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Traffic Flow Control in Automated Highway Systems

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Traffic Flow Control in Automated Highway Systems[1]

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Abstract

Automated Highway Systems (AHS) is a concept intended to increase capacity and safety in current surface transportation systems. The design of control systems for AHS is a challenging problem due to their large scale and hybrid nature. This report addresses the problem of traffic control in the AHS hierarchical architecture of the California PATH program. A link layer controller for the PATH AHS architecture is presented. The controller is derived from a principle of conservation of vehicles. Different topologies of highways are considered, including multiple lane highways in which vehicles have different destinations and types. It is assumed that the velocity and the change of lane can be commanded for vehicles in a stretch of highway. With the use of Lyapunov stability techniques, it is shown that the control laws proposed in this report stabilize the vehicular density and flow around predetermined profiles. This link layer controller exhibits important properties for implementation: it is distributed, in the sense that only local information is used, and avoids highway dynamics inversion. The link layer control schemes were implemented and tested using SmartPath-3 AHS simulation software. Simulation results were in complete agreement with theoretical predictions.

Keywords

Automated Highway Systems, Traffic control, Velocity control, Multi-destination traffic, Multi-type traffic.

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Executive Summary

One of the AHS architectures most used in the California PATH program consists of five hierarchical layers (Varaiya and Shladover, 1991): network, link, coordination, regulation and physical layers. This report addresses the problem of traffic control in the link layer of the AHS hierarchical architecture of the the California PATH program proposed in (Varaiya and Shladover, 1991). The focus is on the control strategies at the macroscopic level of traffic. At this level the important quantities to abstract are the aggregate vehicular density and traffic flow in different sections of the highway.

A link layer controller for the PATH AHS architecture is presented. This controller is assumed to be used in a fully automated highway. No assumptions are made in terms of drivers' reaction to traffic or in terms of any explicit vehicular density-velocity relationship.

The traffic of vehicles with different destination although sharing a multiple lane highway is considered. Assuming that the velocity and the lane change of vehicles in a stretch of highway can be commanded, a set of control laws that stabilize the vehicular traffic flow to predetermined desired profiles of velocity and density is presented.

The controller is derived from a model based on the principle of vehicles conservation and is based on Lyapunov stability results. The more important features of this controller are:

- It is suitable for distributed implementation because it requires only local traffic information.
- It avoids traffic flow dynamics inversion that produce unbounded controls signals for small vehicular densities.
- It tracks the vehicle density profile that minimizes the error along all the stretch of highway, even when there is a mismatch between the desired and real inlet traffic flow.

The desired velocity and density profiles that are considered included the cases in which different desired velocities can be assigned to sections of the highway where lane change is supposed to occur.

Simulation results are presented both in Matlab and in SmartPath (Eskafi *et al.*, 1992). The simulation results indicate both, the validity of the assumptions about the dynamics of the coordination and regulation layers, and the effectiveness of the link layer controller.

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Chapter 1

Introduction

The concept of Automated Highway Systems (AHS) has been proposed to increase capacity and safety in current surface transportation systems (Varaiya, 1993). One of the AHS architectures used in the California PATH program consists of five hierarchical layers (Varaiya and Shladover, 1991): network, link, coordination, regulation and physical layers. (see figure 1.1). There are different abstractions for each layer. In the physical and regulation layers, the abstraction is a continuous time model of the closed loop controlled vehicle dynamics. In the coordination layer, the execution of maneuvers is modeled through finite state machines that incorporate the structured communication between vehicles. The link layer uses a flow model to abstract macroscopic highway vehicular density and traffic flow. The proper abstraction for the network layer is yet to be determined. For examples of these different abstractions the reader is advised to consult, for example, (Hsu *et al.*, 1991; Swaroop *et al.*, 1994; Godbole and Lygeros, 1994; Frankel *et al.*, 1996; Li *et al.*, 1997*a*; Li *et al.*, 1995; Li *et al.*, 1997*b*; Rao and Varaiya, 1994; Papageorgiou, 1990; Papageorgiou *et al.*, 1990).

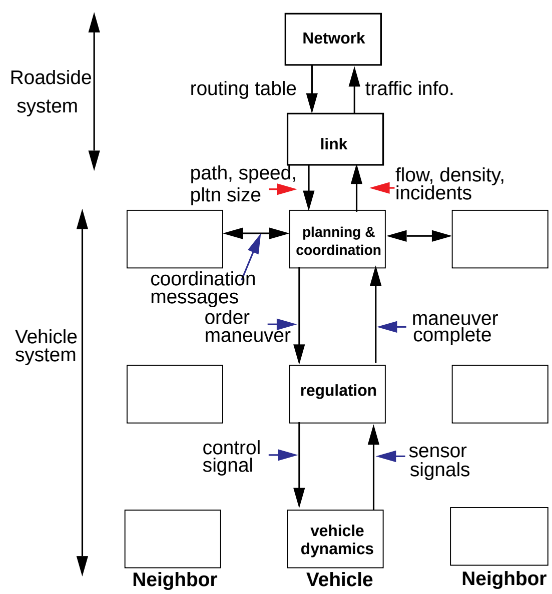

Figure 1.1: Hierarchical architecture of AVHS in the PATH program

This report addresses the problem of traffic control in link layer of the AHS hierarchical architecture of the the California PATH program proposed in (Varaiya and Shladover, 1991). The focus is on the control strategies at the macroscopic level of traffic. At this level the important quantities to abstract are the aggregate vehicular density and traffic flow in different sections of the highway. Some of the relevant control strategies in the literature are reviewed here.

In (Karaaslan *et al.*, 1990), the authors present a detailed traffic flow model based on the behavior of human drivers. They argue that the dynamics caused by the human driver behavior prevent full utilization of the highway in the presence of congestion. With vehicles under automatic control this is not necessarily the case. It is proposed that if one of the terms that describe driver behavior is replaced with a control term intended to homogenize the density profile, then the capacity of the highway can be better realized. This formulation is equivalent to solve the problem of tracking a uniform density profile.

Chien et al. (Chien *et al.*, 1993) generalize this problem to the tracking of an arbitrary density profile. Using a macroscopic traffic model similar to that in (Karaaslan *et al.*, 1990), they derive a controller that commands a desired velocity at each section of the highway such that the density of the entire highway conforms to a specified density profile. Their model is based on the behavior of human drivers. While it is possible to design control laws for automated vehicles so that they behave like those driven by people, this is not the only approach. The developed control law is based upon the inversion of the traffic flow dynamics, which requires a certain traffic flow controllability condition. This condition is violated when the density in any section of the highway becomes very small. The control action at a point in the highway requires information from the entire highway. This problem is alleviated by a dynamic version of the control law that solves the matrix inversion dynamically. No multiple lane or lane change commands are considered in this work.

A description of a link layer controller consistent with the AHS architecture in (Varaiya, 1993), for a highway operating under normal conditions, can be found in (Rao and Varaiya, 1994). The design presented in (Rao and Varaiya, 1994) assumes a fully automated highway and uses a dynamic model of the coordination and regulation layers obtained through extensive simulations under normal operation conditions. Lane change proportions, desired speeds and maximum platoon size are possible control variables, in that order of priority. The only control that has been fully developed and implemented in SmartPath (Eskafi *et al.*, 1992) is the lane change proportions. The control is heuristic and is derived based on four constraints: a) the vehicles should not miss their chosen exits, b) the capacity usage should be maximized, c) lane changes should not result in speed degradation and d) shorter travel times are preferred. These constraints are implemented through three control laws. The first one is intended to balance the traffic across the lanes. The second law specifies which cars must change lanes to reach their exit while maintaining the traffic balance. The third law acts to avoid significant increases in travel times. The speed on any section of the highway is guaranteed to be within certain bounds and the maximum value of the acceleration is also prescribed. It is demonstrated that, even with the use of simple control policies, it is possible to reduce the delays caused by incidents on the highway. The paper does not, however, consider the stability issues that arise when such control policies are implemented, and the results are difficult to generalize to arbitrary cases.

In (Papageorgiou *et al.*, 1990), the parameters for a discrete space traffic flow model of the Southern Boulevard Peripherique of Paris are identified. The vehicles are of course under manual control. The goal here is to stabilize the traffic to a desired density and flow. The only control is on-ramp metering. As in (Karaaslan *et al.*, 1990), the major problem in the absence of feedback control is that congestion and driver behavior prevent realization of the full highway capability. The authors linearize their model and apply a linear quadratic technique to determine the metering control, based on density and flow information at various positions on the highway. In simulations, it is shown that with feedback control, congestion is decreased and the highway is able to sustain an otherwise non realizable capacity. The control action at each on-ramp is mainly determined by the traffic condition local to the on-ramp. Hence, the control can be approximated by a distributed control.

In a previous work (Li *et al.*, 1995; Li *et al.*, 1997*b*) a link layer controller whose goal is to regulate vehicle flow and density in a stretch of highway around desired vehicle density and velocity profiles is presented. It is assumed that for each conceivable scenario (e.g. normal traffic condition, stopped vehicle on highway, blocked or closed lane), a desired behavior of the highway consistent with its capability under that circumstance can be prescribed. Three topologies are investigated: a single lane highway, a discrete lane highway and a dense lane highway. The $\mathcal{L}_2$ stability of the density error along the stretch of highway is proved. The structure of the proposed control laws is distributed and is simple to compute in real time.

This report deals mainly with traffic flow stabilization in different discrete lane highway topologies. Two important additions to the results reported in (Li *et al.*, 1995; Li *et al.*, 1997*b*) are presented here. In (Li *et al.*, 1995; Li *et al.*, 1997*b*) all vehicles were supposed to have the same final destination. The traffic control of a stretch of highway in which there are vehicles with different final destinations is now considered. The change of lane of vehicles is treated in a different way. In (Li *et al.*, 1995; Li *et al.*, 1997*b*), the desired velocity for the two lanes involved in a lane change maneuver was assumed equal. Now this constraint is relaxed and the analysis allows to command lane changes for vehicles in adjacent lanes which have different desired longitudinal velocities. The desired density and flow profiles for the vehicles directed to each destination are again assumed to be prescribed. The derivation and analysis of the multi-destination feedback control law is presented. Simulation results are included.

This report is divided in three chapters and one appendix. Chapter 2 contains the link layer controller design and chapter 3 presents the simulation results for this link layer traffic flow controller. In all cases the results were obtained in Matlab and in SmarPath (Eskafi *et al.*, 1992). Chapter 4 contains the conclusions of the work. Appendix A contains the more general case for the desing of the link layer traffic flow controller.

Chapter 2

Traffic Flow Stabilization

The goal of this chapter is to present a controller that regulates aggregate traffic conditions defined by density and velocity profiles to their appropriate values, while acting within the link layer of the PATH AHS hierarchical architecture.

The approach taken in this report to the problem of designing a control system for the link layer is significantly different from those in the literature reviewed in the introduction. Since a fully automated highway is under consideration, no a-priori behavior of the vehicles of the vehicles is assumed (i.e., there is not attempt to control mixed automated and manual traffic). An important assumption is that the closed loop dynamics of the regulation layer control system has a sufficiently high bandwidth, so that it can adequately track the reference velocity and change lane commands issued by the link layer. For this reason, the coordination and regulation layer dynamics are not included in the link layer model. To derive the link layer controller, a spatially and temporally continuous model of the highway, described by a partial differential equation and obeying only the law of conservation of vehicles, is used. The partition of the this highway model into different sections (spatial partition) as well as the sampling time of the link layer control should also be determined by the bandwidth requirements of the link layer temporal and spatial dynamics. It has to be pointed out that while the assumption on the dynamics of controlling velocity is realistic, the assumption on the change lane maneuvers, which currently take 3 to 6 sec. (Chee and Tomizuka, 1995), has to be analyzed more carefully in the future.

It is considered that for each conceivable scenario (e.g. normal traffic condition, stopped vehicle on highway, blocked or closed lane), a desired traffic condition on the highway consistent with its capability under that circumstance can be prescribed. The desired traffic condition is encoded by the pair consisting of a desired density profile $K_d(x,t)$ and a desired velocity field $V_d(x)$ such that the vehicle flow rate at different positions on the highway is given by: $\phi_d(x,t) = K_d(x,t)V_d(x)$. The desired density profile determines the desired concentration of vehicles in the highway as a function of position and time, whereas the desired velocity field specifies how cars should be maneuvered so as to maintain the desired density profile and flow rate. The specific design of $K_d(x,t)$ and $V_d(x)$ depends on the demand and capability of the highway, as well as on the presence or absence of extraordinary circumstances such as accidents or lane closures.

The link layer is assumed to have a repertoire of (K_d, V_d) pairs each of which encapsulates a strategy to deal with a particular situation on the highway. The link layer control law described below stabilizes the actual density and velocity at the desired values. The determination of the traffic condition involves some form of optimization and it is not a problem pursued in this report. In (Broucke and Varaiya, 1996) a theory that can be used to determine this repertoire of (K_d, V_d) pairs is presented.

Traffic stabilization control for three highway topologies is investigated:

Single lane highway - the highway consists of only one automated lane and the control is the longitudinal velocity of the traffic.

Discrete lane highway - the highway is modeled as a discrete set of automated lanes. The controls in this case are the longitudinal velocities and the rates of proportion of vehicles to change lanes.

Discrete lane highway with multiple destinations - the highway is modeled as in the previous case, although the final destination of vehicles is considered to specify the change lane control action.

In all cases the following assumption is considered

Assumption 2.1

1. *The dynamics of the coordination and regulation layers are sufficiently fast and are capable of achieving velocity regulation.*

2.1 One Lane Highway

Consider a one lane highway which is parameterized by $x \in [0, L]$ and time t, schematically shown in Fig. 2.1.

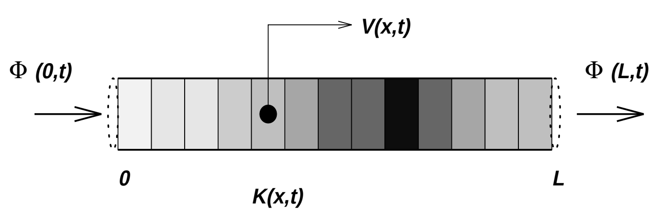

Figure 2.1: One lane model of a highway

The dynamics of the density on the highway is given by a conservation of vehicles principle (Li *et al.*, 1995):

$$\frac{\partial}{\partial t}K(x,t) = -\frac{\partial}{\partial x}\left\{K(x,t)V(x,t)\right\}. \tag{2.1}$$

The control objective is to stabilize traffic flow by commanding a velocity profile $V(x,t)$ such that, after a transient response, the density profile $K(x,t)$ is close to a desired density profile $K_d(x,t)$ and the traffic moves with a desired time-invariat velocity $V_d(x) > 0$. The specification of both, the velocity and density, determines the desired flow rate, $\phi_d(x,t) = K_d(x,t)V_d(x)$. For the one lane highway the following assumption is required.

