Title
ELECTRICAL ENGINEERING REVIEW COURSE - LECTURE XII

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Authors
Martinelli, E.
Kilpatrick, R.
Stone, K.

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SYMBOLS AND UNITS

Units in MKS System

\[ i = \text{instantaneous current--amperes} \]
\[ e = \text{instantaneous voltage--volts} \]
\[ I' = \text{peak current--amperes} \]
\[ E' = \text{peak voltage--volts} \]
\[ I = \text{Root mean square current} = \frac{I'}{\sqrt{2}} \]
\[ E = \text{Root mean square voltage} = \frac{E'}{\sqrt{2}} \]
\[ W = \text{circular frequency--rad/sec} \]
\[ F = \text{frequency--cycles/sec} = \frac{n}{2\pi} \]
\[ WL = \text{Inductive reactance--Ohms} \]
\[ WC = \text{Capacitive reactance--Ohms} \]
\[ Z = \text{Impedance--Ohms} \]
\[ Z_{eq} = \text{Equivalent series impedance of the parallel circuit.} \]
\[ R = \text{Resistance--Ohms} \]
\[ L = \text{Inductance--Henry} \]
\[ C = \text{Capacity--Farad} \]
\[ \omega_0 = \text{Frequency at resonance.} \]
\[ \phi = \text{Phase angle} \]
\[ \phi' = \text{Peak magnetic field strength} \]
\[ \phi = \text{Instantaneous magnetic field strength} \]

PARALLEL CIRCUITS

Definition: A parallel circuit is one in which the line voltage is common to all branches of the circuit.

Consider the simple circuit of Fig. 1 in which the supply voltage is sinusoidal and the circuit constants are as shown.

![Diagram of a parallel circuit with labels for R, L, and C, and currents I0, I1, and I2.]

NOTE: In many practical circuits the resistance in the capacitive branch is negligible and therefore, is not shown above.
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Applying Kirchoff's law at junction "a" gives:

\[ I_1 + I_2 = I_0 \]  
(1)

This is a vector equation since the branch currents are obviously out of phase with each other.

Two further equations can be written by noting that the voltage drop across each branch is the line voltage

\[ L \frac{di_1}{dt} + i_1R = e_0 \]  
(2)

\[ \frac{1}{C} \int i_2 dt = e_0 \]  
(3)

To solve these equations use the complex representations

\[ i_0 = I'e^{j\omega t} \]
\[ i_1 = I'e^{j\omega t} \]
\[ i_2 = I'e^{j\omega t} \]
\[ e_0 = E'e^{j\omega t} \]

Now assume the circuit has reached steady state. This will eliminate constants of integration from solutions of above equations.

Equation (2) becomes:

\[ Lj\omega I_1'e^{j\omega t} + RI_1'e^{j\omega t} = E'e^{j\omega t} \quad \text{OR} \]
\[ I_1' = \frac{E'}{R + j\omega L} \]  
(4)

Equation (3) becomes:

\[ \frac{1}{C} \frac{I_2'e^{j\omega t}}{j\omega} = E'e^{j\omega t} \quad \text{OR} \]
\[ \frac{I_2'}{j\omega C} = E' \]  
(5)

Equation (4) and (5) can be changed to the more usable form involving R.M.'s values simply by multiplying by \( 1/\sqrt{2} \)

Then

\[ I_1 = \frac{E}{R + j\omega L} \]  
(6)

\[ I_2 = \frac{E}{j\omega C} \]  
(7)

and from (1)

\[ I_0 = E \left[ \frac{1}{E + j\omega L} + \frac{1}{j\omega C} \right] \]  
(8)
Define $Z_{eq} = \frac{E}{I_0}$ (8a)

Then

$$\frac{1}{Z_{eq}} = \left[\frac{1}{R + jwL} + jwC\right]$$

$$\frac{1}{Z_{eq}} = jwC + \left(\frac{1}{R + jwL}\right)\left(\frac{R - jwL}{R - jwL}\right)$$

$$\frac{1}{Z_{eq}} = jwC + \frac{R - jwL}{R^2 + w^2L^2}$$

$$\frac{1}{Z_{eq}} = \frac{R}{R^2 + w^2L^2} + j\left[wC - \frac{wL}{R^2 + w^2L^2}\right]$$

