UNIVERSITY OF CALIFORNIA,
IRVINE

Cohomology of Symplectic Manifolds

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

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2024

DEDICATION

To my family, for keeping me anchored to the world outside of math.

TABLE OF CONTENTS

Page

LIST OF TABLES **v**

ACKNOWLEDGMENTS **vi**

VITA **vii**

ABSTRACT OF THE DISSERTATION **viii**

1 Background **1**
1.1 Symplectic Manifolds 1
1.2 Lefschetz Decomposition 2
1.3 Filtered Cohomology 4
1.4 Cone Representation 5
1.5 Computing the Cone Cohomology 13
1.6 Summary of Results 15

2 Maps on Cone Cohomology **20**
2.1 Restriction Map 20
2.2 Inverting the Restriction Map 27
2.3 Filter Reduction Map 28

3 Gysin Sequence for Filtered Cohomology **31**
3.1 Integration Along Fibers for Filtered Forms 32
3.2 Thom Isomorphism 34
3.3 Inverse of the Thom Isomorphism 41
3.3.1 The Gysin Sequence for p-Filtered Cone Complex 42

4 Applying Mayer-Vietoris to Cone Cohomology **44**
4.1 Primitive Case 45
4.2 Higher Filtration 60
4.2.1 Case 1: $p < m - 2$ 63
4.2.2 Case 2: $p = m - 2$ 69
4.2.3 Case 3: $p = m - 1$ 70
4.2.4 Case 4: $p = m$ 71
4.2.5 Case 5: $p > m$ 71

5 Cone Cohomology on Non-Symplectic Manifolds **73**
5.1 Motivating Example 74
5.2 Properties of the Cone Cohomology 76
5.3 Application to Nilmanifolds 79

Bibliography **82**

Appendix A Cone Cohomology of Nilmanifolds **83**

Appendix B Mathematica Code for Computing Cone Cohomology Dimensions **102**

LIST OF TABLES

Page

4.1 Summary of cone cohomologies for M_2, $M_1 \cap M_2$, and $\tilde{M}_2$. 50
4.2 Dimension of kernel and cokernel terms in the primitive $m > 2$ case. 58
4.3 Dimension of kernel and cokernel terms in the primitive $m = 2$ case. 60
4.4 Action of Lefschetz map on a $\mathbb{CP}^{m-1}$ bundle. 62
4.5 Dimension of kernel and cokernel terms in the non-primitive $p < m - 2$ case. 68
4.6 Dimension of kernel and cokernel terms in the non-primitive $p = m - 2$ case. 69
4.7 Dimension of kernel and cokernel terms in the non-primitive $p = m - 1$ case. 70

ACKNOWLEDGMENTS

I would like to thank my advisor Dr. Li-Sheng Tseng for his consistent guidance and encouragement while completing this work.

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ABSTRACT OF THE DISSERTATION

Cohomology of Symplectic Manifolds

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University of California, Irvine, 2024

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Symplectic manifolds have many connections to complex manifolds but lack many of the nice properties that come with a complex structure, especially the fact that symplectic manifolds have no local properties that can differentiate them from other symplectic manifolds. One of the most common global invariants of manifolds is the de Rham cohomology, but this can only detect the topological structure of the manifold and discards the additional symplectic structure. This work expands upon the definition of a cohomology theory specific to symplectic manifolds, specifically based on a viewpoint which comes from the concept of a mapping cone from homological algebra. We demonstrate how common operations on differential forms can be extended to these cone cohomologies with particular focus on the blowup along a symplectic submanifold. Additionally, we show how this approach can be extended to other types of manifolds using nilmanifolds as an example.

Chapter 1

Background

1.1 Symplectic Manifolds

The concept of a symplectic manifold has its origin in physics as a way of generalizing the rules of classical mechanics. Namely, it isolates the properties that a 2-form needs to induce a vector field which describes flow lines through phase space and a corresponding Hamiltonian. In order for the system to have the properties that (1) the Hamiltonian is constant along the flow and (2) the 2-form does not change along the flow it is necessary for the 2-form to be closed and non-degenerate. This determines the definition of a symplectic manifold.

Definition 1.1. *A symplectic manifold consists of a pair* (M, ω) *where* $\omega \in \Omega^2(M)$ *is both closed and non-degenerate. We call* ω *the symplectic form.*

One immediate consequence of the non-degeneracy of ω is that M must have even dimension, say $\dim M = 2n$, and ω^n is a volume form which means M is orientable. Another important result in symplectic manifolds is Darboux's Theorem which states that symplectic manifolds are locally the same.

Theorem 1.2 (Theorem 8.1 of [2]). *Let (M, ω) be a $2n$-dimensional symplectic manifold and p be any point in M. Then there is a coordinate chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that on $\mathcal{U}$*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

Such a chart is called a Darboux chart.

This implies that symplectic manifolds cannot be distinguished by local invariants - only global invariants can tell symplectic manifolds apart.

A common global invariant is the de Rham cohomology. The problem for symplectic manifolds is that de Rham cohomology only contains topological information about the manifold so the choice of symplectic form isn't represented. So if we want to study two different symplectic forms on the same underlying manifold then de Rham cohomology is no help. A different point of view taking into account the symplectic form is necessary to create a useful cohomology.

1.2 Lefschetz Decomposition

The method used to develop the symplectic cohomology is similar to the Dolbeault cohomology on a complex manifold. On a complex manifold we can separate 1-forms into groups of holomorphic and anti-holomorphic forms, which in turn induces a decomposition of the exterior derivative as $d = \partial + \overline{\partial}$. We can then construct a chain complex with differential $\overline{\partial}$ which produces the Dolbeault cohomology. In the symplectic case we use a decomposition based on the symplectic form called the Lefschetz decomposition which produces a similar decomposition of the exterior derivative.

We can define a trio of maps $L = \omega\wedge$, Λ, and H which induce an sl_2 structure on $\Omega(M)$ and allows us to uniquely decompose a general k-form as

$$A_k = \sum_r L^r B_{k-2r} = \sum_r \omega^r \wedge B_{k-2r}$$

where the B_{k-2r} are all primitive forms [5]. We denote the collection of primitive s-forms as $\mathcal{P}^s(M)$ all forms of the type $L^r B_s$ with B_s primitive as $\mathcal{L}^{r,s}(M)$.

In Section 2.2 of [5] it is then shown that the exterior derivative decomposes as

$$d : \mathcal{L}^{r,s} \to \mathcal{L}^{r,s+1} \oplus \mathcal{L}^{r+1,s-1}$$

If we write $d(L^r B_s) = L^r B_{s+1} + L^{r+1} B_{s-1}$ where $B_s \in \mathcal{P}^s(M)$ and $B_{s-1} \in \mathcal{P}^{s-1}(M)$ then we say that $\partial_+ B_s = B_{s+1}$ and $\partial_- B_s = B_{s-1}$. This leads to the decomposition of the exterior derivative

$$d = \partial_+ + L\,\partial_-$$

From here [5] constructs two sets of primitive cohomologies:

$$PH^s_{\partial_+}(M) = \frac{\ker \partial_+ \cap \mathcal{P}^s}{\partial_+ \mathcal{P}^{s-1}}$$

and

$$PH^s_{\partial_-}(M) = \frac{\ker \partial_- \cap \mathcal{P}^s}{\partial_- \mathcal{P}^{s+1}}$$

for all $s < n$.

Unfortunately, these complexes are not elliptic on their own, but Proposition 2.8 of [5] shows they can be connected by the map $\partial_+ \partial_- : \mathcal{P}^n(M) \to \mathcal{P}^n(M)$ to create a combined elliptic complex, proving that they are finite dimensional.

1.3 Filtered Cohomology

The cohomologies from the previous section are called primitive cohomologies since they consist of only primitive forms - we have removed all other information. One could ask if we could extend the ideas from the primitive cohomologies while keeping more than just the primitive forms. The answer is to use a filter p, and increase our view to the set of all forms whose Lefschetz decomposition has no more than a p power of ω. These forms are called p-filtered and the space of all p-filtered k-forms is denoted $F^p\Omega^k(M)$ [4]. Note that this contains primitive forms as $F^0\Omega^k(M) = \mathcal{P}^k(M)$.

Section 2.2 of [4] shows that we can similarly extend the differential operators $\partial_+ : \mathcal{P}^s(M) \to \mathcal{P}^{s+1}(M)$ and $\partial_- : \mathcal{P}^s(M) \to \mathcal{P}^{s-1}(M)$ to operators $d_+ : F^p\Omega^k(M) \to F^p\Omega^{k+1}(M)$ and $d_- : F^p\Omega^k(M) \to F^p\Omega^{k-1}(M)$. As before, a connecting map is needed to create an elliptic complex.

Theorem 1.3 (Theorem 3.1 of [4]). *The following differential complex is elliptic for* $0 \leq p \leq n$.

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & F^p\Omega^0 & \xrightarrow{d_+} & F^p\Omega^1 & \xrightarrow{d_+} & \cdots & \xrightarrow{d_-} & F^p\Omega^{n+p-1} & \xrightarrow{d_+} & F^p\Omega^{n+p} \\ & & & & & & & & & & \big\downarrow \partial_+\partial_- \\ 0 & \longleftarrow & F^p\Omega^0 & \xleftarrow{d_-} & F^p\Omega^1 & \xleftarrow{d_-} & \cdots & \xleftarrow{d_-} & F^p\Omega^{n+p-1} & \xleftarrow{d_-} & F^p\Omega^{n+p} \end{array}$$

We denote these cohomologies by $F^pH^0_+, \ldots, F^pH^{n+p}_+, F^pH^{n+p}_-, \ldots, F^pH^0_-$. Note that since each degree appears twice in the complex we distinguish the cohomologies on the the top part of the complex with H_+ and those on the bottom as H_-. However, Proposition 4.8 of [4] states that in the case that (M, ω) being a closed symplectic manifold we have $F^pH^k_+(M) \cong F^pH^k_-(M)$.

While these cohomologies do satisfy our goal of being dependent on the choice of symplectic form (Section 4 of [5] computes an example where two different symplectic forms on a 6-dimensional nilmanifold produce distinct cohomologies), a few problems remain. For one, they are difficult to compute since we need to know how each form decomposes. They are also not compatible with the wedge product. The fact that the degree of the forms increases from 0 to $n+p$ then decreases back to 0 in particular means that the degrees are incompatible with the wedge product, let alone the filter. This contrasts with the de Rham case with $\Omega(M)$, d, and $\wedge$ where we get a differential graded algebra structure. Section 5 shows that we can define a product on the filtered cohomology but it is quite difficult to compute in practice and fails to be associative. Instead an $\mathcal{A}_\infty$ structure is needed for the filtered case.

1.4 Cone Representation

The answer to the computational issues raised by the filtered cohomology is by using an isomorphic cone cohomology. From homological algebra we get the concept of the mapping cone of a cochain complec. For each filter p we consider the cone due to the map

$$\wedge\omega^{p+1} : \Omega^k(M) \to \Omega^{k+2p+2}(M)$$

Section 3.1 of [3] discusses the details of this construction, but the result is a new cone complex $\mathcal{C}_p(M)$ which consists of spaces

$$\mathcal{C}_p^k(M) := \Omega^k(M) \oplus \Omega^{k-2p-1}(M)$$

with differential $d_{\mathcal{C}}$ (usually simply denoted d when there is no confusion) defined by

$$d_{\mathcal{C}}(A_j, B_{k-2p-1}) = (dA_j + \omega^{p+1} \wedge B_{k-2p-1}, -dB_{k-2p-1})$$

We will use a few notation conventions that make computations less confusing. The simplest is that we will use $j_p = j - 2p - 1$ since this will appear frequently and the standard convention is that a cone element $C_j \in \mathcal{C}^j_p(M)$ has elements (A_j, B_{j_p}) where it is understood that $A_j \in \Omega^j(M)$ and $B_{j_p} \in \Omega^{j_p}(M) = \Omega^{j-2p-1}(M)$. The other notation is that we will write a single element rather than an order pair by writing

$$(A_j, B_{j_p}) = A_j + \theta B_{j_p}$$

where θ is a formal variable of degree $2p+1$ satisfying $d\theta = \omega^{p+1}$. Using this notation the exterior derivative and wedge product act as

$$dC_j = d(A_j + \theta B_{j_p}) = dA_j + \omega^{p+1} B_{j_p} - \theta dB_{j_p}$$

and

$$C_j \wedge C_k = (A_j + \theta B_{j_p}) \wedge (A_k + \theta B_{k_p}) = A_j \wedge A_k + \theta(B_{j_p} \wedge A_k + (-1)^j A_j \wedge B_{k_p})$$

The second equation uses that fact that since θ is odd degree we have $\theta^2 = 0$. This makes the cone complex into a differential graded algebra.

At this point we need to show how this cone complex relates to the filtered complex. Definition 3.4 of [3] describes a pair of maps between the complexes.

Definition 1.4 (Definition 3.4 of [3]). *$f : \mathcal{C}^j_p(M) \to \mathcal{F}^j_p(M)$ is defined as*

$$f(C_j) = \begin{cases} \Pi^p A_j & j \leq n+p \\ -\Pi^p *_r (dL^{-(p+1)} A_j + B_{j_p}) & j > n+p \end{cases}$$

and $g : \mathcal{F}_p^j(M) \to \mathcal{C}_p^j(M)$ *is defined as*

$$\begin{cases} g(\alpha_j) = & \alpha_j \quad -\theta L^{-(p+1)} d\alpha_j \quad j \leq n+p \\ g(\overline{\alpha_l}) = & \qquad -\theta *_r \overline{\alpha_l} \qquad j > n+p, \; l = 2n+2p+1-j \end{cases}$$

Note here that because for $j \geq n+p+1$ a degree j element of $\mathcal{F}_p^j(M)$ is actually an l-form where $l = 2n+2p+1-j$. The bar is used to denote that $\overline{\alpha_l}$ is a degree j element of the complex and not a degree l element.

Remark 1.5. *The notation has been altered from [3] to match the notation used here. The other simplification in* f *is rewriting the* A_j *terms as* $\Pi^{p*} L^p d \Pi^p *_r A_j = L^p \partial_+ \Pi^0 *_r A_j = \Pi^p *_r dL^{-(p+1)} A_j$. *The proof of the final equality comes from:*

Proof. First since $*_r$ is invertible it suffices to show for elements of the form $*_r A_j$. Then we want to show $L^p \partial_+ \Pi^0 A_j = \Pi^p *_r dL^{-(p+1)} *_r A_j$. Now,

$$\begin{aligned} \Pi^p *_r dL^{-(p+1)} *_r A_j &= \Pi^p (*_r d *_r) L^{p+1} A_j \\ &= \Pi^p d_- L^{p+1} A_j \end{aligned}$$

Since $d_- = \partial_- + \partial_+ L^{-1}$ can only decrease the grading by one, we only get something p-graded from the primitive part of A_j and the $\partial_+ L^{-1}$ term of d_-. Thus $\Pi^p d_- L^{p+1} A_j = \partial_+ L^{-1} L^{p+1} \Pi^0 A_j$. Now suppose we have a primitive element β_j, we must show that $\partial_+ L^p \beta_j = \partial_+ L^{-1} L^{p+1} \beta_j$. Then $L^p \beta_j = L^{-1} L^{p+1} \beta_j$ unless we have $L^{p+1} \beta_j = 0$ while $L^p \beta_j \neq 0$. This requires $p+1 > n-j$ and $p \leq n-j$ respectively. Combining we get $p = n-j$. But then $L^p \partial_+ \beta_j = 0$ as well since then $n-(j+1) = n-j-1 = p-1 < p$. $\square$

These provide the quasi-isomorphisms we need:

Lemma 1.6 (Lemma 3.7 of [3])**.** *Both f and g are quasi-isomorphisms and $H^*(\mathcal{F}) \cong H^*(\mathcal{C})$.*

One result is that f and g allow the differential and wedge product on $\mathcal{C}_p$ to induce the differential and product on $\mathcal{F}_p$.

Lemma 1.7. $d_{\mathcal{F}} = f \circ d_{\mathcal{C}} \circ g$

Proof. Consider α_j where $j < n+p$, so

$$\begin{aligned}
f(d_{\mathcal{C}}(g(\alpha_j))) &= f(d_{\mathcal{C}}(\alpha_j - \theta L^{-(p+1)} d\alpha_j)) \\
&= f(d\alpha_j - L^{p+1} L^{-(p+1)} d\alpha_j + \theta d L^{-(p+1)} d\alpha_j) \\
&= \Pi^p (d\alpha_j - L^{p+1} L^{-(p+1)} d\alpha_j) \\
&= \Pi^p d\alpha_j \\
&= d_+ \alpha_j
\end{aligned}$$

If $j = n+p$ then $\alpha_j = L^p \beta_{n-p}$ for β_{n-p} primitive. Then,

$$\begin{aligned}
f(d_{\mathcal{C}}(g(\alpha_j))) &= f(d_{\mathcal{C}}(\alpha_j - \theta L^{-(p+1)} d\alpha_j)) \\
&= f(d\alpha_j - L^{p+1} L^{-(p+1)} d\alpha_j + \theta d L^{-(p+1)} d\alpha_j) \\
&= f(\Pi^p d\alpha_j + \theta \partial_+ \partial_- \beta_{n-p}) \\
&= -\Pi^p *_r (\partial_+ \partial_- \beta_{n-p}) \\
&= -\Pi^p L^p (\partial_+ \partial_- \beta_{n-p}) \\
&= -\partial_+ \partial_- L^p \beta_{n-p} \\
&= -\partial_+ \partial_- \alpha_j
\end{aligned}$$

Finally, if $j > n+p$ we have

$$
\begin{aligned}
f(d_{\mathcal{C}}(g(\overline{\alpha_l}))) &= f(d_{\mathcal{C}}(-\theta *_r \overline{\alpha_l})) \\
&= f(-L^{p+1} *_r \overline{\alpha_l} + \theta d *_r \overline{\alpha_l} \\
&= f(\theta d *_r \overline{\alpha_l}) \\
&= -\Pi^p *_r d *_r \overline{\alpha_l} \\
&= -\Pi^p d_- \overline{\alpha_l} \\
&= -d_- \overline{\alpha_l}
\end{aligned}
$$

All match the definition of $d_{\mathcal{F}}$ so the equality holds. □

Lemma 1.8. $\alpha_j \times \alpha_k = f(g(\alpha_j) \wedge g(\alpha_k))$.

Proof. First suppose that $j, k \leq n+p$. Then

$$
\begin{aligned}
g(\alpha_j) \wedge g(\alpha_k) &= \left(\alpha_j - \theta L^{-(p+1)} d\alpha_j\right) \wedge \left(\alpha_k - \theta L^{-(p+1)} d\alpha_k\right) \\
&= \alpha_j \wedge \alpha_k - \theta[(L^{-(p+1)} d\alpha_j) \wedge \alpha_k + (-1)^j \alpha_j \wedge (L^{-(p+1)} d\alpha_k)]
\end{aligned}
$$

If $j+k \leq n+p$ applying f results in $\Pi^p(\alpha_j \wedge \alpha_k)$, if $j+k > n+p$ we get

$$
\Pi^p *_r [-dL^{-(p+1)}(\alpha_j \wedge \alpha_k) + (L^{-(p+1)} d\alpha_j) \wedge \alpha_k + (-1)^j \alpha_j \wedge (L^{-(p+1)} d\alpha_k)]
$$

Finally if $j > n+p, k \leq n+p$ we have

$$\begin{aligned} f[g(\alpha_j) \wedge g(\alpha_k)] &= f[(-\theta *_r \alpha_j) \wedge (\alpha_k - \theta L^{-(p+1)} d\alpha_k)] \\ &= f[-\theta(*_r\alpha_j) \wedge \alpha_k] \\ &= \Pi^p *_r ((*_r\alpha_j) \wedge \alpha_k) \\ &= *_r((*_r\alpha_j) \wedge \alpha_k) \end{aligned}$$

All cases match the definition of $\alpha_j \times \alpha_k$. □

Since $\mathcal{F}_p$ is an A_∞ structure we need to consider the possibility of higher order maps.

Proposition 1.9 (Theorem 3.8 of [3]). *With g^1 as above, $g^2 = -\theta L^{-(p+1)} m^2_{\mathcal{C}}(g \otimes g)$, and $g^l = 0$ for $l > 2$ form an A_∞ map $g^l : \mathcal{F}_p \to \mathcal{C}_p$.*

We have f^1, and f^2 has been determined to be the below map. The existence of higher order maps are unknown, but are hypthesized to be all zero like g^l.

Theorem 1.10. $f^2 = f\left[m^2_{\mathcal{C}}(G \otimes \frac{1}{2}(1+gf)) + m^2_{\mathcal{C}}(\frac{1}{2}(1+gf) \otimes G\right]$. *Explicitly,*

$$f^2(C_j \otimes C_k) = f\left[GC_j \wedge \frac{1}{2}(1+g\circ f)C_k + (-1)^j \frac{1}{2}(1+g\circ f)C_j \wedge GC_k\right]$$

Proof. For simplicity we will simply write f and g instead of f and g.

(i) From the lemma we can see that we may rewrite

$$f(C_j) \times f(C_k) = f[gf(C_j) \wedge gf(C_k)]$$

(ii)

$$\begin{aligned}d_F f^2(C_j \otimes C_k) &= f[d(GC_j \wedge \frac{1}{2}(1+gf)C_k + (-1)^j \frac{1}{2}(1+gf)C_j \wedge GC_k)] \\ &= f[dGC_j \wedge \frac{1}{2}(1+gf)C_k - (-1)^j GC_j \wedge \frac{1}{2}(1+gf)dC_k \\ &\quad + (-1)^j \frac{1}{2}(1+gf)dC_j \wedge GC_k + \frac{1}{2}(1+gf)C_j \wedge dGC_k]\end{aligned}$$

(iii)

$$f^2(d \otimes 1)(C_j \otimes C_k) = f[GdC_j \wedge \frac{1}{2}(1+gf)C_k - (-1)^j \frac{1}{2}(1+gf)C_j \wedge GC_k]$$

(iv)

$$f^2(1 \otimes d)(C_j \otimes C_k) = f[(-1)^j GC_j \wedge \frac{1}{2}(1+gf)dC_k + \frac{1}{2}(1+gf)C_j \wedge GdC_k]$$

Summing parts (ii), (iii), and (iv) gives

$$\begin{aligned}f[(dG+Gd)C_j \wedge \frac{1}{2}(1+gf)C_k + \frac{1}{2}(1+gf)C_j \wedge (dg+Gd)C_k] \\ &= f[(1-gf)C_j \wedge \frac{1}{2}(1+gf)C_k \\ &\quad + \frac{1}{2}(1+gf)C_j \wedge (1-gf)C_k] \\ &= \frac{1}{2}f[C_j \wedge C_k + C_j \wedge gfC_k - gfC_j \wedge C_k \\ &\quad - gfC_j \wedge gfC_k + C_j \wedge C_k - C_j \wedge gfC_k \\ &\quad + gfC_k \wedge C_k - gfC_j \wedge gfC_k] \\ &= f[C_j \wedge C_k] - f[gfC_j \wedge gfC_k] \\ &= f[C_j \wedge C_k] - f(C_j) \times f(C_k)\end{aligned}$$

as desired. Note the final equality is due to the observation in (i). □

Proposition 1.11. *$f^2(C_j \otimes C_k) = (-1)^{jk} f^2(C_k \otimes C_j)$.*

Proof. Using the same notation as above:

$$\begin{aligned}
f^2(C_j \otimes C_k) &= \frac{1}{2} f\left[GC_j \wedge (1+gf)C_k + (-1)^j (1+gf)C_j \wedge GC_k\right] \\
&= \frac{1}{2} f\left[(-1)^{(j-1)k}(1+gf)C_k \wedge GC_j + (-1)^j(-1)^{j(k-1)} GC_k \wedge (1+gf)C_j\right] \\
&= (-1)^{jk}\frac{1}{2} f\left[GC_k \wedge (1+gf)C_k + (-1)^k (1+gf)C_k \wedge GC_j\right] \\
&= (-1)^{jk} f^2(C_k \otimes C_j)
\end{aligned}$$

□

Proposition 1.12. *$f^2(C_j \otimes C_k) = (-1)^{jk} f^2(C_k \otimes C_j)$.*

Proof. Using the same notation as above:

$$\begin{aligned}
f^2(C_j \otimes C_k) &= \frac{1}{2} f\left[GC_j \wedge (1+gf)C_k + (-1)^j (1+gf)C_j \wedge GC_k\right] \\
&= \frac{1}{2} f\left[(-1)^{(j-1)k}(1+gf)C_k \wedge GC_j + (-1)^j(-1)^{j(k-1)} GC_k \wedge (1+gf)C_j\right] \\
&= (-1)^{jk}\frac{1}{2} f\left[GC_k \wedge (1+gf)C_k + (-1)^k (1+gf)C_k \wedge GC_j\right] \\
&= (-1)^{jk} f^2(C_k \otimes C_j)
\end{aligned}$$

□

1.5 Computing the Cone Cohomology

One may have noticed that the cone complex is very similar to the complex of differential forms on a sphere bundle

$$\begin{array}{ccc} S^{2p+1} & \longrightarrow & E \\ & & \downarrow \\ & & M \end{array}$$

where θ acts like the global angular form. In fact, if ω is integral class this does correspond to a sphere bundle but the cone complex can be constructed for any ω. While we don't always have a sphere bundle, it is possible to resolve the Lefschetz maps via a Gysin sequence. Namely, we have an exact triangle

$$\begin{array}{ccccc} H^*(X) & & \xrightarrow{\omega^{p+1}\wedge} & & H^*(X) \\ & \nwarrow^{\pi_*} & & \swarrow_{\pi^*} & \\ & & H^*\mathcal{C}_p(X) & & \end{array}$$

where the maps are given by

$$\pi^*(\omega) = \omega \qquad\qquad \pi_*(A_j + \theta_X B_{j_p}) = B_{j_p}$$

This is the primary method of computing the cone cohomology when the de Rham cohomology is explicitly known. There is a filtered version of this exact triangle as Theorem 4.2 of [4] which we could also obtain by mapping the cone exact triangle via the maps f and g, but this is much harder to compute.

