_{\lambda\mu}$. Since $n \leq \dim \tilde{M}_\nu$ was arbitrary, we have $\dim \tilde{M}_\nu \leq N^\nu_{\lambda\mu}$, as required. □

Appendix A

The Cauchy transform for Riemann surfaces

During our construction of Segal CFTs in Chapter 4 we relied on two facts about the Hardy space that we did not prove. Our goal in this appendix is to bridge the gap by establishing Theorem A.1 below. First, we recall some notation.

Let $X = (\Sigma, V, \beta, \alpha) \in \mathcal{E}$, and let $H_\Gamma = \bigoplus_{j \in \pi_0(\Gamma)} L^2(S^1, \mathbb{C}^{k(j)})$. Let

$$p_\Gamma = \bigoplus_{j \in \pi_0(\Gamma^0)} (1 - p_+) \oplus \bigoplus_{j \in \pi_0(\Gamma^1)} p_+,$$

and let q_X be the projection of H_Γ onto the Hardy space $H^2(X)$. Recall that we think of K_Σ as an element of $\mathcal{E}$ be giving it boundary parameterizations coming from the pushforwards $(\beta_j|_{S^1})_*$.

Theorem A.1. *Let* $X \in \mathcal{E}$*.*

1. $q_X - p_\Gamma$ *is Hilbert-Schmidt*
2. $H^2(X \otimes K_\Sigma)^\perp = M_{\pm\overline{z}} H^2(X^*)$*, where* $M_{\pm\overline{z}}$ *is multiplication by* $\overline{z}$ *on outgoing components, and multiplication by* $-\overline{z}$ *on incoming components.*

Corollary A.2. *If* $X \in \mathcal{E}_*$*, then*

$$H^2(\check{X})^\perp = \{(f_j)_{j \in \pi_0(\check{\Gamma})} : \exists f_{j^0} = -f_{j^1} \in H^{k(j^0)} \text{ with } (f_j)_{j \in \pi_0(\Gamma)} \in H^2(X)^\perp\}.$$

The main tool will be a generalization of the Cauchy transform to Riemann surfaces. We would like to thank Antony Wassermann for indicating the importance of the Cauchy transform in the study of the free fermion, and for suggesting the reference [Bel92]. The first statement of Theorem A.1 was proven by Wassermann for planar domains, also using the Cauchy transform. Our treatment of Cauchy transforms for Riemann surfaces is motivated by the desire to extend Wassermann's proof to a more general setting.

Let $X = (\Sigma, V, \beta, \alpha) \in \mathcal{E}$. As $H^2(X)$ does not depend on the labeling α, we will omit it for the remainder of this appendix. We will assume at first that V is the trivial bundle $\Sigma \times \mathbb{C}$, and that β acts identically on fibers. If $\gamma = \beta|_{S^1}$, then in this case we will write $H^2(\Sigma, \gamma)$ or $H^2(\Sigma)$ for $H^2(X)$. By $H^2(\Sigma)^\perp$, we mean the orthogonal complement taken in

$$H_\Gamma = \bigoplus_{j \in \pi_0(\Gamma)} H^{k(j)} = \bigoplus_{j \in \pi_0(\Gamma)} L^2(S^1) = L^2(\bigsqcup_{j \in \pi_0(\Gamma)} S^1).$$

The inner product on H_Γ is given by

$$\langle f, g \rangle = \frac{1}{2\pi i} \int_{\bigsqcup S^1} f(z)\overline{g(z)} \frac{dz}{z},$$

with the integral taken with the standard counterclockwise orientation on each circle.

Let ω be a holomorphic 1-form on Σ, and suppose $\gamma^*\omega = g(z)dz$. Suppose that $F \in \mathcal{O}(\Sigma)$ and let f be the pullback $f = \gamma^* F \in H^2(\Sigma)$. Then we have

$$\begin{aligned} \langle f, M_{\pm\overline{z}}\overline{g} \rangle &= \int_{\bigsqcup_{j \in \pi_0 \Gamma^1} S^1} f(z)g(z)dz - \int_{\bigsqcup_{j \in \pi_0 \Gamma^0} S^1} f(z)g(z)dz \\ &= \int_\Gamma F\omega \\ &= 0. \end{aligned}$$

Let $K = (\Sigma, K_\Sigma, \gamma_*) \in \mathcal{E}$. By construction, $g \in H^2(K)$ if $gdz = \gamma^*\omega$ for some $\omega \in \mathcal{O}(K_\Sigma)$. Since $H^2(K^*) = \overline{H^2(K)}$, this computation shows that $M_{\pm\overline{z}} H^2(K^*) \subseteq H^2(\Sigma)^\perp$. We will need to work harder to establish the reverse inclusion.

If Σ is a planar region, then the desired result is [Bel92, Thm. 4.3]. Our proof will mimic the analysis done in the planar case, making adjustments as necessary.

We begin by fixing several pieces of notation. Let $C^\infty(\Sigma)$ be the space of smooth functions on the interior $\mathring{\Sigma}$ that extend, along with all of their derivatives, continuously to the boundary in any chart and let $A^\infty(\Sigma)$ be the subspace of $C^\infty(\Sigma)$ consisting of functions that are holomorphic on the interior $\mathring{\Sigma}$. For technical reasons, we will use $A^\infty(\Sigma)$ instead of the slightly smaller space $\mathcal{O}(\Sigma)$, consisting of holomorphic functions on $\mathring{\Sigma}$ that extend holomorphically past the boundary. In contrast, any function in $C^\infty(\Sigma)$ actually extends smoothly past the boundary in any smooth chart by Borel's lemma.