Assumption 2.2

1. *The velocity profile $V(x,t)$ can be commanded.*
2. *The dynamics of the desired density and velocity profiles satisfies*

$$\frac{\partial}{\partial t}K_d(x,t) = -\frac{\partial}{\partial x}\left\{K_d(x,t)V_d(x)\right\} . \tag{2.2}$$

Define the density error $\tilde{K}(x,t) = K_d(x,t) - K(x,t)$ and consider the control law

$$V(x,t) = V_d(x) + V_f(x,t)\,, \tag{2.3}$$

$$V_f(x,t) = \zeta(x,t)\frac{\partial}{\partial x}\left\{V_d(x)\tilde{K}(x,t)\right\} \tag{2.4}$$

where $\zeta(x,t) \geq 0$.

The dynamics for the density error, obtained after substituting the control law in Eqs. (2.3)-(2.4) into (2.1) and using (2.2), are

$$\frac{\partial\tilde{K}(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left\{\tilde{K}(x,t)V_d(x)\right\} + \frac{\partial}{\partial x}\left\{K(x,t)V_f(x,t)\right\} \tag{2.5}$$

For $u : [0,L] \rightarrow \mathcal{R}$ a real valued function on $[0,L]$, denote the $\mathcal{L}_2$ norm by $||u||_2^2 = \int_0^L u^2 dx$. The following theorem states that with (2.3) and (2.4) as the velocity control, the desired traffic condition is stable in the $\mathcal{L}_2$ sense.

Theorem 2.1 *Consider the single lane highway in* (2.1)*. Suppose that the inlet flow rate is $\phi(0,t) = K_d(0,t)V_d(0)$, then, under assumptions 2.1 and 2.2, the control law in* (2.3) *and* (2.4) *with $\zeta(x,t) \geq 0$ and $\zeta(0,t) = \zeta(L,t) = 0$, is such that the density error $\tilde{K}(x,t) = 0\ \forall x \in [0,L]$ is $\mathcal{L}_2$ stable in time.*

Proof: Consider the following Lyapunov functional:

$$W(t) = \frac{1}{2}\int_0^L \tilde{K}(x,t)^2 V_d(x)dx. \tag{2.6}$$

Differentiating (2.6) with respect to time and using Eq. (2.5),

$$\dot{W} = \int_0^L \tilde{K}\dot{\tilde{K}}V_d dx = -\int_0^L \tilde{K}(x,t)V_d(x)\frac{\partial}{\partial x}\left\{\tilde{K}(x,t)V_d(x)\right\} + \tilde{K}(x,t)V_d(x)\frac{\partial}{\partial x}\left\{K(x,t)V_f(x,t)\right\}dx. \tag{2.7}$$

The first term in (2.7) is an exact differential:

$$\frac{1}{2}\frac{\partial}{\partial x}\left\{\tilde{K}(x,t)V_d(x)\right\}^2 = \tilde{K}(x,t)V_d(x)\frac{\partial}{\partial x}\left\{\tilde{K}(x,t)V_d(x)\right\} . \tag{2.8}$$

Using the Leibnitz rule in in the second term of (2.7) and substituting $V_f(x,t)$ this term becomes

$$\int_0^L \tilde{K}(x,t)V_d(x)\frac{\partial}{\partial x}\left\{K(x,t)V_f(x,t)\right\}dx = \frac{1}{2}\,\zeta(x,t)K(x,t)\frac{\partial}{\partial x}\left(\tilde{K}(x,t)V_d(x)\right)^2\Big|_0^L - \int_0^L \zeta(x,t)K(x,t)\left\{\frac{\partial}{\partial x}\left\{\tilde{K}(x,t)V_d(x)\right\}\right\}^2 dx\ . \tag{2.9}$$

From (2.8) and (2.9) in (2.7)

$$\dot{W}(t) = -\frac{1}{2}\left(\tilde{K}(x,t)V_d(x)\right)^2\Big|_0^L + \frac{1}{2}\,\zeta(x,t)K(x,t)\frac{\partial}{\partial x}\left(\tilde{K}(x,t)V_d(x)\right)^2\Big|_0^L - \int_0^L \zeta(x,t)K(x,t)\left\{\frac{\partial}{\partial x}\left\{\tilde{K}(x,t)V_d(x)\right\}\right\}^2 dx\ . \tag{2.10}$$

By the theorem's assumptions: $\zeta(0,t) = \zeta(L,t) = 0$ and $\phi(0,t) = V_d(0)K_d(0,t)$, $\zeta(0,t) = 0 \Rightarrow V(0,t) = V_d(0)$, and therefore $K(0,t) = K_d(0,t)$. Hence,

$$\dot{W}(t) \le -\frac{1}{2}\int_0^L \zeta(x,t)K(x,t)\left\{\frac{\partial}{\partial x}\left\{\tilde{K}(x,t)V_d(x)\right\}\right\}^2 dx \le 0\ ,$$

since the density $K(x,t) \ge 0$. Then, $W(t) \le W(0)$. Defining $\underline{V_d} = \inf_{x\in[0,L]} V_d(x)$, and $\overline{V_d} = \sup_{x\in[0,L]} V_d(x)$,

$$\underline{V_d}\ \|\tilde{K}(\cdot,t)\|_2^2 \le W(t) \le W(0) \le \overline{V_d}\ \|\tilde{K}(\cdot,0)\|_2^2.$$

Thus, for all $t \ge 0$, $\|\tilde{K}(\cdot,t)\|_2 \le \alpha\|\tilde{K}(\cdot,0)\|_2$ for $\alpha = \sqrt{\overline{V_d}/\underline{V_d}}$. $\mathcal{L}_2$ stability follows. ■

Consider the simplified case when both V_d and K_d are constants. The control becomes:

$$V(x,t) = V_d + \zeta(x,t)V_d\frac{\partial\tilde{K}(x,t)}{\partial x}.$$

Thus, if the density is higher downstream than it is upstream, $\partial\tilde{K}(x,t)/\partial t < 0$, and the control law decreases the velocity. This has the effect of preventing a pile up downstream. The control law can be interpreted as a density homogenizing law. Notice that the control law (2.4) is distributed in the sense that velocity commanded at position x is determined only by the nearby weighted density error.

2.2 Discrete Lanes Highway

Consider now a highway of length L, consisting of n lanes (in Fig. 2.2, $n = 3$). Using a principle of vehicles conservation, the dynamics of the vehicle density satisfy the following partial differential equation:

$$\frac{\partial\mathbf{K}(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left\{\mathbf{V}(x,t)\mathbf{K}(x,t)\right\} + \mathbf{N}(x,t)\mathbf{K}(x,t)\ \ , \tag{2.11}$$

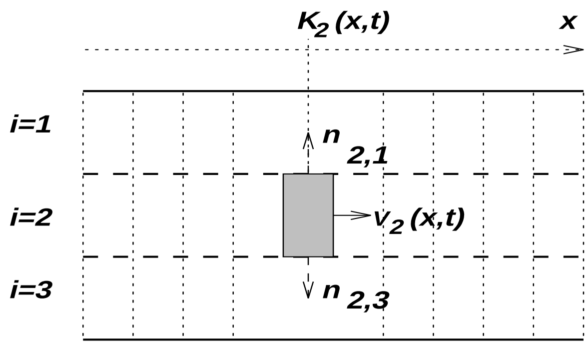

Figure 2.2: Discrete lane highway model.

where $\mathbf{K}(x,t) \in \mathcal{R}^n$, $K_i(x,t)$, the i-th element, is the vehicle density on lane i, position $x \in [0, L] \subset \mathcal{R}$ and time t, $\mathbf{V}(x,t) \in \mathcal{R}^n \times \mathcal{R}^n$ is a diagonal matrix whose i-th diagonal entry, $V_i(x,t)$, is the traffic flow velocity in lane i, and $\mathbf{N}(x,t) \in \mathcal{R}^n \times \mathcal{R}^n$ represents the proportion of vehicle density that is changing lanes per unit time at that particular position x and time t. Traffic flow stabilization for the discrete lane highway is analyzed under the following assumption.

Assumption 2.3

1. *The velocity $\mathbf{V}(x,t)$ and the proportion of lane change $\mathbf{N}(x,t)$ can be commanded.*
2. *Lane change is constrained to occur only between adjacent lanes, therefore the structure of $\mathbf{N}(x,t)$ is*

$$\mathbf{N}(x,t) = \begin{pmatrix} -n_{1,2}(x,t), & n_{2,1}(x,t), & 0, & \cdots 0 \\ n_{1,2}(x,t), & -n_{2,1}(x,t) - n_{2,3}(x,t), & n_{3,2}(x,t), & \cdots 0 \\ 0, & n_{2,3}(x,t), & -n_{3,2}(x,t) - n_{3,4}(x,t), & \cdots 0 \\ & \vdots & & \\ 0, & \cdots & n_{n-1,n}(x,t), & -n_{n,n-1}(x,t) \end{pmatrix},$$

   where $0 \leq n_{i,j}(x,t) \leq 1 \,; \forall i, j \in \{1, \cdots, n\}$ represents the proportion of vehicles changing from lane i to lane j per unit time, $n_{i,j}(x,t) = 0 \,; \forall\, |i-j| > 1$.
3. *The desired velocity, proportion of change lane and density profiles, $\mathbf{V}_d(x)$, $\mathbf{N}_d(x,t)$ and $\mathbf{K}_d(x,t)$, satisfy*

$$\frac{\partial \mathbf{K}_d(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left\{\mathbf{V}_d(x)\mathbf{K}_d(x,t)\right\} + \mathbf{N}_d(x,t)\mathbf{K}_d(x,t)\,, \tag{2.12}$$

   with $\mathbf{K}_d(x,t)$, $\mathbf{V}_d(x)$ and $\mathbf{N}_d(x,t)$ similarly defined to $\mathbf{K}(x,t)$, $\mathbf{V}(x,t)$ and $\mathbf{N}(x,t)$, respectively. The desired velocity can be different between lanes.
4. *Only net changes of lane should be considered when specifying the matrix $\mathbf{N}_d(x,t)$, that is*

$$\textit{if} \;\; n_{d_{i,j}}(x,t) \neq 0 \Rightarrow n_{d_{j,i}}(x,t) = 0 \,; \quad |i-j| = 1 \;\; ; \forall i, j \in \{1, \cdots, n\}.$$

Define the density error vector as

$$\tilde{\mathbf{K}}(x,t) = \mathbf{K}_d(x,t) - \mathbf{K}(x,t)\ , \tag{2.13}$$

and decompose $\mathbf{V}(x,t)$ and $\mathbf{N}(x,t)$ as

$$\mathbf{V}(x,t) = \mathbf{V}_d(x) + \mathbf{V}_f(x,t)\ , \tag{2.14}$$
$$\mathbf{N}(x,t) = \mathbf{N}_d(x,t) + \mathbf{N}_f(x,t)\ . \tag{2.15}$$

By subtracting Eqs. (2.12) and (2.11), the equation for the dynamics of the density error is

$$\begin{aligned}\frac{\partial \tilde{\mathbf{K}}(x,t)}{\partial t} =& -\frac{\partial}{\partial x}\left\{\mathbf{V}_d(x)\tilde{\mathbf{K}}(x,t)\right\} + \mathbf{N}_d(x)\tilde{\mathbf{K}}(x,t)\\ &+ \frac{\partial}{\partial x}\left\{\mathbf{V}_f(x,t)\mathbf{K}(x,t)\right\} - \mathbf{N}_f(x,t)\mathbf{K}(x,t)\ .\end{aligned} \tag{2.16}$$

Define the feedback control law $\mathbf{V}_f(x,t)$ by

$$\mathbf{V}_f(x,t) = \gamma(x,t)\,\mathrm{diag}\left\{\frac{\partial}{\partial x}\{\mathbf{V}_d(x)\tilde{\mathbf{K}}(x,t)\}\right\}\ , \tag{2.17}$$

where $\gamma(x,t) \geq 0$ is a gain with $\gamma(0,t) = \gamma_i(L,t) = 0$.

The matrix $\mathbf{N}_f(x,t)$ has the same structure as $\mathbf{N}(x,t)$ and its elements are defined with the same sign convention. The (i,j)-element of the feedback control law matrix $\mathbf{N}_f(x,t)$ is defined by

$$n_{f_{i,j}}(x,t) = \begin{cases} \zeta_{i,j}(x,t)(\tilde{K}_i(x,t)V_{d_i}(x) - \tilde{K}_j(x,t)V_{d_j}(x)); & |i-j| = 1 \\ & \tilde{K}_i(x,t)V_{d_i}(x) < \tilde{K}_j(x,t)V_{d_j}(x)\ , \\ 0\,; & \text{else} \end{cases} \tag{2.18}$$

where $\zeta_{i,j}(x,t) \geq 0$ is a gain.

Denote the $\mathcal{L}_2$ norm of the density error vector $\tilde{\mathbf{K}}(\cdot,t)$ to be

$$\|\tilde{\mathbf{K}}(\cdot,t)\|_2^2 = \int_0^L \tilde{\mathbf{K}}(x,t)^T\tilde{\mathbf{K}}(x,t)dx\ \ .$$

The following theorem is presented.

Theorem 2.2 *Consider the discrete n-lane highway model in Eq. (2.11). Suppose the inlet flow condition is such that $\tilde{\mathbf{K}}(0,t) = 0$. Then, under assumptions 2.1 and 2.3, the control laws in Eqs. (2.17)-(2.18) are such that the equilibrium $\tilde{\mathbf{K}}(x,t) = 0\ ; \forall x \in [0,L]$ is stable in the $\mathcal{L}_2$ sense.*

Proof: Choose the following Lyapunov function candidate

$$U(t) = \tfrac{1}{2}\int_0^L \tilde{\mathbf{K}}(x,t)^T\mathbf{V}_d(x)\tilde{\mathbf{K}}(x,t)\ dx\ . \tag{2.19}$$

First notice that the argument in (2.19) is positive definite as $\mathbf{V}_d(x)$ is always positive definite.