We end the section with an explicit computation of the cone cohomologies of even dimensional spheres and $\mathbb{CP}^n$ as examples.

Lemma 1.13.

$$H^j\mathcal{C}_p(\mathbb{CP}^n) \cong \begin{cases} \mathbb{R} & 0 \leq j \leq 2p \;\; even \\ \mathbb{R} & 2p+1 \leq j \leq 2n+1 \;\; odd \\ 0 & else \end{cases}$$

Proof. By the resolution sequence we have an exact sequence

$$\cdots \xrightarrow{L^{p+1}} H^j_d(\mathbb{CP}^n) \xrightarrow{\pi^*} H^j\mathcal{C}_p(\mathbb{CP}^n) \xrightarrow{\pi_*} H^{j-2p-1}_d(\mathbb{CP}^n) \xrightarrow{L^{p+1}} \cdots$$

Since powers of the symplectic form generates the cohomology of $\mathbb{CP}^n$ we have that L^{p+1} : $H^{2k}_d(\mathbb{CP}^n) \to H^{2k+2p+2}_d(\mathbb{CP}^n)$ is an isomorphism as long as $k+p<n$. Recall the odd cohomology groups are all zero.

Then for $j=2k$ even we have $H^{2k}\mathcal{C}_p(\mathbb{CP}^n) \cong H^{2k}(\mathbb{CP}^n)/L^{p+1}(H^{2k-2p-2})$. By the above this gives $\mathbb{R}$ when $k \leq p$ and 0 otherwise.

If $j = 2k+1$ odd then $H^{2k+1}\mathcal{C}_p(\mathbb{CP}^n) \cong \ker L^{p+1} : H^{2k-2p}_d(\mathbb{CP}^n) \to H^{2k+2}_d(\mathbb{CP}^n)$. By the above we then get $\mathbb{R}$ when $p \leq k \leq n$ and 0 otherwise. □

Lemma 1.14.

$$H^j\mathcal{C}_p(S^{2n}) \cong \begin{cases} \mathbb{R} & j = 0,\;\, 2n+2p+1 \\ \mathbb{R} & j = 2n,\;\, 2p+1,\;\, n \neq p+1 \\ 0 & else \end{cases}$$

Proof. Since $H^j_d(S^{2n}) = \mathbb{R}$ for $j = 0, 2n$ and is zero otherwise we can only expect non-zero cone cohomology in indices $j = 0, 2n, 2p+1, 2n+2p+1$. For each relevant index j the resolution sequence gives:

$$j=0: \qquad 0 \to H^0(S^{2n}) \to H^0\mathcal{C}_p(S^{2n}) \to 0$$

so $H^0\mathcal{C}_p(S^{2n}) \cong H^0(S^{2n}) \cong \mathbb{R}$.

$$j = 2n + 2p + 1 : \qquad 0 \to H^{2n+2p+1}\mathcal{C}_p(S^{2n}) \to H^{2n}(S^{2n}) \to 0$$

so $H^{2n+2p+1}\mathcal{C}_p(S^{2n}) \cong H^{2n}(S^{2n}) \cong \mathbb{R}$. If $n \neq p + 1$:

$$j = 2n : \qquad 0 \to H^{2n}(S^{2n}) \to H^{2n}\mathcal{C}_p(S^{2n}) \to 0$$

so $H^{2n}\mathcal{C}_p(S^{2n}) \cong H^{2n}(S^{2n}) \cong \mathbb{R}$

$$j = 2p + 1 : \qquad 0 \to H^{2p+1}\mathcal{C}_p(S^{2n}) \to H^0(S^{2n}) \to 0$$

so $H^{2p+1}\mathcal{C}_p(S^{2n}) \cong H^0(S^{2n}) \cong \mathbb{R}$. If $n = p + 1$:

$$j = 2n : \qquad H^0(S^{2n}) \to H^{2n}(S^{2n}) \to H^{2n}\mathcal{C}_p(S^{2n}) \to 0$$

where the first map is an isomorphism. Then $H^{2n}(S^{2n}) \to H^{2n}\mathcal{C}_p(S^{2n})$ must be surjective and the zero map, so $H^{2n}\mathcal{C}_p(S^{2n}) = 0$.

$$j = 2p + 1 : \qquad 0 \to H^{2p+1}\mathcal{C}_p(S^{2n}) \to H^0(S^{2n}) \to H^{2n}(S^{2n})$$

where again the last map is an isomorphism. Then $H^{2p+1}\mathcal{C}_p(S^{2n})$ injects into $H^0(S^{2n})$ via the zero map so $H^{2p+1}\mathcal{C}_p(S^{2n}) = 0$. □

1.6 Summary of Results

In Chapter 2 we will study how common maps such as restrictions and projections extend to the cone cohomology. Proposition 2.1 describes how we can extend the pullback of an embedding $\iota : X \to M$ which preserves the symplectic structure:

Proposition 1.15. *$\iota^* : \mathcal{C}^j_p(N) \to \mathcal{C}^j_p(M)$ is a chain map which commutes with the wedge product.*

Afterward we discuss how this restriction interacts with the resolution sequence and how this result can be translated to a statement on the filtered cohomology. A computation of the induced restriction map on the filtered complex is given in Proposition 2.2.

Lemma 2.2 then describes how a similar process works for a projection map which almost preserves the symplectic structure.

Lemma 1.16. *Let (X, ω_X) and (Y, ω_Y) be symplectic manifolds with $\dim Y > \dim X$. Let $2n = \dim X$ and $2m = \dim Y - \dim X$. Suppose that there is a map $\pi : Y \to X$ such that $\pi^*\omega_X = \omega_Y + d\mu$ for some $\mu \in \Omega^1(Y)$. Then the following defines a chain map $\tilde{\pi} : \mathcal{C}^j_p(X) \to \mathcal{C}^j_p(Y)$:*

$$\tilde{\pi}(A_j + \theta_X B_{j_p}) = \pi^*(A_j) - \mu_{2p+1} \wedge \pi^*(B_{j_p}) + \theta_Y \pi^*(B_{j_p})$$

where $\mu_{2p+1} = \mu \wedge \sum_{j=0}^{p} (\pi^*\omega_X)^{p-j} \wedge \omega_Y^j$ so that $d\mu_{2p+1} = (\pi^*\omega_X)^{p+1} - \omega_Y^{p+1}$

Finally, the chapter ends with Proposition 2.3 and Proposition 2.4 which describe how reducing the filter induces a map on the cone or filtered complex respectively.

Chapter 3 outlines the process for defining a Gysin sequence for the cone cohomology. To start Lemmas 3.1 and 3.2 describe how to integrate over a fiber, for the sphere and disk case respectively. The definition of integrating over a fiber is given by the below:

Lemma 1.17. *$\tilde{\pi}^S_*(A_j + \theta_{E_0} B_{j_p}) = \pi^S_*(A_j + \mu_{2p+1} \wedge B_{j_p}) + \theta_X \pi^S_*(B_{j_p})$ defines a chain map from $\mathcal{C}^j_p(E_0) \to \mathcal{C}^{j-2m+1}_p(X)$.*

where we use π^S_* to denote integration of a standard form over the spherical fiber. Integrating over a disk, denoted $\tilde{\pi}_*^D$, is defined in the same way except with π_*^D which is the standard integration over the disk fiber.

We can then use these maps to create the Thom isomorphism. Proposition 3.2 defines the map

Proposition 1.18. *The map* $\tilde{\pi}_*(C_j, C_{j-1}) = \tilde{\pi}_*^S(C_{j-1}) - \frac{1}{2}\overline{\tilde{\pi}_*^D(C_j)}$ *defines a homomorphism from* $H\mathcal{C}_p^j(E, E_0)$ *to* $H\mathcal{C}_p^{j-2m}(X)$.

where the conjugate is defined so that $\overline{A_j + \theta B_{j_p}} = A_j - \theta B_{j_p}$. Then Theorem 3.1 shows that this map is an isomorphism. This results in the Gysin sequence in Theorem 3.2

Theorem 1.19 (Gysin Sequence for p-Filtered Cone Complex)**.**

$$\cdots \longrightarrow H^{j-2m}\mathcal{C}_p(X) \xrightarrow{\wedge(-1)^j \mathfrak{e}} H^j\mathcal{C}_p(X) \xrightarrow{\tilde{\pi}} H^j\mathcal{C}_p(E_0) \xrightarrow{\tilde{\pi}_*^S} H^{j-2m+1}\mathcal{C}_p(X) \longrightarrow \cdots$$

is an exact sequence of cohomology.

The first map is given by wedging with the Euler class, the second is induced by the projection $\pi : E_0 \to X$, and the third is integration over the sphere fiber.

The objective of Chapter 4 is to establish how the dimension of the cone cohomology changes when blowing-up a manifold along a submanifold. The main primitive case is shown below, and Theorem 4.4 gives the corresponding non-primitive case.

Theorem 1.20. *The cone cohomology of M and $\tilde{M}$ are related by the formulas:*

$$\begin{aligned}
\dim H^2\mathcal{C}_0(\tilde{M}) &= \dim H^2\mathcal{C}_0(M) + 1 - k_1 \\
\dim H^3\mathcal{C}_0(\tilde{M}) &= \dim H^3\mathcal{C}_0(M) + 2g - k_1 - k_2 \\
\dim H^4\mathcal{C}_0(\tilde{M}) &= \dim H^4\mathcal{C}_0(M) + 1 - k_2 \\
\dim H^{2m-1}\mathcal{C}_0(\tilde{M}) &= \dim H^{2m-1}\mathcal{C}_0(M) + k_2' \\
\dim H^{2m}\mathcal{C}_0(\tilde{M}) &= \dim H^{2m}\mathcal{C}_0(M) - 1 + k_1' + k_2' \\
\dim H^{2m+1}\mathcal{C}_0(\tilde{M}) &= \dim H^{2m+1}\mathcal{C}_0(M) + 1 - 2g + k_1' \\
\dim H^k\mathcal{C}_0(\tilde{M}) &= \dim H^k\mathcal{C}_0(M) \qquad \textit{otherwise}
\end{aligned}$$

where k_1, k_2, k_2', k_1' are the rank of the restriction maps from $H^*\mathcal{C}_0(M_1) \to H^*\mathcal{C}_0(M_1 \cap M_2)$ in degrees 2, 3, $2m-1$, and $2m$ respectively.

Finally, in Chapter 5 we will explore how cone cohomology can be considered in a broader context than just symplectic manifolds. Rather than specifically computing with the symplectic form we change focus to see how the cone cohomology changes as we let $d\theta$ vary over an entire cohomology space. Lemma 5.1 creates a bound on the dimension of the cone cohomology.

Lemma 1.21. *Let $\psi \in H^l(M)$. Then*

$$\dim H^k\mathcal{C}(\psi, M) \leq \dim H^k(M) + \dim H^{k-l+1}(M)$$

for all k.

Then Proposition 5.1 specifies some cases where the cone cohomology in certain degrees is equal to the de Rham cohomology.

Proposition 1.22. *Let $\psi \in H^l(M)$. If $k < l$ then $H^k\mathcal{C}(\psi, M) \cong H^k(M)$, and if $k > n$ then $H^k\mathcal{C}(\psi, M) \cong H^{k-l+1}(M)$.*

Corollaries 5.1 and 5.2 use this result to fully determine the cone cohomology of a n- or $n-1$- form on a n-dimensional manifold.

Appendix B describes Mathematica code for computing the cone cohomology of a nilmanifold, and Appendix A lists information about the cone cohomology of small dimension nilmanfolds. Complete information is available for dimensions 3 through 5, as well as the cone cohomology of 1-forms on a 6-dimensional nilmanifolds. Section 5.3 explains the process used to compute this data and observes some patterns for potential future study.

The work in Chapters 2 through 4 is based on collaborations with Poom Lertpinyowong, Chung-Jun Tsai, Li-Sheng Tseng, and Shing-Tung Yau and will appear in future publications.

Chapter 2

Maps on Cone Cohomology

2.1 Restriction Map

The pullback is the primary map used in the study of forms so we'd like to determine how this operation can be extended to cone elements. Consider two symplectic manifolds (M, ω_M) and (N, ω_N), and suppose that $\iota : M \to N$ is a map which preserves the symplectic structure so that $\iota^*(\omega_N) = \omega_M$. We may then extend the pullback ι^* to cone elements by defining $\iota^* : \mathcal{C}_p^j(N) \to \mathcal{C}_p^j(M)$ in the natural way:

$$\iota^*(A_j + \theta_N B_{j_p}) = \iota^*(A_j) + \theta_M \iota^*(B_{j_p})$$

Since the exterior derivative and wedge product commute with ι^* and we assume $\iota^*(\omega_N) = \omega_M$ we get the following proposition.

Proposition 2.1. *$\iota^* : \mathcal{C}_p^j(N) \to \mathcal{C}_p^j(M)$ is a chain map which commutes with the wedge product.*

Of particular interest is the case of an embedding $\iota : X \to M$ of a submanifold X into M. In that case we see ι^* as restricting the domain of forms on M to forms on X, and this extended pullback map as the restriction of cone elements to a submanifold.

Remark 2.2. *This map is also compatible with the restriction map on the filtered cohomology* $r : \mathcal{F}^j_p(M) \to \mathcal{F}^j_p(X)$*. Recall the maps* $f : \mathcal{C}_p(M) \to \mathcal{F}_p(M)$ *and* $g : \mathcal{F}_p(M) \to \mathcal{C}_p(M)$ *which induce the isomorphism of the filtered cohomology and cone cohomology. We can compute that* $r = f \circ \iota^* \circ g$ *which gives the commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_p(M) & \xrightarrow{\iota^*} & \mathcal{C}_p(X) \\ g\uparrow & & \downarrow f \\ \mathcal{F}_p(M) & \xrightarrow{r} & \mathcal{F}_p(X) \end{array}$$

This also demonstrates the benefit of working with the cone cohomology over the filtered version - the cone version of the restriction map is much easier and clearer than the filtered version.

We can explicitly compute the restriction map $r : \mathcal{F}_p(M) \to \mathcal{F}_p(X)$:

Proposition 2.3. $r : \mathcal{F}^j_p \to \mathcal{F}^j_p$ *defined by*

$$\begin{cases} r(\alpha_j) = \Pi^p(\iota^*\alpha_j) & j \leq n+p \\ r(\alpha_j) = \Pi^p *_r \left[\iota^*(L^{-(p+1)}d\alpha_j) - dL^{-(p+1)}(\iota^*\alpha_j)\right] & n+p < j \leq n+m+p \\ r(\overline{\alpha_l}) = *_r[\iota^*(*_r\overline{\alpha_l})] & j > n+m+p \end{cases}$$

where $l = 2(n+m+p)+1-j$*, is a chain map.*

Proof. Since $r = f \circ \iota^* \circ g$ the chain map property is immediate, all we need to do is compute the composition and show that it matches what is listed above. First suppose that

$j \leq n+m+p$. Then

$$\iota^*(g(\alpha_j)) = \iota^*(\alpha_j - \theta_M L^{-(d+1)}\alpha_j) = \iota^*(\alpha_j) - \theta_X \iota^*(L^{-(p+1)} d\alpha_j)$$

If $j < n+p$ then $f(\iota^*(g(\alpha_j))) = \pi^p \iota^*(\alpha_j)$ while if $n+p < j \leq n+m+p$ then

$$f(\iota^*(g(\alpha_j))) = \pi^p *_r \left[\iota^*(L^{-(p+1)} d\alpha_j) - dL^{-(p+1)}(\iota^*(\alpha_j)\right]$$

Now if $j > n+m+p$ we have

$$f(\iota^*(g(\alpha_j))) = f(\iota^*(-\theta_M *_r \alpha_j)) = f(-\theta_X \iota^*(*_r \alpha_j)) = \Pi^p *_r \iota^*(*_r \alpha_j) = *_r \iota^*(*_r \alpha_j)$$

In all cases these match $r(\alpha_j)$. □

Since $\mathcal{F}_p$ is an A_∞ algebra there is the potential for higher order parts of the restriction map. We can compute r^2 from the composition rules for A_∞ morphisms, namely if $g : A \to B$ and $f : B \to C$ are A_∞ morphisms then $(f \circ g)_2 = f \circ g^2 + f^2(g \otimes g)$. This results in

$$r^2 = f \circ \iota^* \circ g^2 + f^2((\iota^* \circ g) \otimes (\iota^* \circ g))$$

Note there is no term for ι^* since $(\iota^*)_2 = 0$. We can then compute each term directly:

$$f(\iota^*(g^2(\alpha_j \otimes \alpha_k))) = \begin{cases} \Pi^p *_r \left[\iota^*(L^{-(p+1)}(\alpha_j \wedge \alpha_k))\right] & j,k \leq n+m+p,\ j+k > n+p+1 \\ 0 & \text{otherwise} \end{cases}$$

$f^2(\iota^*(g(\alpha_j)) \otimes \iota^*(g(\alpha_k))) =$

(a) $j+k \le n+p+1$

$$-L^p \Pi^0 \iota^* \left[(L^{-(p+1)} d\alpha_j) \wedge \alpha_k + (-1)^j \alpha_j \wedge (L^{-(p+1)} d\alpha_k)\right]$$

(b) $j \le n+p$, $k > n+m+p$

$$\frac{1}{2}(-1)^j \Pi^p *_r \Big[dL^{-(p+1)}\left((1+\Pi^p)\iota^*\alpha_j \wedge L^p \iota^*(*_r \alpha_k)\right) \\ - L^{-(p+1)} d\Pi^p \iota^* \alpha_j \wedge L^p \iota^* (*_r \alpha_k\Big]$$

(c) $j,k \le n+p$, $j+k > n+p+1$

$$\frac{1}{2}\Pi^p *_r \Big[(1+\Pi^p)\iota^*\alpha_j \wedge L^{-(p+1)}\iota^*\alpha_k - L^{-(p+1)}\iota^*\alpha_j \wedge (1+\Pi^p)\iota^*\alpha_k \\ + (-1)^j L^p \iota^* (L^{-(p+1)} d\alpha_j) \wedge L^{-(p+1)} d\Pi^p \iota^* \alpha_k) \\ - (-1)^j L^{-(p+1)} d\Pi^p \iota^* \alpha_j \wedge L^p \iota^* L^{-(p+1)} d\alpha_k \\ + dL^{-(p+1)}\big[L^p \iota^* L^{-(p+1)} d\alpha_j \wedge (1+\Pi^p)\iota^*\alpha_k \\ + (-1)^j (1+\Pi^p)\iota^*\alpha_j \wedge L^p \iota^* L^{-(p+1)} d\alpha_k\big]\Big]$$

(d) $j \leq n+p$, $n+p < k \leq n+m+p$, $j+k > n+p+1$

$$-\frac{1}{2}\Pi^p *_r \Big[L^{-(p+1)}\iota^*\alpha_j \wedge \iota^*\alpha_k - (1+\Pi^p)\iota^*\alpha_j \wedge L^{-(p+1)}\iota^*\alpha_k$$
$$+ (-1)^j L^p \iota^* L^{-(p+1)} d\alpha_j \wedge \Pi^{p*}(dL^{-(p+1)}\iota^*\alpha + k - \iota^* L^{-(p+1)} d\alpha_k)$$
$$+ (-1)^j L^{-(p+1)} d\Pi^p \iota^*\alpha_j \wedge L^p \iota^* L^{-(p+1)} d\alpha_k$$
$$- dL^{-(p+1)}(L^p \iota^* L^{-(p+1)} d\alpha_j \wedge (1+\Pi^p)\iota^*\alpha_k$$
$$+ (-1)^j (1+\Pi^p \iota^*\alpha_j \wedge L^p \iota^* L^{-(p+1)} d\alpha_k)\Big]$$

Proof. By the graded anti-symmetry of f^2 it suffices to compute $f^2(C_j \otimes C_k)$ when $j \leq k$. As usual we use f and g instead of f and g for simplicity in these calculations.

(a) First suppose that $j, k \leq n+p$. Then $\iota^* g(\alpha_j)) = \iota^*\alpha_j - \theta\iota^*(L^{-(p+1)} d\alpha_j)$, $G\iota^* g(\alpha_j) = -L^p \iota^*(L^{-(p+1)} d\alpha_j) + \theta L^{-(p+1)}\iota^*\alpha_j$, and $gf(\iota^* g(\alpha_j)) = \Pi^p \iota^*\alpha_j - \theta L^{-(p+1)} d\Pi^p \iota^*\alpha_j$. The same calculations hold for α_k. Note that since in this case we assume $j+k-1 \leq n+p$ we may neglect the θ terms. Then:

$$f^2(C_j \otimes C_k) = \frac{1}{2} f\Big[(-L^p \iota^*(L^{-(p+1)} d\alpha_j)) \wedge ((1+\Pi^p)\iota^*\alpha_k)$$
$$+ (-1)^j ((1+\Pi^p)\iota^*\alpha_j) \wedge (-L^p \iota^*(L^{-(p+1)} d\alpha)k)\Big]$$
$$= -\frac{1}{2}\Pi^p L^p \Big[(\iota^*(L^{-(p+1)} d\alpha_j)) \wedge ((1+\Pi^p)\iota^*\alpha_k)$$
$$+ (-1)^j ((1+\Pi^p)\iota^*\alpha_j) \wedge (\iota^*(L^{-(p+1)} d\alpha)k)\Big]$$
$$= -\frac{1}{2} L^p \Pi^0 \Big[(\iota^*(L^{-(p+1)} d\alpha_j)) \wedge ((1+\Pi^p)\iota^*\alpha_k)$$
$$+ (-1)^j ((1+\Pi^p)\iota^*\alpha_j) \wedge (\iota^*(L^{-(p+1)} d\alpha)k)\Big]$$

While the wedge product of primitive forms may not be primitive, we do know that only the primitive terms can wedge to give a primitive result. Therefore we can simplify by adding or removing higher order terms to the wedge products:

$$= -\frac{1}{2}L^p\Pi^0\left[(\iota^*(L^{-(p+1)}d\alpha_j)) \wedge (2\iota^*\alpha_k) + (-1)^j(2\iota^*\alpha_j) \wedge (\iota^*(L^{-(p+1)}d\alpha)k)\right]$$
$$= -L^p\Pi^0\iota^*\left[(L^{-(p+1)}d\alpha_j) \wedge \alpha_k + (-1)^j\alpha_j \wedge (L^{-(p+1)}d\alpha_k)\right]$$

There is also the possibility that $j = 0$ and $k = n + p + 1$. Note that since α_j is a zero form this forces $L^{-(p+1)}d\alpha_j = 0$ so $\iota^*g(\alpha_j) = \iota^*\alpha_j$ and $G\iota^*g(\alpha_j) = 0$. α_k follows the same relations as above except now $gf(\iota^*g(\alpha_k)) = -\theta\Pi^{p*}\iota^*(*_r\alpha_k)$. Again since $j + k - 1 \leq n + p$ we neglect θ terms so we get

$$f^2(C_j \otimes C_k) = \frac{1}{2}f\left[(1 + \Pi^p)\iota^*\alpha_j \wedge (-L^p\iota^*(L^{-(p+1)}d\alpha_k))\right]$$

Now the same simplification techniques as in the first case apply so this reduces to

$$f^2(C_j \otimes C_k) = -L^p\Pi^0\iota^*\left[\alpha_j \wedge (L^{-(p+1)}d\alpha_k)\right]$$

Finally we notice that this is equivalent to applying the $j, k \leq n + p$ formula taking $j = 0$ since the first term vanishes.

