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GENERAL TREATMENT ON THE MULTIPLE FACTORIZATIONS IN THE DUAL RESONANCE MODELS—AND THE N-REGGEON AMPLITUDES. II.*

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ABSTRACT

We present a compact way of carrying out repeated factorizations on the dual amplitude. Prescriptions are given for writing down the multiply factorized tree amplitudes. As an application of the prescriptions, we derive the N-reggeon amplitude.

I. INTRODUCTION

In this paper, we use the fact that the projective transformation of cross-ratio is invariant under changing of the projective frames (duality) to generalize the multiple-factorization technique developed in a previous paper.[1] We find a neat and compact way of carrying out the multiple factorizations on the dual amplitude. As a consequence of this, we obtain a set of prescriptions which enable us to write down directly the multiply-factorized tree amplitudes by simply examining the corresponding tree diagrams. Applying the prescriptions in a particular case, we obtain the formula for the N-reggeon amplitudes.

In Sec. II we explicitly carry out the third and the fourth factorizations and prove the factorization of the quadruply factorized tree into two triply factorized trees. We thus discuss the application of the quadruply factorized tree to the general double-loop calculation. In Sec. III we give the prescriptions for writing down an arbitrarily multiply factorized tree amplitude. And in Sec. IV we consider a particular case of multiply factorized tree to obtain the N-reggeon amplitude. In the conclusion we discuss the difficulties due to linear dependences.

II. THE THIRD AND FOURTH FACTORIZATIONS

It is convenient to follow the notations of Kikkawa and Sato.[2] They define

$$|p_i) \equiv \left(\frac{p_i}{1^{\frac{1}{2}}}, \frac{p_i}{2^{\frac{1}{2}}}, \frac{p_i}{3^{\frac{1}{2}}}, \cdots \right) ,$$

$$|a^R) \equiv \left(a_1^{\ R}, a_2^{\ R}, a_3^{\ R}, \cdots \right) ,$$

$$(M_+)_{nm} \equiv (\tilde{M}_+)_{mn} = \left(\frac{n}{m}\right)^{\frac{1}{2}} (-)^n \binom{-m}{n} ,$$

$$(\tilde{M}_-)_{nm} \equiv (M_-)_{mn} = \left(\frac{n}{m}\right)^{\frac{1}{2}} (-)^n \binom{m}{n} ,$$

$$(M_0)_{nm} \equiv \delta_{nm} , \tag{1}$$

$$(a|x \equiv \sum_{n=1}^{\infty} a_n x^n ,$$

$$(a|x|b) \equiv \sum_{n=1}^{\infty} a_n x^n b_n ,$$

$$(a|xM_\pm y\ M_\pm z|b) \equiv \sum_{n,m,k=1}^{\infty} a_n x^n (M_\pm)_{nm} y^m (M_\pm)_{mk} z^k b_k .$$

In the Appendix we give all the identities used in this paper. We start with the double factorized tree of Eq. (17) of Ref. 1:

$$\left\langle 0 \left| \tilde{G}^{(2)}_{(W)}(a,b) \right| \begin{matrix}\lambda_a \\ \lambda_b\end{matrix} \right\rangle = \int \pi\, dw_i \{W_M\} \left\langle 0 \right| \exp\Bigg[\sum_{nm} \frac{a_n\, b_m}{(nm)^{\frac{1}{2}}} \left(\frac{w_{L+1} - w_L}{w_{L+1} - w_{L-1}} \right)^n$$

$$\times\; B_{nm}\left(\frac{w_{L-1}}{w_L}\right) w_L{}^m \Bigg] \exp\Bigg[\sum_n \frac{a_n}{n^{\frac{1}{2}}} \sum_{\substack{i=1 \\ (i \neq L)}}^{M} p_i \left(\frac{w_{L+1} - w_L}{w_{L+1} - w_{L-1}} \right)^n \cdot \left(\frac{w_{L-1} - w_i}{w_L - w_i} \right)^n$$

$$+ \sum_m \frac{b_m}{m^{\frac{1}{2}}} \left(\sum_{i=1}^{M-1} p_i\, w_i{}^m \right) \Bigg] \left| \begin{matrix}\lambda_a \\ \lambda_b\end{matrix} \right\rangle , \tag{2}$$

where

$$B_{nm}(x) \equiv \sum_{i=0}^{m} \binom{m}{i} \binom{-n}{m-i} (-)^m m\, x^i , \tag{3}$$

and $w_M = \infty$, $w_{M-1} = 1$, $w_1 = 0$ (Fig. 1). In our new notations, we write the formula as follows:

$$\tilde{G}^{(2)}_{(W)}(a^L, a^M) = \int \pi w_i \{W_M\}$$

$$\times\; \exp\Bigg\{ \sum_{\substack{i=1 \\ (i \neq L)}}^{M} (a^L | P(w_L, w_{L-1}, w_{L+1}, w_i) | p_i) + \sum_{\substack{i=1 \\ (i \neq M)}}^{M} (a^M | P(w_M, w_1, w_{M-1}, w_i) | p_i)$$

$$+ (a^L | P(w_L, w_{L-1}, w_{L+1}, w_M) M\_P(w_{L-1}, w_1, w_L, w_M) \tilde{M}\_P(w_M, w_1, w_{M-1}, w_L) | a^M) \Bigg\} , \tag{4}$$

where $a^L \equiv a$, $a^M \equiv b$, and

$$P(w_i,w_j,w_k,w_m) \equiv \frac{(w_i - w_k)(w_j - w_m)}{(w_i - w_m)(w_j - w_k)} \quad . \tag{5}$$

In Eq. (4), we have omitted the coherent states $|\lambda_a\rangle$, $|\lambda_b\rangle$, and we will continue to do so throughout the paper if no confusion arises. We observe that the cross-rations $P(w_i,w_j,w_k,w_l)$ appear in a rather regular fashion, depending on the dot positions of the excited legs. This turns out to be a very general feature as we go on to higher factorizations.

Let us now do the third factorization. In Eq. (4) we change from the frame $w_M = \infty$, $w_{M-1} = 1$, $w_1 = 0$ to the frame $w_1' = \infty$, $w_2' = 1$, $w_M' = 0$ (Fig. 1), i.e.,

$$w_i = P(w_R',w_{R-1}',w_{R+1}',w_i') \ ;$$

then Eq. (4) becomes

$$\tilde{G}^{(2)}_{(W')}(a^R,a^S) = \int \pi w_i' \{W_M'\}$$

$$\times \exp\left\{ \sum_{\substack{i=1\\(i\neq R)}}^{M} (a^R|P(w_R',w_{R-1}',w_{R+1}',w_i')|p_i) + \sum_{\substack{i=1\\(i\neq S)}}^{M} (a^S|P(w_S',w_{S+1}',w_{S-1}',w_i')|p_i) \right.$$

$$\left. + (a^S|P(w_S',w_{S+1}',w_{S-1}',w_R')M\_P(w_{S+1}',w_{R-1}',w_S',w_R')\tilde{M}\_P(w_R',w_{R-1}',w_{R+1}',w_S')|a^R) \right\} . \tag{6}$$

Divide w_i' into r_i, s_j, and t such that

$$\text{R frame:}\quad r_T=\infty,\quad r_{T-1}=1,\quad r_1=0;\ r_{T+i}=r_i\ ,$$

$$\text{S frame:}\quad s_{T-1}=\infty,\quad s_T=1,\quad s_M=0;\ s_{M+i}=s_{T-2+i}\ , \tag{7}$$

$$\text{W' frame:}\quad w'_1=\infty,\quad w'_2=1,\quad w'_M=0;\ w'_{M+i}=w'_i\ .$$

Thus

$$w'_i = \frac{r_2}{r_i}, \qquad i=1,2,\cdots,T-1,$$

$$w'_j = r_2 t\, s_j\ , \qquad j=T,T+1,\cdots,M\ . \tag{8}$$

We introduce the creation and the destruction harmonic oscillator operatores a^{T+}, a^T to remove t and s_j, $j=T,T+1,\cdots,M$. As discussed in Ref. 1, we have from Eqs. (6), (7), (8),

$$\exp\left\{\sum_{\substack{i=1\\(i\neq R)}}^{M}(a^R|P(w'_R,w'_{R-1},w'_{R+1},w'_i)|p_i)\right\}\Longrightarrow\exp\left\{\sum_{\substack{i=1\\(i\neq R)}}^{T}\left[(a^R|\right.\right.$$

$$\left.\times\ P(r_R,r_{R-1},r_{R+1},r_i)|p_i)\right] + (a^R|P(r_R,r_{R-1},r_{R+1},r_T)M_-$$

$$\left.\times\ P(r_{R-1},r_1,r_R,r_T)\tilde{M}_-P(r_T,r_1,r_{T-1},r_R)|a^T)\right\}\ , \tag{9.a}$$

$$\exp\left\{\sum_{\substack{i=1\\(i\neq S)}}^{M}(a^S|P(w'_S,w'_{S+1},w'_{S-1},w'_i)|p_i)\right\}\Longrightarrow\exp\left\{\sum_{\substack{i=1\\(i\neq S)}}^{T}\left[(a^S|\right.\right.$$

$$\left.\times\ P(r_S,r_{S+1},r_{S-1},r_i)|p_i)\right] + (a^S|P(r_S,r_{S+1},r_{S-1},r_T)M_-$$