Taking the time derivative of Eq. (2.19)

$$\dot{U}(t) = \int_0^L \tilde{\mathbf{K}}(x,t)^T \mathbf{V}_d(x) \frac{\partial \tilde{\mathbf{K}}(x,t)}{\partial t} dx \, .$$

Using Eq. (2.16)

$$\dot{U}(t) = -\int_0^L \tilde{\mathbf{K}}(x,t)^T \mathbf{V}_d(x) \frac{\partial}{\partial x} \{\mathbf{V}_d(x) \tilde{\mathbf{K}}(x,t)\} dx + \int_0^L \tilde{\mathbf{K}}(x,t)^T \mathbf{V}_d(x) \frac{\partial}{\partial x} \{\mathbf{V}_f(x,t) \mathbf{K}(x,t)\} \, dx$$
$$+ \int_0^L \tilde{\mathbf{K}}(x,t)^T \mathbf{V}_d(x) \mathbf{N}_d(x,t) \tilde{\mathbf{K}}(x,t) \, dx - \int_0^L \tilde{\mathbf{K}}(x,t)^T \mathbf{V}_d(x) \mathbf{N}_f(x,t) \mathbf{K}(x,t) \, dx \, . \tag{2.20}$$

The first integral in Eq. (2.20) is an exact differential and the second can be rewritten using Lebnitz's rule. Thus,

$$\dot{U}(t) = - \left. \tilde{\mathbf{K}}(x,t)^T \mathbf{V}_d(x) \mathbf{V}_d(x) \tilde{\mathbf{K}}(x,t) \right|_0^L - \int_0^L \frac{\partial}{\partial x} \left\{ \tilde{\mathbf{K}}(x,t)^T \mathbf{V}_d(x) \right\} \mathbf{V}_f(x,t) \mathbf{K}(x,t) \, dx$$
$$+ \left. \tilde{\mathbf{K}}(x,t)^T \mathbf{V}_d(x) \mathbf{V}_f(x,t) \mathbf{K}(x,t) \right|_0^L + \int_0^L \tilde{\mathbf{K}}(x,t)^T \mathbf{V}_d(x) \mathbf{N}_d(x,t) \tilde{\mathbf{K}}(x,t) \, dx$$
$$- \int_0^L \tilde{\mathbf{K}}(x,t)^T \mathbf{V}_d(x) \mathbf{N}_f(x,t) \mathbf{K}(x,t) \, dx \, . \tag{2.21}$$

Recall that, by theorem assumption, $\tilde{\mathbf{K}}(0,t) = 0$. Select matrices $\mathbf{V}_f(x,t)$ and $\mathbf{N}_f(x,t)$ according to Eqs (2.17)-(2.18). Then Eq. (2.21) becomes

$$\dot{U}(t) \leq \int_0^L \tilde{\mathbf{K}}(x,t)^T \mathbf{V}_d(x) \mathbf{N}_d(x,t) \tilde{\mathbf{K}}(x,t) \, dx \tag{2.22}$$

Assumption 2.3 establishes that $\mathbf{N}_d(x,t)$ has a tri-diagonal structure and requires to specify in it only net changes of lane. Therefore, to analyze the term inside the integral in Eq. (2.22) it is possible to take separately any two pair of adjacent lanes i and j with $|i-j| = 1$. According to this Eq. (2.22) can be rewriten as

$$\begin{aligned} \dot{U}(t) &\leq \sum_{i=1}^{n} \int_0^L R_{i,j}(x,t) \, dx \\ &= \sum_{i=1}^{n} \int_0^L -n_{d_{i,j}}(x,t) \left(\tilde{K}_i(x,t) V_{d_i}(x) - \tilde{K}_j(x,t) V_{d_j}(x) \right) \tilde{K}_i(x,t) \, dx \, , \end{aligned} \tag{2.23}$$

where $|i-j| = 1$ and, without loss of generality, it is assumed that $n_{d_{i,j}}(x,t) > 0$ and $n_{d_{j,i}}(x,t) = 0$.

There are six possible combinations for the sign of $R_{i,j}(x,t)$ in Eq. (2.23) that depend on the signs of $\tilde{K}_i(x,t)$, $\tilde{K}_j(x,t)$ and the sign of the argument $\left(\tilde{K}_i(x,t) V_{d_i}(x) - \tilde{K}_j(x,t) V_{d_j}(x) \right)$. These combinations are:

1. If $\tilde{K}_i(x,t) = 0$ or $\tilde{K}_j(x,t) = 0 \Rightarrow R_{i,j}(x,t) \leq 0$.
2. If $\mathrm{sign}(\tilde{K}_i(x,t)) \neq \mathrm{sign}(\tilde{K}_j(x,t)) \Rightarrow R_{i,j}(x,t) \leq 0$.
3. If $\tilde{K}_i(x,t), \tilde{K}_j(x,t) > 0$ and $\tilde{K}_i(x,t)V_{d_i}(x) > \tilde{K}_j(x,t)V_{d_j}(x) \Rightarrow R_{i,j}(x,t) \leq 0$
4. If $\tilde{K}_i(x,t), \tilde{K}_j(x,t) > 0$ and $\tilde{K}_i(x,t)V_{d_i}(x) < \tilde{K}_j(x,t)V_{d_j}(x) \Rightarrow R_{i,j}(x,t) \geq 0$
5. If $\tilde{K}_i(x,t), \tilde{K}_j(x,t) < 0$ and $\tilde{K}_i(x,t)V_{d_i}(x) < \tilde{K}_j(x,t)V_{d_j}(x) \Rightarrow R_{i,j}(x,t) \leq 0$
6. If $\tilde{K}_i(x,t), \tilde{K}_j(x,t) < 0$ and $\tilde{K}_i(x,t)V_{d_i}(x) > \tilde{K}_j(x,t)V_{d_j}(x) \Rightarrow R_{i,j}(x,t) \geq 0$

Notice that $\dot{U}(t) \leq 0$ for all cases with the exception of items 4 and 6 in the previous list[1]. To show $\mathcal{L}_2$ stability for $\tilde{\mathbf{K}}(\cdot,t) = 0$ for those two cases define L_4 and L_6 to be the union of all the segments of the highway length $[0, L]$ where conditions 4 and 6 hold, respectively[2].

Case 4**,** $\tilde{K}_i(x,t), \tilde{K}_j(x,t) > 0$ **and** $\tilde{K}_i(x,t)V_{d_i}(x) < \tilde{K}_j(x,t)V_{d_j}(x)$

Define $U_{i,j}(t)|_{L_4}(t) = U_i(t)|_{L_4} + U_j(t)|_{L_4}$ as the components of $U(t)$ related with lanes i and j, respectively; $|i-j| = 1$, evaluated for the segments in the set L_4. From Eq. (2.23)

$$
\begin{aligned}
\dot{U}_{i,j}(t)|_{L_4} \leq& - \int_{L_4} n_{d_{i,j}}(x,t)\left(\tilde{K}_i(x,t)V_{d_i}(x) - \tilde{K}_j(x,t)V_{d_j}(x)\right)\tilde{K}_i(x,t)\,dx \\
=& - \int_{L_4} \tilde{K}_i(x,t)^2 V_{d_i}(x) n_{d_{i,j}}(x,t)\,dx - \int_{L_4} \tilde{K}_j(x,t)^2 V_{d_j}(x) n_{d_{i,j}}(x,t)\,dx \\
&+ \int_{L_4} \tilde{K}_j(x,t)^2 V_{d_j}(x) n_{d_{i,j}}(x,t)\,dx + \int_{L_4} \tilde{K}_i(x,t)\tilde{K}_j(x,t) V_{d_j}(x) n_{d_{i,j}}(x,t)\,dx \\
\leq& - 2\underline{n}_{d_{i,j}}(t) U_{i,j}(t)|_{L_4} + \int_{L_4} \tilde{K}_j(x,t)(\tilde{K}_i(x,t) + \tilde{K}_j(x,t)) V_{d_j}(x) n_{d_{i,j}}(x,t)\,dx
\end{aligned} \tag{2.24}
$$

where

$$
\underline{n}_{d_{i,j}}(t) = \inf_{x \in L_4}\{n_{d_{i,j}}(x,t)\}
$$

The desired density is always prescribed to be bounded, i.e., $0 \leq K_{d_i}(x,t), K_{d_j}(x,t) < M < \infty$. As $\tilde{K}_i(x,t), \tilde{K}_j(x,t) > 0$, this implies that $\tilde{K}_i(x,t), \tilde{K}_j(x,t) < M$.

Define

$$
\begin{aligned}
\bar{V}_{d_j} &= \sup_{x \in L_4}\{V_{d_j}(x)\} \\
\bar{n}_{d_{i,j}}(t) &= \sup_{x \in L_4}\{n_{d_{i,j}}(x,t)\}
\end{aligned}
$$

[1]The trivial case when $n_{d_{i,j}}(x,t) = n_{d_{j,i}}(x,t) = 0$ is not considered in the analysis. It does not present any problem from the stability point of view.

[2]Eqs. (2.11)-(2.12) imply that variables in them are continuos and differentiable with respect to time and position. Therefore segments in L_4 and L_6 are well defined.

Then

$$\dot{U}_{i,j}(t)|_{L_4} \leq -\, 2\underline{n}_{d_{i,j}}(t) U_{i,j}(t)|_{L_4} + 2M^2 \bar{V}_{d_j} \bar{n}_{d_{i,j}}(t) \left(\int_{L_4} dx \right)$$
$$\leq -\, 2\underline{n}_{d_{i,j}}(t) U_{L_4}(t) + \alpha \,, \tag{2.25}$$

where

$$\alpha = 2M^2 \bar{V}_{d_j} \bar{n}_{d_{i,j}}(t) \left(\int_{L_4} dx \right)$$

Eq. (2.25) implies that $U_{i,j}(t)|_{L_4}$ can at most grow to the point where

$$U_{i,j}(t)|_{L_4} = \frac{\alpha}{2\underline{n}_{d_{i,j}}(t)} = \frac{M^2 \bar{V}_{d_j} \bar{n}_{d_{i,j}}(t) \left(\int_{L_4} dx \right)}{\underline{n}_{d_{i,j}}(t)} \tag{2.26}$$

and therefore $\mathcal{L}_2$ stability for $\tilde{\mathbf{K}}(\cdot, t) = 0$ follows for this case.

To lower the bound in Eq. (2.26) it is possible to combine Eq. (2.23) and the term contributed by $\mathbf{N}_f(x,t)$. This yields

$$\dot{U}_{i,j}(t)|_{L_4} \leq -\int_{L_4} n_{d_{i,j}}(x,t) \left(\tilde{K}_i(x,t) V_{d_i}(x) - \tilde{K}_j(x,t) V_{d_j}(x) \right) \tilde{K}_i(x,t)$$
$$- \left(\tilde{K}_i(x,t) V_{d_i}(x) - \tilde{K}_j(x,t) V_{d_j}(x) \right)^2 \zeta_{i,j}(x,t) K_i(x,t) \, dx \,. \tag{2.27}$$

A sufficent condition for the argument inside the integral of Eq. (2.27) to be negative semidefinite is

$$\left(\tilde{K}_j(x,t) V_{d_j}(x) - \tilde{K}_i(x,t) V_{d_i}(x) \right) \zeta_{i,j}(x,t) K_i(x,t) \geq n_{d_{i,j}}(x,t) \tilde{K}_i(x,t) \,. \tag{2.28}$$

Therefore using an high enough gain, $\zeta_{i,j}(x,t)$, will suffice to satisfy condition (2.28), with the exception of the cases where $K_i(x,t)$ is small. In particular if $K_i(x,t) = 0$ the lateral control has no ability to compensate the first term inside the integral in Eq. (2.27). Notice, however, that in this case of small $K_i(x,t)$ the value of $\tilde{K}_i(x,t)$, $\tilde{K}_j(x,t)$ is bounded by M. This leads to assure that

$$\tilde{K}_i(x,t), \tilde{K}_j(x,t) < M \;\; ; \forall t \geq 0 \,.$$

Case 6**,** $\tilde{K}_i(x,t), \tilde{K}_j(x,t) < 0$ **and** $\tilde{K}_i(x,t) V_{d_i}(x) > \tilde{K}_j(x,t) V_{d_j}(x)$

The time derivative of the i-th component in Eq. (2.23) is

$$\dot{U}_i(t)|_{L_6} \leq -\int_{L_6} n_{d_{i,j}}(x,t) \tilde{K}_i(x,t)^2 V_{d_i}(x) \, dx \leq -2\underline{n}_{d_{i,j}}(t) U_i(t)|_{L_6} \,, \tag{2.29}$$

where

$$\underline{n}_{d_{i,j}}(t) = \inf_{x\in L_6} \{n_{d_{i,j}}(x,t)\}\,.$$

Eq. (2.29) implies that if $\tilde{K}_i(x,t)$ is initially bounded, it will remain bounded, i.e.,

$$|\tilde{K}_i(x,t)| < M < \infty;\ \ \forall t\,, x \in L_6. \tag{2.30}$$

Similarly, the time derivative of the j-th component in Eq. (2.23) is

$$\dot{U}_j(t)|_{L_6} \leq \int_{L_6} n_{d_{i,j}}(x,t)|\tilde{K}_i(x,t)||\tilde{K}_j(x,t)|V_{d_j}(x)\,dx\,.$$

Using Eq. (2.30)

$$\begin{aligned}\dot{U}_j(t)|_{L_6} \leq& \bar{V}_{d_j}\bar{n}_{d_{i,j}}(t)M\int_{L_6}|\tilde{K}_j(x,t)|\,dx\\ \leq& \bar{V}_{d_j}\bar{n}_{d_{i,j}}(t)M\left(\int_{L_6}dx\right)\left(\int_{L_6}|\tilde{K}_j(x,t)|^2\,dx\right)^{\frac{1}{2}}\end{aligned} \tag{2.31}$$

From the Lyapunov function in Eq. (2.19) it follows that

$$2U_j(t)|_{L_6} = \int_{L_6}\tilde{K}_j(x,t)^2V_{d_j}(x)\,dx \geq \underline{V}_{d_j}\int_{L_6}|\tilde{K}_j(x,t)|^2\,dx\,, \tag{2.32}$$

where

$$\underline{V}_{d_j} = \inf_{x\in L_6}\{V_{d_j}(x)\}\,,$$

From Eq. (2.32) into Eq. (2.31)

$$\begin{aligned}\dot{U}_j(t)|_{L_6} \leq& \frac{\sqrt{2}}{\underline{V}_{d_j}^{\frac{1}{2}}}\bar{V}_{d_j}\bar{n}_{d_{i,j}}(t)M\left(\int_{L_6}dx\right)U_j(t)^{\frac{1}{2}}|_{L_6}\\ \leq& \beta\,U_j(t)^{\frac{1}{2}}|_{L_6}\end{aligned} \tag{2.33}$$

where

$$\beta = \frac{\sqrt{2}}{\underline{V}_{d_j}^{\frac{1}{2}}}\bar{V}_{d_j}\bar{n}_{d_{i,j}}(t)M\left(\int_{L_6}dx\right)$$

From Eqs. (2.29)-(2.33) it follows that

$$\dot{U}_{i,j}(t)|_{L_6} \leq -\,2\underline{n}_{d_{i,j}}(t)U_i(t)|_{L_6} + \beta\,U_j(t)^{\frac{1}{2}}|_{L_6} \tag{2.34}$$

Recall that

$$\frac{d}{dt}\left(U(t)^{\frac{1}{2}}\right) = \frac{1}{2}U(t)^{-\frac{1}{2}}\dot{U}(t)\,. \tag{2.35}$$

From Eq. (2.34) into Eq. (2.35)

$$\frac{d}{dt}\left(U_{i,j}(t)^{\frac{1}{2}}|_{L6}\right) \leq -\frac{\underline{n}_{d_{i,j}}(t)\; U_i(t)|_{L6}}{U_{i,j}(t)^{\frac{1}{2}}|_{L6}} + \frac{\beta\; U_j(t)^{\frac{1}{2}}|_{L6}}{2\; U_{i,j}(t)^{\frac{1}{2}}|_{L6}}$$
$$\leq -\,\underline{n}_{d_{i,j}}(t)\; U_i(t)^{\frac{1}{2}}|_{L6} + \frac{\beta}{2} \tag{2.36}$$

From Eq. (2.36) it follows than $U_{i,j}(t)|_{L6}$ can grow only to the point where

$$U_{i,j}(t)|_{L6} \leq \left(\frac{\beta}{2\underline{n}_{d_{i,j}}(t)}\right)^2 . \tag{2.37}$$

Eqs. (2.30)-(2.37) imply that $|\tilde{K}_j(x,t)|$ is also bounded and therefore, $\mathcal{L}_2$ stability for $\tilde{\mathbf{K}}(\cdot,t) = 0$ hold also for this case.

