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These notes are detailed background material for a set of Lectures given at Lawrence Radiation Laboratory in the Fall of 1968. The final two parts of this series are contained in UCRL-18664 and UCRL-18665.

I.A. Notation and Definitions

The process is shown in Fig. 1.1.

The S matrix is, ignoring isospin,

$$S = 1 - iT(2\pi)^4 \delta^4(\sum p) \quad . \tag{1.1}$$

Defining

$$M^s = \sqrt{s}/2\, f^s = -F/16\pi \quad , \tag{1.2}$$

where F is a matrix element of the T matrix, the c.m. differential cross section is

$$\frac{d\sigma}{d\Omega} = |f|^2 = \frac{|F|^2}{64\pi^2 s} \quad . \tag{1.3}$$

In the following, quantities with no channel label are s-channel quantities. The Mandelstam variables are

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2; \quad t = (p_1 - p_3)^2 = (p_2 - p_4)^2;$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \quad . \tag{1.4}$$

The s-channel c.m. scattering angle and spatial momentum are

$$q^2 \equiv |\vec{q}|^2 = \tfrac{1}{4}(s - 4\mu^2); \quad z \equiv \cos\theta_s = 1 + \frac{2t}{s - 4\mu^2} = -1 - \frac{2u}{s - 4\mu^2} \quad . \tag{1.5}$$

For convenience we define the parameter space

$$x = (s,\ t,\ u;\ p_1^2,\ p_2^2,\ p_3^2,\ p_4^2) \tag{1.6}$$

with the dependence relation

$$s + t + u = \sum_{i=1}^{4} p_i^2 \quad \text{(For physical pions } p^2 = \mu^2 \equiv m_\pi^2\text{)} \ . \tag{1.7}$$

I.B. Isospin, Crossing, Bose Statistics

The full amplitude is

$$M^{dcba}(x) = A(x)\,\delta_{ab}\delta_{cd} + B(x)\,\delta_{ac}\delta_{bd} + C(x)\,\delta_{ad}\delta_{bc} \quad . \qquad (1.10)$$

A, B, C are related by

$$A(s, t, u) = B(t, u, s) = C(u, s, t) \qquad (1.11)$$

and

$$A(s,t,u) = A(s,u,t);\; B(s,t,u) = B(u,t,s);\; C(s,t,u) = C(t,s,u). \qquad (1.12)$$

The isospin amplitudes in the various channels are[1]

$$\begin{aligned}
s:\quad & A_0^{\,s} = 3A + B + C, \\
& A_1^{\,s} = B - C, \\
& A_2^{\,s} = B + C; \\
t:\quad & A_0^{\,t} = 3B + C + A, \\
& A_1^{\,t} = C - A, \\
& A_2^{\,t} = C + A; \\
u:\quad & A_0^{\,u} = 3C + A + B, \\
& A_1^{\,u} = A - B, \\
& A_2^{\,u} = A + B \quad .
\end{aligned} \qquad (1.13)$$

The amplitudes for definite charge states are

$$A(\pi^+\pi^+ \to \pi^+\pi^+) = A_2 = B + C,$$

$$A(\pi^0\pi^0 \to \pi^0\pi^0) = \frac{1}{3}(2A_2 + A_0) = A + B + C,$$

$$A(\pi^+\pi^- \to \pi^+\pi^-) = \frac{1}{3}A_0 + \frac{1}{2}A_1 + \frac{1}{6}A_2 = A + \frac{1}{2}B + \frac{1}{2}C,$$

$$A(\pi^+\pi^- \to \pi^0\pi^0) = \frac{1}{3}(A_2 - A_0) = -A,$$

$$A(\pi^+\pi^0 \to \pi^+\pi^0) = \frac{1}{2}(A_1 + A_2) = B . \qquad (1.14)$$

The $\pi\pi$ crossing matrix $C_{st} = C_{st}^{-1}$ can be written

$$C = \begin{pmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{pmatrix} . \qquad (1.15)$$

I.C. Partial Wave Expansion, Unitarity

$$f_I(s, z) = \sum_{J=0}^{\infty} 2(2J + 1)\, a_J^I(s)\, P_J(z) \quad , \tag{1.20}$$

where the 2 arising from Bose statistics is shown explicitly, and $a_J^I(s)$ is nonvanishing if $(-1)^{J+I} = 1$.

$$a_J^I(s) = \frac{e^{i\delta_J^I(s)} \sin \delta_J^I(s)}{q} = \frac{e^{2i\delta_J^I} - 1}{2iq} \quad . \tag{1.21}$$

Inverting (1.20)

$$a_J^I(s) = \int_{-1}^{+1} dz\, P_J(z)\, f_I(s, z) \quad . \tag{1.22}$$

Near an elastic resonance at $s = s_J = M_J^2$,

$$a_J^I(s) \cong -\frac{1}{q_J} \frac{\Gamma_J^I M_J}{s - s_J + i\frac{1}{2}\Gamma_J^I M_J} \quad . \tag{1.23}$$

As we move away from the resonance we expect[2]

$$\Gamma_J^I(s) \cong \Gamma_J^I(s_J)\, (q/q_J)^{2J+1} \quad . \tag{1.24}$$

The resonant phase shift is then

$$\tan \delta_J^I(s) \cong \frac{\Gamma_J^I(s)\, M_J}{s_J - s} \quad . \tag{1.25}$$

For $4\mu^2 \leq s \leq 16\mu^2$, $\delta_J^{\,I}(s)$ is real because of unitarity. (Because of G conservation only intermediate multipion states with even numbers of pions occur.)

For the case when there is inelasticity, we rewrite (1.23)

$$a_J^{\,I}(s) \cong -\frac{1}{q_J} \frac{\Gamma_J^{\,I}\, x_J^{\,I}\, M_J}{s - s_J + i\,\frac{1}{2}\,\Gamma_J^{\,I}\, M_J} \quad , \tag{1.23a}$$

where $x_J^{\,I}$ measures the amount of inelasticity. $(0 \leq x \leq 1.)$

The unitarity condition

$$S^+S = SS^+ = 1 \tag{1.26}$$

yields the T matrix conditions

$$-i(T^+ - T) = (2\pi)^4 \sum{}' \; T^+T\, \delta^4(\textstyle\sum p) \quad , \tag{1.27}$$

where $\sum{}'$ indicates summation and integration over all states allowed by energy momentum conservation.

In the c.m. system (1.27) yields

$$\mathrm{Im}\, F(s, 0) = 2q\sqrt{s}\; \sigma_{total}(s) \quad . \tag{1.28}$$

On the other hand

$$\frac{d\sigma}{dt} = \frac{|F(s, t)|^2}{16\pi s(s - 4\mu^2)} = \frac{16\pi\, |M(s, t)|^2}{s(s - 4\mu^2)} \; , \tag{1.29}$$

so that

$$[\mathrm{Re}\, F(s, 0)]^2 = 64\pi\, s q^2 \frac{d\sigma}{dt}(s, t)\Big|_{t=0} - 4q^2\, s\, \sigma_{total}(s) \quad . \tag{1.30}$$

We define the real inelasticity parameter η_J $(0 \leq \eta_J \leq 1)$ by

$$a_J(s) = \frac{\eta_J\, e^{2i\delta_J} - 1}{2iq} \quad , \tag{1.31}$$

so that δ_J is also real. The unitarity relation for $a_J(s)$ is then

$$\mathrm{Im}\, a_J(s) = |a_J(s)|^2 + \frac{1}{2}\,[1 - \eta_J^{\,2}(s)] \quad . \tag{1.32}$$