The other cases follow from similar computations but with less simplifications. □

Since the resolution sequence of the Lefschetz map is a common way to compute the cone cohomology using the de Rham cohomology, it is also useful to point out that this cone restriction map is compatible with the resolution sequence. Recall that in cone representation the resolution sequence looks like

$$\begin{array}{ccc} H^*(M) & \xrightarrow{\omega_M^{p+1}\wedge} & H^*(M) \\ {\scriptstyle \pi_*}\nwarrow & & \swarrow{\scriptstyle \pi^*} \\ & H^*\mathcal{C}_p(M) & \end{array}$$

where

$$\pi^*(\omega) = \omega \qquad\qquad \pi_*(A_j + \theta_M B_{j_p}) = B_{j_p}$$

Then the restriction map is compatible with the resolution sequence in the following sense:

Lemma 2.4. *If $\iota : X \to M$ is an inclusion which preserves the symplectic structure, then the following diagram commutes:*

$$\begin{CD} H^j_d(M) @>{\pi^*}>> H^j\mathcal{C}_p(M) @>{\pi_*}>> H_d^{j-2p-1}(M) \\ @VV{\iota^*}V @VV{\iota^*}V @VV{\iota^*}V \\ H^j_d(X) @>{\pi^*}>> H^j\mathcal{C}_p(X) @>{\pi_*}>> H_d^{j-2p-1}(X) \end{CD}$$

Proof. This is immediate from the fact that the restriction map on cone elements acts on each term separately. □

While this is essentially a trivial statement in the cone setting it is worth mentioning that there is an analogous statement about filtered cohomology and r that is still true but much harder to verify computationally. In many cases it makes sense to show statements in the cone cohomology setting and then use f and g to convert to statements about filtered cohomology.

2.2 Inverting the Restriction Map

Returning to the case of a submanifold (X, ω_X) embedded in (M, ω_M), we can choose to identify a tubular neighborhood U of X in M such that $\omega_M|_U = \omega_X + d\mu$ for some $\mu \in \Omega^1(U)$ such that $\mu|_X = 0$. In this case ι^* is an isomorphism of the de Rham cohomologies of X and U, and the Five Lemma combined with the previous lemma implies an isomorphism $H^j\mathcal{C}_p(U) \cong H^j\mathcal{C}_p(X)$. We compute this inverse map in a slightly more general case.

Lemma 2.5. *Let (X, ω_X) and (Y, ω_Y) be symplectic manifolds with $\dim Y > \dim X$. Let $2n = \dim X$ and $2m = \dim Y - \dim X$. Suppose that there is a map $\pi : Y \to X$ such that $\pi^*\omega_X = \omega_Y + d\mu$ for some $\mu \in \Omega^1(Y)$. Then the following defines a chain map $\tilde{\pi} : \mathcal{C}_p^j(X) \to \mathcal{C}_p^j(Y)$:*

$$\tilde{\pi}(A_j + \theta_X B_{j_p}) = \pi^*(A_j) - \mu_{2p+1} \wedge \pi^*(B_{j_p}) + \theta_Y \pi^*(B_{j_p})$$

where $\mu_{2p+1} = \mu \wedge \sum_{j=0}^{p} (\pi^*\omega_X)^{p-j} \wedge \omega_Y^j$ so that $d\mu_{2p+1} = (\pi^*\omega_X)^{p+1} - \omega_Y^{p+1}$

Proof.

$$\begin{aligned}
\tilde{\pi}(dC_j) &= \tilde{\pi}(dA_j + \omega_X^{p+1} \wedge B_{j_p} - \theta_X dB_{j_p}) \\
&= \pi^*(dA_j + \omega_X^{p+1} \wedge B_{j_p}) - \mu_{2p+1} \wedge \pi^*(dB_{j_p}) - \theta_Y \pi^*(dB_{j_p}) \\
&= d\pi^*(A_j) + (\pi^*\omega_X)^{p+1} \wedge \pi^*(B_{j_p}) - \mu_{2p+1} \wedge \pi^*(dB_{j_p}) - \theta_Y \pi^*(dB_{j_p}) \\
&= d\pi^*(A_j) + (\omega_Y^{p+1} + d\mu_{2p+1}) \wedge \pi^*(B_{j_p}) - \mu_{2p+1} \wedge \pi^*(dB_{j_p}) - \theta_Y \pi^*(dB_{j_p}) \\
&= d(\pi^*(A_j) - \mu_{2p+1} \wedge \pi^*(B_{j_p}) + \theta_Y \pi^*(B_{j_p})) \\
&= d\tilde{\pi}(C_j)
\end{aligned}$$

□

2.3 Filter Reduction Map

Another basic operation we can do is to reduce the filtration, in the filtered cohomology case this corresponds to removing terms from the Lefschetz decomposition of a form. Consider a symplectic manifold (M, ω_M). Since the manifold doesn't change but the filter does we will use θ_{p+q} for the formal variable on $\mathcal{C}_{p+q}(M)$ and θ_p for the variable on $\mathcal{C}_p(M)$. We will define the cone representation of this map by $q : \mathcal{C}^j_{p+q}(M) \to \mathcal{C}^j_p(M)$ where

$$q(A_{p+q} + \theta_{p+q} B_{j_{p+q}}) = A_{p+q} + \theta \omega^q_M \wedge B_{j_{p+q}}$$

Proposition 2.6. *q is a chain map which commutes with the wedge product.*

Proof. To show it is a chain map:

$$\begin{aligned} q(dC_j) &= q(dA_{p+q} + \omega^{p+q}_M \wedge B_{j_{p+q}} - \theta_{p+q} dB_{j_{p+q}}) \\ &= dA_{p+q} + \omega^{p+q}_M \wedge B_{j_{p+q}} - \theta_p \omega^q_M \wedge dB_{j_{p+q}} \\ &= d(A_{p+q} + \theta_p \omega^q_M \wedge B_{j_{p+q}}) \\ &= dq(C_j) \end{aligned}$$

For the wedge product, consider cone elements C_j and C_k in $\mathcal{C}_{p+q}(M)$ of degree j and k

respectively. Then,

$$\begin{aligned}
q(C_j \wedge C_k) &= q(A_j \wedge A_k + \theta_{p+q}(B_{j_{p+q}} \wedge A_k + (-1)^j A_j \wedge B_{k_{p+q}})) \\
&= A_j \wedge A_k + \theta_p \omega_M^q \wedge (B_{j_{p+q}} \wedge A_k + (-1)^j A_j \wedge B_{k_{p+q}}) \\
&= A_j \wedge A_k + \theta_p((\omega_M^q \wedge B_{j_{p+q}}) \wedge A_{k_{p+q}} + (-1)^j A_j \wedge (\omega_M^q \wedge B_{k_{p+q}})) \\
&= (A_j + \theta_p \omega_M^q \wedge B_{j_{p+q}}) \wedge (A_k + \theta_p \omega_M^q \wedge B_{k_{p+q}}) \\
&= q(C_j) \wedge q(C_k)
\end{aligned}$$

□

Again we can use this to find the correspoding map on the filtered cohomology.

Proposition 2.7. *$l : \mathcal{F}^j_{p+q}(M) \to \mathcal{F}^j_p(M)$ defined by*

$$\begin{cases}
l(\alpha_j) = \Pi^p(\alpha_j) & j \leq n+p \\
l(\alpha_j) = -\Pi^p *_r dL^{-(p+1)}\alpha_j & n+p < j \leq n+p+q \\
l(\overline{\alpha_l}) = L^{-q}\overline{\alpha_l} & j > n+p+q, \ l = 2(n+p+q)+1-j
\end{cases}$$

is a chain map.

Proof. As with r this is immediate from $l = f \circ q \circ g$ and all we need to do is compute. First suppose that $j \leq n+p+q$ so

$$q(g(\alpha_j)) = q(\alpha_j - \theta_{p+q}L^{-(p+q+1)}d\alpha_j) = \alpha_j - \theta_p L^q L^{-(p+q+1)}d\alpha_j$$

If $j \leq n+p$ then

$$f(q(g(\alpha_j))) = \Pi^p \alpha_j$$

and if $n+p<j\leq n+p+q$

$$f(q(g(\alpha_j))) = \Pi^p *_r \left[L^q L^{-(p+q+1)} d\alpha_j - dL^{-(p+1)}\alpha_j\right] = -\Pi^p *_r dL^{-(p+1)}\alpha_j$$

The first term vanishes since $L^{-(p+q+1)}d\alpha_j$ is only non-zero for terms of the form $L^{p+q}\beta_{j-2p-2q}$ where $\beta_{j-2p-2q}\in\mathcal{P}^{j-2p-2q}(M)$. Then we get

$$\Pi^p *_r L^q \partial_- \beta_{j-2p-2q} = \Pi^p L^{n-j+2p+q+1}\partial_-\beta_{j-2p-2q}$$

Then we only get a non-zero term if $n-j+2p+q+1\leq p$, or $j\geq n+p+q+1$. But then $\alpha_j=0$ since $p+q<p+q+1\leq j-n$.

Now if $j>n+p+q$ we have

$$f(q(g(\alpha_j))) = f(q(-\theta_{p+q} *_r \alpha_j)) = f(-\theta_p L^q *_r \alpha_j) = \Pi^p *_r (L^q *_r \alpha_j) = \Pi^p L^{-q}\alpha_j = L^{-q}\alpha_j$$

since α_j is $p+q$ graded.

All terms match $l(\alpha_j)$. $\square$

As with r^2 we can compute $l^2 = f\circ q\circ g^2 + f^2((q\circ g)\otimes(q\circ g))$. Computing each term gives:

$$f(q(g^2(\alpha_j\otimes\alpha_k))) = \begin{cases} \Pi^p *_r (L^q L^{-(p+1)}(\alpha_j\wedge\alpha_k)) & j,k\leq n+p+q,\ j+k>n+p+1 \\ 0 & \text{otherwise} \end{cases}$$

Chapter 3

Gysin Sequence for Filtered Cohomology

Another potential action we can consider is the idea of mapping a bundle and the base manifold by integrating over fibers. One application of this idea is to use these actions to relate the cohomologies of a base manifold and a sphere bundle, and in this section we develop those techniques for the cone cohomology.

Setting 3.1. *The following notions will be used throughout this section.*

1. (X^{2n}, ω_X) *is a symplectic manifold with* $\mathbb{C}^m$*-vector bundle* $\pi : E \to X$ *such that* $\omega = \pi^*\omega_X + d\mu$
2. $\mu_{2p+1} = \mu \wedge \sum_{j=0}^{p} \omega^{p-j} \wedge \pi^*\omega_X^j$ *so that* $d\mu_{2p+1} = \omega^{p+1} - \pi^*\omega_X^{p+1}$
3. $E_0 = E \setminus X$ *is also a symplectic manifold with* ω
4. *Fix a disk bundle* $D \subset E$*, and let* $S = \partial D$ *be the boundary sphere bundle*
5. $\pi_*^D : \Omega^j(E) \to \Omega^{j-2m}(X)$ *is an operator defined by restriction to* D *then integration*

along fibers. It commutes with the exterior derivative and $\pi_*^D(\pi^*(\tau_k)\wedge C_j = \tau_k \wedge \pi_*^D(C_j)$ *for* $\tau_k \in \Omega^k(X)$, $C_j \in \Omega^j(E)$

6. $\pi_*^S : \Omega^j(E_0) \to \Omega^{j-2m+1}(X)$ *is an operator defined by restriction to* S *then integration along fibers. It commutes with the exterior derivative and* $\pi_*^S(\pi^*(\tau_k)\wedge C_j) = \tau_k \wedge \pi_*^S(C_j)$ *for* $\tau_k \in \Omega^k(X)$, $C_j \in \Omega^j(E_0)$

7. *By Stoke's Theorem* $\pi_*^S(C_j) = \pi_*^D(dC_j) = d\pi_*^D(C_j)$ *for* $C_j \in \Omega^j(E)$

8. *There exists* $\Psi \in \Omega^{2m-1}(E_0)$ *such that* $\pi_*^S(\Psi) = 1$ *and* $d\Psi = -\pi^*\mathfrak{e}$ *for* $\mathfrak{e} \in \Omega^{2m}(X)$

3.1 Integration Along Fibers for Filtered Forms

First we define what it means to integrate cone forms over a fiber, both in the sphere bundle and disk bundle case.

Lemma 3.2. $\tilde{\pi}_*^S(A_j + \theta_{E_0} B_{j_p}) = \pi_*^S(A_j + \mu_{2p+1} \wedge B_{j_p}) + \theta_X \pi_*^S(B_{j_p})$ *defines a chain map from* $\mathcal{C}_p^j(E_0) \to \mathcal{C}_p^{j-2m+1}(X)$.

Proof.

$$\begin{aligned}
d\tilde{\pi}^S_*(C_j) &= d(\pi^S_*(A_j + \mu_{2p+1} \wedge B_{j_p}) + \theta_X \pi^S_*(B_{j_p}) \\
&= \pi^S_*(dA_j + d\mu_{2p+1} \wedge B_{j_p} - \mu_{2p+1} \wedge dB_{j_p}) + \omega_X^{p+1} \wedge \pi^S_*(B_{j_p}) - \theta_X \pi^S_*(dB_{j_p}) \\
&= \pi^S_*(dA_j + \omega^{p+1} \wedge B_{j_p} - \pi^* \omega_X^{p+1} \wedge B_{j_p} - \mu_{2p+1} \wedge dB_{j_p}) \\
&\quad + \omega_X^{p+1} \wedge \pi^S_*(B_{j_p}) - \theta_X \pi^S_*(dB_{j_p}) \\
&= \pi^S_*(dA_j + \omega^{p+1} \wedge B_{j_p} - \mu_{2p+1} \wedge dB_{j_p}) - \theta_X \pi^S_*(dB_{j_p}) \\
&= \tilde{\pi}^S_*(dA_j + \omega^{p+1} \wedge B_{j_p} - \theta_{E_0} dB_{j_p}) \\
&= \tilde{\pi}^S_*(dC_j)
\end{aligned}$$

□

Note that the above proof only uses the properties of $\tilde{\pi}^S_*$ that are shared with $\tilde{\pi}^D_*$. Therefore we also have that

Lemma 3.3. $\tilde{\pi}^D_*(A_j + \theta_E B_{j_p}) = \pi^D_*(A_j + \mu_{2p+1} \wedge B_{j_p}) + \theta_X \pi^D_*(B_{j_p})$ *defines a chain map from* $\mathcal{C}^j_p(E) \to \mathcal{C}^{j-2m}_p(X)$.

Now we define the conjugate of a cone form which will let us form a relation between $\tilde{\pi}^D_*$ and $\tilde{\pi}^S_*$.

Definition 3.4. *For* $C_j \in \mathcal{C}^j_p(M)$ *we define the conjugate* $\overline{C_j} = A_j - \theta_M B_{j_p}$.

Lemma 3.5. $d\overline{C_j} + \overline{dC_j} = 2(dA_j + \theta dB_{j_p})$

Proof.

$$
\begin{aligned}
d\overline{C_j} + \overline{dC_j} &= d(A_j - \theta B_{j_p}) + \overline{dA_j + \omega^{p+1} \wedge B_{j_p} - \theta dB_{j_p}} \\
&= dA_j - \omega^{p+1} \wedge B_{j_p} + \theta dB_{j_p} + dA_j + \omega^{p+1} \wedge B_{j_p} + \theta dB_{j_p} \\
&= 2(dA_j + \theta dB_{j_p})
\end{aligned}
$$

□

Proposition 3.6. $d\overline{\tilde{\pi}_*^D(C_j)} + \overline{\tilde{\pi}_*^D(dC_j)} = 2\tilde{\pi}_*^S(C_j)$

Proof. From the previous lemma and the definition of $\tilde{\pi}_*^D$ we have

$$
\begin{aligned}
d\overline{\tilde{\pi}_*^D(C_j)} + \overline{\tilde{\pi}_*^D(dC_j)} &= 2(d\pi_*^D(A_j + \mu_{2p+1} \wedge B_{j_p}) + \theta_X d\pi_*^D(B_{j_p})) \\
&= 2(\pi_*^S(A_j + \mu_{2p+1} \wedge B_{j_p}) + \theta_X \pi_*^S(B_{j_p})) \\
&= 2\tilde{\pi}_*^S(C_j)
\end{aligned}
$$

□

3.2 Thom Isomorphism

At this point we can follow the usual method of proving the Thom Isomorphism:

Definition 3.7. *The relative p-filtered cone complex is defined to be*

$$
\mathcal{C}_p^j(E, E_0) = \mathcal{C}_p^j(E) \oplus \mathcal{C}_p^{j-1}(E_0)
$$

with differential $d(C_j, C_{j-1}) = (dC_j, C_j - dC_{j-1})$

Lemma 3.8. *The relative p-filtered cone cohomology satisfies the following long exact sequence:*

$$\cdots \to H^j\mathcal{C}_p(E,E_0) \to H^j\mathcal{C}_p(E) \to H^j\mathcal{C}_p(E_0) \to H^{j+1}\mathcal{C}_p(E,E_0) \to \cdots$$

Proof. The proof of this statement is the exactly the same as the de Rham version. See Prop 6.49 of [1] for more details. □

Proposition 3.9. *The map* $\tilde{\pi}_*(C_j, C_{j-1}) = \tilde{\pi}^S_*(C_{j-1}) - \frac{1}{2}\overline{\tilde{\pi}^D_*(C_j)}$ *defines a homomorphism from* $H\mathcal{C}^j_p(E,E_0)$ *to* $H\mathcal{C}^{j-2m}_p(X)$*.*

Proof.

$$\begin{aligned}
\tilde{\pi}_*(d(C_j, C_{j-1})) &= \tilde{\pi}_*(dC_j, C_j - dC_{j-1}) \\
&= \tilde{\pi}^S_*(C_j - dC_{j-1}) - \frac{1}{2}\overline{\tilde{\pi}^D_*(dC_j)} \\
&= \tilde{\pi}^S_*(C_j) - d\tilde{\pi}^S_*(C_{j-1}) + \frac{1}{2}d\overline{\tilde{\pi}^D_*(C_j)} - \tilde{\pi}^S_*(C_j) \\
&= -d\tilde{\pi}^S_*(C_{j-1}) + \frac{1}{2}d\overline{\tilde{\pi}^D_*(C_j)} \\
&= -d\tilde{\pi}_*(C_j, C_{j-1})
\end{aligned}$$

□

The following is a useful lemma for reducing

Lemma 3.10. $H^j(M) = H^{j-2p-1}(M) = 0$ *implies that* $H\mathcal{C}^j_p(M) = 0$*.*

Proof. Suppose $C_j \in H\mathcal{C}^j_p(M)$ is closed, so

$$0 = dC_j = dA_j + \omega^{p+1} \wedge B_{j_p} - \theta_M dB_{j_p}$$

so $dB_{j_p} = 0$ and $dA_j + \omega^{p+1} \wedge B_{j_p} = 0$. Then $B_{j_p} \in H^{j-2p-1}(M)$ and therefore must be exact, say $B_{j_p} = -dB_{j_p-1}$. Now

$$d(A_j - \omega^{p+1} \wedge B_{j-1}) = dA_j - \omega^{p+1} \wedge dB_{j_p-1} = dA_j + \omega^{p+1} \wedge B_{j_p} = 0$$

so $A_j - \omega^{p+1} \wedge B_{j_p-1}$ is also exact and then equal to dA_{j_p-1}. Finally

$$d(A_{j_p-1} + \theta_M B_{j-1}) = A_j - \omega^{p+1} \wedge B_{j_p-1} + \omega^{p+1} \wedge B_{j_p-1} - \theta_M dB_{j_p-1} = A_j + \theta_M B_{j_p} = C_j$$

is exact. □

Lemma 3.11 (Poincaré Lemma for $\mathcal{C}_p$)**.** *Let U be diffeomorphic to a disk with symplectic form ω, then $H^j\mathcal{C}_p(U) = 0$ for all $j \neq 0, 2p+1$. $H^0\mathcal{C}_p(U) = H^{2p+1}\mathcal{C}_p(U) = \mathbb{R}$ with generators 1 and $\omega^p \wedge a - \theta_U$ where $\omega = da$ for some $a \in \Omega^1(U)$.*

Proof. By the de Rham Poincaré Lemma the only non-trivial de Rham cohomology is in degree zero generated by 1. From the previous lemma we then only get non-zero cohomology in degree $j = 0$ and $j = 2p+1$. The $j = 0$ case is immediate from $H^0\mathcal{C}^p(U) = H^0(U)$.

For $j = 2p+1$, $C_j \in H^j\mathcal{C}_p(U)$ means that $B_{j_p} \in \Omega^0(U)$ and therefore proportional to 1. For convenience we will take $B_{j_p} = -1$. Then the closed condition requires $dA_j = \omega^{p+1}$, and $A_j = \omega^p \wedge a$ satisfied this condition giving the generator $\omega^p \wedge a - \theta_U$. Any other solution is cohomologous to $\omega^p \wedge a$ and so gives no new cohomology classes. □

Lemma 3.12. *Let $U \subset X$ be an open set diffeomorphic to a disk. Then $\tilde{\pi}_*$ induces an isomorphism between $H^j\mathcal{C}_p(E|_U, E_0|_U)$ and $H^{j-2m}\mathcal{C}_p(U)$.*

Proof. Let $\omega_X|_U = da$ for $a \in \Omega^1(U)$ and $\mathfrak{e}|_U = db$ for $b \in \Omega^{2m-1}(U)$. By previous lemma we have $\bigoplus_j H^j\mathcal{C}_p(U) = \mathbb{R}^2$ with generators 1 and $\omega^p \wedge a - \theta_U$. Similarly E_U is topologically a disk so it has generators 1 and $\omega^p \wedge (\pi^* a + \mu) - \theta_E$, note the addition of μ since $\omega =$

$\pi^*\omega_X + d\mu = d(\pi^* a + \mu)$.

$E_0|_U$ is a $(2m-1)$-sphere with generator $\Psi + \pi^* b$ for $H^{2m-1}(E_0|_U)$. By the resolution sequence and Lemma 3.10 we may only have non-trivial cohomology in degree 0, $2p+1$, $2m-1$, and $2m+2p$. As before we have generators 1 and $\omega^p \wedge (\pi^* a + \mu) - \theta_{E_0}$. The degree $2m-1$ generator is $\Psi + \pi^* b$, taken from $H^{2m-1}(E_0|_U)$. The final element is the wedge product $(\omega^p \wedge (\pi^* a + \mu) - \theta_{E_0}) \wedge (\Psi + \pi^* b)$.

Now by Lemma 3.11 $\oplus_j H^j \mathcal{C}_p(E|_U, E_0|_U) = \mathbb{R}^2$ with generators in degree $2m$ and $2m+2p+1$ given by those on $E_0|_U$. What remains is to check that these map to the corresponding generators of $\oplus_j H^j \mathcal{C}_p(U)$ under $\tilde{\pi}$ which acts as $\tilde{\pi}^S_*$ on these generators.

$$\tilde{\pi}^S_*(\Psi + \pi^* b) = \pi^S_*(\Psi + \pi^* b) = 1 + b \wedge \pi^S_*(1) = 1$$

for degree reasons.

For the second generator we have

$$\tilde{\pi}^S_*((\omega^p \wedge (\pi^* a + \mu) - \theta_{E_0}) \wedge (\Psi + \pi^* b)) = \pi^S_*((\omega^p \wedge (\pi^* a + \mu) - \mu_{2p+1}) \wedge (\Psi + \pi^* b)) - \theta_U \pi^S_*(\Psi + \pi^* b)$$

By the same calculation as above we immediately see the second term is $-\theta_U$ as desired. It remains to show the first term is equal to $\omega^p \wedge a$.