To lower the bound in Eq. (2.37) in this case it is possible to combine again Eq. (2.23) and the term contributed by $\mathbf{N}_f(x,t)$. This yields to an equivalent condition to inequality (2.28) that is

$$\left(\tilde{K}_i(x,t)V_{d_i}(x) - \tilde{K}_j(x,t)V_{d_j}(x)\right)\zeta_{j,i}(x,t)K_j(x,t) \geq -n_{d_{i,j}}(x,t)\tilde{K}_i(x,t)\,. \tag{2.38}$$

Assumptions for case 6 consider $\tilde{K}_i(x,t), \tilde{K}_j(x,t) < 0$; this implies $K_j(x,t) > 0$. Therefore the gain $\zeta_{j,i}(x,t)$ can be always chosen to guarantee inequality (2.38). This in turn will guarantee that

$$\tilde{K}_i(x,t) \leq \tilde{K}_i(x,0)\ \ ;\forall t \geq 0\,,$$
$$\tilde{K}_j(x,t) \leq \tilde{K}_j(x,0)\ \ ;\forall t \geq 0\,.$$

■

Remark:

1. The control laws (2.4) and (2.17) are similar. For the single lane case, the longitudinal control feedback term in (2.4), V_f, is given by a partial derivative of a weighted density error, $V_d(x)\tilde{K}(x,t)$. For the discrete lanes case, the longitudinal control feedback term in (2.17) is also a partial derivative of a weighted density error, $\tilde{\mathbf{K}}(x,t)^T\mathbf{V}_d(x)$, that reduces to $\partial/\partial x\{V_d(x)\tilde{K}(x,t)\}$ when $n = 1$. The feedback control for lane change $\mathbf{N}_f(x,t)$ is done by comparing the weighted errors in the adjacent lanes. This control can be interpreted as the gradient of $\tilde{\mathbf{K}}(x,t)^T\mathbf{V}_d(x)$ in the transverse direction. Similarly to the single lane case, the control for the discrete lanes highway is distributed: it requires only traffic information near the particular longitudinal displacement along the highway.
2. Theorem 2.2 allows to change lane even when the adjacent lanes have different desired velocities. This was not the case in (Li *et al.*, 1995; Li *et al.*, 1997*b*), where a connectivity constraint was necessary. This connectivity constraint was introduced to allow a diagonal structure in a matrix $\mathbf{A}(x)$ that performed a change of coordinates. Commutation of the product of diagonal matrices was used to guarantee a diagonal structure in the matrix $\mathbf{P}(x) = \mathbf{A}(x)\Sigma(x)\mathbf{A}^{-1}(x)$, where $\Sigma(x)$ was a diagonal matrix expressing the connectivity constraint. In contrast, in Theorem 2.2 the structure of the matrix $\mathbf{N}_f(x,t)$ is used to concluded on the stability of the control laws.

3. When the real number of vehicles in the stretch of highway does not correspond with the desired value, the time derivative of the Lyapunov function in Eq. (2.21 will become zero when the gradient of the weighted density error vanishes in both the longitudinal and lateral directions, that is when

$$\frac{\partial}{\partial x}\{\mathbf{V}_d(x)\tilde{\mathbf{K}}(x,t)\} = 0\,,$$

and

$$\left(\tilde{K}_j(x,t)V_{d_j}(x) - \tilde{K}_i(x,t)V_{d_i}(x)\right) = 0 \quad \forall|i-j| = 1;$$

In this sense, the control laws proposed in Eqs. (2.17)-(2.18) are gradient homogenizing control laws, i.e., they tend to distribute the error evenly along the highway.

2.3 Multi-destination Traffic Discrete Lanes Highway

Consider a n-lane highway in which vehicles with different destination are sharing the lanes. Vehicles with the same final destination can be associated with a particular index or color. Using a principle of vehicles conservation, the dynamics of the vehicle density for each color satisfy the following partial differential equation:

$$\frac{\partial \mathbf{K}^c(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left\{\mathbf{V}^c(x,t)\mathbf{K}^c(x,t)\right\} + \mathbf{N}^c(x,t)\mathbf{K}^c(x,t)\ , \tag{2.39}$$

where $\mathbf{K}^c(x,t) \in \mathcal{R}^n$, $K_i^c(x,t)$, the i-th element, is the vehicle density of color c on lane i, position $x \in [0, L] \subset \mathcal{R}$ and time t, $\mathbf{V}^c(x,t) \in \mathcal{R}^n \times \mathcal{R}^n$ is a diagonal matrix whose i-th diagonal entry is the traffic flow velocity of color c in lane i, and $\mathbf{N}^c(x,t) \in \mathcal{R}^n \times \mathcal{R}^n$ represents the proportion of vehicle density of color c that is changing lanes per unit time at that particular position x and time t. Traffic flow stabilization for the multi-destination discrete lane highway is analyzed under the following assumption.

Assumption 2.4

1. *The velocity $\mathbf{V}^c(x,t)$ and the proportion of lane change $\mathbf{N}^c(x,t)$ can be commanded.*
2. *Lane change is constrained to occur only between adjacent lanes, therefore the structure of $\mathbf{N}^c(x,t)$ is*

$$\mathbf{N}^c(x,t) = \begin{pmatrix} -n^c_{1,2}(x,t) & n^c_{2,1}(x,t) & 0 & \cdots \\ n^c_{1,2}(x,t) & -n^c_{2,1}(x,t) - n^c_{2,3}(x,t) & n^c_{3,2}(x,t) & \cdots 0 \\ 0 & n^c_{2,3}(x,t) & -n^c_{3,2}(x,t) - n^c_{3,4}(x,t) & \cdots 0 \\ & \vdots & & \\ 0 & \cdots & n^c_{n-1,n}(x,t) & -n^c_{n,n-1}(x,t) \end{pmatrix},$$

where $0 \leq n^c_{i,j}(x,t) \leq 1\ ; \forall i,j \in \{1,\cdots,n\}$ represents the proportion of vehicles changing from lane i to lane j per unit time. Notice that $n^c_{i,j}(x,t) = 0\ ; \forall\ |i-j| > 1$.

3. *The desired velocity, proportion of change lane and density profiles,* $\mathbf{V}_d^c(x)$, $\mathbf{N}_d^c(x)$ *and* $\mathbf{K}_d^c(x,t)$, *satisfy*

$$\frac{\partial \mathbf{K}_d^c(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left\{\mathbf{V}_d^c(x)\mathbf{K}_d^c(x,t)\right\} + \mathbf{N}_d^c(x,t)\mathbf{K}_d^c(x,t)\ , \tag{2.40}$$

with $\mathbf{K}_d^c(x,t)$, $\mathbf{V}_d^c(x)$ *and* $N_d^c(x,t)$ *similarly defined to* $\mathbf{K}^c(x,t)$, $\mathbf{V}^c(x,t)$ *and* $N^c(x,t)$, *respectively.*

4. *Only net changes of lane are considered when specifying the matrix* $\mathbf{N}_d^c(x,t)$, *that is*

$$\textit{if}\ \ n_{d_{i,j}}^c(x,t) \neq 0 \Rightarrow n_{d_{j,i}}^c(x,t) = 0\ ;\quad |i-j| = 1\ \ ;\forall i,j \in \{1,\cdots,n\}.$$

Define the vectors and matrices

$$\begin{aligned}
\mathbf{K}_\omega(x,t) &= \left[\mathbf{K}^1(x,t)^T,\cdots,\mathbf{K}^m(x,t)^T\right]^T\ ,\\
\mathbf{V}_\omega(x,t) &= \ \text{diag}\ \left\{\mathbf{V}^1(x,t),\cdots,\mathbf{V}^m(x,t)\right\}\ ,\\
\mathbf{N}_\omega(x,t) &= \ \text{diag}\ \left\{\mathbf{N}^1(x,t),\cdots,\mathbf{N}^m(x,t)\right\}\ ,\\
\mathbf{K}_{\omega_d}(x,t) &= \left[\mathbf{K}_d^1(x,t)^T,\cdots,\mathbf{K}_d^m(x,t)^T\right]^T\ ,\\
\mathbf{V}_{\omega_d}(x) &= \ \text{diag}\ \left\{\mathbf{V}_d^1(x),\cdots,\mathbf{V}_d^m(x)\right\}\ ,\\
\mathbf{N}_{\omega_d}(x,t) &= \ \text{diag}\ \left\{\mathbf{N}_d^1(x,t),\cdots,\mathbf{N}_d^m(x,t)\right\}\ ,
\end{aligned}$$

where m is the total number of different destinations. The vehicle density dynamics for all colors can be expressed as

$$\frac{\partial \mathbf{K}_\omega(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left\{\mathbf{V}_\omega(x,t)\mathbf{K}_\omega(x,t)\right\} + \mathbf{N}_\omega(x,t)\mathbf{K}_\omega(x,t)\ , \tag{2.41}$$

and the desired velocity, proportion of change lane and density profiles for all color as,

$$\frac{\partial \mathbf{K}_{\omega_d}(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left\{\mathbf{V}_{\omega_d}(x)\mathbf{K}_{\omega_d}(x,t)\right\} + \mathbf{N}_{\omega_d}(x,t)\mathbf{K}_{\omega_d}(x,t)\ . \tag{2.42}$$

Define the density error vector as

$$\tilde{\mathbf{K}}_\omega(x,t) = \mathbf{K}_{\omega_d}(x,t) - \mathbf{K}_\omega(x,t)\ , \tag{2.43}$$

and decompose $\mathbf{V}_\omega(x,t)$ and $\mathbf{N}_\omega(x,t)$ as

$$\mathbf{V}_\omega(x,t) = \mathbf{V}_{\omega_d}(x) + \mathbf{V}_{\omega_f}(x,t)\ , \tag{2.44}$$

$$\mathbf{N}_\omega(x,t) = \mathbf{N}_{\omega_d}(x,t) + \mathbf{N}_{\omega_f}(x,t)\ . \tag{2.45}$$

By subtracting Eqs. (2.42) and (2.41) and using (2.44) and (2.45) the equation for the dynamics of the density error is

$$\begin{aligned}\frac{\partial \tilde{\mathbf{K}}_\omega(x,t)}{\partial t} =& -\frac{\partial}{\partial x}\left\{\mathbf{V}_{\omega_d}(x)\tilde{\mathbf{K}}_\omega(x,t)\right\} + \mathbf{N}_{\omega_d}(x)\tilde{\mathbf{K}}_\omega(x,t) \\ &+ \frac{\partial}{\partial x}\left\{\mathbf{V}_{\omega_f}(x,t)\mathbf{K}_\omega(x,t)\right\} - \mathbf{N}_{\omega_f}(x,t)\mathbf{K}_\omega(x,t)\ . \end{aligned} \tag{2.46}$$

To deal with the special conditions imposed by multi-destination traffic control, the following additional assumption is introduced.

Assumption 2.5

1. *The velocity that is commanded to vehicles that are sharing the same lane is independent of their destination, i.e.,*

$$\mathbf{V}(x,t) = \mathbf{V}^1(x,t) = \mathbf{V}^2(x,t) = \cdots = \mathbf{V}^m(x,t)\ .$$

2. *It is allowed to command different proportions of lane changes to vehicles with different destinations that are sharing the same lane, i.e., in general*

$$\mathbf{N}^1(x,t) \neq \mathbf{N}^2(x,t) \neq \cdots \neq \mathbf{N}^m(x,t)$$

Notice that assumption in 2.5.1 is important not only for practical reasons, but also from a safety point of view. It is necessary to avoid that vehicles with different destination are trying to overrun each other in the same lane.

To define the feedback control laws $\mathbf{V}_f(x,t)$ and $\mathbf{N}_f(x,t)$ first define the vectors

$$\begin{aligned}\mathbf{F}_\omega(x,t) &= [\mathbf{F}^1(x,t), \cdots, \mathbf{F}^B(x,t)] \\ &= \frac{\partial}{\partial x}\left\{\tilde{\mathbf{K}}_\omega(x,t)^T\mathbf{V}_{\omega_d}(x)\right\}\ ,\end{aligned} \tag{2.47}$$

and

$$\begin{aligned}\mathbf{H}_\omega(x,t) =& [\mathbf{H}^1(x,t), \cdots, \mathbf{H}^m(x,t)] \\ =& \tilde{\mathbf{K}}_\omega(x,t)^T\mathbf{V}_{\omega_d}(x)\ .\end{aligned} \tag{2.48}$$

The i-th element of the diagonal matrix $\mathbf{V}_{\omega_f}(x,t)$ is given by

$$V_{\omega_{f_i}}(x,t) = \gamma_i(x,t)\left[F_i^1(x,t)K_i^1(x,t) + \cdots + F_i^m(x,t)K_i^m(x,t)\right] \tag{2.49}$$

where $\gamma_i(x,t) \geq 0$ is a gain with $\gamma_i(0,t) = \gamma_i(L,t) = 0$ and $F_i^c(x,t)$ is the i-th element of $\mathbf{F}^c(x,t)$ in Eq. (2.47).