$$\left.\times\ P(r_{S+1},r_1,r_S,r_T)\tilde{M}_-P(r_T,r_1,r_{T-1},r_S)|a^T)\right\}\ , \tag{9.b}$$

and

$$\exp(a^S|P(w'_S,w'_{S+1},w'_{S-1},w'_R)M\_P(w'_{S+1},w'_{R-1},w'_S,w'_R)\tilde{M}\_P(w'_R,w'_{R-1},w'_{R+1},w'_S)|a^R)$$

$$= \exp(a^S|P(r_S,r_{S+1},r_{S-1},r_R)M\_P(r_{S+1},r_{R-1},r_S,r_R)\tilde{M}\_P(r_R,r_{R-1},r_{R+1},r_S)$$

$$\times |a^R) \quad . \qquad (9.c)$$

Substituting Eqs. (8) and (9) in Eq. (6), and defining

$$\tilde{G}^{(2)}_{(W')}(a^R,a^S) = \left\langle {}_T0|\tilde{G}^{(3)}_{(R)}(a^R,a^S,a^T)\; D(R_{a^T},s_3)\; G^{(1)}_{(S)}(a^{T+})|0_T\right\rangle \quad ,$$

we thus obtain the triply factorized tree (Fig. 2):

$$\tilde{G}^{(3)}_{(R)}(a^R,a^S,a^T) = \int \pi r_i\{R_T\}$$

$$\times \exp\Bigg\{\sum_{\substack{i=1\\(i\neq R)}}^{T}(a^R|P_R(i)|p_i) + \sum_{\substack{i=1\\(i\neq S)}}^{T}(a^S|P_S(i)|p_i) + \sum_{\substack{i=1\\(i\neq T)}}^{T}(a^T|P_T(i)|p_i)$$

$$+ (a^R|P_R(S)\; M\_P(R-1,S+1,R,S)\; \tilde{M}\_P_S(R)|a^S)$$

$$+ (a^S|P_S(T)\; M\_P(S+1,1,S,T)\; \tilde{M}\_P_T(S)|a^T)$$

$$+ (a^T|P_T(R)\; M\_P(1,R-1,T,R)\; \tilde{M}\_P_R(T)|a^R)\Bigg\} \quad , \qquad (10)$$

where

$$P_S(i) = P(S,S+1,S-1,i) \; ,$$

$$P_T(i) = P(T,1,T-1,i) \; ,$$

and

$$P_R(i) = P(R,R-1,R+1,i) = \frac{(r_R - r_{R+1})(r_{R-1} - r_i)}{(r_{R-1} - r_{R+1})(r_R - r_i)} \quad . \tag{11}$$

It is clear that the new cross-ratios occur in the same manner as before. It is also interesting to observe, from the identities given in the Appendix, that application of the twist operator, say $\Omega^+(-p_R, a^R)$ on the r_R-leg, is simply equivalent to interchange of r_{R+1} with r_{R-1} everywhere in Eq. (10). This insures that the cross-ratio $P_R(i)$ has the same form whatever the positions of the dots.

We now proceed to write down the fourth factorization; we shall then be able to infer the general prescriptions for an arbitrary number of factorizations on the dual amplitude. Applying techniques identical to[1] those used in Eqs. (6), (7), (8), (9), and (10), we obtain the result for the quadruply factorized tree (Fig. 3):

$$\tilde{G}^{(4)}_{(W)}(a^R, a^S, a^T, a^U) = \int \pi dw_i \{W_U\}$$

$$\times \exp\left\{ \sum_{\substack{i=1 \\ (i\neq U)}}^{U} (a^U|P_U(i)|p_i) + \sum_{\substack{i=1 \\ (i\neq R)}}^{U} (a^R|P_R(i)|p_i) \right.$$

$$+ \sum_{\substack{i=1 \\ (i\neq S)}}^{U} (a^S|P_S(i)|p_i) + \sum_{\substack{i=1 \\ (i\neq T)}}^{U} (a^T|P_T(i)|p_i)$$

$$+ (a^R|P_R(S)\, M\_P(R+1,S-1,R,S)\, \tilde{M}\_P_S(R)|a^S)$$

$$+ (a^S|P_S(T)\, M\_P(S-1,T-1,S,T)\, \tilde{M}\_P_T(S)|a^T)$$

Equation (12) Continued

Equation (12) Continued.

$$+ \ (a^T|P_T(R)\ M\_P(T-1,R+1,T,R)\ \tilde{M}\_P_R(T)|a^R)$$

$$+ \ (a^R|P_R(U)\ M\_P(R+1,1,R,U)\ \tilde{M}\_P_U(R)|a^U)$$

$$+ \ (a^S|P_S(U)\ M\_P(S-1,1,S,U)\ \tilde{M}\_P_U(S)|a^U)$$

$$+ \ (a^T|P_T(U)\ M\_P(T-1,1,T,U)\ \tilde{M}\_P_U(T)|a^U)\Bigg\} \quad , \qquad (12)$$

where $w_U = \infty$, $w_1 = 0$, $w_{U-1} = 1$, and

$$P_U(i) = P(U,1,U-1,i) \quad .$$

By studying Eqs. (10) and (12) it is not hard to infer the general prescriptions for the multiply factorized dual tree. It is also not hard to write down the explicit form of the amplitude with an arbitrary number of excited legs. We will proceed to do so in the next section.

Before leaving this section we discuss the factorization property of Eq. (12). We first express Eq. (12) in the W' frame defined by $w_1' = \infty$, $w_2' = 1$, $w_U' = 0$ (Fig. 4), i.e.,

$$w_i = P(w_V', w_{V+1}', w_{V-1}', w_i') \quad .$$

We then introduce R, S frames as before. Using the inverse relations of Eq. (8),

$$r_i = \frac{w'_{M-1}}{w'_i}, \qquad i < M-1 ,$$

$$s_j = \frac{w'_j}{w'_M}, \qquad j > M , \tag{13}$$

$$t = \frac{w'_M}{w'_{M-1}},$$

and the identities in the Appendix, we can then prove that the following factorized expression is identical to the right-hand side of Eq. (12),

$$\left\langle {}_M0 \left| \tilde{G}^{(3)}_{(R)}(a^S,a^V,a^M)\ D(a^{M-1^+}\ a^M,t)\ \tilde{G}^{(3)}_{(S)}(a^{M-1^+},a^R,a^T) \right| 0_M \right\rangle$$

$$= \left\langle {}_M0 \right| \Bigg\{ \int \pi dr_i \{R_M\} \exp \Bigg[\sum_{\substack{i=1 \\ (i\neq V)}}^{M} (a^V|P_{r_V}(i)|p'_i) + \sum_{\substack{i=1 \\ (i\neq S)}}^{M} (a^S|P_{r_S}(i)|p'_i)$$

$$+ \sum_{\substack{i=1 \\ (i\neq M)}}^{M} (a^M|P_{r_M}(i)|p'_i) + (a^V|P_{r_V}(M)\ M\_P_r(V+1,1,V,M)\ \tilde{M}\_P_{r_M}(V)|a^M)$$

$$+ (a^S|P_{r_S}(M)\ M\_P_r(S-1,1,S,M)\ \tilde{M}\_P_{r_M}(S)|a^M)$$

$$+ (a^V|P_{r_V}(S)\ M\_P_r(V+1,S-1,V,S)\ \tilde{M}\_P_{r_S}(V)|a^S) \Bigg] \Bigg\}$$

$$\times\ D(a^{M-1^+}\ a^M,t)$$

Equation (14) Continued

$$\times \left\{ \int \pi ds_j \{S_{U-M}\} \exp\left[\sum_{\substack{i=M-1 \\ (i\neq R)}}^{U} (a^R|P_{s_R}(i)|p_i'') + \sum_{\substack{i=M-1 \\ (i\neq T)}}^{U} (a^T|P_{s_T}(i)|p_i'') \right.\right.$$

$$+ \sum_{\substack{i=M-1 \\ (i\neq M-1)}}^{U} (a^{M-1^+}|P_{s_{M-1}}(i)|p_i'')$$

$$+ (a^{M-1^+}|P_{s_{M-1}}(R)\ M\_P_s(U,R+1,M-1,R)\ \widetilde{M}\_P_{s_R}(M-1)|a^R)$$

$$+ (a^{M-1^+}|P_{s_{M-1}}(T)\ M\_P_s(U,T-1,M-1,T)\ \widetilde{M}\_P_{s_T}(M-1)|a^T)$$

$$\left.\left. + (a^T|P_{s_T}(R)\ M\_P_s(T-1,R+1,T,R)\widetilde{M}\_P_{s_R}(T)|a^R) \right] \right\} |0_M\rangle , \tag{14}$$

where $r_M = \infty,\quad r_{M-1} = 1,\quad r_1 = 0,$

$s_{M-1} = \infty,\quad s_M = 1,\quad s_U = 0\ ,$

and

$$P_{r_V}(i) = P(r_V, r_{V+1}, r_{V-1}, r_i) \ ,$$

$$P_r(V+1,1,V,M) = P(r_{V+1}, r_1, r_V, r_M) \ ,$$

$$P_{s_T}(i) = P(s_T, s_{T-1}, s_{T+1}, s_i) \ , \tag{15.a}$$

$$P_s(U,R+1,M-1,R) = P(s_U, s_{R+1}, s_{M-1}, s_R) \ , \quad \text{etc.}$$

and

$$\left[a_n^{\ M}, a_m^{\ M-1^+}\right] = \delta_{nm}, \qquad a^M|0_M\rangle = 0, \qquad \langle 0_M|a^{M-1^+} = 0 \ , \tag{15.b}$$

and all other commutators between a's vanish, also

$$
\begin{aligned}
p'_i &= p_i\,, & i &= 1,2,\cdots,M-1\,,\\
p''_j &= p_j\,, & j &= M,\cdots,U\,, \qquad\qquad (15.c)\\
p'_M &= \sum_{j=M}^{U} p''_j = -\sum_{i=1}^{M-1} p'_i = -p''_{M-1}\,.
\end{aligned}
$$

From Eq. (14) it is seen that Eq. (12) does factorize into the product of two triple-factorized trees which were given in Eq. (10). As a consequence of the factorizability of Eq. (12), we assert that if we join any pair of the excited legs we get the loop amplitudes. Namely, if we join the a^S leg with the a^U leg and the a^R leg with the a^T leg, we get the nonplanar double-loop diagram (Fig. 5), of which the planar double-loop diagram[3] is the special case when $S = U+1 = 1$, $R+1 = T$. And if we join the twisted a^R leg with the a^U leg, the twisted a^T leg with the a^S leg, we then obtain the overlapping double-loop diagram (Fig. 6). Thus Eq. (12) contains all possible perturbative unitarity diagrams with three-particle intermediate states, if we join the two pairs of the excited legs.