We will show that the first term maps to $\omega^p \wedge a$ by induction on p. If $p = 0$, then $\mu_{2p+1} = \mu$ and we have

$$\pi^S_*(\pi^* a \wedge (\Psi + \pi^* b)) = a \wedge \pi^S_*(\Psi + \pi^* b) = a$$

Now take $p > 0$. Similar to μ_{2p+1} we can define $\mu_{2p-1} = \mu \wedge \sum_{j=0}^{p-1} \omega^j \wedge \pi^* \omega_X^{p-1-j}$ which satisfies the relation $\omega^p = \pi^* \omega_X^p + d\mu_{2p-1}$. From the similar structure of μ_{2p+1} and μ_{2p-1} we also obtain the relation $\mu_{2p+1} = \pi^*\omega_X \wedge \mu_{2p-1} + \mu \wedge \omega^p$. Then

$$\begin{aligned}
\omega^p \wedge (\pi^* a + \mu) - \mu_{2p+1} &= \omega^p \wedge \pi^* a - \pi^*\omega_X \wedge \mu_{2p-1} \\
&= (\pi^*\omega_X + d\mu) \wedge \omega^{p-1} \wedge \pi^* a - \pi^*\omega_X \wedge \mu_{2p-1} \\
&= \pi^*\omega_X \wedge (\omega^{p-1} \wedge \pi^* a - \mu_{2p-1}) + d\mu \wedge \omega^{p-1} \wedge \pi^* a \\
&= \pi^*\omega_X \wedge (\omega^{p-1} \wedge \pi^* a - \mu_{2p-1}) + d(\mu \wedge \omega^{p-1} \wedge \pi^* a) \\
&\quad + \mu \wedge \omega^{p-1} \wedge \pi^*\omega_X \\
&= \pi^*\omega_X \wedge (\omega^{p-1} \wedge (\pi^* a + \mu) - \mu_{2p-1}) + d(\mu \wedge \omega^{p-1} \wedge \pi^* a)
\end{aligned}$$

Since π_*^S is a chain map and $\Psi + \pi^* b$ is closed, the $d(\mu \wedge \omega^{p-1} \wedge \pi^* a)$ will result in an exact term which can be neglected. What remains is

$$\begin{aligned}
&\pi_*^S[\pi^*\omega_X \wedge (\omega^{p-1} \wedge (\pi^* a + \mu) - \mu_{2p-1}) \wedge (\Psi + \pi^* b)] \\
&\qquad = \omega_X \wedge \pi_*^S[(\omega^{p-1} \wedge (\pi^* a + \mu) - \mu_{2p-1}) \wedge (\Psi + \pi^* b)] \\
&\qquad = \omega_X \wedge \omega_X^{p-1} \wedge a \\
&\qquad = \omega^p \wedge a
\end{aligned}$$

as was desired. □

Theorem 3.13. *Suppose X is of finite type. Then $\tilde{\pi}_* : H^j\mathcal{C}_p(E, E_0) \to H^{j-2m}\mathcal{C}_p(X)$ is an isomorphism.*

Proof. By standard Mayer-Vietoris argument if U and V are open subsets of X we then have a short exact sequence of the relative cone complex

$$0 \to \mathcal{C}_p^j(E|_{U\cup V}, E_0|_{U\cup V}) \to \mathcal{C}_p^j(E|_U, E_0|_U) \oplus \mathcal{C}_p^j(E|_V, E_0|_V) \to \mathcal{C}_p^j(E|_{U\cap V}, E_0|_{U\cap V}) \to 0$$

which induces a long exact sequence of relative cohomology. Similarly there is a corresponding sequence of cone cohomology on $U \cup V$. $\tilde{\pi}_*$ maps between these two sequences. If the diagram commutes, since $\tilde{\pi}_*$ is an isomorphism over $U, V, U \cap V$ the five lemma implies the isomorphism on $U \cup V$. The only non-trivial part to check is with the connecting homomorphisms:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^j\mathcal{C}_p(E|_{U\cap V}, E_0|_{U\cap V}) & \xrightarrow{g_j} & H^{j+1}\mathcal{C}_p(E|_{U\cup V}, E_0|_{U\cup V}) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \tilde{\pi}_*} & & \downarrow{\scriptstyle \tilde{\pi}_*} & \\
\cdots \longrightarrow & H^{j-2m}\mathcal{C}_p(U \cap V) & \xrightarrow{f_{j-2m}} & H^{j-2m+1}\mathcal{C}_p(U \cup V) & \longrightarrow \cdots
\end{array}$$

Let ρ be a smooth function defined on $U \cup V$ which is 1 on $U \setminus V$ and 0 on $V \setminus U$. The connecting homomorphisms are defined by

$$\begin{aligned}
f_{j-2m}(C_{j-2m}) &= d(\rho C_{j-2m}) \\
g_j(C_j, C_{j-1}) &= (-d(\pi^*\rho\ C_j), d(\pi^*\rho\ C_{j-1}) - \pi^*\rho\ C_j)
\end{aligned}$$

Here g_j differs from the usual definition by a minus sign. Note that the result looks like an exact form, but is not the case since a term like ρC_{j-2m} is only defined on V because on $U \setminus V$ C_{j-2m} is undefined but $\rho \neq 0$. Then the result is not a smooth $j-2m$ form on $U \cup V$ and then cannot be exact.

With this setup we need to check that $(f_{j-2m} \circ \tilde{\pi}_* - \tilde{\pi}_* \circ g_j)(C_j, C_{j-1})$ is trivial:

$$\begin{aligned}
(f_{j-2m} \circ \tilde{\pi}_* - \tilde{\pi}_* \circ g_j)(C_j, C_{j-1}) \\
&= d(\rho \tilde{\pi}_*^S(C_{j-1})) - \frac{1}{2} d(\rho \overline{\tilde{\pi}_*^D(C_j)}) \\
&\quad - \tilde{\pi}_*^S(d(\pi^* \rho\ C_{j-1})) + \tilde{\pi}_*^S(\pi^* \rho\ C_j) - \frac{1}{2} \overline{\tilde{\pi}_*^D(d(\pi^* \rho\ C_j))} \\
&= d(\rho \tilde{\pi}_*^S(C_{j-1})) - \frac{1}{2} d(\rho \overline{\tilde{\pi}_*^D(C_j)}) - d\tilde{\pi}_*^S(\pi^* \rho\ C_{j-1}) + \frac{1}{2} d\overline{\tilde{\pi}_*^D(\pi^* \rho\ C_j)} \\
&= d\left[\rho \tilde{\pi}_*^S(C_{j-1}) - \tilde{\pi}_*^S(\pi^* \rho\ C_{j-1}) - \frac{1}{2}(\overline{\rho \tilde{\pi}_*^D(C_j)} - \overline{\tilde{\pi}_*^D(\pi^* \rho\ C_j)})\right] \\
&= d\left[\rho \tilde{\pi}_*^S(C_{j-1}) - \tilde{\pi}_*^S(\pi^* \rho\ C_{j-1}) - \frac{1}{2}(\overline{\rho \tilde{\pi}_*^D(C_j) - \tilde{\pi}_*^D(\pi^* \rho\ C_j)})\right]
\end{aligned}$$

Then it suffices to show $\rho \tilde{\pi}_*^S(C_{j-1}) - \tilde{\pi}_*^S(\pi^* \rho\ C_{j-1})$ and $\rho \tilde{\pi}_*^D(C_j) - \tilde{\pi}_*^D(\pi^* \rho\ C_j)$ are smooth on $U \cup V$.

$$\begin{aligned}
\tilde{\pi}_*^D(\pi^* \rho\ C_j) &= \pi_*^D(\pi^* \rho \wedge (A_j + \mu_{2p+1} B_{j_p}) + \theta_X \pi_*^D(\pi^* \rho \wedge B_{j_p}) \\
&= \rho \pi_*^D(A_j + \mu_{2p+1} B_{j_p}) + \theta_X \rho \pi_*^D(B_{j_p}) \\
&= \rho \tilde{\pi}_*^D(C_j)
\end{aligned}$$

This calculation only requires shared properties of $\tilde{\pi}_*^D$ and $\tilde{\pi}_*^S$ so the same holds for $\tilde{\pi}_*^S$. Therefore in fact $f_{j-2m} \circ \tilde{\pi}_* = \tilde{\pi}_* \circ g_j$.

□

3.3 Inverse of the Thom Isomorphism

Proposition 3.14. *Suppose X is of finite type. Then*

$$\tilde{\pi}^*(C_j) = (d(\tilde{\pi}(C_j) \wedge \Psi), \tilde{\pi}(C_j) \wedge \Psi)$$

is the inverse of $\tilde{\pi}_*$.

Proof. We first note that if C_j is closed, then $\tilde{\pi}^*(C_j)$ must be as well.

Now

$$\tilde{\pi}_* \circ \tilde{\pi}^*(C_j) = \tilde{\pi}_*^S(\tilde{\pi}(C_j) \wedge \Psi) - \frac{1}{2}\overline{d\tilde{\pi}_*^D(\tilde{\pi}(C_j) \wedge \Psi)}$$

Starting with the first term we compute:

$$\begin{aligned}
\tilde{\pi}_*^S(\tilde{\pi}(C_j) \wedge \Psi) &= \tilde{\pi}_*^S([\pi^* A_j - \mu_{2p+1} \wedge \pi^* B_{j_p}] \wedge \Psi + \theta_{E_0}[\pi^* B_{j_p} \wedge \Psi]) \\
&= \pi_*^S([\pi^* A_j - \mu_{2p+1} \wedge \pi^* B_{j_p}] \wedge \Psi + \mu_{2p+1} \wedge \pi^* B_{j_p} \Psi + \theta_X \pi_*^S(\pi^* B_{j_p} \wedge \Psi) \\
&= \pi_*^S(\pi^* A_j \wedge \Psi) + \theta_X \pi_*^S(\pi^* B_{j_p} \wedge \Psi) \\
&= A_j \wedge \pi_*^S(\Psi) + \theta_X B_{j_p} \wedge \pi_*^S(\Psi) \\
&= A_j + \theta_X B_{j_p} \\
&= C_j
\end{aligned}$$

Therefore it suffices to show the second term is zero. By similar calculation as above we have

$$\tilde{\pi}_*^D(\tilde{\pi}(C_j) \wedge \Psi) = A_j \wedge \pi_*^D(\Psi) + \theta_X B_{j_p} \wedge \pi_*^D(\Psi) = 0$$

since $\pi_*^D(\Psi) = 0$ for degree reasons. $\square$

3.3.1 The Gysin Sequence for p-Filtered Cone Complex

Rewriting Lemma 3.8 as a Gysin sequence using the just established isomorphism gives

$$\cdots \to H^{j-2m}\mathcal{C}_p(X) \to H^j\mathcal{C}_p(X) \to H^j\mathcal{C}_p(E_0) \to H^{j-2m+1}\mathcal{C}_p(X) \to \cdots$$

The map $H^j\mathcal{C}_p(E_0) \to H^{j-2m+1}\mathcal{C}_p(X)$ is given by

$$[C_j] \mapsto [\tilde{\pi}_*(C_j, 0)] = [\tilde{\pi}_*^S(C_j)]$$

The map $H^j\mathcal{C}_p(X) \to H^j\mathcal{C}_p(E_0)$ is

$$[C_j] \mapsto [\tilde{\pi}(C_j)]$$

The final map $H^{j-2m}\mathcal{C}_p(X) \to H^j\mathcal{C}_p(X)$ is defined by the negation of the composition of the first component of $\tilde{\pi}^*$ with the pullback $\iota^*(A_j + \theta_E B_{j_p}) = \iota^* A_j + \theta_X \iota^* B_{j_p}$. Then

$$\begin{aligned}
-\iota^*(d(\tilde{\pi}(C_j) \wedge \Psi)) &= -(-1)^j \iota^*(\tilde{\pi}(C_j) \wedge d\Psi) \\
&= -(-1)^j \iota^*([\pi^* A_j - \mu_{2p+1} \wedge \pi^* B_{j_p}] \wedge d\Psi + \theta_E \pi^* B_{j_p} \wedge d\Psi \\
&= (-1)^j \iota^*([\pi^* A_j - \mu_{2p+1} \wedge \pi^* B_{j_p}] \wedge \pi^* \mathfrak{e}) + \theta_X \iota^*(\pi^* B_{j_p} \wedge \pi^* \mathfrak{e}) \\
&= (-1)^j (A_j \wedge \mathfrak{e} + \theta_X B_{j_p} \wedge \mathfrak{e}) \\
&= C_j \wedge (-1)^j \mathfrak{e}
\end{aligned}$$

since $\iota^* \circ \pi^*$ is the identity and $\mu_{2p+1}|_X = 0$.

Therefore we can summarize this Gysin sequence by

Theorem 3.15 (Gysin Sequence for p-Filtered Cone Complex)**.**

$$\cdots \longrightarrow H^{j-2m}\mathcal{C}_p(X) \xrightarrow{\wedge(-1)^j\mathfrak{e}} H^j\mathcal{C}_p(X) \xrightarrow{\tilde{\pi}} H^j\mathcal{C}_p(E_0) \xrightarrow{\tilde{\pi}^S_*} H^{j-2m+1}\mathcal{C}_p(X) \longrightarrow \cdots$$

is an exact sequence of cohomology.

Chapter 4

Applying Mayer-Vietoris to Cone Cohomology

As another example we will demonstrate using the Mayer-Vietoris sequence to compute how the cone cohomology of a manifold changes after blowing up along a submanifold.

Setting 4.1. *In this section we will use the following notation:*

1. (M^{2n+2m}, ω) *is a symplectic manifold*

2. (X^{2n}, ω_X) *is a codimension* $2m$ *submanifold of* M *with symplectic form* $\omega_X = \omega|_X$*, where* $m \geq 1$

3. M_2 *is a tubular neighborhood of* X *in* M

4. $\tilde{M}_2$ *will denote the blowup of* M_2 *along* X

5. M_1 *is* M *without a tubular neighborhood around* X *(we will assume this is a smaller neighborhood than with* M_2 *so* M_1 *and* M_2 *overlap)*

6. $\tilde{M} = M_1 \cup \tilde{M}_2$ *is the blowup of* M *at* X

The goal of this section is to use the Mayer-Vietoris sequence for the cone cohomology to relate the cohomologies of M and $\tilde{M}$. Namely we have two short exact sequences on M and $\tilde{M}$:

$$0 \to \mathcal{C}_p(M) \to \mathcal{C}_p(M_1) \oplus \mathcal{C}_p(M_2) \to \mathcal{C}_p(M_1 \cap M_2) \to 0$$

$$0 \to \mathcal{C}_p(\tilde{M}) \to \mathcal{C}_p(M_1) \oplus \mathcal{C}_p(\tilde{M}_2) \to \mathcal{C}_p(M_1 \cap \tilde{M}_2) \to 0$$

We will simplify to the case where X is 2-dimensional so $n = 1$ and we know all information about the de Rham cohomology of X. From that information we can compute the de Rham cohomologies of M_2, $\tilde{M}_2$, and $M_1 \cap M_2 = M_1 \cap \tilde{M}_2$ by noting that they deformation retract to X, a S^{2m-1} bundle over X, and a $\mathbb{CP}^{m-1}$ bundle over X respectively.

4.1 Primitive Case

We will consider separately the primitive case where $p = 0$. Then the de Rham cohomologies of the basic component parts are:

$$\begin{aligned} H^*_d(M_2) &\cong H^*_d(X) \\ H^*_d(M_1 \cap M_2) &\cong H^*_d(X) \otimes \mathbb{R}\langle 1, \psi_{2m-1} \rangle \\ H^*_d(\tilde{M}_2) &\cong H^*_d(X) \otimes \mathbb{R}\langle 1, \omega_F, \dots, \omega_F^{m-1} \rangle \end{aligned}$$

where ψ_{2m-1} is the global angular form of the sphere bundle and ω_F is the symplectic form on the $\mathbb{CP}^{m-1}$ fibers.

Lemma 4.2. *The primitive cone cohomology of X is*

$$H^0\mathcal{C}_0(X) = H^0(X)$$

$$H^1\mathcal{C}_0(X) = H^1(X)$$

$$H^2\mathcal{C}_0(X) = \theta_X H^1(X)$$

$$H^3\mathcal{C}_0(X) = \theta_X H^2(X)$$

all other cohomology groups are zero.

Proof. The resolution sequence immediately gives $H^0\mathcal{C}_0(X) \cong H^0(X)$ and $H^3\mathcal{C}_0(X) \cong H^2(X)$. In fact $H^0\mathcal{C}_0(X) = H^0(X)$ and $H^3\mathcal{C}_3(X) = \theta_X H^2(X)$. For the other indices:

$$j = 1: \qquad 0 \to H^1(X) \to H^1\mathcal{C}_0(X) \to H_0(X) \to H_2(X)$$

The last map is an isomorphism so we have $H^1(X)$ injects into $H^1\mathcal{C}_0(X)$ which then maps to zero in $H_0(X)$. In fact have $H^1\mathcal{C}_0(X) = H^1(X) \cong \mathbb{R}^{2g}$.

$$j = 2: \qquad H^0(X) \to H^2(X) \to H^2\mathcal{C}_0(X) \to H^1(X) \to 0$$

As before the first map is an isomorphism so $H^2(X)$ maps to zero in $H^2\mathcal{C}_0(X)$ which surjects onto $H^1(X)$. Therefore $H^2\mathcal{C}_0(X) = \theta_X H^1(X) \cong \mathbb{R}^{2g}$. □

Lemma 4.3. *If $m > 2$, then*

$$
\begin{aligned}
H^0\mathcal{C}_0(M_1 \cap M_2) &\cong H^0(X) \\
H^1\mathcal{C}_0(M_1 \cap M_2) &\cong H^1(X) \\
H^2\mathcal{C}_0(M_1 \cap M_2) &\cong \theta_{M_1\cap M_2} H^1(X) \\
H^3\mathcal{C}_0(M_1 \cap M_2) &\cong \theta_{M_1\cap M_2} H^2(X) \\
H^{2m-1}\mathcal{C}_0(M_1 \cap M_2) &\cong \psi_{2m-1} \wedge H^0(X) \\
H^{2m}\mathcal{C}_0(M_1 \cap M_2) &\cong \psi_{2m-1} \wedge H^1(X) \\
H^{2m+1}\mathcal{C}_0(M_1 \cap M_2) &\cong \theta_{M_1\cap M_2} \psi_{2m-1} \wedge H^1(X) \\
H^{2m+2}\mathcal{C}_0(M_1 \cap M_2) &\cong \theta_{M_1\cap M_2} \psi_{2m-1} \wedge H^2(X)
\end{aligned}
$$

and are otherwise zero.

Proof. We can only have non-zero cohomologies in indices $0 \leq j \leq 3$ and $2m-1 \leq j \leq 2m+2$. Since $m \geq 3$, ψ_{2m-1} is at least a degree 5 form. This is sufficient to so that the $H^j(X)$ terms and $\psi_{2m-1} \wedge H^j(X)$ terms do not interact in the resolution sequence. Then the $0 \leq j \leq 3$ indices compute exactly as above, and the $2m-1 \leq j \leq 2m+2$ terms are the same but shifted by ψ_{2m-1}. □

The assumption that $m > 2$ is necessary to ensure the $H^*(X)$ and $\theta_{M_1\cap M_2} H^*(X)$ parts of $H^*_d(M_1 \cap M_2)$ do not interfere with each other. We will later handle the case when $m = 1$ and $m = 2$ which corresponds to M being 4- or 6-dimensional.

Lemma 4.4. *The primitive cone cohomology of $\tilde{M}_2$ is given by*

$$\begin{aligned} H^0\mathcal{C}_0(\tilde{M}_2) &\cong H^0(X) \\ H^1\mathcal{C}_0(\tilde{M}_2) &\cong H^1(X) \\ H^2\mathcal{C}_0(\tilde{M}_2) &\cong H^2(X) \\ H^{2m-1}\mathcal{C}_0(\tilde{M}_2) &\cong \theta_{\tilde{M}_2}(\omega_F^{m-1} - \omega_X \wedge \omega_F^{m-2}) \wedge H^0(X) \\ H^{2m}\mathcal{C}_0(\tilde{M}_2) &\cong \theta_{\tilde{M}_2}\omega_F^{m-1} \wedge H^1(X) \\ H^{2m+1}\mathcal{C}_0(\tilde{M}_2) &\cong \theta_{\tilde{M}_2}\omega_F^{m-1} \wedge H^2(X) \end{aligned}$$

and zero otherwise.

Proof. First note that the de Rham cohomologies are of the form

$$\begin{aligned} H^{2k}(\tilde{M}_2) &\cong \omega_F^k \wedge H^0(X) + \omega_F^{k-1} \wedge H^2(X) \\ H^{2k+1}(\tilde{M}_2) &\cong \omega_F^k \wedge H^1(X) \end{aligned}$$

depending on whether the index is even or odd. Then the Lefschetz map acts as

$$\begin{aligned} (\omega_X + \omega_F) \wedge (\omega_F^k \wedge \eta_0 + \omega_F^{k-1} \wedge \eta_2) &= \omega_X \wedge \omega_F^k \wedge \eta_0 + \omega_F^{k+1} \wedge \eta_0 + \omega_F^k \wedge \eta_2 \\ (\omega_X + \omega_F) \wedge (\omega_F^k \wedge \eta_1) &= \omega_F^{k+1} \wedge \eta_1 \end{aligned}$$

where we take $\eta_k \in H^k(X)$. From this we see the Lefschetz map is an isomorphism for indices $1 \leq j \leq 2m-3$ (note that there is no η_2 term when $j = 0$ and no η_0 term when $j = 2m$). For $3 \leq j \leq 2m-2$ we have

$$H^{j-2}(\tilde{M}_2) \xrightarrow{\sim} H^j(\tilde{M}_2) \xrightarrow{0} H^j\mathcal{C}_0(\tilde{M}_2) \xrightarrow{0} H^{j-1}(\tilde{M}_2) \xrightarrow{\sim} H^{j+1}(\tilde{M}_2)$$

So $H^j\mathcal{C}_0(\tilde{M}_2) = 0$ when $3 \leq j \leq 2m-2$. Therefore we only have nonzero cohomology when $j = 0, 1, 2, 2m-1, 2m, 2m+1$. In the following we denote $\theta_{\tilde{M}_2}$ as simply θ for convenience.

For the relevant indices we have:

$$j = 0 : \qquad 0 \to H^0(\tilde{M}_2) \to H^0\mathcal{C}_0(\tilde{M}_2) \to 0$$

so $H^0\mathcal{C}_0(\tilde{M}_2) = H^0(\tilde{M}_2) \cong H^0(X)$.

$$j = 1 : \qquad 0 \to H^1(\tilde{M}_2) \to H^1\mathcal{C}_0(\tilde{M}_2) \to H^0(\tilde{M}_2) \to H^2(\tilde{M}_2)$$

The final map is injective so $H^1\mathcal{C}_0(\tilde{M}_2) = H^1(\tilde{M}_2) \cong H^1(X)$.

$$j = 2 : \qquad H^0(\tilde{M}_2) \to H^2(\tilde{M}_2) \to H^2\mathcal{C}_0(\tilde{M}_2) \to H^1(\tilde{M}_2) \xrightarrow{\sim} H^3(\tilde{M}_2)$$

Since the last map is an isomorphism then $H^2(\tilde{M}_2)$ surjects onto $H^2\mathcal{C}_0(\tilde{M}_2)$. The kernel of that map is elements of the form $(\omega_X + \omega_F) \wedge \eta_0$, so since ω_X is an isomorphism of $H^0(X)$ and $H^2(X)$ we can represent any element of $H^2\mathcal{C}_0(\tilde{M}_2)$ uniquely by an element of $H^2(X)$. So $H^2\mathcal{C}_0(\tilde{M}_2) \cong H^2(X)$.

$$j = 2m-1 : \qquad H^{2m-3}(\tilde{M}_2) \xrightarrow{\sim} H^{2m-1}(\tilde{M}_2) \to H^{2m-1}\mathcal{C}_0(\tilde{M}_2) \to H^{2m-2}(\tilde{M}_2) \to H^{2m}(\tilde{M}_2)$$

Since the first map is an isomorphism then $H^{2m-1}\mathcal{C}_0(\tilde{M}_2)$ injects to $H^{2m-2}(\tilde{M}_2)$ and is isomorphic to the kernel of the Lefschetz map from $H^{m-2}(\tilde{M}_2)$ to $H^m(\tilde{M}_2)$. Then $H^{2m-1}\mathcal{C}_0(\tilde{M}_2) \cong \theta(\omega_F^{m-1} - \omega_X \wedge \omega_F^{m-2}) \wedge H^0(X)$.

$$j = 2m : H^{2m-2}(\tilde{M}_2) \to H^{2m}(\tilde{M}_2) \to H^{2m}\mathcal{C}_0(\tilde{M}_2) \to H^{2m-1}(\tilde{M}_2) \to 0$$

Since the first map is surjective this gives $H^{2m}\mathcal{C}_0(\tilde{M}_2) = \theta H^{2m-1}(\tilde{M}_2) \cong \theta\omega_F^{m-1} \wedge H^1(X)$

$$j = 2m+1 : 0 \to H^{2m+1}\mathcal{C}_0(\tilde{M}_2) \to H^{2m}(\tilde{M}_2) \to 0$$

Then $H^{2m+1}\mathcal{C}_0(\tilde{M}_2) = H^{2m}(\tilde{M}_2) \cong \theta\omega_F^{m-1} \wedge H^2(X)$ □

The following table summarizes the above results:

	$H^0\mathcal{C}_0$	$H^1\mathcal{C}_0$	$H^2\mathcal{C}_0$	$H^3\mathcal{C}_0$
M_2	$H^0(X)$	$H^1(X)$	$\theta H^1(X)$	$\theta H^2(X)$
$M_1 \cap M_2$	$H^0(X)$	$H^1(X)$	$\theta H^1(X)$	$\theta H^2(X)$
$\tilde{M}_2$	$H^0(X)$	$H^1(X)$	$H^2(X)$	0
	$H^{2m-1}\mathcal{C}_0$	$H^{2m}\mathcal{C}_0$	$H^{2m+1}\mathcal{C}_0$	$H^{2m+2}\mathcal{C}_0$
M_2	0	0	0	0
$M_1 \cap M_2$	$H^0(X)\theta_{2m-1}$	$H^1(X)\theta_{2m-1}$	$\theta H^1(X)\theta_{2m-1}$	$\theta H^2(X)\theta_{2m-1}$
$\tilde{M}_2$	$\theta H^0(X)\omega_F^{m-2}(\omega_F - \omega_X)$	$\theta H^1(X)\omega_F^{m-1}$	$\theta H^2(X)\omega_F^{m-1}$	0

Table 4.1: Summary of cone cohomologies for M_2, $M_1 \cap M_2$, and $\tilde{M}_2$.