In the case of the matrix $\mathbf{N}_{\omega_f}(x,t)$, from (2.45) it follows that its structure is

$$\mathbf{N}_{\omega_f}(x,t) = \text{ diag }\left\{\mathbf{N}_f^1(x,t), \cdots, \mathbf{N}_f^m(x,t)\right\}\ .$$

According with assumption 2.5, the matrices $\mathbf{N}^c_f(x,t)$ must have the same structure of $\mathbf{N}^c_d(x,t)$;$\forall c \in \{1,\cdots,m\}$ to allow changes of lane only between adjacent lanes. Their (i,j)-element is defined as

$$n^c_{f_{i,j}}(x,t) = \begin{cases} -\zeta^c_{i,j}(x,t)(h^c_i(x,t) - h^c_j(x,t)); & |i-j| = 1\ , \\ & h^c_i(x,t) > h^c_j(x,t) \\ 0 & \text{else} \end{cases} \tag{2.50}$$

where $\zeta^c_{i,j}(x,t) \geq 0$ is a gain and $h^c_j(x,t)$ is the j-th element of $\mathbf{H}^c(x,t)$ in Eq. (2.48).

Denote the $\mathcal{L}_2$ norm of the density error vector $\tilde{\mathbf{K}}_\omega(\cdot,t)$ to be

$$\|\tilde{\mathbf{K}}_\omega(\cdot,t)\|_2^2 = \int_0^L \tilde{\mathbf{K}}_\omega(x,t)^T\tilde{\mathbf{K}}_\omega(x,t)dx$$

The main result of this chapter is now presented.

Theorem 2.3 *Consider the discrete n-lane highway model in Eq. (2.41). Suppose the desired highway conditions satisfy Eq. (2.42). Suppose the inlet flow condition is such that $\tilde{\mathbf{K}}_\omega(0,t) = 0$. Then, under assumptions 2.1, 2.4 and 2.5, the control law in Eqs. (2.49)-(2.50) is such that the equilibrium $\tilde{\mathbf{K}}_\omega(x,t) = 0$;$\forall x \in [0,L]$ is $\mathcal{L}_2$ stable.*

Proof: Choose the following Lyapunov function candidate

$$U_\omega(t) = \tfrac{1}{2}\int_0^L \tilde{\mathbf{K}}_\omega(x,t)^T\mathbf{V}_{\omega_d}(x)\tilde{\mathbf{K}}_\omega(x,t)\ dx\ . \tag{2.51}$$

Taking the time derivative of Eq. (2.51)

$$\dot{U}_\omega(t) = \int_0^L \tilde{\mathbf{K}}_\omega(x,t)\mathbf{V}_{\omega_d}(x)\frac{\partial\tilde{\mathbf{K}}_\omega(x,t))}{\partial t}dx\ .$$

Using Eq. (2.46)

$$\begin{aligned}\dot{U}_\omega(t) = &-\int_0^L \tilde{\mathbf{K}}_\omega(x,t)^T\mathbf{V}_{\omega_d}(x)\frac{\partial}{\partial x}\{\mathbf{V}_{\omega_d}(x)\tilde{\mathbf{K}}_\omega(x,t)\}dx \\ &+\int_0^L \tilde{\mathbf{K}}_\omega(x,t)^T\mathbf{V}_{\omega_d}(x)\frac{\partial}{\partial x}\left\{\mathbf{V}_{\omega_f}(x,t)\mathbf{K}_\omega(x,t)\right\}dx \\ &+\int_0^L \tilde{\mathbf{K}}_\omega(x,t)^T\mathbf{V}_{\omega_d}(x)\mathbf{N}_{\omega_d}(x,t)\tilde{\mathbf{K}}_\omega(x,t)\ dx \\ &-\int_0^L \tilde{\mathbf{K}}_\omega(x,t)^T\mathbf{V}_{\omega_d}(x)\mathbf{N}_{\omega_f}(x,t)\mathbf{K}_\omega(x,t)\ dx\ .\end{aligned} \tag{2.52}$$

The first integral in Eq. (2.52) is an exact differential and the second can be rewritten using Lebnitz's rule. Thus,

$$\begin{aligned}\dot{U}_\omega(t) = &-\tilde{\mathbf{K}}_\omega(x,t)^T\mathbf{V}_{\omega_d}(x)\mathbf{V}_{\omega_d}(x)\tilde{\mathbf{K}}_\omega(x,t)\Big|_0^L - \int_0^L \frac{\partial}{\partial x}\left\{\tilde{\mathbf{K}}_\omega(x,t)^T\mathbf{V}_{\omega_d}(x)\right\}\mathbf{V}_{\omega_f}(x,t)(x,t)dx \\ &+\tilde{\mathbf{K}}_\omega(x,t)^T\mathbf{V}_{\omega_d}(x)\mathbf{V}_{\omega_f}(x,t)\mathbf{K}_\omega(x,t)\Big|_0^L + \int_0^L \tilde{\mathbf{K}}_\omega(x,t)^T\mathbf{V}_{\omega_d}(x)\mathbf{N}_{\omega_d}(x,t)\tilde{\mathbf{K}}_\omega(x,t)\ dx \\ &-\int_0^L \tilde{\mathbf{K}}_\omega(x,t)^T\mathbf{V}_{\omega_d}(x)\mathbf{N}_{\omega_f}(x,t)\mathbf{K}_\omega(x,t)\ dx\ .\end{aligned} \tag{2.53}$$

Use the fact that, by assumption, $\tilde{\mathbf{K}}_\omega(0,t) = 0$, and select matrices $\mathbf{V}_{\omega_f}(x,t)$ and $\mathbf{N}_{\omega_f}(x,t)$ according to Eqs (2.49)-(2.50). Then procede as in the proof of Theorem 2.2 to conclude on the $\mathcal{L}_2$ stability for $\tilde{\mathbf{K}}_\omega(\cdot,t) = 0$. ∎

Remark: Theorem 2.3 allows lane change even when the adjacent lanes have different desired velocity. Theorem 2.3 also allows the distributed implementation of the link layer controller. Only information from neighbor postions is required.

Appendix A contains results for the case in which the type of vehicle is also allowed to vary.

Chapter 3

Link Layer Simulation Results

To illustrate the effectiveness of the link layer controller, some simulation results are shown. The first set of results corresponds to the implementation on SmartPath (Eskafi *et al.*, 1992) of the control laws in Eqs. (2.49) and (2.50), for the case of one destination. The second set corresponds to Matlab simulations results for the case of vehicles multiple destinations.

3.1 SmartPath simulation results

SmartPath (Eskafi *et al.*, 1992) is a comprehensive simulator for the hierarchical PATH AHS architecture. Simulations are executed based on a user provided file that contains information about the highway topology. Highways are partitioned into different sections. The geometry of each section has to be specified: length, curvature, banking, number of lanes, width of lanes, etc. A set pointers defines the order in which the different sections and lanes are connected.

SmartPath includes routines to populate the highway with vehicles. These routines can create an initial set to vehicles to perform the simulation, or create vehicles during the course of the simulation. In any case, the maximum platoon size must be provided.

Vehicles are the unit of simulation in SmartPath. Independent coordination and regulation layers are created for each vehicle. The maximum number of vehicles in a single simulation is only limited by the time the corresponding simulation takes to complete. One important reason to test the link layer controller in SmartPath is that, as the coordination and regulation layers are included in it, it is possible to validate the assumption on the dynamics of these layers that were made while deriving the link layer model.

SmartPath includes a communications module that deals with all the information interchange between vehicles and between vehicles and the highway infrastructure. All the layers are assumed to used this module to place or retrieve information.

There is a routine to obtain the number of vehicles in each lane and section that simulates the sensor for detecting vehicles on a highway. The vehicular density is determined based on the information provided by this routine and the length of the section. The real velocity of the leader of a platoon is obtained by a simulated radio receptor that is hypothetically placed between sections of the highway. The actual number of changes of lane that is being executed in a given lane and section is supposed to be broadcasted by the platoon leaders to the highway infrastructure.

To establish the desired vehicular density and velocity profiles , special routines were developed for these simulations. These routines have to be substituted in a near future by a generator of desired traffic profiles, based on (Broucke and Varaiya, 1996).

Once the control laws in Eqs. (2.49) and (2.50) are used to calculate the value of the control for the longitudinal velocity of the platoon leaders and the proportion of vehicles to change lane, these commands are broadcasted to the coordination layer using the communication module. To deal with the discrete event nature of the lane change command, the link layer controller keeps track of the number of required lane changes and the number of lane changes that is under execution. Only the difference between these two numbers is commanded to avoid duplication.

SmartPath simulations were performed for one and two lane highways. In both cases an oval shaped track was used. The length of the oval is approximately of $5\ km$. There are about 100 vehicles per lane traveling at a nominal speed of $25\ m/s$. The circulation is in the counterclockwise direction.

The objective in the one lane simulation is to test the ability of the link layer controller to empty sections of highway. This capability is important in AHS systems because, for example, it provides space for vehicles entry to the AHS. The desired density distribution is illustrated in figure 3.1.

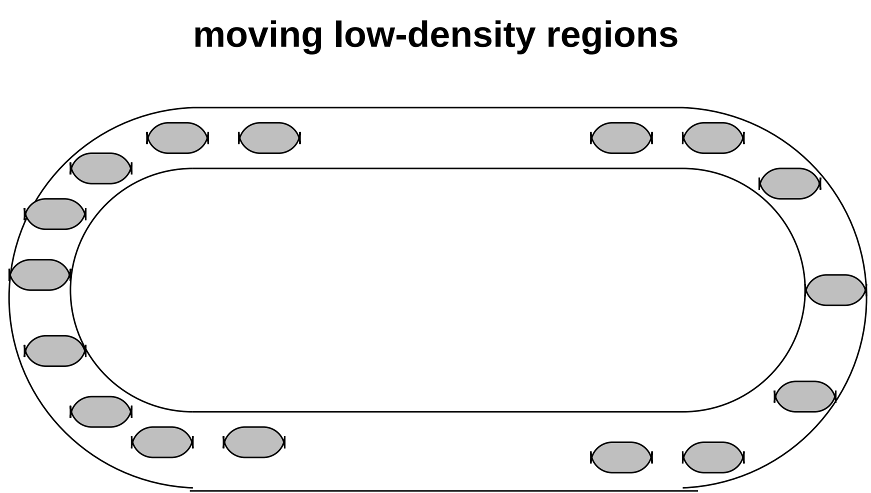

Figure 3.1: SmartPath one lane simulation. Desired low density region

Figures 3.2-3.4 show the simulation results for a one lane highway. Each block on figures 3.2-3.4 represents a platoon of vehicles including the headway of its leader, $60\ m$ in these simulations. As figure 3.4 clearly illustrates, after 160 s there are large empty sections of highway in the two straight sections of the oval highway. It should be noticed that there is a reduction in the number of platoons, and therefore in the occupancy of the highway, due to the regulation and coordination layer control laws that enforce the occurrence of joins. The size of the empty sections is much larger that the one that can be obtained without the use of the link layer controller here proposed.

In the case of the two lane SmartPath simulation, the link layer controller was required to perform two different tasks. The first task, that is to take place in the lower straight section of the oval highway, consist on homogenizing the vehicle density on both lanes. The second task is to

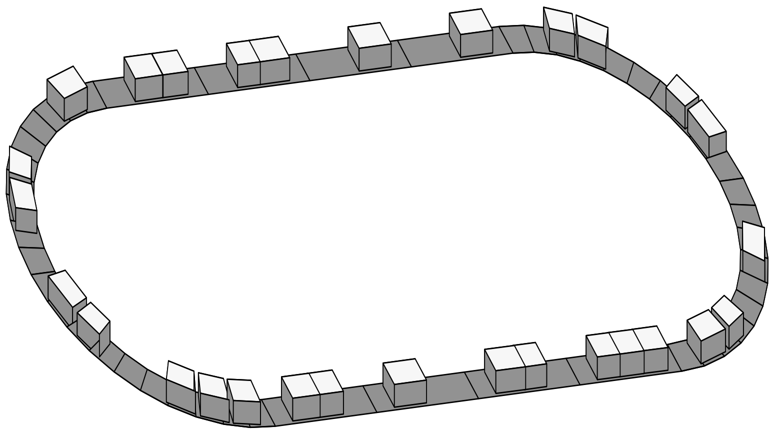

Figure 3.2: SmartPath one lane simulation results $t = 0\ s$.

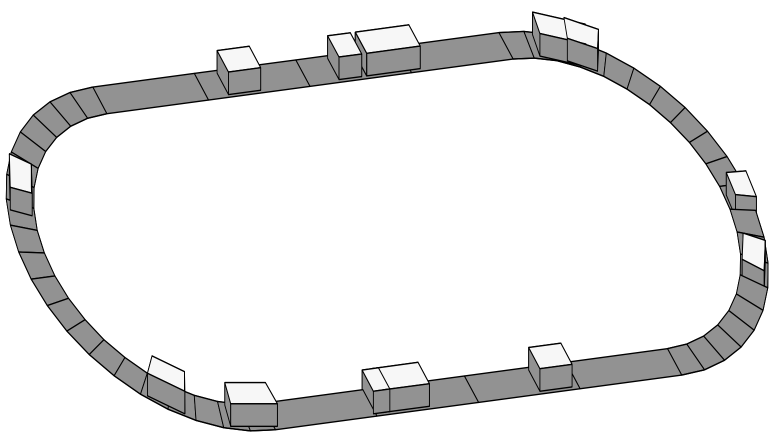

Figure 3.3: SmartPath one lane simulation results $t = 80\ s$.