Let

$$R_{eq} = \frac{R}{R^2 + w^2L^2}$$

$$X_{eq} = \left[wC - \frac{wL}{R^2 + w^2L^2}\right]$$

Substituting in (10)

$$\frac{1}{Z_{eq}} = R_{eq} + jX_{eq}$$

$$Z_{eq} = \left(\frac{1}{R_{eq} + jX_{eq}}\right)\left(R_{eq} - jX_{eq}\right) = \frac{R_{eq} - jX_{eq}}{R_{eq}^2 + X_{eq}^2}$$

$$Z_{eq} = \frac{R_{eq}}{R_{eq}^2 + X_{eq}^2} - j\frac{X_{eq}}{R_{eq}^2 + X_{eq}^2}$$

(13)

The magnitude of $Z_{eq}$ is found by taking the square root of the sum of the squares of the real parts of the complex impedance given by equation (13).

$$|Z_{eq}| = \sqrt{\frac{R_{eq}^2}{(R_{eq}^2 + X_{eq}^2)^2} + \frac{X_{eq}^2}{(R_{eq}^2 + X_{eq}^2)^2}}$$

OR

$$|Z_{eq}| = \frac{1}{\sqrt{R_{eq}^2 + X_{eq}^2}}$$

Substituting for $R$ and $X$

$$|Z_{eq}| = \frac{1}{\sqrt{\frac{R^2}{R^2 + w^2L^2} + \frac{wL}{R^2 + w^2L^2}}}$$

(14)

From equation (13) the line phase angle ($\theta_0$) is found.

$$\theta_0 = \tan^{-1} \frac{\text{Reactive Component}}{\text{Resistive Component}} = \tan^{-1} \frac{X}{R}$$

OR
\[ \phi = \tan^{-1}\left[ \frac{\frac{wC}{R^2} + \frac{wL}{R^2 + 2w^2L^2}}{\frac{wL}{R^2 + 2w^2L^2}} \right] \] 

(16)

**RESONANCE IN PARALLEL CIRCUIT**

The parallel circuit is similar to the series circuit in that it has a resonant frequency \( w_0 \), but at this frequency the current is at minimum in the parallel circuit. At resonance, \( Z_{eq} \) is therefore a maximum.

Consider the case where \( R \ll wL \)

Then \( R^2 \ll w^2L^2 \)

And equation (14) can be rewritten as:

\[ Z_{eq} = \frac{1}{\sqrt{\frac{R^2}{w^4L^4} + \left( \frac{wC}{wL} \right)^2}} \]

(17)

It is seen that \( |Z_{eq}| \) is maximum when \( wC - \frac{1}{wL} = 0 \)

Solving for the frequency which fits this condition gives the resonant frequency.

\[ w_0 = \frac{1}{\sqrt{LC}} \]

(18)

At this frequency

\[ |Z_{eq}|_o = \frac{1}{\sqrt{\frac{R^2}{w_0^4L^4} + \left( \frac{w_0L}{wL} \right)^2}} = \frac{w_0^2L^2}{wL} \]

and since

\[ \frac{w_0L}{R} = Q \]

\[ |Z_{eq}|_o = w_0LQ \]

(19)

Note that at resonance, the capacitive reactance \((1/wC)\) and the inductive reactance \((wL)\) of the two branch circuits are equal and the parallel circuit looks like a pure resistance to the line current. With \( R \) small relative to the reactances, the branch currents are approximately 180° out of phase and circulate around the parallel circuit without entering the line.
The variation of $I_0$ with change in frequency is shown in Fig. 2.

Variation of $|Z_{eq}|$ with $W$ is shown in Fig. 3.