Note that in this table θ always represents the appropriate formal variable for each space.

This also provides the information we need about the restriction maps from M_2 and $\tilde{M}_2$ into $M_1 \cap M_2$, defined by $r(A_j + \theta B_{j_p}) = \iota^*(A_j) + \theta\iota^*(B_{j_p})$ where ι is the specified inclusion map.

Lemma 4.5. *The restriction map $r : H^*\mathcal{C}_0(M_2) \to H^*\mathcal{C}_0(M_1 \cap M_2)$ is an isomorphism for degree $0 \leq j \leq 3$ and is the zero map otherwise.*

Proof. This follows from the definition of the restriction map. □

Lemma 4.6. *The restriction map $r : H^*\mathcal{C}_0(\tilde{M}_2) \to H^*\mathcal{C}_0(M_1 \cap M_2)$ is an isomorphism in degrees 0, 1, $2m-1$, $2m$ and is the zero map otherwise.*

Proof. First we consider degrees $0 \leq j \leq 2$. Here $H^j\mathcal{C}_0$ doesn't depend on the fibers at all so the restriction map on the cone cohomology is given by the restriction of de Rham cohomology from which we see that this is an isomorphism in degree 0 and 1 but the zero map in degree 2.

The interesting case occurs in degrees $2m-1$ through $2m+1$. The complication here is that now the cone cohomology depends on the fibers and we have considered $M_1 \cap M_2$ as its deformation retract to a sphere bundle. Therefore we also have to take this retract into consideration when computing the restriction map. The key observation is how to handle the global angular form ψ_{2m-1}, namely that a de Rham cohomology element $\eta \wedge \psi_{2m-1}$ represents an de Rham cohomology element of $M_1 \cap M_2$ which becomes η after integrating over the fibers. Hence we first need to restrict to $M_1 \cap M_2$ and then integrate over the fibers to compute the coefficient form in front of ψ_{2m-1}.

After restricting to $M_1 \cap M_2$, ω_F becomes exact so the symplectic form of $M_1 \cap M_2$ can be written $\omega = \omega_X + d\mu$. Then the map π_*^S given by integrating over the fibers of $M_1 \cap M_2$ over X induces a corresponding chain map

$$\tilde{\pi}_*^S(A_j + \theta B_{j_p}) = \pi_*^S(A_j + \mu \wedge B_{j_p}) + \theta \pi_*^S(B_{j_p})$$

per Lemma 3.2. Note that this map decreases the degree by $2m-1$.

First we compute the map in degree $2m+1$. An element of $H^{2m+1}\mathcal{C}_0(\tilde{M}_2)$ is of the form $\theta\eta_2 \wedge \omega_F^{m-1}$ where $\eta_2 \in H^2(X)$. Then

$$\tilde{\pi}_*^S(\theta\eta_2 \wedge \omega_F^{m-1}) = \pi_*^S(\mu \wedge \eta_2 \wedge \omega_F^{m-1}) + \theta\pi_*^S(\eta_2 \wedge \omega_F^{m-1})$$

Immediately we have $\pi_*^S(\eta_2 \wedge \omega_F^{m-1}) = \eta_2 \wedge \pi_S^*(\omega_F^{m-1}) = 0$ since ω_F^{m-1} has degree less than $2m-1$. To compute the first term we note that we can consider the sphere bundle as the boundary of a disk bundle in $\tilde{M}_2$ over X, and we can denote integration over those disks by a map π_*^D. By Stokes' Theorem these are related by $\pi_*^S(C_j) = \pi_*^D(dC_j)$. Therefore

$$\pi_*^S(\mu \wedge \eta_2 \wedge \omega_F^{m-1}) = \pi_*^D(d\mu \wedge \eta_2 \wedge \omega_F^{m-1}) = \pi_*^D(\eta_2 \wedge \omega_F^m) = \eta_2 \wedge \pi_*^D(\omega_F^m)$$

But ω_F is the symplectic form on the fibers and so ω_F^m is the volume form which means $\pi_*^D(\omega_F^m)$ is simply the volume of the fibers. Hence the image of this map is a scalar multiple of $\eta_2\psi_{2m-1}$. But since ω_X is an isomorphism of $H^0(X)$ and $H^2(X)$ we can find $\eta_0 \in H^0(X)$ such that $\omega_X\eta_0 = \eta_2$ which also implies that $d(\theta\eta_0 \wedge \psi_{2m-1}) = \eta_2 \wedge \theta_{2m-1} \in H^{2m+1}\mathcal{C}_0(M_1 \cap M_2)$ so this is a trivial element. Hence the restriction map is trivial as well.

Now consider degree $2m$. An element of $H^{2m}\mathcal{C}_0(\tilde{M}_2)$ is of the form $\theta\eta_1 \wedge \omega_F^{m-1}$. Then

$$\tilde{\pi}_*^S(\theta\eta_1 \wedge \omega_F^{m-1}) = \pi_*^S(\mu \wedge \eta_1 \wedge \omega_F^{m-1}) + \theta\pi_*^S(\eta_1 \wedge \omega_F^{m-1})$$

Then $\pi_*^S(\eta_1 \wedge \omega_F^{m-1}) = \eta_1 \wedge \pi_*^S(\omega_F^{m-1}) = 0$ for degree reasons. As before we pass to the disk bundle to compute

$$\pi_*^S(\mu \wedge \eta_1 \wedge \omega_F^{m-1}) = \pi_*^D(\eta_1 \wedge \omega_F^m) = \eta_1 \wedge \pi_*^D(\omega_F^m)$$

so for the same reasoning above the restriction map results in a scalar multiple of $\eta_1\theta_{2m-1}$ and therefore the restriction map is an isomorphism.

Finally an element of $H^{2m-1}\mathcal{C}_0(\tilde{M}_2)$ is of the form $\theta(\eta_0 \wedge \omega_F^{m-1} - \omega_X \wedge \eta_0 \wedge \omega_F^{m-2})$ for some $\eta_0 \in H^0(X)$. When integrating over fibers the second term will vanish for degree reasons and the first term computes to a multiple of η_0 analogously to the previous case and again we have an isomorphism. $\square$

Theorem 4.7. *The cone cohomology of M and $\tilde{M}$ are related by the formulas:*

$$\begin{aligned}
\dim H^2\mathcal{C}_0(\tilde{M}) &= \dim H^2\mathcal{C}_0(M) + 1 - k_1 \\
\dim H^3\mathcal{C}_0(\tilde{M}) &= \dim H^3\mathcal{C}_0(M) + 2g - k_1 - k_2 \\
\dim H^4\mathcal{C}_0(\tilde{M}) &= \dim H^4\mathcal{C}_0(M) + 1 - k_2 \\
\dim H^{2m-1}\mathcal{C}_0(\tilde{M}) &= \dim H^{2m-1}\mathcal{C}_0(M) + k_2' \\
\dim H^{2m}\mathcal{C}_0(\tilde{M}) &= \dim H^{2m}\mathcal{C}_0(M) - 1 + k_1' + k_2' \\
\dim H^{2m+1}\mathcal{C}_0(\tilde{M}) &= \dim H^{2m+1}\mathcal{C}_0(M) + 1 - 2g + k_1' \\
\dim H^k\mathcal{C}_0(\tilde{M}) &= \dim H^k\mathcal{C}_0(M) \qquad \textit{otherwise}
\end{aligned}$$

where k_1, k_2, k_2', k_1' are the rank of the restriction maps from $H^*\mathcal{C}_0(M_1) \to H^*\mathcal{C}_0(M_1 \cap M_2)$ in degrees 2, 3, $2m-1$, and $2m$ respectively.

Proof. The primary tool for these computations is the Mayer-Vietoris sequence for both M and $\tilde{M}$. From the two long exact sequences combined with the Five Lemma we immediately get that if $r : H^j\mathcal{C}_0(\tilde{M}_2) \to H^j\mathcal{C}_0(M_2)$ is an isomorphism and the same holds for degree $j-1$ then $\dim H^j\mathcal{C}_0(\tilde{M}) = \dim H^j\mathcal{C}_0(M)$. This holds for $j = 0, 1, 5, \ldots, 2m-2, 2m+3$.

We will handle the remaining indices one at a time:

$j = 2$: Since both $H^1\mathcal{C}_0(\tilde{M}_2)$ and $H^1\mathcal{C}_0(M_2)$ surject onto $H^1\mathcal{C}_0(M_1 \cap M_2)$ the connecting homomorphism is trivial in both cases. Then $H^2\mathcal{C}_0(M)$ and $H^2\mathcal{C}_0(\tilde{M})$ are isomorphic to the kernel of $H^2\mathcal{C}_0(M_1) \oplus H^2(M_2)$ and $H^2\mathcal{C}_0(M_1) \oplus H^2\mathcal{C}_0(\tilde{M}_2)$ respectively into $H^2\mathcal{C}_0(M_1 \cap M_2)$.

Then

$$\dim H^2\mathcal{C}_0(M) = \dim H^2\mathcal{C}_0(M_1)$$
$$\dim H^2\mathcal{C}_0(\tilde{M}) = 1 + \dim\ker r : H^2\mathcal{C}_0(M_1) \to H^2\mathcal{C}_0(M_1 \cap M_2)$$

so that

$$\dim H^2\mathcal{C}_0(\tilde{M}) = \dim H^2\mathcal{C}_0(M) + 1 - k_1$$

where $k_1 = \mathrm{rank}(r : H^2\mathcal{C}_0(M_1) \to H^2\mathcal{C}_0(M_1 \cap M_2)$.

$j = 3$: Since $r : H^2\mathcal{C}_0(M_2) \to H^2\mathcal{C}_0(M_1 \cap M_2)$ is surjective, we have $H^3\mathcal{C}_0(M)$ is isomorphic to its image in $H^3\mathcal{C}_0(M_1) \oplus H^3(M_2)$. This is equal to $\ker r : H^3\mathcal{C}_0(M_1) \oplus H^3\mathcal{C}_0(M_2) \to H^3\mathcal{C}_0(M_1 \cap M_2)$. As $H^3\mathcal{C}_0(M_2)$ already is surjective, we have $\dim H^3\mathcal{C}_0(M) = \dim H^3\mathcal{C}_0(M_1)$.

In the blow-up case we get

$$H^2\mathcal{C}_0(M_1) \oplus H^2\mathcal{C}_0(\tilde{M}_2) \to H^2\mathcal{C}_0(M_1 \cap M_2) \to H^3\mathcal{C}_0(\tilde{M}) \to H^3\mathcal{C}_0(M_1) \to H^3\mathcal{C}_0(M_1 \cap M_2)$$

Then

$$\dim H^3\mathcal{C}_0(\tilde{M}) = \dim\ker r : H^3\mathcal{C}_0(M_1) \to H^3\mathcal{C}_0(M_1 \cap M_2) + 2g$$
$$- \mathrm{rank}\, r : H^2\mathcal{C}_0(M_1) \to H^2\mathcal{C}_0(M_1 \cap M_2)$$

and so

$$\dim H^3\mathcal{C}_0(\tilde{M}) = \dim H^3\mathcal{C}_0(M) + 2g - k_1 - k_2$$

where $k_2 = \mathrm{rank}\, r : H^3\mathcal{C}_0(M_1) \to H^3\mathcal{C}_0(M_1 \cap M_2)$.

$j = 4$: Since $H^3\mathcal{C}_0(M_2) \to H^3\mathcal{C}_0(M_1 \cap M_2)$ is surjective and $H^4\mathcal{C}_0(M_1 \cap M_2) = H^4\mathcal{C}_0(M_2) = 0$

then $\dim H^4\mathcal{C}_0(M) = \dim H^4\mathcal{C}_0(M_1)$.

In the blow-up case we have

$$H^3\mathcal{C}_0(M_1) \to H^3\mathcal{C}_0(M_1 \cap M_2) \to H^4\mathcal{C}_0(\tilde{M}) \to H^4\mathcal{C}_0(M_1) \to 0$$

Then

$$\dim H^4\mathcal{C}_0(\tilde{M}) = \dim H^4\mathcal{C}_0(M_1) + 1 - \operatorname{rank} r : H^3\mathcal{C}_0(M_1) \to H^3\mathcal{C}_0(M_1 \cap M_2)$$

so

$$\dim H^4\mathcal{C}_0(\tilde{M}) = \dim H^4\mathcal{C}_0(M) + 1 - k_2$$

$j = 2m-1$: Since $H^{2m-2}\mathcal{C}_0(M_1 \cap M_2) = 0$ we have $H^{2m-1}\mathcal{C}_0(M)$ is isomorphic to its image in $H^{2m-1}\mathcal{C}_0(M_1)$. Therefore $\dim H^{2m-1}\mathcal{C}_0(M) = \dim\ker r : H^{2m-1}\mathcal{C}_0(M_1) \to H^{2m-1}\mathcal{C}_0(M_1 \cap M_2)$. For $\tilde{M}$ the same logic applies but we have to account for the fact that $H^{2m-1}\mathcal{C}_0(\tilde{M}_2)$ exists and the restriction to $M_1 \cap M_2$ is an isomorphism. So $\dim H^{2m-1}\mathcal{C}_0(\tilde{M}) = \dim\ker r : H^{2m-1}\mathcal{C}_0(M_1) \to H^{2m-1}\mathcal{C}_0(M_1 \cap M_2) + \dim\operatorname{im} r : H^{2m-1}\mathcal{C}_0(M_1) \to H^{2m-1}\mathcal{C}_0(M_1 \cap M_2)$. This second term is k_2', thus $\dim H^{2m-1}\mathcal{C}_0(\tilde{M}) = \dim H^{2m-1}\mathcal{C}_0(M) + k_2'$.

$j = 2m$: We have

$$H^{2m-1}\mathcal{C}_0(M_1) \oplus H^{2m-1}\mathcal{C}_0(M_2) \to H^{2m-1}\mathcal{C}_0(M_1 \cap M_2) \to H^{2m}\mathcal{C}_0(M) \to$$

$$\to H^{2m}\mathcal{C}_0(M_1) \oplus H^{2m}\mathcal{C}_0(M_2) \to H^{2m}\mathcal{C}_0(M_1 \cap M_2)$$

and similar with $\tilde{M}_2$ and $\tilde{M}$. Then

$$\dim H^{2m}\mathcal{C}_0(M) = \dim\ker H^{2m}\mathcal{C}_0(M_1) \oplus H^{2m}\mathcal{C}_0(M_2) \to H^{2m}\mathcal{C}_0(M_1 \cap M_2)$$
$$+ \dim\operatorname{coker} H^{2m-1}\mathcal{C}_0(M_1) \oplus H^{2m-1}\mathcal{C}_0(M_2) \to H^{2m-1}\mathcal{C}_0(M_1 \cap M_2)$$

and again a corresponding equation for the blow-up. Since the restriction maps from M_2 into $M_1 \cap M_2$ is zero in degrees $2m-1, 2m$, we can neglect the M_2 parts of that equation. However the restriction from $\tilde{M}_2$ to $M_1 \cap M_2$ is an isomorphism in these degrees so $\dim H^{2m}\mathcal{C}_0(\tilde{M}) = \dim\ker r : H^{2m}\mathcal{C}_0(M_1) \oplus H^{2m}\mathcal{C}_0(\tilde{M}_2) \to H^{2m}\mathcal{C}_0(M_1 \cap M_2) + \dim\operatorname{coker} H^{2m-1}\mathcal{C}_0(M_1) \oplus H^{2m-1}\mathcal{C}_0(\tilde{M}_2) \to H^{2m-1}\mathcal{C}_0(M_1 \cap M_2)$. Therefore

$$\dim H^{2m}\mathcal{C}_0(\tilde{M}) = \dim H^{2m}\mathcal{C}_0(M) - 1 + k_1' + k_2'$$

$j = 2m+1$: The same reasoning applies as in the $j = 2m$ case but with degree $2m$ and $2m+1$ instead, and we get

$$\dim H^{2m+1}\mathcal{C}_0(\tilde{M}) = \dim H^{2m+1}\mathcal{C}_0(M) + 1 - 2g + k_1'$$

$j = 2m+2$: Since $H^{2m+2}\mathcal{C}_0(M_2) = H^{2m+2}\mathcal{C}_0(\tilde{M}_2) = 0$ the image of the restriction map acting on $H^{2m+2}\mathcal{C}_0(M)$ and $H^{2m+2}\mathcal{C}_0(\tilde{M})$ are the same so any difference in dimension comes from the image of the connecting homomorphism. But then $H^{k-1}\mathcal{C}_0(M_2)$ and $H^{k-1}\mathcal{C}_0(\tilde{M}_2)$ both restrict trivially so the connecting homomorphisms are the same as well. Hence $\dim H^{2m+2}\mathcal{C}_0(\tilde{M}) = \dim H^{2m+2}\mathcal{C}_0(M)$. □

We can also prove the same result by using the fact that

$$H^j\mathcal{C}_0(M) \cong \ker \omega\wedge : H^j(M) \to H^{j+2}(M) \oplus \operatorname{coker} \omega\wedge : H^{j-1}(M) \to H^{j+1}(M)$$

and then computing how the dimensions of the kernel and cokernel of the $\omega\wedge$ map change between M and $\tilde{M}$ will also provide the result.

Alternate Proof. From the two Mayer-Vietoris sequences combined with the Five Lemma we immediately get that if $r : H^j\mathcal{C}_0(\tilde{M}_2) \to H^j\mathcal{C}_0(M_2)$ is an isomorphism and the same holds for degree $j-1$ then $\dim H^j\mathcal{C}_0(\tilde{M}) = \dim H^j\mathcal{C}_0(M)$. This holds for $j = 0, 1, 5, \ldots, 2m-2, 2m+3$.

Now from the Mayer-Vietoris sequence we immediately get that

$$\begin{aligned}\dim H^j\mathcal{C}_0(M) = \dim \operatorname{coker} r &: H^{j-1}\mathcal{C}_0(M_1) \oplus H^{j-1}\mathcal{C}_0(M_2) \to H^{j-1}\mathcal{C}_0(M_1 \cap M_2) \\ &+ \dim \ker r : H^j\mathcal{C}_0(M_1) \oplus H^j\mathcal{C}_0(M_2) \to H^j\mathcal{C}_0(M_1 \cap M_2)\end{aligned}$$

and the same for $\tilde{M}$ and $\tilde{M}_2$. Therefore we can compute the change in dimension by finding the change in dimension of the above kernel and cokernel terms.

In the following N will represent M_2 or $\tilde{M}_2$. The restriction maps from $H^*\mathcal{C}_0(N) \to H^*\mathcal{C}_0(M_1 \cap M_2)$ are either the zero map or an isomorphism. Then the restriction maps on the direct product have images

- Zero Map: $\operatorname{im} r : H^j\mathcal{C}_0(M_1) \to H^j\mathcal{C}_0(M_1 \cap M_2)$
- Isomorphism: $H^j\mathcal{C}_0(M_1 \cap M_2)$

which implies that the cokernel dimensions are

- Zero Map: $\dim H^j\mathcal{C}_0(M_1 \cap M_2) - \operatorname{rank} r : H^j\mathcal{C}_0(M_1) \to H^j\mathcal{C}_0(M_1 \cap M_2)$
- Isomorphism: 0

and the kernels have dimensions

- Zero Map: $\dim \ker r : H^j\mathcal{C}_0(M_1) \to H^j\mathcal{C}_0(M_1 \cap M_2) + \dim H^j\mathcal{C}_0(N)$
- Isomorphism: $\dim \ker r : H^j\mathcal{C}_0(M_1) \to H^j\mathcal{C}_0(M_1 \cap M_2) + \operatorname{rank} r : H^j\mathcal{C}_0(M_1) \to H^j\mathcal{C}_0(M_1 \cap M_2)$

Note that in the kernel case the $\dim \ker r : H^j\mathcal{C}_0(M_1) \to H^j\mathcal{C}_0(M_1 \cap M_2)$ term shows in both cases and doesn't depend on N so it contributes no change in dimension. Therefore the change in dimensions are:

j	Kernel Term (degree j)	Cokernel Term (degree $j-1$)	Total
2	$1-k_1$	0	$1-k_1$
3	$-k_2$	$2g-k_1$	$2g-k_1-k_2$
4	0	$1-k_2$	$1-k_2$
$2m-1$	k_2'	0	k_2'
$2m$	k_1'	$k_2'-1$	$-1+k_1'+k_2'$
$2m+1$	1	$k_1'-2g$	$1-2g+k_1'$
$2m+2$	0	0	0

Table 4.2: Dimension of kernel and cokernel terms in the primitive $m > 2$ case.

which completes the proof. □

We now discuss the $m = 2$ case where X is embedded in a 6-dimensional manifold. The difference here is that the cone cohomology $H^*\mathcal{C}_0(M_1 \cap M_2)$ changes slightly due to having two contributions in degree 3:

Lemma 4.8. *For $m = 2$, $H^*\mathcal{C}_0(M_1 \cap M_2) \cong H^*\mathcal{C}_0(X) \otimes \mathbb{R}\langle 1, \psi_3 \rangle$.*

Remark 4.9. *The change here is that $\dim H^3\mathcal{C}_0(M_1 \cap M_2) = 2$ since $2m-1 = 3$ so there are two contributions to this index.*

Theorem 4.10. *Suppose $m = 2$. The cone cohomology of M and $\tilde{M}$ are related by the formulas:*

$$\dim H^2\mathcal{C}_0(\tilde{M}) = \dim H^2\mathcal{C}_0(M) + 1 - k_1$$
$$\dim H^3\mathcal{C}_0(\tilde{M}) = \dim H^3\mathcal{C}_0(M) + 2g - k_1 + k_2 - k_2'$$
$$\dim H^4\mathcal{C}_0(\tilde{M}) = \dim H^4\mathcal{C}_0(M) + k_1' + k_2 - k_2'$$
$$\dim H^5\mathcal{C}_0(\tilde{M}) = \dim H^5\mathcal{C}_0(M) + 1 - 2g + k_1'$$
$$\dim H^k\mathcal{C}_0(\tilde{M}) = \dim H^k\mathcal{C}_0(M) \qquad \textit{otherwise}$$

where k_1 and k_1' are the rank of the restriction maps from $H^*\mathcal{C}_0(M_1) \to H^*\mathcal{C}_0(M_1 \cap M_2)$ in degrees 2 and 4 respectively and k_2 and k_2' are the rank of the restriction map from $H^3\mathcal{C}_0(M_1)$ to $H^3\mathcal{C}_0(M_1 \cap M_2)$ in the $H^0(X)\theta_3$ and $H^2(X)\theta$ components respectively.

