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On the problem of deformation invariance of plurigenera

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The dissertation of Iacopo Brivio is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California San Diego

2020

DEDICATION

To my family

EPIGRAPH

Guarda non è che c'è tanto da vedere,

con uno stile di moto così, come vedi qua,

cioè uno stile di fisico del genere,

può accompagnare solo.

— William

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The dissertation author was the primary investigator and author of this material.

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ABSTRACT OF THE DISSERTATION

On the problem of deformation invariance of plurigenera

by

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Doctor of Philosophy in Mathematics

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Professor James M^cKernan, Chair

In this dissertation, we study the problem of the deformation invariance of plurigenera of algebraic varieties using techniques from the Mori and Iitaka Programs. In characteristic zero, we study the problem for smooth families $X \longrightarrow T$, such that $\kappa(X/T) \geq 0$. In this case, we reformulate a famous theorem of Siu in terms of a condition on the central fiber of certain special models of the relative Iitaka fibration $X \dashrightarrow \underline{\mathbf{Proj}}_T R(K_{X/T})/T$. Under some additional assumptions, this gives algebraic proofs of deformation invariance of plurigenera, generalizing results of Nakayama and Kawamata. In positive and mixed characteristic, we construct examples of families of smooth surfaces where all sufficiently divisible plurigenera fail to be constant, answering a quesion of Katsura and Ueno [KU85]. In particular, invariance of plurigenera does not follow from the MMP and Abundance Conjecture. Lastly, we show that invariance of all sufficiently divisible plurigenera holds for families of quasi-elliptic surfaces and certain families of log Calabi-Yau fibrations of small relative dimension.

Chapter 1

Introduction

One of the most important guiding principles of higher dimensional algebraic geometry is that one should understand the geometry of a smooth projective variety X in terms of the positivity properties of its *canonical divisor* K_X. Suppose, for example, that X is a curve: its most important invariant is its *genus* $g(X) = h^0(K_X)$, i.e. the number of independent holomorphic 1-forms. Then, the Riemann-Roch Theorem yields the following trichotomoy

- $g(X) = 0, -K_X$ is ample (Fano case);
- $g(C) = 1, K_X$ is trivial (Calabi-Yau case); and
- $g(C) \geq 2, K_X$ is ample (general type case).

Moreover, when $g \geq 2$, there exists a *moduli space* for genus g stable curves, in this case a projective DM-stack $\bar{\mathcal{M}}_g \longrightarrow \mathrm{Spec}\mathbb{Z}$, parametrizing stable families of genus g

curves. In higher dimensions the situation becomes more complicated: in particular, the genus does not detect the positivity of K_X anymore.

Example 1.0.1. Let $V = \{x^5 + y^5 + z^5 + w^5 = 0\} \subset \mathbb{P}^3$, let ζ be a primitive fifth root of 1, and consider the group action $\langle\zeta\rangle \circlearrowleft V$ via $[x : y : z : w] \mapsto [\zeta x : \zeta^2 y : \zeta^3 z : \zeta^4 w]$, and let $X = V/\langle\zeta\rangle$ be the quotient. Then $h^0(K_X) = h^0(K_{\mathbb{P}^2}) = 0$, but K_X is ample while $K_{\mathbb{P}^2}$ is anti-ample

Since ampleness is an asymptotic condition, it is natural to consider *all* multiples of the canonical divisor.

Definition 1.0.2. Let X be a smooth projective variety and m a positive integer. Then, $P_m(X) = h^0(mK_X)$ defined to be the m^{th}-*plurigenus of* X. The *Kodaira dimension of* X is defined as

$$\kappa(X) = \begin{cases} \limsup_{m\to\infty} \frac{\log P_m(X)}{\log m} & \text{if } |K_X|_{\mathbb{Q}} \neq \emptyset \\ -\infty & \text{otherwise} \end{cases}$$

If $\kappa(X) = \dim X$, we say that X is of *general type.*

Remark 1.0.1. Equivalently, one can define $\kappa(X)$ as $\max_{m\geq 0} \dim \mathrm{Image}\phi_{mK_X}$, when $|K_X|_{\mathbb{Q}}$ is not empty, and $-\infty$ otherwise (see [Laz04a, Definition 2.1.3, Corollary 2.1.37]).

By the Riemann-Roch Theorem, $\kappa(X) \in \{-\infty, 0, 1, ..., \dim X\}$. When X is a curve, we can still recover the trichotomy given by the genus: $g(X) = 0, 1$ or ≥ 2 if and only if $\kappa(X) = -\infty, 0$ or 1 respectively. Furthermore, we can now distingish between the surfaces of Example 1.0.1: $\kappa(\mathbb{P}^2) = -\infty$ and $\kappa(X) = 2$.

On the other hand, constructing moduli spaces for higher-dimensional varieties is considerably more complicated than in the case of curves. In particular, while any birational map between smooth curves is an isomorphism, there are in infinitely many non-isomorphic birational manifolds. This suggests that moduli spaces of higher-dimensional varieties should parametrize birational equivalence classes or, more precisely, "nice representatives" of each equivalence class. A recipe for the construction of such models is given by the Minimal Model Program (MMP), a partially conjectural algorithm aimed at providing a birational classification of algebraic varieties, generalizing the theory of minimal surfaces developed by the Italian school of algebraic geometry. We refer the reader to Section 2.3 for a more complete introduction to the MMP, here we just recall the fundamental definitions and conjectures.

Definition 1.0.3. Let X be a $\mathbb{Q}$-factorial projective variety X with terminal singularities. If K_X is nef, we say that X is a *minimal model*. If there exists a contraction $f : X \longrightarrow Z$ such that $\dim Z < \dim X$ and the general fiber is a Fano variety of Picard rank 1, we say X (or f) is a *Mori fiber space*.

The following are the main Conjectures of the MMP: they are known in dimension ≤ 3 by work of Kawamata, Kollár, Mori, and Shokurov [Kaw84, Kol84, KM92, Mor88, Sho93], and for all varieties of general type, by work of Birkar, Cascini, Hacon, and McKernan [BCHM10].

Conjecture 1.0.1 (Minimal Model)**.** *Let X be a smooth complex projective variety. Then there exist a birational contraction $X \dashrightarrow X'$ where*

- *if K_X is pseudo-effective, X' is a minimal model; or*
- *if K_X is not pseudo-effective, X' has a Mori fiber space structure.*

Conjecture 1.0.2 (Abundance)**.** *If X is a minimal model, then K_X is semi-ample.*

It would be tempting to construct moduli spaces for minimal models, however it turns out that this leads to non-separated moduli spaces (see [Kol17, Example 1.28]). Furthermore, when $\dim X \geq 3$ a minimal model is never unique. The solution is to consider the *canonical model of* X, that is

$$X^{\text{can}} := \text{Proj} R(K_X)$$

where $R(K_X) := \bigoplus_{m \geq 0} H^0(X, mK_X)$ is the *canonical ring of* X. By [BCHM10, Theorem 1.2, Corollary 1.1.1] $R(X)$ is a finitely generated $\mathbb{C}$-algebra. Suppose now that X is of general type: then, for $m > 0$ sufficiently divisible, we have a birational map $\phi_{mK_X} : X \dashrightarrow X^{\text{can}}$, and $K_{X^{\text{can}}}$ is ample. Usually X^{can} will not be smooth, but will have canonical singularities (see Definition 2.2.2).

Note that the canonical model is always unique, and it depends only on the birational equivalence class of X. It turns out that canonical models, and their non-normal generalization, *stable varieties*, are the right object to consider in order to have a moduli space that is separated and projective, and very recently it has been showed by Kollár

[Kol19] that a good moduli theory exists for stable pairs in characteristic zero. The next result, and its generalizations, play a fundamental role in the construction of moduli of varieties of general type (see [HMX18]). Roughly speaking, since we are interested in parametrizing families of canonical models, it is desirable that forming the canonical model commutes with the restriction to a fiber of the family.

Theorem 1.0.4 (Invariance of plurigenera, [Siu98])**.** *Let $X \longrightarrow T$ be a smooth family over a smooth affine curve. Then, the plurigenera $P_m(X_t)$ do not depend on $t \in T$, for all $m \geq 0$.*

In other words, the restriction map $R(K_X) \longrightarrow R(K_{X_t})$ is surjective for all $t \in T$ hence, if $\phi : X \dashrightarrow X^{\mathrm{can}}$ is the relative canonical model of X/T, then the restriction $\phi_t : X_t \dashrightarrow (X^{\mathrm{can}})_t$ is the canonical model of X_t for all $t \in T$. It is interesting to note that the only known proof of this result is analytic, relying on deep results from complex analysis such as the Ohsawa-Takegoshi extension theorem [OT87], and there is currently no completely algebraic proof of Theorem 1.0.4. A notable exception is when K_X is assumed to be big over T: then Siu's argument can be reduced to the Nadel vanishing theorem, as shown by Kawamata [Kaw99] (see also [Laz04b, Section 11.5]). Moreover, it is known by results of Nakayama [Nak86, Theorem 8], that Theorem 1.0.4 follows from Conjectures 1.0.1 and 1.0.2.

As one can expect, the situation is even more mysterious in positive or mixed characteristic, even just for families of surfaces. By results of Katsura-Ueno and Egbert-Hacon [KU85, EH16] Conjectures 1.0.1 and 1.0.2 hold for log smooth families of klt surfaces pair $(X, \Delta) \longrightarrow \mathrm{Spec} R$, where R is a positive or mixed characteristic DVR. As a consequence, the Kodaira dimension is invariant. Furthermore, it is showed in [EH16] that, for all sufficiently divisible $m \geq 0$, $h^0(m(K_{X_s} + \Delta_s))$ does not depend on the geometric point $s \in \mathrm{Spec} R$ unless, possibly, if $|m(K_X + \Delta)|$ induces an elliptic fibration. On the other hand, it is known that deformation invariance of *all* plurigenera can fail: in [KU85] the authors give examples of families of elliptic surfaces where P_1 is not invariant. Furthermore, in [Suh08] it is shown that the same can happen even when K_X is ample. Hence, it is natural to ask the following.

Conjecture 1.0.3 (Asymptotic invariance of log plurigenera)**.** *Let* $(X, \Delta) \longrightarrow \mathrm{Spec} R$ *be a log smooth family of klt pairs over the spectrum of a DVR. Then,*

$$h^0(m(K_{X_{\bar{K}}} + \Delta_{\bar{K}})) = h^0(m(K_{X_k} + \Delta_k))$$

for all sufficiently divisible $m \geq 0$.

1.1 Statement of main results

The thesis is divided into two parts: in the first one we work in characteristic zero, and in the second over a DVR of positive or mixed characteristic. A recurring theme in both parts will be the reduction of Theorem 1.0.4 to certain properties of the relative Iitaka fibration of the family.

1.1.1 Characteristic zero

In the first part we study (algebraically) the problem of deformation invariance of plurigenera for a smooth, one-parameter family of complex varieties $X \longrightarrow T$ such that $\kappa(X/T) \geq 0$. Albeit not being a birational geometry problem, invariance of plurigenera is very susceptible to the positivity properties of the canonical divisor: for example, if K_X is ample and $m \gg 0$, Serre vanishing yields $P_m(X_t) = \chi(mK_{X_t})$, and the latter is constant. When K_X is semi-ample, a similar argument shows that $P_m(X_t)$ is invariant for all sufficiently divisible $m \geq 0$ (see Proposition 4.0.1). Hence, it is natural to approach the problem using techniques from the Minimal Model Program. The results of Kawamata and Nakayama [Kaw99, Nak86] both assume rather strong positivity properties of K_X over T, such as bigness and semi-ampleness. It is then interesting to (a) weaken those hypothesis, and/or (b) assume positivity of just K_{X_0} instead. In this paper, we make progress in both these directions: we study algebraically the problem of invariance of plurigenera for families $X \longrightarrow T$ such that $\kappa(X/T) \geq 0$. Although we are unable to prove invariance of plurigenera under just this assumption, we manage to reformulate it in terms of a condition on a "nice" model of the relative Iitaka fibration of X/T (see Theorem 1.1.2).

Outline of the argument

We begin by showing that deformation invariance of *all* plurigenera is equivalent to deformation invariance of *all sufficiently divisible* plurigenera.

Theorem 1.1.1. *Let $X \longrightarrow T$ be a smooth family and suppose that $P_m(X_t)$ is independent of $t \in T$ for all sufficiently divisible $m \geq 0$. Then, $P_m(X_t)$ is independent of t for all $m \geq 0$.*

This was also proven by Nakayama, assuming the MMP and the Abundance Conjecture. We give an alternative proof, based on a torsion-freeness result for higher direct images of m-pluricanonical forms twisted by a suitable multiplier ideal sheaf (see Theorem 3.3.3). Let now X be a smooth projective variety such that $\kappa(X) \geq 0$. Modulo passing to a higher birational model, we may assume that the Iitaka fibration is a morphism $f : X \longrightarrow Z$. Furthermore, by results of Fujino and Mori ([FM00], see also Theorem 2.4.6), there exists a boundary Δ on Z, and positive integers a and b, such that $K_Z + \Delta$ is klt and $R(K_X)^{(a)} \cong R(K_Z + \Delta)^{(b)}$. In particular, (Z, Δ) is a klt pair of log general type. Then we can take the log canonical model $\rho : (Z, \Delta_Z) \dashrightarrow (\bar{Z}, \bar{\Delta})$, so that we still have $R(K_X)^{(a)} \cong R(K_{\bar{Z}} + \bar{\Delta})^{(b)}$, but now $K_{\bar{Z}} + \bar{\Delta}$ is ample. Suppose now that the same result holds for families of varieties. That is, given $X \longrightarrow T$ a smooth family, with $\kappa(X/T) \geq 0$, there exists a family of pairs $(\bar{Z}, \bar{\Delta}) \longrightarrow T$, and positive integers a, b, such that $K_{\bar{Z}} + \bar{\Delta}$ is ample, and $R(K_{X_t})^{(a)} \cong R(K_{\bar{Z}_t} + \bar{\Delta}_t)^{(b)}$ for all $t \in T$. Then, by Serre vanishing, it follows immediately that $P_m(X_t)$ is independent of t for all sufficiently divisible $m \geq 0$.

Unfortunately, we are unable to prove such a result, since several technical issues arise in the relative case. First, the relative Iitaka fibration is only a rational map, not a morphism. When dealing with one variety, this does not cause problems, as one can always pass to a resolution of indeterminacies. However if $X \longrightarrow T$ is a family of varieties, and $Y \longrightarrow X$ is a higher birational model, the induced map $Y \longrightarrow T$ will almost never be a family of varieties. Second, it is not clear whether the construction of Δ commutes with the restriction to a fiber X_t (see Remark 2.4.2). Lastly, even after constructing $(Z, \Delta) \longrightarrow T$, and taking its relative canonical model $(\bar{Z}, \bar{\Delta}) \longrightarrow T$, it is not true in general that $(\bar{Z}_t, \bar{\Delta}_t)$ is the canonical model of (Z_t, Δ_t); indeed, this last property holds if and only if, letting $\psi : (Z, \Delta) \dashrightarrow (Z^{\min}, \Delta^{\min})$ be the $(K_Z + \Delta)$-MMP over T, $\psi_t : (Z_t, \Delta_t) \dashrightarrow (Z^{\min,t}, \Delta^{\min,t})$ is a $(K_{Z_t} + \Delta_t)$-MMP, for all $t \in T$.

However, we are able to solve these issues, at least in part. Our main technical result is that, given a smooth family $X \longrightarrow T$ such that $\kappa(X/T) \geq 0$, there exists a *T-adapted model*, i.e. a morphism $\bar{f} : \bar{X} \longrightarrow (\bar{Z}, \bar{\Delta})/T$, such that $(\bar{Z}, \bar{\Delta})$ is a good minimal model of log general type over T and $R(K_{\bar{X}})^{(a)} \cong R(K_{\bar{Z}} + \bar{\Delta})^{(b)}$. This allows us to reformulate invariance of plurigenera in terms of a condition on the central fiber of these models.

Theorem 1.1.2. *Let $X \longrightarrow T$ be a smooth family with $\kappa(X/T) \geq 0$. Let $f : \bar{X} \longrightarrow \bar{Z}/T$ be a T-adapted model of the relative Iitaka fibration of X/T. Then, the following are equivalent:*

(1) $P_m(X_t)$ is independent of t for all $m \geq 0$;

(2) $P_{am}(\bar{X}_0) = h^0(bm(K_{\bar{Z}_0} + \bar{\Delta}_0))$ for some $a, b \geq 1$ and all sufficiently divisible $m \geq 0$.

As a corollary, we have that, when $\kappa(X/T) \geq 0$, invariance of plurigenera follows from the MMP and Abundance Conjecture for varieties of Kodaira dimension zero, generalizing [Nak86, Theorem 8].

Corollary 1.1.3. *Let $X \longrightarrow T$ be a smooth family such that $\kappa(X/T) \geq 0$. Suppose that the general fiber of the Iitaka fibration of $X_{\bar{\eta}}$ has a good minimal model. Then $P_m(X_t)$ is independent of $t \in T$ for all $m \geq 0$.*

Finally, Theorem 1.1.2 and Corollary 1.1.3 imply that, when the general fiber of the Iitaka fibration of X_0 has a good minimal model, invariance of plurigenera is equivalent to invariance of Kodaira dimension.

Corollary 1.1.4. *Let $X \longrightarrow T$ be a smooth family. Assume $\kappa(X_t)$ is independent of t, and that the general fiber of the Iitaka fibration of X_0 has a good minimal model. Then:*

(1) the general fiber of the Iitaka fibration of $X_{\bar{\eta}}$ has a good minimal model; and

(2) $P_m(X_t)$ is independent of $t \in T$ for all $m \geq 0$.

1.1.2 Positive and mixed characteristic

In the second part we replace T by the spectrum of a positive or mixed characteristic DVR R, with residue and fraction fields k and K, respectively. First, we answer Conjecture 1.0.3 negatively: we construct a family of smooth minimal surfaces of Kodaira dimension one $X \longrightarrow \mathrm{Spec} R$ such that $P_m(X_k) > P_m(X_{\bar{K}})$ for all $m \in 3\mathbb{N}$ (see Section 4.1). In particular, Conjecture 1.0.3 does not follow from the MMP and Abundance Conjectures, unlike the characteristic zero case. However, we are able to recover the following extension result.

Theorem 1.1.5. *Let* $(X,\Delta) \longrightarrow \mathrm{Spec}R$ *be an abundant family of pairs over a DVR of positive or mixed characteristic. Then, there exists a non-negative integer* r *such that, for all sufficiently divisible* $m \geq 0$ *and all* $s \in H^0(X_k, m(K_{X_k}+\Delta_k))$*, the section* s^{p^r} *lifts to a section of* $H^0(X, mp^r(K_X+\Delta))$*.*

Both the construction of counterexamples and the proof of Theorem 1.1.5 rely on the following observation: if $(X,\Delta) \longrightarrow \mathrm{Spec}R$ is an abundant family of pairs and $f : X \longrightarrow \mathrm{Proj}R(K_X+\Delta)/\mathrm{Spec}R$ is the relative Iitaka fibration, the failure of invariance of all sufficiently divisible plurigenera is equivalent to the existence of a factorization

$$f_k : X_k \xrightarrow{g_k} \mathrm{Proj}R(K_{X_k}+\Delta_k) \xrightarrow{h_k} \mathrm{Proj}\left(R(K_X+\Delta)\otimes_R k\right)$$

where h_k is purely inseparable (see Proposition 4.0.2 and Corollary 4.0.3). As a consequence, we are able to give sufficient conditions for Conjecture 1.0.3.

Theorem 1.1.6. *Let* $(X,\Delta) \longrightarrow \mathrm{Spec}R$ *be an abundant family of pairs and let* $f : X \longrightarrow Z := \mathrm{Proj}R(K_X+\Delta)$ *be the relative Iitaka fibration. Let* $F_{\bar{K}}$ *be a general fiber of* $f_{\bar{K}}$*, let* $d = \dim F_{\bar{K}}$*, and let*

$$\iota : \mathrm{Pic}(X_{\bar{K}})^{\times d} \longrightarrow \mathbb{Z}$$

be the map given by $\iota([D_1],...,[D_d]) := D_1 \cdot ... \cdot D_d \cdot F_{\bar{K}}$*. If* $\mathrm{Image}(\iota) \not\subset p\mathbb{Z}$*, then*

$$h^0(m(K_{X_{\bar{K}}}+\Delta_{\bar{K}})) = h^0(m(K_{X_k}+\Delta_k))$$

for all sufficiently divisible $m \geq 0$*.*

As a consequence of the proof of the above Theorem, we have that Conjecture 1.0.3 holds for families of quasi-elliptic surfaces.

Theorem 1.1.7. *Let* $X \longrightarrow \mathrm{Spec}R$ *be a family of quasi-elliptic surfaces, then* $P_m(X_k) = P_m(X_{\bar{K}})$ *for all sufficiently divisible* $m \geq 0$*.*

We are also able to prove Conjecture 1.0.3 for certain families of log Calabi-Yau fibrations.

Theorem 1.1.8. *Let* $(X, \Delta) \longrightarrow \mathrm{Spec}R$ *be an abundant family of pairs over the spectrum of a DVR, such that*

(a) $\kappa((X, \Delta)/R) = \dim X - 1$*; and*

(b) Δ_k *is ample over* $\mathrm{Proj}R(K_{X_k} + \Delta_k)$*.*

Then, there exists an $m_0 \in \mathbb{N}$ *such that* $h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_{\bar{K}}} + \Delta_{\bar{K}}))$ *for all* $m \in m_0\mathbb{N}$*.*

Theorem 1.1.9. *For every* $\epsilon > 0$ *there exists* p_0 *such that, if* $(X, \Delta) \longrightarrow \mathrm{Spec}R$ *is an abundant family of pairs over the spectrum of a DVR with residue characteristic* $p > p_0$ *satisfying:*

(a) $\dim X_{\bar{K}} - \kappa(X_{\bar{K}}) = 2$*;*

(b) Δ_k *is ample over* $\mathrm{Proj}R(K_{X_k} + \Delta_k)$*; and*

(c) the general fiber of the Iitaka fibration of (X_s, Δ_s) *is* ϵ*-lc for all geometric points* $s \in \mathrm{Spec} R$*;*

then $h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_{\bar{K}}} + \Delta_{\bar{K}}))$ *for all sufficiently divisible* $m \geq 0$.