For $4\mu^2 \leq s \leq 16\mu^2$, $\eta_J(s) = 1$. Comparing (1.31) and (1.23a), at resonance,

$$x_J = \frac{1}{2}\,[1 + \eta_J(s_J)] \quad . \tag{1.33}$$

II. Model for $\pi\pi$ Scattering - General Remarks[3]

We take, for the $I = 2$, s channel amplitude,

$$A_2^s(s, t, u) = g \frac{\Gamma(1 - \alpha(t))\, \Gamma(1 - \alpha(u))}{\Gamma(1 - \alpha(t) - \alpha(u))} , \qquad (2.1)$$

where

$$\alpha(x) = a + bx , \qquad (2.2)$$

and we choose

$$0 \leq a \leq 1 - 4\mu^2 b , \qquad b > 0 . \qquad (2.3)$$

We also define, for convenience,

$$F_K(x, y) = \frac{\Gamma(K - x)\, \Gamma(K - y)}{\Gamma(K - x - y)} . \qquad (2.4)$$

From (1.11) - (1.13),

$$X^s \equiv \begin{pmatrix} A_0^s \\ \\ A_1^s \\ A_2^s \end{pmatrix} = g \begin{pmatrix} -\frac{1}{2} F_0[\alpha(t), \alpha(u)] + \frac{3}{2} F_0[\alpha(s), \alpha(t)] + \frac{3}{2} F_0[\alpha(s), \alpha(u)] \\ \\ F_0[\alpha(s), \alpha(t)] - F_0[\alpha(s), \alpha(u)] \\ F_0[\alpha(t), \alpha(u)] \end{pmatrix} \qquad (2.5)$$

As we will discuss in detail below, $F_0[\alpha(t), \alpha(u)]$ has no singularities for $s \geq 4\mu^2$. Therefore the choice (2.1) implies there are no $I = 2$ poles. The most general eigenfunction of the crossing operator O,

$$OX^s = X^t , \tag{2.6}$$

if X is expressed as linear combinations of $F_0\big(\alpha(s), \alpha(t)\big)$, $F_0\big(\alpha(s), \alpha(u)\big)$, $F_0\big(\alpha(t), \alpha(u)\big)$, is

$$X^s = g\begin{pmatrix} c_1\left[\frac{3}{2} F_0\big(\alpha(s), \alpha(u)\big) + \frac{3}{2} F_0\big(\alpha(s), \alpha(t)\big) - \frac{1}{2} F_0\big(\alpha(t), \alpha(u)\big)\right] \\ + c_2[3F_0\big(\alpha(t), \alpha(u)\big) + F_0\big(\alpha(s), \alpha(u)\big) + F_0\big(\alpha(s), \alpha(t)\big)] \\ (c_1 - c_2)[F_0\big(\alpha(s), \alpha(t)\big) - F_0\big(\alpha(s), \alpha(u)\big)] \\ c_1 F_0\big(\alpha(t), \alpha(u)\big) + c_2[F_0\big(\alpha(s), \alpha(u)\big) + F_0\big(\alpha(s), \alpha(t)\big)] \end{pmatrix} \tag{2.7}$$

Equation (2.5) arises on choosing $c_2 = 0$ and $c_1 = 1$ in (2.7).

The function $F_0(x, y)$ is one of an infinite class which have the properties:

(i) $F_0(x, y) = F_0(y, x)$.

(ii) $\lim_{\substack{|x|\to\infty \\ \text{fixed } y}} F_0(x, y) = x^y f_0(y, \phi)$; $|\phi \equiv \arg x| > 0$.

(iii) For y fixed, the only singularities of $F_0(x, y)$ in x, excepting the behavior at infinity, are simple poles at x = positive definite integers. (There are no double poles at points where both x and y are positive integers.)

(iv) The residue of a pole in $F_0(x, y)$ at $x = N$, $\gamma(N, y)$, is a polynomial in y of order $\leq N$.

The property (i) enables one to construct an eigenfunction of the crossing operator as outlined above; (ii) is just Regge asymptotic behavior; (iii) will be referred to below as the narrow resonance approximation (NRA),[4] and will be seen to violate unitarity; (iv) guarantees that any pole can be written as a sum over a finite number of poles having definite angular momentum.

In Fig. 2.1, we show the possible places in the (N, L) $\left(N = \alpha(s)\right)$ plane where the integral

$$H(N, L) = \int_{-1}^{+1} dz\, P_L(z)\, \gamma(N, y) \qquad (2.8)$$

can be nonzero. [It is assumed that y is a linear function of z, and that (iv) holds.]

Clearly, multiplying F_0 by, for example, a properly chosen rational function need not conflict with (i) - (iv). We return to this point below.

We will always identify (x, y) with the Mandelstam variables by setting, for example,

$$x = a + bs \left(\equiv \alpha(s)\right) , \qquad (2.9)$$

so that

$$\alpha(t) = z\left[\frac{1}{2}\left(\alpha(s) - D\right) + a\right] - \frac{1}{2}\left(\alpha(s) - D\right) , \qquad (2.10)$$

where, using $\lambda \equiv 4\mu^2 b$, we have introduced

$$\alpha(s) + \alpha(t) + \alpha(u) = 3a + 4\mu^2 b = 3a + \lambda = D \ . \qquad (2.11)$$

(We will not attempt to define here a procedure for going off mass shell with an external leg, so all $p_i^2 = \mu^2$.)

There is one important condition missing above, namely that all the widths of the $\pi\pi$ resonances in the model are ≥ 0. As we shall see this can be written as

$$\text{(v)} \quad \hat{H}(N, L) \geq 0 \quad (\text{all } N, L) \ . \qquad (2.12)$$

III. PROPERTIES OF $F_0(x, y) = \dfrac{\Gamma(1 - x)\,\Gamma(1 - y)}{\Gamma(1 - x - y)}$.

III.A. Pole Residues

For $y < 0$, we can expand F_0 as[5]

$$F_0(x, y) = \sum_{K=1}^{\infty} \frac{\Gamma(K + y)}{\Gamma(K)\,\Gamma(y)}\,\frac{1}{x - K} \quad , \tag{3.1}$$

explicitly exhibiting the residues

$$\gamma(K, y) = \frac{\Gamma(K + y)}{\Gamma(K)\,\Gamma(y)} = \frac{1}{\Gamma(K)}\, y(y + 1)\cdots(y + K - 1) \quad . \tag{3.2}$$

We will define

$$T_K(y) = \Gamma(K + y)/\Gamma(y) \quad . \tag{3.3}$$

The polynomials $T_K(y)$ have interesting properties. We can write

$$(-1)^N\, T_N(x) = \sum_{m=0}^{N} A_m(N)\, x^m \quad , \tag{3.4}$$

with

$$A_m(N) = (-1)^m\, S_N^{(m)} \tag{3.5}$$

and the $S_N^{(m)}$ are Stirling's numbers of the first kind, the sum of the products of the negative integers $-1, -2, \cdots, -(N - 1)$, taken m at a time in all possible combinations.

Defining

$$x' = x + \frac{1}{2}(N - 1) \quad , \qquad (3.6)$$

we have

$$T_N'(x') = T_N\left(x' - \frac{1}{2}(N - 1)\right) = \left(x' - \frac{1}{2}(N - 1)\right)\left(x' - \frac{1}{2}(N - 1) + 1\right)$$

$$\cdots\left(x' + \frac{1}{2}(N - 1)\right) \quad , \qquad (3.7)$$

so that

$$T_N'(x') = (-1)^N \, T_N'(-x') \ , \qquad (3.8)$$

and $T_N(x)$ is symmetric or antisymmetric about the point $x = -\frac{1}{2}(N - 1)$ depending on whether N is even or odd.