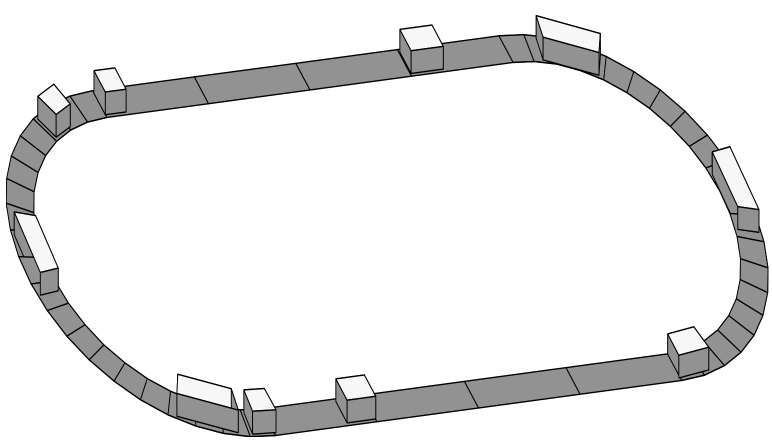

Figure 3.4: SmartPath one lane simulation results $t = 160\ s$.

empty the inner lane of the highway at the end of the upper straight section of the oval. The desired situation is illustrated in figures 3.5 and 3.6

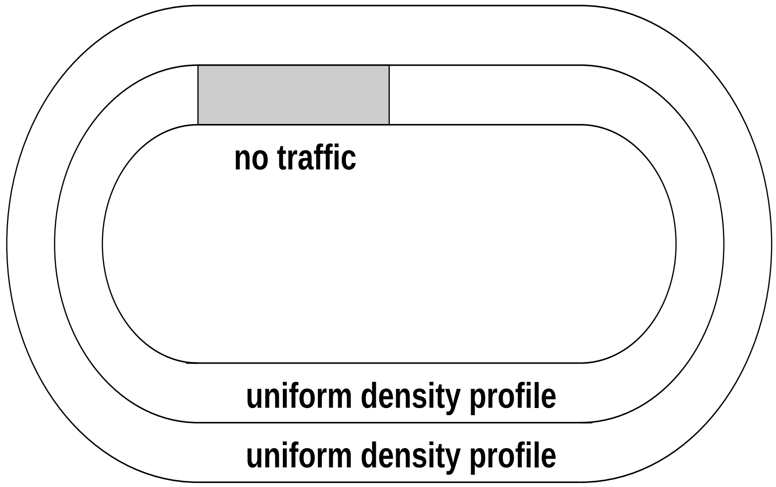

Figure 3.5: SmartPath two lane simulation. Desired density regions

The results in figures 3.7-3.9, that correspond to $40s$, $80s$ and $120s$ of simulation time respectively, indicate that the link layer controller performed the two tasks successfully.

3.2 Matlab simulation results

The last set of results corresponds to Matlab simulations of the the control laws in Eqs. (2.49) and (2.50) for the case when vehicles which have two different destinations are traveling on a two lane highway. Figures 3.10 and 3.11 show the desired density profiles for the two colors. At the beginning of the highway the desired behavior corresponds to vehicles of both colors mixed homogeneously in both lanes. At the middle of the stretch of highway, each lane should only contain vehicles of one color. At the end of the stretch of highway, the desired conditions are the same as those in the beginning. Figure 3.12 show the initial state for the simulation, which corresponds to an homogeneous mixture of the two colors along all the stretch of highway. Figures 3.13-3.14 illustrate the simulation results after $t = 9$ units of simulation time. It is clear from these figures that the multi-destination link layer controller achieved a color density profile very close to the desired one.

It should be noticed that in a real highway there are many different destinations. However, it is possible to tag vehicles traveling to distant exits with just one color. Following the suggestion in (Rao and Varaiya, 1994) to allow vehicles only one change of lane per highway section, the required number of different colors is related to the number of lanes in the AHS. In this case the computational complexity required to implement the multi-destination link layer controller remains small.

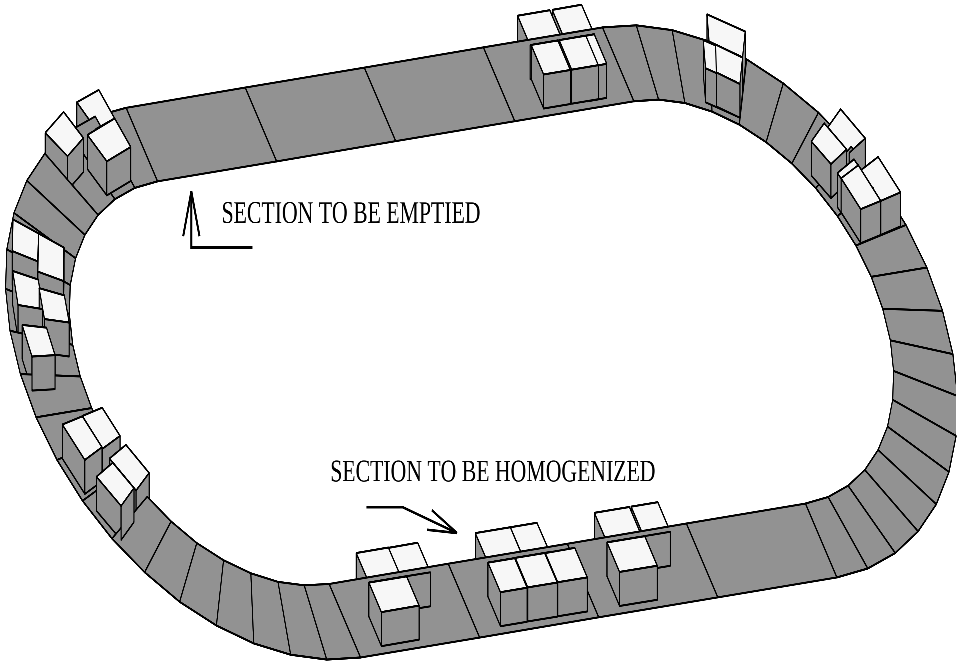

Figure 3.6: SmartPath two lane simulation results $t = 0\ s$.

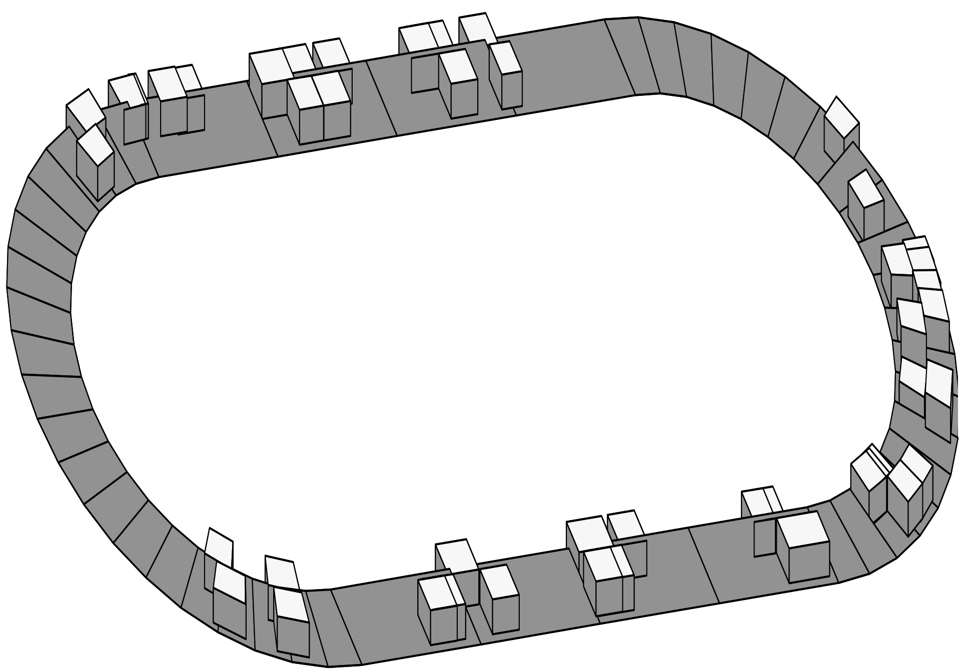

Figure 3.7: SmartPath two lane simulation results $t = 40\ s$.

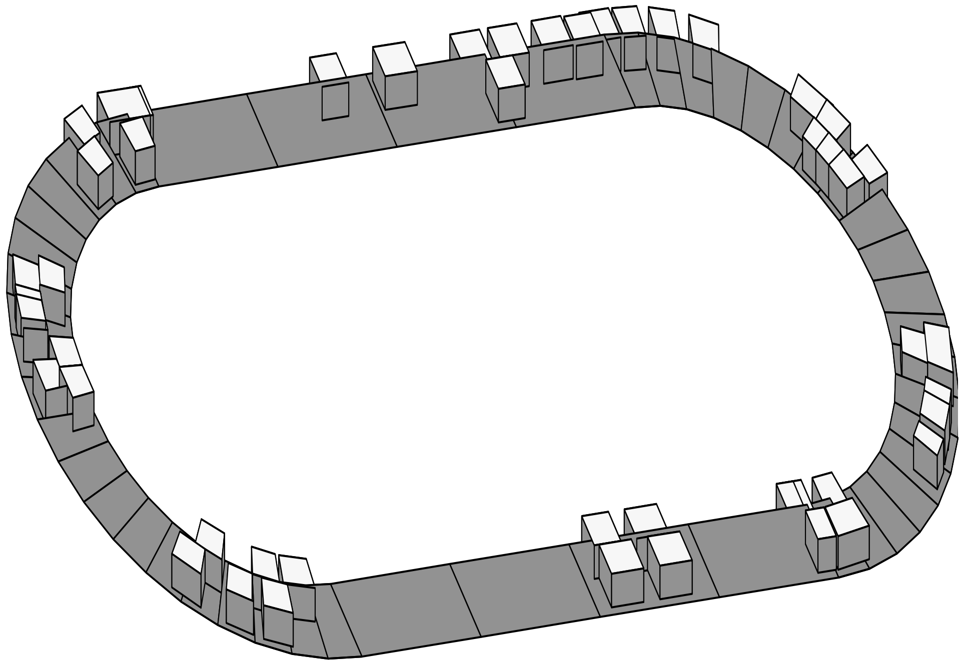

Figure 3.8: SmartPath two lane simulation results $t = 80\ s$.

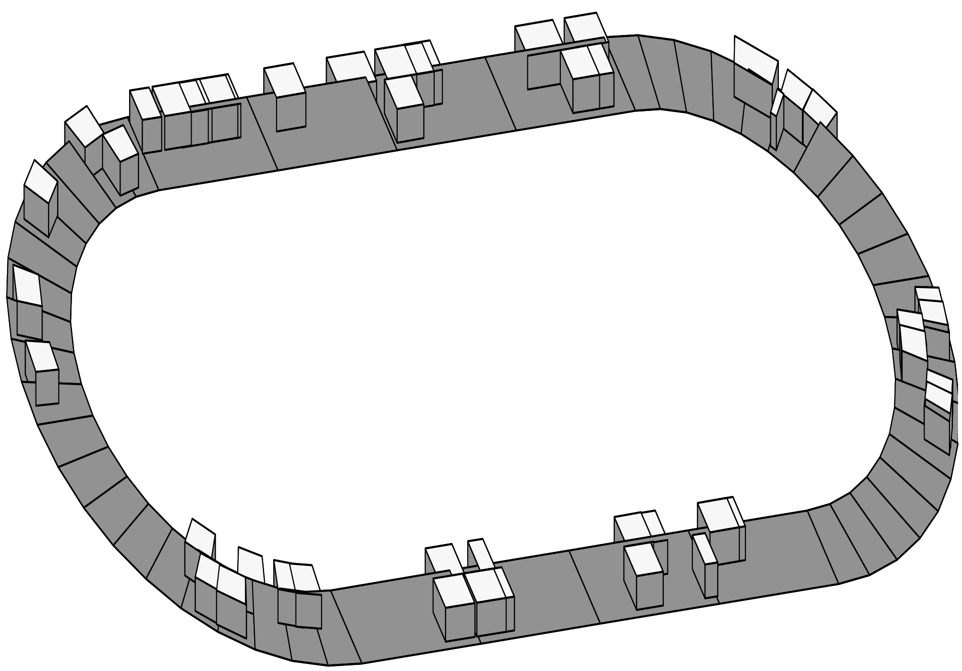

Figure 3.9: SmartPath two lane simulation results $t = 120\ s$.

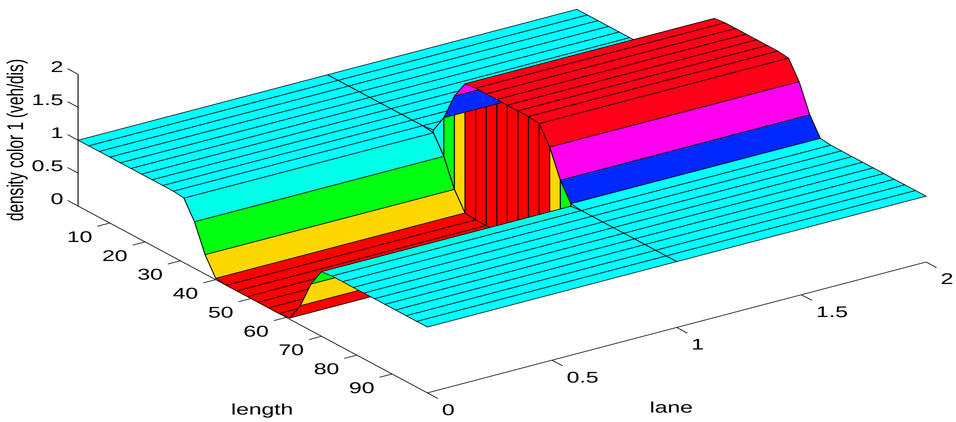

Figure 3.10: Matlab two lane simulation results. Color 1 desired density.

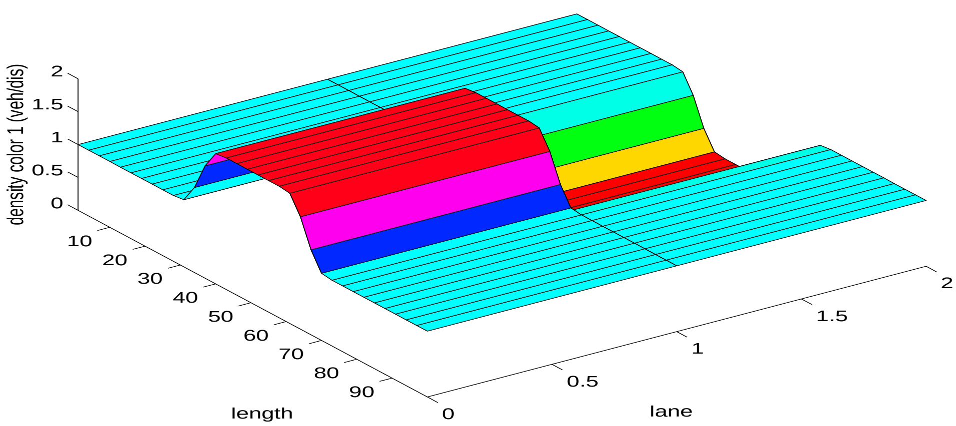

Figure 3.11: Matlab two lane simulation results. Color 2 desired density.