We denote by $C^\infty(\mathring{\Sigma})$ and $A^\infty(\mathring{\Sigma})$ the smooth (resp. holomorphic) functions on the interior of Σ.

Fix a holomorphic embedding of Σ into an open Riemann surface $\tilde{\Sigma}$, neighborhoods U_j in $\tilde{\Sigma}$ of each $j \in \pi_0(\Gamma)$, and a biholomorphic map

$$g : \sqcup_{j \in \mathcal{A}_\epsilon} \to U_j$$

as in Section 3.1. If g_j is the restriction of g to the annulus labeled by j, then g_j is orientation preserving if and only if j is outgoing.

By [GN67], there exists a holomorphic immersion $\rho : \tilde{\Sigma} \to \mathbb{C}$. The differential $d\rho$ is a non-vanishing holomorphic 1-form, and induces an area form

$$dA = \frac{1}{-2i} d\rho \wedge d\overline{\rho}.$$

Using this result, Scheinberg [Sch78] showed the existence of a meromorphic function $q(s,t) : \tilde{\Sigma} \times \tilde{\Sigma} \to \mathbb{C}$ such that $q(s,t)$ has a simple pole of residue 1 at $s = t$, and is holomorphic elsewhere. Additionally, $q(s,t) - (\rho(s) - \rho(t))^{-1}$ is holomorphic on $U \times U$ for any open U on which ρ is injective. We can assume that $q(s,t) = -q(t,s)$ by replacing q with $\frac{1}{2}q(s,t) - \frac{1}{2}q(t,s)$. Let $\omega_t(s) = q(s,t)d\rho(s)$.

We call q a Cauchy kernel on $\tilde{\Sigma}$, which is justified by the following Cauchy integral formula.

Proposition A.3 ([Sch78, Prop. 7.1])**.** *Let U be an open set in $\tilde{\Sigma}$ with $\overline{U}$ compact, and with a piecewise C^1 oriented boundary Γ_U. If $u \in C^1(\overline{U})$, then for every $z \in U$,*

$$u(t) = \frac{1}{2\pi i} \int_{\Gamma_U} u\omega_t - \frac{1}{2\pi i} \int_U \overline{\partial} u \wedge \omega_t.$$

Definition A.4. If $u \in C^\infty(\Gamma)$, then define its Cauchy transform $\mathcal{C}u \in \mathcal{O}(\mathring{\Sigma})$ by

$$(\mathcal{C}u)(t) = \frac{1}{2\pi i} \int_\Gamma u\omega_t.$$

Note that $\mathcal{C}$ depends on the choice of ρ and q, so we will regard these as fixed. We will now show that $\mathcal{C}u \in A^\infty(\Sigma)$, but first we need the following version of [Bel92, Thm. 2.2].

Theorem A.5. *Suppose $v \in C^\infty(\Sigma)$. Then the function u defined by*

$$u(t) = \frac{1}{2\pi i} \int_\Sigma v\omega_t \wedge d\overline{\rho}$$

for $t \in \Sigma$ satisfies $\overline{\partial} u = v d\overline{\rho}$ and $u \in C^\infty(\Sigma)$.

Proof. We first check that the integral defining u makes sense. Fix $t_0 \in \Sigma$, and let V be a neighborhood of t_0 in Σ on which ρ is injective. Let $z_0 = \rho(t_0)$, and let $\tau = (\rho|_V)^{-1}$. For $z \in \rho(V)$ we have an identity of 1-forms on $\rho(V)$

$$\tau^* \omega_{\tau(z)} = \frac{dw}{w - z} + f(z,w)dw,$$

where f is holomorphic and w is the standard global parameter for $\mathbb{C}$. If we give $\rho(V)$ the orientation coming from V and ρ, we have for $z \in V$

$$\begin{aligned} u(\tau(z)) = \frac{1}{2\pi i} \int_{\Sigma \setminus V} v\omega_{\tau(z)} \wedge d\overline{\rho} + \frac{1}{2\pi i} \int_{\rho(V)} \frac{v(\tau(w))dw \wedge d\overline{w}}{w - z} + \\ + \frac{1}{2\pi i} \int_{\rho(V)} v(\tau(w)f(z,w)dw \wedge d\overline{w} \\ := u_1(z) + u_2(z) + u_3(z). \end{aligned}$$

Both u_1 and u_3 are clearly smooth in a neighborhood of z_0. From [Bel92, Thm 2.2], u_2 is well-defined and $u_2 \in C^\infty(\rho(V))$. Thus u is smooth in a neighborhood of t_0, and since t_0 was arbitrary $u \in C^\infty(\Sigma)$.

Differentiating under the integral, we see that

$$\frac{\partial}{\partial \overline{z}} u \circ \tau = \frac{\partial}{\partial \overline{z}} u_2 = v \circ \tau$$

by [Bel92, Thm 2.2]. Pulling back by ρ gives $\overline{\partial} u = v d\overline{\rho}$ on V, and hence on Σ. □

As a corollary, we can show that $\mathcal{C}u$ extends smoothly to the boundary.

Theorem A.6. *The Cauchy transform maps $C^\infty(\Gamma)$ into $A^\infty(\Sigma)$.*

Proof. Let $u \in C^\infty(\Gamma)$ and let $\tilde{u}$ be a function in $C^\infty(\Sigma)$ which is equal to u on Γ. The Cauchy integral formula says

$$\tilde{u}(t) = (\mathcal{C}u)(z) - \frac{1}{2\pi i} \int_{\mathring{\Sigma}} \overline{\partial} \tilde{u} \wedge \omega_t.$$

We can write $\overline{\partial}\tilde{u} = v d\overline{\rho}$ for some $v \in C^\infty(\Sigma)$, so by the preceding theorem, the integral term is in $C^\infty(\Sigma)$. Hence $\mathcal{C}u \in C^\infty(\Sigma)$ as well. □

By restriction, we can consider $\mathcal{C}$ as a map from $C^\infty(\Gamma)$ into itself. The Cauchy integral formula says that $\mathcal{C}$ is idempotent.