Note that, when $\dim_R X = 2$, Theorem 1.1.8 follows also from [EH16, Theorem 1.1].

Chapter 2

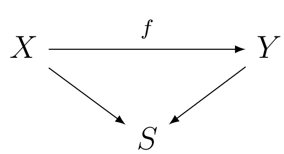
Preliminaries

In this chapter we collect some notation and preliminary facts that will be freely used in the rest of the dissertation.

2.1 Notations and conventions

- A *variety* will be an integral and separated scheme of finite type over an algebraically closed field k. Our varieties will usually be quasi-projective and normal. If X is a variety or scheme, we denote by $\mathrm{Sing}(X)$ its *singular locus*.

- Let S be any scheme and let X, Y be S-schemes. A morphism $f : X \longrightarrow Y$ is *over* S if the following diagram is commutative



In this case we write $f : X \longrightarrow Y/S$. An analogous definition holds when $f : X \dashrightarrow Y$ is just a rational map.

- Let $\phi : X \dashrightarrow Z$ and $\phi' : X' \dashrightarrow Z'$ be dominant rational maps of normal, quasi-projective varieties. We say that ϕ and ϕ' are *birational* if there exists a commutative diagram

$$\begin{array}{ccc} X' & \dashrightarrow & X \\ {\scriptstyle \phi'} \downarrow & & \downarrow {\scriptstyle \phi} \\ Z' & \dashrightarrow & Z \end{array}$$

  where the horizontal arrows are birational. When the horizontal arrows are morphisms, we say ϕ' is an *higher birational model of* ϕ'.

- Let $\mathbb{K}$ be any of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$: a $\mathbb{K}$-*(Weil) divisor* is a formal linear combination of codimension one subvarieties of X with coefficients in $\mathbb{K}$. An analogous definition holds for $\mathbb{K}$-Cartier divisors. By *divisor* we just mean a $\mathbb{Z}$-Weil divisor.

- If D_1 and D_2 are $\mathbb{K}$-divisors on X, we say they are $\mathbb{K}$-*linearly equivalent* if their difference is a $\mathbb{K}$-linear combination of principal divisors on X. In this case, we write $D_1 \sim_{\mathbb{K}} D_2$. Furthermore, given a morphism $f : X \longrightarrow S$, we say they are $\mathbb{K}$-*linearly equivalent over* S if $D_1 - D_2 \sim_{\mathbb{K}} f^*A$, for some $\mathbb{K}$-divisor A on S.

- Let $\pi : X \longrightarrow S$ be a projective morphism of normal varieties. Let D be an $\mathbb{K}$-divisor on X: its *relative* $\mathbb{K}$-*linear system* is defined to be $|D/S|_{\mathbb{K}} := \{D' \text{ s.t. } 0 \leq D' \sim_{\mathbb{K},S} D\}$. The *relative stable base locus of* D is $\mathbb{B}(D/S) = \bigcap_{D' \in |D/S|_{\mathbb{R}}} \mathrm{Supp} D'$, the *relative diminished base locus of* D is $\mathbb{B}_{-}(D/S) = \bigcup_{\epsilon>0} \mathbb{B}(D + \epsilon A/S)$, where A is any π-ample divisor. We define the *relative fixed part of* D $\mathbb{F}\mathrm{ix}(D/S)$ to be the biggest effective divisor F such that $D' - F \geq 0$ for all $D' \in |D/S|_{\mathbb{R}}$, and the *relative mobile part* as $\mathbb{M}\mathrm{ov}(D/S) := D - \mathbb{F}\mathrm{ix}(D/S)$. Note that $\mathbb{F}\mathrm{ix}(D/S)$ is the codimension one part of the stable base locus of D, and $\mathbb{F}\mathrm{ix}(\mathbb{M}\mathrm{ov}(D/S)/S) = 0$. IWe will omit the word "relative" whenever S is affine.

- A *contraction* is a projective morphism $f : X \longrightarrow Y$ of normal varieties such that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. Let P be a prime divisor on X. We say that P is *f-horizontal*, or *horizontal over* Y, if it dominates Y. Otherwise we say that P is *f-vertical*, or *vertical over* Y. If D is an $\mathbb{R}$-divisor on X, we have a unique decomposition $D = D^{\mathrm{hor}} + D^{\mathrm{ver}}$, where every component of D^{hor} is f-horizontal and every component of D^{ver} is f-vertical. We call D^{hor} and D^{ver} the *f-horizontal* and *f-vertical parts of* D, respectively.

- Let D be a $\mathbb{R}$-divisor on a normal variety: we have a unique decomposition $D = D^+ - D^-$, where D^+ and D^- are effective and $D^+ \wedge D^- = 0$. We call D^+ and D^- the *positive* and *negative part* of D, respectively.

- If $D = \sum a_i D_i$ is a $\mathbb{R}$-divisor, we define its *round down* $\lfloor D \rfloor = \sum \lfloor a_i \rfloor D_i$. Analogous definitions hold for the *round up* $\lceil D \rceil$ and the *fractional part* $\{D\}$. Its *support* is defined as $\mathrm{Supp} D = \bigcup_{a_i \neq 0} D_i$. If $D' = \sum b_i D_i$ is another $\mathbb{R}$-divisor, we define $D \wedge D'$ as $\sum \min(a_i, b_i) D_i$. If P is a prime divisor, we denote by $\mathrm{coeff}_P(D)$ the coefficient of P in D.

- If X is a variety, we say E is a divisor *over* X if it is a divisor on some higher birational model $\mu : X' \longrightarrow X$. We will denote by $c_X(E) := \mu(E)$ the *center of* E *on* X'.

- A *sub-pair* (X, Δ) consists of a normal variety X and a $\mathbb{Q}$-divisor Δ, such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. If furthermore $\Delta \geq 0$, we say (X, Δ) is a *pair*.

- A *pair* $(X, \Delta = \sum a_i D_i)$ is *normal crossing* (NC) if X is smooth and, in étale-local coordinates x_i, we have $\mathrm{Supp}\Delta = \{\prod_{i \in I} x_i = 0\}$. If furthermore D_i is smooth for all i, then the pair is said to be *simple normal crossing* (SNC) or *log smooth*.

- If (X, Δ) is a log smooth pair, a *stratum* of (X, Δ) is either X or an irreducible component of $\bigcap_{j \in J} D_j$ where J is a subset of the components of Δ having coefficient one.

- For a birational morphism $\mu : X \longrightarrow Y$, we define $\mathrm{Exc}(\mu)$ to be the *exceptional locus of f*, i.e. the set of points of X where μ is not an isomorphism. Its codimension one part is called the *exceptional divisor of* μ. if D is a divisor on Y, we denote by $\widetilde{D}$ or $\mu_*^{-1}D$ the *strict transform of* D *on* X.

- A *log resolution of* (X, Δ) is a birational morphism $\mu : Y \longrightarrow X$, which is an isomorphism away from $\mathrm{Sing}(X) \cup \mathrm{Sing}(\mathrm{Supp}\Delta)$, $\mathrm{Exc}(\mu)$ is a divisor, and $(Y, \mu_*^{-1}D + \mathrm{Exc}(\mu))$ is log smooth. When $\mathrm{char} k = 0$ and k is algebraically closed, log resolutions exists by Hironaka's desingularization theorem [Hir64]. If X is a regular scheme of dimension ≤ 3 over an excellent ring, a log resolution exists by [CP09].

- Let $\pi : X \longrightarrow S$ be a morphism of schemes or algebraic spaces. A *simultaneous resolution of* π is a commutative diagram

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\mu} & X \\ & {\scriptstyle\bar{\pi}} \searrow \quad \swarrow {\scriptstyle\pi} & \\ & S & \end{array}$$

such that μ is proper, $\bar{\pi}$ is smooth, and $\mu_s : \bar{X}_s \longrightarrow X_s$ is birational for all $s \in S$. Simultaneous resolutions do not exists in general; a notable exception is the case of families of surfaces with DuVal singularities (see Theorem **??**).

- Let $\pi : X \longrightarrow S$ be a projective morphism of normal varieties and let (X, Δ) be a lc pair. The *relative log canonical ring of* (X, Δ) *over* S is

$$R(K_X + \Delta/S) := \bigoplus_{m \geq 0} \mathcal{O}_X(\lfloor K_X + \Delta \rfloor)$$

In characteristic zero $R(K_X + \Delta/S)$ is a finitely generated graded $\mathcal{O}_S$-algebra by [BCHM10, Theorem 1.2]. The same holds, regardless of the characteristic, when $K_X + \Delta$ is semi-ample over S. When $R(K_X + \Delta/S)$ is finitely generated, we define the *relative log canonical model of* (X, Δ) *over* S as $Z := \underline{\mathbf{Proj}}_S R(K_X + \Delta/S) \longrightarrow S$. When $m \geq 0$ is sufficiently divisible we the rational map $\phi_{m(K_X+\Delta)} : X \dashrightarrow Z/S$ is called the *relative Iitaka fibration*. By [Laz04a, Theorem 2.1.32], if $X' \longrightarrow X$ is a resolution of indeterminacies of $\phi_{m(K_X+\Delta)}$, the induced morphism $X' \longrightarrow Z$ is a contraction. The *Kodaira dimension of* (X,Δ) *over* S is $\kappa((X,\Delta)/S) = \dim_S Z$.

- We will denote by $T = (T, 0)$ the germ of a smooth affine curve. We will denote by $\bar{\eta}$ its geometric generic point, respectively. Since we can freely shrink T around 0, we will often write "$t \neq 0$" instead of "$t \in T$ general".

- if R is a DVR of positive or mixed characteristic, we denote by K, k, ϖ and p its fraction field, its residue field, the uniformizer, and the characteristic of k respectively. We say it is *nice* if K is perfect and k is algebraically closed. If $X \longrightarrow \mathrm{Spec}R$ is a flat R-scheme and V is a subscheme, we will denote by $V_\daleth$ the fiber product $V \times_{\mathrm{Spec}R} \mathrm{Spec}\daleth$, where $\daleth = k, K$ or $\bar{K}$.

- Let B be either the germ of a smooth affine curve, or the spectrum of a DVR: a *family of pairs over* B is a flat projective morphism $(X,\Delta) \longrightarrow B$ where X is an algebraic space[1], Δ a $\mathbb{Q}$-divisor, such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and all the geometric fibers over B are pairs. The family is said to be *log smooth* if all the strata of $(X, \mathrm{Supp}\Delta)$ are smooth over B. A *family of varieties over* B is a family of pairs over B such that $\Delta = 0$. A family of pairs $(X,\Delta) \longrightarrow B$ is *abundant* if $K_X + \Delta$ is semi-ample over B.

- if X is a variety over field of characeristic $p > 0$, we will denote by $\mathrm{Fr}^e : X^{(p^{-e})} \longrightarrow X$ the geometric Frobenius of X.

[1]This is just required in positive and mixed characteristic. When char$k = 0$ one can assume that X is a variety

- Let $\pi : X \longrightarrow S$ be a projective morphism of normal varieties. We will denote by $\overline{\mathrm{NE}}(X/S) \subset N_1(X/S)_{\mathbb{R}}$ the cone of effective curves of X over S. We will denote by $\mathrm{Amp}(X/S), \mathrm{Nef}(X/S), \mathrm{Big}(X/S), \mathrm{Eff}(X/S)$ and $\overline{\mathrm{Eff}}(X/S) \subset N^1(X)_{\mathbb{R}}$ the relative ample, nef, big, effective, and pseudo-effective cone of X over S, respectively.

2.2 Singularities of pairs and linear systems

Suppose X is a normal, quasi-projective variety: then $\mathrm{Sing}(X)$ has codimension at least two in X, thus the invertible sheaf $\omega_{X\setminus\mathrm{Sing}(X)}$ determines a divisor $K_{X\setminus\mathrm{Sing}(X)}$, unique up to linear equivalence. Its closure in X gives a Weil divisor (class) on X, called the *canonical divisor of* X, and denoted by K_X. When (X, Δ) is a sub-pair, the $\mathbb{Q}$-divisor $K_X + \Delta$ is called the *log canonical divisor* of the pair.

Definition 2.2.1. A normal, quasi-projective variety X is ($\mathbb{Q}$-)*Gorenstein* if K_X is ($\mathbb{Q}$-)Cartier. If every Weil divisor is $\mathbb{Q}$-Cartier, we say that X is $\mathbb{Q}$-*factorial*.

Given a sub-pair (X, Δ), its singularities are measured by comparing its log canonical divisor with that of a resolution. More precisely, let $\mu : Y \longrightarrow X$ be a log resolution of (X, Δ) and write

$$K_Y + \mu_*^{-1}\Delta = \mu^*(K_X + \Delta) + \sum b_E E$$

where the sum runs over all prime μ-exceptional divisors. The *log discrepancy of* E *with respect to* (X, Δ) is

$$a(E; X, \Delta) := 1 + b_E$$

If P is a prime divisor on X, we define $a(P; X, \Delta) = 1 - \operatorname{coeff}_P(\Delta)$. The log discrepancy function can be extended to all geometric valuations of $\mathbb{C}(X)$: if v is such a valuation, then there is a prime Cartier divisor E on a higher birational model of X, such that v is the valuation associated to E. One then defines $a(v; X, \Delta)$ as $a(E; X, \Delta)$: it is easy to check that the definition depends only on v and not on the model of X that realizes it geometrically. The *log discrepancy of a pair* (X, Δ) is

$$\operatorname{logdiscrep}(X, \Delta) := \inf\{a(v; X, \Delta) \text{ s.t. } v \text{ is a geometric valuation of } k(X)\}$$

Definition 2.2.2. A pair (X, Δ) is said to be

- *terminal*, if $a(E; X, \Delta) > 1$ for all divisors E exceptional over X;
- *canonical*, if $a(E; X, \Delta) \geq 1$ for all divisors E exceptional over X;
- *Kawamata log terminal* (klt), if $\operatorname{logdiscrep}(X, \Delta) > 0$; and
- *log canonical* (lc), if $\operatorname{logdiscrep}(X, \Delta) \geq 0$.

There is also a generalization of the lc condition to non-normal schemes. Recall that a scheme is *demi-normal* if it is S_2 and its codimension one points are either regular or nodal, i.e. étale-locally isomorphic to $\{z_1 - z_2 z_3 = 0\} \subset \mathbb{A}^{n+1}$.

Definition 2.2.3. Let X be a demi-normal separated scheme of finite type over $\mathbb{C}$ and let Δ be a $\mathbb{Q}$-divisor whose support does not contain any component of the conductor $D \subset X$. We say (X, Δ) is *semi-log canonical (slc)* if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $K_{X^\nu} + \Delta^\nu + D^\nu$ is lc, where $\nu : X^\nu \longrightarrow X$ is the normalization, Δ^ν is the divisorial component of $\nu^{-1}\Delta$, and D^ν is the conductor on X^ν.

Theorem 2.2.4 (Inversion of Adjunction, [BCHM10], Corollary 1.4.5)**.** *Let (X, Δ) be a pair, and let $\nu : S^\nu \longrightarrow S$ of a component S of Δ of coefficient one. If Θ is defined by adjunction*

$$\nu^*(K_X + \Delta) = K_{S^\nu} + \Theta$$

then the log discrepancy of $K_{S^\nu} + \Theta$ is equal to the minimum of the log discrepancies with respect to $K_X + \Delta$ of any valuation v such that $c_X(v) \cap S \neq \emptyset$ and $\mathrm{codim}_X c_X(v) \geq 2$. In particular, if (X, Δ) is lc and we define Δ_S by the adjunction $K_S + \Delta_S := (K_X + \Delta)|_S$, we have that (S, Δ_S) is slc.

If (X, Δ) is a sub-pair, $f : X \longrightarrow S$ a contraction, and $V \subset S$ a subvariety with generic point η_V, we say (X, Δ) is *log canonical over* η_V if no divisor E over X with $a(E; X, \Delta) = 0$ dominates V.

Definition 2.2.5. Let $f : X \longrightarrow S$ be a projective morphism of normal varieties, let (X, Δ) be a pair, let D be an $\mathbb{Q}$-Cartier divisor on X, and let $V \subset S$ be a subvariety. The *log canonical threshold of* (X, Δ) *with respect to* D *over* η_V is

$$\mathrm{lct}_V(X, \Delta; D) = \sup\{u \text{ s.t. } (X, \Delta + uD) \text{ is lc over } \eta_V\}$$

2.2.1 Multiplier ideal sheaves

In this section we recall the definition and basic properties of multiplier ideal sheaves. Multiplier ideals can be associated to $\mathbb{Q}$-divisors, linear systems, and ideal sheaves. They provide with a measure of the singularities of the object they are attached to: the deeper the multiplier ideal, the more singular the object.

Let X be a smooth complex variety, and let $\mathfrak{a} \subset \mathcal{O}_X$ be an ideal sheaf. A *log resolution of* $\mathfrak{a}$ is a projective birational morphism $\mu : Y \longrightarrow X$ from a smooth variety, such that $\mathrm{Exc}(\mu)$ is a divisor, $\mu^{-1}\mathfrak{a} = \mathcal{O}_Y(-F)$ for an effective divisor F, $F + \mathrm{Exc}(\mu)$ has SNC support, and is an isomorphism away from the cosupport of $\mathfrak{a}$.

Definition 2.2.6. Let $c > 0$ be a real number, let $\mathfrak{a}$ be a sheaf of ideals on a smooth variety X, and let $\mu : Y \longrightarrow X$ be a log resolution of $\mathfrak{a}$. The *multiplier ideal sheaf associated to* $c \cdot \mathfrak{a}$ is defined as

$$\mathcal{I}(X, c \cdot \mathfrak{a}) := \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor cF \rfloor)$$

When D is an effective $\mathbb{Q}$-divisor, we define

$$\mathcal{I}(X, cD) := \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor \mu^*(cD) \rfloor)$$

When $V \subset H^0(X, L)$ is a linear system, we define

$$\mathcal{I}(X, c \cdot V) := \mathcal{I}(X, c \cdot \mathfrak{a})$$

where $\mathfrak{a} = \mathfrak{b}(V)$ is the base ideal of the linear system V.

We will often write $\mathcal{I}(c \cdot \mathfrak{a})$ (resp. $\mathcal{I}(cD)$ and $\mathcal{I}(c \cdot V)$) rather than $\mathcal{I}(X, c \cdot \mathfrak{a})$ (resp. $\mathcal{I}(X, cD)$ and $\mathcal{I}(X, c \cdot V)$), when no confusion is likely. One can check easily that the above definitions do not depend on the choice of the resolution μ.

Definition 2.2.7. A *graded sequence of ideals* is a collection $\mathfrak{a}_\bullet = \{\mathfrak{a}_p\}_{p \in \mathbb{N}}$ of ideal sheaves of $\mathcal{O}_X$, such that $\mathfrak{a}_p \cdot \mathfrak{a}_q \subset \mathfrak{a}_{p+q}$ for all $p, q \in \mathbb{N}$.

Example 2.2.8. Suppose that L is a divisor on a smooth projective variety X, such that $h^0(mL) \neq 0$ for some $m \geq 0$. Then $\{\mathfrak{a}_p := \mathfrak{b}(|pL|)\}_{p \in \mathbb{N}}$ gives a graded sequence of ideals.

If $\mathfrak{a}_\bullet$ is a graded sequence of ideals, for any fixed positive real number c we can consider the multiplier ideals sheaves $\mathcal{I}\left(\frac{c}{p} \cdot \mathfrak{a}_p\right)$, for $p \in \mathbb{N}$. Then, when $p \leq q$ we have inclusions

$$\mathcal{I}\left(\frac{c}{p} \cdot \mathfrak{a}_p\right) \subset \mathcal{I}\left(\frac{c}{q} \cdot \mathfrak{a}_q\right)$$

Hence, the sequence of multiplier ideals eventually stabilizes.

Definition 2.2.9. The *asymptotic multiplier ideal sheaf associated to* $c \cdot \mathfrak{a}_\bullet$ is defined as

$$\mathcal{I}(c \cdot \mathfrak{a}_\bullet) := \lim_{p \to \infty} \mathcal{I}\left(\frac{c}{p} \cdot \mathfrak{a}_p\right)$$

When $\mathfrak{a}_\bullet = \{\mathfrak{b}(|kL|)\}_{k\in\mathbb{N}}$ for some divisor L, we denote by $\mathcal{I}(||cL||) = \mathcal{I}(X, ||cL||)$ the asymptotic multiplier ideal sheaf associated to $c \cdot \mathfrak{a}_\bullet$.

Multiplier ideal sheaves satisfy a number of useful properties. We recall some of them, which will be used in the following Chapter.

Proposition 2.2.10 ([Laz04b], Proposition 9.2.26)**.** *Let X be a smooth variety, let $V \subset H^0(X, L)$ be a linear system, and fix a positive real number c. Let $k > c$ be a natural number, let $A_1, ..., A_k \in V$ be general divisors, and set $D = \frac{1}{k}(A_1 + ... + A_k)$. Then,*

$$\mathcal{I}(c \cdot V) = \mathcal{I}(cD)$$

.