III. PRESCRIPTIONS OF THE MULTIPLY FACTORIZED TREE AMPLITUDE

Generalizing the result of the previous section, we now state explicitly the following set of rules in writing down the multiply factorized tree amplitude by simply inspection of the tree diagrams.

Rule 1. We assign to each leg (scalar or excited) one Koba-Nielsen variable w_i, $i = 1,2,\cdots,M$, and an incoming four-momentum p_i. Corresponding to each excited leg, we assign one destruction (creation) operator $a_{\mu,n}^{R}$ $(a_{\mu,n}^{R+})$, where μ is space-time indices $\mu = 1,2,3,4$; n is the excited harmonic oscillator mode in question, $n = 1,2,\cdots,\infty$, and the superscript R coincides with the labeling of the Koba-Nielsen variables, $R < M$.

Rule 2. The scalar part of the multiply factorized tree is the ordinary Koba-Nielsen representation[4] or the Donini-Sciuto representation[5] of the M-point dual amplitude, namely $\int \pi dw_i \{W_M\}$.

Rule 3. All destruction (creation) operators a^R (a^{R+}) appear only in the exponents. The exponential factors can be divided into two classes. One class involves the scalar product $a_\mu^R \cdot p_i^\mu$, the other class involves the scalar product of $a_\mu^R \cdot a_\mu^S$; $R,S \leqslant M$. The factors in the first class have the form

$$\exp\left\{\sum_{\substack{R \\ R\leqslant M}} \sum_{\substack{i=1 \\ (i\neq R)}}^{M} (a^R|P_R(w_i)|p_i)\right\} , \tag{16}$$

and those in the second class take the form

$$\exp\left\{\frac{1}{2}\sum_{\substack{R,S\leq M\\(R\neq S)}}(a^R|P_R(S)\;M_-P(R\pm 1,S\pm 1,R,S)\;\tilde{M}_-P_S(R)|a^S)\right\}\;, \quad (17)$$

where

$$P_R(i) = \begin{cases} P(w_R, w_{R-1}, w_{R+1}, w_i), & \text{if the dot of } w_R \text{ leg lies between } w_R \text{ and } w_{R+1} \\ P(w_R, w_{R+1}, w_{R-1}, w_i), & \text{if the dot of } w_R \text{ leg lies between } w_R \text{ and } w_{R-1}, \end{cases} \quad (18.a)$$

and

$$P(R\pm 1,S\pm 1,R,S) = \begin{cases} P(w_{R\pm 1}, w_{S\pm 1}, w_R, w_S), & \text{if the dot of } w_R \text{ leg lies between } w_R \text{ and } w_{R\mp 1}, \text{ and the dot of } w_S \text{ leg lies between } w_S \text{ and } w_{S\mp 1}, \\ P(w_{R\pm 1}, w_{S\mp 1}, w_R, w_S), & \text{if the dot of } w_R \text{ leg lies between } w_R \text{ an } w_{R\mp 1}, \text{ and the dot of } w_S \text{ leg lies between } w_S \text{ and } w_{S\pm 1}. \end{cases} \quad (18.b)$$

Rule 4. When two excited legs, say w_R and w_{R+1}, are adjacent to each other, i.e., there is no external scalar leg between them, then we must take appropriate limits of the relevant terms in Eq. (17). The limits are as follows:

<u>Rule 4a</u>. When the w_R dot lies between w_{R-1} and w_R, and the w_{R+1} dot lies between w_R and w_{R+1} (Fig. 7a), then

$$(a^{R+1}|P_{R+1}(R)\ M_-P(R+2,R+1,R+1,R)\ \tilde{M}_-P_R(R+1)|a^R)$$

$$\longrightarrow (a^{R+1}|M_-\ P(R+2,R-1,R+1,R)|a^R)\quad . \qquad (19.a)$$

<u>Rule 4b</u>. When the w_R dot lies between w_{R-1} and w_R, and the w_{R+1} dot lies between w_{R+1} and w_{R+2} (Fig. 7b), then

$$(a^{R+1}|P_{R+1}(R)\ M_-\ P(R,R+1,R+1,R)\ \tilde{M}_-\ P_R(R+1)|a^R)$$

$$\longrightarrow (a^{R+1}|P(R+2,R-1,R+1,R)|a^R)\quad . \qquad (19.b)$$

<u>Rule 4c</u>. When the w_R dot and the w_{R+1} dot both lie between w_R and w_{R+1} (Fig. 7c), then

$$(a^{R+1}|P_{R+1}(R)\ M_-\ P(R+2,R-1,R+1,R)\ \tilde{M}_-\ P_R(R+1)|a^R)$$

$$\longrightarrow (a^{R+1}|\ M_-\ P(R+2,R-1,R+1,R)\ \tilde{M}_-|a^R)\quad . \qquad (19.c)$$

However, in this case we need an asymmetrical propagator[1] for the w_R leg (or w_{R+1} leg),

$$D'(R_{a^R}, R_{a^{R+1}}, w_{R+2};\ P_R^{\,2}) = \int_0^1 dt\ t^{-\alpha(p_R)-1+R_{a^R}} \times (1-t)^{a-1}$$

$$\left(\frac{1-t}{1-t\cdot P_R(w_{R+2})}\right)^{R_{a^{R+1}}-\alpha(p_{R+1})}\quad , \qquad (20)$$

where

$$P_R(w_{R+2}) = P(w_R, w_{R-1}, w_{R+1}, w_{R+2}) \cdot$$

In the cases of rules 4a and 4b, we do not modify the propagator at all.

We note that rules 4a and 4b are related to rule 4c by the twisting operations.

Rule 5. The two amplitudes with the dot on the opposite sides of the excited leg, say the w_R leg, are related to each other by the twist operator $\Omega^+(-p_R;\, a^R)$, or equivalently, by the interchange of w_{R+1} with w_{R-1} everywhere in Eqs. (16) and (17). The only additional complication to this rule is the case stated in rule 4c, where we need an asymmetrical propagator.

By using the above prescriptions and by directly inspecting the factorized tree diagram, we can write down the general formula for the multiply factorized tree amplitude

$$\left\langle 0 \middle| G^{(N)}_{(W)}(a^R) \middle| \text{all } \lambda_a \right\rangle = \int \pi dw_i \{W_M\} \left\langle 0 \middle| \exp\left\{ \sum_{\substack{R \\ R\leq M}} \sum_{\substack{i=1 \\ (i\neq R)}}^{M} (a^R|P_R(i)|p_i) \right.\right.$$

$$\left.\left. + \frac{1}{2} \sum_{\substack{R,S\leq M \\ (R\neq S)}} (a^R|P_R(S)\; M_-\; P(R\pm 1, S\pm 1, R, S)\; \tilde{M}_-\; P_S(R)|a^S) \right\} \middle| \text{all } \lambda_a \right\rangle, \quad (21)$$

where M is the total number of excited legs plus the external scalar legs. Also, $w_{M+i} \equiv w_i$, $a^{M+R} \equiv a^R$. We also give an alternative recursion formula for Eq. (21), which relates the integrand of the

Nth-factorized tree amplitude to the integrand of the (N-1)th-factorized tree amplitude (Fig. 8)

$$\left\langle 0 \left| G^{(N)}_{(W)}(a^R) \right| \text{all } \lambda_a \right\rangle = \int \pi dw_i \{W_M\} \left\langle 0 \right| \exp \left\{ \sum_{\substack{i=1 \\ (i \neq R)}}^{M} (a^R | P_R(i) | p_i) \right.$$

$$\left. + \sum_{\substack{S \leq M-1 \\ (S \neq R)}} (a^R | P_R(S)\, M_- \, P(R\pm 1, S\pm 1, R, S)\, \tilde{M}_- \, P_S(R) | a^S) \right\}$$

$$\left\{ \mathcal{G}^{(N-1)}_{(W)}(a^T;\ T \neq R) \right\} \left| \text{all } \lambda_a \right\rangle \quad , \qquad (22)$$

where $\mathcal{G}^{(N-1)}_{(W)}(a^T;\ T \neq R)$ is the operator part of the (N-1)th-factorized tree integrand, and the index R is the Koba-Nielsen label of the first-factorized excited leg. Equation (22) is useful in obtaining the N-reggeon amplitude. We will proceed to do so in the next section.