Our next step is to identify a formal adjoint for $\mathcal{C}$ defined on $C^\infty(\Gamma)$. We will need the following technical lemmas, which are a generalization of [Bel92, Lem. 2.3 and Thm 3.4].

Lemma A.7. *Suppose that $v \in C^\infty(\Sigma)$. Then there exists a function $\Phi \in C^\infty(\Sigma)$ which vanishes on Γ and satisfies $\overline{\partial}\Phi|_\Gamma = \overline{\partial} v|_\Gamma$.*

Proof. Recall that we have neighborhoods U_j in $\tilde{\Sigma}$ of each $j \in \pi_0(\Gamma)$, and biholomorphic maps $g_j : \mathcal{A}_\epsilon \to U_j$. To prove the lemma, it suffices to construct a family $\Phi_j \in C^\infty(\Sigma \cap U_j)$ satisfying the conclusion of the lemma such that the support of Φ_j is bounded away from ∂U_j. We may then define our desired Φ by extending the Φ_j by zero.

That Φ_j vanish on j and satisfy

$$\overline{\partial}(\Phi_j)|_j = \overline{\partial} v|_j$$

is equivalent to the requirement that for all $w \in S^1$ we have

$$\frac{\partial \Psi_j}{\partial \overline{w}}(w) = \frac{\partial (v \circ g_j)}{\partial \overline{w}}(w),$$

and

$$\Psi_j(w) = 0,$$

where $\Psi_j = \Phi_j \circ g_j \in C^\infty(g_j^{-1}(\Sigma \cap U_j))$. The existence of such a Ψ_j is given by [Bel92, Lem. 2.3], and we may use a smooth cutoff function to keep the support of Φ_j away from ∂U_j. □

Lemma A.8. *Suppose that $u \in C^\infty(\Gamma)$. There is a $\Psi \in C^\infty(\Sigma)$ with $\overline{\partial}\Psi|_\Gamma = 0$ such that the boundary values of $\mathcal{C}u$ are expressed by*

$$(\mathcal{C}u)(t) = u(t) + \frac{1}{2\pi i}\int_{\mathring{\Sigma}} \overline{\partial}\Psi \wedge \omega_t,$$

for all $t \in \Gamma$. If $(\overline{\partial}\Psi \wedge \omega_t)(s) = h(s,t)dA(s)$, then h is continuous on $\Sigma \times \Gamma$.

Proof. Let $\tilde{u}$ be an element of $C^\infty(\overline{M})$ with boundary values u. Let $\Phi \in C^\infty(\Sigma)$ be a function from Lemma A.7 that vanishes on Γ such that $\overline{\partial}\Phi|_\Gamma = \overline{\partial}\tilde{u}|_\Gamma$. Let

$$\Psi = \tilde{u} - \Phi.$$

Applying the Cauchy integral formula to Ψ yields

$$\Psi(t) = (\mathcal{C}u)(t) - \frac{1}{2\pi i}\int_{\mathring{\Sigma}} \overline{\partial}\Psi \wedge \omega_t.$$

Since $\Psi = u$ on the boundary, we have established the desired boundary value formula for $\mathcal{C}u$.

The function $h(s,t)$ is clear continuous at all points not of the form (t_0, t_0) with $t_0 \in \Gamma$. Fix a neighborhood V of t_0 on which ρ is injective and set $z = \rho(t)$ and $\tau = \rho^{-1}$. We have

$$\tau^*(\overline{\partial}\Psi \wedge \omega_t)(w) = \left(\frac{\partial_{\overline{w}}(\phi \circ \tau)(w)}{w - z} + \text{smooth}\right) d\overline{w} \wedge dw$$

for $(w, z) \in \rho(V) \times \rho(V)$ with $w \neq z$. Since $\partial_{\overline{w}}(\phi \circ \tau)$ is smooth and vanishes on $\rho(\Gamma \cap V)$, the above expression defines a continuous function on $\rho(V) \times \rho(V \cap \Gamma)$. $\square$

Define a bilinear form $[\cdot,\cdot]$ on $C^\infty(\Gamma)$ by

$$[u, v] = \frac{1}{2\pi i}\int_\Gamma uv d\rho.$$

Theorem A.9. *For $u, v \in C^\infty(\Gamma)$, we have $[\mathcal{C}u, v] = [u, (\mathbf{1} - \mathcal{C})v]$.*

Proof. By Lemma A.8, for $t \in \Gamma$ we have $\mathcal{C}u = u + \mathcal{I}$ where

$$\mathcal{I}(t) = \frac{1}{2\pi i}\int_{\mathring{\Sigma}} \overline{\partial}\Psi \wedge \omega_t$$

and Ψ is as in the lemma. By the continuity of the integrand of $\mathcal{I}$, we may apply Fubini's theorem to compute

$$\begin{aligned}\int_\Gamma I(t)v(t)d\rho(t) &= \int_\Gamma \left(\frac{1}{2\pi i}\int_{\mathring{\Sigma}} \overline{\partial}\Psi(s)\wedge\omega_t(s)\right) v(t)d\rho(t)\\ &= \int_{\mathring{\Sigma}} \left(\frac{1}{2\pi i}\int_\Gamma -q(s,t)v(t)d\rho(t)\right) d\rho(s)\wedge\overline{\partial}\Psi(s)\\ &= \int_{\mathring{\Sigma}} \left(\frac{1}{2\pi i}\int_\Gamma q(t,s)v(t)d\rho(t)\right) d\rho(s)\wedge\overline{\partial}\Psi(s)\\ &= \int_{\mathring{\Sigma}} (\mathcal{C}v)(s)d\rho(s)\wedge\overline{\partial}\Psi(s).\end{aligned}$$