Proof. The only change from the above is in degrees 3 and 4, the rest stay the same. This is due to the fact that $H^3\mathcal{C}_0(M_1 \cap M_2)$ is now two-dimensional, which also adds the additional complication that the image of $H^3\mathcal{C}_0(M_1)$ may be split between the $H^0(X)\theta_3$ and $H^2(X)\theta$ components of $H^3\mathcal{C}_0(M_1 \cap M_2)$. We will use a slightly different method of calculation for this case, where we know that we can write both $\dim H^*\mathcal{C}_0(M)$ and $\dim H^*\mathcal{C}_0(\tilde{M})$ in terms of the kernel and cokernel of the restriction maps in the proper degrees. Then we need only compute how the dimension of the kernel and cokernel change when moving from M to $\tilde{M}$ and sum to get the total difference in dimension. We list the change in dimension in the following table.

Since $H^3(M_2)$ and $H^3(\tilde{M}_2)$ restrict isomorphically onto $H^2(X)\theta$ and $H^0(X)\theta_3$ respectively. This gives that

j	Kernel Term (degree j)	Cokernel Term (degree $j-1$)	Total
3	$k_2 - k_2'$	$2g - k_1$	$2g - k_1 + k_2 - k_2'$
4	k_1'	$k_2 - k_2'$	$k_1' + k_2 - k_2'$

Table 4.3: Dimension of kernel and cokernel terms in the primitive $m = 2$ case.

which completes the proof. □

Finally we compute the In this case the 2-dimensional manifold X is embedded into the 4-dimensional manifold M.

Theorem 4.11. *When $m = 1$ the cone cohomology of M and $\tilde{M}$ are isomorphic.*

Proof. When $m = 1$, $H^*(\tilde{M}_2) \cong H^*(X) \cong H^*(M_2)$. Then the cone cohomologies will be the same and so $H^*\mathcal{C}_0(M)$ and $H^*\mathcal{C}_0(\tilde{M})$ satisfy the same long exact sequence and therefore are isomorphic. □

4.2 Higher Filtration

In this section we assume that $p > 0$.

Lemma 4.12. *If $p > 0$ then $H^*\mathcal{C}_p(X) = H^*(X) \otimes \mathbb{R}\langle 1, \theta_X \rangle$.*

Proof. For degree reasons all Lefschetz maps are zero, so the resolution sequence splits into

$$0 \to H^k(X) \to H^k\mathcal{C}_p(X) \to H^{k-2p-1}(X) \to 0$$

Since $2p + 1 \geq 3$ only one of the de Rham cohomology terms is non-zero at a time. □

Lemma 4.13. *In this lemma θ will refer to $\theta_{\tilde{M}_2}$. If $m > p+1$ then*

$$H^k\mathcal{C}_p(\tilde{M}_2) = \begin{cases} H^k(\tilde{M}_2) & 0 \le k \le 2p+1 \\ H^2(X)\omega_F^p & k = 2p+2 \\ 0 & 2p+2 < k < 2m-1 \\ \theta \ker L^{p+1}(H^{2m-2p-2}(\tilde{M}_2)) & k = 2m-1 \\ \theta H^{k-2p-1}(\tilde{M}_2) & 2m \le k \le 2m+2p+1 \end{cases}$$

If $m = p+1$ then

$$H^k\mathcal{C}_p(\tilde{M}_2) = \begin{cases} H^k(\tilde{M}_2) & 0 \le k \le 2m-1 \\ \theta H^{k-2p-1}(\tilde{M}_2) & 2m \le k \le 2m+2p+1 \end{cases}$$

If $m < p+1$ then $H^*\mathcal{C}_p(\tilde{M}_2) = H^*(\tilde{M}_2) \otimes \mathbb{R}\langle 1, \theta\rangle$.

Proof. From above we know that L is an isomorphism for indices $1 \le j \le 2m-3$, injective in index 0, and surjective in index $2m-2$. L is the zero map for higher indices. This generalizes to L^{p+1} being injective in degree 0, isomorphism from degree 1 to degree $2m-2p-3$, surjective in degree $2m-2p-2$, and zero map in higher degrees.

First we consider the case when $m < p+1$, so all Lefschetz maps are zero which gives $H^*\mathcal{C}_p(\tilde{M}_2) = H^*(\tilde{M}_2) \otimes \mathbb{R}\langle 1, \theta\rangle$. When $m = p+1$ all Lefschetz maps are zero except $L^{p+1} : H^0(\tilde{M}_2) \to H^{2m}(\tilde{M}_2)$ which is an isomorphism. In degrees $k \ne 2m-1, 2m$ the resolution sequence becomes

$$0 \to H^k(\tilde{M}_2) \to H^k\mathcal{C}_p(\tilde{M}_2) \to H^{k-2p-1}(\tilde{M}_2) \to 0$$

Only one of the de Rham cohomology terms can be zero for degree reasons so $H^k\mathcal{C}_p(\tilde{M}_2) = H^k(\tilde{M}_2)$ for $0 \le k < 2m-1$ and $H^k\mathcal{C}_p(\tilde{M}_2) = \theta H^{k-2p-1}\mathcal{C}_p(\tilde{M}_2)$ for $2m < k \le 2m+2p+1$. For the last two degrees we have

$$H^0(\tilde{M}_2) \xrightarrow{\sim} H^{2m}(\tilde{M}_2) \to H^{2m}\mathcal{C}_p(\tilde{M}_2) \to H^1(\tilde{M}_2) \to 0$$

and

$$0 \to H^{2m-1}(\tilde{M}_2) \to H^{2m-1}\mathcal{C}_p(\tilde{M}_2) \to H^0(\tilde{M}_2) \xrightarrow{\sim} H^{2m}(\tilde{M}_2)$$

which imply $H^{2m-1}\mathcal{C}_p(\tilde{M}_2) = H^{2m-1}(\tilde{M}_2)$ and $H^{2m}\mathcal{C}_p(\tilde{M}_2) = \theta H^1(\tilde{M}_2)$.

Now suppose $m > p+1$. We compute $H^k\mathcal{C}_p(\tilde{M}_2)$ by

$$H^k\mathcal{C}_p(\tilde{M}_2) \cong \ker L^{p+1} : H^{k-2p-1}(\tilde{M}_2) \to H^{k+1}(\tilde{M}_2) \oplus \operatorname{coker} L^{p+1} : H^{k-2p-2}(\tilde{M}_2) \to H^k(\tilde{M}_2)$$

Then L^{p+1} acts as follows in these degrees:

degree k	kernel	cokernel
0	0	$H^2(X)\omega_F^p$
$1 \le k \le 2m-2p-3$	0	0
$2m-2p-2$	$\ker L^{p+1}(H^{2m-2p-2}(\tilde{M}_2))$	0
$k > 2m-2p-2$	$H^k(\tilde{M}_2)$	0

Table 4.4: Action of Lefschetz map on a $\mathbb{CP}^{m-1}$ bundle.

which gives the desired results. □

Lemma 4.14. $H^k\mathcal{C}_p(M_1 \cap M_2) = H^k(M_1 \cap M_2) \oplus \theta_{M_1 \cap M_2} H^{k-2p-1}(M_1 \cap M_2)$.

Proof. All Lefschetz maps are zero so the resolution sequence splits as

$$0 \to H^k(M_1 \cap M_2) \to H^k\mathcal{C}_p(M_1 \cap M_2) \to H^{k-2p-1}(M_1 \cap M_2) \to 0$$

from which the direct sum formula follows. Note that overlap only happens when $m - 2 \leq p \leq m$. □

We now split into cases based on the filtration.

4.2.1 Case 1: $p < m - 2$

Lemma 4.15. *The restriction map $r : H^*\mathcal{C}_p(M_2) \to H^*\mathcal{C}_p(M_1 \cap M_2)$ is an isomorphism in degrees 0, 1, 2, $2p+1$, $2p+2$, and $2p+3$ and the zero map otherwise.*

The restriction map $r : H^*\mathcal{C}_p(\tilde{M}_2) \to H^*\mathcal{C}_p(M_1 \cap M_2)$ is an isomorphism in degrees 0, 1, $2m-1$, and $2m$ and surjective in degrees 2 and $2m+1$. It is the zero map in all other degrees.

Proof. The restriction map on M_2 follows immediately from the definition.

In degrees 0 through $2p + 2$ the restriction map on $\tilde{M}_2$ is the inclusion map which gives the desired results. In degrees $2m - 1$ through $2m + 2p + 1$ there is the added complication of restricting to the sphere bundle and then integrating over fibers. Recall that this occurs because ω_F is exact on $M_1 \cap M_2$ so the symplectic form becomes $\omega = \omega_X + d\mu$, which induces a form μ_{2p+1} such that $d\mu_{2p+1} = \omega^{p+1} - \omega_X^{p+1}$. Then integration over fibers is defined as

$$\tilde{\pi}_*^S(\theta B_{j_p}) = \pi_*^S(\mu_{2p+1} \wedge B_{j_p}) + \theta\pi_*^S(B_{j_p})$$

In degree $2m-1$, $H^{2m-1}\mathcal{C}_p(\tilde{M}_2) = \theta \ker L^{p+1}(H^{2m-2p-2}(\tilde{M}_2))$, which has elements of the form $\theta(\eta_0 \wedge \omega_F^{m-p-1} - \omega_X \wedge \eta_0 \wedge \omega_F^{m-p-2})$ for $\eta_0 \in H^0(X)$. Then because integrating over the fiber decreases the degree by $2m-1$ we have

$$
\begin{aligned}
\tilde{\pi}_*^S(\theta(\eta_0 \wedge \omega_F^{m-p-1} - \omega_X \wedge \eta_0 \wedge \omega_F^{m-p-2})) &= \pi_*^S(\mu_{2p+1} \wedge (\eta_0 \wedge \omega_F^{m-p-1} \\
&\quad - \omega_X \wedge \eta_0 \wedge \omega_F^{m-p-2})) \\
&= \pi_*^S(\mu_{2p+1} \wedge \eta_0 \wedge \omega_F^{m-p-1}) \\
&= \pi_*^D(d\mu_{2p+1} \wedge \eta_0 \wedge \omega_F^{m-p-1}) \\
&= \pi_*^D(\eta_0 \wedge \omega_F^m) \\
&= \eta_0 \pi_*^D(\omega_F^m)
\end{aligned}
$$

which is a scalar multiple of η_0 since ω_F^m is the volume form on the fibers. Then the restriction map is an isomorphism.

In degree $2m$ we have $H^{2m}\mathcal{C}_p(\tilde{M}_2)$ has elements of the form $\theta\eta_1\omega_F^{m-p-1}$ so

$$
\begin{aligned}
\tilde{\pi}_*^S(\theta\eta_1 \wedge \omega_F^{m-p-1}) &= \pi_*^S(\mu_{2p+1} \wedge \eta_1 \wedge \omega_F^{m-p-1}) \\
&= \pi_*^D(d\mu_{2p+1} \wedge \eta_1 \wedge \omega_F^{m-p-1}) \\
&= \pi_*^D(\eta_1 \wedge \omega_F^m) \\
&= \eta_1 \pi_*^D(\omega_F^m)
\end{aligned}
$$

which is a scalar multiple of η_1 therefore the restriction is an isomorphism.

$H^{2m+1}\mathcal{C}_p(\tilde{M}_2)$ consists of elements of the form $\eta_0 \wedge \omega_F^{m-p} + \eta_2 \wedge \omega_F^{m-p-1}$ where $\eta_0 \in H^0(X)$

and $\eta_2 \in H^2(X)$.

$$
\begin{aligned}
\tilde{\pi}_*^S(\theta(\eta_0 \wedge \omega_F^{m-p} + \eta_2 \wedge \omega_F^{m-p-1})) &= \pi_*^S(\mu_{2p+1} \wedge (\eta_0 \wedge \omega_F^{m-p} + \eta_2 \wedge \omega_F^{m-p-1})) \\
&= \pi_*^D(d\mu_{2p+1} \wedge (\eta_0 \wedge \omega_F^{m-p} + \eta_2 \wedge \omega_F^{m-p-1})) \\
&= \pi_*^D((\omega_X \wedge \eta_0 + \eta_2) \wedge \omega_F^m + \eta_0 \wedge \omega_F^{m+1}) \\
&= (\omega_X \wedge \eta_0 + \eta_2)\pi_*^D(\omega_F^m) + \eta_0 \wedge \pi_*^D(\omega_F^{m+1})
\end{aligned}
$$

Since $H^{2m+1}\mathcal{C}_p(M_1 \cap M_2) = H^2(X)\theta_{2m-1}$ by setting $\eta_0 = 0$ we see that the restriction map is surjective by choosing the proper η_2. □

Theorem 4.16. *When $p < m-2$ the cone cohomology of M and $\tilde{M}$ are related by:*

$$\begin{aligned}
\dim H^2\mathcal{C}_p(\tilde{M}) &= \dim H^2\mathcal{C}_p(M) + 1\\
\dim H^3\mathcal{C}_p(\tilde{M}) &= \dim H^3\mathcal{C}_p(M) + 2g\\
\dim H^4\mathcal{C}_p(\tilde{M}) &= \dim H^4\mathcal{C}_p(M) + 2\\
\dim H^5\mathcal{C}_p(\tilde{M}) &= \dim H^5\mathcal{C}_p(M) + 2g\\
&\vdots\\
\dim H^{2p+1}\mathcal{C}_p(\tilde{M}) &= \dim H^{2p+1}\mathcal{C}_p(M) + 2g - k_1\\
\dim H^{2p+2}\mathcal{C}_p(\tilde{M}) &= \dim H^{2p+2}\mathcal{C}_p(M) + 2 - k_1 - k_2\\
\dim H^{2p+3}\mathcal{C}_p(\tilde{M}) &= \dim H^{2p+3}\mathcal{C}_p(M) + 2g - k_2 - k_3\\
\dim H^{2p+4}\mathcal{C}_p(\tilde{M}) &= \dim H^{2p+4}\mathcal{C}_p(M) + 1 - k_3\\
\dim H^{2m-1}\mathcal{C}_p(\tilde{M}) &= \dim H^{2m-1}\mathcal{C}_p(M) + k_3'\\
\dim H^{2m}\mathcal{C}_p(\tilde{M}) &= \dim H^{2m}\mathcal{C}_p(M) - 1 + k_2' + k_3'\\
\dim H^{2m+1}\mathcal{C}_p(\tilde{M}) &= \dim H^{2m+1}\mathcal{C}_p(M) + 1 - 2g + k_1' + k_2'\\
\dim H^{2m+2}\mathcal{C}_p(\tilde{M}) &= \dim H^{2m+2}\mathcal{C}_p(M) - 1 + 2g + k_1'\\
&\vdots\\
\dim H^{2m+2p-2}\mathcal{C}_p(\tilde{M}) &= \dim H^{2m+2p-2}\mathcal{C}_p(M) + 2g\\
\dim H^{2m+2p-1}\mathcal{C}_p(\tilde{M}) &= \dim H^{2m+2p-1}\mathcal{C}_p(M) + 2\\
\dim H^{2m+2p}\mathcal{C}_p(\tilde{M}) &= \dim H^{2m+2p}\mathcal{C}_p(M) + 2g\\
\dim H^{2m+2p+1}\mathcal{C}_p(\tilde{M}) &= \dim H^{2m+2p+1}\mathcal{C}_p(M) + 1\\
\dim H^{k}\mathcal{C}_p(\tilde{M}) &= H^{k}\mathcal{C}_p(M) \qquad \textit{otherwise}
\end{aligned}$$

where k_1, k_2, k_3 and k_3', k_2', k_1' are the ranks of the restriction maps $H^*\mathcal{C}_p(M_1) \to H^*\mathcal{C}_p(M_1 \cap M_2)$ in degrees $2p+1$, $2p+2$, $2p+3$ and $2m-1$, $2m$, and $2m+1$ respectively.

Proof. From the Mayer-Vietoris sequence we get that the difference between $\dim H^*\mathcal{C}_p(M)$ and $\dim H^*\mathcal{C}_p(\tilde{M})$ can be computed by finding the difference in dimension of the kernel and cokernel of the restriction maps

$$r_k : H^k\mathcal{C}_p(M_1) \oplus H^k\mathcal{C}_p(M_2) \to H^k\mathcal{C}_p(M_1 \cap M_2)$$

and

$$\tilde{r}_k : H^k\mathcal{C}_p(M_1) \oplus H^k\mathcal{C}_p(\tilde{M}_2) \to H^k\mathcal{C}_p(M_1 \cap M_2)$$

This uses information about the restrictions from M_1, M_2, and $\tilde{M}_2$ into $M_1 \cap M_2$. r'_k will be used to denote the restriction map $r'_k : H^k\mathcal{C}_p(M_1) \to H^k\mathcal{C}_p(M_1 \cap M_2)$.

There are two possibilities for $\dim\ker r_k$ and $\dim\operatorname{coker} r_k$. If k=0, 1, 2, $2p+1$, $2p+2$, $2p+3$ then $H^k\mathcal{C}_p(M_2)$ restricts isomorphically onto $H^k\mathcal{C}_p(M_1 \cap M_2)$ in that case $\dim\ker r_k = \dim\ker r'_k + \operatorname{rank} r'_k$ and $\dim\operatorname{coker} r_k = 0$. Otherwise, $H^k\mathcal{C}_p(M_2) = 0$ so $\dim\ker r_k = \dim\ker r'_k$ and $\dim\operatorname{coker} r_k = \dim H^k\mathcal{C}_p(M_1 \cap M_2) - \operatorname{rank} r'_k$. Note that this cokernel term is only non-zero when $H^k\mathcal{C}_p(M_1 \cap M_2)$ is non-zero.

Now we wish to compute $\dim\ker \tilde{r}_k$ and $\dim\operatorname{coker} \tilde{r}_k$. In degrees 0, 1, $2m-1$, and $2m$ $H^k\mathcal{C}_p(\tilde{M}_2)$ restricts isomorphically onto $H^k\mathcal{C}_p(M_1 \cap M_2)$ so again we get that $\dim\ker \tilde{r}_k = \dim\ker r'_k + \operatorname{rank} r'_k$ and $\dim\operatorname{coker} \tilde{r}_k = 0$. Similarly when $H^k\mathcal{C}_p(\tilde{M}_2)$ restricts trivially again we get $\dim\ker \tilde{r}_k = \dim\ker r'_k + \dim H^k\mathcal{C}_p(\tilde{M}_2)$ and $\dim\operatorname{coker} \tilde{r}_k = \dim H^k\mathcal{C}_p(M_1 \cap M_2) - \operatorname{rank} r'_k$. The final case is in degrees 2 and $2m+1$ where $H^k\mathcal{C}_p(\tilde{M}_2)$ is only surjective. As before $\dim\operatorname{coker} \tilde{r}_k = 0$, but $\dim H^k\mathcal{C}_p(M_1 \cap M_2) = 1$ so $\dim\ker \tilde{r}_k = \dim\ker r'_k + 1 + \operatorname{rank} r'_k$.

The change in dimension is then given by the following table, excluding degrees where the change in dimension is zero.

k	Kernel Term (degree k)	Cokernel Term (degree $k-1$)	Total
2	1	0	1
3	$2g$	0	$2g$
4	2	0	2
5	$2g$	0	$2g$
$\vdots$			$\vdots$
$2p+1$	$2g-k_1$	0	$2g-k_1$
$2p+2$	$1-k_2$	$1-k_1$	$2-k_1-k_2$
$2p+3$	$-k_3$	$2g-k_2$	$2g-k_2-k_3$
$2p+4$	0	$1-k_3$	$1-k_3$
$2m-1$	k_3'	0	k_3'
$2m$	k_2'	$-1+k_3'$	$-1+k_2'+k_3'$
$2m+1$	$1+k_1'$	$-2g+k_2'$	$1-2g+k_1'+k_2'$
$2m+2$	$2g$	$-1+k_1'$	$-1+2g+k_1'$
$\vdots$			$\vdots$
$2m+2p-2$	$2g$	0	$2g$
$2m+2p-1$	2	0	2
$2m+2p$	$2g$	0	$2g$
$2m+2p+1$	1	0	1

Table 4.5: Dimension of kernel and cokernel terms in the non-primitive $p < m-2$ case.

$\square$

4.2.2 Case 2: $p = m - 2$

Theorem 4.17. *When $p = m - 2$ the cone cohomology of M and $\tilde{M}$ is the same as in Theorem 4.16 but with the change that*

$$\dim H^{2m-1}\mathcal{C}_p(\tilde{M}) = \dim H^{2m-1}\mathcal{C}_p(M) + 2g - k_2 + k_3 - k_3'$$
$$\dim H^{2m}\mathcal{C}_p(\tilde{M}) = \dim H^{2m}\mathcal{C}_p(M) + k_2' + k_3 - k_3'$$

where now we define k_3 and k_3' to be the rank of the restriction map from $H^{2m-1}\mathcal{C}_p(M_1)$ onto the $\theta_{2m-1}H^0(X)$ and $\theta H^2(X)$ components of $H^{2m-1}\mathcal{C}_p(M_1 \cap M_2)$ respectively. Note also that since $2p + 3 = 2m - 1$ and $2p + 4 = 2m$ there are no longer separate cohomologies in these degrees.

Proof. The change from the previous case is due to the fact that $2p + 3 = 2m - 1$ so $\dim H^{2m-1}\mathcal{C}_p(M_1 \cap M_2) = 2$. This only affects the calculation of degree $2m - 1$ and $2m$. Redoing these cases gives

k	Kernel Term (degree k)	Cokernel Term (degree $k-1$)	Total
$2m-1$	$k_3 - k_3'$	$2g - k_2$	$2g - k_2 + k_3 - k_3'$
$2m$	k_2'	$k_3 - k_3'$	$k_2' + k_3 - k_3'$

Table 4.6: Dimension of kernel and cokernel terms in the non-primitive $p = m - 2$ case.

which gives the desired result. □

4.2.3 Case 3: $p = m - 1$

Theorem 4.18. *When $p = m - 1$ then cone cohomology of M and $\tilde{M}$ are related the same as in Theorem 4.16 in degrees less than $2m - 1$ and degrees above $2m + 2$. In the middle we have*

$$\begin{aligned} H^{2m-1}\mathcal{C}_p(\tilde{M}) &= H^{2m-1}\mathcal{C}_p(M) + 2g - k_1' \\ H^{2m}\mathcal{C}_p(\tilde{M}) &= H^{2m}\mathcal{C}_p(M) + 1 - k_1' + k_2 - k_2' \\ H^{2m+1}\mathcal{C}_p(\tilde{M}) &= H^{2m+1}\mathcal{C}_p(M) + 1 + k_2 - k_2' + k_3 - k_3' \\ H^{2m+2}\mathcal{C}_p(\tilde{M}) &= H^{2m+2}\mathcal{C}_p(M) + 2g - 1 + k_3 - k_3' \end{aligned}$$

where k_1, k_2, k_3 are the rank of the restriction map from $H^*\mathcal{C}_p(M_1)$ to the $\psi_{2m-1}H^*(X)$ components of $H^*\mathcal{C}_p(M_1 \cap M_2)$ in degrees $2m - 1$, $2m$, and $2m + 1$ respectively. Likewise k_1', k_2', and k_3' are the rank onto the $\theta H^*(X)$ components of $H^*\mathcal{C}_p(M_1 \cap M_2)$.

Proof. Much like the $p = m - 2$ there is overlap of the $2m - 1$, $2m$, and $2m + 1$ terms and the $2p + 1$, $2p + 2$, and $2p + 3$ terms. Recomputing these terms we get

k	Kernel Term (degree k)	Cokernel Term (degree $k - 1$)	Total
$2m - 1$	$2g - k_1'$	0	$2g - k_1'$
$2m$	$k_2 - k_2'$	$1 - k_1'$	$1 - k_1' + k_2 - k_2'$
$2m + 1$	$1 + k_3 - k_3'$	$k_2 - k_2'$	$1 + k_2 - k_2' + k_3 - k_3'$
$2m + 2$	$2g$	$-1 + k_3 - k_3'$	$2g - 1 + k_3 - k_3'$

Table 4.7: Dimension of kernel and cokernel terms in the non-primitive $p = m - 1$ case.