$T_N(x)$ has N zeros at $x = 0, -1, \cdots -N + 1$ and it oscillates with linearly growing amplitude as $|x'|$ increases. A plot of $T_8(x)$ is shown in Fig. 3.1, where it can be seen that these polynomials are essentially zero, for $|x'| < \frac{1}{2}(N - 1)$.

For N even and >0,

$$T_N(x) = (-1)^{N/2} \cdot \frac{1}{\pi} \, \Gamma\left(\frac{1}{2}(N + 1) - x'\right) \Gamma\left(\frac{1}{2}(N + 1) + x'\right) \cos \pi x'$$

$$= (x'^2 - \tfrac{1}{4})(x'^2 - \tfrac{9}{4}) \cdots \left(x'^2 - \tfrac{1}{4}(N - 1)^2\right) \quad . \qquad (3.9)$$

For N odd,

$$T_N(x) = (-1)^{\frac{N-1}{2}} \frac{1}{\pi} \Gamma\left(\frac{N+1}{2} - x'\right) \Gamma\left(\frac{N+1}{2} + x'\right) \sin \pi x'$$

$$= x'(x'^2 - 1)\cdots\left(x'^2 - \frac{1}{4}(N-1)^2\right) \quad , \qquad (3.10)$$

where we have used

$$\Gamma(z)\,\Gamma(1 - z) = \pi/\sin \pi z \quad , \qquad (3.11)$$

$$\Gamma(\tfrac{1}{2} + z)\,\Gamma(\tfrac{1}{2} - z) = \pi/\cos \pi z \quad , \qquad (3.12)$$

$$\Gamma(z + 1) = z\,\Gamma(z) \quad . \qquad (3.13)$$

III.B. Asymptotic Behavior of $F_0(x, y)$

We want to compute $\lim_{\substack{|x|\to\infty \\ \text{fixed } y}} F_0(x, y)$, where $|\arg x| > 0$. To do this we will need the asymptotic expansion of the ratio of Γ functions,[6]

$$R_{\alpha\beta}(z) = \Gamma(z + \alpha)/\Gamma(z + \beta) \overset{|z|\to\infty}{\sim} \sum_{n=0}^{\infty} c_n(\alpha - \beta, \beta) z^{\alpha-\beta-n} , \quad (3.14)$$

where (α, β) are constant complex quantities, and where $z \neq (-\alpha, -\alpha - 1, \cdots; -\beta, -\beta - 1, \cdots)$, while the complex z plane is cut along any curve connecting $z = 0$ with $z = \infty$.

Specifically, the c_i are

$$\begin{aligned} c_0 &= 1, \\ c_1 &= \tfrac{1}{2}(\alpha - \beta)(\alpha + \beta - 1), \\ c_2 &= \tfrac{1}{12}\binom{\alpha - \beta}{2} [3(\alpha + \beta - 1)^2 - \alpha + \beta - 1], \\ &\vdots \end{aligned} \quad (3.15)$$

$$c_n(\gamma, \beta) = \frac{1}{n} \sum_{m=0}^{n-1} \left[\binom{\gamma - m}{n - m + 1} - (-1)^{n+m} \gamma \beta^{n-m} \right] \cdot c_m(\gamma, \beta) , \quad (3.16)$$

where $\gamma \equiv \alpha - \beta$.

Now

$$F_0(x, y) = \frac{\pi\Gamma(x + y)}{\Gamma(x)\,\Gamma(y)}\left[\frac{\sin \pi(x + y)}{\sin \pi x \sin \pi y}\right]$$

$$= \pi\, B^{-1}(x, y)[\cot \pi x + \cot \pi y] \quad . \qquad (3.17)$$

Suppose we take $x \to +\infty$. Then (3.14) tells us

$$B^{-1}(x, y) \sim x^y/\Gamma(y) + O(x^{y-1}) \quad . \qquad (3.18)$$

However, $\cot \pi x$ oscillates, so we must go slightly off the real axis,

$$x \to +\infty + iI, \text{ and then } \cot \pi x \sim -i \quad ,$$

$I \to \infty$ faster than logarithmically, giving

$$F_0(x, y) \underset{x \to +\infty + iI}{\sim} \frac{\pi\, e^{-i\pi y}}{\sin \pi y\, \Gamma(y)}\, x^y \quad . \qquad (3.19)$$

Because the poles lie on the positive real axis, just where we would like to go, this asymptotic behavior is true in an average sense only. Without making an effort to go past the narrow resonance approximation this problem is unavoidable.

The other Mandelstam variable, $w = \alpha(u)$, satisfies

$$x + y + w = D \quad . \qquad (3.20)$$

We can therefore rewrite F_0 as

$$F_0(x, y) = \frac{\Gamma(1 - x)\, \Gamma(1 - D + x + w)}{\Gamma(1 - D + w)} . \tag{3.21}$$

Equation (3.20) gives, just as before,

$$F_0(x, y) \underset{\substack{w \to \infty + iI \\ \text{fixed } x}}{\sim} \frac{\pi w^x}{\sin \pi x\, \Gamma(x)} . \tag{3.22}$$

However, for fixed w,

$$F_0(x, y) \underset{\substack{x \to \infty + iI \\ \text{fixed } w}}{\sim} \frac{\pi}{\sin \pi x} \frac{x^{w+1-D}}{\Gamma(w + 1 - D)} \to 0 \quad \text{(faster than any power)} . \tag{3.23}$$

Another equivalent way of stating this average asymptotic behavior is that $\mathrm{Im}\, \alpha(s) \geq 0$, but $\cong 0$ for finite s, and as $s \to \infty$, $\mathrm{Im}\, \alpha(s) \to \infty$ faster than $\ln s$, so that any power growth is overcome by the $\mathrm{Im}\, \alpha(s)$ piece, in the fixed $w(u)$ direction.

The asymptotic behavior of $F_0\big(\alpha(s), \alpha(t)\big)$ is shown in Fig. 3.2. For large z_t, we also have the following expressions

$$\lim_{\substack{z_t \to \infty \\ \text{fixed } \alpha(t)}} F_0\Big(\alpha(s), \alpha(t)\Big) \sim \Gamma\Big(1 - \alpha(t)\Big)\, z_t^{\alpha(t)-D}\, e^{i\pi\left(\alpha(t)-D\right)} , \tag{3.24}$$

$$\lim_{\substack{z_t \to \infty \\ \text{fixed } \alpha(t)}} F_0\Big(\alpha(s), \alpha(u)\Big) \sim 0 , \tag{3.25}$$

$$\lim_{\substack{z_t \to \infty \\ \text{fixed } \alpha(t)}} F_0\Big(\alpha(t),\, \alpha(u)\Big) \sim \Gamma\Big(1 - \alpha(t)\Big)\, z_t^{\alpha(t)-D} \quad , \qquad (3.26)$$

where we have used (2.10).

III.C. Poles and Zeros of $F_0(x, y)$

The zeros of $F_0(x, y)$ occur along the lines $1 - x - y = 0, -1, -2, \cdots$. Excepting the line $x + y = 1$, each line of zeros passes through the region $(x, y) \geq 0$ and cancels the double poles that would otherwise appear, as shown in Fig. 3.3.

Lovelace[7] has noticed that if $\alpha(s \approx \mu^2)$ is near 1/2, then a zero appears in all the isospin amplitudes near threshold. Lovelace identifies this zero with the zero of the Adler consistency condition, which makes the off-shell statement[8]

$$A(x_A) = B(x_A) = C(x_A) = 0 \quad , \tag{3.30}$$

where

$$x_A = (\mu^2, \mu^2, \mu^2; 0, \mu^2, \mu^2, \mu^2) \quad . \tag{3.31}$$

Actually, if one requires a zero in all the isospin amplitudes one has

$$F_0\Big(\alpha(s_0), \alpha(t_0)\Big) = F_0\Big(\alpha(s_0), \alpha(u_0)\Big) = F_0\Big(\alpha(t_0), \alpha(u_0)\Big) \quad . \tag{3.32}$$

Intersections between the zeros of all three functions occur as shown in Fig. 3.4.