Recall that $\Psi|_\Gamma = u|_\Gamma$. Since ρ and $\mathcal{C}v$ are holomorphic,

$$d(\Psi(\mathcal{C}v)d\rho) = -(\mathcal{C}v)d\rho\wedge\overline{\partial}\Psi$$

and we may apply Stokes' theorem to obtain

$$\begin{aligned}\int_{\mathring{\Sigma}} (\mathcal{C}v)d\rho\wedge\overline{\partial}\Psi &= -\int_\Gamma \Psi(\mathcal{C}v)d\rho\\ &= -\int_\Gamma u(\mathcal{C}v)d\rho.\end{aligned}$$

As a result of the calculation, we have

$$\int_\Gamma (\mathcal{C}u)vd\rho = \int_\Gamma u(v-\mathcal{C}v)d\rho,$$

which was to be shown. □

Let $\tilde{u},\tilde{v}\in C^\infty(\Gamma)$, and let $u=\gamma^*\tilde{u}$ and $v=\gamma^*\tilde{v}\in H_\Gamma$. Let $M_{\pm\overline{z}}$ be the operator on H_Γ given by multiplication by the function $\overline{z}$ on direct summands indexed by $j\in\pi_0(\Gamma^1)$, and multiplication by $-\overline{z}$ on the complement. If $r\in C^\infty(\bigsqcup S^1)$ is given by $\gamma^*d\rho = rdz$, we have

$$\begin{aligned}\langle u, M_{\pm\overline{z}}\overline{rv}\rangle &= \frac{1}{2\pi i}\int_{\bigsqcup S^1} \pm u(z)v(z)r(z)dz\\ &= \frac{1}{2\pi i}\int_\Gamma u(t)v(t)d\rho(t)\\ &= [\tilde{u},\tilde{v}].\end{aligned}$$

Hence

$$\langle \mathcal{C}u, M_{\pm\overline{z}}\overline{rv}\rangle = \langle u, M_{\pm\overline{z}}\overline{r(\mathbf{1}-\mathcal{C})v}\rangle.$$

If we formally define

$$(\mathcal{C}^*v)(z) := v(z) - \overline{\pm zr(z)\mathcal{C}(M_{\pm\overline{z}}r^{-1}\overline{v})(z)},$$

we have $\langle \mathcal{C}u, v\rangle = \langle u, \mathcal{C}^* v\rangle$ when u and v are smooth.

We now establish the Kerzman-Stein formula

$$q_\Sigma(\mathbf{1} + \mathcal{A}) = \mathcal{C} \tag{A.1}$$

where q_Σ is the orthogonal projection of H_Γ onto $H^2(\Sigma)$, and

$$\mathcal{A} = \mathcal{C} - \mathcal{C}^*.$$

For the moment, we will regard these as unbounded operators with dense invariant domain $C^\infty(\bigsqcup S^1)$. The importance of equation (A.1) is that $\mathcal{A}$ is given by an integral operator with smooth kernel. In particular, it follows that $\mathcal{C}$ is bounded and that $\mathcal{C}^*$ is its adjoint.

Proposition A.10. *If u is a smooth function in H_Γ, then*

$$q_\Sigma(I + \mathcal{A})u = \mathcal{C}u.$$

Proof. For $v \in H^2(\Sigma)$ we have

$$\langle(\mathbf{1} - \mathcal{C}^*)u, v\rangle = \langle u, v\rangle - \langle \mathcal{C}^* u, v\rangle = \langle u, v\rangle - \langle u, \mathcal{C}v\rangle = 0.$$

Thus $(\mathbf{1} - \mathcal{C}^*)u$ is orthogonal to any smooth function in $H^2(\Sigma)$. By construction, such functions are dense in $H^2(\Sigma)$ so we have $q_\Sigma(\mathbf{1} - \mathcal{C}^*)u = 0$. We now have

$$q_\Sigma(\mathbf{1} + \mathcal{A})u = q_\Sigma \mathcal{C}u = \mathcal{C}u.$$

□

Our proof that $\mathcal{A}$ is an integral operator with smooth kernel follows [Bel92, Ch. 4-5]. If $t_0 \in \Gamma$, let V be a neighborhood of t_0 in Σ on which ρ is injective, and let

$$\Gamma_\epsilon = \Gamma \setminus V \cup \{t \in \Gamma : |\rho(t) - \rho(t_0)| \geq \epsilon\}.$$

Observe that for a different choice of V, the resulting sets Γ_ϵ coincide for sufficiently small ϵ. Define the Hilbert transform $\mathcal{H}u$ for $u \in C^\infty(\Gamma)$ by

$$(\mathcal{H}u)(t_0) = \mathbf{P}.\,\mathrm{V}.\ \frac{1}{2\pi i}\int_\Gamma u\omega_{t_0}$$

where the principal value is defined by

$$\lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{\Gamma_\epsilon} u\omega_{t_0}$$

We will now establish the Plemelj formula relating the Cauchy and Hilbert transforms.

