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Brownian Motion in Zp is a Limit of Discrete Time Random Walks

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Brownian Motion in $\mathbb{Z}_p$ is a Limit of Discrete Time Random Walks

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by

Tyler J. Pierce

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ABSTRACT OF THE DISSERTATION

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by

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Doctor of Philosophy, Graduate Program in Mathematics
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Dr. David Weisbart, Chairperson

The p-adic diffusion equation is a pseudo-differential equation that is formally analogous to the real diffusion equation. Where the real diffusion equation involves the Laplace operator, the p-adic diffusion equation involves the Vladimirov operator. The fundamental solution to the diffusion equation in both settings gives rise to a convolution semigroup of probability density functions that determines a Brownian motion. In both the real and p-adic settings, Brownian motion is a scaling limit.

Vladimirov used the Vladimirov operator to define an analogous operator on balls in $\mathbb{Q}_p$. Kochubei gave a probabilistic interpretation of this operator, the *Vladimirov-Kochubei operator*, and showed that it generates a real-time Brownian motion in the ring of p-adic integers. The main result of this thesis is the proof that the process valued in $\mathbb{Z}_p$ that is generated by a Vladimirov-Kochubei operator is a limit of discrete time random walks.

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Chapter 1

Introduction

1.1 A brief history of the diffusion equation

No description of the diffusion equation would be complete without a discussion of its origins. The first approach to studying diffusion that we would recognize today comes from Thomas Graham (1805-1869), a chemist. Graham was one of the first to recognize that the mixture of different gases does not separate by their densities. His work gave us what is called Graham's Law,

$$\frac{r_1}{r_2} = \sqrt{\frac{M_2}{M_1}}.$$

where r_1 and M_1 are the rate of effusion of gas 1 and the molar mass of gas 1, and r_2 and M_2 are similarly defined for gas 2. Graham's Law connected diffusion with the random motion of atoms. Later, Graham conducted experiments on the diffusion of salts. In his studies, he noticed the diffusion in liquids was several thousand times slower than in gases.

A major advance in the study of diffusion came from Adolf Fick (1829-1901), a

German physiologist. Suppose a thin tube is motionless and contains water with dye. The water and dye are initially at rest and are at a consistent uniform temperature. The tube is marked with a signed length scale. The concentration of the dye at time t and position x is $u(t,x)$. The dye flux $\phi(t,x)$ describes the mass of dye per unit time that moves across the point x at time t. He postulated the following: Dye flows from higher density to lower density. More precisely, there is a positive function D so that

$$\phi(t,x) = -D(x)\frac{\partial u}{\partial x}(t,x).$$

This law, Fick's first law, comes from experimental observation.

Since dye is neither created nor destroyed, the change in the amount of dye in a region depends only on the flow at the endpoints of the region, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} u(t,x)\,\mathrm{d}x = -(\phi(t,x_2) - \phi(t,x_1)).$$

The above equation is called the conservation equation. The Leibniz integration formula implies that for any x and any positive quantity Δx,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x}^{x+\Delta x} u(t,z)\,\mathrm{d}z = \int_{x}^{x+\Delta x} \frac{\partial u}{\partial t}(t,z)\,\mathrm{d}z.$$

The conservation equation together with the Leibniz integration formula implies that

$$\int_{x}^{x+\Delta x} \frac{\partial u}{\partial t}(t,z)\,\mathrm{d}z = -(\phi(t,x+\Delta x) - \phi(t,x)),$$

so that

$$\frac{1}{\Delta x}\int_x^{x+\Delta x}\frac{\partial u}{\partial t}(t,z)\,\mathrm{d}z = -\frac{1}{\Delta x}(\phi(t,x+\Delta x)-\phi(t,x)).$$

Take the limit as Δx approaches zero from the right and suppose that D is a constant to obtain the equalities

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) &= -\frac{\partial \phi}{\partial x}(t,x)\\ &= -\frac{\partial}{\partial x}\Big(-D\frac{\partial \phi}{\partial x}(t,x)\Big) = D\frac{\partial^2 u}{\partial x^2}(t,x),\end{aligned}$$

hence,

$$\frac{\partial u}{\partial t}(t,x) = D\frac{\partial^2 u}{\partial x^2}(t,x). \tag{1.1}$$

Equation (1.1) is Fick's second law.

Robert Brown (1773-1858) was a Scottish Botanist. Brown first studied medicine but his studies were interrupted by military service. He was stationed in Ireland where he had ample time to gaze at local flora. As the years went on Brown became a respected botanist, even though he never received formal training or a degree. In fact, he was selected for a scientific expedition to travel to "New Holland"—what we now call Australia—to collect as many different plant, insect, and bird species as possible. Brown's love of classifying plants often had him analyzing pollen suspended in water under a microscope. Within the grains of pollen he noticed smaller particles in seemingly random motion. Brown was not the first to notice this type of motion. A Roman poet, Lucretius, wrote about the random movement of dust particles in the air, but what he observed was more likely due to convection and turbulence.

It was not until 1918 when the mathematician Norbert Wiener gave a modern definition of Brownian motion. Namely, Brownian is a collection of random variables $\{X_t : t \in [0, \infty)\}$ with state space $\mathbb{R}$ such that:

1. $X_0 = 0$ (with probability 1).
2. X_t has stationary increments. That is, for any $s, t \in [0, \infty)$ with $s < t$, the distribution of $X_t - X_s$ is the same as the distribution of X_{t-s}.
3. X has independent increments. This is to say that for any $t_1, t_2, \ldots, t_n \in [0, \infty)$ with $t_1 < t_2 < \cdots < t_n$, the random variables $X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.
4. X_t is normally distributed with mean 0 and variance t for each $t \in (0, \infty)$.
5. With probability 1, $t \mapsto X_t$ is continuous on $[0, \infty)$.

The study of the convergence of discrete time random walks on scaled integer lattices to Brownian motion is both classical and foundational to the subject of probability. For any given positive real number D, the diffusion equation in the real setting with diffusion constant D is the partial differential equation

$$\frac{\partial \phi}{\partial t}(t, x) = \frac{D}{2} \frac{\partial^2 \phi}{\partial x^2}(t, x). \tag{1.2}$$

The fundamental solution, ρ, is given by

$$\rho(t, x) = \frac{1}{\sqrt{2\pi D t}} \exp\Big(- \frac{x^2}{2Dt} \Big).$$

The function ρ is a probability density function on the set of real numbers, $\mathbb{R}$, and gives rise to a Wiener measure W, a probability measure on the set $C([0,\infty):\mathbb{R})$ of continuous functions defined on $[0,\infty)$ and valued in $\mathbb{R}$ that gives full measure to the set of paths that are initially at zero at time zero. The stochastic process X that maps each non-negative real t to the random variable X_t, where X_t acts on the probability space $(C([0,\infty):\mathbb{R}), W)$ by

$$X_t(\omega) = \omega(t) \quad \text{with} \quad \omega \in C([0,\infty):\mathbb{R}),$$

is a Brownian motion. For each positive t, the random variable X_t is mean free with variance Dt. The space of continuous paths is a closed subspace of the Skorohod space $D([0,\infty):\mathbb{R})$, the set of cádlág functions mapping $[0,\infty)$ to $\mathbb{R}$ endowed with the Skorohod metric. A central result in the theory of convergence of stochastic processes is that Wiener measure is the weak limit of probability measures on $D([0,\infty):\mathbb{R})$ that are concentrated on the step functions and associated to a sequence of discrete time stochastic processes known as discrete time random walks. In this sense, Brownian motion is a limit of discrete time random walks [6].

1.2 Context and goals

The study of p-adic diffusion processes and of integration over p-adic path spaces has a long history and is ongoing. Vladimirov introduced in [18] a pseudo-differential operator that is, in many respects, an analog of the classical Laplacian in the p-adic setting and further investigated the spectral properties of this operator in [19]. Vladimirov and Volovich together initiated the study of quantum systems in the p-adic setting with their seminal article [20]. Ismagilov studied the spectra of self-adjoint operators in [12] in the setting of $L^2(K)$, where K is a local field. In [23], Zelenov studied Feynman integrals with p-adic valued paths. Kochubei gave not only the fundamental solution to the p-adic analog of the diffusion equation in [14], using the operator introduced by Vladimirov, but also developed a theory of p-adic diffusion equations and a Feynman-Kac formula for the operator semigroup with a p-adic Schrödinger type operator as its infinitesimal generator. Albeverio and Karwowski further investigated diffusion in the p-adic setting in [1], constructing a continuous time random walk on $\mathbb{Q}_p$, computing its transition semigroup and infinitesimal generator, and showing among other things that the associated Dirichlet form is of jump type.

We follow the approach of [17], in which Varadarajan discussed an analog to the diffusion equation in the non-Archimedean setting in the general context where the functions have domains contained in $[0, \infty) \times \mathcal{S}$, where $\mathcal{S}$ is a finite dimensional vector space over a division ring which is finite dimensional over a local field of arbitrary characteristic. The results of [17] specialize to show that the fundamental solutions to certain pseudo-differential equations formally analogous to the diffusion equation in the p-adic setting give rise to measures on the Skorohod space $D([0, \infty) : \mathbb{Q}_p)$ of cádlág functions defined on $[0, \infty)$ and valued in $\mathbb{Q}_p$. Given such a measure P on $D([0, \infty) : \mathbb{Q}_p)$, the associated stochastic process X that maps each non-negative real t to the random variable X_t, where X_t acts on $D([0, \infty) : \mathbb{Q}_p)$ by

$$X_t(\omega) = \omega(t) \quad \text{with} \quad \omega \in D([0, \infty) : \mathbb{Q}_p),$$

is a process with independent increments and a p-adic analog of a Brownian motion. The probability density function f_t for the random variable X_t is a solution to the pseudo-differential equation that gives rise to P. We show that the well-known convergence in the real setting of discrete time random walks to Brownian motion has an analog in the p-adic setting, demonstrating that the analogy between the p-adic diffusion equation and the real diffusion equation is the result of a general principal of convergence of discrete time random walks on grids to a continuum limit. Whereas earlier articles such as [5] and [4] discuss the convergence of sequences of continuous time random walks on grids in local fields to a continuum limit, this current work differs in that it studies the convergence of discrete time random walks. Discrete time random walks approximating p-adic diffusion offer greater intuition about their continuum limit but are, as one should expect, more difficult to study than the continuous time approximations.

The last two decades have seen considerable interest in p-adic mathematical physics. The book [21] seems to be the first textbook on p-adic mathematical physics and is still a standard reference in the field. The more recent article [8] gives a comprehensive overview of p-adic mathematical physics, as of the year 2009, and a detailed list of references that document the development of the subject. While there is intrinsic interest in the study of non-Archimedean analogs of Brownian motion, these analogs are also of interest because of their potential application to the study of physical systems. Ultrametricity arises naturally in the theory of complex systems and many study an area in which the present work should find direct application, for example, in the works [3] [2] of Avetisov, Bikulov, Kozyrev, and Osipov dealing with p-adic models for complex systems. There are also potential applications of discrete time p-adic random walks in the study of the fractal properties of p-adic spaces. For instance, [10] and [11] investigate p-adic fractal strings and their complex dimensions. The p-adic Brownian path spaces offer a new setting in which to study the theory of complex dimension and discrete time p-adic random walks already appear useful in developing intuition about dimension in this context. Discrete time random walks that converge to real Brownian motion give intuition about the properties of real Brownian motion and give insight into these more complicated processes.

Studying diffusion on $\mathbb{Q}_p$ is of particular importance to mathematical physics. In particular, the structural differences between diffusion over $\mathbb{Q}_p$ and diffusion over $\mathbb{Z}_p$ may give insight into other diffusion processes since $\mathbb{Z}_p$ is a cone over a projective limit. There has been little investigation into diffusion over ultrametric compact sets. A foundational question in the study of diffusion processes is whether a continuous time process is a scaling limit. In 2019, Bakken and Weisbart showed p-adic Brownian motion is a limit of discrete time random walks [5]. Then, in 2020 Weisbart showed p-adic Brownian motion is a scaling limit. In 2024, Weisbart and collaborators showed Brownian motion in a vector space over a local field is a scaling limit [22, 16]. It is natural to ask the question *"Is Brownian motion over $\mathbb{Z}_p$ a scaling limit?"* In this article, we get one step closer to providing an answer to that question by showing Brownian motion over $\mathbb{Z}_p$ is a limit of discrete time random walks.

We suspect that there may be a way of understanding Brownian motion in $\mathbb{Z}_p$ as a scaling limit, but we require a creative leap in the discretization of $\mathbb{Z}_p$. When we discretize $\mathbb{Z}_p$ we get different probabilities, so we do not have a single primitive process with associated probabilities that embed into the underlying groups.

Chapter 2

Path spaces and convergence of measures

2.1 Paths and processes

Assume throughout this section that the set $\mathcal{S}$ is a Polish space, a completely metrizable, separable topological space.