Theorem 2.2.11 (Relative Nadel Vanishing, [Laz04b], Theorem 9.4.1)**.** *Let D be an $\mathbb{Q}$-divisor on a smooth variety X, and let $\mu : Y \longrightarrow X$ be a log resolution of (X, D). Then*

$$R^i\mu_*\mathcal{O}_Y(K_{Y/X} - \lfloor \mu^* D \rfloor) = 0 \qquad \text{for all } i > 0$$

Theorem 2.2.12 (Restriction Theorem, [Laz04b], Theorem 9.5.1)**.** *Let X be a smooth variety, D an effective $\mathbb{Q}$-divisor, and $Z \subset X$ a smooth hypersurface not contained in* $\mathrm{Supp}D$. *Then, the following inclusion holds*

$$\mathcal{I}(Z, D|_Z) \subset \mathcal{I}(X, D)|_Z$$

Theorem 2.2.13 ([Laz04b], Theorem 11.1.19). *Let $\mathfrak{a}_\bullet$ be a graded sequence of ideals on a smooth variety X and let c be a positive real number. Then*

(a) for all $m \in \mathbb{N}$ we have $\mathcal{I}(cm \cdot \mathfrak{a}_\bullet) = \mathcal{I}\left(\frac{c}{p} \cdot \mathfrak{a}_{pm}\right)$, for all sufficiently divisible $p \in \mathbb{N}$;

(b) if $d \geq c$ then $\mathcal{I}(c \cdot \mathfrak{a}_\bullet) \supset \mathcal{I}(d \cdot \mathfrak{a}_\bullet)$;

(c) for all $l, m \in \mathbb{N}$ we have $\mathfrak{a}_m \cdot \mathcal{I}(\mathfrak{a}_\bullet^l) \subset \mathcal{I}(\mathfrak{a}_\bullet^{m+l})$.

Proposition 2.2.14 ([Laz04b], Corollary 11.2.4). *Let X be a smooth projective variety, let L be a divisor on X such that $h^0(kL) \neq 0$ for some $k \geq 1$, and let c be a positive real number. Then*

$$\mathcal{I}(c \cdot ||(m+k)L||) \subset \mathcal{I}(c \cdot ||mL||) \cdot \mathcal{I}(c \cdot ||kL||)$$

for all positive integers l, m.

Proposition 2.2.15 ([Laz04b], Corollary 9.6.13). *Let X be a smooth variety, $\mathfrak{a} \subset \mathcal{O}_X$ a sheaf of ideals, and c a positive real number. Then, the multiplier ideal $\mathcal{I}(c \cdot \mathfrak{a}) \subset \mathcal{O}_X$ is integrally closed.*

2.2.2 Nakayama-Zariski Decomposition

We will need some definitions and results from [Nak04].

Lemma 2.2.16. *Let $\pi : X \longrightarrow S$ be a projective morphism, where X is smooth and S is normal, and let D be a π-big $\mathbb{R}$-divisor on X, and let P be any prime divisor. The* *function*

$$\sigma_P(D/S) := \inf\{\mathrm{coeff}_P D' : 0 \le D' \sim_{\mathbb{Q},S} D\}$$

is continuous on $\mathrm{Big}(X/S)$*. Suppose now* D *is only pseudo-effective, let* A *be a* π*-ample divisor and set* $\sigma_P(D/S) := \lim_{\epsilon\to 0}\sigma_P(D+\epsilon A/S)$*. Then* $\sigma_P(D)$ *exists and it is independent of* A*. Furthermore, there are finitely many prime divisors* P *such that* $\sigma_P(D) > 0$*.*

Definition 2.2.17. Let $\pi : X \longrightarrow S$ be a projective morphism, where X is smooth and S is normal, and let D be a pseudo-effective $\mathbb{R}$-divisor on X. Let $N_\sigma(D/S) := \sum_P \sigma_P(D/S)C$ and $P_\sigma(D/S) := D - N_\sigma(D/S)$. The equality $D = P_\sigma(D/S) + N_\sigma(D/S)$ is called the *Nakayama-Zariski decomposition* or the σ*-decomposition of* D *over* S. The divisors $P_\sigma(D/S)$ and $N_\sigma(D/S)$ are called the *positive* and *negative part* of the σ-decomposition of D over S.

Note that $\mathrm{Supp}N_\sigma(D/S)$ is exactly the codimension one part of the relative diminished base locus $\mathbb{B}_-(D/S)$. The σ-decomposition is usually not easy to compute, however we can say when certain divisors coincide with their negative part.

Definition 2.2.18. Let $f : X \longrightarrow Z$ be a projective contraction of normal varieties, and let D be an effective $\mathbb{R}$-divisor on X. We say that D is of *insufficient fiber type for* f if $\mathrm{codim}_Z f(\mathrm{Supp}D) = 1$ and there exists a prime divisor $P \subset f^{-1}(f(\mathrm{Supp}D))$ such that $P \wedge D = 0$ and $\mathrm{codim}_Z f(P) = 1$. An effective $\mathbb{R}$-divisor on X is said to be *f-degenerate* if it is either f-exceptional or of insufficient fiber type for f.

Lemma 2.2.19 ([Nak04], Ch. III, Lemma 5.7)**.** *With notation as above, let $D \geq 0$ be an f-degenerate $\mathbb{R}$-divisor. Then, $D = N_\sigma(D/Z)$.*

2.3 The Minimal Model Program

In this section we give a description of the MMP algorithm in more detail. Let $X \longrightarrow S$ be a projective morphism of normal varieties, and let (X, Δ) be a pair with lc singularities.

Definition 2.3.1. A birational map $\phi : X \dashrightarrow Y/S$ is a *contraction* if the inverse ϕ^{-1} restricts to a birational map on every prime divisor of Y.

Definition 2.3.2. Let $\phi : X \dashrightarrow Y/S$ be a birational contraction, and let D be an $\mathbb{R}$-Cartier divisor on X such that $\phi_* D$ is also $\mathbb{R}$-Cartier. The map ϕ is said to be D-non-positive (resp. D-negative) if there is a common resolution with maps $p : W \longrightarrow X$ and $q : W \longrightarrow Y$ such that

$$p^* D = q^* D + E$$

where $E \geq 0$ is q-exceptional (resp. and it contains the strict transform of the ϕ-exceptional divisor).

Definition 2.3.3. Let $(X, \Delta) \longrightarrow S$ a lc pair over S. A $(K_X + \Delta)$-*minimal model over* S is a pair $(X', \Delta') \longrightarrow S$, with a birational contraction $\phi : X \dashrightarrow X'/S$ such that

- $K_{X'} + \Delta'$ is nef over S;

- $\phi_*\Delta = \Delta'$; and

- ϕ is $(K_X + \Delta)$-negative.

When $K_X + \Delta$ is not nef over S, we have the following description of the cone of curves.

Theorem 2.3.4 (Cone Theorem, [Fuj11], Theorem 3.3, Remark 4.4)**.** *Let (X, Δ) be a klt pair and let $f : X \longrightarrow S$ be a contraction of normal varieties. Then, there exists countably many rational curves $\Gamma_i \subset X$, contained in fibres of f, such that*

$$\overline{\mathrm{NE}}(X/S) = \overline{\mathrm{NE}}(X/S)_{K_X+\Delta\geq 0} \oplus \bigoplus_i \mathbb{R}_+[\Gamma_i] \qquad -2\dim X \leq (K_X + \Delta)\cdot\Gamma_i < 0 \quad \forall i$$

Furthermore, for any divisor H ample over S, and any $\epsilon > 0$, only finitely many of the Γ_i are $(K_X + \Delta + \epsilon H)$-negative.

Let Γ be a rational curve spanning a $(K_X+\Delta)$-negative ray: dually, it corresponds to a codimension one face $F = [\Gamma]^\perp \cap \overline{\mathrm{NE}}(X/S)$ of the nef cone of X over S. Let now $[D]$ be a class in the interior of F, and suppose for simplicity that D is $\mathbb{Q}$-Cartier: then $\mathbb{R}_+[\Gamma] = [D]^\perp \cap \overline{\mathrm{NE}}(X/S)$. In paricular, for $0 < \epsilon \ll 1$ the divisor $D - \epsilon(K_X + \Delta)$ is ample over S, by Kleiman Criterion.

Theorem 2.3.5 (Base-point-free Theorem, [Fuj11], Theorem 4.1)**.** *Let $(X, \Delta) \longrightarrow S$ be a klt pair over S. Let D be a $\mathbb{Q}$-Cartier divisor, nef over S, and such that for some $a > 0$, $aD - (K_X + \Delta)$ is ample over S. Then there exists a positive integer k such that kD is base* *point free over S, i.e. there exists a projective morphism of S-varieties $f : X \longrightarrow Y/S$, and a divisor A ample over S, such that $kD \sim_S f^*A$.*

The Base-point-free Theorem implies that every $(K_X + \Delta)$-negative curve Γ as above corresponds to a contraction mapping Γ to a point. More precisely, we have the following result.

Theorem 2.3.6 (Contraction Theorem, [Fuj11], Theorem 3.3)**.** *Let $F \subset \overline{\mathrm{NE}}(X/S)$ be a $(K_X+\Delta)$-negative face of dimension d. Then, there exists a unique contraction morphism $\mathrm{cont}_F : X \longrightarrow Y/S$ to a variety Y, projective over S, such that for every effective curve C on X, $\mathrm{cont}_{F,*}C = 0$ if and only if $[C] \in F$. Moreover, we have the following exact sequence*

$$0 \longrightarrow N^1(Y/S) \longrightarrow N^1(X/S) \longrightarrow \mathbb{Z}^d \longrightarrow 0$$

where the first map is pullback via cont_F^* and the second map is given by the intersection with classes in the linear span of F inside $\overline{\mathrm{NE}}(X/S)$.

When $d = 1$, i.e. $F = \mathbb{R}_+[\Gamma_i]$ as in the Cone Theorem, the corresponding contraction $\mathrm{cont} : X \longrightarrow Y$ has Picard rank 1. In this case we have a complete classification of the possible contraction types:

- *divisorial contraction*: cont_F is birational, $\mathrm{Exc}(\mathrm{cont}_F)$ is an irreducible divisor;
- *small contraction*: cont_F is birational, $\mathrm{codim}_X \mathrm{Exc}(\mathrm{cont}_F) \geq 2$; or
- *fiber-type contraction* or *Mori fiber space*: $\dim Y < \dim X$.

Note that, if $\mathrm{cont}_{\mathbb{R}_+[\Gamma]} : X \longrightarrow Y$ is a small contraction, then K_Y is never $\mathbb{Q}$-Cartier. If it were then, since $\mathrm{cont}_{\mathbb{R}_+[\Gamma]}$ is an isomorphism in codimension one, we would have $K_X = \mathrm{cont}^*_{\mathbb{R}_+[\Gamma]} K_Y$ contradicting $K_X \cdot \Gamma < 0$. As explained in the next subsection, this is the main source of trouble when running the MMP for varieties of dimension ≥ 3. Note also that, if $X \longrightarrow Y$ is a Mori fiber space and F is a general fiber, then $(F, \Delta|_F)$ is a log Fano pair (i.e. $-(K_F + \Delta|_F)$ is ample) of Picard rank one.

Conjecture 2.3.1 (Minimal Model Conjecture)**.** *Let $(X, \Delta) \longrightarrow S$ be a lc pair over S. Then, there exists a $(K_X + \Delta)$-negative birational contraction $\phi : (X, \Delta) \dashrightarrow (X', \Delta' := \phi_* \Delta)$, such that either:*

- *(X', Δ') is a $(K_X + \Delta)$-minimal model over S; or*
- *there is a Mori fiber space structure $(X', \Delta') \longrightarrow Y$.*

Conjecture 2.3.2 (Abundance Conjecture)**.** *Let $(X, \Delta) \longrightarrow S$ be a lc pair over S and suppose that $K_X + \Delta$ is relatively nef. Then $K_X + \Delta$ is semi-ample.*

Remark 2.3.1. When K_X is a nef divisor, the *numerical Kodaira dimension of* X is denoted by $\nu(X)$ and defined as follows: let H be an ample divisor on X, then

$$\nu(X) := \max\{l \geq 0 \text{ s.t. } K_X^l \cdot H^{\dim X - l} \neq 0\}$$

By the main result of [Kaw13], Conjecture 2.3.2 is equivalent to the equality $\kappa(K_X + \Delta) = \nu(K_X + \Delta)$, whenever (X, Δ) is a minimal model.

2.3.1 MMP run

We now describe the MMP algorithm: let (X, Δ) be a $\mathbb{Q}$-factorial klt pair and let $X \longrightarrow S$ a projective contraction of normal varieties. The upshot is to construct a series of birational transformations over S

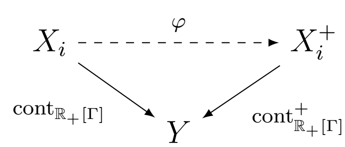
$$(X, \Delta) =: (X_0, \Delta_0) \overset{\phi_1}{\dashrightarrow} (X_1, \Delta_1 := \phi_{1,*}\Delta_0) \overset{\phi_2}{\dashrightarrow} ... \overset{\phi_l}{\dashrightarrow} (X_l, \Delta_l := \phi_{l,*}\Delta_{l-1})/S$$

such that (X_i, Δ_i) is a $\mathbb{Q}$-factorial klt pair, projective over S, and (X_l, Δ_l) is either a $(K_X + \Delta)$-minimal model, or a Mori fiber space.

Step 1_i If $K_{X_i} + \Delta_i$ is nef over S, the algorithm stops. Otherwise, go to Step 2_i.

Step 2_i Let $\mathbb{R}_+[\Gamma]$ be a $(K_{X_i} + \Delta_i)$-negative extremal ray, and let $\mathrm{cont}_{\mathbb{R}_+[\Gamma]} : X_i \longrightarrow Y$ be the corresponding contraction.

- $\mathrm{cont}_{\mathbb{R}_+[\Gamma]}$ is divisorial: set $(X_{i+1}, \Delta_{i+1}) = (Y, \mathrm{cont}_{\mathbb{R}_+[\Gamma],*}\Delta_i)$ and go to step 1_{i+1}.
- $\mathrm{cont}_{\mathbb{R}_+[\Gamma]}$ is small: since Y is never $\mathbb{Q}$-factorial, the strategy is to construct a commutative diagram



such that φ is a birational map which is an isomorphism in codimension one, X_i^+ is $\mathbb{Q}$-factorial and $K_{X_i^+} + \varphi_*\Delta_i$ is ample over Y. The above diagram is usually referred to as the *flip of* $\mathrm{cont}_{\mathbb{R}_+[\Gamma]}$. In this case, we set $(X_{i+1}, \Delta_{i+1}) := (X_i^+, \varphi_*\Delta_i)$ and go to Step 1_{i+1}. The *flipping locus* is the closed subset of X_i where φ is not an isomorphism. Similarly, the closed subset of X_i^+ where φ^{-1} is not an isomorphism is called the *flipped locus*.

– $\mathrm{cont}_{\mathbb{R}_+[\Gamma]}$ is of fiber-type: then $X_i \longrightarrow Y$ is a Mori fiber space over S, and the algorithm stops.

The Negativity Lemma [KM98, Lemma 3.39] implies that if $\phi_i : X_i \dashrightarrow X_{i+1}/S$ is a birational contraction of an $(K_X + \Delta)$-MMP over S, then it is a $(K_{X_i} + \Delta_i)$-negative contraction in the sense of Definition **??**. In particular, it does not make singularities worse.

Lemma 2.3.7 ([KM98], 3.42, 3.43, 3.44)**.** *With notation as above, X_i is $\mathbb{Q}$-factorial, and $a(E; X_i, \Delta_i) \leq a(E; X_{i+1}, \Delta_{i+1})$ for any divisor E over X_i, and the strict inequality holds if and only if the center of E is contained in* $\mathrm{Exc}(\phi_i)$*, if ϕ_i is a divisorial contraction, or in the flipping or flipped locus, if ϕ_i is a flip. In particular, if (X_i, Δ_i) is lc/klt,, then so is (X_{i+1}, Δ_{i+1}). Furthermore, if (X_i, Δ_i) is terminal, resp. canonical, and $E \wedge \Delta_i = 0$, then (X_{i+1}, Δ_{i+1}) is terminal, resp. canonical.*

Note that, if $\phi : X \longrightarrow Y$ is a divisorial contraction, the Picard rank of X drops by one. Hence, there can only be finitely many divisorial contractions in a MMP-run. Suppose now ϕ is small: then, it follows from the definition that if the flip exists it is equal to $X^+ = \underline{\mathbf{Proj}}_Y \bigoplus_{m \geq 0} \phi_* \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)$. The existence of flips is a consequence of a celebrated theorem of Birkar, Cascini, Hacon and McKernan [BCHM10], where the authors prove Conjecture 2.3.1 for klt pairs of general type.

Theorem 2.3.8 ([BCHM10], Theorem 1.2)**.** *Let* $(X, \Delta) \longrightarrow S$ *be a klt pair, projective over* S*, and assume that either* Δ *is big over* S *and* $K_X + \Delta$ *is pseudo-effective over* S*, or* $K_X + \Delta$ *is big over* S*. Then,*

(1) there is a $(K_X + \Delta)$*-minimal model over* S*;*

(2) if $K_X + \Delta$ *is big over* S*, there is a* $(K_X + \Delta)$*-log canonical model;*

(3) if $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier, the* $\mathcal{O}_S$*-algebra*

$$\bigoplus_{m \geq 0} \phi_* \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)$$

is finitely generated.

Note that the Abundance Conjecture for minimal models of log general type is a consequence of the Base-point-free Theorem. Hence the only condition left to check is finiteness of flips. This is currently an open problem

Conjecture 2.3.3 (Termination of flips)**.** *Let* $(X, \Delta) \longrightarrow S$ *be a log canonical pair over* S*. Then,any sequence of* $(K_X + \Delta)$*-flips/*S *terminates.*

Remark 2.3.2. In [BCHM10] the authors show that flips terminate for a special type of MMP, called *MMP with scaling.* Let $\pi : (X, \Delta) \longrightarrow S$ be a klt pair over S, let A be a π-ample $\mathbb{Q}$-divisor, and let $\lambda := \min\{t \geq 0 \text{ s.t. } K_X + \Delta + tA \text{ is nef}\}$. If $\lambda = 0$ then

$K_X + \Delta$ is nef, and we stop. Else, the Base-point-free Theorem yields a contraction $\phi_{m(K_X+\Delta+\lambda A)} : X \longrightarrow X_1$. If ϕ is of fiber-type, we stop. If ϕ is divisorial, let $\lambda_1 := \min\{t \geq 0 \text{ s.t. } K_{X_1} + \phi_*\Delta + t\phi_* A \text{ is nef}\}$ and repeat the process. If ϕ is small, let $\varphi : (X, \Delta) \dashrightarrow (X^+, \Delta^+)$ be the flip, let $\lambda^+ := \min\{t \geq 0 \text{ s.t. } K_{X^+} + \varphi_*\Delta^+ + t\varphi_* A \text{ is nef}\}$ and repeat the process. This algorithm is referred to as a $(K_X + \Delta)$-*MMP over* S *with scaling of* A.

The MMP and Abundance Conjecture have been proved for log canonical pairs of dimension ≤ 3 by the work of several mathematicians [Mor88, Miy88, KMM94, Sho93, Kol92].

We conclude with the following useful consequence of Theorem 2.3.8.

Lemma 2.3.9. *Let* $X \longrightarrow S$ *be a projective contraction of normal varieties and* (X, Δ) *be a* $\mathbb{Q}$*-factorial klt pair such that* $K_X + \Delta$ *is pseudo-effective over* S*. Then, there exists a birational* $(K_X + \Delta)$*-negative contraction* $\phi : (X, \Delta) \dashrightarrow (X', \Delta' = \phi_*\Delta)$ *such that* (X', Δ') *is* $\mathbb{Q}$*-factorial klt, and* $N_\sigma(K_{X'} + \Delta'/S) = 0$*.*

Proof. Let H be a general ample divisor and $0 < \lambda \ll 1$ so that $\mathrm{Supp} N_\sigma(K_X + \Delta/S) = \mathrm{Supp}\mathbb{F}\mathrm{ix}(K_X + \Delta + \lambda H/S)$. By [Kol97, Theorem 4.8] we can find an effective klt boundary $\Delta_\lambda \sim_{\mathbb{R},S} \Delta + \lambda H$. Let now $\phi : X \dashrightarrow X'$ be the $(K_X + \Delta_\lambda)$-MMP/S with scaling of an ample divisor: since every step of this MMP is also a step of the $(K_X + \Delta)$-MMP/S we have that $(X', \Delta' := \phi_*\Delta)$ is still $\mathbb{Q}$-factorial klt by Lemma 2.3.7. Since ϕ contracts $\mathbb{F}\mathrm{ix}(K_X + \Delta_\lambda/S)$ we have $N_\sigma(K_{X'} + \Delta'/S) = 0$. □

In particular, running a $(K_X + \Delta)$-MMP$/S$ with scaling of an ample divisor, we will obtain a klt $\mathbb{Q}$-factorial model (X', Δ') such that $N_\sigma(K_{X'} + \Delta'/S) = 0$.

2.4 Log Calabi-Yau fibrations

Definition 2.4.1. Let (X, Δ) be a sub-pair, and let $f : X \longrightarrow Z$ be a projective contraction to a normal variety. The morphism f is a *log Calabi-Yau fibration* if

(a) for all birational models $\mu : X' \longrightarrow X$

$$\mathrm{rank} f_* \mathcal{O}_X \left(\lceil K_{X'} - \mu^*(K_X + \Delta) + \sum_{E : a(E; X, \Delta) = 0} E \rceil \right) = 1;$$

(b) (X, Δ) is sub-lc over the generic point of Z; and

(c) $K_X + \Delta \sim_{\mathbb{R}, Z} 0$.