Lemma A.11. *The limit defining $(\mathcal{H}u)(t_0)$ exists and*

$$(\mathcal{C}u)(t_0) = \frac{1}{2}u(t_0) + (\mathcal{H}u)(t_0).$$

Proof. We first prove the theorem in the case where u is a constant function. Let

$$C_\epsilon = \{t \in V : |\rho(t) - \rho(t_0)| = \epsilon\},$$

oriented so that $\Gamma_\epsilon \cup C_\epsilon$ is an oriented curve (i.e. so that C_ϵ is oriented negatively around t_0). We give $\rho(C_\epsilon)$ the orientation coming from C_ϵ, and use the holomorphicity of $u\omega_t$ and the fractional residue formula to compute

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\Gamma_\epsilon} u\omega_{t_0}d\rho &= -\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{C_\epsilon} u\omega_{t_0}d\rho \\ &= \pm\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\rho(C_\epsilon)}\frac{u(\tau(w))}{w-\rho(t_0)}dw \\ &= \frac{1}{2}u(t_0).\end{aligned}$$

The sign $\pm$ depends on whether ρ preserves or reverses orientation near t_0, but it is canceled in the application of the fractional residue formula.

We now return to arbitrary $u \in C^\infty(\Gamma)$, but we assume without loss of generality that $u(z_0) = 0$. Hence the integrand in the Hilbert and Cauchy transforms $u\omega_{t_0}$ is continuous at t_0, and thus on Σ. In this case $(\mathcal{H}u)(t_0)$ is given by the ordinary integral

$$(\mathcal{H}u)(t_0) = \frac{1}{2\pi i}\int_\Gamma u\omega_{t_0},$$

and the same for $(\mathcal{C}u)(t_0)$. □

We can also define $\mathcal{H}u$ for u a smooth function $u \in H_\Gamma$, under the identification

$$\Gamma \cong \bigsqcup_{j\in\pi^0(\Gamma)} S^1.$$

More precisely, let g be our holomorphic identification of a neighborhood of Γ in $\tilde{\Sigma}$ with $\bigsqcup \mathcal{A}_\epsilon$ and let $h = g^{-1}$. Under this identification we have $(\mathcal{H}u)(z) = \mathcal{H}(u\circ h)(g(z))$

We will now use the Hilbert transform to write the Kerzman-Stein operator as an integral operator.

Theorem A.12. *For $u \in H_\Gamma$, the Kerzman-Stein operator is given by the formula*

$$(\mathcal{A}u)(z) = \frac{1}{2\pi i}\int_\Gamma A(w,z)u(w)\frac{dw}{iw}$$

for a smooth function $A : \bigsqcup S^1 \times \bigsqcup S^1 \to \mathbb{C}$.

Proof. Applying Lemma A.11, we have

$$(\mathcal{A}u)(z) = \frac{1}{2\pi i}\,\mathbf{P}.\,\mathrm{V}. \int_\Gamma u(h(s))q(s,g(z))d\rho(s) + \\ + \overline{\frac{1}{2\pi i} \pm zr(z)\,\mathbf{P}.\,\mathrm{V}. \int_\Gamma \pm\overline{h(s)}r(h(s))^{-1}\overline{u(h(s))}q(s,g(z))d\rho(s)}$$

where the two $\pm$ are determined by whether the boundary near s and t is incoming or outgoing, respectively. It is clear that the kernel of $\mathcal{A}$ is smooth in any neighborhood of (s,t) when s and t lie on distinct components of Γ. Thus in order to simplify notation, we will assume that Γ has a single connected component, and the general case is no different. When restricting to s and t on the same connected component, the signs $\pm$ cancel. We continue the computation

$$\begin{aligned}(\mathcal{A}u)(z) &= \frac{1}{2\pi i}\,\mathbf{P}.\,\mathrm{V}. \int_{S^1} wu(w)q(g(w),g(z))r(s)\frac{dw}{w} + \\ &\quad + \overline{\frac{1}{2\pi i} zr(z)\,\mathbf{P}.\,\mathrm{V}. \int_{S^1} \overline{u(w)}q(g(w),g(z))\frac{dw}{w}} \\ &= \frac{1}{2\pi i}\,\mathbf{P}.\,\mathrm{V}. \int_{S^1} wu(w)q(g(w),g(z))r(s)\frac{dw}{w} + \\ &\quad + \frac{1}{2\pi i}\,\mathbf{P}.\,\mathrm{V}. \int_{S^1} \overline{zr(z)q(g(w),g(z))}u(w)\frac{dw}{w} \\ &= \frac{1}{2\pi i}\,\mathbf{P}.\,\mathrm{V}. \int_{S^1} A(w,z)u(w)\frac{dw}{iw}\end{aligned}$$

where

$$A(w,z) = iwr(w)q(g(w),g(z)) - \overline{izr(z)q(g(w),g(z))}.$$

Clearly A is smooth away from $w = z$, so we fix z and consider when $w - z$ is small. In this scenario, we may write

$$A(w,z) = \frac{iwr(w)}{\rho(g(w)) - \rho(g(z))} - \frac{\overline{izr(z)}}{\overline{\rho(g(w))} - \overline{\rho(g(z))}} + \text{ smooth.}$$

Proving that such $A(w,z)$ is smooth is precisely what is required in the planar case [Bel92, Ch. 5]. □

Theorem A.13. *The Cauchy transform $\mathcal{C}$ extends to a bounded operator on H_Γ and $q_\Sigma - \mathcal{C}$ is Hilbert-Schmidt. We have $H^2(\Sigma)^\perp = \overline{r}M_{\pm\overline{z}}\overline{H^2(\Sigma)}$, where r satisfies $\gamma^* d\rho = rdz$.*

Proof. The fact that $\mathcal{C}$ is bounded follows immediately from the formula of Proposition A.10 and the fact that $\mathcal{A}$ is bounded. Rewriting this formula as $q_\Sigma - \mathcal{C} = -q_\Sigma\mathcal{A}$ we can see that $q_\Sigma - \mathcal{C}$ is Hilbert-Schmidt.