The only triple zero close to threshold lies on $\alpha(s) + \alpha(t) = 1$. If all isospin amplitudes are to have their zeros there, then $D = 3/2 = 3a + 4\mu^2 b$ is required, which is in the neighborhood of the phenomenological values

$$1.5 \lesssim D \lesssim 1.9 \quad . \tag{3.32}$$

This argument is not terribly convincing, because, as we will see below, the zero along $x + y = 1$ can always be moved by multiplying and dividing by polynomials in x and y.

It is amusing to note that the beta functions have no such extraneous line of zeros. In fact

$$F_0(x, y) = \frac{\Gamma(1 - x)\,\Gamma(1 - y)}{\Gamma(1 - x - y)} = -\frac{xy}{x + y} B(-x, -y) \quad , \tag{3.33}$$

explicitly exhibiting the fact that we have destroyed two lines of poles at $x = 0$ and $y = 0$, at the same time as removing the zero along $x + y = 0$.[9]

Now we anticipate a piece of a result below, by remarking $F_0(x, y)$ could be changed by a factor

$$F_0{}'(x, y) = \frac{Q - x - y}{1 - x - y} F_0(x, y) \tag{3.34}$$

without affecting any of its basic properties. More generally, we can take

$$F_0{}''(x, y) = \frac{pxy + q(x + y) + r}{1 - x - y} F_0(x, y) \quad , \tag{3.35}$$

so that the line of zeros $x + y - 1 = 0$ is replaced by a curve $pxy + q(x + y) + r = 0$. We return to this point below.

III.D. Positivity of Resonance Widths[10]

Define the $\pi\pi$ width of an s channel pole at $\alpha(s) = N$ by $\Gamma(N, L)$. Then, from (1.22) and (1.23a)

$$M_N \Gamma(N, L) = q_N \, 2a_L(s_N) = q_N \int_{-1}^{+1} dz \, P_L(z) \, f(s, z) \quad . \tag{3.36}$$

Using (1.2), (2.5), and (1.10)

$$2M_N\Gamma(N, L) = \frac{3}{2} g\left(\frac{2q_N}{M_N}\right) b^{-1} \cdot \int_{-1}^{+1} dz \, \frac{P_L(z)}{2} \Big\{ [\alpha(s) - N] \cdot [F_0(\alpha(s), \alpha(t)) + F_0(\alpha(s), \alpha(u))] \Big\}_{s=s_N} , \tag{3.37}$$

for L even, $I = 0$, and

$$2M_N\Gamma(N, L) = g\left(\frac{2q_N}{M_N}\right) b^{-1} \cdot \int_{-1}^{+1} dz \, \frac{P_L(z)}{2} \Big\{ [\alpha(s) - N] \cdot [F_0(\alpha(s), \alpha(t)) - F_0(\alpha(s), \alpha(u))] \Big\}_{s=s_N} , \tag{3.38}$$

for L odd, $I = 1$. (The factor $2q_N/M_N = \sqrt{1 - 4\mu^2/s_N} \cong 1$.)

From (3.1) and (3.2), and the definition (2.8),

$$2M_N\Gamma(N, L) = R_I \, g \, b^{-1}(1 - 4\mu^2/s_N)^{\frac{1}{2}} \, H(N, L) \quad , \tag{3.39}$$

where $R_0 = 3/2$, $R_1 = 1$.

Working out the first few $H(N, L)$ we have

$$H(0,0) = 0 \quad , \tag{3.40a}$$

$$H(1,0) = D - 1 \quad , \tag{3.40b}$$

$$H(1,1) = \frac{1}{3}(1 + 2a - D) \quad , \tag{3.40c}$$

$$H(2,0) = \frac{2}{3}(1 + a - D/2)^2 + D(D/2 - 1) \quad , \tag{3.40d}$$

$$H(2,1) = \frac{2}{3}(D - 1)(1 + a - D/2) \quad , \tag{3.40e}$$

$$H(2,2) = \frac{4}{15}(1 + a - D/2)^2 \quad . \tag{3.40f}$$

If we write, using (2.10), with $\alpha(s) = N$,

$$T_N\Big(\alpha(t)\Big) = \Gamma\Big(\alpha(t) + N\Big)/\Gamma\Big(\alpha(t)\Big) = \sum_{K=0}^{N} r_K(N)\, z^K \quad , \tag{3.41}$$

we have the explicit expressions

$$r_N(N) = 2^{-N}(N - D + 2a)^N \quad , \tag{3.42}$$

$$r_{N-1}(N) = 2^{-N}\, N(D - 1)(N - D + 2a)^{N-1} \quad , \tag{3.43}$$

$$r_{N-2}(N) = 2^{-N}\,\frac{N(N - 1)}{6}\,(N - D + 2a)^{N-2}(3D^2 - 6D + 2 - N) \quad . \tag{3.44}$$

Now we see from (3.40c) and (3.43) that $H(1, 1) \geqslant 0$ implies $H(N, N) \geqslant 0$, for all N. This condition is just

$$1 + 2a - D = 1 - a - \lambda \geqslant 0 \quad , \tag{3.45}$$

as in (2.3). It is also the statement that the $N = L = 1$ state, (the ρ) not be bound,

$$m_\rho^{\ 2} \geqslant 4\mu^2 \quad , \tag{3.46}$$

$$\alpha(m_\rho^{\ 2}) = a + b\, m_\rho^{\ 2} = 1 \leqslant a + \lambda \quad . \tag{3.47}$$

From (3.43) we see $H(N, N - 1) \geqslant 0$ if (3.45) holds and if $D - 1 \geqslant 0$. It turns out that $H(N, N - 2) \geqslant H(2,0) \geqslant 0$ then assures $H(N, L) \geqslant 0$ for all N and L. From (3.40d) this can be written

$$a \geqslant \frac{1}{2} - \frac{2}{3}\lambda^2 + O(\lambda^4)\cdots . \tag{3.48}$$

We will show here that $H(2,0) \geqslant 0$ implies $H(N,0) \geqslant 0$, if $N \geqslant 2$ and even; and $H(N,0) \geqslant 0$, if $D - 1 \geqslant 0$, and $N \geqslant 1$ and odd.