Log Calabi-Yau fibrations appear naturally as an output of an MMP run: if (X, Δ) is an lc pair such that $K_X + \Delta$ is nef, then Conjecture 2.3.2 (Abundance) predicts that the Iitaka fibration $f := \phi_{mK_X} : X \longrightarrow Z$ is a morphism. In particular, f is a log Calabi-Yau fibration. Another example are Mori fiber spaces: if $f : (X, \Delta) \longrightarrow Z$ is a $(K_X + \Delta)$-negative fiber-type contraction, then $-(K_X + \Delta)$ is f-ample. Let $D \in |-(K_X + \Delta)|_{\mathbb{Q}}$ be a general member, so that $K_X + \Delta + D$ is still lc, by [Kol97, Theorem 4.8]. Then $f : (X, \Delta + D) \longrightarrow Z$ is a log Calabi-Yau fibration.

We will be interested in a particularly simple type of log Calabi-Yau fibration.

Definition 2.4.2. Let $f : X \longrightarrow Z$ be a projective contraction of normal varieties. We say that f is a *κ-trivial* fibration if X is canonical, and the general fiber of f has Kodaira dimension zero.

Example 2.4.3. Suppose that X is a smooth projective variety, such that $\kappa(X) \geq 0$, let $\phi : X \dashrightarrow Z$ be its Iitaka fibration, and let $\mu : Y \longrightarrow X$ be a resolution of indeterminacy of ϕ. The induced morphism $Y \longrightarrow Z$ is then a κ-trivial fibration.

Every κ-trivial fibration has a natural log Calabi-Yau fibration structure, thanks to the following result.

Proposition 2.4.4 ([FM00], Proposition 2.2, Theorem 4.5.(iii))**.** *Let $f : X \longrightarrow Z$ be a κ-trivial fibration, let F be its general fiber, and let $b := \min\{u \in \mathbb{N} : |uK_F| \neq \emptyset\}$. Then, there exists a $\mathbb{Q}$-divisor D_Z on Z inducing an isomorphism of graded $\mathcal{O}_Z$-algebras*

$$\bigoplus_{m\geq 0} \mathcal{O}_Z(\lfloor mdD_Z \rfloor) \cong \bigoplus_{m \geq 0} f_*\mathcal{O}_X(mdK_{X/Z})^{**}$$

Furthermore, D_Z is unique up to linear equivalence, and the above isomorphism induces an equality of $\mathbb{Q}$-divisors

$$K_X + R_X = f^*(K_Z + D_Z) \tag{2.1}$$

where R_X^+ is f-exceptional and $f_*\mathcal{O}_X(mR_X^-) = \mathcal{O}_Z$ for all sufficiently divisible $m \geq 0$.

Lastly, there exists a positive integer a such that

$$R(K_X)^{(a)} \cong R(K_Z + D)^{(a)}$$

Proof. The sheaf $f_*\mathcal{O}_X(bK_{X/Z})^{**}$ is rank one and reflexive on the normal variety Z, hence $f_*\mathcal{O}_X(bK_{X/Z})^{**} = \mathcal{O}_Z(D_Z)$ for some Weil divisor D_Z, unique up to linear equivalence. Since taking the double dual is a codimension two operation, we have an inclusion $f^*\mathcal{O}_Z(D_Z) \hookrightarrow \mathcal{O}_X(bK_{X/Z})$ over some open set $U \subset Z$ such that $\operatorname{codim}_Z(Z \setminus U) \geq 2$. In particular, $K_{X/Z} = f^*D_Z + R_X^-$, where $f_*\mathcal{O}_X(iR_X^-) = \mathcal{O}_Z$ for all $i > 0$ sufficiently divisible. Extending the inclusion to X we conclude. The isomorphism $R(K_X)^{(a)} \cong R(K_Z + D_Z)^{(a)}$ is a consequence of [FM00, Theorem 4.5.(iii)]. □

If $X \longrightarrow Z$ is a κ-trivial fibration, we will always denote by $(X, R_X) \longrightarrow Z$ the log Calabi-Yau fibration structure induced by Proposition 2.4.4.

Remark 2.4.1.

- The divisor R_X determines the $\mathbb{Q}$-linear equivalence class of D_Z. More precisely, suppose to have equalities $K_{X/Z} + R_X^j = f^*(D_j)$ such that $R_X^{j,+}$ is f-exceptional and $f_*\mathcal{O}_X(mR_X^{j,-}) = \mathcal{O}_Z$ for $j = 1, 2$ and all sufficiently divisible $m \geq 0$. Then $D_1 \sim_{\mathbb{Q}} D_2$. To see this, by pushing forward and restricting to the complement of a codimension two set we obtain

$$\mathcal{O}_Z(mD_1) = \mathcal{O}_Z(mD_1) \otimes f_*\mathcal{O}_X(mR_X^{1,-}) \cong$$

$$\cong f_*\mathcal{O}_X(bK_{X/Z}) \cong \mathcal{O}_Z(mD_2) \otimes f_*\mathcal{O}_X(mR_X^{2;-}) = \mathcal{O}_Z(mD_2)$$

hence we conclude $D_1 \sim_{\mathbb{Q}} D_2$.

- The construction of D_Z and R_X is local over Z that is, if $U \subset Z$ is an open set, the divisors $D_Z|_U$ and $R_X|_{f^{-1}U}$ just depend on $f|_{f^{-1}U}$.

Let $f : (X, \Delta) \longrightarrow Z$ be a log Calabi-Yau fibration, let D_Z be a $\mathbb{Q}$-divisor on Z such that $K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Z + D_Z)$ and define a $\mathbb{Q}$-divisor $B_Z := \sum_P s_P P$ on Z, where $s_P = 1 - \mathrm{lct}_P(X, R_X; f^*P)$ for every prime divisor $P \subset Z$. Set $M_Z := D_Z - B_Z$: the divisors B_Z and M_Z are called the *boundary* and *moduli part* of f. The induced $\mathbb{Q}$-linear equivalence

$$K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Z + M_Z + B_Z)$$

is called the *canonical bundle formula for f.* Note that B_Z is a well defined $\mathbb{Q}$-divisor, while M_Z is only determined up to $\mathbb{Q}$-linear equivalence. Conjecurally, (X, Δ) and (Z, B_Z) are in the same class of singularities (see [Amb99] for a detailed discussion). For our purposes, the following will be sufficient.

Lemma 2.4.5 ([Bir18], Lemma 6.8)**.** *Let (X, Δ) be a projective sub-pair, let $f : X \longrightarrow Z$ be a contraction such that (X, Δ) is lc over the generic point of Z and $K_X + \Delta \sim_{\mathbb{Q},Z} 0$. Let B_Z and M_Z be the discriminant and moduli parts of f: then, for any open subset $U \subset Z$, $(f^{-1}U, \Delta|_{f^{-1}U})$ is sub-lc if and only if $(U, B_Z|_U)$ is sub-lc.*

We will also need the following result of Fujino-Mori.

Theorem 2.4.6 ([FM00], Theorem 5.2)**.** *Let $f : (X, R_X) \longrightarrow Z$ be a κ-trivial fibration. Then, there exists an higher model $f' : X' \longrightarrow Z'$, a positive rational ϵ, and a klt boundary Δ', such that*

$$(1 + \epsilon)(K_{X'} + R_{X'}) = f'^*(K_{Z'} + \Delta')$$

where $R^+_{X'}$ is f'-exceptional, and $f'_*\mathcal{O}_{X'}(iR^-_{X'}) = \mathcal{O}_{Z'}$ for all $i > 0$ sufficiently divisible.

Furthermore, there are positive integers a, b and an isomorphism

$$R(K_X)^{(a)} \cong R(K_Z + \Delta)^{(b)}$$

The next definition is not standard, but it helps simplifying the exposition.

Definition 2.4.7. Let $f : X \longrightarrow Z$ be a κ-trivial fibration, and let ϵ be a positive rational number. An *ϵ-canonical bundle formula for f* (ϵ-CBF) is an expression

$$(1 + \epsilon)(K_X + R_X) \sim_{\mathbb{Q}} f^*(K_Z + \Delta)$$

such that $K_Z + \Delta$ is klt, R^+_X is f-exceptional, and $f_*\mathcal{O}_X(iR^-_X) = \mathcal{O}_Z$ for all sufficiently divisible $i > 0$.

In particular, by Remark 2.4.1, $(1+\epsilon)^{-1}(K_Z+\Delta) \sim_{\mathbb{Q}} K_Z+D$, where D is the divisor given by Proposition 2.4.4. The following Lemma gives a geometric characterization to the negative part of R^-_X.

Lemma 2.4.8. *Let $f : X \longrightarrow Z$ be a contraction, let R be an effective $\mathbb{Q}$-divisor on X such that $f_*\mathcal{O}_X(mR) = \mathcal{O}_Z$ for all $m \geq 0$ sufficiently divisible, and suppose $R^{\mathrm{ver}} \neq 0$. Then, R is f-degenerate. In particular, if $(1+\epsilon)(K_X + R_X) = f^*(K_Z + \Delta)$ is an ϵ-CBF, then $\mathrm{Supp}R_X^- \subset \mathrm{Supp}\mathbb{F}\mathrm{ix}(K_X/Z)$.*

Proof. Suppose by contradiction that R^{ver} is not f-degenerate. Then, after removing a closed subset of codimension ≥ 2 from Z, we may assume there is a prime divisor $P \subset Z$ such that $\mathrm{Supp}f^*P \subset \mathrm{Supp}R^{\mathrm{ver}}$. When $m \geq 0$ is sufficiently divisible we then have $\mathcal{O}_X(f^*(mP)) \hookrightarrow \mathcal{O}_X(mR)$. By the projection formula and the left-exactness of f_* we then obtain an inculsion $\mathcal{O}_Z(mP) \hookrightarrow \mathcal{O}_Z$, a contradiction. In particular, when $(1+\epsilon)(K_X+R_X) = f^*(K_Z+\Delta)$ is an ϵ-CBF for a κ-trivial fibration, $R_X^{-,\mathrm{ver}}$ is f-degenerate if it is nonzero. As the inclusion $|K_X/Z|_{\mathbb{Q}} \hookrightarrow |K_X + R_X^+/Z|_{\mathbb{Q}}$ is an isomorphism, we have $\mathbb{F}\mathrm{ix}(K_X + R_X^+/T) = \mathbb{F}\mathrm{ix}(K_X/T) + R^+$. On the other hand,

$$\mathbb{F}\mathrm{ix}(K_X + R_X^+/T) = f^*\mathbb{F}\mathrm{ix}(K_Z + D_Z/T) + \mathbb{F}\mathrm{ix}(K_X + R_X^+/Z) = f^*\mathbb{F}\mathrm{ix}(K_Z + D_Z/T) + R_X^-$$

and, as $R_X^+ \wedge R_X^- = 0$ we have $\mathrm{Supp}R_X^- \subset \mathrm{Supp}\mathbb{F}\mathrm{ix}(K_X/T)$. As $R_X^{-,\mathrm{ver}}$ is f-degenerate and $R_X^{-,\mathrm{hor}} = \mathbb{F}\mathrm{ix}(R_X^{-,\mathrm{hor}}/Z)$, then $\mathrm{Supp}R_X^- \subset \mathrm{Supp}\mathbb{F}\mathrm{ix}(K_X/Z)$. □

2.4.1 Families of log Calabi-Yau fibrations

In this subsection $T = (T, 0)$ denotes the germ of a smooth affine curve.

Definition 2.4.9. Let (X, Δ) be a sub-pair, projective over T, and let Z be a projective

T-variety. A morphism $f : (X, \Delta) \longrightarrow Z/T$ is a *family of log Calabi-Yau fibrations over T* if $f_t : (X_t, \Delta_t) \longrightarrow Z_t$ is a log Calabi-Yau fibration for all $t \in T$. Similarly, a *family of κ-trivial fibrations* is a morphism of projective T-varieties $f : X \longrightarrow Z/T$ such that $f_t : X_t \longrightarrow Z_t$ is a κ-trivial fibration for all $t \in T$.

Example 2.4.10. Let $(X, \Delta) \longrightarrow T$ be an abundant family of klt pairs. Then, the relative Iitaka fibration $f : (X, \Delta) \longrightarrow Z := \mathrm{Proj}R(K_X + \Delta)/T$ is a family of log Calabi-Yau fibrations.

Note that if $f : (X, \Delta) \longrightarrow Z/T$ is a family of κ-trivial fibrations then it is a κ-trivial fibration itself. The converse however is not so clear, at least to an *algebraic* geometer, as explained in the following elementary but important remark.

Remark 2.4.2. Let $X \longrightarrow T$ be a smooth projective family such that $\kappa(X/T) \geq 0$, and consider the relative Iitaka fibration $X \dashrightarrow Z/T$. Assume, for simplicity, that there is a simultaneous resolution of indeterminacies $Y \longrightarrow X$, so that Y_t is a smooth birational model of X_t for all $t \in T$, and let $f : Y \longrightarrow Z$ be the induced morphism. While f is a κ-trivial fibration, we can *not* conclude that it is a family of κ-trivial fibrations. Indeed, condition (a) in Definition 2.4.1 implies the deformation invariance of the Kodaira dimension, and the only known proof of this is as a corollary of Theorem 1.0.4. In particular, if $(1 + \epsilon)(K_Y + R_Y) \sim_{\mathbb{Q}} f^*(K_Z + \Delta)$ is an ϵ-CBF for f, then we can *not* conclude that the restriction $(1 + \epsilon)(K_{Y_0} + R_Y|_{Y_0}) = f_0^*(K_{Z_0} + \Delta_0)$ is an ϵ-CBF for f_0. However, upon shrinking T, we may always assume that $Y \times_T (T \backslash 0) \longrightarrow Z \times_T (T \backslash 0)/(T \backslash 0)$ is a family of κ-trivial fibrations over $T \setminus 0$ and $(1 + \epsilon)(K_{Y_t} + R_Y|_{Y_t}) \sim_{\mathbb{Q}} f_t^*(K_{Z_t} + \Delta_t)$ is an ϵ-CBF for all $t \neq 0$.

Remark 2.4.3. If $f : X \longrightarrow Z/T$ is a family of κ-trivial fibrations, Proposition 2.4.4 yields equalities

$$K_X + R_X = f^*(K_Z + D_Z) \quad \text{and} \quad K_{X_t} + R_{X_t} = f_t^*(K_{Z_t} + D_{Z_t})$$

for all $t \in T$. It is natural to ask how does the restriction to X_t of the former compares to the latter. Suppose $t \neq 0$: the proof of Proposition 2.4.4 then implies

$$D_{Z_t} = D_Z|_{Z_t} \quad \text{and} \quad R^{\pm}_{X_t} = R^{\pm}_X|_{X_t}$$

Let P be a prime divisor on Z. Since t is general, its restriction $P_t := P|_{Z_t}$ is a prime divisor on Z_t, and $\mathrm{lct}_P(X, R_X; f^*P) = \mathrm{lct}_{P_t}(X_t, R_{X_t}; f_t^*P_t)$. In particular, we have

$$B_Z|_{Z_t} = B_{Z_t} \quad \text{and} \quad M_Z|_{Z_t} = M_{Z_t}$$

On the other hand, by upper semi-continuity of cohomology, all we can say about the central fiber is

$$D_Z|_{Z_0} \leq D_{Z_0} \quad R^-_X|_{X_0} \geq R^-_{X_0} \quad R^+_X|_{X_0} \leq R^+_{X_0} \tag{2.2}$$

The following Definition will come in handy in the next Chapters

Definition 2.4.11. Let $f : (X, \Delta) \longrightarrow Z/B$ be a family of log Calabi-Yau fibrations, where B is either the spectrum of a nice DVR (of any characteristic) or the germ of a smooth affine curve. Up to a finite base change $B' \longrightarrow B$, we may assume that there is a section Σ of $Z \longrightarrow B$, passing through general closed geometric points of Z_K and Z_k. The morphism $F := X \times_Z \Sigma \longrightarrow B$ is called a *family of general fibers of* f.

Chapter 3

Characteristic zero

In this Chapter we work over the field of complex numbers; $T = (T, 0)$ will denote the germ of a smooth affine curve with geometric generic point $\bar{\eta}$. We study, algebraically, the deformation invariance of plurigenera for smooth families $X \longrightarrow T$ such that $\kappa(X/T) \geq 0$.

3.1 Preliminary results

In this section we prove two simple but useful lemmas, which will be used throughout the rest of the Chapter. First, recall the following result.

Lemma 3.1.1 ([Gro61], Lemme 8.11.1)**.** *Let U, V be two schemes, $h : U \longrightarrow V$ a proper, surjective morphism such that $h_*\mathcal{O}_U = \mathcal{O}_V$. Let W be any scheme: then, the natural map* $\mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(U, W)$ *is a bijection onto the subset of morphisms $f : U \longrightarrow W$*

such that, for any $x \in U$, the restriction $f|_{h^{-1}(x)}$ is constant.

Lemma 3.1.2. *Let $X \longrightarrow T$ be a flat family of normal varieties, let $Z \longrightarrow T$ be a proper morphism, and let $\phi : X \dashrightarrow Z/T$ be a dominant rational map over T. Let I be the indeterminacy locus of ϕ: then, $\mathrm{codim}_{X_t} I \cap X_t \geq 2$ for all $t \in T$.*

Proof. Since Z is proper over T, we have $\mathrm{codim}_X I \geq 2$, hence the thesis holds for $t \neq 0$. Suppose by contradiction that $\mathrm{codim}_{X_0} I \cap X_0 = 1$: without loss of generality, we can then assume that X is smooth over T and I consists of a unique smooth and irreducible component contained in X_0. Note that the restriction $\phi_0 := \phi|_{X_0}$ admits an extension at the generic point of I, and denote this by $\bar{\phi}_0 : X_0 \longrightarrow Z_0$. Hence, we obtain a continuous extension $\bar{\phi} : X \longrightarrow Z/T$ given by

$$\bar{\phi}(x) = \begin{cases} \phi(x) & \text{if } x \notin X_0 \\ \bar{\phi}_0(x) & \text{otherwise} \end{cases} \tag{3.1}$$

We now have to show that $\bar{\phi}$ is actually a morphism. Let $\mu : U \longrightarrow X$ be a resolution of indeterminacies and let $f : U \longrightarrow Z/T$ be the induced morphism. Since μ is an isomorphism over $X \setminus I$, we have that $f = \bar{\phi} \circ \mu$, since both sides are continuous and agree on a dense open. But then, Lemma 3.1.1 implies that f descends to a morphism on X, which equals $\bar{\phi}$ since they agree on a dense open subset. □

Lemma 3.1.3. *Let $\phi : X \dashrightarrow Z/T$ be a rational map between normal varieties, projective over T. Let $\mu : W \longrightarrow X$ be a resolution of indeterminacy for ϕ, let $g : W \longrightarrow Z$ be the* *induced morphism, and let D be a pseudo-effective $\mathbb{R}$-divisor on W. Then, $N_\sigma(D/X) \leq N_\sigma(D/Z)$.*

Proof. Let A be an ample divisor on X' and let P be a prime divisor on X': by definition $\mathrm{coeff}_P N_\sigma(D/X) = \lim_{\epsilon \longrightarrow 0} \mathrm{mult}_P |D + \epsilon A/X|_{\mathbb{R}}$, and an analogous formula holds for $N_\sigma(D/Z)$. It is then enough to show $|L/Z|_{\mathbb{R}} \subset |L/X|_{\mathbb{R}}$, for any effective $\mathbb{R}$-divisor L on X'. Take $G \in |L/Z|_{\mathbb{R}}$: by definition $G = L + g^* G_Z + (\varphi)$, where G_Z is some $\mathbb{R}$-divisor on Z and φ is a rational function on X'. As X is normal, ϕ is defined in codimension one, thus we also have the equality $G = L + \mu^* G_X + (\varphi)$, where $G_X := \phi^* G_Z$. In particular $G \in |L/X|_{\mathbb{R}}$. □

3.2 Adapted models for the relative Iitaka fibration

In this section we construct nice models of the relative Iitaka fibration of a smooth family $X \longrightarrow T$. As the relative Iitaka fibration $\phi : X \dashrightarrow Z/T$ is just a rational map in general, it will be necessary to pass to a resolution of indeterminacies $\mu : X' \longrightarrow X$ in order to have a well defined morphism $g : X' \longrightarrow Z/T$. However, $X' \longrightarrow T$ will typically not be a family of varieties, as the central fiber X'_0 might become reducible.

Definition 3.2.1. Let $X \longrightarrow T$ be a smooth family. A contraction $\bar{f} : \bar{X} \longrightarrow \bar{Z}/T$ is a *T-adapted model of the relative Iitaka fibration of X/T* if

- it is birational to the relative Iitaka fibration of X over T;
- $\bar{X} \longrightarrow T$ is a family of varieties with canonical singularities, and $\bar{X}$ is $\mathbb{Q}$-factorial;

- $\bar{Z} \longrightarrow T$ is a family of varieties, and $\bar{Z}$ is $\mathbb{Q}$-factorial; and

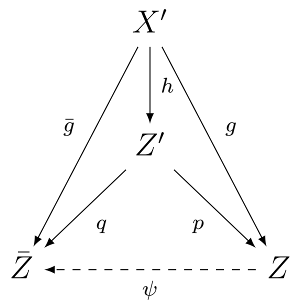
- we have an ϵ-CBF $(1+\epsilon)(K_{\bar{X}} + R_{\bar{X}}) = \bar{f}^*(K_{\bar{Z}} + \bar{\Delta})$, where $K_{\bar{Z}} + \bar{\Delta}$ is semi-ample.