IV. THE N-REGGEON AMPLITUDES

As we have discussed in Ref. 1, to get the pure N-reggeon amplitude we should consider the case when $M = N+1$ in Eq. (21) or Eq. (22). Then we have the tree amplitude with one external scalar leg (the w_N leg) together with N excited legs. For simplicity we choose the dot positions of the N excited legs as shown in Fig. 9. Corresponding to Fig. 9, we apply the prescriptions in Sec. III, or use Eq. (22), to get the amplitude of N excited legs plus one external scalar leg (the w_N leg),

$$G^{(N)}_{(W)}(a^1,a^2,\cdots,a^{N-1},a^{N+1}) = \int \prod_{\substack{i\neq N+1 \\ N \\ 1}} dw_i \{W_{N+1}\}$$

$$\times \left\{ \mathcal{G}^{(N)}_{(W)}(a^1,a^2,\cdots,a^{N-1},a^{N+1}) \right\}$$

$$= \int \pi dw_i \{W_{N+1}\} \mathcal{G}^{(N-1)}_{(W)}(a^1,a^2,\cdots,a^{N-2},a^{N+1})$$

$$\times \exp \left\{ \sum_{\substack{i=1 \\ (i\neq N-1)}}^{N+1} (a^{N-1}|P_{N-1}(i)|p_i) \right.$$

$$\left. + \sum_{\substack{R=1 \\ (R\neq N,N-1)}}^{N+1} \left[(a^{N-1}|P_{N-1}(R)\, M_-\, P(N-2,R+1,N-1,R)\, \widetilde{M}_- P_R(N-1)|a^R) \right] \right\}, \tag{23}$$

where $P_{N-1}(i) = P(N-1,N-2,N,i)$ and we choose the frame

$$w_{N+1} = \infty, \quad w_N = 1, \quad w_1 = 0 \ .$$

Now analogously to Ref. 1, we consider the scalar w_N leg attached to the excited w_{N+1} leg. In other words, we want to write Eq. (23) in the form $W^{(N)}(a^R) \cdot D^{(N)} \cdot V(p_N)$ and then to remove $V(p_N) = \exp(a^{+N+1}|p_N) \times \exp(a^{N+1}|p_N)$ by letting $p_N \to 0$ and modifying the spectrum of the relevant trajectories, as we did[6] in Ref. 1. Guided by the tricks used in Eqs. (33), (34), and (35) of Ref. 1, where we have taken down the common factor $[(1 - w_3)/(1 - w_2)]^{R_c} \cdot (w_3)^{R_d}$ from the exp $\{\cdots\}$ of Eq. (33) to the propagator D' of Eq. (35), we now observe, from the factors $P_{N-1}(w_i)$ and $P_{N+1}(w_i)$ in Eq. (23), that we can bring down the similar common factor

$[(w_N - w_{N-1})/(w_N - w_{N-2})]^{R_{a^{N-1}}} \cdot (w_{N-1})^{R_{a^{N+1}}}$ from the exp $\{\cdots\}$ of Eq. (23) to the propagator $D^{(N)}$. It is this common factor that causes the factorization of w_{N-2} and t to be unfeasible, and presumably it is closely connected with the linear dependence problem.[7]

After setting $p_N \to 0$ and redefining $a^{N+1} = a^N$, $p_{N+1} = p_N$ and reintroducing the Koba-Nielsen variables[1] $w_N = \infty$, $w_{N-1} = 1$, $w_1 = 0$ in Eq. (23), we find the modified propagator[8] of the new w_N leg:

$$D^{(N)}(a^N, a^{N-1}, p_N) = \int_0^1 dt\, t^{-\alpha(p_N)-1+R_{a^N}} (1 - t)^{a-1}$$

$$\times \left(\frac{1 - t}{1 - t\, P(w_N, w_1, w_{N-1}, w_{N-2})} \right)^{R_{a^{N-1}} - \alpha(p_{N-1})} \qquad (24)$$

and the N-reggeon amplitude (Fig. 10)

$$W^{(N)}(a^1,a^2,\cdots,a^N) = \int \pi dw_i \{W_N\}$$

$$\exp\left\{ \sum_{\substack{R=1\\(R\neq N-1)}}^{N} \sum_{\substack{i=1\\(i\neq R)}}^{N} (a^R|P_R(i)|p_i) + \sum_{\substack{i=1\\(i\neq N-1)}}^{N} (a^{N-1}|P(N-1,N-2,N,i)|p_i)\right.$$

$$+ \sum_{\substack{R=1\\(R\neq N-1,N)}}^{N} \Big[(a^R|M\_P(R+1,R-2,R,R-1)|a^{R-1})$$

$$+ (a^{N-1}|P(N,N-3,N-1,N-2)|a^{N-2}) + (a^N|M\_P(1,N-2,N,N-1)\ \tilde{M}\_|a^{N-1})\Big]$$

$$+ \sum_{\substack{R=1\\(R\neq N-1,1)}}^{N} \Big[(a^R|P_R(R-2)\ M\_P(R+1,R-1,R,R-2)\ \tilde{M}\_P_{R-2}(R)|a^{R-2})$$

$$+ (a^{N-1}|P(N-1,N-2,N,N-3)M\_P(N-2,N-2,N-1,N-3)\tilde{M}\_P(N-3,N-2,N-4,N-1)|a^{N-3})$$

$$+ (a^1|P(1,2,N,N-1)M\_P(2,N-2,1,N-1)\tilde{M}\_P(N-1,N-2,N,1)|a^{N-1})\Big]$$

$$+ \cdots$$

$$+ \sum_{\substack{R=1\\(R\neq N-1,K-1)}}^{N} \Big[(a^R|P_R(R-K)M\_P(R+1,R-K+1,R,R-K)\tilde{M}\_P_{R-K}(R)|a^{R-K})$$

$$+ (a^{N-1}|P(N-1,N-2,N,N-K-1)M\_P(N-2,N-K,N-1,N-K-1)$$

$$\times\ \tilde{M}\_P(N-K-1,N-K,N-K-2,N-1)$$

Equation (25) Continued

$$\times \; |a^{N-K-1}) + (a^{K-1}|P_K(N-1)M\_P(K,N-2,K-1,N-1)\tilde{M}\_P(N-1,N-2,N,K-1)|a^{N-1})\Big]$$

$$+ \; \cdots$$

$$+ \sum_{\substack{R=1 \\ (R\neq N-1,\frac{N}{2}-1)}}^{N} \Big[\tfrac{1}{2}(a^R|P_R(R-\tfrac{N}{2})M\_P(R+1,R-\tfrac{N}{2}+1,R,R-\tfrac{N}{2})\tilde{M}\_P_{R-\frac{N}{2}}(R)|a^{R-\frac{N}{2}})$$

$$+ \; (a^{N-1}|P(N-1,N-2,N,\tfrac{N}{2}-1)M\_P(N-2,\tfrac{N}{2},N-1,\tfrac{N}{2}-1)\tilde{M}\_P_{\frac{N}{2}-1}(N-1)|a^{\frac{N}{2}-1})\Big]\Big\} \quad (25)$$

where $a^{N+i} \equiv a^i$, $P_R(S) \equiv P(w_R,w_{R+1},w_{R-1},w_S)$, and we have assumed N = even integer. For N = odd integer, we then replace $\frac{N}{2}$ by $\frac{N-1}{2}$ and drop the factor $\frac{1}{2}$ in the last two terms of Eq. (25). We now also can release the frame dependent constraint, since all terms in Eq. (25) are projectively invariant cross-ratios. Hence our result of N-reggeon amplitude is manifestly dual.[9]

We can further obtain the completely symmetrical N-reggeon amplitude by applying the twist operator $\Omega^+(-p_{N-1},a^{N-1})$ to the right of Eq. (25) times Eq. (24), and passing it to the left of the propagator $D^{(N)}(a^N,a^{N-1},p_N)$. Using the exact agruments given by Caneschi and Schwimmer,[12] we simply drop the factor

$$\left(\frac{1-t}{1-t\,P(N,1,N-1,N-2)}\right)^{R} a_{N-1}^{-\alpha(p_{N-1})}$$

in Eq. (24), and apply the twist operator $\Omega^+(-p_{N-1},a^{N-1})$ directly to Eq. (25). We finally get the completely symmetrical N-reggeon amplitude (Fig. 11):

$$W^{(N)}_{\text{sym.}}(a^1,a^2,\cdots,a^N) = \int \pi dw_i \{W_N\} \exp\left\{ \sum_{R=1}^{N} \sum_{\substack{i=1 \\ (i\neq R)}}^{N} (a^R|P_R(i)|p_i) \right.$$

$$+ \sum_{R=1}^{N} (a^R|M\_P(R+1,R-2,R,R-1)|a^{R-1})$$

$$+ \sum_{R=1}^{N} (a^R|P_R(R-2)M\_P(R+1,R-1,R,R-2)\tilde{M}\_P_{R-2}(R)|a^{R-2})$$

$$+ \cdots$$

$$+ \sum_{R=1}^{N} (a^R|P_R(R-K)M\_P(R+1,R-K+1,R,R-K)\tilde{M}\_P_{R-K}(R)|a^{R-K})$$

$$+ \cdots$$

$$\left. + \frac{1}{2} \sum_{R=1}^{N} (a^R|P_R(R-\tfrac{N}{2})M\_P(R+1,R-\tfrac{N}{2}+1,R,R-\tfrac{N}{2})\tilde{M}\_P_{R-\frac{N}{2}}(R)|a^{R-\frac{N}{2}}) \right\}, \qquad (26)$$

where $a^{N+R} \equiv a^R$; $P_R(S) \equiv P(R,R+1,R-1,S)$, etc.; and N = even integer. For N = odd integer, we replace $\frac{N}{2}$ by $\frac{N-1}{2}$ and drop the factor $\frac{1}{2}$ in the last term of Eq. (26). And the propagator of the w_N leg, $D^{(N)}(\cdots)$, becomes the ordinary one.[11]