Definition 2.1.1. *A continuous time interval I is an interval in $\mathbb{R}$ with left endpoint included and equal to zero. A discrete time interval I is a discrete subset of $[0, \infty)$ that contains zero. A time interval is either a continuous or discrete time interval.*

Definition 2.1.2. *Let I be a time interval and denote by $F(I : \mathcal{S})$ the set of all functions ω with $\omega : I \to \mathcal{S}$. The set $F(I : \mathcal{S})$ is the space of all paths in $\mathcal{S}$ with domain I.*

Definition 2.1.3. *A set $\mathcal{P}$ is a path space of $\mathcal{S}$ if there is a time interval I such that $\mathcal{P}$ is a subset of $F(I : \mathcal{S})$.*

Definition 2.1.4. *An epoch for a path space $\mathcal{P}$ with time interval I is a strictly increasing finite sequence e that is valued in $I \backslash \{0\}$.*

Definition 2.1.5. *A set h is a history for a path space $\mathcal{P}$ with time interval I if there is a natural number k and an epoch e with*

$$e = (t_1, \ldots, t_k),$$

and Borel subsets $U_0, \ldots, U_k$ of $\mathcal{S}$ such that

$$h = ((0, U_0), (t_1, U_1), \ldots, (t_k, U_k)).$$

The finite sequence U with

$$U = (U_0, U_1, \ldots, U_k)$$

is said to be the route of h. If h is a history, then denote by e_h the epoch associated to h, by U_h the route associated to h, and by $\ell(h)$ the number of places of e_h, the length of h.

The language established in the above definitions allows us to say that a history is the pairing of an epoch and an initial time point with a route with one more place than the number of places of the epoch, a starting location that is paired with the initial time.

Definition 2.1.6. *Let H be the set of all histories for $\mathcal{P}$. Define a function $\mathscr{C}$ that associates to each h in H a set $\mathscr{C}(h)$ by*

$$\mathscr{C}(h) = \bigcap_{i \in \{0, \ldots, \ell(h)\}} \{\omega \in \Omega(I) : \omega(e_h(i)) \in U_h(i)\}.$$

The set $\mathscr{C}(H)$ is said to be the set of all simple cylinder sets of $\mathcal{P}$ and a set is said to be a simple cylinder set if it is an element of $\mathscr{C}(H)$.

Remark 2.1.7. *Since the route may take the empty set as a value, the set $\mathscr{C}(H)$ contains the empty set and forms a π-system, though not an algebra. Any two probability measures on the σ-algebra that is generated by the simple cylinder sets that agree on the simple cylinder sets also agree on the σ-algebra that simple cylinder sets generate.*

Central to the present study are the probability measures on the σ-algebra generated by the π-system of simple cylinder sets in Skorohod space $D([0,\infty) : \mathcal{S})$, the set of càdlàg functions from $[0,\infty)$ to $\mathcal{S}$ equipped with the Skorohod metric.

Definition 2.1.8. *Take $\Omega(I)$ to be either the space $F(I : \mathcal{S})$ or $D([0,\infty) : \mathcal{S})$, and Y to be the function that acts on any pair (t, ω) in $I \times \Omega(I)$ by*

$$Y(t, \omega) = \omega(t).$$

Compress notation by writing Y_t rather than $Y(t, \cdot)$, so that for any path ω in $\Omega(I)$,

$$Y_t(\omega) = \omega(t).$$

The probabilities associated to the simple cylinder sets are the finite dimensional distributions of Y. For any probability measure P on $\Omega(I)$, any t in I, and any B in the Borel set, $\mathcal{B}(S)$, the equality,

$$\mathrm{P}_t(B) = \mathrm{P}(Y_t^{-1}(B))$$

defines a family $(\mathrm{P}_t)_{t\in I}$ of probability measures on $\mathcal{S}$. The current work involves paths in a more restrictive setting where $\mathcal{S}$ is, additionally, a locally compact, Abelian group with Haar measure μ. In this setting, the family $(\mathrm{P}_t)_{t\in I}$ forms a convolution semigroup of probability measures on $\mathcal{S}$ that gives rise to a measure P on a space of paths in $\mathcal{S}$. The finite dimensional distributions of the measure are, for any history h, given by

$$\mathscr{C}(h) = \int_{U_h(0)} \cdots \int_{U_h(\ell(h))} \prod_{i=1}^{\ell(h)} \mathrm{dP}_{e_h(i)-e_h(i-1)}(x_i - x_{i-1}) \tag{2.1}$$

and $\mathrm{P}(\mathscr{C}(h))$ is nonzero only if $U_h(0)$ contains 0. In the case of a Brownian motion on $\mathcal{S}$, the probability measures come from a convolution semigroup of probability density functions $(\rho(t,\cdot))_{t>0}$, so that (1.3) becomes

$$\mathscr{C}(h) = \int_{U_h(0)} \cdots \int_{U_h(\ell(h))} \prod_{i=1}^{\ell(h)} \rho_{e_h(i)-e_h(i-1)}(x_i - x_{i-1})\, \mathrm{d}\mu(x_1)\cdots \mathrm{d}\mu(x_{\ell(h)}). \tag{2.2}$$

The formal construction of finite dimensional distributions for a process based on some physical intuition about the process typically precedes the verification that there is a process with the proposed finite dimensional distributions. For this reason, it is useful to refer to the construction of the finite dimensional distributions for a stochastic process as the construction of an abstract stochastic process $\tilde{Y}$ and the construction of the law for a random variable without the specification of its domain as the construction of an abstract random variable $\tilde{X}$. A model for the abstract random variable $\tilde{X}$ is a random variable with the same law as $\tilde{X}$. A model for the abstract stochastic process $\tilde{Y}$ is a stochastic process with the same finite dimensional distributions as $\tilde{Y}$. To distinguish the probabilities associated with the abstract random variable $\tilde{X}$ from those associated with a model for $\tilde{X}$, write $\text{Prob}(\tilde{X} \in A)$ to mean the probability that $\tilde{X}$ takes a value in some Borel set A. Adopt a similar notation for abstract stochastic processes. It is also helpful to adopt a notation that distinguishes between the discrete-time and continuous-time stochastic processes that the present work involves. In the discrete-time setting, the time interval is $\mathbb{N}_0$ and the paths are paths in a discrete space, so $F(\mathbb{N}_0 : \mathcal{S})$ is the space of continuous paths. In this case, write $(F(\mathbb{N}_0 : \mathcal{S}), \mathrm{P}, \mathcal{S})$ to denote the stochastic process rather than $(\Omega(I : \mathcal{S}), \mathrm{P}, Y)$, where for each n in $\mathbb{N}_0$,

$$S_n(\omega) = \omega(n).$$

For any history h for paths in $F(\mathbb{N}_0 : \mathcal{S})$,

$$\mathrm{P}(\mathscr{C}(h)) = \sum_{x_0 \in U_h(0)} \cdots \sum_{x_{\ell(h)} \in U_h(\ell(h))} \prod_{i=1}^{\ell(h)} \mathrm{P}(S_{e_h(i)} - S_{e_h(i-1)} = x_i - x_{i-1}),$$

and is nonzero only if $U_h(0)$ contains 0.

Here we provide the reader with some theorems and results required in this paper. The following theorem is what allows us to get a measure on the σ-algebra of cylinder sets of $F(I : \mathcal{S})$, which agrees with our premeasure.

Theorem 2.1.9 (Kolmogorov Extension Theorem). *Given a consistent family* $\{\mathrm{P}^{t_1,\ldots,t_k}\}$ *of probability measures on the finite products* $\prod_{1 \le i \le k} X_{t_i}$*, each* X_t *being polish, there is a unique probability measure* P *on* $\mathcal{S}(M)$ *which induces the* $\mathrm{P}^{t_1,\ldots,t_k}$ *under the natural projections* $X \to \prod_{1 \le i \le k} X_t$.

This next theorem allows us to get a version of the process with paths in Skorohod space. That is, we get another process whose finite dimensional distributions agree with our original ones.

Theorem 2.1.10 (Čentsov criteria)**.** *Suppose* $(X(t))$ *is a stochastic process such that*

(a) X_t *is stochastically right continuous at all* $t \in [0, T)$ *and stochastically left continuous at* T.

(b) There are constants $C, a, b, c > 0$ *such that for all* $0 \leq t_1 < t_2 < t_3 \leq T$ *we have*

$$\mathbb{E}\Big[|X(t_1) - X(t_2)|^a |X(t_2) - X(t_3)|^b\Big] \leq C|t_1 - t_3|^{1+c}$$

Then there exist a unique probability measure P *on* $D(I \colon \mathcal{S})$ *such that for all* $k \geq 1, \quad t_1, \ldots, t_k \in [0, T]$, *the measure* $\mathrm{P}^{t_1, \ldots, t_k}$ *is the joint probability measures of* $(X_{t_1}, \ldots, X_{t_k})$.

This next theorem bridges the gap from the above theorem to convergence of probability measures. It shows that the uniformity in the moment constant implies tightness of measures, and with convergence of finite dimensional distributions we get weak convergence.

Theorem 2.1.11. *Suppose* P *and, for any* n *in* $\mathbb{N}$, P_n *are probability measures on* $D(I \colon \mathcal{S})$ *such that*

(a) $P_n^{t_1, \ldots, t_k} \to P^{t_1, \ldots, t_k}$ *for all* $k \geq 1, \quad t_1, \ldots, t_k \in [0, T]$

(b) and are constants $C, a, b, c > 0$ *such that for all* $0 \leq t_1 < t_2 < t_3 \leq T$ *we have*

$$\mathbb{E}_{P_n}\Big[|X(t_1) - X(t_2)|^a |X(t_2) - X(t_3)|^b\Big] \leq C|t_1 - t_3|^{1+c}$$

then

$$\mathrm{P}_n \to \mathrm{P}.$$

Note that the convergence in Theorem 2.1.11 is the weak convergence of probability measures. For any sequence (P_n) of probability measures on $D(I\colon \mathcal{S})$ and any probability measure P_n on $D(I\colon \mathcal{S})$, to say that (P_N) converges weakly to P means that for any continuous functional f on $D(I\colon \mathcal{S})$,

$$\int_{D(I\colon \mathcal{S})} f(\omega)\,\mathrm{dP}_n \to \int_{D(I\colon \mathcal{S})} f(\omega)\,\mathrm{dP}.$$

2.2 Classical example

We now turn to a guiding example which will provide a framework for the new results in this article.

Take $\tilde{X}$ to be an abstract random variable with this law:

$$\begin{cases} \mathrm{Prob}(\tilde{X} = -1) = \frac{1}{2} \\ \mathrm{Prob}(\tilde{X} = 1) = \frac{1}{2}. \end{cases}$$

Take $\left(\tilde{X}_i\right)_{i\in\mathbb{N}}$ to be a sequence of independent identically distributed random variables, each with the same distribution as $\tilde{X}$. Let $\tilde{X}_0$ be equal to zero with probability one and define $\tilde{S}_n$ the random variable

$$\tilde{S}_n = \tilde{X}_0 + \tilde{X}_1 + \cdots + \tilde{X}_n.$$

Given that k is an integer, the distribution for $\tilde{S}_n$ is given by

$$\text{Prob}(\tilde{S}_n = k) = \begin{cases} \frac{\binom{n}{\frac{n-k}{2}}}{2^n} & \text{if } n-k \text{ is even,} \\ \\ 0 & \text{if } n-k \text{ is odd.} \end{cases}$$

The following figure is a helpful visual for the random walk on $\mathbb{Z}$.

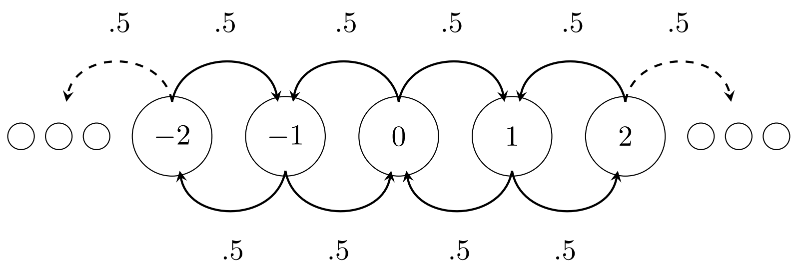

Figure 2.1: A Symmetric Random Walk on $\mathbb{Z}$

Additivity of the mean and independence of the $\left(\tilde{X}_i\right)$ imply that

$$\begin{aligned} \mathbb{E}[\tilde{S}_n] &= \mathbb{E}[\tilde{X}_0 + \tilde{X}_1 + \cdots + \tilde{X}_n] \\ &= \mathbb{E}[\tilde{X}_0] + \mathbb{E}[\tilde{X}_1] + \cdots + \mathbb{E}[\tilde{X}_n] \\ &= 0 + n\mathbb{E}[\tilde{X}] = 0 \end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}[\tilde{S}_n] &= \mathrm{Var}[\tilde{X}_0 + \tilde{X}_1 + \cdots + \tilde{X}_n] \\ &= \mathrm{Var}[\tilde{X}_0] + \mathrm{Var}[\tilde{X}_1] + \cdots + \mathrm{Var}[\tilde{X}_n] \\ &= n\mathrm{Var}[\tilde{X}] = n.\end{aligned}$$

The sequence of random variables $\tilde{S}$, with $\tilde{S} : n \to \tilde{S}_n$ for $n \in \mathbb{N}_0$ is a $\mathbb{Z}$–valued stochastic process.