Proposition 3.2.2. *Let $X \longrightarrow T$ be a smooth family such that $\kappa(X/T) \geq 0$. Then, modulo a finite base-change $T' \longrightarrow T$, there exists a T-adapted model of the relative Iitaka fibration of X/T.*

Proof. Let $\phi : X \dashrightarrow Z/T$ be the relative Iitaka fibration of X/T, let $\mu : X' \longrightarrow X$ be a resolution of indeterminacy, and let g be the induced morphism $X' \longrightarrow Z/T$. Upon replacing Z with its $\mathbb{Q}$-factorialization [BCHM10, Corollary 1.4.3], we may assume that Z is $\mathbb{Q}$-factorial. Since taking the $\mathbb{Q}$-factorialization is an isomorphism in codimension one, we have that $Z \longrightarrow T$ is a family of varieties if and only if Z_0 is normal. We have a canonical bundle formula $K_{X'} + R_{X'} = g^*(K_Z + M_Z + B_Z)$; moreover, by Theorem 2.4.6 we can find $\widehat{g} : \widehat{X} \longrightarrow \widehat{Z}$ a sufficienlty high birational model of g, where we have an ϵ-CBF

$$(1+\epsilon)(K_{\widehat{X}} + R_{\widehat{X}}) = \widehat{g}^*(K_{\widehat{Z}} + \widehat{\Delta})$$

Let now Δ be the pushforward of $\widehat{\Delta}$, so that (Z, Δ) is a $\mathbb{Q}$-factorial klt pair of general type. By Theorem 2.3.8 we may run $\psi : Z \dashrightarrow \bar{Z}$ a $(K_Z + \Delta)$-MMP$/T$ with scaling of an ample divisor, which terminates with a good minimal model $(\bar{Z}, \bar{\Delta})$. Upon replacing X' with an higher model, we may assume that there is a commutative diagram



where Z' is a common resolution of Z and $\bar{Z}$. As ψ is $(K_Z + \Delta)$-negative, we have the equality $p^*(K_Z + \Delta) = q^*(K_{\bar{Z}} + \bar{\Delta}) + F$, where F is an effective q-exceptional divisor, in particular $q_*\mathcal{O}_{Z'}(mF) = \mathcal{O}_{\bar{Z}}$ for all sufficiently divisible $m > 0$, hence we have the following ϵ-CBF

$$(1+\epsilon)(K_{X'} + \widetilde{R_{X'}}) = g^*(K_{\bar{Z}} + \bar{\Delta}) \tag{3.2}$$

where $\widetilde{R_{X'}} := R_{X'} - h^*F$.

Let now E be the reduced μ-exceptional divisor, and let E^{hor} and E^{ver} denote the T-horizontal and T-vertical components. By the the semi-stable reduction theorem [KKMSD73, Chapter II], after shrinking T and replacing it with a finite cover $T' \longrightarrow T$, we may assume that $X'_0 = \widetilde{X_0} + E^{\mathrm{ver}}$ is a reduced SNC divisor. Cosider the ramification formula $\mu^*K_X + \sum a(E_i; X)E_i = K_{X'}$: since X is smooth, it is terminal and K_X is Cartier, hence we have $a(E_i; X) \in \mathbb{Z}_{\geq 1}$ for all i. More precisely, $a(E_i; X) = 1$ if and only if $\mathrm{codim}_X c_X(E_i) = 2$; by Lemma 3.1.2, all such E_i are horizontal over T. Thus, we have the following ramification formula

$$\mu^*(K_X + X_0) + \sum_{E_i \text{hor.}/T} a(E_i; X)E_i + \sum_{E_i \text{ver.}/T} (a(E_i; X) - 1)E_i = K_{X'} + \widetilde{X_0}$$

By Lemmas 3.1.3 and 2.2.19 we see that E is contained in the support of $N_\sigma(K_{X'}/S)$. Let now $\varphi : X' \dashrightarrow \bar{X}/Z$ be a $(K_{X'} + \widetilde{X_0})$-MMP$/Z$, with scaling of an ample divisor, and let $\bar{f} : \bar{X} \longrightarrow \bar{Z}$ be the induced morphism: by Lemma 2.3.9 $N_\sigma(K_{X'} + \widetilde{X_0}/Z)$ is contracted, and since φ does not extract divisors, we have that $\bar{X}_t$ is irreducible for all t. Furthermore, by Lemma 2.4.8 $\mathrm{Supp}\widetilde{R^-_{X'}} \subset \mathrm{Supp}\mathbb{F}\mathrm{ix}(K_{X'}/\bar{Z})$ and the difference of their supports is some g-exceptional divisors. In particular, since φ contracts only g-exceptional divisors or components of $\widetilde{R^-_{X'}}$, we have that the ϵ-CBF (3.2) pushes forward to an ϵ-CBF

$$(1 + \epsilon)(K_{\bar{X}} + \widetilde{R_{\bar{X}}}) = \bar{f}^*(K_{\bar{Z}} + \bar{\Delta}) \tag{3.3}$$

In particular, $K_{\bar{Z}} + \bar{\Delta}$ is semi-ample. For $t \neq 0$, the restriction φ_t is a $K_{X'_t}$-MMP$/Z_t$ with scaling of an ample divisor, hence $\bar{X}_t$ is terminal. On the other hand, since $(X', \widetilde{X_0})$ is canonical and φ is $(K_{X'} + \bar{X}_0)$-negative, we have that $(\bar{X}, \bar{X}_0)$ is canonical too, by Lemma 2.3.7. Hence, $\bar{X}_0$ is canonical by Theorem 2.2.4. Finally, Lemma 2.4.5 yields that (Z, Z_t) is lc for all t. By Theorem 2.2.4, Z_t is slc. As Y_t is normal, we have a factorization $f_t : Y_t \longrightarrow Z_t^\nu \xrightarrow{\nu} Z_t$. If ν were not the identity, then it would be 2-to-1 on the codimension one part of $\mathrm{Sing}(Z_t)$, contradicting the connectedness of the fibers of f_t, hence Z_t is lc for all $t \in T$. □

Corollary 3.2.3. *With the same notation as in Proposition 3.2.2, suppose furthermore that the general fiber of ϕ has a good minimal model. Then, there exists a T-adapted model $\bar{f} : \bar{X} \longrightarrow \bar{Z}/T$ with an ϵ-CBF $(1+\epsilon)K_{\bar{X}} = \bar{f}^*(K_{\bar{Z}} + \bar{\Delta})$.*

Proof. Keep the same notation as in the proof of Proposition 3.2.2, and let $\bar{Z}^0 := \bar{Z} \setminus \bar{Z}_0$ and $\bar{X}_0 := \bar{f}^{-1}(\bar{Z}^0)$. By the main result of [HX13], $\bar{X}^0 \longrightarrow \bar{Z}^0$ is a relative good minimal model, in particular φ contracts $\mathrm{Supp} R_{X'}|_{X'^0}$. Since $R_{X'}$ is horizontal over T, we conclude. □

Remark 3.2.1. Note that although, $\bar{f} : \bar{X} \longrightarrow \bar{Z}$ is a κ-trivial fibration, the construction does not show that it is a *family* of κ-trivial fibrations. Now, assume invariance of Kodaira dimension for smooth families; then $\bar{f}$ is a family of κ-trivial fibrations, hence it is natural to ask how does the canonical bundle formula behave under restriction, i.e. whether

$$(1+\epsilon)(K_{\bar{X}_t} + R_{\bar{X}}|_{\bar{X}_t}) = \bar{f}_t^*(K_{\bar{Z}_t} + \bar{\Delta}_t)$$

is an ϵ-CBF for $\bar{f}_t$. From the proof of Proposition 2.4.4 it is easy to see that the above equality holds for $t \neq 0$ hence, by the uniqueness part of Proposition 2.4.4, it is enough to show: (a) $R^+_{\bar{X}}|_{\bar{X}_0}$ is $\bar{f}_0$-exceptional and (b) $\bar{f}_{0,*}\mathcal{O}_{\bar{X}_t}(mR^-_{\bar{X}}|_{\bar{X}_0}) = \mathcal{O}_{Z_0}$ for all $m \geq 0$ sufficiently divisible. Condition (a) is equivalent to showing that all components of $R^+_{\bar{X}}$ are T-horizontal, and this follows from the fact that all the fibers of $X \longrightarrow T$ are irreducible. Condition (b) does not have a completely clear geometric interpretation, however it implies $\bar{f}_{0,*}\mathcal{O}_{\bar{X}_0}(mR^{-,\mathrm{hor}}_{\bar{X}}|_{\bar{X}_0})$ is a rank one sheaf on Z_0, i.e. invariance of Kodaira dimension, and

$\bar{f}_{0,*}\mathcal{O}_{\bar{X}_t}(mR^{-,\mathrm{ver}}_{\bar{X}}|_{\bar{X}_0}) = \mathcal{O}_{Z_0}$, i.e. f-degenerate divisors specialize to f_0-degenerate divisors.

3.3 A torsion-freeness theorem

The main reference for this section is [Laz04b, Chapters 9-11]. The main result of this Section is Theorem 3.3.3, which is the key for reducing invariance of all plurigenera to invariance of all the sufficiently divisible ones. First, we recall the definition of the trace asymptotic multiplier ideal sheaf.

Definition 3.3.1. Let $X \longrightarrow T$ be a smooth family, let L be a divisor on X and denote by L_t its restriction to the fiber X_t. For all integers $k \geq 0$, let V_k be the linear series given by Image$[H^0(X, kL) \longrightarrow H^0(X_t, kL_t)]$, and let $\mathfrak{b}_k = \mathfrak{b}(V_k)$. We denote by $\mathcal{I}(||L||_t)$ the asymptotic multiplier ideal $\mathcal{I}(\mathfrak{b}_\bullet) \subset \mathcal{O}_{X_t}$.

The trace multiplier ideal sheaf is important for us, since these sheaves control the sections of kL_t which can be lifted to the whole of X. From the definition we see that a section $s_0 \in H^0(X_0, nK_{X_0})$ extends to a section on X, then it vanishes along $\mathcal{I}(||nK_X||_0)$. In particular, it vanishes along $\mathcal{I}(||(n-1)K_X||_0)$ by Theorem 2.2.13. When K_X is big, we have a partial converse.

Lemma 3.3.2 ([Laz04b], Lemma 11.5.5)**.** *Let $X \longrightarrow T$ be a smooth family and let L be a big divisor on X. Then, for every $t \in T$, the following inclusion holds:*

$$H^0(X_t, \mathcal{I}(||L||_t)(K_{X_t} + L_t)) \subset \mathrm{Image}\left[H^0(X, K_X + L) \longrightarrow H^0(X_t, K_{X_t} + L_t)\right]$$

The following result is a generalization of Lemma 3.3.2. We think it is well known to experts, but we could not find a reference with a non-analytic proof of it, so we include it for completeness.

Theorem 3.3.3. *Let $\delta : X \longrightarrow T$ be a smooth family, such that $\kappa(X/T) \geq 0$. Then, the sheaves*

$$R^q\delta_*\mathcal{I}(||(m-1)K_X||)(mK_X)$$

are torsion-free, for all $m \geq 1$ and $q \geq 0$.

Theorem 3.3.3 is a consequence of the following result:

Theorem 3.3.4 ([Fuj11], Theorem 6.3)**.** *Let Y be a smooth variety and let B be a boundary $\mathbb{R}$-divisor such that* Supp$B$ *is simple normal crossing. Let $f : Y \longrightarrow X$ be a projective morphism and let L be a Cartier divisor on Y such that $L - (K_Y + B)$ is f-semi-ample.*

(1) For all $q \geq 0$, every associated prime of $R^qf_*\mathcal{O}_Y(L)$ is the generic point of $f(S)$, where $S \subset Y$ is a stratum of (Y, B).

(2) Let $\delta : X \longrightarrow S$ be a projective morphism. Assume that $L - (K_X + B) \sim_{\mathbb{R}} f^*H$ for some δ-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H on X. Then

$$R^p\delta_*R^qf_*\mathcal{O}_Y(L) = 0$$

for every $p \geq 1$ and $q \geq 0$.

Proof of Theorem 3.3.3. Let $k \gg 0$ be a sufficiently divisible integer such that $\mathcal{I}(||(m-1)K_X||) = \mathcal{I}\left(\frac{1}{k} \cdot |k(m-1)K_X|\right)$, let $\mu : X' \longrightarrow X$ be a log resolution of the base ideal of the linear series $|k(m-1)K_X|$, and let $\delta' : X' \longrightarrow T$ be the induced map. In particular, we can write $\mu^*|k(m-1)K_X| = |U| + F$, where U is a free Cartier divisor on X' and F is an effective divisor with simple normal crossing support. By definition of the multiplier ideal then, it is enough to prove that the sheaves

$$R^q\delta_*(\mu_*\mathcal{O}_{X'}(K_{X'/X} - \lfloor F/k \rfloor + \mu^*(mK_X)))$$

are torsion-free for all $q \geq 0$ and $m \geq 1$. On the other hand,

$$R^p\mu_*\mathcal{O}_X(K_{X'/X} - \lfloor F/k \rfloor + \mu^*(mK_X)) \cong R^p\mu_*\mathcal{O}_X(K_{X'/X} - \lfloor F/k \rfloor) \otimes \mathcal{O}_X(mK_X) = 0$$

as $R^p\mu_*\mathcal{O}_X(K_{X'/X} - \lfloor F/k \rfloor) = 0$, by local Nadel vanishing 2.2.11. Hence the Grothendieck-Leray spectral sequence yields isomorphisms

$$R^q\delta_*(\mu_*\mathcal{O}_{X'}(K_{X'/X} - \lfloor F/k \rfloor + \mu^*(mK_X))) \cong R^q\delta'_*\mathcal{O}_{X'}(K_{X'/X} - \lfloor F/k \rfloor + \mu^*(mK_X))$$

for all $q \geq 0$. Let now $L = K_{X'/X} - \lfloor F/k \rfloor + \mu^*(mK_X) \sim_{\mathbb{Q}} K_{X'} + \{F/k\} + U/k$: picking U/k general enough in its $\mathbb{Q}$-linear system, and setting $\Delta' = \{F/k\} + U/k \geq 0$, we have that (X', Δ') is klt and $L - (K_{X'} + \Delta') \sim_{\mathbb{Q}} 0$ is δ-semi-ample. In particular, thanks to point (1) of Theorem 3.3.4 above, we conclude that $R^q\delta_*\mathcal{I}(||(m-1)K_X||)(mK_X) \cong$

$R^q\delta_*(\mu_*\mathcal{O}_{X'}(K_{X'/X} - \lfloor F/k \rfloor + \mu^*(mK_X)))$ is torsion-free. □

Remark 3.3.1. Several results generalizing Theorem 3.3.3 have been proven by Fujino and Matsumura (see [Fuj11], [Fuj17] and [FM16]), using Hodge-theoretic and analytic techniques.

We conclude this section with an easy generalization of Theorem 2.2.12.

Lemma 3.3.5. *Let $X \longrightarrow T$ be a smooth family, and let L be a divisor on X. Then, the inclusion $\mathcal{I}(||mL||_t) \subset \mathcal{I}(||mL||)|_{X_t}$ holds for all $t \in T$ and all m.*

Proof. Let $t \in T$ be a point and let V_k be the linear series Image$[H^0(X, mkL) \longrightarrow H^0(X_t, mkL_t)]$, so that $V_\bullet$ yields a graded linear series. Let $p \gg 0$, so that $\mathcal{I}(||mL||_t) = \mathcal{I}(p^{-1} \cdot V_p)$ and $\mathcal{I}(||mL||) = \mathcal{I}(p^{-1} \cdot |pmL|)$. Let now D be a general element in $|pmL|$. By the definition of asymptotic multiplier ideal sheaf and Proposition 2.2.10 we have

$$\mathcal{I}(||mL||_t) = \mathcal{I}(D/p|_{X_t}) \quad \mathcal{I}(||mL||)|_{X_t} = \mathcal{I}(D/p)|_{X_t}$$

We then conclude by Theorem 2.2.12. □

3.4 Proofs

We are now ready to prove the main result of this Chapter. First, we reduce invariance of all plurigenera to invariance of all sufficiently divisible plurigenera.

Theorem 3.4.1 (also Theorem 1.1.1)**.** *Let $X \longrightarrow T$ be a smooth projective family and suppose that $P_m(X_t)$ is independent of t for all sufficiently divisible $m \geq 0$. Then $P_m(X_t)$ is independent of t for all $m \geq 0$.*

Proof of Theorem 3.4.1. Suppose first that, for all m sufficiently divisible, $P_m(X_t) = 0$ is independent of t: then $\kappa(X_t) = -\infty$ for all t, in particular $P_m(X_t) = 0$ for all t and all m. Thus, we may assume $0 \leq \kappa(X_t) \leq \dim X_t$. Let $n \geq 2$ and let $s_0 \in H^0(X_0, \mathcal{O}_{X_0}(nK_{X_0}))$. By [Har77, Theorem III.12.11] and Theorem 3.3.3, if we show that s_0 vanishes along $\mathcal{I}(||(n-1)K_X||)|_{X_0}$ we then have that s_0 extends to a section of $\mathcal{O}_X(nK_X)$. Let $m_0 \in \mathbb{N}$ be an integer such that $P_m(X_t)$ is independent of t for all $m \in m_0\mathbb{N}$. Consider the following equation

$$n(l+1) = m_0 k$$

There are infinitely many values of $k \in \mathbb{N}$ such that the above equation has a solution in l. For all those values of l, consider the section $s_0^{l+1} \in H^0(X_0, \mathcal{O}_{X_0}(n(l+1)K_{X_0}))$. As $P_{n(l+1)}(X_t)$ is independent of t, we have that s_0^{l+1} vanishes along $\mathcal{I}(||n(l+1)K_X||_0)$. We also have the following chain of inclusions

$$\mathcal{I}(||n(l+1)K_X||_0) \subset \mathcal{I}^{l+1}(||nK_X||_0) \subset \mathcal{I}^{l}(||nK_X||_0) \subset \mathcal{I}^{l}(||(n-1)K_X||_0)$$

where the first inclusion follows by Proposition 2.2.14 , and the last one by Theorem 2.2.13. In particular, s_0^{l+1} vanishes along $\mathcal{I}^l(||(n-1)K_X||_0)$ for infinitely many l. By [Laz04b, Example 11.5.6] we then have that s_0 vanishes along the integral closure of $\mathcal{I}(||(n-$

$1)K_X||_0)$. By Proposition 2.2.15 we have that multiplier ideal sheaves are integrally closed, thus s_0 vanishes along $\mathcal{I}(||(n-1)K_X||_0)$. By Lemma 3.3.5, we have the inclusion $\mathcal{I}(||(n-1)K_X||_0) \subset \mathcal{I}(||(n-1)K_X||)|_{X_0}$, thus we conclude. □

Theorem 3.4.2 (also Theorem 1.1.2)**.** *Let $X \longrightarrow T$ be a smooth family with $\kappa(X/T) \geq 0$. Let $\bar{f} : \bar{X} \longrightarrow \bar{Z}/T$ be a T-adapted model of the relative Iitaka fibration of X/T. Then, the following are equivalent:*

(1) $P_m(X_t)$ is independent of t for all $m \geq 0$;

(2) $P_{am}(\bar{X}_0) = h^0(bm(K_{\bar{Z}_0} + \bar{\Delta}_0))$ for some $a, b \geq 1$ and all sufficiently divisible $m \geq 0$.

Proof of Theorem 3.4.2. (1)⇒(2). As the Kodaira dimension is invariant, $f : \bar{X} \longrightarrow \bar{Z}/T$ is a family of κ-trivial fibrations by Remark 2.4.2. As $\bar{X}_t$ is canonical for all $t \in T$, we have that $P_m(\bar{X}_t) = P_m(X_t)$ for all t and all sufficiently divisible $m \geq 0$; in particular, for all such m, $P_m(\bar{X}_t)$ is independent of t. As $f : \bar{X} \longrightarrow (\bar{Z}, \bar{\Delta})/T$ is a T-adapted model of the relative Iitaka fibration of X/T, we have an ϵ-CBF

$$(1+\epsilon)(K_{\bar{X}} + R_{\bar{X}}) = f^*(K_{\bar{Z}} + \bar{\Delta})$$

where $0 < \epsilon \ll 1$. Let a and b be coprime positive integers, such that $1 + \epsilon = a/b$. The claim is then a consequence of the following chain of inequalities, where m is a sufficiently divisible positive integer:

$$
\begin{aligned}
P_{am}(\bar{X}_0) =& h^0(am(K_{\bar{X}_0} + R^+_{\bar{X}_0})) \geq \\
& \geq h^0(am(K_{\bar{X}_0} + R^+_{\bar{X}}|_{\bar{X}_0})) = \\
& = h^0(\bar{f}_{0,*}\mathcal{O}_{\bar{X}_0}(amR^-_{\bar{X}}|_{\bar{X}_0}) \otimes \mathcal{O}_{\bar{Z}_0}(bm(K_{\bar{Z}_0} + \bar{\Delta}|_{\bar{Z}_0})))) \geq \\
& \geq h^0((\bar{f}_*\mathcal{O}_{\bar{X}}(amR^-_{\bar{X}}) \otimes \mathcal{O}_{\bar{Z}}(bm(K_{\bar{Z}} + \bar{\Delta})))|_{\bar{Z}_0}) \geq \\
& \geq h^0((\bar{f}_*\mathcal{O}_{\bar{X}}(amR^-_{\bar{X}}) \otimes \mathcal{O}_{\bar{Z}}(bm(K_{\bar{Z}} + \bar{\Delta})))|_{\bar{Z}_t}) = \\
& = h^0(bm(K_{\bar{Z}_t} + \bar{\Delta}_t)) = P_{am}(\bar{X}_t)
\end{aligned}
$$

The first inequality and the third equality follow from Remark 2.4.3. The second and third inequalities follow from $\bar{f}_*\mathcal{O}_{\bar{X}}(mR^-_{\bar{X}}) = \mathcal{O}_{\bar{Z}}$ for all sufficiently divisible $m \geq 0$, and upper semi-continuity of cohomology. The equalities $\bar{f}_*\mathcal{O}_{\bar{X}}(mR^-_{\bar{X}}) = \mathcal{O}_{\bar{Z}}$ also implies

$$
h^0((\bar{f}_*\mathcal{O}_{\bar{X}}(amR^-_{\bar{X}}) \otimes \mathcal{O}_{\bar{Z}}(bm(K_{\bar{Z}} + \bar{\Delta}))|_{\bar{Z}_0}) = h^0(bm(K_{\bar{Z}_0} + \bar{\Delta}_0))
$$

By hypothesis $P_{am}(\bar{X}_0) = P_{am}(\bar{X}_t)$, hence the above chain consists only of equalities. In particular

$$
P_{am}(\bar{X}_0) = h^0(bm(K_{\bar{Z}_0} + \bar{\Delta}_0))
$$

for all sufficiently divisible m.