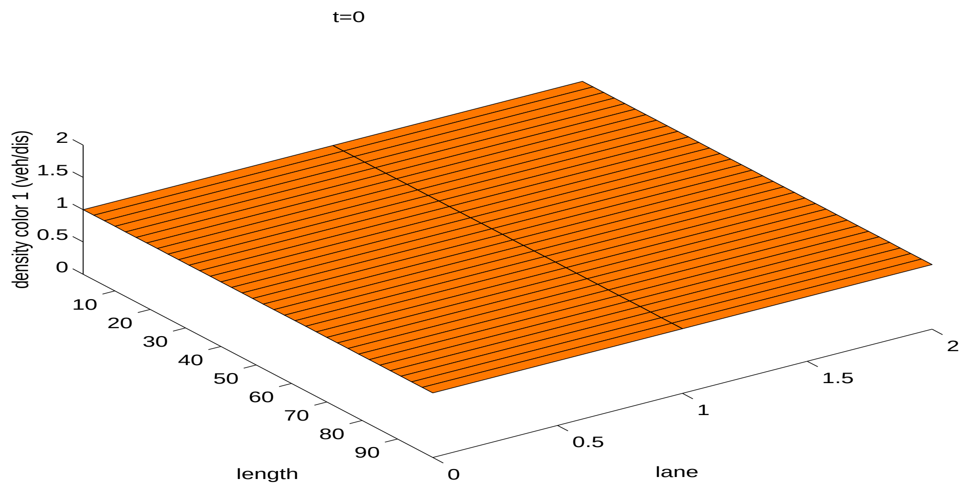

Figure 3.12: Matlab two lane simulation results. Colors 1 and 2, $t = 0$.

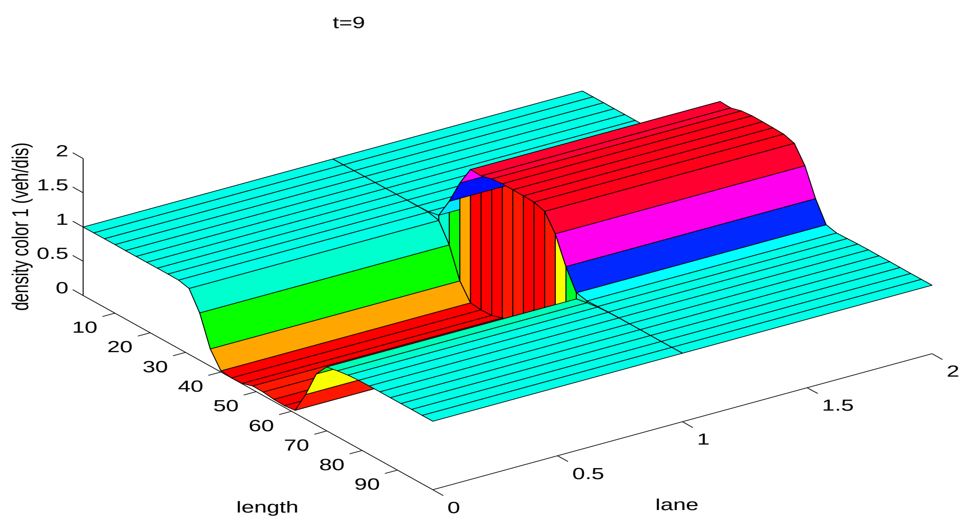

Figure 3.13: Matlab two lane simulation results. Color 1, $t = 9$.

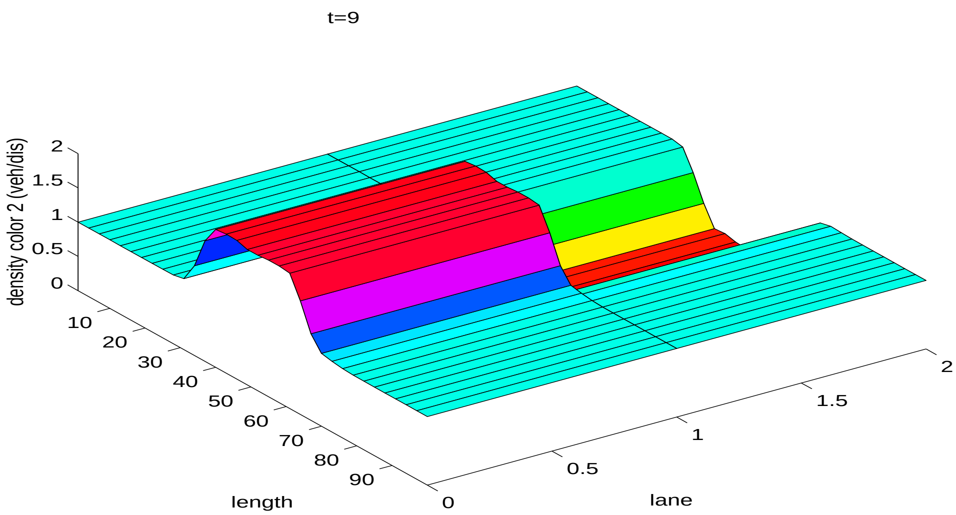

Figure 3.14: Matlab two lane simulation results. Color 2, $t = 9$.

Chapter 4

Conclusions

This report presents results on the traffic control of the hierarchical architecture of the PATH AHS presented in (Varaiya and Shladover, 1991). The main contributions are to traffic flow stabilization in the link layer level of this architecture.

A link layer controller for the PATH AHS architecture is presented. This controller is assumed to be used in a fully automated highway. No assumptions are made in terms of drivers' reaction to traffic or in terms of any explicit vehicular density-velocity relationship.

The traffic of vehicles with different destination although sharing a multiple lane highway is considered. Assuming that the velocity and the lane change of vehicles in a stretch of highway can be commanded, a set of control laws that stabilize the vehicular traffic flow to predetermined desired profiles of velocity and density is presented.

The controller is derived from a model based on the principle of vehicles conservation and is based on Lyapunov stability results. The more important features of this controller are:

- It is suitable for distributed implementation because it requires only local traffic information.
- It avoids traffic flow dynamics inversion that produce unbounded controls signals for small vehicular densities.
- It tracks the vehicle density profile that minimizes the error along all the stretch of highway, even when there is a mismatch between the desired and real inlet traffic flow.

The desired velocity and density profiles that are considered included the cases in which different desired velocities can be assigned to sections of the highway where lane change is supposed to occur.

Simulation results are presented both in Matlab and in SmartPath (Eskafi *et al.*, 1992). The simulation results indicate both, the validity of the assumptions about the dynamics of the coordination and regulation layers, and the effectiveness of the link layer controller.

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Appendix A

Traffic Flow Control: the general case

In this appendix a more general formulation for the traffic stabilization problem is presented. Vehicles are assumed to travel on a discrete lane highway and can have different destination and type.

A.1 Notation

Let the density and velocity of vehicles in the highway be denoted by

$$K_{t,x,y,l,c} \text{ and } V_{t,x,y,l,c} \,,$$

where

- t: time.
- x: position along the highway, $x \in [0, L]$.
- y: lane on the highway, $y \in \{1, \cdots, n\}$.
- l: vehicle type, $l = 1 \Rightarrow$ *leader* and $l = 2 \Rightarrow$ *follower*.
- c: vehicle color or destination, $c \in \{1, \cdots, m\}$.

Whenever an index is omitted in a variable, it is meant that the variable is not a function of that omitted argument.

The desired density and velocity are denoted by

$$\hat{K}_{t,x,y,l,c} \text{ and } \hat{V}_{x,y,l,c} \,,$$

while the density and velocity error are defined as

$$\tilde{K}_{t,x,y,l,c} = \hat{K}_{t,x,y,l,c} - K_{t,x,y,l,c}$$

and

$$\tilde{V}_{t,x,y,l,c} = \hat{V}_{x,y,l,c} - V_{t,x,y,l,c} \,,$$

Tensor notation for summations will be used, e.g.

$$K_{t,x,y,l,c} = N^{y}_{t,x,y,l,c} \, K_{t,x,y,l,c}$$

indicates

$$K_{t,x,y,l,c} = \sum_{j=1}^{n} N^{j}_{t,x,y,l,c} \, K_{t,x,j,l,c} \,.$$

A.2 Traffic modeling

Consider a n-lane highway in which vehicles with different destination and types are sharing the lanes. Using a principle of vehicles conservation, the dynamics of the vehicle density satisfy the following partial differential equation:

$$\frac{\partial K_{t,x,y,l,c}}{\partial t} = -\nabla \cdot \left(K_{t,x,y,l,c} \, \mathbf{V}_{t,x,y,l,c} \right) , \tag{A.1}$$

where $\mathbf{V}_{t,x,y,l,c}$ is a generalized velocity vector field defined by

$$\mathbf{V}_{t,x,y,l,c} = [V_{t,x,y,l,c}, n^y_{t,x,y,l,c}, n^l_{t,x,y,l,c}, n^c_{t,x,y,l,c}]^T ,$$

with $n_{t,x,y,l,c}(y)$, $n_{t,x,y,l,c}(l)$ and $n_{t,x,y,l,c}(c)$ indicating proportion of vehicles changing lane, type or color per unit time, respectively.

Eq. (A.1) can be rewritten as

$$\begin{aligned} \frac{\partial K_{t,x,y,l,c}}{\partial t} &= -\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial l}, \frac{\partial}{\partial c} \right] \begin{bmatrix} K_{t,x,y,l,c} \, V_{t,x,y,l,c} \\ K_{t,x,y,l,c} \, n_{t,x,y,l,c}(y) \\ K_{t,x,y,l,c} \, n_{t,x,y,l,c}(l) \\ K_{t,x,y,l,c} \, n_{t,x,y,l,c}(c) \end{bmatrix} \\ &= -\frac{\partial}{\partial x}(K_{t,x,y,l,c} \, V_{t,x,y,l,c}) - \frac{\partial}{\partial y}(K_{t,x,y,l,c} \, n_{t,x,y,l,c}(y)) \\ &\quad - \frac{\partial}{\partial l}(K_{t,x,y,l,c} \, n_{t,x,y,l,c}(l)) - \frac{\partial}{\partial l}(K_{t,x,y,l,c} \, n_{t,x,y,l,c}(c)) . \end{aligned} \tag{A.2}$$

The derivatives in the y, l, and c generalized directions are defined by

$$\begin{aligned} &\frac{\partial}{\partial y}(K_{t,x,y,l,c} \, n_{t,x,y,l,c}(y)) = K_{t,x,y+1,l,c} \, n_{t,x,y+1,l,c}(y) - K_{t,x,y,l,c} \, n_{t,x,y,l,c}(y) \\ &\frac{\partial}{\partial l}(K_{t,x,y,l,c} \, n_{t,x,y,l,c}(l)) = K_{t,x,y,l+1,c} \, n_{t,x,y,l+1,c}(l) - K_{t,x,y,l,c} \, n_{t,x,y,l,c}(l) \\ &\frac{\partial}{\partial c}(K_{t,x,y,l,c} \, n_{t,x,y,l,c}(c)) = K_{t,x,y,l,c+1} \, n_{t,x,y,l,c+1}(c) - K_{t,x,y,l,c} \, n_{t,x,y,l,c}(c) , \end{aligned}$$

or any other discrete approximation of the derivatives. Using this definitions it is possible to write Eq. (A.2) as

$$\begin{aligned} \frac{\partial K_{t,x,y,l,c}}{\partial t} &= -\frac{\partial}{\partial x}(V_{t,x,y,l,c} \, K_{t,x,y,l,c}) + N^y_{t,x,y,l,c} \, K_{t,x,y,l,c} \\ &\quad + N^l_{t,x,y,l,c} \, K_{t,x,y,l,c} + N^c_{t,x,y,l,c} \, K_{t,x,y,l,c} , \end{aligned} \tag{A.3}$$

with $N^y_{t,x,y,l,c}$, $N^l_{t,x,y,l,c}$ and $N^c_{t,x,y,l,c}$ appropriately defined.

For the analysis we use the same assumption as in (Li *et al.*, 1995; Li *et al.*, 1997*b*): there exists a prescribed profile for the densities and velocities on the highway that satisfy

$$\begin{aligned} \frac{\partial \hat{K}_{t,x,y,l,c}}{\partial t} &= -\frac{\partial}{\partial x}(\hat{V}_{x,y,l,c} \, \hat{K}_{t,x,y,l,c}) + \hat{N}^y_{x,y,l,c} \, \hat{K}_{t,x,y,l,c} \\ &\quad + \hat{N}^l_{x,y,l,c} \, \hat{K}_{t,x,y,l,c} + \hat{N}^c_{x,y,l,c} \, \hat{K}_{t,x,y,l,c} , \end{aligned} \tag{A.4}$$

where $\hat{N}^y_{x,y,l,c}$, $\hat{N}^l_{x,y,l,c}$ and $\hat{N}^c_{x,y,l,c}$ represent the desired proportion of vehicles changing lane, type or color per unit time, respectively. An order is supposed to exist for lanes, colors and types is such a way that desired changes of lane, color of type are only allowed between adjacent members on their respective orders. This constraint implies that if $\hat{N}^y_{x,y,l,c}$, $\hat{N}^l_{x,y,l,c}$ and $\hat{N}^c_{x,y,l,c}$ are placed as elements of matrices according with the given order, the only elements different form zero in these matrices are in the diagonal, first super-diagonal and first sub-diagonal; the algebraic sum of the terms in any column should be zero.

A.3 Error dynamics

Define

$$\begin{aligned} \tilde{N}^y_{t,x,y,l,c} &= \hat{N}^y_{t,x,y,l,c} - N^y_{t,x,y,l,c} \, , \\ \tilde{N}^l_{t,x,y,l,c} &= \hat{N}^l_{t,x,y,l,c} - N^l_{t,x,y,l,c} \, , \\ \tilde{N}^c_{t,x,y,l,c} &= \hat{N}^c_{t,x,y,l,c} - N^c_{t,x,y,l,c} \, , \end{aligned} \tag{A.5}$$

then from Eqs. (A.3) and (A.4)

$$\begin{aligned} \frac{\partial \tilde{K}_{t,x,y,l,c}}{\partial t} = &- \frac{\partial}{\partial x}(\hat{V}_{x,y,l,c}\, \tilde{K}_{t,x,y,l,c}) - \frac{\partial}{\partial x}(\tilde{V}_{t,x,y,l,c}\, K_{t,x,y,l,c}) \\ &+ \hat{N}^y_{t,x,y,l,c}\, \tilde{K}_{t,x,y,l,c} + \tilde{N}^y_{t,x,y,l,c}\, K_{t,x,y,l,c} \\ &+ \hat{N}^l_{t,x,y,l,c}\, \tilde{K}_{t,x,y,l,c} + \tilde{N}^l_{t,x,y,l,c}\, K_{t,x,y,l,c} \\ &+ \hat{N}^c_{t,x,y,l,c}\, \tilde{K}_{t,x,y,l,c} + \tilde{N}^c_{t,x,y,l,c}\, K_{t,x,y,l,c} \, , \end{aligned} \tag{A.6}$$

From Eq. (A.6) it is possible to define the control problem in a precise form: determine $\tilde{V}_{t,x,y,l,c}$, $\tilde{N}^y_{t,x,y,l,c}$, $\tilde{N}^l_{t,x,y,l,c}$ and $\tilde{N}^c_{t,x,y,l,c}$ in such a way that the real density profile $K_{t,x,y,l,c}$ converges to the desired density profile $\hat{K}_{x,y,l,c}$. In the next sections the necessary steps to solve this problem are detailed.