□

4.2.4 Case 4: $p = m$

Theorem 4.19. *When $p = m$,* $\dim H^k\mathcal{C}_p(\tilde{M}) =$

$$\begin{cases} \dim H^k\mathcal{C}_p(M) + \dim H^k(\tilde{M}_2) - \dim H^k(M_2) & 0 \leq k \leq 2m+1 \\ \dim H^k\mathcal{C}_p(M) + \dim H^{k-2p-1}(\tilde{M}_2) - \dim H^{k-2p-1}(M_2) & 2m+2 \leq k \leq 2m+2p+3 \end{cases}$$

Proof. The only non-zero Lefschetz map in this case is the one from degree 0 to degree $2m+2$. Then for $k < 2m+1$ we have $H^k\mathcal{C}_p(M) = H^k\mathcal{C}_p(M)$ and for $k > 2m+2$ we have $H^k\mathcal{C}_p(M) = \theta H^{k-2p-1}(M)$. For $k = 2m+1$ we have

$$0 \to H^{2m+1}(M) \to H^{2m+1}\mathcal{C}_p(M) \to H^0(M) \to H^{2m+2}$$

Since the last map is an isomorphism we have $H^{2m+1}\mathcal{C}_p(M) = H^{2m+1}(M)$. For $k = 2m+2$ we have

$$H^0(M) \to H^{2m+2}(M) \to H^{2m+2}\mathcal{C}_p(M) \to H^1(M) \to 0$$

Since the first map is an isomorphism we get $H^{2m+1}\mathcal{C}_p(M) = \theta H^1(M)$. The same calculations hold for $\tilde{M}$ so the difference in dimension is due to the difference in de Rham cohomology. □

4.2.5 Case 5: $p > m$

Theorem 4.20. *When $p > m$,* $\dim H^k\mathcal{C}_p(\tilde{M}) =$

$$\begin{cases} \dim H^k\mathcal{C}_p(M) + \dim H^k(\tilde{M}_2) - \dim H^k(M_2) & 0 \leq k \leq 2m+2 \\ \dim H^k\mathcal{C}_p(M) + \dim H^{k-2p-1}(\tilde{M}_2) - \dim H^{k-2p-1}(M_2) & 2p+1 \leq k \leq 2m+2p+3 \end{cases}$$

for all indices k.

Proof. For degree reasons all Lefschetz maps are zero in M and $\tilde{M}$ so

$$H^*\mathcal{C}_p(\tilde{M}) = H^*(\tilde{M}) \otimes \mathbb{R}\langle 1, \theta \rangle$$
$$H^*\mathcal{C}_p(M) = H^*(M) \otimes \mathbb{R}\langle 1, \theta \rangle$$

Hence the change in dimension of the cone cohomology is purely due to the change in dimension of the de Rham cohomology. □

Chapter 5

Cone Cohomology on Non-Symplectic Manifolds

Recall that the definition of the filtered cohomology required having a symplectic form so we could take advantage of the Lefschetz decomposition to build a cohomology that depends on the symplectic structure. However, the cone representation only requires that $d\theta$ is a closed form. Note that $d\theta = 0$ is a valid choice but the result always has dimension twice that of the de Rham cohomology of the same degree. The result is that we could consider the cone cohomology over any manifold with respect to any closed form. For symplectic structure it only made sense to consider the cone over the symplectic form, or a power of the symplectic form, otherwise we wouldn't be studying the symplectic structure. However, with a general manifold there is no such distinct form. So rather than studying a single form, instead we will study how the cone cohomology changes as we vary over an entire cohomology space.

5.1 Motivating Example

Nilmanifolds are a useful class of manifolds for this work because their forms have a relatively simple structure. As an example let's use the 4-dimensional nilmanifold M with signature $(0, 0, 0, 12)$. This means we have a basis of one forms e_1, e_2, e_3, e_4 satisfying the relations

$$de_1 = 0, \quad de_2 = 0, \quad de_3 = 0, \quad de_4 = e_1 \wedge e2$$

Let's see how the cone cohomology changes as we vary over $H^1(M)$ which is generated by e_1, e_2, and e_3. The cone cohomology with respect to a 1-form will be non-zero in degrees 0 through 4. The degree 0 space will be 1-dimensional and generated by 1 as with de Rham cohomology. Additionally, the cone dimensions will be symmetric so it suffices to compute what happens in degree 1 and 2.

We want to compute the cone cohomology with respect to a general closed 1-form of the form $\psi = a_1e_1 + a_2e_2 + a_3e_3$, we denote this $H^1\mathcal{C}(\psi, M)$ or simply $H^1\mathcal{C}(\psi)$. $\psi \neq 0$ implies one of a_1, a_2, and a_3 are non-zero. Now $\mathcal{C}^1(\psi) = \Omega^1(M) + \theta\Omega^1(M)$ and θ is considered to be a zero form since it has degree one less than ψ. Now

$$H^1\mathcal{C}(\psi) \cong \ker \psi\wedge : H^1(M) \to H^2(M) \oplus \operatorname{coker} \psi\wedge : H^0(M) \to H^1(M)$$

The second term is always the same, the image of $\psi\wedge : H^0(M) \to H^1(M)$ will always be the subspace generated by ψ, so the cokernel has dimension 2 regardless of the choice of coefficients. But something does change in the kernel term. Regardless of ψ, the kernel will always be at least 1-dimensional since $\psi^2 = 0$. There is another way for a one-form η to be in the kernel of $\psi\wedge$ and that is when $\psi \wedge \eta = e_1 \wedge e_2$ and becomes exact. This cannot happen when a_3 is non-zero but will occur otherwise. Therefore the cone dimension jumps at $a_3 = 0$. To summarize,

$$\dim H^1\mathcal{C}(a_1e_1 + a_2e_2) = 4$$
$$\dim H^1\mathcal{C}(a_1e_1 + a_2e_2 + a_3e_3) = 3 \quad (\text{for } a_3 \neq 0)$$

There are two pieces of information that can be derived from this, both how much the dimension jumps and the location of the subspace where the dimension jumps.

$H^2\mathcal{C}(\psi)$ is very similar in nature. Again,

$$H^2\mathcal{C}(\psi) \cong \ker \psi\wedge : H^2(M) \to H^3(M) \oplus \operatorname{coker} \psi\wedge : H^1(M) \to H^2(M)$$

If we uses the bases

$$\{e_1, e_2, e_3\} \in H^1(M)$$
$$\{e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4\} \in H^2(M)$$
$$\{e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\} \in H^3(M)$$

Then the matrix of $\psi\wedge : H^2(M) \to H^3(M)$ is given by

$$\begin{bmatrix} 0 & -a_2 & 0 & a_1 \\ 0 & -a_3 & 0 & 0 \\ 0 & 0 & 0 & -a_3 \end{bmatrix}$$

and the matrix of $\psi\wedge : H^1(M) \to H^2(M)$ is

$$\begin{bmatrix} -a_3 & 0 & a_1 \\ 0 & 0 & 0 \\ 0 & -a_3 & a_2 \\ 0 & 0 & 0 \end{bmatrix}$$

The result is that $\dim \ker \psi\wedge : H^2(M) \to H^3(M)$ is 2 if $a_3 \neq 0$ and 3 if $a_3 = 0$ (remember that at least one coefficient must be non-zero), and $\dim \operatorname{coker} \psi\wedge : H^1(M) \to H^2(M)$ is 2 if $a_3 \neq 0$ and 3 if $a_3 = 0$. Then

$$\dim H^2\mathcal{C}(a_1e_1 + a_2e_2) = 6$$
$$\dim H^2\mathcal{C}(a_1e_1 + a_2e_2 + a_3e_3) = 4 \quad (\text{for } a_3 \neq 0)$$

Again the jump happens when $a_3 = 0$ because without e_3 we can utilize $e_1 \wedge e_2$ being exact.

5.2 Properties of the Cone Cohomology

One immediate observation we can make is that as we restrict to smaller subsets the cone dimension is non-decreasing. This happens because when the cone cohomology changes dimension it is because the kernel of the $\psi\wedge$ map is changing. Then when we drop to a smaller subset anything that was in the kernel remains in the kernel, but we may have also gained new elements of the kernel. Therefore we hit a limit with the trivial element.

Lemma 5.1. *Let $\psi \in H^l(M)$. Then*

$$\dim H^k\mathcal{C}(\psi, M) \leq \dim H^k(M) + \dim H^{k-l+1}(M)$$

for all k.

Proof. From the observation we made above we just need to show that the right hand side of the inequality is $\dim H^k\mathcal{C}(0,M)$, where we consider $0 \in H^l(M)$. But then the differential on $\mathcal{C}^*(M)$ is simply the differential on each component and so $H^k\mathcal{C}(0,M) \cong H^k(M) \oplus H^{k-l+1}(M)$. □

The other primary limitation on the cone dimension is that the low and high degree cohomologies are determined by the de Rham cohomology. We saw this effect with the blowup calculations - for high filtration which corresponds to a higher degree ψ we got two copies of the de Rham cohomology. The same happens here, as the degree of ψ decreases the two copies of de Rham cohomology overlap more and more and there is less determined structure.

Proposition 5.2. *Let $\psi \in H^l(M)$. If $k < l$ then $H^k\mathcal{C}(\psi,M) \cong H^k(M)$, and if $k > n$ then $H^k\mathcal{C}(\psi,M) \cong H^{k-l+1}(M)$.*

Proof. This relies on the isomorphism

$$H^k\mathcal{C}(\psi,M) \cong \ker\psi\wedge : H^{k-l+1}(M) \to H^{k+1}(M) \oplus \operatorname{coker}\psi\wedge : H^{k-l}(M) \to H^k(M)$$

If $k < l$ then $k - l + 1 \leq 0$ which means the kernel term is zero and the cokernel term is $H^k(M)$. On the other hand, if $k > n$ then the kernel term is $H^{k-l+1}(M)$ and the cokernel term is zero. □

This in turn completely defines the cone cohomology of high degree forms.

Corollary 5.3. *Let M be compact and orientable with $\dim M = n$ and $\psi \in H^n(M)$. Then*

$$\dim H^k\mathcal{C}(\psi,M) = \begin{cases} \dim H^k(M) & k < n \\ \dim H^{k-n+1}(M) & k \geq n \end{cases}$$

Proof. The $k < n$ and $k > n$ are handled by the previous proposition with $l = n$. For $k = n$ we have

$$H^n\mathcal{C}(\psi, M) \cong \ker \psi\wedge : H^1(M) \to H^{n+1}(M) \oplus \operatorname{coker} \psi\wedge : H^0(M) \to H^n(M)$$

Then $\psi\wedge$ is an isomorphism between $H^0(M)$ and $H^n(M)$ so the cokernel term is zero. In the kernel term $\psi\wedge$ is the zero map and therefore $H^n\mathcal{C}(\psi, M) \cong H^1(M)$ as desired. □

Corollary 5.4. *Let M be compact and orientable with $\dim M = n$ and $\psi \in H^{n-1}(M)$. Then*

$$\dim H^k\mathcal{C}(\psi, M) = \begin{cases} \dim H^k(M) & k < n-1 \\ 2\dim H^1(M) - 2 & k = n-1 \\ \dim H^{k-n+2}(M) & k > n-1 \end{cases}$$

Proof. The proposition handles $k < n-1$ and $k > n$. Similarly to before we get $H^n\mathcal{C}(\psi, M) \cong H^2(M)$. Finally,

$$H^{n-1}\mathcal{C}(\psi, M) \cong \ker \psi\wedge : H^1(M) \to H^n(M) \oplus \operatorname{coker} \psi\wedge : H^0(M) \to H^{n-1}(M)$$

Then the kernel term has dimension $\dim H^1(M) - 1$ since $\dim H^n(M) = 1$, and the kernel term has the same dimension since the image of $\psi\wedge$ is 1-dimensional and $H^{n-1}(M) \cong H^1(M)$. □

One question we could ask is if there is a relationship between cone cohomologies of two forms and that of their wedge product, namely if $\psi \in H^l(M)$ and $\phi \in H^m(M)$ is there a relationship between $H^*\mathcal{C}(\psi, M)$, $H^*\mathcal{C}(\phi, M)$, and $H^*\mathcal{C}(\psi \wedge \phi, M)$? The answer is unclear, but there is potential by adapting the filter reducing map to this more general setting.

Lemma 5.5. *The map* $q : \mathcal{C}^k(\psi \wedge \phi, M) \to \mathcal{C}^k(\psi, M)$ *defined by*

$$q(A_k + \theta_{l+m} B_{k_{l+m}}) = A_k + \theta_l \phi \wedge B_{k_{l+m}}$$

is a chain map.

Proof. The proof is identical to the filter reducing map case replacing ω^q with ϕ. □

5.3 Application to Nilmanifolds

In this section we will describe how to apply the cone cohomology to small dimension nilmanifolds. Nilmanifolds are a convenient area for examples because they have a basis of 1-forms that also generate a basis for all higher degree forms. This will allow us to write the $\psi\wedge$ map as a matrix with respect to this convenient basis which allows us to determine how the kernel and cokernel change. Appendix B has complete Mathematica code, but here's the basic process:

1. Use the nilmanifold class to define the action of the differential
2. Use the basis to write forms as vectors and maps as matrices
3. Define closed forms to be those vectors in the kernel of the d matrix
4. Define exact forms as those that can be written as a linear combination of d of the basis one degree less
5. Compute the de Rham cohomology by finding a basis for the kernel of the d matrix and converting back to a form
6. Define ψ as an arbitrary linear combination of the cohomology basis elements

7. Write $\psi\wedge$ in matrix form by first applying it to each cohomology basis element and writing the result as a linear combination of cohomology elements plus an exact term, then discard the exact term

8. Determine the dimension of the kernel and cokernel in terms of the coefficients in ψ

The challenging part is the last step because we need to determine the kernel and cokernel of a matrix with variable entries. This becomes more complex as the de Rham cohomology dimension increases since there are more variables to ψ and the matrices become larger. The method outlined in the code of Appendix B is to compute the minors of the matrix and find what conditions on the variables makes all minors of a certain size vanish. This corresponds to changing the rank of the matrix, and then after identifying these conditions we can explicitly compute the cone dimensions from a single element of the set.

Appendix A contains all cone cohomology information for nilmanifolds of dimension 3 through 5, and the 1-form cone cohomology information in dimension 6. There are a few observations we can make right away. First, there are some nilmanifolds where the cone cohomology is fixed for all forms of the same degree, here these are only the nilmanifolds $(0,0,12)$, $(0,0,12,13)$, and $(0,0,12,13,23)$. Notably there is one of each dimension, and there are possible candidates in 6 dimensions. The question is if this pattern of one such manifold per dimension continues and if there is any significance to these nilmanifolds.

One can also ask if, in the case we have a symplectic manifold, we can derive more information about the manifold than just computing with a symplectic form. As an example, consider the Kodaira-Thurston manifold which is a nilmanifold with signature $(0,0,0,12)$. $H^2(M)$ then has 4 generators - $e_1\wedge e_3$, $e_1\wedge e_4$, $e_2\wedge e_3$, and $e_2\wedge e_4$. The symplectic forms are generated by two symplectic forms: $e_1 \wedge e3 + e_2 \wedge e_4$ and $e_1 \wedge e_4 + e_2 \wedge e_3$. From the information in the appendix, we see that these forms will have cone dimensions 1, 3, 4, 4, 3, 1. However, the cone dimension of a 2-form will jump to 1, 3, 6, 6, 3, 1 if there are no $e_1 \wedge e_4$ and $e_2 \wedge e_4$

terms, but this cannot happen with a symplectic form. So indeed, the considering the cone cohomology over the space of all 2-forms will in general give more information than just considering the symplectic structure.

Another observation is that we do have some cases where the de Rham cohomology of two nilmanifolds is isomorphic but the cone dimensions differ or change on different subsets. For example, in 5 dimensions we have $(0, 0, 12, 13, 14)$ and $(0, 0, 12, 13, 23)$ which both have de Rham cohomologies of dimensions 1, 2, 3, 3, 2, 1 but the first has a 1-form cone cohomology jump when the coeffient on e_2 vanishes but the second does not. Additionally, the 1-form cone cohomologies all have different dimension.

Finally, we'll identify a few patterns in the cone cohomology dimensions. For one, there seems to be a contingent of nilmanifolds where their 1-form cone dimensions are the same as the corresponding de Rham dimensions, but there are also plenty of cases where that doesn't happen. Interestingly, when the cone dimensions don't match the de Rham case, it is also quite common for the cone dimensions to match the de Rham or cone dimensions for other nilmanifolds. As an example, the 1-form cone cohomology for $(0, 0, 12, 13)$ doesn't match its de Rham cohomology but it does match the cone cohomology for $(0, 0, 0, 12)$. There may be a connection between nilmanifolds that is suggested by the cone cohomology.

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Appendix A

Cone Cohomology of Nilmanifolds

We will use specific notation and organization to simplify the data provided here. Each nilmanifold will have its own table with the list of degrees across the top followed by the de Rham cohomology dimensions, and finally distinct sections that correspond to the cone cohomology dimensions due to a 1-form, 2-form, and so on. At the top of each section is the general case which refers to any form that does not fall into any of the subsets listed after. Form are always assumed to be non-zero so at least one coefficient is non-zero. When describing subsets we use the notation a_1, a_2, etc. for the coefficient on the cohomology classes of e_1, e_2, etc. which are implicit from the class of the nilmanifold. For higher degrees we use notation of the form a_{12} to represent the coefficient on the cohomology class of $e1 \wedge e2$ and something of the form a_{12-34} for the coefficient on the class of $e_1 \wedge e_2 - e_3 \wedge e_4$. For example, on $(0, 0, 12)$ the subset $a_3 = 0$ corresponds to forms of the type $a_1 e_1 + a_2 e_2$ while the general case (assuming there are no other subsets) corresponds to those forms of the type $a_1 e_1 + a_2 e_2 + a_3 e_3$ where $a_3 \neq 0$.

A.1 Dimension 3

A.1.1 Nilmanifold Class (0,0,12)

Degree	0	1	2	3	4	5
de Rham	1	2	2	1		
General 1-Form	1	3	3	1		
General 2-Form	1	2	2	2	1	
General 3-Form	1	2	2	2	2	1

Table A.1: Cone Dimensions on (0,0,12)

A.2 Dimension 4

A.2.1 Nilmanifold Class (0,0,0,12)

Degree	0	1	2	3	4	5	6	7
de Rham	1	3	4	3	1			
General 1-Form	1	3	4	3	1			
$a_3 = 0$	1	4	6	4	1			
General 2-Form	1	3	4	4	3	1		
$a_{14} = a_{24} = 0$	1	3	6	6	3	1		
General 3-Form	1	3	4	4	4	3	1	
General 4-Form	1	3	4	3	3	4	3	1

Table A.2: Cone Dimensions on (0,0,0,12)

A.2.2 Nilmanifold Class (0,0,12,13)

Degree	0	1	2	3	4	5	6	7
de Rham	1	2	2	2	1			
General 1-Form	1	3	4	3	1			
General 2-Form	1	2	3	3	2	1		
General 3-Form	1	2	2	2	2	2	1	
General 4-Form	1	2	2	2	2	2	2	1

Table A.3: Cone Dimensions on (0,0,12,13)

A.3 Dimension 5

A.3.1 Nilmanifold Class (0,0,0,0,12)

Degree	0	1	2	3	4	5	6	7	8	9
de Rham	1	4	7	7	4	1				
General 1-Form	1	4	7	7	4	1				
$a_3 = a_4 = 0$	1	5	10	10	5	1				
General 2-Form	1	4	6	6	6	4	1			
$a_{34} = 0$	1	4	7	8	7	4	1			
General 3-Form	1	4	7	6	6	7	4	1		
$a_{135}a_{245} - a_{145}a_{235} = 0$	1	4	7	8	8	7	4	1		
$a_{125} = a_{145} = a_{235} = a_{135} = 0$	1	4	7	10	10	7	4	1		
General 4-Form	1	4	7	7	6	7	7	4	1	
General 5-Form	1	4	7	7	4	4	7	7	4	1

Table A.4: Cone Dimensions on (0,0,0,0,12)

A.3.2 Nilmanifold Class (0,0,0,0,12+34)

Degree	0	1	2	3	4	5	6	7	8	9
de Rham	1	4	5	5	4	1				
General 1-Form	1	4	7	7	4	1				
General 2-Form	1	4	8	10	8	4	1			
General 3-Form	1	4	5	4	4	5	4	1		
$a_{135}a_{245} - a_{145}a_{235} + a_{345-125}^2 = 0$	1	4	5	6	6	5	4	1		
General 4-Form	1	4	5	5	6	5	5	4	1	
General 5-Form	1	4	5	5	4	4	5	5	4	1

Table A.5: Cone Dimensions on (0,0,0,0,12+34)

A.3.3 Nilmanifold Class (0,0,0,12,13+24)

Degree	0	1	2	3	4	5	6	7	8	9
de Rham	1	3	4	4	3	1				
General 1-Form	1	3	4	4	3	1				
$a_3 = 0$	1	4	6	6	4	1				
General 2-Form	1	3	4	4	4	3	1			
$a_{25-34} = 0$	1	3	5	6	5	3	1			
$a_{14} = a_{25-34} = 0$	1	3	6	8	6	3	1			
General 3-Form	1	3	4	4	4	4	3	1		
$a_{145} = a_{245-135} = 0$ 1	3	4	6	6	4	3	1			
General 4-Form	1	3	4	4	4	4	4	3	1	
General 5-Form	1	3	4	4	3	3	4	4	3	1

Table A.6: Cone Dimensions on (0,0,0,12,13+24)

A.3.4 Nilmanifold Class (0,0,0,12,13)

Degree	0	1	2	3	4	5	6	7	8	9
de Rham	1	3	6	6	3	1				
General 1-Form	1	4	7	7	4	1				
$a_2 = a_3 = 0$	1	5	9	9	5	1				
General 2-Form	1	3	5	6	5	3	1			
$a_{14} = a_{15} = a_{24}a_{35} - a_{25+34}^2$	1	3	6	8	6	3	1			
$a_{14} = a_{24} = a_{25+34} = 0$	1	3	6	8	6	3	1			
$a_{15} = a_{35} = a_{25+34} = 0$	1	3	6	8	6	3	1			
$a_{24} = a_{35} = a_{25+34} = 0$	1	3	6	8	6	3	1			
$a_{14} = a_{15} = a_{24} = a_{35} = a_{25+34} = 0$	1	3	8	12	8	3	1			
General 3-Form	1	3	6	6	6	6	3	1		
$a_{145} = 0$	1	3	6	8	8	6	3	1		
General 4-Form	1	3	6	6	4	6	6	3	1	
General 5-Form	1	3	6	6	3	3	6	6	3	1

Table A.7: Cone Dimensions on (0,0,0,12,13)

A.3.5 Nilmanifold Class (0,0,0,12,14)

Degree	0	1	2	3	4	5	6	7	8	9
de Rham	1	3	4	4	3	1				
General 1-Form	1	3	4	4	3	1				
$a_3 = 0$	1	4	7	7	4	1				
General 2-Form	1	3	5	6	5	3	1			
$a_{15} = a_{24} = 0$	1	3	6	8	6	3	1			
General 3-Form	1	3	4	4	4	4	3	1		
$a_{145} = a_{245} = 0$	1	3	4	6	6	4	3	1		
General 4-Form	1	3	4	4	4	4	4	3	1	
General 5-Form	1	3	4	4	3	3	4	4	3	1

Table A.8: Cone Dimensions on (0,0,0,12,14)

A.3.6 Nilmanifold Class (0,0,12,13,14)

Degree	0	1	2	3	4	5	6	7	8	9
de Rham	1	4	5	5	4	1				
General 1-Form	1	3	4	4	3	1				
$a_2 = 0$	1	3	5	5	3	1				
General 2-Form	1	2	2	2	2	2	1			
$a_{34-25} = 0$	1	2	3	4	3	2	1			
$a_{15} = a_{34-25} = 0$	1	2	4	6	4	2	1			
General 3-Form	1	2	3	4	4	3	2	1		
General 4-Form	1	2	3	3	2	3	3	2	1	
General 5-Form	1	2	3	3	2	2	3	3	2	1

Table A.9: Cone Dimensions on (0,0,12,13,14)

A.3.7 Nilmanifold Class (0,0,12,13,23)

Degree	0	1	2	3	4	5	6	7	8	9
de Rham	1	2	3	3	2	1				
General 1-Form	1	3	6	6	3	1				
General 2-Form	1	2	4	6	4	2	1			
General 3-Form	1	2	3	4	4	3	2	1		
General 4-Form	1	2	3	3	2	3	3	2	1	
General 5-Form	1	2	3	3	2	2	3	3	2	1