For any history h, define the probability of $\mathscr{C}(h)$ by

$$\begin{aligned}\mathrm{P}(\mathscr{C}(h)) = \sum_{x_0 \in U_h(0)} \cdots \sum_{x_k \in U_h(\ell(h))} &\mathrm{Prob}\big(\tilde{S}_{e_h(\ell(h))} - \tilde{S}_{e_h(\ell(h)-1)} = x_{e_h(\ell(h))} - x_{e_h(\ell(h)-1)}\big) \\ &\cdots \mathrm{Prob}\big(\tilde{S}_{e_h(1)} - \tilde{S}_{e_h(0)} = x_{e_h(1)} - x_0\big)\mathrm{Prob}\big(\tilde{S}_0 = x_0\big).\end{aligned}$$

The Kolmogorov Extension theorem guarantees the existence of a measure on $F(\mathbb{N}_0 : \mathbb{Z})$ that assigns these probabilities to the simple cylinder sets. Define for each n in $\mathbb{N}$ and each ω in $F(\mathbb{N}_0 : \mathbb{Z})$ the random variable S_n by

$$S_n(\omega) = \omega(n).$$

The following maps make it possible to view the primitive random walk as modeling the position of a particle moving in time throughout a grid in $\mathbb{R}$. For any sequence of spatial scales (δ_m) and time scales (τ_m), denote by $\imath_m$ a real spatiotemporal embedding, that is, a function that for any (n, z) in $\mathbb{N}_0 \times \mathbb{Z}$ are given by

$$\Gamma_m(z) = \delta_m z \text{ and } \imath_m(n, z) = (n\tau_m, \Gamma_m(z)).$$

The embedding $\imath_m$ induces a mapping from the random process S to a stochastic process indexed by $\mathbb{R}_{\geq 0}$ and valued in $\mathbb{R}$. Define for each non-negative t the abstract random process

$$\tilde{Y}_t^m = \delta_m\big(\tilde{S}_n\big) = \delta_m\big(\tilde{X}_0 + \cdots + \tilde{X}_{\lfloor \frac{t}{\tau_m} \rfloor}\big).$$

Now, take any epoch $(t_1, \ldots, t_k)$, associated route $(U_0, \ldots, U_k)$, and h the history with this epoch and route define

$$\mathrm{P}^m(\mathscr{C}(h)) = \sum_{x_0 \in U_0 \cap \delta_m \mathbb{Z}} \cdots \sum_{x_k \in U_k \cap \delta_m \mathbb{Z}} \mathrm{Prob}(\tilde{Y}_{t_k}^m - \tilde{Y}_{t_{k-1}}^m = x_k - x_{k-1}) \cdots \mathrm{Prob}(\tilde{Y}_0^m = x_0).$$

The Kolmogorov Extension theorem extends our pre-measure to a measure, P^m, so that $(F([0, \infty) : \mathbb{R}), \mathrm{P}^m, Y)$ is a model for $\tilde{Y}^m$. The sample space $(F([0, \infty) : \mathbb{R})$ is too large of a space to be very useful and it is important to demonstrate the existence of a version of the process in a smaller sample space. To this end, we need the following proposition.

Denote by $\mathbb{E}^m$ the expected value with respect to P^m.

Proposition 2.2.12. *There exist positive constants $C, \epsilon, a,$ and b so that for any epoch (t_1, t_2, t_3),*

$$\mathbb{E}^m\left[|Y_{t_2} - Y_{t_1}|^a |Y_{t_3} - Y_{t_2}|^b\right] \leq C(t_3 - t_1)^{1+\epsilon}.$$

Proof. For any natural numbers a, b, i, and j, if $i \neq j$, then,

$$\mathbb{E}\big[\tilde{X}_i^a \tilde{X}_j^b\big] = \begin{cases} 0 & \text{if either } a \text{ or } b \text{ is odd,} \\ \\ 1 & \text{if both } a \text{ and } b \text{ are even.} \end{cases}$$

Moreover, if a is even then,

$$\mathbb{E}[\tilde{X}_i^a] = 1,$$

and so if $m_2 > m_1$ then,

$$\begin{aligned} \mathbb{E}[(\tilde{S}_{m_2} - \tilde{S}_{m_1})^2] &= \mathbb{E}[(\tilde{X}_{m_1+1} + \cdots + \tilde{X}_{m_2})^2] \\ &= \mathbb{E}[(\tilde{X}_{m_1+1})^2 + \cdots + (\tilde{X}_{m_2})^2] \\ &= \mathbb{E}[(\tilde{X}_{m_1+1})^2] + \cdots + \mathbb{E}[(\tilde{X}_{m_2})^2] = m_2 - m_1. \end{aligned}$$

For any epoch (t_1, t_2, t_3) in I with,

$$\left\lfloor \frac{t_1}{\tau} \right\rfloor \leq \left\lfloor \frac{t_2}{\tau} \right\rfloor \text{ and } \left\lfloor \frac{t_2}{\tau} \right\rfloor \leq \left\lfloor \frac{t_3}{\tau} \right\rfloor,$$

independence of the increments of the process $\tilde{Y}$ implies that,

$$\begin{aligned} \mathbb{E}\big[|Y_{t_2} - Y_{t_1}|^2 |Y_{t_3} - Y_{t_2}|^2\big] &= \mathbb{E}\big[|\tilde{Y}_{t_2} - \tilde{Y}_{t_1}|^2 |\tilde{Y}_{t_3} - \tilde{Y}_{t_2}|^2\big] \\ &= \mathbb{E}\big[\big(|\tilde{Y}_{t_2} - \tilde{Y}_{t_1}|^2\big] \mathbb{E}\big[|\tilde{Y}_{t_3} - \tilde{Y}_{t_2}|^2\big] \\ &= \mathbb{E}[\delta^2 \big(\tilde{X}_{\lfloor \frac{t_1}{\tau} \rfloor + 1} + \cdots + \tilde{X}_{\lfloor \frac{t_2}{\tau} \rfloor}\big)^2 \cdot \mathbb{E}[\delta^2 (\tilde{X}_{\lfloor \frac{t_2}{\tau} \rfloor + 1} + \cdots + \tilde{X}_{\lfloor \frac{t_3}{\tau} \rfloor}\big)^2] \\ &= \delta^4 \mathbb{E}\big[\big(\tilde{X}_{\lfloor \frac{t_1}{\tau} \rfloor + 1} + \cdots + \tilde{X}_{\lfloor \frac{t_2}{\tau} \rfloor}\big)^2 (\tilde{X}_{\lfloor \frac{t_2}{\tau} \rfloor + 1} + \cdots + \tilde{X}_{\lfloor \frac{t_3}{\tau} \rfloor}\big)^2\big]. \end{aligned}$$

If the assumption on (t_1, t_2, t_3) is not valid, then

$$\mathbb{E}\big[|Y_{t_2} - Y_{t_1}|^2 |Y_{t_3} - Y_{t_2}|^2\big] = 0.$$

If (t_1, t_2, t_3) the above inequality, then

$$\mathbb{E}\big[|Y_{t_2} - Y_{t_1}|^2 |Y_{t_3} - Y_{t_2}|^2\big] = \delta^4 \left(\left\lfloor \frac{t_2}{\tau} \right\rfloor - \left\lfloor \frac{t_1}{\tau} \right\rfloor \right) \left(\left\lfloor \frac{t_3}{\tau} \right\rfloor - \left\lfloor \frac{t_2}{\tau} \right\rfloor \right),$$

and so

$$\mathbb{E}\big[|Y_{t_2} - Y_{t_1}|^2 |Y_{t_3} - Y_{t_2}|^2\big] \leq \delta^4 \left(\frac{t_3}{\tau} - \frac{t_1}{\tau} \right)^2 = \left(\frac{\delta^2}{\tau} \right)^2 (t_3 - t_1).$$

The final inequality needs some justification. It is not certain that

$$\left\lfloor \frac{t_3}{\tau} \right\rfloor - \left\lfloor \frac{t_2}{\tau} \right\rfloor \leq \frac{t_3 - t_1}{\tau} \quad \text{or} \quad \left\lfloor \frac{t_2}{\tau} \right\rfloor - \left\lfloor \frac{t_1}{\tau} \right\rfloor \leq \frac{t_3 - t_1}{\tau}.$$

However, if the first inequality fails to hold, then

$$\left\lfloor \frac{t_2}{\tau} \right\rfloor - \left\lfloor \frac{t_1}{\tau} \right\rfloor = 0.$$

If the second fails to hold, then

$$\left\lfloor \frac{t_3}{\tau} \right\rfloor - \left\lfloor \frac{t_2}{\tau} \right\rfloor = 0.$$

So, in any case,

$$\mathbb{E}\big[|Y_{t_2} - Y_{t_1}|^2 |Y_{t_3} - Y_{t_2}|^2\big] = 0.$$

∎

Proposition (2.2.12) implies that the process Y satisfies the Čentsov Criterion, which implies the important theorem.

Theorem 2.2.13. *There is a version of the process Y with sample paths in $D([0,\infty):\mathbb{R})$.*

Let's compress notation, denote by Y the concrete process for $\tilde{Y}$ with sample paths in $D([0,\infty):\mathbb{R})$ and with corresponding measure $\mathrm{P}_{(\delta,\tau)}$. The jumps of the process Y are the values in the range of $Y_t - Y_s$ where t and s are non-negative real numbers with t greater than s.

Proposition 2.2.14. *The subset of paths in $D([0,\infty):\mathbb{R})$ that, for each natural number m, are constant on the intervals $[(m-1)\tau, m\tau)]$ and that are $\delta\mathbb{Z}$-valued have full measure.*

Proof. For any natural number k and for any epoch $(t_1,\ldots,t_k)$,

$$\mathrm{P}_{(\delta,\tau)}(\{\omega \in \mathrm{D}([0,\infty):\mathbb{R}) : \omega(\mathrm{t_i}) \in \delta\mathbb{Z}, 1 \leq \mathrm{i} \leq \mathrm{k}\}) = 1.$$

Take $\{A_i : i \in \mathbb{N}\}$ to be a set of strictly increasing nested finite subsets of $\mathbb{Q}$ whose union is $\mathbb{Q}$. Continuity from above of the measure implies that

$$\mathrm{P}_{(\delta,\tau)}(\{\omega \in \mathrm{D}([0,\infty):\mathbb{R}) : \omega(\mathrm{s}) \in \delta\mathbb{Z}, \forall \mathrm{s} \in \mathrm{A_i}\}) = 1.$$

The right continuity of the paths implies that the set of paths that take values only in $\delta\mathbb{Z}$ has full measure. To prove that the paths can only change values at time points in $\tau\mathbb{N}$, take t to be any point that is not in $\tau\mathbb{Z}$ and (I_i) to be an increasing sequence of finite subsets of $\left[\tau\lfloor\frac{t}{\tau}\rfloor, \tau\lceil\frac{t}{\tau}\rceil\right) \cap \mathbb{Q}$ with the property that

$$\bigcup_{i\in\mathbb{N}} I_i = \left[\tau\left\lfloor\frac{t}{\tau}\right\rfloor, \tau\left\lceil\frac{t}{\tau}\right\rceil\right) \cap \mathbb{Q}.$$

For any pair of real numbers s_1, s_2, if $\lfloor s_1\rfloor = \lfloor s_2\rfloor$, then

$$\begin{aligned}\mathrm{Prob}(Y_{s_1} - Y_{s_2} = 0) &= \mathrm{Prob}\big(\tilde{Y}_{s_1} - \tilde{Y}_{s_2} = 0\big)\\ &= \mathrm{Prob}\big(\tilde{X}_{\lfloor s_1\rfloor} - \tilde{X}_{\lfloor s_2\rfloor} = 0\big) = 1.\end{aligned}$$

Continuity from above of the measure implies that

$$\mathrm{P}\left(\left\{\omega \in \mathrm{D}([0,\infty):\mathbb{R}) : |\omega(\mathrm{t}) - \omega(\mathrm{s})| = 0,\ \forall \mathrm{s} \in \left[\tau\left\lfloor\frac{\mathrm{t}}{\tau}\right\rfloor, \tau\left\lceil\frac{\mathrm{t}}{\tau}\right\rceil\right) \cap \mathbb{Q}\right\}\right)$$

$$= \lim_{i\to\infty} \mathrm{P}\left(\left\{\omega \in \mathrm{D}([0,\infty):\mathbb{R}) : |\omega(\mathrm{t}) - \omega(\mathrm{s})| = 0,\ \forall \mathrm{s} \in \left[\tau\left\lfloor\frac{\mathrm{t}}{\tau}\right\rfloor, \tau\left\lceil\frac{\mathrm{t}}{\tau}\right\rceil\right) \cap \mathrm{I_i}\right\}\right)$$

The right continuity of the paths implies that

$$\mathrm{P}_{(\delta,\tau)}\left(\left\{\omega \in \mathrm{D}([0,\infty):\mathbb{R}) : |\omega(\mathrm{t}) - \omega(\mathrm{s})| = 0,\ \forall \mathrm{s} \in \left[\tau\left\lfloor\frac{\mathrm{t}}{\tau}\right\rfloor, \tau\left\lceil\frac{\mathrm{t}}{\tau}\right\rceil\right)\right\}\right) = 1.$$

■

Denote by (P^m) the measures associated to (Y^m), so that

$$\mathbb{E}[Y_t^m] = 0 \text{ and } \mathrm{Var}[Y^m] = \delta_m^2\left\lfloor\frac{t}{\tau_m}\right\rfloor.$$

The sequence of stochastic processes (Y^m) converge to a process Y if Y has sample paths in $D([0,\infty):\mathbb{R})$, associated measure P, and if the sequence of measures (P^m) converge weakly to P. This is to say, for any bounded continuous functional on $D([0,\infty):\mathbb{R})$

$$\int_{D([0,\infty):\mathbb{R})} f(\omega)dP^m \to \int_{D([0,\infty):\mathbb{R})} f(\omega)\,\mathrm{dP}.$$

If these processes are to converge to some limiting stochastic process Y with

$$\mathrm{Var}[Y] = D$$

then

$$\frac{\delta_m^2}{\tau_m} \to D,$$

then the estimate given by Proposition (2.2.12) is uniform in m, which guarantees the uniform tightness of (P^m). Uniform tightness and the convergence on the simple cylinder sets of (P^m) to P together imply the weak convergence of (P^m) to P.