$Q$ of the circuit can be determined experimentally by driving with constant voltage, measuring line voltage and line current while the frequency is varied through known values near the resonant point. Then from a plot of $|Z_{eq}| (=E/I)$ versus the frequency, the phase shift ($\Delta w$) indicated in Fig. 3 is found and inserted in the following formula:

$$Q = \frac{\omega_0}{2\Delta w} \quad (20)$$

**TRANSFORMERS**

**Definition:** A transformer is a set of coupled circuits used to transfer electrical energy from one alternating current circuit to another without change in frequency. This transference is usually, but not always, accompanied by a change in voltage.
R\textsubscript{1} is resistance of primary and R\textsubscript{2} is the combined secondary and load. Load is assumed to be pure resistance. Note: Subscript 1 refers to primary and subscript 2 refers to secondary. Assume that the transformer is a perfect coupling, i.e., all flux links both circuits also neglect affects of distributed capacitive. Considering voltage drop across primary:

\[ i_1 R_1 + L_{11} \frac{di_1}{dt} + L_{12} \frac{di_2}{dt} = e_1 \] (21)

\[ i_2 R_2 + L_{21} \frac{di_1}{dt} + L_{22} \frac{di_2}{dt} = 0 \] (22)

Double subscripts: L\textsubscript{21} means the inductance of the secondary due to the change in current in primary, etc.

In all practical transformers the resistance is very small with respect to the inductive reactance.

Symbolically \( R_1 \ll \omega L_{11} \)

Primary amp. turns \( N_1 \phi \)

Second amp. Turns \( N_2 \phi \)

Ignoring losses in primary circuit, \( i_1 R_1 \) terms

\[ \frac{N_1}{C} \frac{d\phi}{dt} = e_1 = L_{11} \frac{di_1}{dt} + L_{12} \frac{di_2}{dt} \] (23)

\[ \frac{N_2}{C} \frac{d\phi}{dt} + i_2 R_2 = 0 = i_2 R_2 + L_{21} \frac{di_1}{dt} + L_{22} \frac{di_2}{dt} \] (24)

Let

\[ \phi = \phi e^{jwt} \] (25)

\[ e_1 = E e^{jwt} \] (26)

Differentiating (25) and substituting for \( \frac{d\phi}{dt} \) in (23)

\[ jw \frac{N_1}{C} \phi' = e_1 \]

\[ \phi' = \frac{e_1}{jw \frac{N_1}{C}} \] (27)
Following the same procedure on (25) and (26)

\[ \frac{N_2}{c} jw\phi' + i_2R_2 = 0 \]

Combining (27) and (28)

\[ \frac{N_2}{N_1} e_1 = - i_2R_2 = - e_2 \]

or

\[ \frac{E_1}{E_2} = - \frac{N_1}{N_2} \]

Note: Ratio of input and output voltage is equal to ratio of primary and secondary terms. This ideal condition is approached in transformer with low flux leakage. To find equivalent series circuit equations (21) and (22) give:

\[ L_{12} = \sqrt{L_{11}L_{22}} \]

Mutual inductance of two perfectly coupled circuits.

or

\[ L_{12} = k\sqrt{L_{11}L_{22}} \]

where \( k \) is loss in coupling, \( k < 1 \)

\[ e_1 \left[ R_2 + jwL_2 \right] = jwL_1R_2i_1 \]

(29)

by definition

\[ e_1 \left( \frac{1}{Z} \right) = i_1 \]

(30)

and substituting (30) in (29) we get

\[ \frac{1}{Z} = \frac{1}{jwL_1} + \frac{1}{R^2} \frac{L_1}{L_2} \]

\( Z = \text{Primary impedance of resistance loaded transformers.} \)

Equivalent circuit

\[ \text{Diagram of equivalent circuit} \]
For a good transformer \( wL_1 \) is low so input line sees resistance load times

\[
\left( \frac{N_1}{N_2} \right)^2
\]

For transformer with leakage

\[ X_{R_1} \quad X_{R_2} \]

\( X_R = \) Leakage reactance.

Find \( X_{R_1} \) on open circuit test.

Find \( X_{R_2} \) on short circuit test.