A.4 Stability of the control laws

Define the longitudinal feedback velocity term as

$$\tilde{V}_{t,x,y,l,c} = -\Gamma_{t,x,y,l,c} \frac{\partial}{\partial x}(\hat{V}_{x,y,l,c}\tilde{K}_{t,x,y,l,c}) \, , \tag{A.7}$$

where $\Gamma_{t,x,y,l,c} \geq 0$ is gain with $\Gamma_{t,0,y,l,c} = 0$ and $\Gamma_{t,L,y,l,c} = 0$.

It should be noticed that it is necessary to impose to $\tilde{N}^y_{t,x,y,l,c}$, $\tilde{N}^l_{t,x,y,l,c}$ and $\tilde{N}^c_{t,x,y,l,c}$ the same constraint that was imposed to $\hat{N}^y_{x,y,l,c}$, $\hat{N}^l_{x,y,l,c}$ and $\hat{N}^c_{x,y,l,c}$, i.e., changes of lane, type or color are allowed only between elements that are adjacent in their respective order. By Eq. (A.5) this constraint also guarantees that $N^y_{t,x,y,l,c}$, $N^l_{t,x,y,l,c}$ and $N^c_{t,x,y,l,c}$ will have the same structure.

Introduce the auxiliary quantity

$$F_{t,x,y,l,c} = \hat{V}_{x,y,l,c}\tilde{K}_{t,x,y,l,c} \, , \tag{A.8}$$

and let the feedback terms $\tilde{N}^y_{t,x,y,l,c}$, $\tilde{N}^l_{t,x,y,l,c}$ and $\tilde{N}^c_{t,x,y,l,c}$ be defined by

$$\tilde{N}^y_{t,x,y,l,c} = \begin{cases} -\xi_{t,x,y,l,c}(F_{t,x,y,l,c} - F_{t,x,y_1,l,c}); & |y - y_1| = 1 \ , F_{t,x,y,l,c} > F_{t,x,y_1,l,c} \\ 0 & \text{else} \end{cases} \quad , \tag{A.9}$$

$$\tilde{N}^l_{t,x,y,l,c} = \begin{cases} -\zeta_{t,x,y,l,c}(F_{t,x,y,l,c} - F_{t,x,y,l_1,c}); & |l - l_1| = 1 \ , F_{t,x,y,l,c} > F_{t,x,y,l,c} \\ 0 & \text{else} \end{cases} \quad , \tag{A.10}$$

$$\tilde{N}^c_{t,x,y,l,c} = \begin{cases} -\varsigma_{t,x,y,l,c}(F_{t,x,y,l,c} - F_{t,x,y,l,c_1}); & |c - c_1| = 1 \ , F_{t,x,y,l,c} > F_{t,x,y,l,c_1} \\ 0 & \text{else} \end{cases} \quad , \tag{A.11}$$

where the gains $\xi_{t,x,y,l,c}$, $\zeta_{t,x,y,l,c}$ and $\varsigma_{t,x,y,l,c}$ are non-negative.

The $\mathcal{L}_2$ norm of the density error $\tilde{K}_{y,l,c}$ is defined to be

$$\|\tilde{K}_{t,y,l,c}\|_2^2 = \int_0^L \tilde{K}^2_{t,x,y,l,c} dx$$

The main result of this appendix is stated in the following theorem.

Theorem A.1 *Consider the highway model of Eq. (A.2) and suppose the desired highway conditions satisfy Eq. (A.4). Assume the inlet flow condition is such that $\tilde{K}_{t,0,y,l,c} = 0$. Then, under the control laws in Eqs. (A.7) and (A.9)-(A.11) the equilibria $\tilde{K}_{t,x,y,l,c} = 0$ and $\tilde{K}_{t,x} = 0 \;\; \forall x \in [0, L]$ are $\mathcal{L}_2$ stable.*

Proof: Choose the following Lyapunov candidate

$$W_{y,l,c} = \frac{1}{2} \int_0^L \hat{V}_{x,y,l,c} \, (\tilde{K}_{t,y,l,c})^2 dx \ . \tag{A.12}$$

Taking the time derivative of Eq. (A.12) and using Eq. (A.6)

$$\begin{aligned} \dot{W}_{y,l,c} &= \int_0^L \tilde{K}_{t,y,l,c} \, \hat{V}_{x,y,l,c} \, \frac{\partial \tilde{K}_{t,y,l,c}}{\partial t} \, dx \\ &= - \int_0^L \hat{V}_{x,y,l,c} \tilde{K}_{t,x,y,l,c} \, \frac{\partial}{\partial x} \left(\hat{V}_{x,y,l,c} \, \tilde{K}_{t,x,y,l,c} \right) \, dx \\ &\quad - \int_0^L \hat{V}_{x,y,l,c} \tilde{K}_{t,x,y,l,c} \frac{\partial}{\partial x} (\tilde{V}_{t,x,y,l,c} \, K_{t,x,y,l,c}) \, dx \\ &\quad + \int_0^L \hat{V}_{x,y,l,c} \tilde{K}_{t,x,y,l,c} \left\{ \hat{N}^y_{t,x,y,l,c} + \hat{N}^l_{t,x,y,l,c} + \hat{N}^c_{t,x,y,l,c} \right\} \tilde{K}_{t,x,y,l,c} \, dx \\ &\quad + \int_0^L \hat{V}_{x,y,l,c} \tilde{K}_{t,x,y,l,c} \left\{ \tilde{N}^y_{t,x,y,l,c} + \tilde{N}^l_{t,x,y,l,c} + \tilde{N}^c_{t,x,y,l,c} \right\} K_{t,x,y,l,c} \, dx \ . \end{aligned} \tag{A.13}$$

The first term in Eq. (A.13) is an exact differential in x and the second can be written using

Leibnitz's rule again, thus

$$
\begin{aligned}
\dot{W}_{y,l,c} = & -\frac{1}{2}\left(\hat{V}_{x,y,l,c}\tilde{K}_{t,x,y,l,c}\right)^2\Big|_{x=0}^{x=L} + \int_0^L \frac{\partial}{\partial x}(\hat{V}_{x,y,l,c}\tilde{K}_{t,x,y,l,c})\tilde{V}_{t,x,y,l,c}\, K_{t,x,y,l,c}\, dx \\
& - \hat{V}_{x,y,l,c}\tilde{K}_{t,x,y,l,c}\, \tilde{V}_{t,x,y,l,c}\, K_{t,x,y,l,c}\Big|_{x=0}^{x=L} \\
& + \int_0^L \hat{V}_{x,y,l,c}\tilde{K}_{t,x,y,l,c}\left\{\hat{N}^y_{t,x,y,l,c} + \hat{N}^l_{t,x,y,l,c} + \hat{N}^c_{t,x,y,l,c}\right\}\tilde{K}_{t,x,y,l,c}\, dx \\
& + \int_0^L \hat{V}_{x,y,l,c}\tilde{K}_{t,x,y,l,c}\left\{\tilde{N}^y_{t,x,y,l,c} + \tilde{N}^l_{t,x,y,l,c} + \tilde{N}^c_{t,x,y,l,c}\right\} K_{t,x,y,l,c}\, dx\ . \qquad \text{(A.14)}
\end{aligned}
$$

Consider that $\tilde{K}_{t,0,y,l,c} = 0$, by assumption, and choose $\tilde{V}_{t,x,y,l,c}$, $\tilde{N}^y_{t,x,y,l,c}$, $\tilde{N}^l_{t,x,y,l,c}$ and $\tilde{N}^c_{t,x,y,l,c}$ according with Eqs. (A.7) and (A.9)-(A.11). Using the same argument as in the proof of Theorem 2.2, it is possible to conclude on the $\mathcal{L}_2$ stability of $\tilde{K}_{t,x} = 0$ follows. ■

Remark: If the stabilizing control law in Eq. (A.7) is to have physical meaning, vehicles on the same lane should have the same velocity at any particular position x, regardless of the vehicle destination or type, i.e., it is necessary to ensure that

$$V_{t,x,y,l,c} = V_{t,x,y}\ ; \qquad \hat{V}_{x,y,l,c} = \hat{V}_{x,y}\ ,$$

this corresponds to modify the control law in Eq. (A.7) to

$$\tilde{V}_{t,x,y} = -\Gamma_{t,x,y}\sum_{\forall l,c}\frac{\partial}{\partial x}(\hat{V}_{x,y,l,c}\tilde{K}_{x,y,l,c})K_{t,x,y,l,c}\ .$$

A.5 Output mappings

To derive measures of highway performance it is possible to operate on the highway states. For example, the total density at a given time t and position x, $K_{t,x}$, is given by

$$K_{t,x} = \sum_{y,l,c} K_{t,x,y,l,c}\ ,$$

while highway occupancy between $x = x_i$ and $x = x_f$ at time t, $O_t(x_i, x_f)$, is determined from

$$O_t(x_i, x_f) = \frac{1}{x_f - x_i}\int_{x_i}^{x_f}\sum_l\left(\sum_{y,c} K_{t,x,y,l,c}\right) S_l\, dx\ ,$$

where S_l is the length associated with vehicle type l.

A.6 Entry and exit of vehicles

Vehicles entry and exit produce a discontinuity in the density, $K_{t,x,y,l,c}$. When a vehicle reaches its final destination and leaves the road, it actually disappears from the highway and therefore the total vehicular density must decrease. Conversely, when a vehicle enters to the highway with a given final destination, the total density must increase. The following theorem establishes a bound on the density error in order for the entry and exit of vehicles to preserve the stability of the traffic flow stabilizer controller if the density error at the section intersections satisfies

$$\tilde{K}^2_{t,L_i,y,l,c} \le \tilde{K}^2_{t,L_i^-,y,l,c} \quad \forall i \in 1,\cdots,p-1 \;, \tag{A.15}$$

where the highway is partition as

$$\begin{aligned} [0,L] =& [0,L_1]\cup[L_1,L_2]\cup\cdots\cup[L_{p-1},L] \\ & L_i \le L_{i+1}\ \forall i\in\{1,\cdots,p-1\}\;. \end{aligned} \tag{A.16}$$

Define $L_0 = 0$ and $L_p = L$.

Theorem A.2 *Consider the highway model of Eq. (A.2) and suppose the desired highway conditions satisfy Eq. (A.4). Assume the inlet flow condition is such that* $\tilde{K}_{t,0,y,l,c} = 0$ *and that for the partition in Eq. (A.16) the conditions in Eq. (A.15) are satisfied. Then, under the control laws in Eqs. (A.7) and (A.9)-(A.11) the equilibria* $\tilde{K}_{t,x,y,l,c} = 0$ *and* $\tilde{K}_{t,x} = 0 \;\; \forall x \in [0,L]$ *are* $\mathcal{L}_2$ *stable.*

Proof: Define

$$W_{y,l,c} = \frac{1}{2}\sum_{i=0}^{p-1}\int_{L_i}^{L_{i+1}} \hat{V}_{x,y,l,c}\,(G_{t,x,y,l,c})^2 dx \;.$$

Proceeding as in the proof of Theorem A.1, the following expression will be satisfied

$$\begin{aligned} \dot{W}_{y,l,c} \le& -\sum_{i=0}^{p-1}\left\{ \frac{1}{2}\left(\hat{V}_{x,y,l,c}\tilde{K}_{t,x,y,l,c}\right)^2\Bigg|_{x=L_i}^{x=L_{i+1}} \right. \\ &\left. + \int_{L_i}^{L_{i+1}} \hat{V}_{x,y,l,c}\tilde{K}_{t,x,y,l,c}\left\{\hat{N}^y_{t,x,y,l,c} + \hat{N}^l_{t,x,y,l,c} + \hat{N}^c_{t,x,y,l,c}\right\}\tilde{K}_{t,x,y,l,c}\,dx \right\} \;, \end{aligned} \tag{A.17}$$

The effect on stability of the term inside the integral in Eq. (A.17) was already analyzed in the proof of Theorem A.1. To analyze the other term, without loss of generality, take sections 1 and 2 of the highway and suppose there is entry and exit of vehicles between these two sections of the highway. The other term in the right hand side of Eq. (A.17) would be in this case

$$\begin{aligned} 2\dot{W}_{y,l,c} \le& \left(\hat{V}_{0,y,l,c}\tilde{K}_{t,0,y,l,c}\right)^2 - \left(\hat{V}_{L_1,y,l,c}\tilde{K}_{t,L_1^-,y,l,c}\right)^2 \\ &+ \left(\hat{V}_{L_1,y,l,c}\tilde{K}_{t,L_1,y,l,c}\right)^2 - \left(\hat{V}_{L_2,y,l,c}\tilde{K}_{t,L_2^-,y,l,c}\right)^2 \;, \end{aligned}$$

where $\tilde{K}_{t,L_1^-,y,l,c}$ and $\tilde{K}_{t,L_1,y,l,c}$ denote the density error before and after the entry-exit in section 1. As the desired velocity is supposed to be continuous at $x = L_1$, then for stability to be preserved in the p sections case it is necessary to guarantee (A.15), that is

$$\left(\tilde{K}_{t,L_i,y,l,c}\right)^2 \le \left(\tilde{K}_{t,L_i^-,y,l,c}\right)^2 \quad \forall i \in 1,\cdots,p-1 \;.$$

This will ensure that

$$2\dot{W}_{y,l,c} \leq \left(\hat{V}_{0,y,l,c}\tilde{K}_{t,0,y,l,c}\right)^2 - \left(\hat{V}_{L_1,y,l,c}\tilde{K}_{t,L_1^-,y,l,c}\right)^2 + \left(\hat{V}_{L_1,y,l,c}\tilde{K}_{t,L_1,y,l,c}\right)^2 - \left(\hat{V}_{L_2,y,l,c}\tilde{K}_{t,L_2^-,y,l,c}\right)^2 \leq 0\,.$$

The stability result follows. ∎