Chapter 3

The $\mathbb{Q}_p$ and $\mathbb{Z}_p$ settings

3.1 Analysis in $\mathbb{Q}_p$

See the book by Gouvéa [9] for further background on the p-adic numbers that this subsection summarizes. Refer to the book by Vladimirov, Volovich, and Zelenov [23] for both an introduction to p-adic mathematical physics and for many helpful examples of integration over $\mathbb{Q}_p$. Fix a prime number, p, and denote by $\mathbb{Q}_p$ the field of p-adic numbers, the completion of the rational numbers with respect to the p-adic valuation $|\cdot|_p$. For each x in $\mathbb{Q}_p$ and each integer k, denote by $B_k(x)$ and $S_k(x)$ the sets

$$B_k(x) = \{y \in \mathbb{Q}_p : |y - x| \leq p^k\} \text{ and } S_k(x) = \{y \in \mathbb{Q}_p : |y - x| = p^k\}.$$

Denote by $\mathbb{Z}_p$ the ring of integers, the set $B_0(0)$. Let μ be the Haar measure on the additive group $\mathbb{Q}_p$, normalized so that the measure of $\mathbb{Z}_p$ is equal to one. Uniquely associated to each x in $\mathbb{Q}_p$ is a function

$$a_x : \mathbb{Z} \to \{0, 1, \ldots, p-1\} \quad \text{and} \quad x = \sum_{k \in \mathbb{Z}} a_x(k) p^k$$

and for some natural number $N,\ k > N \Rightarrow a_x(-k) = 0$.

For each x in $\mathbb{Q}_p$, define $\{x\}$ by

$$\{x\} = \sum_{k<0} a_x(k) p^k \quad \text{and} \quad \chi(x) = \mathrm{e}^{2\pi i \{x\}}.$$

The function χ is a rank zero character on $\mathbb{Q}_p$, this is to say that χ is identically equal to one on $\mathbb{Z}_p$ and is not identically one on any ball centered at the origin with radius larger than one. The locally compact group $\mathbb{Q}_p$ is self dual for any character ϕ on $\mathbb{Q}_p$, there is an α in $\mathbb{Q}_p$ so that for all x in $\mathbb{Q}_p$,

$$\phi(x) = \chi(\alpha x).$$

Denote by $\mathcal{F}$ the Fourier transform on $L^2(\mathbb{Q}_p)$, the unitary extension to $L^2(\mathbb{Q}_p)$ of the operator initially defined as the unitary operator mapping $L^1(\mathbb{Q}_p) \cap L^2(\mathbb{Q}_p)$ to $L^2(\mathbb{Q}_p)$ by

$$(\mathcal{F}f)(x) = \int_{\mathbb{Q}_p} \chi(-xy) f(y) \,\mathrm{d}y,$$

Denote by $\mathcal{F}^{-1}$ the inverse Fourier transform. For all f in $L^1(\mathbb{Q}_p) \cap L^2(\mathbb{Q}_p)$,

$$(\mathcal{F}^{-1}f)(y) = \int_{\mathbb{Q}_p} \chi(xy) f(x) \,\mathrm{d}x.$$

3.2 Brownian motion in $\mathbb{Q}_p$

Fix b to be in $(0,\infty)$ and denote by $SB(\mathbb{Q}_p)$ the Schwartz-Bruhat space of compactly supported, locally constant, $\mathbb{C}$-valued functions. Note that $SB(\mathbb{Q}_p)$ is invariant under the Fourier transform. Take $\mathcal{M}$ to be the multiplication operator that acts on $SB(\mathbb{Q}_p)$ by

$$(\mathcal{M}f)(x) = |x|^b f(x).$$

Denote by Δ_b' the unique self-adjoint extension of the essentially self-adjoint operator that acts on any f in $SB(\mathbb{Q}_p)$ by

$$(\Delta_b' f)(x) = (\mathcal{F}^{-1}\mathcal{M}\mathcal{F}f)(x).$$

For any $\mathbb{C}$-valued function g with domain $\mathbb{R}_+ \times \mathbb{Q}_p$ and any t in $\mathbb{R}_+$, take g_t to be the function that is defined for every x in $\mathbb{Q}_p$ by

$$g_t(x) = g(t,x).$$

Denote by $\mathcal{D}(\Delta_b)$ the set of all such g so that for all t in $\mathbb{R}_+$, g_t is in the domain of Δ_b'. Take Δ_b to be the Vladimirov operator with exponent b that acts on any g in $\mathcal{D}(\Delta_b)$ by

$$(\Delta_b g)(t,x) = (\Delta_b g_t)(x).$$

Similarly extend the Fourier transform to act on $\mathbb{C}$-valued functions on $\mathbb{R}_+ \times \mathbb{Q}_p$ that are square integrable for each positive t and the operator $\frac{\mathrm{d}}{\mathrm{d}t}$ to act on $\mathbb{C}$-valued functions on

$\mathbb{R}_+ \times \mathbb{Q}_p$ that for any x in $\mathbb{Q}_p$ are differentiable in the t variable. Fix σ to be in $(0, \infty)$. The function ρ that is for any (t, x) in $\mathbb{R}_+ \times \mathbb{Q}_p$ given by

$$\rho(t, x) = (\mathcal{F}^{-1} \mathrm{e}^{-\sigma t |\cdot|^b})(x) \tag{3.1}$$

is the fundamental solution for the pseudo-differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\sigma \Delta_b u.$$

Minor modification of Varadarajan's arguments [17] show that

$$\rho(t, x) = \sum_{k \in \mathbb{Z}} p^k \big(\mathrm{e}^{-\sigma t p^{kb}} - \mathrm{e}^{-\sigma t p^{(k+1)b}}\big) \mathbb{1}_{B_{-k}}(x).$$

The set $\{\rho(t, \cdot) : t \in \mathbb{R}_+\}$ is a convolution semigroup of probability density functions that determines the probabilities of simple cylinder sets in the same way as in the real case, by (1.3). These probabilities determine a probability measure P on $D([0, \infty) : \mathbb{Q}_p)$ that is concentrated on the paths that are initially at 0.

3.3 Brownian motion in $\mathbb{Z}_p$

Take $SB(\mathbb{Z}_p)$ to be the locally constant functions on $\mathbb{Z}_p$. Define the function $\sharp$ from $SB(\mathbb{Z}_p)$ to $SB(\mathbb{Q}_p)$ by

$$\sharp f(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{Z}_p \\ \\ 0 & \text{otherwise.} \end{cases}$$

For any function f that acts on $\mathbb{Q}_p$ and any subset X of $\mathbb{Q}_p$, denote by $f\big|_X$ the restriction of f to X. Vladimirov and Kochubei both worked in the slightly more general setting of p-adic balls [19]. In the context of $\mathbb{Z}_p$, define $\Delta'_{b,0}$ to act on any f in $SB(\mathbb{Z}_p)$ by

$$(\Delta'_{b,0} f)(x) = (\Delta_b \circ \sharp f)\Big|_{\mathbb{Z}_p}(x).$$

The *Vladimirov-Kochubei operator* $\Delta_{b,0}$ is the self-adjoint closure of $\Delta'_{b,0}$. Kochubei gave a probabilistic interpretation of this operator and studied its properties [15]. The current section specializes to the ball $\mathbb{Z}_p$, but includes the general diffusion coefficient D.

The Pontryagin dual of $\mathbb{Z}_p$ is the discrete group $\mathbb{Q}_p/\mathbb{Z}_p$. Take $\langle \cdot \rangle$ to be the quotient map from $\mathbb{Q}_p$ to $\mathbb{Q}_p/\mathbb{Z}_p$ that is given for each x in $\mathbb{Q}_p$ by

$$\langle x \rangle = x + \mathbb{Z}_p.$$

Define the absolute value $|\cdot|$ on $\mathbb{Q}_p/\mathbb{Z}_p$ by

$$|\langle x\rangle| = \begin{cases} |x| & \text{if } \langle x\rangle \neq \langle 0\rangle \\ \\ 0 & \text{if } \langle x\rangle = \langle 0\rangle. \end{cases}$$

Take μ_0 to be the counting measure on $\mathbb{Q}_p/\mathbb{Z}_p$, which is the unique Haar measure that gives $\mathbb{Z}_p$ unit measure. Use the canonical inclusion map that takes $\mathbb{Z}_p$ into $\mathbb{Q}_p$ to define the dual pairing $\langle\cdot,\cdot\rangle$. Namely,

$$\langle\cdot,\cdot\rangle\colon \mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p) \to \mathbb{S}^1 \quad \text{by} \quad \langle x, \langle y\rangle\rangle = \chi(xy).$$

For any $(\langle x\rangle, w, y)$ in $(\mathbb{Q}_p/\mathbb{Z}_p) \times \mathbb{Z}_p \times \mathbb{Z}_p$, the product wy is in $\mathbb{Z}_p$, and so

$$\chi((x+w)y) = \chi(xy)\chi(wy) = \chi(xy).$$

The definition of the pairing is, therefore, independent of the choice of representative.

Define the Fourier transform $\mathcal{F}_0$ that takes $L^2(\mathbb{Z}_p)$ to $L^2(\mathbb{Q}_p/\mathbb{Z}_p)$ and the inverse Fourier transform $\mathcal{F}_0^{-1}$ that takes $L^2(\mathbb{Q}_p/\mathbb{Z}_p)$ to $L^2(\mathbb{Z}_p)$ to be given for any f in $L^2(\mathbb{Z}_p)$ and any $\tilde{f}$ in $L^2(\mathbb{Q}_p/\mathbb{Z}_p)$ by

$$(\mathcal{F}_0 f)(y) = \int_{\mathbb{Z}_p} \langle -x, \langle y\rangle\rangle f(x)\,\mathrm{d}\mu(x) \quad \text{and} \quad (\mathcal{F}_0^{-1}\tilde{f})(x) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} \langle x, \langle y\rangle\rangle \tilde{f}(\langle y\rangle)\,\mathrm{d}\mu_0(\langle y\rangle).$$

Take β to be the quantity

$$\beta = \frac{p^{b+1} - 1}{p^b(p-1)} \tag{3.2}$$

and $\mathcal{M}_{b,0}$ to be the multiplication operator that acts on any compactly supported function f with domain $\mathbb{Q}_p/\mathbb{Z}_p$ by

$$(\mathcal{M}_{b,0}f)(\langle y\rangle) = -(|\langle y\rangle|^b - \beta^{-1})f(\langle y\rangle).$$

The Vladimirov-Kochubei operator is the self-adjoint closure of the operator that is given for any function f in $SB(\mathbb{Z}_p)$ by

$$(\Delta_{b,0}f)(x) = (\mathcal{F}^{-1}\mathcal{M}_{b,0}\mathcal{F}f)(x).$$

The diffusion equation in $\mathbb{Z}_p$ is the equation given by (3.1), but with the new operator $\Delta_{b,0}$ replacing Δ_b. The Fourier transform of the fundamental solution ρ to the diffusion equation on $\mathbb{Z}_p$ is ϕ, where

$$\phi(t, \langle y\rangle) = \mathrm{e}^{-D\left(|\langle y\rangle|^b - \beta^{-1}\right)t}. \tag{3.3}$$

The ordered collection of functions $(\rho(t,\cdot))_{t>0}$ forms a convolution semigroup of probability density functions on $\mathbb{Z}_p$ that gives rise to a probability measure P on $D([0,\infty)\colon \mathbb{Z}_p)$ and the triple $(D([0,\infty)\colon \mathbb{Z}_p), \mathrm{P}, Y)$ is a Brownian motion in $\mathbb{Z}_p$.

Chapter 4

Diffusion in the finite groups

4.1 A sequence of finite state spaces

Take G_m to be the discrete group $\mathbb{Z}_p/p^m\mathbb{Z}_p$ and $\tilde{G}_m$ its Pontryagin dual, the discrete group $p^{-m}\mathbb{Z}_p/\mathbb{Z}_p$. It will be useful to write the elements of these groups in a way that both clearly indicates their membership and also their relationship to the elements of $\mathbb{Q}_p$. For this reason, define the projections $[\cdot]$ and $\langle\cdot\rangle$ for any x in $\mathbb{Z}_p$ and for any y in $p^{-m}\mathbb{Z}_p$ respectively by

$$[x] = x + p^m\mathbb{Z}_p \quad \text{and} \quad \langle y\rangle = y + \mathbb{Z}_p,$$

so that

$$[\cdot]\colon \mathbb{Z}_p \to G_m = \mathbb{Z}_p/p^m\mathbb{Z}_p \quad \text{and} \quad \langle\cdot\rangle\colon p^{-m}\mathbb{Z}_p \to \tilde{G}_m = p^{-m}\mathbb{Z}_p/\mathbb{Z}_p.$$

The p-adic absolute value induces a norm on both G_m and on $\tilde{G}_m$, respectively, by

$$\begin{cases} |[x]| = |x| \quad \text{if } [x] \neq [0] \\ \\ |[0]| = 0 \end{cases} \quad \text{and} \quad \begin{cases} |\langle y \rangle| = |y| \quad \text{if } \langle y \rangle \neq \langle 0 \rangle \\ \\ |\langle 0 \rangle| = 0. \end{cases}$$

For any ℓ in $\{1, \ldots, m\}$, take $\mathbb{B}_m(\ell)$ and $\mathbb{S}_m(\ell)$, respectively, to be the ball and circle of radius $p^{\ell-m}$ in G_m with center $p^m\mathbb{Z}_p$, and take

$$\mathbb{B}_m(0) = \mathbb{S}_m(0) = p^m\mathbb{Z}_p.$$

Take μ to be the Haar measure on G_m that is normalized to give G_m unit measure, so that the volume of balls and circles are given by

$$\begin{cases} \mathrm{Vol}(\mathbb{B}_m(k)) = p^{k-m} \\ \\ \mathrm{Vol}(\mathbb{B}_m(0)) = p^{-m} \end{cases} \quad \text{and} \quad \begin{cases} \mathrm{Vol}(\mathbb{S}_m(k)) = \big(1 - p^{-1}\big)p^{k-m} \\ \\ \mathrm{Vol}(\mathbb{S}_m(0)) = \mathrm{Vol}(\mathbb{B}_m(0)) = p^{-m}. \end{cases} \tag{4.1}$$

The Haar measure on $\tilde{G}_m$ that has value 1 on $\mathbb{Z}_p$ is the counting measure. Denote by $\tilde{\mathbb{B}}_m(\ell)$ and $\tilde{\mathbb{S}}_m(\ell)$ the ball and circle in $\tilde{G}_m$ of radius p^ℓ, respectively. For any k in $\{1, 2, 3, \ldots, m\}$,

$$\mathrm{Vol}(\tilde{\mathbb{B}}_m(k)) = p^k \quad \text{and} \quad \mathrm{Vol}(\mathbb{S}_m(k)) = \big(1 - p^{-1}\big)p^k. \tag{4.2}$$

Note that the ball and circle of radius 0 coincide.