It is also interesting to see that the symmetrical N-reggeon amplitude, Eq. (26), can be obtained directly by using the prescriptions of Sec. III and letting all the Koba-Nielsen variables associated with

with the external scalar legs vanish. One could also similarly use this illegal method to recover Eq. (25). However, we then lose the asymmetrical behavior of the propagator in the w_N leg. This again indicates the complications raised by the problem of linear dependence.[12,13]

V. CONCLUSION

We see that the multiple factorizations on the dual amplitude can be carried out in a rather neat and compact way. We also see that the N-reggeon amplitude takes a fairly simple form, which is manifestly dual. However, the intrusive problem of linear dependence still remains unsolved. In fact, it is clear that the N-reggeon amplitude itself should be factorizable. But in directly proving the factorization of Eq. (25) or (26), we encounter the unequal-mass problem, for which we do not have the bootstrap condition $a + bp_i^2 = 0$. Therefore the degeneracy of the spectrum of the trajectories appears to increase, though we know that it cannot do so. In order to decrease the degeneracy of the spectrum of the trajectories and at the same time prove the factorization of the N-reggeon amplitude, we shall have to take into account the existence of linear dependences among the excited states in the dual amplitude. Once we achieve the proof of factorization of the N-reggeon amplitude, we may be able to construct Nambu's[14] field theoretical-type dynamical equation of motion.

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APPENDIX

The notations in the following equations have been explained in Eq. (1). The function $\frac{B_{nm}(x)}{(nm)^{\frac{1}{2}}} = \sum_{i=0}^{m} \binom{n}{i}\binom{-n}{-m-i} (-1)^m m\, x^i$ defined in Ref. 1 is equal to $[M_-(1 - x)\tilde{M}_-]_{nm}$. We further denote

$$(xM_\pm yM_\pm z)_{nm} = x^n(M_\pm)_{ni}\, y^i(M_\pm)_{im} z^m , \quad \text{(sum over } i) ,$$

then we have the following set of identities

$$M_-\tilde{M}_- = M_+ , \tag{a}$$

$$M_+\tilde{M}_- = M_- , \tag{b}$$

$$M_-M_- = M_0 , \tag{c}$$

$$M_+xM_- = \frac{1}{1 - x} M_+ \frac{1}{1 - 1/x} , \tag{d}$$

$$M_-xM_- = (1 - x) M_- \frac{1}{1 - 1/x} , \tag{e}$$

$$M_+(1 - x)\tilde{M}_- = M_-(1 - 1/x) \tilde{M}_-x , \tag{f}$$

$$xM_+(1 - x)\tilde{M}_- = xM_-(1 - 1/x) \tilde{M}_-x = M_-(1 - x)M_+x . \tag{g}$$

The identities involving the twist operator are

$$(a|M_- = (\bar{a}| , \tag{h}$$

$$\tilde{M}_-|a) = |\bar{a}) , \tag{i}$$

$$(p|a) = -(p|\bar{a}) , \tag{j}$$

$$(p|x|a) = -(p|\bar{a}) + (p|(1-x)|\bar{a}) \; , \tag{k}$$

$$\bar{a} = (T_a^+)^{-1} \, a \, T_a^+ , \qquad T_a = :\exp\,(a^+|M_-|a): \; . \tag{ℓ}$$

The identities involving the cross-ratios are

$$P(x,y,z,w) = \frac{(x-z)(y-w)}{(x-w)(y-z)} \; , \tag{m}$$

$$1 - P(x,y,z,w) = P(x,z,y,w) \; , \tag{n}$$

$$\frac{1}{P(x,y,z,w)} = P(x,y,w,z) \; , \tag{o}$$

$$\begin{aligned} P(x,y,z,w) &= P(y,x,w,z) = P(w,z,y,x) \\ &= P(z,w,x,y) \; , \end{aligned} \tag{p}$$

$$P(x,y,z,w)\, P(x,y,u,z) = P(x,y,u,w) \; . \tag{q}$$

FOOTNOTES AND REFERENCES

* This work was supported in part by the U. S. Atomic Energy Commission.

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FIGURE CAPTIONS

Fig. 1. The double-factorized tree. The w_i refers to the frame defined by $w_M = \infty$, $w_1 = 0$, $w_{M-1} = 1$, and w_i' refers to a new frame $w_M' = 0$, $w_1' = \infty$, $w_2' = 1$ for the third factorization.

Fig. 2. The triply factorized tree.

Fig. 3. The quadruply factorized tree.

Fig. 4. The factorization of quadruply factorized tree into two triply factorized trees. The r_i and s_j refer to two new frames defined by $r_M = \infty$, $r_{M-1} = 1$, $r_1 = 0$ and $s_{M-1} = \infty$, $s_M = 1$, $s_U = 0$.

Fig. 5. Nonplanar double-loop.

Fig. 6. Overlapping double-loop.

Fig. 7a. The w_R dot lies between w_R and w_{R-1}, and the w_{R+1} dot lies between w_R and w_{R+1}.

Fig. 7b. The w_R dot lies between w_R and w_{R-1}, and the w_{R+1} dot lies between w_{R+1} and w_{R+2}.

Fig. 7c. The w_R dot lies between w_R and w_{R+1}, and the w_{R+1} dot lies between w_{R+1} and w_R.

Fig. 8. The $\underline{N}$th-factorized tree.

Fig. 9. The N excited legs plus one scalar leg.

Fig. 10. Twisted w_{N-1} leg of the N-reggeon amplitude.

Fig. 11. The symmetrical N-reggeon amplitude.