Again using (2.10), we see the physical region in $\alpha(t)$ is

$$-N + 2a + \lambda \leqslant \alpha(t) \leqslant a \quad , \tag{3.49}$$

so that, changing variables,

$$H(N,0) = \frac{2}{3H(1,1)} \int_{-N+2a+\lambda}^{a} T_N(y)\, dy \quad . \tag{3.50}$$

Since we insist that $H(1,1) \geqslant 0$ we need consider the integral

$$h(N) = \int_{-N+2a+\lambda}^{a} T_N(y)\, dy \quad , \tag{3.51}$$

only. As shown above, $T_N(y)$ is odd about $y = -\frac{1}{2}(N - 1)$ for odd N, and we have

$$h(N) = \int_{1-2a-\lambda}^{a} dy\, T_N(y) \quad , \qquad N \text{ odd} , \tag{3.52}$$

implying

$$H(N,0) \geqslant 0 \text{ for odd } N \text{ if } 3a + \lambda - 1 = D - 1 \geqslant 0 \ . \tag{3.53}$$

For even N, things are more complicated. We have

$$h(N) = \int_{-N+2a+\lambda}^{-N+2} dy\, T_N(y) + \int_{-N+2}^{-1} dy\, T_N(y) + \int_{-1}^{a} dy\, T_N(y)$$

$$= I_1(N) + I_2(N) + I_3(N) \qquad (N \text{ even}) \ . \tag{3.54}$$

Since $T_N(y)$ is now symmetric about $y = -\frac{1}{2}(N - 1)$

$$I_1(N) = \int_{-1}^{1-2a-\lambda} T_N(y)\, dy \quad , \tag{3.55}$$

$$I_2(N) = 2 \int_{-\frac{1}{2}(N-1)}^{-1} T_N(y)\, dy \quad . \tag{3.56}$$

However, we know the oscillations of $T_N(y)$ increase as $|y + \frac{1}{2}(N - 1)|$ increases, and therefore $I_2(N) \geqslant 0$ for $N \geqslant 2$ (N even). Furthermore

$$T_N(x) = \Gamma(N + x)/\Gamma(x) = T_{N-4}(x + 4)\; T_4(x) \quad . \tag{3.57}$$

$T_{N-4}(x + 4)$ is monotonic and positive for $-1 \leqslant x \leqslant +1$, while $T_4(x)$ is monotonic and passes through 0 at $x = 0$. Therefore, for $N \geqslant 4$

$$I_1(N) \geqslant \int_{-1}^{1-2a-\lambda} T_4(x)\, dx \quad , \tag{3.58}$$

$$I_3(N) \geqslant \int_{-1}^{a} T_4(x)\, dx \quad , \tag{3.59}$$

and doing the integrals explicitly we discover $I_1(N) + I_3(N) > 0$ if $a \geqslant \frac{1}{2} - \frac{2}{3}\lambda^2$.

If we set $\lambda = 0 = \mu^2$, $a = \frac{1}{2}$, $D = \frac{3}{2}$, the first few widths, from (3.39), are, in units of $M_1\Gamma(L,1)$, and $\Gamma(1,1)$,

Table I.1.

(N,L)	$M_N\Gamma(N,L)$	$\Gamma(N,L)$
(0,0)	0	0
(1,1)	1	1
(1,0)	9/2	9/2
(2,2)	27/20	$9\sqrt{3}/20$
(2,1)	3/2	$\sqrt{3}/2$
(2,0)	0	0

where

$$M_{N'}/M_N = [(2N' - 1)/(2N - 1)]^{\frac{1}{2}} . \tag{3.60}$$

Using[11]

$$\int_0^1 x^\lambda P_{2K}(x)\, dx = \frac{(-1)^K \Gamma(K - \frac{1}{2}\lambda)\, \Gamma(\frac{1}{2} + \frac{1}{2}\lambda)}{2\Gamma(-\frac{1}{2}\lambda)\, \Gamma(K + \frac{3}{2} + \frac{1}{2}\lambda)} , \tag{3.61}$$

for Re $\lambda > -1$

$$\int_0^1 x^\lambda P_{2K+1}(x)\, dx = \frac{(-1)^K \Gamma(K - \frac{1}{2}\lambda + \frac{1}{2})\, \Gamma(1 + \frac{1}{2}\lambda)}{2\Gamma(\frac{1}{2} - \frac{1}{2}\lambda)\, \Gamma(K + 2 + \frac{1}{2}\lambda)} , \tag{3.62}$$

for Re $\lambda > -2$

we have

$$H(N,N) = r_N(N)/\Gamma(N) \cdot \Gamma(\tfrac{1}{2} N + \tfrac{1}{2})\, \Gamma(\tfrac{N}{2} + 1)/\Gamma(N + \tfrac{3}{2}) , \tag{3.63}$$

$$H(N,N-1) = r_{N-1}(N)/\Gamma(N) \cdot \Gamma(\tfrac{1}{2} N)\ \Gamma(\tfrac{N}{2} + \tfrac{1}{2})/\Gamma(N + \tfrac{1}{2}) \quad , \tag{3.64}$$

so that, for $a = 1/2 = D/3, \quad \mu = 0,$

$$M_N \Gamma(N,N) = \frac{\frac{1}{2} R_I \sqrt{\pi}\ \ 2^{-2N}(N - \frac{1}{2})^N}{\Gamma(N + \frac{3}{2})}\ \frac{N}{2} \quad , \tag{3.65}$$

$$M_N \Gamma(N,N-1) = \frac{\frac{1}{2} R_I \sqrt{\pi}\ \ 2^{-2N}(N - \frac{1}{2})^{N-1}}{\Gamma(N + \frac{1}{2})}\ \frac{N}{2} \quad , \tag{3.66}$$

where we have used the doubling formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\ \Gamma(z)\ \Gamma(z + \tfrac{1}{2}) \quad . \tag{3.67}$$

We note, for future reference, that inserting $\Gamma_{\rho\pi\pi} \cong 112$ MeV and $M_\rho \cong 764$ MeV, with $b \cong 1$ GeV^{-2}, yields $g \cong 1.0$.

The integrals over $T_N(x)$ may be related to the generalized Bernoulli polynomials, $B_n^{(\ell)}(z)$, by

$$\int_{-y}^{-y+1} dx\ T_N(x) = (-1)^N\ B_N^{(N)}\ (y) \quad , \tag{3.68}$$

(Cf. Ref. 14 [19.7(31)])

III.E. Behavior of $\pi\pi$ Resonance widths, $\Gamma(N,L)$, as Functions of (N,L)[12]

The behavior of $\Gamma(N,L)$ as a function of (N,L) is contained in $H(N,L)$ defined as in (2.8),

$$H(N,L) = \int_{-1}^{+1} dz\, P_L(z) \frac{\Gamma(N + y)}{\Gamma(N)\, \Gamma(y)} dz \quad , \tag{2.8}$$

where, using (2.10),

$$y = z\left(\frac{1}{2}(N - D) + a\right) - \frac{1}{2}(N - D) \quad . \tag{3.70}$$

As remarked above, in Section III.A. (see Fig. 3.1), $T_N(y) = \Gamma(N + y)/\Gamma(y)$ essentially consists of a huge forward peak, between

$$0 \leqslant y \leqslant a \qquad \text{or} \quad 1 - \frac{2a}{N} \lesssim z \leq 1 \quad , \tag{3.71}$$

for

$$N \gg D, a, \text{and } \lambda.$$

We will therefore approximate $H(N,L)$ by

$$H(N,L) \cong \int_{1-\frac{2a}{N}}^{1} dz \frac{P_L(z)\, \Gamma(N + y)}{\Gamma(N)\, \Gamma(y)} \quad . \tag{3.72}$$

We will work here with N,L large, and will consider the following regions:

(i) $L \ll \sqrt{N}$,

(ii) $L \cong \sqrt{N}$,

(iii) $L \gg \sqrt{N}$.

For large L, and small $1 - z$,[13]

$$P_L(z) = J_0\left((2L+1)\sqrt{\frac{1-z}{2}}\right) + \left(\frac{1-z}{2}\right)\left[\frac{J_1(\eta)}{2\eta} - J_2(\eta) + \frac{\eta}{6} J_3(\eta)\right] + 0\left((1-z)^2\right) \cong J_0\left(2L\sqrt{\frac{1-z}{2}}\right) \qquad (3.73)$$

$[\eta = (2L+1)\sqrt{(1-z)/2}]$.