The character χ on $\mathbb{Q}_p$ induces a dual pairing $\langle \cdot, \cdot \rangle$ on $G_m \times \tilde{G}_m$ that is given for any $([x], \langle y \rangle)$ in $G_m \times \tilde{G}_m$ by

$$\langle [x], \langle y \rangle \rangle = \chi(xy).$$

Although the pairing makes use of a specific choice of representative for the group elements, since χ is additive and rank 0, the pairing is independent of the choice of representative. Define the Fourier transform $\mathcal{F}_m$ on G_m for any function f on G_m and any $\langle y \rangle$ in $\tilde{G}_m$ by

$$\mathcal{F}_m : f \mapsto \hat{f} : \langle y \rangle \mapsto \int_{G_m} \chi(xy) f([x]) \, \mathrm{d}\mu([x]).$$

Denote by $\mathcal{F}_m^{-1}$ the inverse Fourier transform that is given for any function g on $\tilde{G}_m$ by

$$\mathcal{F}_m^{-1} : g \mapsto \check{g} : [x] \mapsto \int_{\tilde{G}_m} \chi(-xy) g(\langle x \rangle) \, \mathrm{d}\tilde{\mu}(\langle y \rangle).$$

Both $\mathcal{F}_m$ and $\mathcal{F}_m^{-1}$ are unitary with respect to the L^2-norm on the respective space.

The following lemmas are useful for making calculations that involve integrals of the dual pairing. The proof follows from the proof in the $\mathbb{Q}_p$ setting.

Lemma 4.1.15. *For any i in $\{1, \ldots, m\}$,*

$$\int_{\mathbb{B}_m(i)} \chi(xy) \, \mathrm{d}\mu([x]) = \mathrm{Vol}(\mathbb{B}_m(i)) \cdot \mathbb{1}_{\tilde{\mathbb{B}}_{m(m-i)}}(\langle y \rangle)$$

and for any $\langle y\rangle$ *in* $\tilde{G}_m$,

$$\int_{\mathbb{B}_m(0)} \chi(xy)\,\mathrm{d}\mu([x]) = p^{-m}.$$

Lemma 4.1.16. *For any* i *in* $\{1,\ldots,m\}$,

$$\int_{\tilde{\mathbb{B}}_m(i)} \chi(xy)\,\mathrm{d}\mu(\langle y\rangle) = \mathrm{Vol}(\mathbb{B}_m(i)) \cdot \mathbb{1}_{\mathbb{B}_m(m-i)}([x])$$

and for any $[x]$ *in* G_m,

$$\int_{\tilde{\mathbb{B}}_m(0)} \chi(xy)\,\mathrm{d}\mu(\langle y\rangle) = p^{-m}.$$

4.2 Random walks in the finite state spaces

Take $X^{(m)}$ to be the G_m-valued abstract random variable with probability mass function $\rho_{X^{(m)}}$ that is constant on each circle in G_m and that satisfies the equalities

$$\begin{cases} \mathrm{Prob}(X^{(m)} \in \mathbb{S}_m(\ell)) = \frac{C_m}{p^{\ell b}} \quad \text{if } \ell \in \{1,2,\ldots,m\} \\ \mathrm{Prob}(X^{(m)} = [0]) = 0. \end{cases}$$

To simplify notation, from now on we'll write X rather than $X^{(m)}$ unless it is necessary to specify the natural number m. The equality

$$1 = \left\{ \frac{1}{p} + \frac{1}{p^{2b}} + \cdots + \frac{1}{p^{mb}} \right\} \cdot c_m = \frac{1}{p^{mb}} \cdot \frac{p^{mb} - 1}{p^b - 1} \cdot c_m$$

implies that

$$c_m = \frac{p^{mb}}{p^{mb} - 1} (p^b - 1).$$

The constancy of ρ_X on each circle implies that

$$\begin{cases} \rho_X([x]) = \frac{1}{\mathrm{Vol}(\mathbb{S}_m(k))} \cdot \frac{c_m}{p^{kb}} & \text{if } |[x]| = p^{-m+k} \\ \rho_X([0]) = 0. \end{cases}$$

Denote by ϕ_X the characteristic function of X and use the decomposition of $\mathbb{Z}_p$ into a countable union of circles to obtain the equalities

$$\begin{aligned} \phi_X(\langle y \rangle) &= (\mathcal{F}_m \rho_X)(\langle y \rangle) \\ &= \int_{G_m} \chi(xy) \rho_X([x]) \, \mathrm{d}\mu([x]) \\ &= \int_{G_m} \chi(xy) \rho_X([x]) \, \mathrm{d}\mu([x]) + \sum_{\ell=1}^{m} \int_{\mathbb{S}_m(-m+\ell)} \chi(xy) \rho_X([x]) \, \mathrm{d}\mu([x]). \end{aligned}$$

For each i in $\{0, 1, \ldots, m\}$, simplify notation by writing

$$\rho(i) = \begin{cases} 0 & \text{if } i = 0 \\ \frac{1}{\mathrm{Vol}(\mathbb{S}_m(i))} \cdot \frac{c_m}{p^{ib}} & \text{if } i \in \{1, \ldots, m\} \end{cases}$$

to obtain the equalities

$$\begin{aligned}\phi_X(\langle y\rangle) &= \sum_{i=1}^{m}\rho(i)\int_{\mathbb{S}_m(i)}\chi(xy)\,\mathrm{d}\mu([x])\\ &= \sum_{i=1}^{m}\rho(i)\biggl(\int_{\mathbb{B}_m(i)}\chi(xy)\,\mathrm{d}\mu([x]) - \int_{\mathbb{B}_m(i)}\chi(xy)\,\mathrm{d}\mu([x])\biggr)\\ &= -\rho(1)\int_{\mathbb{B}_m(-m)}\chi(xy)\,\mathrm{d}\mu([x]) + \sum_{i=1}^{m-1}(\rho(i)-\rho(i+1))\int_{\mathbb{B}_m(i)}\chi(xy)\,\mathrm{d}\mu([x])\\ &\qquad + \rho(m)\int_{\mathbb{B}_m(m)}\chi(xy)\,\mathrm{d}\mu([x]).\end{aligned}$$

Since ϕ_X is the Fourier transform of a probability mass function,

$$\phi_X(\langle 0\rangle) = 1.$$

Denote by β the quantity

$$\beta = \frac{p^{b+1}-1}{p^b(p-1)}.$$

Proposition 4.2.17. *The characteristic function for X, ϕ_X, is given for any $\langle y\rangle$ not equal to $\langle 0\rangle$ by*

$$\phi_X(\langle y\rangle) = 1 - \frac{\beta\Bigl(|\langle y\rangle|^b\bigl(1+\frac{1}{p^{mb}}\bigr) - \beta^{-1}\Bigr)}{p^{mb}} + \frac{1}{p^{mb}(p^{mb}-1)}\left(1 - \frac{\beta|\langle y\rangle|^b}{p^{mb}}\right), \tag{4.3}$$

and

$$\phi_X(\langle 0\rangle) = 1.$$

Proof. For any $\langle y\rangle$ in $\tilde{G}_m$, either $|\langle y\rangle|$ is equal to 0, or there is a k in $\{1, 2, \ldots, m\}$ so that

$|\langle y\rangle|$ is equal to p^k. Since ϕ_X is the Fourier transform of a probability density function, it takes on the value 1 at $\langle 0\rangle$. Use the expression for ϕ_X to obtain the equalities

$$\phi_X(\langle y\rangle) = -\rho(1)\cdot p^{-m} + \sum_{i=1}^{m-1}\Big\{(\rho(i)-\rho(i+1))\,\mathrm{Vol}(\mathbb{B}_m(i))\mathbb{1}_{\tilde{\mathbb{B}}_{m(m-i)}}(\langle y\rangle)\Big\}$$
$$+\,\rho(m)\cdot\mathrm{Vol}(\mathbb{B}_m(m))\cdot\mathbb{1}_{\langle 0\rangle}(\langle y\rangle).$$

For each i in $\{1,2,\ldots,m\}$,

$$\begin{aligned}\rho(i)-\rho(i+1) &= \frac{1}{\mathrm{Vol}\big(\mathbb{S}_m(i)\big)}\cdot\frac{c_m}{p^{ib}} - \frac{1}{\mathrm{Vol}\big(\mathbb{S}_m(i+1)\big)}\cdot\frac{c_m}{p^{(i+1)b}}\\ &= \frac{1}{\mathrm{Vol}\big(\mathbb{S}_m(i)\big)}\cdot\frac{c_m}{p^{ib}}\cdot\left(1-\frac{1}{p\cdot p^b}\right)\end{aligned}$$

and so

$$\begin{aligned}&(\rho(i)-\rho(i+1))\cdot\mathrm{Vol}\big(\mathbb{B}_m(i)\big)\\ &\quad= \frac{c_m}{p^{ib}}\cdot\frac{1}{\mathrm{Vol}\big(\mathbb{S}_m(i)\big)}\cdot\left(1-\frac{1}{p\cdot p^b}\right)\cdot\mathrm{Vol}\big(\mathbb{B}_m(i)\big)\\ &\quad= \frac{c_m}{p^{ib}}\cdot\frac{1}{1-\frac{1}{p}}\cdot\left(1-\frac{1}{p\cdot p^b}\right) = \frac{c_m}{p^{ib}}\cdot\frac{p}{p-1}\cdot\frac{p^{b+1}-1}{p^{b+1}}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\rho(1)\cdot p^{-m} &= \mathrm{Prob}\big(X\in\mathbb{S}_m(1)\big)\cdot\frac{1}{\mathrm{Vol}\big(\mathbb{S}_m(1)\big)}\cdot p^{-m}\\ &= \frac{c_m}{p^b}\cdot\frac{1}{p^{-m}(1-\frac{1}{p})\cdot p}\cdot p^{-m} = \frac{c_m}{p^b}\cdot\frac{1}{p-1},\end{aligned}$$

and so

$$\phi_X(\langle y\rangle) = -\frac{c_m}{p^b}\cdot\frac{1}{p-1} + c_m C(p,b)\Bigg\{\frac{1}{p^b}\cdot \mathbb{1}_{\tilde{\mathbb{B}}_m(m-1)}(\langle y\rangle]) + \frac{1}{p^{2b}}\cdot \mathbb{1}_{\tilde{\mathbb{B}}_m(m-2)}(\langle y\rangle)$$
$$+\cdots+\frac{1}{p^{ib}}\cdot \mathbb{1}_{\tilde{\mathbb{B}}_m(m-i)}(y) + \cdots + \frac{1}{p^{(m-1)b}}\cdot \mathbb{1}_{\tilde{\mathbb{B}}_m(1)}(\langle y\rangle)\Bigg\}$$
$$+\frac{1}{1-\frac{1}{p}}\cdot\frac{c_m}{p^{mb}}\cdot \mathbb{1}_{\tilde{\mathbb{B}}_m(0)}([y]),$$

where $C(p,b)$ denotes the quantity

$$C(p,b) = \frac{p}{p-1}\cdot\frac{p^{b+1}-1}{p^{b+1}}.$$

For any k in $\{1,2,\ldots,m-1\}$,

$$p^k = p^{m-i} \iff m-k=i.$$

Take $|\langle y\rangle|$ to be equal to p^k, so that

$$\begin{aligned}\phi_X(\langle y\rangle) &= -\frac{c_m}{p^b}\cdot\frac{1}{p-1} + c_m C(p,b)\cdot\left\{\frac{1}{p^b}+\frac{1}{p^{2b}}+\cdots+\frac{1}{p^{(m-k)b}}\right\}\\ &= \frac{p^{mb}}{p^{mb}-1}\left\{1-\frac{p^{b+1}-1}{p^b(p-1)}\cdot\frac{p^{kb}}{p^{mb}}\right\} = \frac{p^{mb}}{p^{mb}-1}\left\{1-\frac{p^{b+1}-1}{p^b(p-1)}\cdot\frac{|\langle y\rangle|^b}{p^{mb}}\right\}.\end{aligned}$$

Use the series expansion of α that is given by

$$\alpha = \frac{p^{mb}}{p^{mb}-1} = \frac{1}{1-\frac{1}{p^{mb}}} = 1 + \frac{1}{p^{mb}} + \frac{1}{p^{2mb}} + \frac{1}{p^{3mb}} + \cdots$$

to rewrite $\phi_X(\langle y\rangle)$ for any $\langle y\rangle$ not equal to $\langle 0\rangle$ as

$$\begin{aligned}\phi_X(\langle y\rangle) &= \left(1+\frac{1}{p^{mb}}+\frac{1}{p^{2mb}}+\frac{1}{p^{3mb}}+\cdots\right)\left(1-\frac{\beta|\langle y\rangle|^b}{p^{mb}}\right)\\ &= 1-\frac{\beta\Big(|\langle y\rangle|^b\big(1+\frac{1}{p^{mb}}\big)-\beta^{-1}\Big)}{p^{mb}}+\frac{1}{p^{mb}(p^{mb}-1)}\left(1-\frac{\beta|\langle y\rangle|^b}{p^{mb}}\right)\end{aligned}$$

■

The Kolmogorov Extension theorem guarantees that there is a measure P^*_m on $F(\mathbb{N}_0\colon G_m)$ and a stochastic process $(F(\mathbb{N}_0\colon G_m),\mathrm{P}^*_m,S)$ so that the increments of S are independent and have the same law as X. For any natural number n, take (X_i) to the sequence of increments of the process this process, so that for each n in $\mathbb{N}_0$,

$$S_n = S_0 + X_1 + \cdots + X_n$$

and S_0 is almost surely equal to $[0]$. Take $\rho(n,\cdot)$ to be the law for S_n. To compress notation, for any i in $\{1,\dots,m\}$, write

$$\phi^m(i)=\phi^m(\langle y\rangle),\quad \text{where}\quad \langle y\rangle = p^i,$$

and write $\phi(0)$ rather than $\phi_X(\langle 0\rangle)$.