(2)⇒(1). As $K_{\bar{Z}} + \bar{\Delta}$ is klt, big and nef, it is semi-ample, by the Base-point-free Theorem 2.3.5. Hence, there is a morphism $g : \bar{Z} \longrightarrow \bar{Z}^{\text{can}}/T$ and an ample $\mathbb{Q}$-divisor A

on $\bar{Z}^{\text{can}}$ such that $K_{\bar{Z}} + \bar{\Delta} \sim_{\mathbb{Q}} g^*A$. If m is divisible enough, we have equalities for all $t \in T$

$$h^0(bm(K_{\bar{Z}_t} + \bar{\Delta}_t)) = h^0(bmA|_{V_t}) = \chi(bmA|_{V_t})$$

where the first equality follows by the projection formula, and the second by Serre vanishing. As the Euler characteristic of a line bundle is constant in a flat family, the rightmost term of the above chain of equalities is independent of t. By hypothesis

$$P_{am}(\bar{X}_t) = h^0(bm(K_{\bar{Z}_t} + \bar{\Delta}_t))$$

for all t and all sufficiently divisible m. In particular, for all t and all such m, we have $P_{am}(X_t) = P_{am}(\bar{X}_t)$ is independent of t, hance we conclude by Theorem 1.1.1. □

Corollary 3.4.3 (also Corollary 1.1.3)**.** *Let $X \longrightarrow T$ be a smooth family such that $\kappa(X/T) \geq 0$. Suppose that the general fiber of the Iitaka fibration of $X_{\bar{\eta}}$ has a good minimal model. Then $P_m(X_t)$ is independent of $t \in T$ for all $m \geq 0$.*

Proof of Corollary 3.4.3. By Corollary 3.2.3, we can find a T-adapted model of the relative Iitaka fibration of X/T, $\bar{f} : \bar{X} \longrightarrow \bar{Z}/T$ and an ϵ-CBF $(1+\epsilon)K_{\bar{X}} = \bar{f}^*(K_{\bar{Z}} + \bar{\Delta})$. Let a and b be coprime positive integers, such that $1+\epsilon = a/b$. Then, restriction to $\bar{X}_0$ yields $aK_{\bar{X}_0} = \bar{f}^*(b(K_{\bar{Z}_0} + \Delta_0))$. Thus, we conclude by Theorem 3.4.2. □

Corollary 3.4.4 (also Corollary 1.1.4)**.** *Let $X \longrightarrow T$ be a smooth projective family. Assume $\kappa(X/T) = \kappa(X_0)$, and that the general fiber of the Iitaka fibration of X_0 has a good minimal model. Then:*

(1) the general fiber of the Iitaka fibration of $X_{\bar{\eta}}$ has a good minimal model; and

(2) $P_m(X_t)$ is independent of $t \in T$ for all $m \geq 0$.

Proof of Corollary 3.4.4. Note that (2) is implied by (1) and Corollary 3.4.3. Let $F \longrightarrow T$ be a family of general fibers of f: by [HMX18, Lemma 3.2], we can run a K_F-MMP/T, $F \dashrightarrow \bar{F}$, so that $\bar{F}_0$ is a semi-ample model of F_0 and $\bar{F}_t$ is a minimal model of F_t, for all $t \neq 0$. In particular, $\nu(\bar{F}_0) = 0$. As the numerical Kodaira dimension is deformation invariant, $\nu(\bar{F}_t)$ equals zero too. By the main result of [Kaw13], we have that $\bar{F}_t$ is a good minimal model for all $t \in T$. □

Portions of the work in the above chapter is being prepared for submission for publication.

Brivio, Iacopo "On algebraic deformation invariance of plurigenera".

The dissertation author was the primary investigator and author of this material.

Chapter 4

Positive and mixed characteristic

Throughout this Chapter, unless otherwise stated, R will denote a nice DVR, with residue field k of characteristic $p > 0$, fraction field K, and uniformizer ϖ. We study Conjecture 1.0.3 for abundant families $(X, \Delta) \longrightarrow \mathrm{Spec}R$. In characteristic zero this is not very interesting, as shown by the following result.

Proposition 4.0.1. *Let $(X, \Delta) \longrightarrow \mathrm{Spec}R$ be an abundant family of pairs, over a nice DVR of equicharacteristic zero. Then, there exists $m_0 \in \mathbb{N}$ such that*

$$h^0(m(K_{X_{\bar{K}}} + \Delta_{\bar{K}})) = h^0(m(K_{X_k} + \Delta_k))$$

for all $m \in m_0\mathbb{N}$.

Proof. Let $f : X \longrightarrow Z := \mathrm{Proj}R(K_X + \Delta)/\mathrm{Spec}R$ be the relative Iitaka fibration of (X, Δ), so that $K_X \sim_{\mathbb{Q}} f^*A$ for some ample $\mathbb{Q}$-divisor A on Z. Then, for all sufficiently divisible $m \geq 0$ and all geometric points $s \in \mathrm{Spec}R$, we have

$$h^0(m(K_{X_s} + \Delta_s)) = h^0(\mathcal{O}_{Z_s}(mA_s) \otimes f_{t,*}\mathcal{O}_{X_s}) = h^0(mA_s) = \chi(mA_s)$$

where the first equality follows by the projection formula, the second by the connectedness of the fibers of f_s, and the third by Serre vanishing. As the Euler characteristic of a line bundle is invariant in a flat family, we conclude. □

Note that the above argument breaks down when $p > 0$: the condition $f_{k,*}\mathcal{O}_{X_k} = \mathcal{O}_{Z_k}$ is more restrictive than requiring f_k to have connected fibers, due to the existence of inseparable morphisms. The situation is summed up in the following result.

Proposition 4.0.2. *Let $(X, \Delta) \longrightarrow \mathrm{Spec}R$ be an abundant family of pairs over a nice DVR of positive or mixed characteristic. Let $f : X \longrightarrow Z := \mathrm{Proj}R(K_X + \Delta)/\mathrm{Spec}R$ be the relative Iitaka fibration of (X, Δ). Then, there exists a factorization*

$$f_k : X_k \xrightarrow{g_k} S_k \xrightarrow{h_k} Z_k$$

where $g_{k,*}\mathcal{O}_{X_k} = \mathcal{O}_{S_k}$, and h_k is birational or purely inseparable.

Proof. Let $g_k : X_k \longrightarrow S_k := \mathrm{Proj}R(K_{X_k} + \Delta_k)$ be the Iitaka fibration of (X_k, Δ_k); since for all sufficiently divisible $m \geq 0$ we have an inclusion

$$H^0(X, mK_X) \otimes_R k \longrightarrow H^0(X_k, mK_{X_k})$$

this induces a generically finite morphism $h_k : S_k \longrightarrow Z_k$, such that $f_k = h_k \circ g_k$. Since f_k has connected fibers, h_k is generically one-to-one, thus it can only be either birational or purely inseparable. $\square$

As an immediate corollary, we have:

Corollary 4.0.3. *With the same notation of Proposition 4.0.2, h_k is birational if and only if $h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_{\bar{K}}} + \Delta_{\bar{K}}))$ for all $m \geq 0$ sufficiently divisible.*

Proof. If h_k is birational, then $f_{k,*}\mathcal{O}_{X_k} = \mathcal{O}_{Z_k}$. In particular, the same argument as in the proof of Proposition 4.0.1 yields deformation invariance all sufficiently divisible log plurigenera. On the other hand, if all sufficienlty divisible log plurigenera are invariant, then $S_k = Z_k$ by definition, hence h_k is the identity. $\square$

4.1 Failure of invariance of plurigenera

In this Section, we give a negative answer to Conjecture 1.0.3. More precisely, we prove:

Theorem 4.1.1. *There exist families of pairs $(X, \Delta) \longrightarrow \mathrm{Spec}R$ such that*

- *R is a nice DVR of positive or mixed characteristic, with $\mathrm{char}k = 2$;*
- *(X_s, Δ_s) is log smooth and terminal for all geometric points $s \in \mathrm{Spec}R$;*
- *$K_{X_s} + \Delta_s$ is semi-ample; and*

- $h^0(m(K_{X_{\bar{K}}} + \Delta_{\bar{K}}) < h^0(m(K_{X_k} + \Delta_k))$ *for all* $m > 0$ *divisible enough.*

Furthermore, one can take $\Delta = 0$.

Note that, by [EH16, Theorem 1.1], if (X, Δ) is as in Theorem 4.1.1 and $\Delta \neq 0$, then Δ_k must be vertical over $\mathrm{Proj}R(K_{X_k} + \Delta_k)$.

Let $\mathcal{L}$ be a line bundle on $\mathbb{P}^1_R$, let $\mathcal{E} := \mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}_{\mathbb{P}^1_R}$, and let $[X_0 : X_1 : X_2]$ be the global projective coordinate system on $\mathbb{P}\mathcal{E}$. A *(generalized) Weierstrass fibration* is a smooth family of elliptic surfaces $g : X \longrightarrow \mathbb{P}^1_R/\mathrm{Spec}R$, where

$$X := \{X_0X_2^2 + A_1X_0X_1X_2 + A_3X_0^2X_2 - X_1^3 - A_2X_0X_1^2 - A_4X_0^2X_1 - A_6X_0^3 = 0\} \subset \mathbb{P}\mathcal{E}$$

and $A_i \in H^0(\mathbb{P}^1_R, \mathcal{L}^{-i})$.

We will also need the following result on simultaneous resolutions for families of surfaces with DuVal singularities.

Theorem 4.1.2 ([Art74, Bri71]). *Let* $\pi : (X, x) \longrightarrow (S, 0)$ *be a flat morphism of pointed schemes or algebraic spaces, such that* X_0 *is a surface with a DuVal singularity at* x. *Then, there exists a finite surjective morphism,* $S' \longrightarrow S$ *such that* $\pi' : X' := X \times_S S' \longrightarrow S'$ *has a simultaneous resolution*

$$\begin{array}{ccc} \overline{X'} & \xrightarrow{\mu} & X' \\ & {\scriptstyle \overline{\pi'}} \searrow \quad \swarrow {\scriptstyle \pi'} & \\ & S' & \end{array}$$

where μ *is projective and* $\mu_{s'} : \overline{X'}_{s'} \longrightarrow X'_{s'}$ *is the minimal resolution for all* $s' \in S'$.

Suppose furthermore that $\pi : X \longrightarrow (S,0)$ is projective: then there exists $S' \longrightarrow S$ and simultaneous resolution as above if and only if $K^2_{X_s}$ is independent of $s \in S$.

The idea behind the construction of the counterexamples in Theorem 4.1.1 is the same in mixed or positive characteristic, however the techniques used are different, hence we split the proof in two parts.

Proof of Theorem 4.1.1: mixed characteristic R. We begin with the case $\Delta \neq 0$. Let $E \longrightarrow \mathrm{Spec}R$ be an elliptic curve over R, such that there exists a non-trivial, 2-torsion line bundle $\mathcal{M}$ on E (in other words, E_k is an ordinary elliptic curve). Consider $\mathbb{P}^1_R$ with homogeneous coordinates T and S, and let $Y := E \times_R \mathbb{P}^1_R$ with pr_i the projection morphisms, for $i = 1,2$. We will construct X as a 2-to-1 Galois cover $p : X \longrightarrow Y$, branched over $Y_k \cup E \times_R \{0, \infty\}$. Consider the line bundle $\mathcal{L} := \mathrm{pr}_1^*\mathcal{M} \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1_R}(1)$ on Y, let $\mathbb{L} := \underline{\mathbf{Spec}}_Y \bigoplus_{m\geq 0} \mathcal{L}^{-m} \xrightarrow{\pi} Y$ be the total space bundle of $\mathcal{L}$, and let $\Lambda \in H^0(\mathbb{L}, \pi^*\mathcal{L})$ be the tautological section. Finally, let $\sigma := TS \in H^0(Y, \mathcal{L}^2)$: then we define X to be the 2-to-1 cyclic cover of Y branched along σ, i.e.

$$\mathbb{L} \supset X := \{\Lambda^2 - \sigma = 0\} \xrightarrow{p:=\pi|_X} Y$$

Note that, locally on E, we have

$$\left(X \xrightarrow{p} Y\right) = \left(\mathrm{Spec}\frac{\mathcal{O}_E[t, \lambda_t]}{(\lambda_t^2 - t)} \longrightarrow \mathrm{Spec}\mathcal{O}_E[t]\right)$$

if $S \neq 0$, and

$$\left(X \xrightarrow{p} Y\right) = \left(\mathrm{Spec}\frac{\mathcal{O}_E[s,\lambda_s]}{(\lambda_s^2 - s)} \longrightarrow \mathrm{Spec}\mathcal{O}_E[s]\right)$$

if $T \neq 0$, where λ_t and λ_s denote local trivializations of Λ over $S \neq 0$ and $T \neq 0$, respectively. In the first case

$$\Omega_{X/R} = \frac{\mathrm{pr}_1^*\Omega_{E/R} \oplus \Omega_{\mathbb{A}^2_R/R}}{(2\lambda_t \mathrm{d}\lambda_t - \mathrm{d}t)},$$

in the second case

$$\Omega_{X/R} = \frac{\mathrm{pr}_1^*\Omega_{E/R} \oplus \Omega_{\mathbb{A}^2_R/R}}{(2\lambda_s \mathrm{d}\lambda_s - \mathrm{d}s)}.$$

In particular, $X \longrightarrow \mathrm{Spec}R$ is smooth. Let now $f : X \longrightarrow \mathbb{P}^1_R/\mathrm{Spec}R$ be the induced morphism: then $f_{\bar{K}} : X_{\bar{K}} \longrightarrow \mathbb{P}^1_{\bar{K}}$ is an isotrivial elliptic surface, with two multiple fibers over 0 and ∞, both isomorphic to $2E_{\bar{K}}$, and general fiber isomorphic to the étale double cover of $E_{\bar{K}}$ induced by $\mathcal{M}_{\bar{K}}$. On the other hand, the central fiber fits in the following Cartesian square

$$\begin{array}{ccc} X_k & \xrightarrow{p_k} & E_k \times \mathbb{P}^1_k \\ {\scriptstyle q_k}\downarrow & \searrow^{f_k} & \downarrow{\scriptstyle \mathrm{pr}_2} \\ \mathbb{P}^1_k & \xrightarrow{\mathrm{Fr}} & \mathbb{P}^1_k \end{array} \tag{4.1}$$

Note that $q_{k,*}\mathcal{O}_{X_k} = \mathcal{O}_{\mathbb{P}^1_k}$. Let now $A_1, ..., A_6 \subset \mathbb{P}^1_R$ be disjoint sections of the morphism $\mathbb{P}^1_R \longrightarrow \mathrm{Spec}R$, let

$$\Delta := f^*\left(\frac{A_1 + ... + A_6}{3}\right),$$

and consider the family of pairs $(X, \Delta) \longrightarrow \mathrm{Spec}R$. We have canonical bundle formulae

$$K_{X_{\bar{K}}} + \Delta_{\bar{K}} = f^*_{\bar{K}}\left(K_{\mathbb{P}^1_{\bar{K}}} + \frac{[0] + [\infty]}{2} + \frac{A_{1,\bar{K}} + ... + A_{6,\bar{K}}}{3}\right) \sim_{\mathbb{Q}} f^*_{\bar{K}}\mathcal{O}_{\mathbb{P}^1_{\bar{K}}}(1)$$

$$K_{X_k} + \Delta_k = q^*_k\left(K_{\mathbb{P}^1_k} + \frac{2A_{1,k} + ... + 2A_{6,k}}{3}\right) \sim_{\mathbb{Q}} q^*_k\mathcal{O}_{\mathbb{P}^1_k}(2)$$

hence, by the projection formula,

$$h^0(m(K_{X_{\bar{K}}} + \Delta_{\bar{K}})) = h^0(\mathcal{O}_{\mathbb{P}^1_{\bar{K}}}(m)) = m + 1$$

$$h^0(m(K_{X_k} + \Delta_k)) = h^0(\mathcal{O}_{\mathbb{P}^1_k}(2m)) = 2m + 1$$

for all $m \in 3\mathbb{N}$. Note that the geometric fibers of $(X, \Delta) \longrightarrow \mathrm{Spec}R$ are log smooth terminal pairs, although the *family* itself is not log smooth.

Now we consider the case $\Delta = 0$: let $g : Y \longrightarrow \mathbb{P}^1_R/\mathrm{Spec}R$ be a family of elliptic surfaces over $\mathrm{Spec}R$, such that

(1) $\kappa(Y/\mathrm{Spec}R) = 1$;

(2) g is a Weierstrass fibration with at-worst-nodal singular fibers;

(3) g is smooth in a neighborhood 0 and ∞; and

(4) there exists a non-trivial line bundle $\mathcal{M}$ on Y, which is 2-torsion over $\mathbb{P}^1_R$.

Since g is a Weierstrass fibration, it admits a section $S \subset Y$: in particular, if $\mathcal{M}^2 = g^*\mathcal{O}_{\mathbb{P}^1_R}(n)$, by intersecting with S we see $2 \mid n$. Thus, after replacing $\mathcal{M}$ by $\mathcal{M} \otimes g^*\mathcal{O}_{\mathbb{P}^1_R}(-n/2)$, we may assume that $\mathcal{M}$ is 2-torsion. Let $\mathcal{L} := \mathcal{M} \otimes g^*\mathcal{O}_{\mathbb{P}^1_R}(1)$, let $\mathbb{L} := \underline{\mathbf{Spec}}_Y \bigoplus_{m\geq 0} \mathcal{L}^{-m} \xrightarrow{\pi} Y$ be the total space bundle of $\mathcal{L}$, and let $\Lambda \in H^0(\mathbb{L}, \pi^*\mathcal{L})$ be the tautological section. Lastly, let $\sigma := TS \in H^0(Y, \mathcal{L}^2)$. As before, we define

$$\mathbb{L} \supset X' := \{\Lambda^2 - \sigma = 0\} \xrightarrow{p:=\pi|_{X'}} Y.$$

Let $f' : X' \longrightarrow \mathbb{P}^1_R/\mathrm{Spec}R$ be the induced morphism and note that, again, the central fiber X'_k fits in the Cartesian square

$$\begin{array}{ccc} X'_k & \xrightarrow{p_k} & Y_k \\ {\scriptstyle q'_k}\downarrow & \searrow^{f'_k} & \downarrow{\scriptstyle g_k} \\ \mathbb{P}^1_k & \xrightarrow{\mathrm{Fr}} & \mathbb{P}^1_k \end{array} \tag{4.2}$$

and $q'_{k,*}\mathcal{O}_{X'_k} = \mathcal{O}_{\mathbb{P}^1_k}$. Since g is a Weierstrass fibration, when $S \neq 0$ and locally over Y we have

$$\left(X' \xrightarrow{p} Y\right) = \left(\mathrm{Spec}\frac{R[t, x_i, x_j, \lambda_t]}{(c(t, x_i, x_j), \lambda_t^2 - t)} \longrightarrow \frac{\mathrm{Spec}R[t, x_i, x_j]}{(c(t, x_i, x_j))}\right)$$

while, when $T \neq 0$ and locally over Y we have

$$\left(X' \xrightarrow{p} Y\right) = \left(\mathrm{Spec}\frac{R[s,x_i,x_j,\lambda_s]}{(c(s,x_i,x_j),\lambda_s^2 - s)} \longrightarrow \mathrm{Spec}\frac{R[s,x_i,x_j]}{(c(s,x_i,x_j))}\right)$$

where λ_t and λ_s denote trivializations of Λ when $S \neq 0$ and $T \neq 0$ respectively, and c is a polynomial of degree 3 in x_i and x_j. In particular

$$\Omega_{X'/R} = \frac{\Omega_{\mathbb{A}^4_R/R}}{(\partial_t c\mathrm{d}t + \partial_{x_i} c\mathrm{d}x_i + \partial_{x_j} c\mathrm{d}x_j, 2\lambda_t \mathrm{d}\lambda_t - \mathrm{d}t)}$$

when $S \neq 0$ and

$$\Omega_{X'/R} = \frac{\Omega_{\mathbb{A}^4_R/R}}{(\partial_s c\mathrm{d}s + \partial_{x_i} c\mathrm{d}x_i + \partial_{x_j} c\mathrm{d}x_j, 2\lambda_s \mathrm{d}\lambda_s - \mathrm{d}s)}$$

Thus, X'_0 fails to be smooth over the singular points of the singular fibers of $g_k : Y_k \longrightarrow \mathbb{P}^1_k$. Let $y \in Y_k$ be such a point: since the fibers of g_k are at-worst-nodal, formally-locally around y we have

$$\left(X'_k \xrightarrow{p_k} Y_k \xrightarrow{g_k} \mathbb{P}^1_k\right) = \left(\mathrm{Spec}\frac{k[[t,x_i,x_j]][\lambda]}{(x_i x_j - t, \lambda^2 - t)} \longrightarrow \mathrm{Spec}\frac{k[[t,x_i,x_j]]}{(x_i x_j - t)} \longrightarrow \mathrm{Spec}k[[t]]\right)$$

and, since $\mathrm{Spec}\frac{k[[t,x_i,x_j]][\lambda]}{(x_i x_j - t,\lambda^2 - t)} \simeq \mathrm{Spec}\frac{k[[x_i,x_j,\lambda]]}{(\lambda^2 - x_i x_j)}$, we conclude that the only singularities of X'_k over $\{S \neq 0\} \subset \mathbb{P}^1_k$ are A_1-singularities. An analogous computation shows that the same holds over $\{T \neq 0\} \subset \mathbb{P}^1_k$. In particular, X'_k has only DuVal singularities. Since $K^2_{X'_K} = K^2_{X'_k} = 0$, by Theorem 4.1.2 we can find an extension $\tilde{R} \supset R$ and a simultaneous resolution

$$\mu : X \longrightarrow X' \times_R \mathrm{Spec}\tilde{R}/\mathrm{Spec}\tilde{R}$$

In particular, $X \longrightarrow \mathrm{Spec}\tilde{R}$ is a smooth abundant family of surfaces with Kodaira dimension one. Let $f : X \longrightarrow \mathbb{P}^1_{\tilde{R}}/\mathrm{Spec}\tilde{R}$ be the induced morphism, and let $f_k : X_k \xrightarrow{q_k} \mathbb{P}^1_k \xrightarrow{\mathrm{Fr}} \mathbb{P}^1_k$ be the factorization induced by the Cartesian square 4.2, and note that $q_{k,*}\mathcal{O}_{X_k} = \mathcal{O}_{\mathbb{P}^1_k}$. Let now A be an ample $\mathbb{Q}$-divisor on $\mathbb{P}^1_{\tilde{R}}$ such that $K_X \sim_{\mathbb{Q}} f^*A$. Then,

$$K_{X_{\bar{K}}} \sim_{\mathbb{Q}} f^*_{\bar{K}} A_{\bar{K}}$$

$$K_{X_k} \sim_{\mathbb{Q}} f^*_k A_k \sim_{\mathbb{Q}} 2q^*_k A_k$$

hence, by the projection formula

$$h^0(mK_{X_{\bar{K}}}) = h^0(\mathcal{O}_{\mathbb{P}^1_{\bar{K}}}(m \deg_{\mathbb{P}^1_{\bar{K}}} A_{\bar{K}}))$$

$$h^0(mK_{X_k}) = h^0(\mathcal{O}_{\mathbb{P}^1_k}(2m \deg_{\mathbb{P}^1_k} A_k)$$

for all sufficiently divisible $m \geq 0$. Since $\deg_{\mathbb{P}^1_{\bar{K}}} A_{\bar{K}} = \deg_{\mathbb{P}^1_k} A_k$, we conclude. □

Remark 4.1.1. Before moving on to the equicharacteristic case we point out two aspects of the above construction which do not appear in the equicharacteristic case. Roughly speaking, they are the reason we need to define our cover $X \longrightarrow Y$ in a different way, when $\mathrm{char}R = 2$.