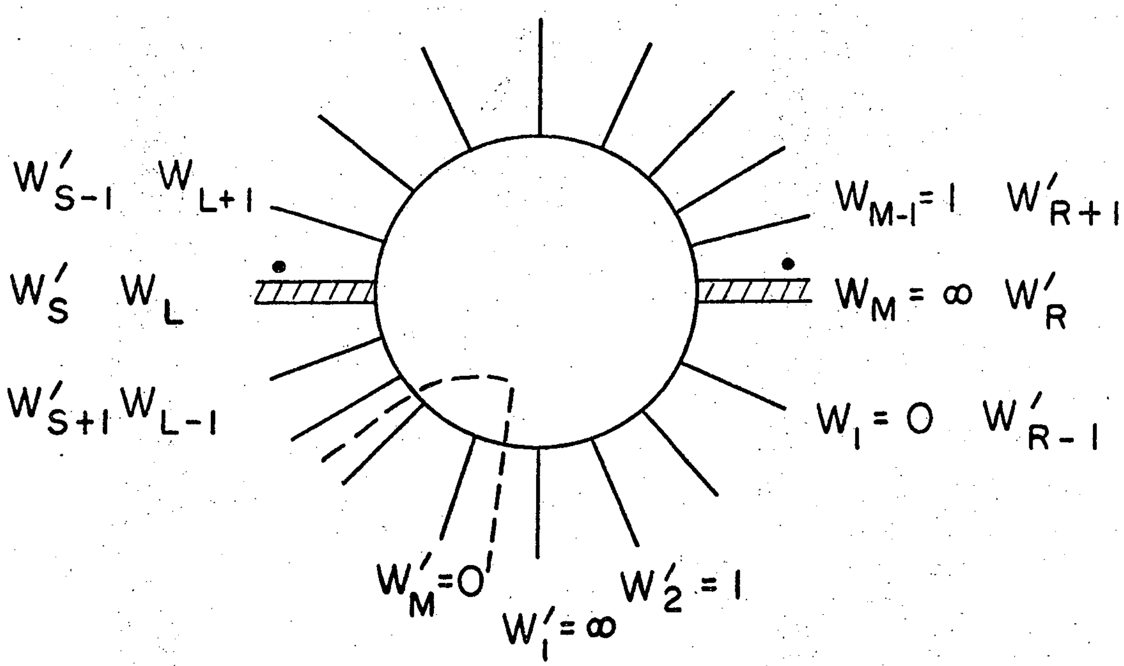

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Fig. 1

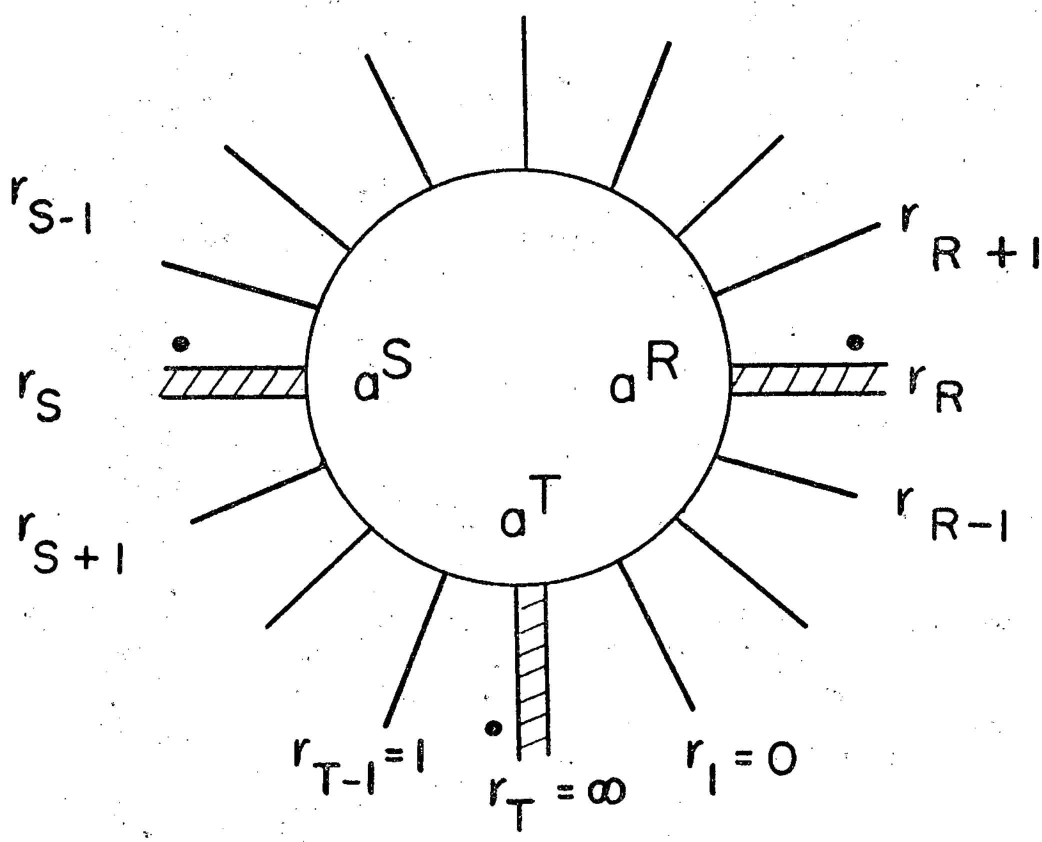

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Fig. 2

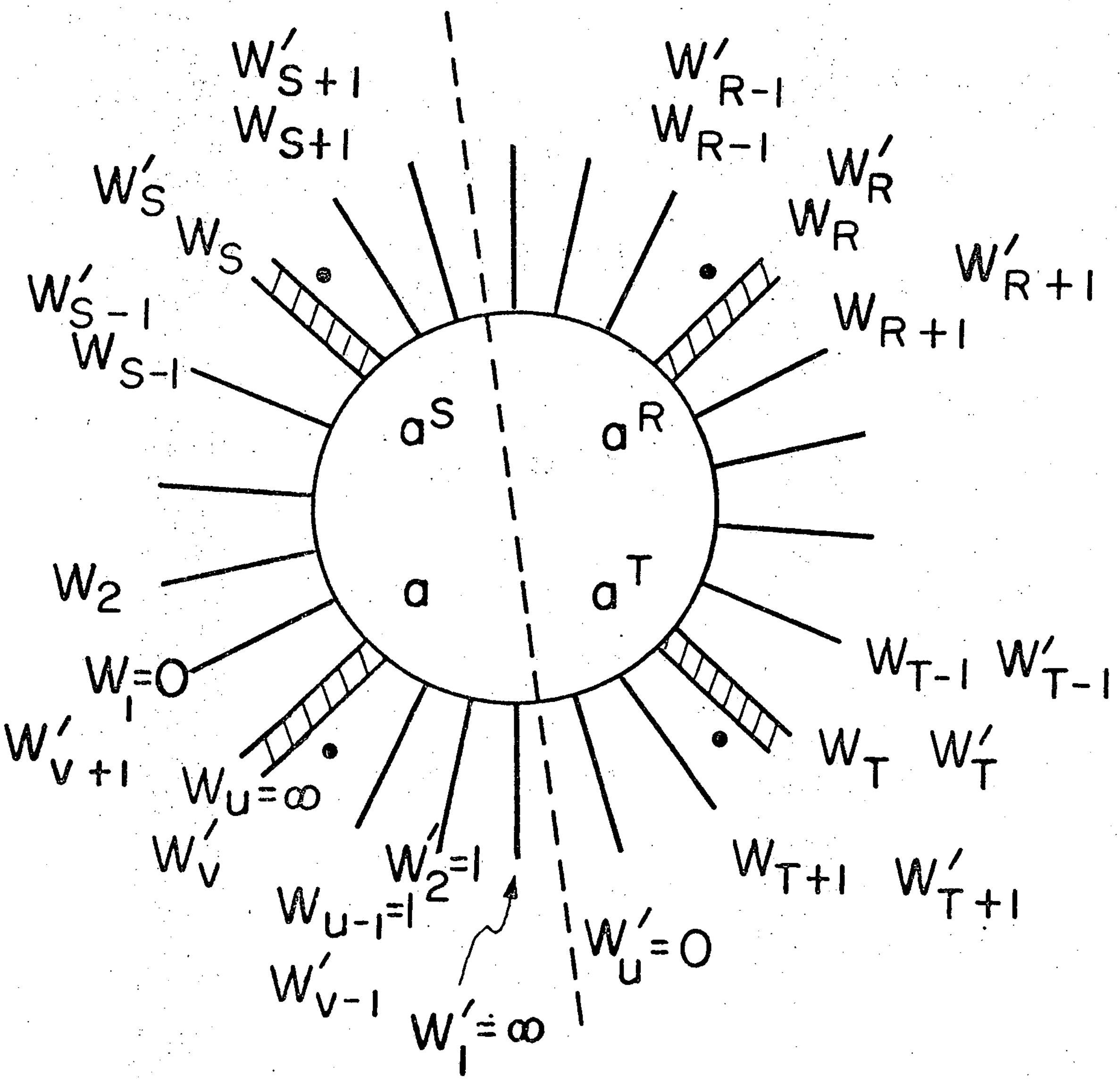

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Fig. 3

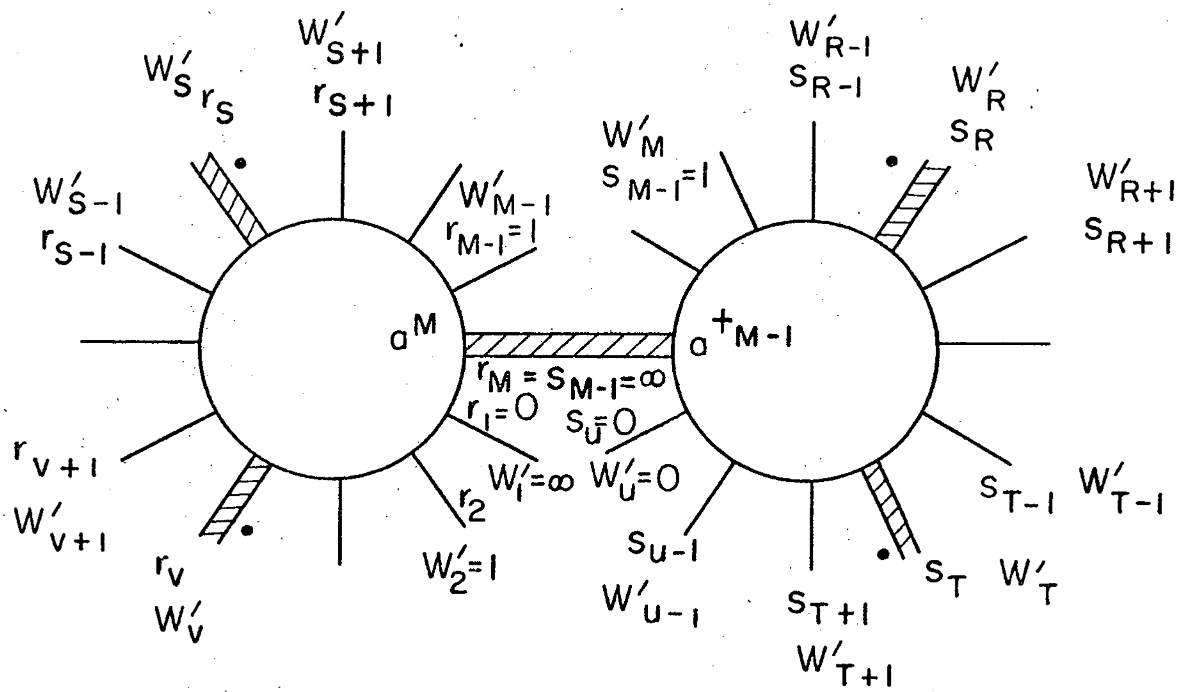