Proposition 4.2.18. *For any natural number n,*

$$\rho(n,[x])=\sum_{i=0}^{m-1}\Big(\phi(i)^n-\phi(i+1)^n\Big)p^i\mathbb{1}_{\mathbb{B}_m(m-i)}([x])+\phi(m)^n\mathbb{1}_{\mathbb{B}_m(0)}([x]).$$

Proof. Take the inverse Fourier transform of the n-fold product of the characteristic function of X to obtain the equalities

$$\begin{aligned}
\rho(n,[x]) &= \big(\mathcal{F}^{-1}(\phi(\cdot)^n\big)([x]) \\
&= \int_{p^{-m}\mathbb{Z}_p/\mathbb{Z}_p} \chi(xy)\phi(\langle y\rangle)^n \,\mathrm{d}\mu(\langle y\rangle) \\
&= \int_{\mathbb{Z}_p} \chi(xy)\phi(\langle y\rangle)^n \,\mathrm{d}\mu(\langle y\rangle) + \int_{\tilde{\mathbb{S}}_m(1)} \chi(xy)\phi(\langle y\rangle)^n \,\mathrm{d}\mu(\langle y\rangle) \\
&\qquad + \cdots + \int_{\tilde{\mathbb{S}}_m(m)} \chi(xy)\phi(\langle y\rangle)^n \,\mathrm{d}\mu(\langle y\rangle) \\
&= \sum_{i=0}^{m} \phi(i)^n \int_{\tilde{\mathbb{S}}_m(i)} \chi(xy)\,\mathrm{d}\mu(\langle y\rangle).
\end{aligned}$$

Rearrange terms to obtain the equalities

$$\begin{aligned}
\rho(n,[x]) = &\int_{\tilde{\mathbb{B}}_m(0)} \chi(xy)\,\mathrm{d}\mu(\langle y\rangle) \\
&\quad + \sum_{i=1}^{m} \phi(i)^n \left(\int_{\tilde{\mathbb{B}}_m(i)} \chi(xy)\,\mathrm{d}\mu(\langle y\rangle) - \int_{\tilde{\mathbb{B}}_m(i-1)} \chi(xy)\,\mathrm{d}\mu(\langle y\rangle) \right) \\
= &\,(1-\phi(1)^n) \int_{\tilde{\mathbb{B}}_m(0)} \chi(xy)\,\mathrm{d}\mu(\langle y\rangle) \\
&\quad + \sum_{i=1}^{m-1} (\phi(i)^n - \phi(i+1)^n) \int_{\tilde{\mathbb{B}}_m(i)} \chi(xy)\,\mathrm{d}\mu(\langle y\rangle) \\
&\quad + \phi(m)^n \int_{\tilde{\mathbb{B}}_m(m)} \chi(xy)\,\mathrm{d}\mu(\langle y\rangle),
\end{aligned}$$

which together with Lemma 4.1.16 imply the given formula for $\rho(n,[x])$. ∎

4.3 Isometric embeddings

Take Γ_m to be the injection from G_m to $\mathbb{Z}_p$ that is given for any x in $\mathbb{Z}_p$ by

$$\Gamma_m([x]) = \sum_{i \in \mathbb{N}_0} a_x(i) p^i \mathbb{1}_{[0,m]}(i).$$

For any g in G_m, there is an x in $\mathbb{Z}_p$ so that g is equal to $[x]$. If x' is any other such element of $\mathbb{Z}_p$, the $\Gamma_m([x])$ is equal to $\Gamma_m([x']$, and so this injection is independent of choice of representative. The function Γ_m is an isometry because the metric on $\mathbb{Z}_p$ is an ultrametric, however, Γ_m is obviously not a group homomorphism. Because of this, it is helpful to first construct random walks in the G_m, and then use Γ_m together with a time scaling to transform the G_m-valued random walks into random walks on a discrete subset of $\mathbb{Z}_p$.

Chapter 5

Convergence

5.1 Moment estimates for processes in the finite groups

To work with the formula for ϕ_X that Proposition 4.2.17 provides requires some determination of the sign of various associated quantities. The first required estimate is Proposition 5.1.19.

First, we have

$$0 < \beta - 1 = \frac{1}{p-1}\frac{p^b - 1}{p^b} < 1.$$

The inequality

$$\frac{d}{db}\left(p^{2b+1} - p^{2b}\right) > \frac{d}{dp}\left(p^{b+1} - 1\right)$$

together with the equality of $p^{2b+1} - p^{2b}$ and $p^{b+1} - 1$ when b is equal to 0 implies that

$$\frac{\beta}{p^b} = \frac{p^{b+1} - 1}{p^{2b+1} - p^{2b}} < 1.$$

This implies the following proposition.

Proposition 5.1.19. *For any positive real number b, both $\beta - 1$ and $\frac{\beta}{p^b}$ are in $(0,1)$.*

It is helpful to list some formulas that the proof of Theorem 5.2.20 requires and analyze the positivity of the various terms. For any i in $\{1,\ldots,m\}$,

$$\phi(i) - \phi(i+1) = \beta(p^b-1)\frac{p^{ib}}{p^{mb}} + \beta(p^b-1)\frac{p^{ib}}{p^{mb}}\frac{1}{p^{mb}} + \beta(p^b-1)\frac{p^{ib}}{p^{mb}}\frac{1}{p^{mb}(p^{mb}-1)}, \quad (5.1)$$

which shows that $(\phi(i))$ is increasing in i. The equality

$$\phi(m) = (1-\beta)\left(1 + \frac{1}{p^{mb}} + \frac{1}{p^{mb}(p^{mb}-1)}\right)$$

implies that $\phi(m)$ is negative, however a similar calculation shows that $\phi(m-1)$ is positive. Notice that the formula for $\phi(i)$ only agrees with $\phi(0)$ in limit, as the latter is equal to 1, however, formally substituting 0 for i in the given formula for $\phi(0)$ gives the value $\psi(0)$, where

$$\psi(0) = 1 - p^{-mb}\left(\beta - 1 + \beta p^{-mb}\right) + p^{-mb}(p^{mb}-1)^{-1}(1-\beta p^{-mb}),$$

and so is bounded above by 1 if and only if β is bounded below by 1, which it is. Since $\phi(i)$ is decreasing in i, $\phi(i)$ is in $[0,1]$ for every i in $\{1,\ldots,m-1\}$.

For any m and every i in $\{1,\ldots,m-1\}$,

$$\phi(i)^n = \left(1 - \frac{\beta\left(p^{ib}(1+\frac{1}{p^{mb}}) - \beta^{-1}\right)}{p^{mb}} + \frac{1}{p^{mb}(p^{mb}-1)}\left(1 - \frac{\beta p^{ib}}{p^{mb}}\right)\right)^n, \quad (5.2)$$

and so

$$\phi(i)^n \leq \left(1 - \frac{\beta\Big(p^{ib} - \beta^{-1}\Big)}{p^{mb}} + \frac{1}{p^{mb}(p^{mb}-1)}\right)^n = \left(1 - \frac{\beta p^{ib} - \frac{p^{mb}}{p^{mb}-1}}{p^{mb}}\right)^n. \tag{5.3}$$

Finally, since $\phi(0)$ is not given by the same formula as the other $\phi(i)$, it will be important to separately determine the quantity $\phi(0)^n - \phi(1)^n$, which is given by

$$\begin{aligned} \phi(0) - \phi(1)^n &= 1 - \phi(1)^n & (5.4) \\ &\leq 1 - \left(1 - \frac{\beta p^b\big(1 + \frac{1}{p^{mb}}\big)}{p^{mb}}\right)^n. & (5.5) \end{aligned}$$

5.2 Moment estimates

Denote by E^*_m the expected value with respect to the measure P^*_m. Henceforth, take $M(p,b)$ to be a natural number with the property that if m is greater than $M(p,b)$, then

$$\beta \geq \frac{p^{mb}}{p^{mb}-1}.$$

Some basic manipulations reveal that the given inequality is valid as long as

$$m \geq \frac{1}{b}\log_p\left(\frac{p^{b+1}-1}{p^b-1}\right),$$

so if b is equal to 1, and p is greater than 2, then the condition is non-restrictive. If p is 2, then m must be at least 2. Since the arguments below are simplified if m is at least 2, for sake of convenience take $M(p,b)$ to be given, from now on, by

$$M(p,b) = 1 + \left\lceil \frac{1}{b} \log_p \left(\frac{p^{b+1}-1}{p^b - 1} \right) \right\rceil,$$

where $\lceil \cdot \rceil$ represents the *ceiling function.*

Theorem 5.2.20. *For any natural number n and any positive real number r, there are constants K and C that are independent of m so that for any m that is greater than $M(p,b)$,*

$$\mathbb{E}^*_m[|S_n|^r] \leq K n^{\frac{r}{b}} p^{-mr} + \left(1 - \left(1 - \frac{\beta p^b \left(1 + \frac{1}{p^{mb}}\right)}{p^{mb}}\right)^n\right) \frac{p-1}{p}.$$

Proof. Use the expression for $\rho^{(m)}(n,[x])$ given by Proposition 4.2.18 to obtain the equalities

$$\begin{aligned} \mathbb{E}^*_m[|S_n|^r] &= \int_{G_m} |[x]|^r \rho(n,[x]) \,\mathrm{d}\mu([x]) \\ &= \sum_{i=0}^{m-1} \Big(\big(\phi^m(i)\big)^n - \big(\phi^m(i+1)\big)^n \Big) p^i \int_{G_m} |[x]|^r \mathbb{1}_{\mathbb{B}_m(m-i)}([x]) \,\mathrm{d}\mu([x]). \end{aligned} \tag{5.6}$$

For each i in $\{0,1,\ldots,m\}$, use the decomposition of an integral over G_m into an integral over circles to obtain the equalities

$$\begin{aligned} \int_{G_m} |[x]|^r \mathbb{1}_{\mathbb{B}_m(m-i)}([x]) \,\mathrm{d}\mu([x]) &= \sum_{\ell=i}^{m} \int_{G_m} |[x]|^r \mathbb{1}_{\mathbb{S}_m(m-\ell)}([x]) \,\mathrm{d}\mu([x]) \\ &= \sum_{\ell=i}^{m-1} p^{-\ell r} \int_{G_m} \mathbb{1}_{\mathbb{S}_m(m-\ell)}([x]) \,\mathrm{d}\mu([x]) \\ &= \sum_{\ell=i}^{m-1} p^{-\ell r} p^{-\ell} \left(1 - \frac{1}{p}\right) \\ &= \left(1 - \frac{1}{p}\right) p^{-r(m-1)} p^{-(m-1)} \left\{ \frac{p^{r(m-i)} p^{(m-i)} - 1}{p^r p^1 - 1} \right\}. \end{aligned} \tag{5.7}$$

Multiply by p^i to see that

$$p^i \int_{G_m} |[x]|^r \mathbb{1}_{\mathbb{B}_m(-i)}([x])\, \mathrm{d}\mu([x]) = (p-1)\, p^r p^i \left\{ \frac{p^{-ir} p^{-i} - p^{-rm} p^{-m}}{p^{r+1}-1} \right\}. \tag{5.8}$$

Equations (5.7) and (5.8) together imply that

$$\begin{aligned}
\mathbb{E}[|S_n|^r] &= \sum_{i=0}^{m-1} \Big(\phi(i)^n - \phi(i+1)^n\Big)\, (p-1)\, p^r p^i \left\{ \frac{p^{-i(r+1)} - p^{-m(r+1)}}{p^{r+1}-1} \right\} \\
&= \sum_{i=1}^{m-1} \Big(\phi(i)^n - \phi(i+1)^n\Big)\, (p-1)\, p^r p^i \left\{ \frac{p^{-i(r+1)} - p^{-m(r+1)}}{p^{r+1}-1} \right\} \\
&\qquad + \Big(\phi(0)^n - \phi(1)^n\Big)\, (p-1)\, p^r \left\{ \frac{1 - p^{-m(r+1)}}{p^{r+1}-1} \right\} \\
&= \sum_{i=1}^{m-1} \Big(\phi(i)^n - \phi(i+1)^n\Big)\, (p-1)\, p^r p^i \left\{ \frac{p^{-i(r+1)} - p^{-m(r+1)}}{p^{r+1}-1} \right\} \\
&\qquad + \Big(1 - \phi(1)^n\Big)\, (p-1)\, p^r \left\{ \frac{1 - p^{-m(r+1)}}{p^{r+1}-1} \right\}.
\end{aligned} \tag{5.9}$$