Since $\mathcal{C}$ is an idempotent with image $H^2(\Sigma)$, we have that $I - \mathcal{C}^*$ is an idempotent with image $H^2(\Sigma)^\perp$. Since $\mathcal{C}$ is bounded, the formula for the formal adjoint from Theorem A.9 indeed gives the adjoint. It follows immediately that $H^2(\Sigma)^\perp = \overline{r}M_{\pm\overline{z}}\overline{H^2(\Sigma)}$. □

Corollary A.14. *Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are two Cauchy transforms for (Σ, γ) coming from different choices of ρ and q. Then $\mathcal{C}_1 - \mathcal{C}_2$ is Hilbert-Schmidt.*

For $j \in \pi_0(\Gamma)$, let p_j be the projection of H_Γ onto the copy of $L^2(S^1)$ indexed by j, and let $p_{j,+}$ be the projection of H_Γ onto the corresponding copy of $H^2(S^1)$. Let

$$p_\Gamma = \bigoplus_{j \in \pi_0(\Gamma^0)} \mathbf{1} - p_{j,+} \oplus \bigoplus_{j \in \pi_0(\Gamma^1)} p_{j,+}$$

To prove the first statement of Theorem A.1, we must show that $q_\Sigma - p_\Gamma$ is Hilbert-Schmidt. By Theorem A.13, it suffices to show that $\mathcal{C} - p_\Gamma$ is Hilbert-Schmidt or equivalently that

- $p_j \mathcal{C} p_j - p_{j,+}$ is Hilbert-Schmidt when $j \in \pi_0(\Gamma^1)$,
- $p_j \mathcal{C} p_j - (\mathbf{1} - p_{j,+})$ is Hilbert-Schmidt when $j \in \pi_0(\Gamma^0)$,
- $p_j \mathcal{C} p_k$ is Hilbert-Schmidt when $j \neq k$.

When $j \neq k$, $p_j \mathcal{C} p_k$ is an integral operator with smooth kernel, so the final point is easy. We will be able to understanding the behavior of $p_j \mathcal{C} p_j$ using the fact that the Cauchy transform is a local operation. The locality of the Cauchy transform is expressed by the property that $p_j \mathcal{C} p_j$ only depends on a germ of Σ around j. This property can most easily be observed from Lemma A.11. This behavior should be contrasted with the behavior of $p_j q_\Sigma p_j$, which has no a priori locality property. As a consequence of the locality of the Cauchy transform, we can conclude a fortiori that $p_j q_\Sigma p_j$ only depends on a germ of Σ near j, up to Hilbert-Schmidt operators.

The following theorem was proven by Wassermann in the case where Σ is a planar domain, and our proof parallels his.

Theorem A.15. *$q_\Sigma - p_\Gamma$ is Hilbert-Schmidt.*

Proof. Fix a holomorphic embedding $\Sigma \subset \tilde{\Sigma}$, neighborhoods U_j of each $j \in \pi_0(\Gamma)$, and holomorphic extensions g_j of γ_j identifying U_j with $\mathcal{A}_\epsilon$. Fix ρ and q, and let $\mathcal{C}$ be the corresponding Cauchy transform for Σ. By the preceding remarks, it suffices to prove that $p_j \mathcal{C} p_j - p_{j,+}$ is Hilbert-Schmidt when $j \in \pi_0(\Gamma^1)$, and that $p_j \mathcal{C} p_j - (\mathbf{1} - p_{j,+})$ is Hilbert-Schmidt when $j \in \pi_0(\Gamma^0)$. We first consider when $j \in \pi_0(\Gamma^1)$.

Let

$$K = \{z \in \mathbb{C} : 1 - \epsilon' \leq |z| \leq 1\}$$

for some small ϵ'. There is a Cauchy transform $\mathcal{C}_K$ for K given by $(g_j \circ \rho)$ and $q(g(z), g(w))$, and we have

$$p_j \mathcal{C} p_j = p_{S^1} \mathcal{C}_K p_{S^1}.$$

On the other hand, we have the standard Cauchy transform $\mathcal{C}_{st}$ on K given by $\frac{1}{w-z}$, and by Corollary A.14 $\mathcal{C}_K - \mathcal{C}_{st}$ is Hilbert-Schmidt. Hence $p_j\mathcal{C}p_j - p_{S^1}\mathcal{C}_{st}p_{S^1}$ is Hilbert-Schmidt as well. But

$$p_{S^1}\mathcal{C}_{st}p_{S^1} = p_{S^1,+} = p_{j,+},$$

which completes the proof in the case $j \in \pi_0(\Gamma^1)$.

If $j \in \pi_0(\Gamma^0)$, we use the same argument with

$$K = \{z \in \mathbb{C} : 1 \leq |z| \leq 1 + \epsilon'\}.$$

The only modification in this case is that $p_{S^1}\mathcal{C}_{st}p_{S^1}$ and $(\mathbf{1} - p_{S^1,+})$ do not quite coincide, but instead differ by a rank 1 operator. □

We can now prove the main theorem of Appendix A.

Proof of Theorem A.1. We have proven that $q_X - p_\Gamma$ is Hilbert-Schmidt when $V = \Sigma\times\mathbb{C}$ and the boundary parameterizations act identically on fibers. It suffices to consider when Σ is connected. Taking direct sums establishes the result when $V = \Sigma \times \mathbb{C}^k$, and by Propositions 4.5 and 2.21 we may relax the requirement that boundary parameterizations act identically. Since every holomorphic vector bundle on an open Riemann surface is trivializable, the result holds in general by conformal invariance.