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Fig. 4

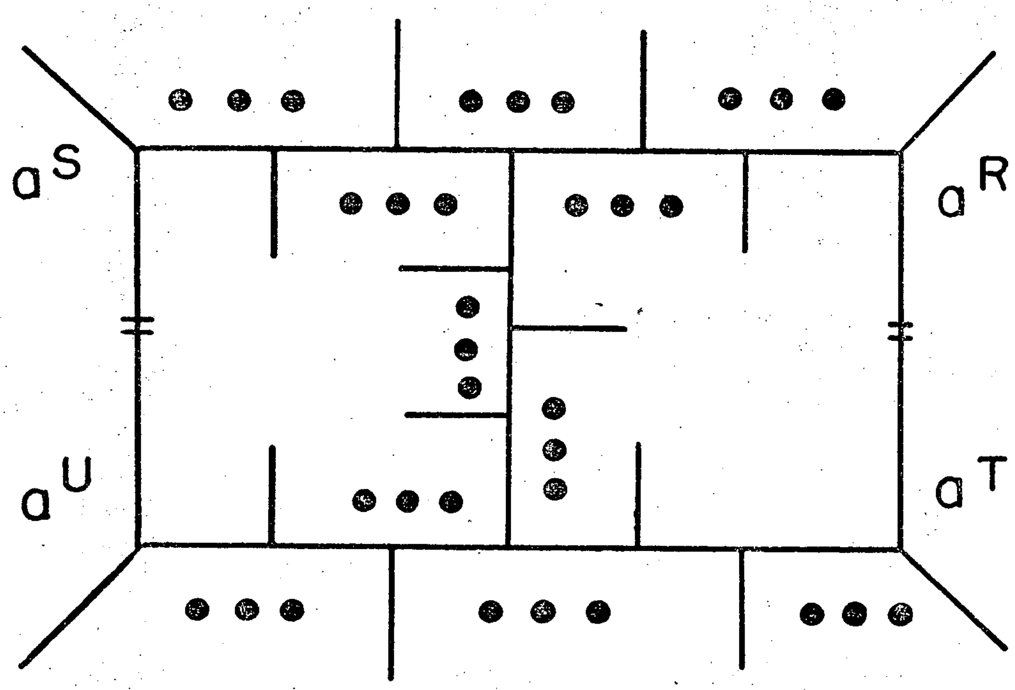

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Fig. 5

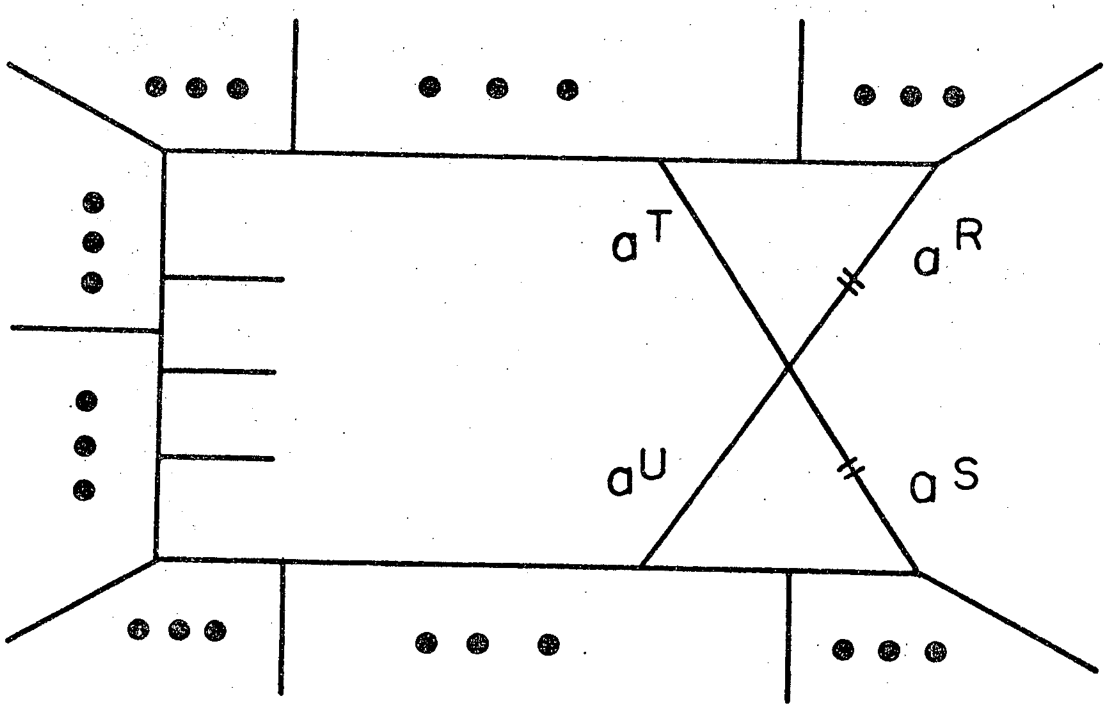

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Fig. 6

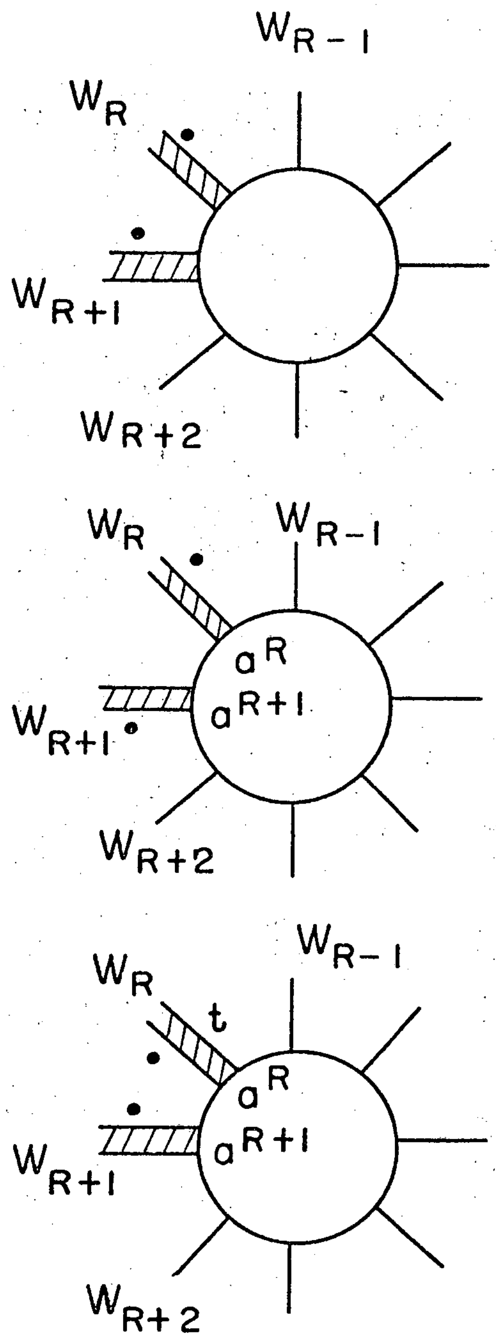

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Fig. 7 a, b, c

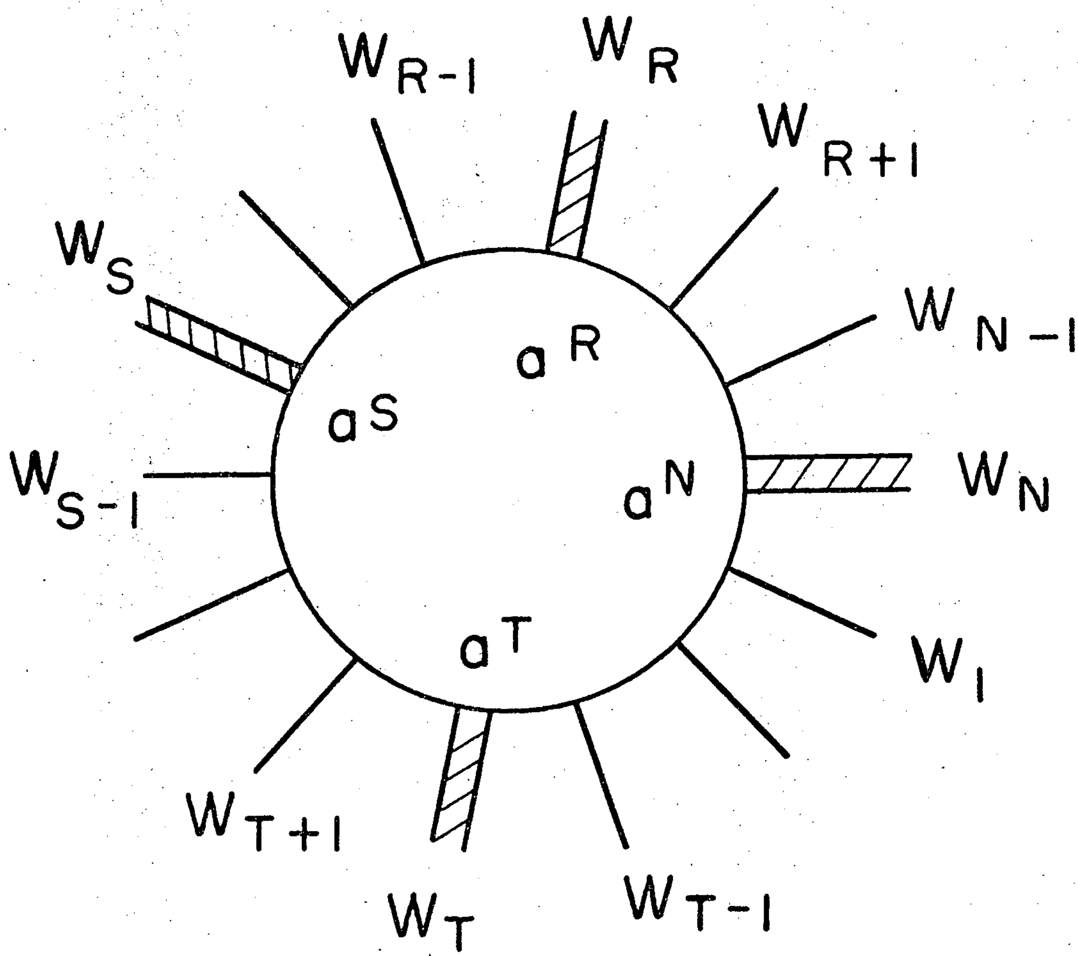