- Let $A := \mathcal{O}_{\mathbb{P}^1_R, \eta_{\mathbb{P}^1_k}}$, let $\mathbb{X} := X \times A$ and $\mathbb{Y} := Y \times A$. Then $\mathbb{X} \longrightarrow \mathbb{Y}$ is a $\underline{\mu}_2$-torsor over $\mathbb{Y}$. In particular, since $\mathrm{char} K = 0$, we have that $\mathbb{X}_K$ is a smooth K-scheme. On the other hand, $\underline{\mu}_2$ is a non-reduced group scheme in characteristic two.

- in general, p-to-1 cyclic covers of a variety in characteristic p do not live in the total space of some line bundle, but they are controlled by Artin-Schreier theory.

Proof of Theorem 4.1.1: equicharacteristic R. We begin with the case $\Delta \neq 0$. Let $\pi : E' \longrightarrow E$ be a 2-to-1 étale morphism between elliptic curves over k. Then E' is a $\mathbb{Z}/2\mathbb{Z}$-torsor over E. Recall that, by Artin-Schreier theory, affine-locally on E we have $\mathcal{O}_{E'} \simeq \mathcal{O}_E[\lambda]/(\lambda^2 + \lambda + a)$, where a is a regular function such that $a \neq b^2 + b$ for all $b \in \mathcal{O}_E$. In particular, the data of $\pi : E' \longrightarrow E$ is equivalent to:

(a) an affine open covering $\{U_i\}_{i \in I}$ of E;

(b) glueing commutative diagrams

$$\begin{array}{ccc} \frac{\mathcal{O}_E(U_{ij})[\lambda_i]}{(\lambda_i^2+\lambda_i+a_i)} & \xrightarrow{\psi_{ij}} & \frac{\mathcal{O}_E(U_{ij})[\lambda_j]}{(\lambda_j^2+\lambda_j+a_j)} \\ \uparrow & & \uparrow \\ \mathcal{O}_E(U_{ij}) & \xrightarrow{\phi_{ij}} & \mathcal{O}_E(U_{ij}) \end{array}$$

where $U_{ij} = U_i \cap U_j$, such that the horizontal arrows are isomorphisms, and they satisfy the cocycle condition on $U_{ijk} = U_i \cap U_j \cap U_k$ for all $i, j, k \in I$.

Consider now

$$Y := E \times \mathbb{P}^1_{TS} \times \mathbb{A}^1_u$$

as a family of elliptic surfaces over $\mathbb{A}^1_u$: we will construct X as a 2-to-1 Galois cover $p : X \longrightarrow Y$, ramified over $E \times \mathbb{P}^1 \times 0 \cup E \times \{S = 0\} \times \mathbb{A}^1_u$.

Denote by $\mathbb{A}^1_t, \mathbb{A}^1_s \subset \mathbb{P}^1_{TS}$ the two standard affine coordinate charts and let $\tau : k[t^{\pm 1}] \longrightarrow k[s^{\pm 1}]$ be the isomorphism sending t to s^{-1}. Consider the open covering of Y given by $\{V_\alpha\}_{\alpha \in A}$, where each V_α is of the form $U_i \times \mathbb{A}^1_z \times \mathbb{A}^1_u$, for $i \in I$ and $z = t, s$. Then, X is given locally over each V_α by the morphism

$$\mathcal{O}_X(p^{-1}(V_\alpha)) := \frac{\mathcal{O}_E(U_i)[u,t,\lambda_i]}{(\lambda_i^2 + u\lambda_i + ua_i + t)} \longleftarrow \mathcal{O}_Y(V_\alpha) = \mathcal{O}_E(U_i)[u,t]$$

if $V_\alpha = U_i \times \mathbb{A}^1_t \times \mathbb{A}^1_u$, and

$$\mathcal{O}_X(p^{-1}(V_\alpha)) := \frac{\mathcal{O}_E(U_i)[u,s,\xi_i]}{(\xi_i^2 + us\xi_i + us^2 a_i + s)} \longleftarrow \mathcal{O}_Y(V_\alpha) = \mathcal{O}_E(U_i)[u,s]$$

if $V_\alpha = U_i \times \mathbb{A}^1_s \times \mathbb{A}^1_u$.

The glueing diagrams on the intersections $V_{\alpha\beta}$ are given as follows:

- If $V_{\alpha\beta} = U_{ij} \times \mathbb{A}^1_t \times \mathbb{A}^1_u$

$$\begin{array}{ccc} \frac{\mathcal{O}_E(U_{ij})[u,t,\lambda_i]}{(\lambda_i^2+u\lambda_i+ua_i+t)} & \xrightarrow{\gamma^t_{ij}} & \frac{\mathcal{O}_E(U_{ij})[u,t,\lambda_j]}{(\lambda_j^2+u\lambda_j+ua_j+t)} \\ \uparrow & & \uparrow \\ \mathcal{O}_E(U_{ij})[u,t] & \xrightarrow{\phi_{ij} \otimes \mathrm{id}_{k[u,t]}} & \mathcal{O}_E(U_{ij})[u,t] \end{array}$$

with $\gamma_{ij}^t(\lambda_i) := \lambda_j$ if ψ_{ij} is trivial and $\gamma_{ij}^t(\lambda_i) := \lambda_j + u$ if ψ_{ij} is not trivial.

- If $V_{\alpha\beta} = U_{ij} \times \mathbb{A}_s^1 \times \mathbb{A}_u^1$

$$\begin{array}{ccc} \frac{\mathcal{O}_E(U_{ij})[u,s,\xi_i]}{(\xi_i^2+us\xi_i+us^2a_i+s)} & \xrightarrow{\gamma_{ij}^s} & \frac{\mathcal{O}_E(U_{ij})[u,s,\xi_j]}{(\xi_j^2+us\xi_j+us^2a_j+s)} \\ \uparrow & & \uparrow \\ \mathcal{O}_E(U_{ij})[u,s] & \xrightarrow{\phi_{ij}\otimes \mathrm{id}_{k[u,s]}} & \mathcal{O}_E(U_{ij})[u,s] \end{array}$$

with $\gamma_{ij}^s(\xi_i) := \xi_j$ if ψ_{ij} is trivial and $\gamma_{ij}^s(\xi_i) := \xi_j + us$ if ψ_{ij} is not trivial.

- If $V_{\alpha\beta} = U_{ij} \times (\mathbb{A}_t^1 \cap \mathbb{A}_s^1) \times \mathbb{A}_u^1$

$$\begin{array}{ccc} \frac{\mathcal{O}_E(U_{ij})[u,t^{\pm1},\lambda_i]}{(\lambda_i^2+u\lambda_i+ua_i+t)} & \xrightarrow{\gamma_{ij}^{ts}} & \frac{\mathcal{O}_E(U_{ij})[u,s^{\pm1},\xi_j]}{(\xi_j^2+us\xi_j+us^2a_j+s)} \\ \uparrow & & \uparrow \\ \mathcal{O}_E(U_{ij})[u,t^{\pm1}] & \xrightarrow{\phi_{ij}\otimes \tau} & \mathcal{O}_E(U_{ij})[u,s^{\pm1}] \end{array}$$

with $\gamma_{ij}^{ts}(\lambda_i) := \frac{\xi_j}{s}$ if ψ_{ij} is trivial and $\gamma_{ij}^{ts}(\lambda_i) := \frac{\xi_j}{s} + u$ if ψ_{ij} is not trivial, and

$$\begin{array}{ccc} \frac{\mathcal{O}_E(U_{ij})[u,s^{\pm1},\xi_i]}{(\xi_i^2+us\xi_i+us^2a_i+s)} & \xrightarrow{\gamma_{ij}^{st}} & \frac{\mathcal{O}_E(U_{ij})[u,t^{\pm1},\lambda_j]}{(\lambda_j^2+u\lambda_j+ua_j+t)} \\ \uparrow & & \uparrow \\ \mathcal{O}_E(U_{ij})[u,s^{\pm1}] & \xrightarrow{\phi_{ij}\otimes \tau^{-1}} & \mathcal{O}_E(U_{ij})[u,s^{\pm1}] \end{array}$$

with $\gamma_{ij}^{st}(\xi_i) := \frac{\lambda_j}{t}$ if ψ_{ij} is trivial and $\gamma_{ij}^{st}(\xi_i) := \frac{\lambda_j}{t} + \frac{u}{t}$ if ψ_{ij} is not trivial.

One can check that the cocycle condition on intersections $V_{\alpha\beta\gamma}$ is satisfied: this boils down to the fact that $\gamma^t_{ij}, \gamma^s_{ij}$ and γ^{ts}_{ij} are defined in terms of ψ_{ij}, and that $\psi_{ij}, \phi_{ij}, \tau$ satisfy the cocycle condition themselves.

Over $U_i \times \mathbb{A}^1_t \times \mathbb{A}^1_u$, resp. $U_i \times \mathbb{A}^1_s \times \mathbb{A}^1_u$, we have

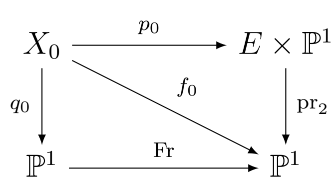
$$\Omega_{X/\mathbb{A}^1_u} = \frac{\Omega_{E\times\mathbb{A}^3_{ut\lambda}/\mathbb{A}^1_u}}{(u\mathrm{d}\lambda + u\mathrm{d}a_i + \mathrm{d}t)}$$

resp.

$$\Omega_{X/\mathbb{A}^1_u} = \frac{\Omega_{E\times\mathbb{A}^3_{us\xi}/\mathbb{A}^1_u}}{(u(\xi\mathrm{d}s + s\mathrm{d}\xi) + us^2\mathrm{d}a + \mathrm{d}s)}$$

In particular, possibly after restricting to an open $U \subset \mathbb{A}^1_u$ containing 0 and replacing X by $X \times_{\mathbb{A}^1_u} U$, the morphism $X \longrightarrow U$ is smooth.

Let $f : X \longrightarrow \mathbb{P}^1 \times U/U$ be the induced morphism: then, the central fiber fits in the following Cartesian square



Note that $q_{0,*}\mathcal{O}_{X_0} = \mathcal{O}_{\mathbb{P}^1}$. For $u \neq 0$ the morphism $f_u : X_u \longrightarrow \mathbb{P}^1$ is an isotrivial elliptic fibration, with general fiber E' and a singular fiber $2E$ over $S = 0$.

Let now $A_1, ..., A_6 \subset \mathbb{P}^1 \times U$ be disjoint sections of the map $\mathbb{P}^1 \times U \longrightarrow U$, let

$$\Delta := f^* \left(\frac{A_1 + ... + A_6}{3} \right)$$

and consider the family of log pairs $(X, \Delta) \longrightarrow U$. We have canonical bundle formulae (see [BM77, Theorem 2])

$$K_{X_u} + \Delta_u = f_u^* \left(K_{\mathbb{P}^1_k} + L + \frac{a[0]}{2} + \frac{A_{1,u} + ... + A_{6,u}}{3} \right) \sim_{\mathbb{Q}} f_u^* \mathcal{O}_{\mathbb{P}^1_k}(a/2 + \deg L)$$

$$K_{X_0} + \Delta_0 = q_0^* \left(K_{\mathbb{P}^1_k} + \frac{2A_{1,0} + ... + 2A_{6,0}}{3} \right) \sim_{\mathbb{Q}} q_0^* \mathcal{O}_{\mathbb{P}^1_k}(2)$$

By [KU85, 8.3] we have $a = 0$ and $\deg L = 1$ hence, by the projection formula,

$$h^0(m(K_{X_u} + \Delta_u)) = h^0(\mathcal{O}_{\mathbb{P}^1_k}(m))$$

$$h^0(m(K_{X_0} + \Delta_0)) = h^0(\mathcal{O}_{\mathbb{P}^1_k}(2m))$$

for all $u \neq 0$ and $m \in 3\mathbb{N}$. Note that the geometric fibers (X_u, Δ_u) are log smooth terminal pairs, although the family $(X, \Delta) \longrightarrow U$ is not log smooth. $\square$

Remark 4.1.2. Alternatively, the morphism $p : X \longrightarrow Y$ can be given as follows: fix $i \in I$

and consider the field extension

$$k(Y) \hookrightarrow L := k(Y)[\lambda_i]/(\lambda_i^2 + u\lambda_i + ua_i + t)$$

Define X to be the normalization of Y in L: then

$$\mathcal{O}_X(p^{-1}(U_j \times \mathbb{A}^1_t \times \mathbb{A}^1_u)) = \mathcal{O}_Y(U_j \times \mathbb{A}^1_t \times \mathbb{A}^1_u)[\lambda_j]/(\lambda_j^2 + u\lambda_j + ua_j + t)$$

for all $j \in I$. On the other hand, over $\mathbb{A}^1_s$, the integral closure of $\mathcal{O}_Y(U_j \times \mathbb{A}^1_s \times \mathbb{A}^1_u)$ in L is given by

$$\mathcal{O}_X(p^{-1}(U_j \times \mathbb{A}^1_s \times \mathbb{A}^1_u)) := \mathcal{O}_Y(U_j \times \mathbb{A}^1_s \times \mathbb{A}^1_u)[\xi_j]/(\xi_j^2 + us\xi + us^2a_j + s)$$

where $\xi_j = s\lambda_j$.

We now consider the case $\Delta = 0$: let $g_1 : Y_1 \longrightarrow \mathbb{P}^1$ be a smooth elliptic surface, such that:

(1) $\kappa(Y_1) = 1$;

(2) g_1 is a Weierstrass fibration with at-worst-nodal singular fibers;

(3) g_1 is smooth over a neighborhood of $\{TS = 0\}$; and

(4) letting $E := g_1^{-1}(T = 0)$, there is a 2-to-1 étale cover $\pi : E' \longrightarrow E$.

By Artin-Schreier theory, the morphism π corresponds to a field extension

$$k(E) \hookrightarrow \frac{k(E)[\lambda]}{(\lambda^2 + \lambda + a)}$$

where $a \in k(E)^*$ satisfies $a \neq b^2 + b$ for all $b \in k(E)$.

Consider the family of elliptic surfaces

$$g := g_1 \times \mathrm{id}_{\mathbb{A}^1_u} : Y := Y_1 \times \mathbb{A}^1_u \longrightarrow \mathbb{P}^1 \times \mathbb{A}^1_u / \mathbb{A}^1_u$$

As in the previous section, we construct a 2-to-1 Galois cover $p : X' \longrightarrow Y$, branched over $Y \times 0 \cup g_1^{-1}\{S = 0\} \times \mathbb{A}^1_u$. Note that, since g_1 is a Weierstrass fibration, we have

$$k(Y_1) = Q\left(\frac{k[t, x_i, x_j]}{(c(t, x_i, x_j))}\right)$$

where c is of degree 3 in x_i, x_j, it is smooth for general t, and it has at-worst-nodal singularities over $\mathbb{A}^1_t$ (an analogous picture holds over $\mathbb{A}^1_s \subset \mathbb{P}^1_{TS}$).

Let $\alpha \in \mathcal{O}_{Y_1, \eta_E} \subset k(Y_1)$ such that $\alpha|_E = a$ and α does not depend on t. Consider then the field extension

$$k(Y) \hookrightarrow L := \frac{k(Y)[\lambda]}{(\lambda^2 + u\lambda + u\alpha + t)}$$

and let X be the normalization of Y in L. Over $\mathbb{A}^1_t \subset \mathbb{P}^1_{TS}$ and locally on Y we have

$$\left(Y \xrightarrow{g_1\times \mathrm{pr}_2} \mathbb{P}^1\times\mathbb{A}^1_u\right) = \left(\mathrm{Spec}\frac{k[u,t,x_i,x_j]}{(c(t,x_i,x_j))} \longrightarrow \mathbb{A}^1_t\times\mathbb{A}^1_u\right)$$

Hence

$$\left(X' \xrightarrow{p} Y\right) = \left(\mathrm{Spec}\frac{k[u,t,x_i,x_j,\lambda]}{(c(t,x_i,x_j),\lambda^2+u\lambda+u\alpha+t)} \longrightarrow \mathrm{Spec}\frac{k[u,t,x_i,x_j]}{(c(t,x_i,x_j))}\right)$$

In particular

$$\Omega_{X'/\mathbb{A}^1_u} = \frac{\Omega_{\mathbb{A}^5_{utx_ix_j\lambda}/\mathbb{A}^1_u}}{(\partial_t c\mathrm{d}t+\partial_{x_i}c\mathrm{d}x_i+\partial_{x_j}c\mathrm{d}x_j, u\mathrm{d}\lambda+u\mathrm{d}\alpha+\mathrm{d}t)}$$

hence X'_0 fails to be smooth over the singular points of the singular fibers of $g_0 : Y_0 \longrightarrow \mathbb{P}^1$. Let $y \in Y_0$ be such a point: by condition (3), formally-locally around y we have

$$\left(X'_0 \xrightarrow{p_0} Y_0 \xrightarrow{g_0} \mathbb{P}^1\right) = \left(\mathrm{Spec}\frac{k[[t,x_i,x_j]][\lambda]}{(x_ix_j-t,\lambda^2-t)} \longrightarrow \mathrm{Spec}\frac{k[[t,x_i,x_j]]}{(x_ix_j-t)} \longrightarrow \mathrm{Spec}k[[t]]\right)$$

and, since $\mathrm{Spec}\frac{k[[t,x_i,x_j]][\lambda]}{(x_ix_j-t,\lambda^2-t)} \simeq \mathrm{Spec}\frac{k[[x_i,x_j,\lambda]]}{(\lambda^2-x_ix_j)}$, we conclude that the only singularities of X'_0 over $\mathbb{A}^1_t \subset \mathbb{P}^1_{TS}$ are A_1-singularities.

We now check what happens over the chart $\mathbb{A}^1_s \subset \mathbb{P}^1_{TS}$: since $\mathcal{O}_{X'}$ is defined to be the integral closure of $\mathcal{O}_Y$ in L, locally over Y we have

$$\left(X' \xrightarrow{p} Y\right) = \left(\mathrm{Spec}\frac{k[u,s,x_i,x_j,\xi]}{(c(s,x_i,x_j),\xi^2+us\xi+us^2\alpha+s)} \longrightarrow \mathrm{Spec}\frac{k[u,s,x_i,x_j]}{(c(s,x_i,x_j))}\right)$$

where $\xi = s\lambda$, and c is of degree 3 in x_i, x_j and is smooth for general s and $s = 0$.