We now show that $H^2(X)^\perp = M_{\pm\bar{z}}H^2((K_\Sigma \otimes X^d)^*)$, where K_Σ is understood as the element of $\mathcal{E}$ given by $(\Sigma, K_\Sigma, \gamma_*)$ with $\gamma = \beta|_{S^1}$. Since both sides take direct sums to direct sums and are covariant under fiberwise reparameterization, we may assume that V is a line bundle. Let $f = F \circ \gamma$ with F a non-vanishing holomorphic section of X, and let ρ be a holomorphic immersion of Σ as before. Choose r so that $rdz = g^*(d\rho|_\Gamma)$, and observe that $r \in H^2(K_\Sigma, \gamma)$. By Proposition 4.5,

$$M_{\pm\bar{z}}H^2((K_\Sigma \otimes X^d)^*) = \overline{f^{-1}rM_{\pm z}H^2(\Sigma,\gamma)} = \overline{f^{-1}}H^2(\Sigma,\gamma)^\perp.$$

On the other hand,

$$H^2(X)^\perp = (fH^2(\Sigma,\gamma))^\perp = \overline{f^{-1}}H^2(\Sigma,\gamma)^\perp,$$

which completes the proof. □

Before proving Corollary A.2, we need the corresponding statement that relates $H^2(X)$ and $H^2(\check{X})$ for $X \in \mathcal{E}_*$.

Lemma A.16. *Suppose $X \in \mathcal{E}_*$. Then*

$$H^2(\check{X}) = \{(f_j)_{j\in\pi_0(\check{\Gamma})} : \exists f_{j^0} = f_{j^1} \in H^{k(j^0)} \text{ with } (f_j)_{j\in\pi_0(\Gamma)} \in H^2(X)\}.$$

Proof. By the same observations as in the proof of Theorem A.1, we may assume that $X = (\Sigma, \Sigma\times\mathbb{C}, \gamma, \alpha, j^0, j^1)$ where γ acts identically on fibers. We will use the same notation as the proof of Theorem A.15.

Let ρ and q be a Cauchy kernel for X, and let $\mathcal{C}$ be the corresponding Cauchy transform. The Cauchy integral of f defines a holomorphic function F on $\mathring{\Sigma}$. For $j \in \pi_0(\Gamma)$ and $\delta > 0$ sufficiently small, define $f_{j,\delta} \in C^\infty(S^1)$ by $f_{j,\delta}(z) = F(g_j((1-\delta)z))$. We would like that $f_{j,\delta} \to f_j$ in L^2 as $\delta \to 0$, from which the conclusion would follow from Morera's theorem by a standard argument.

We have $f_j = (\mathcal{C}f)_j = \mathcal{C}_{st} f_j + (\mathcal{D}f)_j$, where $\mathcal{D}$ is an integral operator with smooth kernel. Hence $(\mathcal{D}f)_{j,\delta} \to (\mathcal{D}f)_j$ uniformly, and in particular in L^2. From [Bel92, Thm 6.3], we have $(\mathcal{C}_{st} f_j)_\delta \to (\mathcal{C}_{st} f_j)$, which completes the proof. □

Now Corollary A.2 follows from Theorem A.1, Lemma A.16 and Proposition 4.5.

Bibliography

[AS60] Lars V. Ahlfors and Leo Sario. *Riemann surfaces*. Princeton Mathematical Series, No. 26. Princeton University Press, Princeton, N.J., 1960.

[Ati71] Michael F. Atiyah. Riemann surfaces and spin structures. *Ann. Sci. École Norm. Sup. (4)*, 4:47–62, 1971.

[BDH13] Arthur Bartels, Christopher L. Douglas, and André Henriques. Conformal nets I: coordinate-free nets, 2013, arXiv:1302.2604.

[Bel92] Steven R. Bell. *The Cauchy transform, potential theory, and conformal mapping*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[BK01] Bojko Bakalov and Alexander Kirillov, Jr. *Lectures on tensor categories and modular functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001.

[Bor86] Richard E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. *Proc. Nat. Acad. Sci. U.S.A.*, 83(10):3068–3071, 1986.

[DHR69a] Sergio Doplicher, Rudolf Haag, and John E. Roberts. Fields, observables and gauge transformations. I. *Comm. Math. Phys.*, 13:1–23, 1969.

[DHR69b] Sergio Doplicher, Rudolf Haag, and John E. Roberts. Fields, observables and gauge transformations. II. *Comm. Math. Phys.*, 15:173–200, 1969.

[dlHJ95] Pierre de la Harpe and Vaughan Jones. An introduction to C*-algebras. Université de Genève, July 1995.

[DM99] Pierre Deligne and John W. Morgan. Notes on supersymmetry (following Joseph Bernstein). In *Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997)*, pages 41–97. Amer. Math. Soc., Providence, RI, 1999.

[FFRS06] Jens Fjelstad, Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. TFT construction of RCFT correlators. V. Proof of modular invariance and factorisation. *Theory Appl. Categ.*, 16:No. 16, 342–433, 2006.

[FLM88] Igor Frenkel, James Lepowsky, and Arne Meurman. *Vertex operator algebras and the Monster*, volume 134 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[For81] Otto Forster. *Lectures on Riemann surfaces*, volume 81 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1981. Translated from the German by Bruce Gilligan.