The first zero of $J_0(x)$ occurs at $x \cong 2.41$. Therefore, the first zero of $P_L(z)$, by (3.73) occurs near

$$L_0 \cong 1.2\sqrt{N/a} \quad . \qquad (3.74)$$

For $L \ll \sqrt{N}$, $H(N,L)$ will thus be roughly constant, slowly decreasing with increasing L, since the entire forward peak in $T_N(y)$ is included, as shown in Fig. 3.5.[12]

As L gets larger than L_0, $H(N,L)$ will begin to drop sharply. A numerical computation for $N = 50$ is shown in Fig. 3.6. Taking $N \gg y$ and expanding $\Gamma^{-1}(y)$ in a power series about $y = a$,

$$H(N,L) \cong \int_{1-2a/N}^{1} dz\, J_0\left(2L\sqrt{\frac{1-z}{a}}\right) \frac{N^y}{\Gamma(y)}$$

$$\cong 2N^a \int_0^\infty dx\, e^{-N(\log N)x}\, J_0(2L\sqrt{x}) \sum_{j=0}^{\infty} c_j^{(N)} x^j$$

$$\cong 2N^a \sum_{j=0}^{\infty} c_j^{(N)} \frac{\Gamma(j+1)}{(N \log N)^{j+1}}\; {}_1F_1\left(j+1;\, 1;\, \frac{-L^2}{N \log N}\right)$$

$$= 2N^a \sum_{j=0}^{\infty} c_j^{(N)}\, \Gamma(j+1)(N \log N)^{-j-1}\, e^{-L^2/N \log N} \cdot L_j\left(\frac{L^2}{N \log N}\right), \qquad (3.75)$$

where $L_j(x)$ is a Laguerre polynomial.[14] Now

$$\frac{1}{\Gamma(y)} \cong \frac{1}{\Gamma\ a(1 - \frac{xN}{2})} = \Gamma\left(1 - a + \frac{aNx}{2}\right) \sin \pi\, a\left(1 - \frac{Nx}{2}\right). \qquad (3.76)$$

Since we care about $0 \leq x \leq 2/N$, we can set

$$\frac{1}{\Gamma(y)} \leq \Gamma(1-a) \sin \pi a\left(1 - \frac{Nx}{2}\right)$$

$$= \Gamma(1-a)\left\{\sin \pi a \sum_{j=0}^{\infty} \frac{\varphi^{2j}}{\Gamma(2j+1)} - \cos \pi a \sum_{j=0}^{\infty} \frac{\varphi^{2j+1}}{\Gamma(2j+2)} (-1)^j\right\},$$

where $\varphi \equiv \frac{1}{2} Nx$. $\qquad (3.77)$

Fejer's formula for $L_j(z)$ gives,[15] for fixed z,

$$\lim_{j\to\infty} L_j(z) = \sqrt{\pi}\, e^{\frac{1}{2}z} z^{-1/4} \cos[2\sqrt{jz} - \pi/4] + O(j^{-1/4}) \quad . \tag{3.78}$$

From (3.77) and (3.78) we see that the series (3.75) converges for fixed and large L,N. For $L \ll \sqrt{N}$, the argument of the Laguerre polynomial becomes small and (3.78) is no longer applicable. Instead we have

$$e^{-\frac{1}{2}z} L_j(z) \sim J_0\left([2(2j+1)z]^{\frac{1}{2}}\right) + O(j^{-\frac{3}{4}}) \quad , \tag{3.79}$$

and (3.75) still clearly converges, with each term dying exponentially. For $L \ll \sqrt{N}$, $H(N,L) \sim e^{-L^2/N \log N}$ and, as we have mentioned above, is relatively insensitive to L.

Finally, for $L \gg \sqrt{N}$, a much more complicated set of arguments[15] leads once more to the conclusion that the series (3.75) converges, and the sum drops off exponentially as L increases.

From (3.75), note that the first term in the series gives $H(N,L) \sim \dfrac{N^{a-1}}{\log N}$ for large N.

From (3.75), [see Fig. 3.6 also] we see that for large, fixed, N, $\Gamma(N, L)$ is a monotonically decreasing function of L. This means that the positivity of $\Gamma(N, 1)$ and $\Gamma(N, 0)$ guarantees $\Gamma(N, L) \geq 0$, for large N.

III.F. Threshold Behavior of $F_0(x,y)$

At threshold $(s = 4\mu^2,\ t = u = 0)$, our amplitude (2.5) becomes

$$X^s(x_T) = g\begin{pmatrix} -\frac{1}{2}F_0(a,a) + F_0(a+\lambda,\ a) \\ 0 \\ F_0(a,a) \end{pmatrix} . \tag{3.80}$$

The s-wave scattering lengths are related to the amplitude by

$$a_I = -A_I{}^s/2\mu \ . \tag{3.81}$$

The two quantities of interest are

$$\mu L = \frac{1}{6}(2a_0 - 5a_2)\mu = g[-F_0(a,a) + F_0(a+\lambda,\ a)] \tag{3.82}$$

and

$$R = a_0/a_2 = -\frac{1}{2} + 3F_0(a+\lambda,\ a)/F_0(a,a) \ . \tag{3.83}$$

For $a = 1/2$, $F_0(a,a) \sim \frac{1}{\Gamma(0)} = 0$, and $a_2 = 0$. We set $b \cong 1\ \text{GeV}^{-2}$ so that $\lambda \cong 0.08$. For $|a - \frac{1}{2}| > \lambda$, $R \cong 5/2$, which is also the value A_0/A_2 has at the symmetry point $s = t = u = 4/3\ \mu^2$. Taking $a = 1/2$, we have, if we use $\Gamma_{\rho\pi\pi} \cong 112$ MeV, and $M_\rho \cong 764$ MeV to fix $g \cong 1$, as in Sect. III.D,

$$a_0 \cong -3F_0(1/2 + \lambda,\ 1/2)/2\mu$$

$$= \frac{3\Gamma(1/2)\ \Gamma(1/2 - \lambda)}{\Gamma(-\lambda)\ 2\mu} \cong +0.4/\mu \quad , \qquad (3.84)$$

and therefore L, which is relatively insensitive to the zero, is

$$L \cong 0.13/\mu \quad . \qquad (3.85)$$

(We will return to L and its relation to the Adler $\pi\pi$ sum rule[16] below.) If the amplitude is reasonable, the partial-wave amplitudes $a_J(s)$ should have a threshold behavior like q^{2J}. Recalling (1.22), we have

$$a_J^I(s) = \int_{-1}^{+1} dz\ P_J(z)\ f_I(s,z) \quad . \qquad (1.22)$$

As long as $a < 1 - \lambda$ our amplitudes have no singularities at threshold and they are also invariant functions of (s,t). This automatically means $a_J^I(s) \sim q^{2J}$ or faster as $q^2 \to 0$. Said another way, $f_I(s,z)$ is expansible in a power series in $q^2 = \frac{1}{4}(s - 4\mu^2)$ around $q^2 = 0$,

$$f_I(s,z) = \sum_{p=0}^{\infty} c_p(z)\ q^{2p} \quad . \qquad (3.86)$$

The threshold behavior is right if $c_K(z)$

$$c_K(z) = \sum_{j=0}^{K} c_{Kj} z^j , \qquad (3.87)$$

can be expanded in a power series in z whose highest power is K. This is always true if there is no singularity in t at threshold, because z always appears in the combination $q^2 z$.

III.G. Angular Behavior at High Energies

As shown in Section III.B, the average asymptotic behavior of $F_0(x,y)$ as $|x| \to \infty$ for fixed y is

$$F_0(x,y) \sim \frac{\pi\ x^y\ e^{-i\pi y}}{\sin \pi y\ \Gamma(y)} . \tag{3.88}$$

We want to relate this to the statements in Section III.A about the forward and backward peaking of the pole residues.