In particular

$$\Omega_{X'/\mathbb{A}^1_u} = \frac{\Omega_{\mathbb{A}^5_{usx_ix_j\xi}/\mathbb{A}^1_u}}{(\partial_s c\mathrm{d}s + \partial_{x_i} c\mathrm{d}x_i + \partial_{x_j} c\mathrm{d}x_j, us\mathrm{d}\xi + u\xi\mathrm{d}s + us^2\mathrm{d}\alpha + \mathrm{d}s)}$$

hence, again, X'_0 fails to be smooth over the singular points of the singular fibers of $g_0 : Y_0 \longrightarrow \mathbb{P}^1$. Let $y \in Y_0$ be such a point: by condition (3), formally-locally around y we have

$$\left(X'_0 \xrightarrow{p_0} Y_0 \xrightarrow{g_0} \mathbb{P}^1\right) = \left(\mathrm{Spec}\frac{k[[s,x_i,x_j]][\xi]}{(x_ix_j - s, \xi^2 - s)} \longrightarrow \mathrm{Spec}\frac{k[[s,x_i,x_j]]}{(x_ix_j - s)} \longrightarrow \mathrm{Spec}k[[s]]\right)$$

and, since $\mathrm{Spec}\frac{k[[s,x_i,x_j]][\xi]}{(x_ix_j-s,\xi^2-s)} \simeq \mathrm{Spec}\frac{k[[x_i,x_j,\xi]]}{(\xi^2-x_ix_j)}$, we conclude that the only singularities of X'_0 over $\mathbb{A}^1_s \subset \mathbb{P}^1_{TS}$ are A_1-singularities as well.

Note that, as in the $\Delta \neq 0$ case, letting $f' : X' \longrightarrow \mathbb{P}^1 \times \mathbb{A}^1_u/\mathbb{A}^1_u$ be the induced morphism, the central fiber X'_0 fits in the following Cartesian square:

$$\begin{array}{ccc} X_0' & \xrightarrow{p_0} & Y_0 \\ {\scriptstyle q_0'}\downarrow & \overset{f_0'}{\searrow} & \downarrow{\scriptstyle g_0} \\ \mathbb{P}^1 & \xrightarrow{\text{Fr}} & \mathbb{P}^1 \end{array}$$

where $q'_{0,*}\mathcal{O}_{X'_0} = \mathcal{O}_{\mathbb{P}^1}$.

Since X'_0 has only DuVal singularities and $K^2_{X'_u} = 0$ for all u, we can find a finite morphism $(C, 0) \longrightarrow (\mathbb{A}^1_u, 0)$ and a simultaneous resolution

$$\mu : X \longrightarrow X \times_{\mathbb{A}^1_u} C/C$$

so that $X \longrightarrow C$ is a smooth family of minimal surfaces with Kodaira dimension one. Let $f : X \longrightarrow \mathbb{P}^1 \times C$ be the induced morphism and note that the above Cartesian square induces a factorization

$$f_0 : X_0 \xrightarrow{q_0} \mathbb{P}^1 \xrightarrow{\text{Fr}} \mathbb{P}^1$$

where $q_{0,*}\mathcal{O}_{X_0} = \mathcal{O}_{\mathbb{P}^1}$. Let now A be an ample $\mathbb{Q}$-divisor on $\mathbb{P}^1 \times C$ such that $K_X \sim_{\mathbb{Q}} f^*A$. Then,

$$K_{X_c} \sim_{\mathbb{Q}} f_c^* A_c \text{ if } c \in C \setminus 0$$

$$K_{X_0} \sim_{\mathbb{Q}} f_0^* A_0 \sim_{\mathbb{Q}} 2q_0^* A_0$$

hence, by the projection formula

$$h^0(mK_{X_c}) = h^0(\mathcal{O}_{\mathbb{P}^1}(m \deg_{\mathbb{P}^1} A_c)) \text{ if } c \in C \setminus 0$$

$$h^0(mK_{X_0}) = h^0(\mathcal{O}_{\mathbb{P}^1}(2m \deg_{\mathbb{P}^1} A_0))$$

for all sufficiently divisible $m \geq 0$. Since $\deg_{\mathbb{P}^1} A_c$ is independent of $c \in C$ by Proposition 4.2.4, we conclude.

4.2 Sufficient conditions

In this Section we give sufficient conditions for the invariance of all sufficiently divisible log plurigenera for abundant families of pairs. Consider the Iitaka fibration $f_{\bar{K}} : X_{\bar{K}} \longrightarrow Z_{\bar{K}}$, let $F_{\bar{K}}$ be a general fiber, and let $d = \dim F_{\bar{K}}$. Consider the map

$$\iota : \operatorname{Pic}(X_{\bar{K}})^{\times d} \longrightarrow \mathbb{Z}$$

$$([D_1], ..., [D_d]) \longmapsto D_1 \cdot ... \cdot D_d \cdot F_{\bar{K}}$$

Theorem 4.2.1. *If* $\operatorname{Image}(\iota) \not\subset p\mathbb{Z}$*, then* $h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_{\bar{K}}} + \Delta_{\bar{K}}))$ *for all sufficiently divisible* $m \geq 0$*.*

Before proceeding with the proof, we recall some results from intersection theory.

Lemma 4.2.2. *Let* $f : X \longrightarrow Y$ *be a finite surjective morphism of normal projective varieties over an algebraically closed field. Let* Z *be any* 0*-cycle on* Y*. Then,*

$$\deg f^* Z = \deg f \cdot \deg Z.$$

Proof. Let $[X]$ and $[Y]$ denote the fundamental classes of X and Y respectively: by the projection formula [Ful98, Ch. 2, Proposition 2.3]

$$\deg(f^* Z) = \deg(f^* Z \cdot [X]) = \deg(Z \cdot f_*[X]) = \deg(\deg(f) \cdot Z) = \deg(f) \cdot \deg(Z)$$

□

We also recall the definition and basic properties of intersection numbers over an arbitrary field.

Definition 4.2.3. [Deb01, Definition 1.7], Let X be a proper scheme over a field. Let $D_1, ..., D_r$ be Cartier divisor, where $r \geq \dim X$. Then the intersection number $D_1 \cdot ... \cdot D_r$ is defined as the coefficient of $m_1 \cdot ... \cdot m_r$ in the polynomial $\chi(X, m_1 D_1 + ... + m_r D_r)$.

Intersection of *Cartier* divisors has integer values: this follows from the equality

$$D_1 \cdot ... \cdot D_d \cdot Y = D_1|_Y \cdot ... \cdot D_d|_Y$$

where $Y \subset X$ is a closed subscheme of dimension $\leq d$, and the fact that $D_i \sim A_i^+ - A_i^-$, for ample divisors $A_i^{\pm}$.

As the Euler characteristic of a line bundle is constant in a flat family, we have the following

Proposition 4.2.4. *Let $X \longrightarrow \mathrm{Spec}R$ be a flat projective morphism of relative dimension d and let $D_1, ..., D_d$ be Cartier divisors on X. Then*

$$D_{1,K} \cdot ... \cdot D_{d,K} = D_{1,k} \cdot ... \cdot D_{d,k}$$

More generally we have

Proposition 4.2.5. *Let $X \longrightarrow \mathrm{Spec}R$ be a flat projective morphism, let $F \subset X$ be a regularly embedded subscheme, flat over R and of relative dimension d, and let $D_1, ..., D_d$ be Cartier divisors on X. Then*

$$D_{1,K} \cdot ... \cdot D_{d,K} \cdot F_K = D_{1,k} \cdot ... \cdot D_{d,k} \cdot F_k$$

Proof. For every $j = 1, ..., d$ we can write

$$D_j \sim A_j^+ - A_j^-$$

where $A_j^{\pm}$ are sufficiently ample divisors on X, such that the restriction $H^0(X, A_j^{\pm}) \longrightarrow H^0(X_0, A_j^{\pm}|_{X_0})$ is surjective and $A_j^{\pm}$ is general in its linear system for all j. Up to replacing the D_j with $A_j^+ - A_j^-$, we may then assume that $\mathcal{J} := \sum_j \mathcal{I}_{D_j} + \mathcal{I}_F$ is a local complete intersection ideal sheaf. Let P be the zero-dimensional R-scheme defined by $\mathcal{J}$: then we need to show

$$\mathrm{length}(P_K) = \mathrm{length}(P_k)$$

Since $\mathcal{J}$ is a local complete intersection ideal, P is Cohen-Macaulay, hence $P \longrightarrow \mathrm{Spec}R$ is flat by [Sta18, Lemma 10.127.1]. In particular, $\chi(\mathcal{I}_{P_K}) = \chi(\mathcal{I}_{P_k})$. But $\chi(\mathcal{I}_{P_s}) = \mathrm{length}(P_s)$ for all geometric points $s \in \mathrm{Spec}R$, thus we conclude. □

Proof of Theorem 4.2.1. Suppose there is $([D_1], ..., [D_d]) \in \mathrm{Pic}(X_{\bar{K}})^{\times d}$ such that p does not divide $\iota([D_1], ..., [D_d])$. Modulo a finite extension of R, we may assume D_i is defined over K for all i. Let $\bar{D}_i$ be the closure of D_i in X and let $F \longrightarrow \mathrm{Spec}R$ be a family of general fibers of f. By Proposition 4.2.5 $\bar{D}_{1,k} \cdot ... \cdot \bar{D}_{d,k} \cdot F_k = \bar{D}_{1,K} \cdot ... \cdot \bar{D}_{d,K} \cdot F_K$. Since $F_k = f^*(\mathrm{point}) = g_k^* h_k^*(\mathrm{point}) = g_k^*(p^r \cdot \mathrm{point})$, Lemma 4.2.2 implies h_k cannot be purely inseparable, hence we conclude by Corollary 4.0.3. □

We also have the following general extension result

Theorem 4.2.6. *Let $(X, \Delta) \longrightarrow \mathrm{Spec}R$ be an abundant family of pairs over a nice DVR of positive or mixed characteristic. Then, there exists a non-negative integer r such that, for all sufficiently divisible $m \geq 0$ the restriction map*

$$H^0(X, \mathcal{O}_X(p^r m(K_X + \Delta))) \longrightarrow H^0(X_k, \mathcal{O}_X(p^r m(K_{X_k} + \Delta_k)))$$

is surjective on $\{s^{p^r} \text{ s.t. } s \in H^0(m(K_{X_k} + \Delta_k))\}$.

Proof. Let $f : X \longrightarrow Z := \mathrm{Proj}R(K_X + \Delta)/\mathrm{Spec}R$ be the relative Iitaka fibration, so that $K_X + \Delta \sim_{\mathbb{Q}} f^*A$ for some ample $\mathbb{Q}$-divisor on Z. Consder the factorization $f_k : X_k \xrightarrow{g_k} S_k \xrightarrow{h_k} Z_k$ given by Proposition 4.0.2, and let $h_k = h_k'' \circ h_k'$ be the Stein factorization, so that h'_k is birational and h''_k is purely inseparable and finite, or it is the identity. Up to replacing g_k with $h'_k \circ g_k$, we may assume that h_k is finite and purely inseparable. Let now m be a sufficiently divisible positive integer, and let $s \in H^0(X_k, m(K_{X_k} + \Delta_k))$: then, s lifts if and only if its is in the image of the pullback map $H^0(Z_k, mA_k) \longrightarrow H^0(X_k, m(K_{X_k} + \Delta_k))$. Let $B_k := h_k^* A_k$: by the projection formula, s corresponds to a section of $H^0(S_k, mB_k)$. Letting $p^r := \deg h_k$ we have then $k(S_k)^{p^r} \subset k(Z_k)$: since every line bundle is a subsheaf of the sheaf of rational functions we see that the image of the pullback map $H^0(Z_k, p^r mA_k) \longrightarrow H^0(S_k, p^r mB_k)$ are exactly the p^r-th powers of elements of $H^0(S_k, mB_k)$. □

We now study more closely two special types of families, namely quasi-elliptic surfaces, or pairs where the log pluricanonical map is a $\mathbb{P}^1$ or del Pezzo fibration.

4.2.1 Quasi-elliptic surfaces

Quasi-elliptic surfaces only exist over fields of characteristic 2 or 3 (see [BM77]), hence we will restrict ourselves to equicharacteristic R. Without loss of generality we can also assume R is complete hence, by Cohen's Structure Theorem (see [Coh46]), $R = k[[t]]$.

Definition 4.2.7. Let X be a smooth surface over an algebraically closed field of positive characerstic. We say X is *quasi-elliptic* if it admits a morphism to a smooth curve $X \longrightarrow Z$ such that the general fiber is a cuspidal rational curve.

All quasi-elliptic surfaces we will consider will be minimal and of Kodaira dimension one. In particular, if $X \longrightarrow \mathrm{Spec}R$ is a family of quasi-elliptic surfaces, K_X will be semi-ample.

Definition 4.2.8. Let $f : X \longrightarrow Z$ be a quasi-elliptic surface and let Γ_0 be a locally closed subset X such that, for all $x \in \Gamma_0$, the fiber $f^{-1}(f(x))$ has an ordinary cusp at x. The *line of cusps of* X is the closure Γ of Γ_0 in X.

Lemma 4.2.9 ([BM77], Proposition 3)**.** *If $f : X \longrightarrow Z$ be a quasi-elliptic surface, then $\Gamma^{(p)} \cong Z$. In particular, if F is a general fiber, we have $\Gamma \cdot F = p$.*

Let now $X \longrightarrow \mathrm{Spec}R$ be a family quasi-elliptic surfaces, and let $\Gamma_{\bar{K}}$ be the line of cusps of $X_{\bar{K}}$: up to a finite extension of R we may assume that $\Gamma_{\bar{K}}$ is actually defined over K. Let Γ be the closure of Γ_K in X:

Lemma 4.2.10. *With notation as above, Γ_k is the line of cusps of X_k.*

Proof. Let $f : X \longrightarrow Z := \mathrm{Proj}R(K_X)/\mathrm{Spec}R$ be the relative Iitaka fibration, let $F \longrightarrow \mathrm{Spec}R$ be a family of general fibers and let $P_{\bar{K}} \in F_{\bar{K}}$ be the cusp point. Modulo a finite extension of R, we may assume $P_{\bar{K}}$ is defined over K. Let P be the closure of P_K in F: it is then enough to show that P_k is the cusp of $F_{k,\mathrm{red}}$. The normalization morphism $\nu_K : \mathbb{P}^1_K \longrightarrow F_K$ corresponds to a K-point $[\nu_K]$ of the Hom-scheme $\mathrm{Hom}_K(\mathbb{P}^1_K, F_K)$. As Hom-schemes are proper, by the valuative criterion for properness we can extend $[\nu_K]$ to an R-point $[\nu] \in \mathrm{Hom}_R(\mathbb{P}^1_R, F)$. Let now $Q_K \in \mathbb{P}^1_K$ be the unique point mapping to P_K and let $Q \subset \mathbb{P}^1_R$ be its closure: note that P and Q are section of $F \longrightarrow \mathrm{Spec}R$ and

$\mathbb{P}^1_R \longrightarrow \mathrm{Spec}R$ respectively. Identifying P with its image in X, we further have that P is a section of $X \longrightarrow \mathrm{Spec}R$. Consider the morphism

$$\psi : \mathbb{P}^1_R \longrightarrow F \hookrightarrow X/\mathrm{Spec}R$$

Let u be a local formal coordinate on $\mathbb{P}^1_T$ and let x, y be local formal coordinates on X, and suppose that Q and P are given by $u = 0$ and $x = y = 0$ respectively. In these coordinates, the map ψ_K corresponds to a morphism of complete $k((t))$-algebras

$$\psi^\sharp_K : k((t))[[x, y]] \longrightarrow k((t))[[u]]$$

$$(x, y) \longmapsto (a(t)u^2, b(t)u^3)$$

where $a(t), b(t) \in k((t))$. Upon rescaling $a(t)$ and $b(t)$ by some positive power of t, we may assume $a(t), b(t) \in k[[t]]$ and $a(0), b(0) \in k^\times$: this allows to define a map $\psi^\sharp_0$ on the central fiber. It is then clear from the equations for $\psi^\sharp_0$ that P_0 is the cusp of $F_{0,\mathrm{red}}$. □

We can now show that Conjecture 1.0.3 holds for families of quasi-elliptic surfaces.

Theorem 4.2.11. *Let $X \longrightarrow \mathrm{Spec}R$ be a family of quasi-elliptic surfaces. Then, $P_m(X_{\bar{K}}) = P_m(X_k)$ for all sufficiently divisible $m \geq 0$.*

Proof. Let $f : X \longrightarrow Z := \mathrm{Proj}R(K_X)/\mathrm{Spec}R$ be the relative Iitaka fibration. By contradiction, suppose that h_k is purely inseparable, and let $F \longrightarrow T$ be a family of general fibers. Up to a finite extension of R, we may assume that $\Gamma_{\bar{K}}$ is defined over K.

Lemmas 4.2.9 and 4.2.10 imply $\Gamma_K \cdot F_K = p$ and $\Gamma_k \cdot F_k = p^e(\Gamma_k \cdot F_{k,\mathrm{red}}) = p^{e+1}$, where $e \geq 1$. Proposition 4.2.5 then yields $p = p^{e+1}$, a contradiction. □

4.2.2 Fano-type fibrations

In [EH16] the authors prove invariance of log plurigenera for log smooth families of klt surfaces $(X, \Delta) \longrightarrow \mathrm{Spec}R$ such that Δ_k is ample over $\mathrm{Proj}R(K_{X_k} + \Delta_k)$, i.e. when the log pluricanonical map is a $\mathbb{P}^1$-fibration. In the case of abundant families, their result generalizes to higher dimensions.

Theorem 4.2.12. *Let* $(X, \Delta) \longrightarrow \mathrm{Spec}R$ *be an abundant family of pairs, such that* $\kappa((X, \Delta)/R) = \dim X - 1$*,* Δ_k *is ample over* $\mathrm{Proj}R(K_{X_k} + \Delta_k)$*. Then, there exists an* $m_0 \in \mathbb{N}$ *such that* $h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_{\bar{K}}} + \Delta_{\bar{K}}))$ *for all* $m \in m_0\mathbb{N}$*.*

Proof. Let $f : X \longrightarrow Z := \mathrm{Proj}R(K_X + \Delta)/\mathrm{Spec}R$ be the relative Iitaka fibration and $F \longrightarrow \mathrm{Spec}R$ a family of general fibers. Since K_{F_k} is ample, so are K_{F_K} and $K_{F_{\bar{K}}}$ by [Laz04a, Theorem 1.2.17], in particular $F_{\bar{K}} = \mathbb{P}^1_{\bar{K}}$, and $F_k = \mathbb{P}^1_k$. Proposition 4.2.5 then yields $-2 = K_{X_{\bar{K}}} \cdot F_{\bar{K}} = K_{X_k} \cdot F_k = p^e(K_{X_k} \cdot F_{k,\mathrm{red}}) = -2p^e$ with $e \geq 1$, a contradiction. □

Provided $p \gg 0$, an analogous result holds when the log pluricanonical map is a del Pezzo fibration.

Theorem 4.2.13. *For every* $\epsilon > 0$*, there exists* p_0 *such that, if* $(X, \Delta) \longrightarrow \mathrm{Spec}R$ *is an abundant family of pairs over a nice DVR with residue characteristic* $p > p_0$ *satisfying:*

(a) $\kappa(K_X + \Delta/R) = \dim_R X - 2$*;*

(b) Δ_k *is ample over* $\mathrm{Proj}R(K_{X_k}+\Delta_k)$*; and*

(c) the general fiber of the Iitaka fibration of $K_{X_s}+\Delta_s$ *is* ϵ*-klt for all geometric points* $s\in \mathrm{Spec}R$*;*

then there exists $m_0\in\mathbb{N}$ *such that the equality* $h^0(m(K_{X_k}+\Delta_k)) = h^0(m(K_{X_{\bar{K}}}+\Delta_{\bar{K}}))$ *holds for all* $m\in m_0\mathbb{N}$*.*

Proof. Let $f : X \longrightarrow Z := \mathrm{Proj}R(K_X+\Delta)/\mathrm{Spec}R$ be the relative Iitaka fibration of (X,Δ), and let $F\longrightarrow \mathrm{Spec}R$ be a family of general fibers. Then $K_{F_k} = K_{X_k}|_{F_k} \sim_{\mathbb{Q}} -\Delta_k$ is anti-ample, and so are K_{F_K} and $K_{F_{\bar{K}}}$ by [Laz04a, Theorem 1.2.17]. Since $F_{k,\mathrm{red}}$ is a general fiber of the Iitaka fibration of $K_{X_k}+\Delta_k$, we have that $F_{\bar{K}}$ and $F_{k,\mathrm{red}}$ are ϵ-klt del Pezzos, which are bounded over $\mathbb{Z}[\frac{1}{30}]$ by [CTW17, Lemma 3.1]. In particular, for $l=k,\bar{K}$, the sets $V_l^\epsilon = \{(-K_Y)^2 \text{ s.t. } Y \text{ is an } \epsilon-\text{klt del Pezzo}/l\}\subset\mathbb{Q}_{>0}$ are finite. Let now p_0 be the smallest prime number such that $p_0 > \max V^\epsilon_{\bar{K}}/\min V^\epsilon_k$, and assume from now on that $p>p_0$. Consider the factorization $f_k : X_k \xrightarrow{g_k} S_k \xrightarrow{h_k} Z_k$ and suppose, by contradiction, that h_k is purely inseparable of degree p^r with $r\geq 1$. Up to a finite extension of R, we may assume that $F_{\bar{K}}$ is defined over K: then Proposition 4.2.5 yields

$$\max V^\epsilon_{\bar{K}} \geq (-K_{F_K})^2 = (-K_{X_K})^2\cdot F_K = (-K_{X_k})^2\cdot F_k =$$

$$= p^r(-K_{X_k})^2\cdot F_{k,\mathrm{red}} = p^r(-K_{F_k})^2 \geq p^r \min V^\epsilon_k > \max V^\epsilon_{\bar{K}}$$

a contradiction. □

Portions of the work in the above chapter is being prepared for submission for publication.

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