[FRS89] K. Fredenhagen, K.-H. Rehren, and B. Schroer. Superselection sectors with braid group statistics and exchange algebras. I. General theory. *Comm. Math. Phys.*, 125(2):201–226, 1989.

[FRS92] Klaus Fredenhagen, Karl-Henning Rehren, and Bert Schroer. Superselection sectors with braid group statistics and exchange algebras. II. Geometric aspects and conformal covariance. *Rev. Math. Phys.*, (Special Issue):113–157, 1992. Special issue dedicated to R. Haag on the occasion of his 70th birthday.

[FRS02] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. TFT construction of RCFT correlators. I. Partition functions. *Nuclear Phys. B*, 646(3):353–497, 2002.

[FRS04a] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. TFT construction of RCFT correlators. II. Unoriented world sheets. *Nuclear Phys. B*, 678(3):511–637, 2004.

[FRS04b] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. TFT construction of RCFT correlators. III. Simple currents. *Nuclear Phys. B*, 694(3):277–353, 2004.

[FRS05] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. TFT construction of RCFT correlators. IV. Structure constants and correlation functions. *Nuclear Phys. B*, 715(3):539–638, 2005.

[FRS10] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. Twenty five years of two-dimensional rational conformal field theory. *J. Math. Phys.*, 51(1):015210, 19, 2010.

[FZ92] Igor B. Frenkel and Yongchang Zhu. Vertex operator algebras associated to representations of affine and Virasoro algebras. *Duke Math. J.*, 66(1):123–168, 1992.

[Gaw00] Krzysztof Gawędzki. Conformal field theory: a case study. In *Conformal field theory (Istanbul, 1998)*, volume 102 of *Front. Phys.*, page 55. Adv. Book Program, Perseus Publ., Cambridge, MA, 2000.

[GN67] R. C. Gunning and Raghavan Narasimhan. Immersion of open Riemann surfaces. *Math. Ann.*, 174:103–108, 1967.

[HK64] Rudolf Haag and Daniel Kastler. An algebraic approach to quantum field theory. *J. Mathematical Phys.*, 5:848–861, 1964.

[HL94] Yi-Zhi Huang and James Lepowsky. Tensor products of modules for a vertex operator algebra and vertex tensor categories. In *Lie theory and geometry*, volume 123 of *Progr. Math.*, pages 349–383. Birkhäuser Boston, Boston, MA, 1994.

[Hua97] Yi-Zhi Huang. *Two-dimensional conformal geometry and vertex operator algebras*, volume 148 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1997.

[Hua98] Yi-Zhi Huang. Genus-zero modular functors and intertwining operator algebras. *Internat. J. Math.*, 9(7):845–863, 1998.

[Kac98] Victor Kac. *Vertex algebras for beginners*, volume 10 of *University Lecture Series*. American Mathematical Society, Providence, RI, second edition, 1998.

[KLM01] Yasuyuki Kawahigashi, Roberto Longo, and Michael Müger. Multi-interval subfactors and modularity of representations in conformal field theory. *Comm. Math. Phys.*, 219(3):631–669, 2001.

[KR87] V. G. Kac and A. K. Raina. *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, volume 2 of *Advanced Series in Mathematical Physics*. World Scientific Publishing Co., Inc., Teaneck, NJ, 1987.

[LM89] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.

[MS89] Gregory Moore and Nathan Seiberg. Classical and quantum conformal field theory. *Comm. Math. Phys.*, 123(2):177–254, 1989.

[Pos03] Hessel B. Posthuma. *Quantization of Hamiltonian loop group actions*. PhD thesis, University of Amsterdam, 2003.

[PS86] Andrew Pressley and Graeme Segal. *Loop groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1986. Oxford Science Publications.

[Sch78] Stephen Scheinberg. Uniform approximation by functions analytic on a Riemann surface. *Ann. of Math. (2)*, 108(2):257–298, 1978.

[Seg81] Graeme Segal. Unitary representations of some infinite-dimensional groups. *Comm. Math. Phys.*, 80(3):301–342, 1981.

[Seg04] Graeme Segal. The definition of conformal field theory. In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 421–577. Cambridge Univ. Press, Cambridge, 2004.

[ST04] Stephan Stolz and Peter Teichner. What is an elliptic object? In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 247–343. Cambridge Univ. Press, Cambridge, 2004.

[Tha92] Bernd Thaller. *The Dirac equation*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992.

[TK88] Akihiro Tsuchiya and Yukihiro Kanie. Vertex operators in conformal field theory on $\mathbf{P}^1$ and monodromy representations of braid group. In *Conformal field theory and solvable lattice models (Kyoto, 1986)*, volume 16 of *Adv. Stud. Pure Math.*, pages 297–372. Academic Press, Boston, MA, 1988.

[TUY89] Akihiro Tsuchiya, Kenji Ueno, and Yasuhiko Yamada. Conformal field theory on universal family of stable curves with gauge symmetries. In *Integrable systems in quantum field theory and statistical mechanics*, volume 19 of *Adv. Stud. Pure Math.*, pages 459–566. Academic Press, Boston, MA, 1989.

[Was98] Antony Wassermann. Operator algebras and conformal field theory. III. Fusion of positive energy representations of $\mathrm{LSU}(N)$ using bounded operators. *Invent. Math.*, 133(3):467–538, 1998.

[Was11] Antony Wassermann. Kac-Moody and Virasoro algebras. Course notes, University of Cambridge, 2011.

[Xu00] Feng Xu. Jones-Wassermann subfactors for disconnected intervals. *Commun. Contemp. Math.*, 2(3):307–347, 2000.

[Zhu96] Yongchang Zhu. Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, 9(1):237–302, 1996.