From (1.3), (1.2). (3.88), and (1.29),

$$\lim_{s\to\infty} \frac{d\sigma}{dt}(\pi^+\pi^- \to \pi^0\pi^0) \sim \frac{g^2{}_\pi{}^2 (bs)^{2\alpha(t)}\ \sin^2\left(\frac{\pi}{2}\,\alpha(t)\right)\ 64\pi s^{-2}}{\Gamma^2\left(\alpha(t)\right)\ \sin^2 \pi\alpha(t)} . \tag{3.89}$$

Parameterizing this in the usual way, as

$$\frac{d\sigma}{dt} \sim e^{c+dt} , \tag{3.90}$$

we have, for $a \cong 1/2$ and $b \cong 1\ \text{GeV}^{-2}$, $d \cong 12\ \text{GeV}^{-2}$ at $s = 30\ \text{GeV}^2$, which is in the reasonable range. (Slopes range from $\approx 5\ \text{GeV}^{-2}$ for $K^-p \to \bar{K}^0 n$ to $\approx 12\ \text{GeV}^{-2}$ for NN charge exchange,[17] $|t| > 0.2\ \text{GeV}^2$.)

The author has benefited from discussions with G. F. Chew, S. Mandelstam, and J. Shapiro.

FOOTNOTES AND REFERENCES*

* This work was supported in part by the U.S. Atomic Energy Commission.

1. We follow G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960); and Nuovo Cimento 19, 752 (1961).

2. For a general discussion of resonance phenomenology, see J. D. Jackson, Nuovo Cimento 34, 1644 (1964).

3. The model under discussion here was first suggested by G. Veneziano, Nuovo Cimento 57, 190 (1968).

4. See S. Mandelstam, Phys. Rev. 166, 1539 (1968); H. Cheng and D. Sharp, Ann. Phys. (N.Y.) 22, 481 (1963); Phys. Rev. 132, 1854 (1963).

5. Whittaker and Watson, Modern Analysis (Cambridge University Press) p. 260, Ex. 9.

6. F. C. Tricomi and A. Erdelyi, P. J. Math. 1, 133 (1942).

7. C. Lovelace, CERN report (1968), unpublished.

8. See N. N. Khuri, Phys. Rev. 153, 1477 (1967); S. L. Adler, Phys. Rev. 137, 1022 (1968).

9. The extra factor in (3.33) destroys the ghost at $\alpha(x) = 0$.

10. See J. Shapiro, University of California preprint (1968), to be published.

11. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (Academic Press, 1965), (7.231(1,2)). Note the misprint in the latter equation: read n for m.

12. The arguments of this section should be compared with those of M. Kugler, Phys. Rev. Letters 21, 570 (1968).

13. Ref. 11 [8.722(1)]. (Read μ for ν in the exponent of the bracket on the left.)

14. See A. Erdelyi et al., Tables of Integral Transforms [4.14(33)], [4.14(35)], (McGraw-Hill 1954).

15. A. Erdelyi et al., Higher Transcendental Functions (McGraw-Hill 1954), Sect. 10.15.

16. S. L. Adler, Phys. Rev. 140, B736 (1965).

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FIGURE CAPTIONS

Fig. 1.1. $\pi\pi$ scattering with momenta and isospin labels.

Fig. 2.1. The plane (N, L), ($N = \alpha(s)$, L = orbital angular momentum in the s channel) showing possible nonzero $\pi\pi$ couplings arising from functions of the type discussed in the text. Note the absence of states at $N = L = 0$, (a possible "ghost" state) and in the "ancestor" region $L > N$.

Fig. 3.1. Plot of $T_N(x) = \Gamma(x + N)/\Gamma(x)$ for $N = 8$. This is proportional to the residue of the pole at $\alpha(s) = 8$ in $F_0[\alpha(s), \alpha(t)]$. Curve b is a blowup of the central portion of curve a. The scale on the left is for a, the one on the right for b. The range of integration indicated is the physical region for $a = 0.48$ and the physical π and ρ masses. Notice the strong forward peaking, and the growing size of the oscillations as one goes away from $x = -3.5$.

Fig. 3.2. Average asymptotic behavior of $F_0[\alpha(u), \alpha(t)]$ on the Mandelstam plot, showing zeros and poles.

Fig. 3.3. Poles and zeros of $F_0(x,y) = \Gamma(1 - x)\,\Gamma(1 - y)/\Gamma(1 - x - y)$. Poles are solid lines, zeros dotted lines.

Fig. 3.4. Intersections of lines of zeros in $F_0[\alpha(s), \alpha(t)]$; $F_0[\alpha(s), \alpha(u)]$; $F_0[\alpha(t), \alpha(u)]$.

Fig. 3.5. $T_4(y)$ vs $P_4(z)$, $P_1(z)$, and $P_0(z)$.

Fig. 3.6. $\Gamma(50, L)$ as function of L.

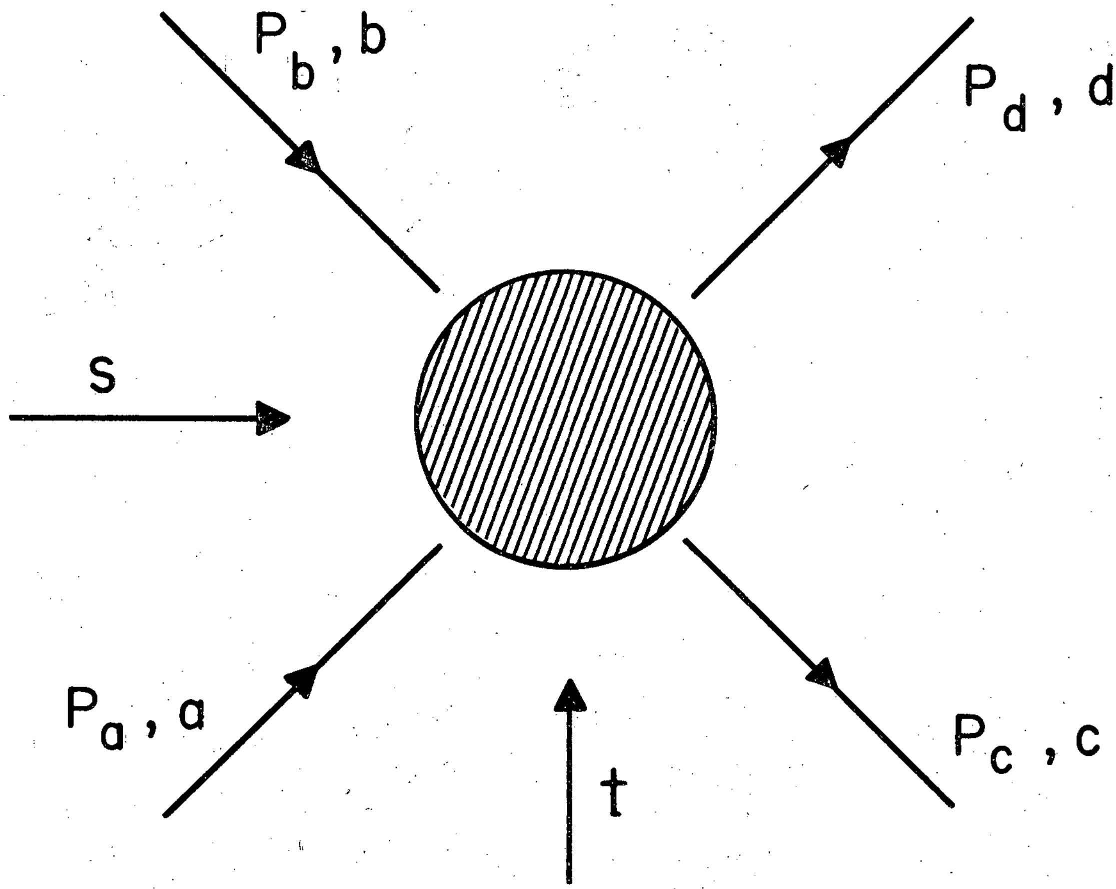

Fig. 1.1.