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Fig. 8

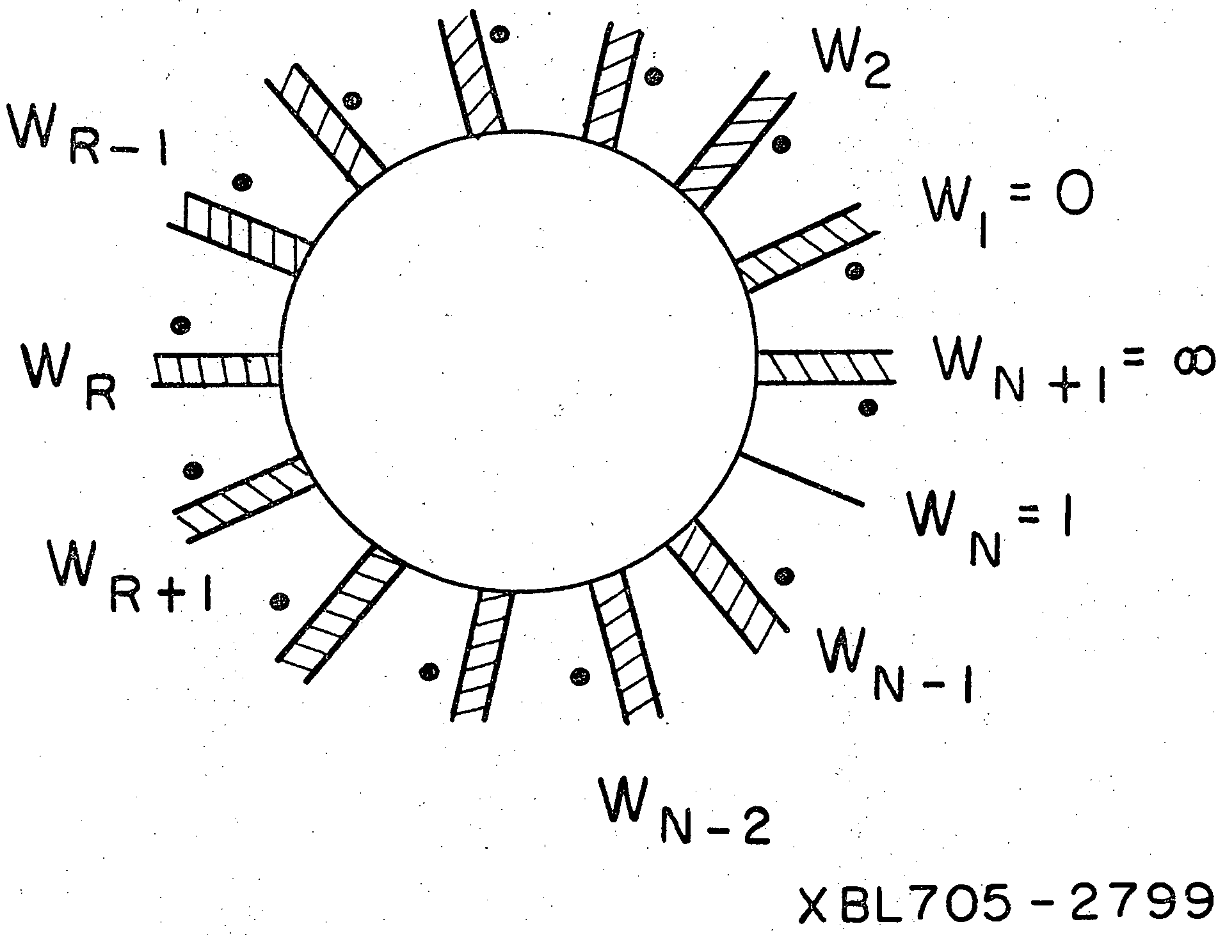

Fig. 9

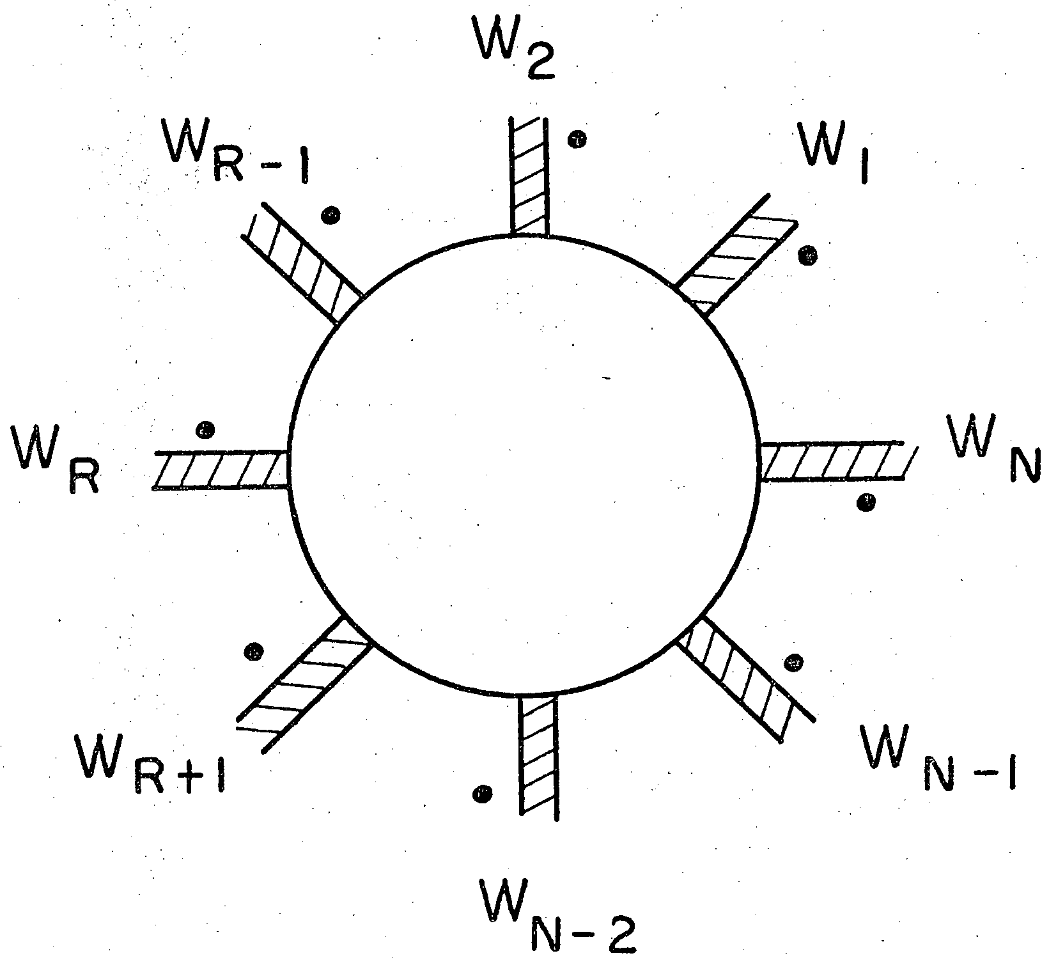

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Fig. 10

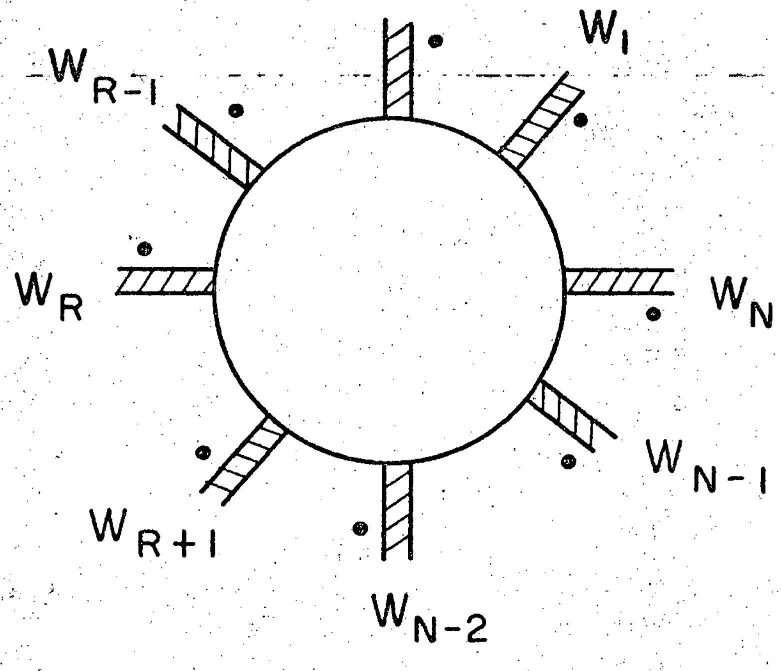

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Fig. 10

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