Denote by $I(m,n)$ the quantity

$$\begin{aligned}
I(m,n) &= \sum_{i=1}^{m-1} \Big(\big(\phi(i)\big)^n - \big(\phi(i+1)\big)^n\Big)\, (p-1)\, p^r p^i \left\{ \frac{p^{-i(r+1)} - p^{-r(m+1)}}{p^{r+1}-1} \right\} \\
&< \sum_{i=1}^{m-1} \Big(\big(\phi(i)\big)^n - \big(\phi(i+1)\big)^n\Big)\, (p-1)\, p^r p^i \left\{ \frac{p^{-i(r+1)}}{p^{r+1}-1} \right\} \\
&= \frac{p^r (p-1)}{p^{r+1}-1} \sum_{i=1}^{m-1} \Big(\phi(i)^n - \phi(i+1)^n\Big) p^{-ir}.
\end{aligned}$$

Use the fact that $(\phi(i)^n)$ is a decreasing finite sequence in $[0,1]$ in the index i to write the difference of successive terms as

$$\phi(i)^n - \phi(i+1)^n = n \int_{\phi(i+1)}^{\phi(i)} s^{n-1} \, \mathrm{d}s.$$

Since the integrand is an increasing function on $[0, 1]$,

$$\begin{aligned} \phi(i)^n - \phi(i+1)^n &< n\phi(i)^{n-1}\big(\phi(i) - \phi(i+1)\big) \\ &= n\phi(i)^{n-1}\beta(p^b - 1)\frac{p^{ib}}{p^{mb}}\left\{1 + \frac{1}{p^{mb}} + \frac{1}{p^{mb}(p^{mb} - 1)}\right\} \\ &= c(m)n\phi(i)^{n-1}\frac{p^{ib}}{p^{mb}}, \end{aligned} \tag{5.10}$$

where

$$c(m) = \beta(p^b - 1)\left(1 + \frac{1}{p^{mb}} + \frac{1}{p^{mb}(p^{mb} - 1)}\right).$$

Equations (5.3) and (5.10) together imply that

$$\begin{aligned} I(m, n) &\leq \frac{c(m)p^r(p-1)}{p^{r+1} - 1} \sum_{i=1}^{m-1} np^{-ir}\phi(i)^{n-1}\frac{p^{ib}}{p^{mb}} \\ &= \frac{c(m)p^r(p-1)}{p^{r+1} - 1} \sum_{i=1}^{m-1} np^{-ir}\left(1 - \frac{\beta p^{ib} - \frac{p^{mb}}{p^{mb}-1}}{p^{mb}}\right)^{n-1}\frac{p^{ib}}{p^{mb}} \end{aligned} \tag{5.11}$$

For each i in $\{0, 1, \ldots, m-1\}$ write

$$x_i = \frac{\beta p^{ib} - \frac{p^{mb}}{p^{mb}-1}}{p^{mb}}.$$

Notice it is not initially certain that x_0 is non-negative. However, x_0 is non-negative as long as m is taken to be large enough so that

$$\beta \geq \frac{p^{mb}}{p^{mb}-1},$$

and so this is valid under the hypotheses of the theorem. For each interval $I(i)$ with endpoints x_{i-1} and x_i, the length of $I(i)$, denoted $\Delta(i)$, is given by

$$\Delta(i) = x_i - x_{i-1} = \frac{1}{p^b}\beta(p^b-1)\frac{p^{ib}}{p^{mb}}.$$

Use the expressions for x_i and $\Delta(i)$ to rewrite (5.11) and obtain the inequality

$$I(m,n) \leq \frac{c(m)p^r(p-1)}{\beta p^b(p^b-1)(p^{r+1}-1)} \sum_{i=1}^{m-1} np^{-ir}\left(1-x_i\right)^{n-1}\Delta(i). \tag{5.12}$$

Rewrite p^{-ir} in terms of x_i to obtain the equality

$$p^{-ir} = \left(x_i + \frac{1}{p^{mb}-1}\right)^{-\frac{r}{b}} p^{-rm}\beta^{\frac{r}{b}} < x_i^{-\frac{r}{b}} p^{-rm}\beta^{\frac{r}{b}},$$

which implies together with (5.12) that

$$I(m,n) \leq Kp^{-rm}\sum_{i=1}^{m-1} nx_i^{-\frac{r}{b}}\left(1-x_i\right)^{n-1}\Delta(i), \tag{5.13}$$

where

$$K = \frac{c(m)p^r(p-1)}{p^b(p^b-1)(p^{r+1}-1)}\beta^{\frac{r-b}{b}}.$$

The sum in (5.13) is a lower Riemann sum approximation of the integral

$$\int_{x_1}^{x_{m-1}} nx^{-\frac{r}{b}}(1-x)^{n-1}\,\mathrm{d}x < \int_0^1 nx^{-\frac{r}{b}}(1-x)^{n-1}\,\mathrm{d}x.$$

Follow the earlier arguments in the $\mathbb{Q}_p$ and local field settings[22] to obtain the equality

$$\sum_{i=1}^{m-1} nx_i^{-\frac{r}{b}}\,(1-x_i)^{n-1}\,\Delta(i-1) \leq n^{-\frac{b-r}{b}}\left(\frac{n+\frac{b-r}{b}}{n}\right)^{\frac{r}{b}}, \tag{5.14}$$

which together with (5.13) implies that

$$I(m,n) \leq Kp^{-rm}n^{\frac{r}{b}}\left(\frac{n+\frac{b-r}{b}}{n}\right)^{\frac{r}{b}} \tag{5.15}$$

Take C to be equal to $K\left(\frac{n+\frac{b-r}{b}}{n}\right)^{\frac{r}{b}}$ to obtain the inequality

$$I(m,n) \leq Cp^{-rm}n^{\frac{r}{b}}. \tag{5.16}$$

The inequality given by (5.5) gives an upper bound for the right hand term in (5.9), namely

$$\begin{aligned} A(n) &= \left(1-\phi(1)^n\right)(p-1)\,p^r\left\{\frac{1-p^{-m(r+1)}}{p^{r+1}-1}\right\} \\ &\leq \left(1-\left(1-\frac{\beta p^b\left(1+\frac{1}{p^{mb}}\right)}{p^{mb}}\right)^n\right)\frac{p-1}{p}. \end{aligned}$$

■

5.3 Time scaling and the approximating processes

Given any null sequence (τ_m), a sequence of *time scales*, take (ι_m) to be a sequence of functions so that for each m,

$$\iota_m \colon \mathbb{N}_0 \times G_m \to [0, \infty) \times \mathbb{Z}_p \quad \text{by} \quad \iota(n, [x]) = (\tau_m n, \Gamma_m([x])).$$

For typographical efficiency, take λ_m to be the reciprocal of τ_m. The function ι_m acts on the stochastic process $(F(\mathbb{N}_0 \colon G_m), \mathrm{P}^*_m, S)$ to produce a continuous time process in the following way. For any real number t, define the abstract random variable $\tilde{Y}_t$ by

$$\tilde{Y}_t = \Gamma_m\big(S_0^{(m)} + X_1^{(m)} + \cdots + X_{\lfloor t\lambda_m \rfloor}^{(m)}\big).$$

The Kolmogorov Extension theorem guarantees the existence of a stochastic process $(F([0, \infty) \colon \mathbb{Z}_p), \mathrm{P}_m, Y)$ that has the same finite dimensional distributions as $\tilde{Y}$. Equivalently, directly define the pre-measure P_m on any simple cylinder set $C(h)$ by

$$\mathrm{P}_m(C(h)) = \mathrm{P}^*_m \left(\bigcap_{i \in \{0,1,\ldots,\ell(h)\}} \left(S^{(m)}_{\lfloor e_h(i)\lambda \rfloor} \right)^{-1} (\Gamma_m^{-1}(U(i)) \right). \tag{5.17}$$

The Kolmogorov extension theorem guarantees that this pre-measure on a π-system extends to a measure on the σ-algebra generated by the simple cylinder sets in $F([0, \infty) \colon \mathbb{Z}_p)$.

Specialize (τ_m) so that for any positive real number σ,

$$\lambda_m = \sigma p^{mb},$$

and take P_m to be the measure that is given by (5.17). Take $\mathbb{E}_m$ to be the expected value with respect to the measure P_m.

Lemma 5.3.21. *For any natural number m that is greater than $M'(p,b)$ and any real number s in $(0,1)$, there is a constant K so that*

$$\left(1-\left(1-\frac{\beta p^b\left(1+\frac{1}{p^{mb}}\right)}{p^{mb}}\right)^{t_m\sigma p^{mb}}\right)\leq K t_m^s.$$

Proof. For any natural number n and any x in $[0,n]$,

$$1-x\leq\left(1-\frac{x}{n}\right)^n \quad\text{and so}\quad x\geq 1-\left(1-\frac{x}{n}\right)^n. \tag{5.18}$$

Change variables and take m to be large enough so that

$$\beta p^b\left(1+\frac{1}{p^{mb}}\right)\leq p^{mb}$$

to see that (5.18) implies that

$$2p^b\beta\sigma t_m\geq\left(1-\left(1-\frac{\beta p^b\left(1+\frac{1}{p^{mb}}\right)}{p^{mb}}\right)^{t_m\sigma p^{mb}}\right). \tag{5.19}$$

The quantity on the right hand side of (5.19) is bounded above by 1, and for any t_m in $[0,1]$, t_m^s is at least t_m, so

$$\max(2p^b\beta\sigma,1)t_m^s\geq\left(1-\left(1-\frac{\beta p^b\left(1+\frac{1}{p^{mb}}\right)}{p^{mb}}\right)^{t_m\sigma p^{mb}}\right). \tag{5.20}$$

■

Denote by $N(p,b)$ the maximum of $M(p,b)$ and $M'(p,b)$.

Proposition 5.3.22. *For any natural number m that is greater than $M(p,b)$ and any positive real number t, there is a constant C that is independent of t so that for any r in $(0,b)$,*

$$\mathbb{E}_m[|Y_t|^r] \le Ct^{\frac{r}{b}}.$$

Proof. For any positive real number t, there is an increasing sequence (t_m) so that

$$t_m\lambda_m = \lfloor t\lambda_m \rfloor \quad \text{and} \quad \lim_{m\to\infty} t_m = t.$$

The equality

$$\mathbb{E}_m[|Y_t|^r] = \mathbb{E}_m^*[|S_{t_m\lambda_m}|^r]$$

together with Theorem 5.2.20, with n replaced by $t_m\lambda_m$, and Lemma 5.3.21 implies that

$$\begin{aligned}
\mathbb{E}_m[|Y_t|^r] &\le K_1(t_m\lambda_m)^{\frac{r}{b}}p^{-mr} + \left(1 - \left(1 - \frac{\beta p^b\big(1+\frac{1}{p^{mb}}\big)}{p^{mb}}\right)^{t_m\sigma p^{mb}}\right)\frac{p-1}{p} \\
&\le K_1(t_m\sigma p^{mb})^{\frac{r}{b}}p^{-mr} + K_2(t_m\sigma)^{\frac{r}{b}} \\
&\le Kt_m^{\frac{r}{b}}\sigma^{\frac{r}{b}} \le Kt^{\frac{r}{b}}\sigma^{\frac{r}{b}},
\end{aligned}$$

since t_m is bounded above by t. ■

5.4 Uniform tightness

Proposition 5.4.23. *The stochastic process* $(F([0,\infty)\colon \mathbb{Z}_p), \mathrm{P}_m, Y)$ *has a version with paths in* $D([0,\infty)\colon \mathbb{Z}_p)$*, a process* $(D([0,\infty)\colon \mathbb{Z}_p), \mathrm{P}_m, Y)$*. The sequence of measures* (P_m) *with paths in* $D([0,\infty)\colon \mathbb{Z}_p)$ *is uniformly tight.*

Proof. For any strictly increasing finite sequence (t_1, t_2, t_3) in $[0,\infty)$, the independence of the increments of the process $(F(\mathbb{N}_0\colon G_m), \mathrm{P}^*_m, S)$ implies that

$$\begin{aligned}\mathbb{E}_m\left[\left|Y_{t_3} - Y_{t_2}\right|^r \left|Y_{t_2} - Y_{t_1}\right|^r\right] &= \mathbb{E}^*_m\left[\left|Y_{t_3} - Y_{t_2}\right|^r\right] \mathbb{E}^*_m\left[\left|Y_{t_2} - Y_{t_1}\right|^r\right] \\ &\leq C(t_3 - t_2)^{\frac{r}{b}} C(t_2 - t_1)^{\frac{r}{b}} \leq C^2(t_3 - t_1)^{\frac{2r}{b}}.\end{aligned}$$

Take r to be any real number in $\left(\frac{b}{2}, b\right)$ to verify that $(F([0,\infty)\colon \mathbb{Z}_p), \mathrm{P}^m, Y)$ satisfies the criterion of Chentsov, which implies that the stochastic process $(F([0,\infty)\colon \mathbb{Z}_p), \mathrm{P}^m, Y)$ has a version with paths in $D([0,\infty)\colon \mathbb{Z}_p)$. Uniformity of the constant C in both m and the triple of time points implies that (P^m) is a uniformly tight sequence of probability measures [7]. ∎

Continuity from above of the measure P^m together with the right continuity of the paths implies Proposition 5.4.24.