Table A.10: Cone Dimensions on (0,0,12,13,23)

A.3.8 Nilmanifold Class (0,0,12,13,14+23)

Degree	0	1	2	3	4	5	6	7	8	9
de Rham	1	2	3	3	2	1				
General 1-Form	1	3	4	4	3	1				
$a_2 = 0$	1	3	5	5	3	1				
General 2-Form	1	2	2	2	2	2	1			
$a_{25-34} = 0$	1	2	3	4	3	2	1			
$a_{25-34} = a_{15+24} = 0$	1	2	4	6	4	2	1			
General 3-Form	1	2	3	4	4	3	2	1		
General 4-Form	1	2	3	3	2	3	3	2	1	
General 5-Form	1	2	3	3	2	2	3	3	2	1

Table A.11: Cone Dimensions on (0,0,12,13,14+23)

A.4 Dimension 6 (Only 1-Forms)

A.4.1 Nilmanifold Class (0,0,12,13,14,15)

Degree	0	1	2	3	4	5	6
de Rham	1	2	3	4	3	2	1
General 1-Form	1	3	5	6	5	3	1
$a_2 = 0$	1	3	6	8	6	3	1

Table A.12: Cone Dimensions on (0,0,12,13,14,15)

A.4.2 Nilmanifold Class (0,0,12,13,14,34+52)

Degree	0	1	2	3	4	5	6
de Rham	1	2	2	2	2	2	1
General 1-Form	1	3	4	4	4	3	1

Table A.13: Cone Dimensions on (0,0,12,13,14,34+52)

A.4.3 Nilmanifold Class (0,0,12,13,14,23+15)

Degree	0	1	2	3	4	5	6
de Rham	1	2	3	4	3	2	1
General 1-Form	1	3	5	6	5	3	1
$a_2 = 0$	1	3	6	8	6	3	1

Table A.14: Cone Dimensions on (0,0,12,13,14,23+15)

A.4.4 Nilmanifold Class (0,0,12,13,23,14)

Degree	0	1	2	3	4	5	6
de Rham	1	2	4	6	4	2	1
General 1-Form	1	3	6	8	6	3	1
$a_2 = 0$	1	3	7	10	7	3	1

Table A.15: Cone Dimensions on (0,0,12,13,23,14)

A.4.5 Nilmanifold Class (0,0,12,13,23,14-25)

Degree	0	1	2	3	4	5	6
de Rham	1	2	4	6	4	2	1
General 1-Form	1	3	6	8	6	3	1

Table A.16: Cone Dimensions on (0,0,12,13,23,14-25)

A.4.6 Nilmanifold Class (0,0,12,13,23,14+25)

Degree	0	1	2	3	4	5	6
de Rham	1	2	4	6	4	2	1
General 1-Form	1	3	6	8	6	3	1

Table A.17: Cone Dimensions on (0,0,12,13,23,14+25)

A.4.7 Nilmanifold Class (0,0,12,13,14+23,34+52)

Degree	0	1	2	3	4	5	6
de Rham	1	2	2	2	2	2	1
General 1-Form	1	3	4	4	4	3	1

Table A.18: Cone Dimensions on (0,0,12,13,14+23,34+52)

A.4.8 Nilmanifold Class (0,0,12,13,14+23,24+15)

Degree	0	1	2	3	4	5	6
de Rham	1	2	3	4	3	2	1
General 1-Form	1	3	5	6	5	3	1
$a_2 = 0$	1	3	6	8	6	3	1

Table A.19: Cone Dimensions on (0,0,12,13,14+23,24+15)

A.4.9 Nilmanifold Class (0,0,0,12,13,14+35)

Degree	0	1	2	3	4	5	6
de Rham	1	3	5	6	5	3	1
General 1-Form	1	4	7	8	7	4	1
$a_2 = a_3 = 0$	1	5	9	10	9	5	1

Table A.20: Cone Dimensions on (0,0,0,12,13,14+35)

A.4.10 Nilmanifold Class (0,0,0,12,13,14+23)

Degree	0	1	2	3	4	5	6
de Rham	1	3	6	8	6	3	1
General 1-Form	1	4	7	8	7	4	1
$a_3 = 0$	1	4	9	12	9	4	1
$a_2 = a_3 = 0$	1	5	11	14	11	5	1

Table A.21: Cone Dimensions on (0,0,0,12,13,14+23)

A.4.11 Nilmanifold Class (0,0,0,12,13,24)

Degree	0	1	2	3	4	5	6
de Rham	1	3	6	8	6	3	1
General 1-Form	1	4	7	8	7	4	1
$a_3 = 0$	1	4	9	12	9	4	1
$a_2 = a_3 = 0$	1	5	10	12	10	5	1

Table A.22: Cone Dimensions on (0,0,0,12,13,24)

A.4.12 Nilmanifold Class (0,0,0,12,13,14)

Degree	0	1	2	3	4	5	6
de Rham	1	3	6	8	6	3	1
General 1-Form	1	4	7	8	7	4	1
$a_3 = 0$	1	4	9	12	9	4	1
$a_2 = a_3 = 0$	1	5	11	14	11	5	1

Table A.23: Cone Dimensions on (0,0,0,12,13,14)

A.4.13 Nilmanifold Class (0,0,0,12,13,23)

Degree	0	1	2	3	4	5	6
de Rham	1	3	8	12	8	3	1
General 1-Form	1	5	11	14	11	5	1

Table A.24: Cone Dimensions on (0,0,0,12,13,23)

A.4.14 Nilmanifold Class (0,0,0,12,14,15+23)

Degree	0	1	2	3	4	5	6
de Rham	1	3	5	6	5	3	1
General 1-Form	1	3	5	6	5	3	1
$a_3 = 0$	1	4	7	8	7	4	1
$a_2 = a_3 = 0$	1	4	8	10	8	4	1

Table A.25: Cone Dimensions on (0,0,0,12,14,15+23)

A.4.15 Nilmanifold Class (0,0,0,12,14,15+23+24)

Degree	0	1	2	3	4	5	6
de Rham	1	3	5	6	5	3	1
General 1-Form	1	3	5	6	5	3	1
$a_3 = 0$	1	4	7	8	7	4	1
$a_2 = a_3 = 0$	1	4	8	10	8	4	1

Table A.26: Cone Dimensions on (0,0,0,12,14,15+23+24)

A.4.16 Nilmanifold Class (0,0,0,12,14,15+24)

Degree	0	1	2	3	4	5	6
de Rham	1	3	5	6	5	3	1
General 1-Form	1	3	5	6	5	3	1
$a_3 = 0$	1	4	7	8	7	4	1
$a_2 = a_3 = 0$	1	4	8	10	8	4	1

Table A.27: Cone Dimensions on (0,0,0,12,14,15+24)

A.4.17 Nilmanifold Class (0,0,0,12,14,15)

Degree	0	1	2	3	4	5	6
de Rham	1	3	5	6	5	3	1
General 1-Form	1	3	5	6	5	3	1
$a_3 = 0$	1	4	7	8	7	4	1
$a_2 = a_3 = 0$	1	4	8	10	8	4	1

Table A.28: Cone Dimensions on (0,0,0,12,14,15)

A.4.18 Nilmanifold Class (0,0,0,12,14,24)

Degree	0	1	2	3	4	5	6
de Rham	1	3	5	6	5	3	1
General 1-Form	1	3	5	6	5	3	1
$a_3 = 0$	1	4	9	12	9	4	1

Table A.29: Cone Dimensions on (0,0,0,12,14,24)

A.4.19 Nilmanifold Class (0,0,0,12,14,13+42)

Degree	0	1	2	3	4	5	6
de Rham	1	3	5	6	5	3	1
General 1-Form	1	3	5	6	5	3	1
$a_3 = 0$	1	4	9	12	9	4	1

Table A.30: Cone Dimensions on (0,0,0,12,14,13+42)

A.4.20 Nilmanifold Class (0,0,0,12,14,23+24)

Degree	0	1	2	3	4	5	6
de Rham	1	3	5	6	5	3	1
General 1-Form	1	3	5	6	5	3	1
$a_3 = 0$	1	4	9	12	9	4	1

Table A.31: Cone Dimensions on (0,0,0,12,14,23+24)

A.4.21 Nilmanifold Class (0,0,0,12,23,14+35)

Degree	0	1	2	3	4	5	6
de Rham	1	3	5	6	5	3	1
General 1-Form	1	4	7	8	7	4	1
$a_1 = a_3 = 0$	1	5	8	8	8	5	1
$a_2 = a_1 + a_3 = 0$	1	4	8	10	8	4	1
$a_2 = a_1 - a_3 = 0$	1	4	8	10	8	4	1

Table A.32: Cone Dimensions on (0,0,0,12,23,14+35)

A.4.22 Nilmanifold Class (0,0,0,12,23,14-35)

Degree	0	1	2	3	4	5	6
de Rham	1	3	5	6	5	3	1
General 1-Form	1	4	7	8	7	4	1
$a_1 = a_3 = 0$	1	5	8	8	8	5	1

Table A.33: Cone Dimensions on (0,0,0,12,23,14-35)

A.4.23 Nilmanifold Class (0,0,0,12,14-23,15+34)

Degree	0	1	2	3	4	5	6
de Rham	1	3	4	4	4	3	1
General 1-Form	1	3	5	6	5	3	1
$a_3 = 0$	1	4	6	6	6	4	1
$a_2 = a_3 = 0$	1	4	7	8	7	4	1

Table A.34: Cone Dimensions on (0,0,0,12,14-23,15+34)

A.4.24 Nilmanifold Class (0,0,0,12,14+23,13+42)

Degree	0	1	2	3	4	5	6
de Rham	1	3	5	6	5	3	1
General 1-Form	1	3	5	6	5	3	1
$a_3 = 0$	1	4	9	12	9	4	1

Table A.35: Cone Dimensions on (0,0,0,12,14+23,13+42)

A.4.25 Nilmanifold Class (0,0,0,0,12,15+34)

Degree	0	1	2	3	4	5	6
de Rham	1	4	6	6	6	4	1
General 1-Form	1	4	7	8	7	4	1
$a_3 = a_4 = 0$	1	5	9	10	9	5	1

Table A.36: Cone Dimensions on (0,0,0,0,12,15+34)

A.4.26 Nilmanifold Class (0,0,0,0,12,15)

Degree	0	1	2	3	4	5	6
de Rham	1	4	7	8	7	4	1
General 1-Form	1	4	7	8	7	4	1
$a_3 = a_4 = 0$	1	5	11	14	11	5	1

Table A.37: Cone Dimensions on (0,0,0,0,12,15)

A.4.27 Nilmanifold Class (0,0,0,0,12,14+25)

Degree	0	1	2	3	4	5	6
de Rham	1	4	7	8	7	4	1
General 1-Form	1	4	7	8	7	4	1
$a_3 = a_4 = 0$	1	5	10	12	10	5	1
$a_1 = a_3 = a_4 = 0$	1	5	11	14	11	5	1

Table A.38: Cone Dimensions on (0,0,0,0,12,14+25)

A.4.28 Nilmanifold Class (0,0,0,0,12,14+23)

Degree	0	1	2	3	4	5	6
de Rham	1	4	8	10	8	4	1
General 1-Form	1	4	9	12	9	4	1
$a_3 = a_4 = 0$	1	5	11	14	11	5	1

Table A.39: Cone Dimensions on (0,0,0,0,12,14+23)

A.4.29 Nilmanifold Class (0,0,0,0,12,34)

Degree	0	1	2	3	4	5	6
de Rham	1	4	8	10	8	4	1
General 1-Form	1	4	9	12	9	4	1
$a_1 = a_2 = 0$	1	5	11	14	11	5	1
$a_3 = a_4 = 0$	1	5	11	14	11	5	1

Table A.40: Cone Dimensions on (0,0,0,0,12,34)

A.4.30 Nilmanifold Class (0,0,0,0,12,13)

Degree	0	1	2	3	4	5	6
de Rham	1	4	9	12	9	4	1
General 1-Form	1	4	9	12	9	4	1
$a_4 = 0$	1	5	11	14	11	5	1
$a_2 = a_3 = a_4 = 0$	1	6	14	18	14	6	1

Table A.41: Cone Dimensions on (0,0,0,0,12,13)

A.4.31 Nilmanifold Class (0,0,0,0,13+42,14+23)

Degree	0	1	2	3	4	5	6
de Rham	1	4	8	10	8	4	1
General 1-Form	1	4	9	12	9	4	1

Table A.42: Cone Dimensions on (0,0,0,0,13+42,14+23)

A.4.32 Nilmanifold Class (0,0,0,0,0,12+34)

Degree	0	1	2	3	4	5	6
de Rham	1	5	9	10	9	5	1
General 1-Form	1	5	9	10	9	5	1
$a_5 = 0$	1	5	11	14	11	5	1

Table A.43: Cone Dimensions on (0,0,0,0,0,12+34)

A.4.33 Nilmanifold Class (0,0,0,0,0,12)

Degree	0	1	2	3	4	5	6
de Rham	1	5	11	14	11	5	1
General 1-Form	1	5	11	14	11	5	1
$a_3 = a_4 = a_5 = 0$	1	6	15	20	15	6	1

Table A.44: Cone Dimensions on (0,0,0,0,0,12)

A.4.34 Nilmanifold Class (0,0,0,0,0,0)

Degree	0	1	2	3	4	5	6
de Rham	1	6	15	20	15	6	1
General 1-Form	1	6	15	20	15	6	1

Table A.45: Cone Dimensions on (0,0,0,0,0,0)

Appendix B

Mathematica Code for Computing Cone Cohomology Dimensions

```
(** Clear all data to prepare for defining a new manifold. **)
ResetForms[] := Module[{},
    BasisForms = {};
    Bases = <||>;
    dRelations = <||>;
    dMatrix = <||>;
    Cohomology = <||>;
];

(** Generate basis forms, differential map, and cohomology
for a 'dim' dimensional manifold where 'dRelList' defines
the differential of the basis of 1-forms. **)
Setup[dim_, dRelList_] := Module[{},
    ResetForms[];
```

```
    GenerateBasis[dim];
    GenerateBases[];
    For[i = 1, i <= Length[dRelList], i++,
     AppendTo[dRelations, BasisForms[[i]]  → dRelList[[i]]]];
    GeneratedMatrix[];
    GenerateCohomology[];
];

(** Defines what it means to be a basis form and a
general form. **)
BasisFormQ[form_] := MemberQ[BasisForms, form];
FormQ[a_Wedge] := True;
FormQ[a_Symbol] := BasisFormQ[a];
FormQ[a_ + b_] := FormQ[a] && FormQ[b];
FormQ[r_?NumericQ*a_] := FormQ[a];

(** Generates the basis forms in each degree. **)
GenerateBasis[dim_Integer?NonNegative] :=
  For[i = 1, i <= dim, i++,
   AppendTo[BasisForms, Symbol["e" <> ToString[i]]]];
GenerateBases[] :=
  For[k = 0, k <= Length[BasisForms] + 1, k++,
   AppendTo[Bases, k  → GenerateBasisInDegree[k]]];
GenerateBasisInDegree[n_] := Module[{list, flist},
    If[n == 0, Return[{1}]];
    If[n == 1, Return[BasisForms]];
    If[n > Length[BasisForms], Return[{0}]];
```

```
    list = Subsets[BasisForms, {n}];
    flist = {};
    For[i = 1, i <= Length[list], i++,
     AppendTo[flist, Wedge @@ list[[i]]]];
    Return[flist];
];

(** Creates the matrix representation of the differential
map with respect to the basis forms. **)
GeneratedMatrix[] :=
  For[k = 0, k <= Length[BasisForms], k++,
   AppendTo[dMatrix, k  -> dMatrixInDegree[k]]];
dMatrixInDegree[k_] :=
  Transpose[ToMatrix[d[#], k + 1] & /@ Bases[k]];

(** Defines the wedge product and its properties. **)
Wedge[a_?FormQ] := a;
Wedge[a___?FormQ, x_?FormQ + y_?FormQ, z___?FormQ] :=
  Wedge[a, x, z] + Wedge[a, y, z];
Wedge[a___?FormQ, s_?NumericQ*x_?FormQ, z___?FormQ] :=
  s Wedge[a, x, z];
Wedge[a___?FormQ, r_?NumericQ, z___?FormQ] := r Wedge[a, z];
Wedge[a___?BasisFormQ, x_?BasisFormQ, b___?BasisFormQ, x_,
    c___?BasisFormQ] := 0;
Wedge[args__?BasisFormQ] /; ! OrderedQ[{args}] :=
  Signature[{args}] Wedge @@ Sort[{args}];
SetAttributes[Wedge, {Listable, Flat}];
```

```
(** Defines the degree of a form and its properties. **)
Deg[a_?BasisFormQ] := 1;
Deg[r_?NumericQ] := 0;
Deg[a_?FormQ + b_?FormQ] := Deg[a];
Deg[a_?FormQ ∧ b_?FormQ] := Deg[a] + Deg[b];
Deg[r_?NumericQ*f_?FormQ] := Deg[f];

(** Defines the exterior derivative and its properties. **)
d[a___?FormQ, x_?FormQ + y_?FormQ, z___?FormQ] :=
  d[a, x, z] + d[a, y, z];
d[r_?NumericQ*f_?FormQ] := r d[f];
d[r_?NumericQ] := 0;
d[a_?FormQ ∧ b_?FormQ] :=
  d[a] ∧ b + (-1)^Deg[a] a ∧ d[b];
d[a_?BasisFormQ] := dRelations[a];

(** Defines what it means for a form to be closed. **)
ClosedQ[form_?FormQ | form_?NumericQ] := d[form] === 0;

(** Defines what it means for a form to be exact. Checks to
see if the form can be written as a linear combination of the
differential of lower degree forms. If the result is a List
that means that it found a solution and the form is exact. **)
ExactQ[form_?FormQ] := Head[Exact[form]] === List;
ExactQ[n_?NumericQ] := False;
Exact[form_?FormQ] :=
```

```
  Quiet@Check[
    LinearSolve[dMatrix[Deg[form]-1], ToMatrix[form, Deg[form]]],
    False, LinearSolve::nosol];

(** Computes a basis for the de Rham cohomology of the manifold
in each degree. **)
GenerateCohomology[] :=
  For[k = 0, k <= Length[BasisForms] + 1, k++,
   AppendTo[Cohomology, k  -> GenerateCohomologyInDegree[k]]];
GenerateCohomologyInDegree[k_] :=
  Module[{formList, matList = {{}}, cohList = {}, dup},
    If[k > Length[BasisForms], Return[{0}]];
    formList = Reverse[Bases[k] . # & /@ NullSpace[dMatrix[k]]];
    For[i = 1, i <= Length[formList], i++,
      dup =
        Quiet@Check[
          LinearSolve[Join[dMatrix[k - 1], matList, 2],
            ToMatrix[formList[[i]], k]], False, LinearSolve::nosol];
      If[Head[dup] =!= List,
        AppendTo[cohList, formList[[i]]];
        matList =
          Join[matList, Transpose[{ToMatrix[formList[[i]], k]}], 2];
        ];
      ];
    Return[cohList];
];
```

```
(** Returns a list of the de Rham cohomology dimensions. **)
CohomologyDims[] :=
  Length[Cohomology[#]] & /@ Range[0, Length[BasisForms]];

(** Converts a form of degree 'deg' to a vector form in the basis
of either the basis forms of that degree or the cohomology
elements of that degree. **)
ToMatrix[form_, deg_] :=
  If[0 < deg && deg <= Length[BasisForms],
   Coefficient[form, #] & /@ Bases[deg], {form}];
MatrixInCohomology[form_, deg_] :=
  LinearSolve[
    Join[Transpose[ToMatrix[#, deg] & /@ Cohomology[deg]],
     dMatrix[deg - 1], 2], ToMatrix[form, deg]][[
   1 ;; Length[Cohomology[deg]]]];

(** Writes the 'form' ∧ map on the index 'ind' cohomology. **)
CohomologyWedgeMatrix[form_, ind_] :=
  Transpose[
   MatrixInCohomology[Wedge[form, #], Deg[form] + ind] & /@
    Cohomology[ind]
  ];

(** Computes the dimension of the kernel or cokernel. **)
DimKer[mat_] := Length[NullSpace[mat]];
DimCoker[mat_] :=
  Dimensions[mat][[1]] - Dimensions[mat][[2]] + DimKer[mat];
```

```
(** Compute the kernel/cokernel matrix of the 'form' ∧
map that appears in the definition of the index 'ind' cone
cohomology. **)
KernelMatrix[psi_, ind_] :=
  CohomologyWedgeMatrix[psi, ind + 1 - Deg[psi]];
CokernelMatrix[psi_, ind_] :=
  CohomologyWedgeMatrix[psi, ind - Deg[psi]];

(** Compute the dimension of the cone cohomology with respect
to 'psi' in index 'ind'. **)
ConeCohomologyDim[psi_, ind_] := Module[{total = 0},
    If[ind + 1 - Deg[psi] >= 0,
     total +=
        If[ind >= Length[BasisForms],
         Length[Cohomology[ind + 1 - Deg[psi]]],
         DimKer[CohomologyWedgeMatrix[psi, ind + 1 - Deg[psi]]]];
     ];
    If[ind <= Length[BasisForms],
     total +=
        If[ind - Deg[psi] < 0, Length[Cohomology[ind]],
         DimCoker[CohomologyWedgeMatrix[psi, ind - Deg[psi]]]];
     ];
    Return[total];
];

(** Lists all cone cohomology dimensions with respect to
```

```
the form 'psi'. **)
ConeCohomologyDims[psi_] :=
  ConeCohomologyDim[psi, #] & /@
   Range[0, Length[BasisForms] + Deg[psi] - 1];

(** Abbreviation for GerKerMatrix[1, ind]. **)
GenKerMatrix[ind_] :=
  Sum[If[ClosedQ[BasisForms[[i]]],
     Symbol["a"<>ToString[i]] KernelMatrix[BasisForms[[i]], ind],
     0], {i, 1, Length[BasisForms]}];

(** Abbreviation for GenCokerMatrix[1, ind]. **)
GenCokerMatrix[ind_] :=
  Sum[If[ClosedQ[BasisForms[[i]]],
     Symbol["a"<>ToString[i]] CokernelMatrix[BasisForms[[i]], ind],
     0], {i, 1, Length[BasisForms]}];

(** Computes the matrix of ψ∧ needed to compute the
kernel part of the index 'ind' cone cohomology with respect to
ψ, where ψ is a general linear combination of the
degree 'deg' cohomology basis forms. **)
GenKerMatrix[deg_, ind_] :=
  Sum[Symbol["a"<>ToString[i]] KernelMatrix[Cohomology[deg][[i]],
      ind], {i, 1, Length[Cohomology[deg]]}];

(** Computes the matrix of ψ∧ needed to compute the
cokernel part of the index 'ind' cone cohomology with respect
```

```
to ψ, where ψ is a general linear combination of the
degree 'deg' cohomology basis forms. **)
GenCokerMatrix[deg_, ind_] :=
  Sum[Symbol["a"<>ToString[i]]
     CokernelMatrix[Cohomology[deg][[i]],
     ind], {i, 1, Length[Cohomology[deg]]}];

(** A 4-Dimensional Example with ψ a 2-form **)
Setup[4, {0, 0, 0, e1 ∧ e2}]

CohomologyDims[]
(** Output: {1, 3, 4, 3, 1} **)

Cohomology[2]
(** Output: {e1 ∧ e3, e1 ∧ e4, e2 ∧ e3, e2 ∧ e4} **)

ind = 2;
GK = GenKerMatrix[2, ind];
GC = GenCokerMatrix[2, ind];

For[i = 1, i <= Min[Dimensions[GK]], i++,
 Print["Rank >= ", i, ": ",
  Solve[DeleteDuplicates[Flatten[Minors[GK, i]]] == 0, {}]]]
(** Output:
Rank >= 1: {{a2 → 0,a4 → 0}}
Rank >= 2: {{a2 → 0,a4 → 0}}
Rank >= 3: {{}}
```

```
**)

For[i = 1, i <= Min[Dimensions[GC]], i++,
  Print["Rank >= ", i, ": ",
   Solve[DeleteDuplicates[Flatten[Minors[GC, i]]] == 0, {}]]]
(** Output:
Rank >= 1: {{a1 → 0, a2 → 0, a3 → 0, a4 → 0}}
**)
```