Proposition 5.4.24. *The measure* P^m *is concentrated on the subset of* $\Gamma_m(G_m)$*-valued paths in* $D([0,\infty)\colon \mathbb{Z}_p)$ *that are constant on each interval in* $\left\{\left[(n-1)\tau_m, n\tau_m\right)\colon n \in \mathbb{N}\right\}$.

Proof. For any strictly increasing finite sequence $(t_1, \ldots, t_k)$ in $[0,\infty)$, paths in $D([0,\infty) : \mathbb{Z}_p)$ are P^m-almost surely in $\Gamma_m(G_m)$ at any given place of $(t_1, \ldots, t_k)$. Finite intersections of almost sure events are almost sure and so

$$\mathrm{P}^m(\{\omega \in D([0,\infty) : \mathbb{Z}_p) : \omega(t_i) \in \Gamma_m(G_m), 1 \leq i \leq k\}) = 1$$

For any sequence (A_i) of strictly increasing nested finite subsets of $\mathbb{Q}$ whose union is $\mathbb{Q}$, continuity from above of P^m implies that

$$\mathrm{P}^m\big(\{\omega \in D([0,\infty) : \mathbb{Z}_p) : \omega(s) \in \Gamma_m(G_m), \forall s \in \mathbb{Q}\}\big)$$
$$\lim_{i\to\infty} \mathrm{P}^m\big(\{\omega \in D([0,\infty) : \mathbb{Z}_p) : \omega(s) \in \Gamma_m(G_m),\ \forall s \in A_i\}\big) = 1.$$

The right continuity of the paths implies that the set of $\Gamma_m(G_m)$-valued paths has full measure with respect to the measure P^m.

For any interval I in $[0,\infty)$ that does not intersect the set $\tau_m\mathbb{N}$, take (V_i) to be a nested sequence of finite subsets of I whose union is $I \cap \mathbb{Q}$. For any pair of real numbers s_1, s_2 in I, $\lfloor s_1\lambda(m)\rfloor$ is equal to $\lfloor s_2\lambda(m)\rfloor$ and so

$$\mathrm{P}^m\big(Y_{s_1} - Y_{s_2} = 0\big) = \mathrm{Prob}\big(\tilde{Y}^m_{s_1} - \tilde{Y}^m_{s_2} = 0\big)$$
$$\mathrm{Prob}\big(\tilde{S}^m_{\lfloor s_1(m)\rfloor} - \tilde{S}^m_{\lfloor s_1(m)\rfloor} = 0\big) = 1.$$

Continuity from above of the measure P^m implies that for any t in I,

$$\mathrm{P}^m\big(\{\omega \in D([0,\infty) : \mathbb{Z}_p) : |\omega(t) - \omega(s)| = 0,\ \forall s \in I \cap \mathbb{Q}\}\big)$$
$$\lim_{i\to\infty} \mathrm{P}^m\big(\{\omega \in D([0,\infty) : \mathbb{Z}_p) : |\omega(t) - \omega(s)| = 0,\ \forall s \in I \cap V_i\}\big) = 1.$$

The right continuity of the paths implies that

$$\mathrm{P}^m\big(\{\omega \in D([0,\infty):\mathbb{Z}_p) : |\omega(t)-\omega(s)| = 0,\ \forall s,t \in I\}\big) = 1.$$

Take (I_n) to be a sequence of disjoint intervals that do not intersect $\tau_m\mathbb{N}$ and whose union is $\mathbb{Q}\setminus\tau_m\mathbb{N}$. Continuity from above of the measure P^m implies that

$$\mathrm{P}^m\Bigg(\bigcap_{n\in\mathbb{N}}\{\omega \in D([0,\infty):\mathbb{Z}_p) : |\omega(t)-\omega(s)| = 0,\ \forall s,t \in I_n\}\Bigg) = 1.$$

■

5.5 Weak convergence

It is helpful to use more precise notation since there are two different types of quotient maps involved in this section. Again denote by $\langle\cdot\rangle$ the quotient map from $\mathbb{Q}_p$ to $\mathbb{Q}_p/\mathbb{Z}_p$ and by $\langle\cdot\rangle_m$ the quotient map from $p^{-m}\mathbb{Z}_p$ to $\tilde{G}_m$. View $p^{-m}\mathbb{Z}_p$ as a subset of $\mathbb{Q}_p$ and extend functions from $p^{-m}\mathbb{Z}_p$ to all of $\mathbb{Q}_p$ by using cutoff functions. Similarly, view $\tilde{G}_m$ as a subset of $\mathbb{Q}_p/\mathbb{Z}_p$ and extend functions similarly. For any positive real number D, write σ as

$$\sigma = \frac{D}{\beta}.$$

This way of writing σ gives a direct specification of the time scaling in order to have a diffusion constant D for the limiting process.

Denote by E_m the function that is defined for any $(t, \langle y\rangle)$ in $(0,\infty)\times \mathbb{Q}_p/\mathbb{Z}_p$ by

$$E_m(t,\langle y\rangle) = \begin{cases} (\phi_m(\langle y\rangle_m))^{\lfloor t\lambda(m)\rfloor} & \text{if } |\langle y\rangle| \leq p^m \\ 0 & \text{if } |\langle y\rangle| > p^m. \end{cases}$$

Lemma 5.5.25. *For any t in $(0,\infty)$,*

$$\lim_{m\to\infty} \int_{\mathbb{Q}_p/\mathbb{Z}_p} \left| E_m(t,\langle y\rangle) - \mathrm{e}^{-D(|\langle y\rangle|^b - \beta^{-1})t} \right| \mathrm{d}\langle y\rangle = 0.$$

Proof. For any z in $[0,1]$, the inequality

$$0 \leq 1 - z \leq \mathrm{e}^{-z}$$

implies that for any m and any x in $[0,m]$,

$$0 \leq \left(1 - \frac{x}{m}\right)^m \leq \left(\mathrm{e}^{-\frac{x}{m}}\right)^m = \mathrm{e}^{-x}. \tag{5.21}$$

Since the sequence on the left-hand side of (5.21) is increasing, for any positive real number R, Dini's theorem implies that $\left(1-\frac{x}{m}\right)^m$ converges uniformly to e^{-x} in $[0,R]$. For any $\langle y\rangle$, if $|\langle y\rangle|$ is less than p^m the Proposition 5.1.19 implies that

$$\beta|\langle y\rangle|^b \leq p^{mb}, \tag{5.22}$$

and so (5.21) together with the integrability of the exponential function over $(-\infty,0]$ implies that there is a divergent, increasing, positive sequence (R_m) in $(0, p^{m-1}]$ so that

$$\lim_{m\to\infty} \int_{|\langle y\rangle|\le R_m} \left| E_m(t, \langle y\rangle) - \mathrm{e}^{-\sigma(|\langle y\rangle|^b - \beta^{-1})t} \right| \, \mathrm{d}\mu(\langle y\rangle) = 0. \tag{5.23}$$

Denote by A_m the set

$$A_m = \{\langle y\rangle \in \mathbb{Q}_p/\mathbb{Z}_p \colon R_m < |\langle y\rangle| < p^m \text{ or } |\langle y\rangle| > p^m\}.$$

The inequality (5.22) implies that for any $\langle y\rangle$ in A_m,

$$\left| E_m(t, \langle y\rangle) - \mathrm{e}^{-\sigma(|\langle y\rangle|^b - \beta - 1)t} \right| \le \mathrm{e}^{-\sigma(|\langle y\rangle|^b - \beta^{-1})t},$$

and so

$$\lim_{m\to\infty} \int_{A_m} \left| E_m(t, \langle y\rangle) - \mathrm{e}^{-\sigma(|\langle y\rangle|^b - \beta^{-1})t} \right| \, \mathrm{d}\mu(y) = 0. \tag{5.24}$$

Since $|1 - \beta + p^{-mb}|$ is in $[0, 1)$ if m is large enough,

$$\begin{aligned} &\lim_{m\to\infty} \int_{|\langle y\rangle| = p^m} \left| E_m(t, \langle y\rangle) - \mathrm{e}^{-\sigma(|\langle y\rangle|^b - \beta^{-1})t} \right| \, \mathrm{d}\mu(\langle y\rangle) \\ &\qquad \le \lim_{m\to\infty} p^{md} \left| (1 - \beta - p^{-mb})^{\lfloor t\lambda_m \rfloor} - \mathrm{e}^{-\sigma(|\langle y\rangle|^b - \beta^{-1})t} \right| = 0. \end{aligned} \tag{5.25}$$

Decompose the integral in the statement of the lemma into three regions, the ball of radius less than R_m, the set A_m, and the circle of radius p^m and use (5.23), (5.24), and (5.25) to obtain the desired limit. ∎

The set H_R of *restricted histories* for paths in $D([0, \infty) \colon \mathbb{Z}_p)$ is the set of all histories whose route is a finite sequence of balls.

Proposition 5.5.26. *For any restricted history h in H_R,*

$$\mathrm{P}^m(\mathscr{C}(h)) \to \mathrm{P}(\mathscr{C}(h)).$$

Proof. For any ball B of radius p^{-m} in $\mathbb{Z}_p$, if x is in B, then $[x]_m$ is the unique element of G_m so that $\Gamma_m([x]_m)$ is in B, and so (5.17) implies that

$$\mathrm{P}^m(Y_t \in B) = \mathrm{P}_m\big(S_{t_m\lambda_m} \in \Gamma_m^{-1}(B)\big).$$

Denote by ρ^m the function that for each t in $(0, \infty)$ is given by

$$\rho^m(t, z) = \sum_{[x]\in G_m} \rho_m\big(t_m\lambda_m, [x]\big)\mathbb{1}_{x+p^m\mathbb{Z}_p}(z), \tag{5.26}$$

so that for any t in $(0, \infty)$ and any set U in $\mathbb{Z}_p$ that is a disjoint union of balls that each have radius at least p^{-m},

$$\mathrm{P}^m(Y_t \in U) = \int_{\mathbb{Z}_p} \rho^m(t, x)\,\mathrm{d}x = \mathrm{P}_m(S_{t_m\lambda_m} \in [U]).$$

Take ρ to be the probability mass function for the $\mathbb{Z}_p$-valued process. The absolute value of the difference between ρ^m and ρ is given by

$$\begin{aligned} |\rho^m(t,x) - \rho(t,x)| &\le \int_{\mathbb{Q}_p/\mathbb{Z}_p} \left| \chi(xy) E_m(t, \langle y \rangle) - \chi(xy) \mathrm{e}^{-\sigma(t|\langle y \rangle|^b - \beta^{-1})} \right| \mathrm{d}\langle y \rangle \\ &= \int_{\mathbb{Q}_p/\mathbb{Z}_p} \left| E_m(t, \langle y \rangle) - \mathrm{e}^{-\sigma t_m \lambda_m(|\langle y \rangle|^b - \beta^{-1})} \right| \mathrm{d}\langle y \rangle = \varepsilon_m(t) \to 0. \quad (5.27) \end{aligned}$$

The sequence $(\varepsilon_m(t))$ is independent of $[x]$, which implies that $(\rho_m(t,\cdot))$ converges uniformly on $\mathbb{Z}_p$ to $\rho(t,\cdot)$.

Take h to be any restricted history and without loss in generality suppose that $U_h(0)$ is the set $\{0\}$. Simplify the notation by writing

$$e_h = (t_0, \dots, t_k) \quad \text{and} \quad U_h = (\{0\}, U_1, \dots, U_k).$$

For any i in $\{0, \dots, k\}$, denote by r_i the radius of U_i. For any m so that

$$p^{-m} < \min\{r_1, \dots, r_k\},$$

the uniform convergence given by (5.27) implies that

$$\begin{aligned} \mathrm{P}^m(\mathscr{C}(h)) &= \mathrm{P}^m(Y_{t_1} \in U_1, Y_{t_2} \in U_2, \dots, Y_{t_n} \in U_n) \\ &= \int_{U_1} \cdots \int_{U_k} \prod_{i \in \{1,\dots,k\}} \rho_m(t_i - t_{i-1}, x_i - x_{i-1}) \, \mathrm{d}x_k \cdots \mathrm{d}x_1 \\ &\to \int_{U_1} \cdots \int_{U_k} \prod_{i \in \{1,\dots,k\}} \rho(t_i - t_{i-1}, x_i - x_{i-1}) \, \mathrm{d}x_k \cdots \mathrm{d}x_1 = \mathrm{P}(\mathscr{C}(h)). \end{aligned}$$

■

The intersection of any two balls in $\mathbb{Z}_p$ is again a ball in $\mathbb{Z}_p$, and so $\mathscr{C}(H_R)$ is a π-system that generates the σ-algebra of cylinder sets of paths in $D([0,\infty)\colon \mathbb{Z}_p)$. The uniform tightness of the family of measures $\{\mathrm{P}^m \colon m \in \mathbb{N}_0\}$ that Proposition 5.4.23 guarantees together with the convergence for any restricted history h of $(\mathrm{P}^m(\mathscr{C}(h)))$ to $\mathrm{P}(\mathscr{C}(h))$ implies our main Theorem 5.5.27.

Theorem 5.5.27. *The sequence of measures* (P^m) *converges weakly to* P.

As in the real case and the p-adic case [5, 22], the scaling factor for the time scales is proportionate to a power of the scaling factor for the space scales.

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