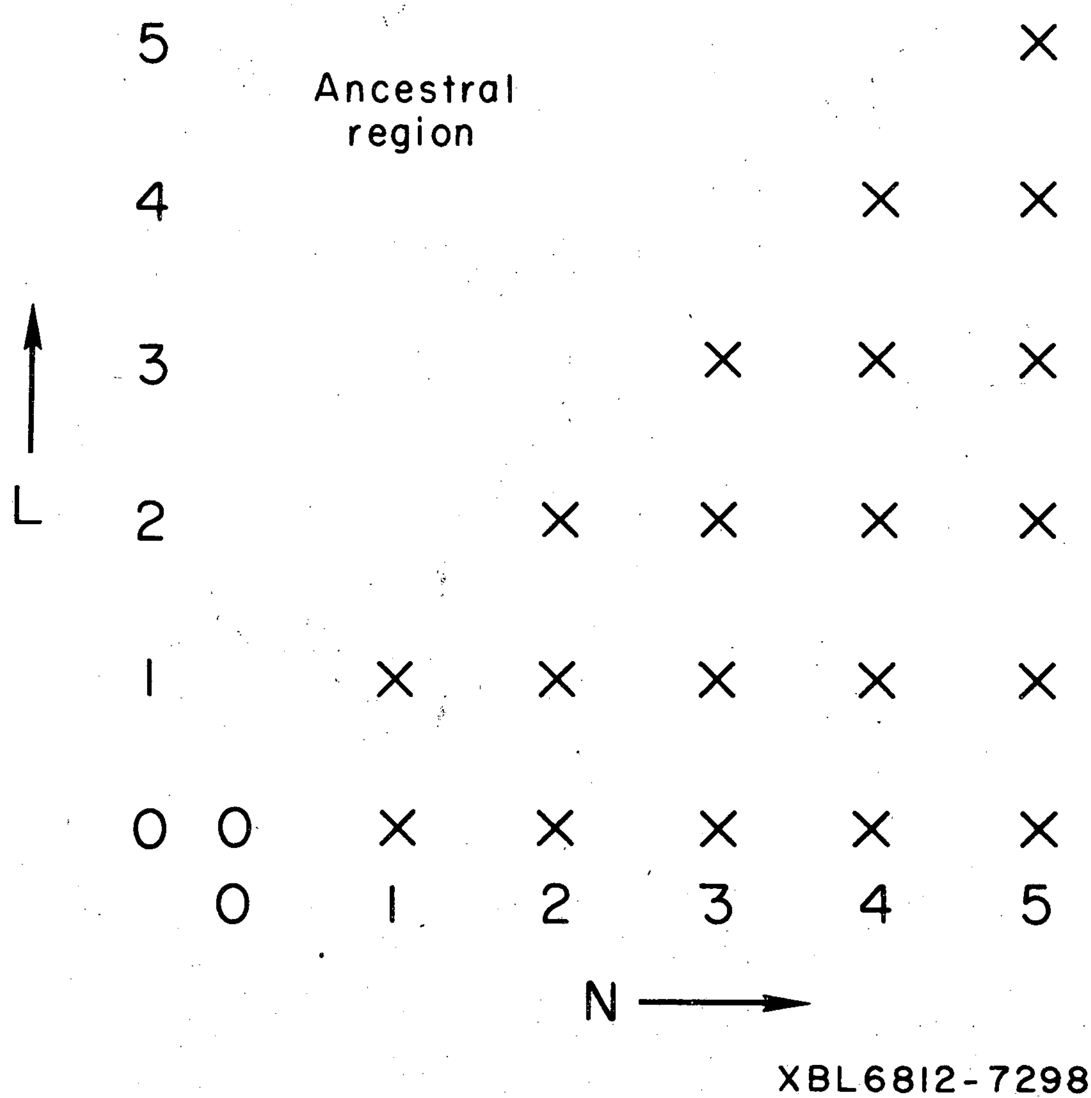

Fig. 2.1.

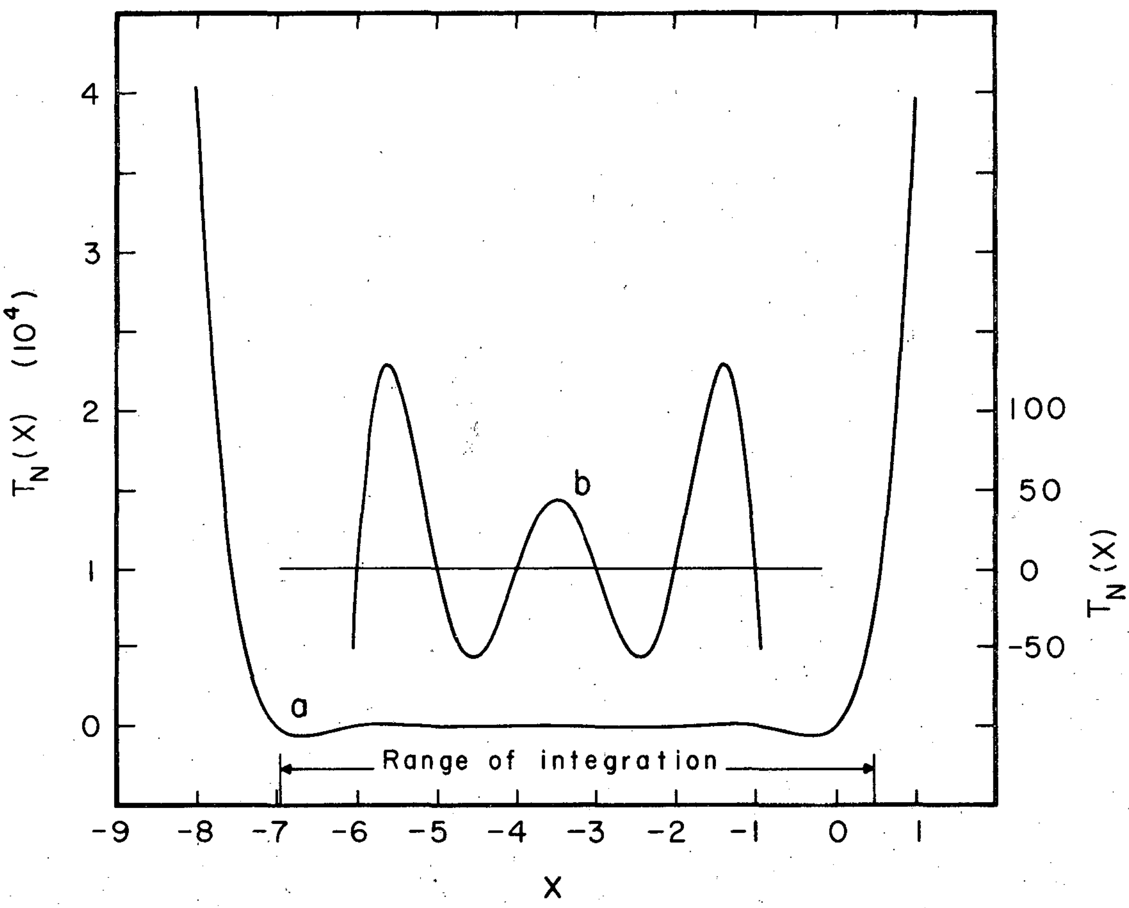

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Fig. 3.1.

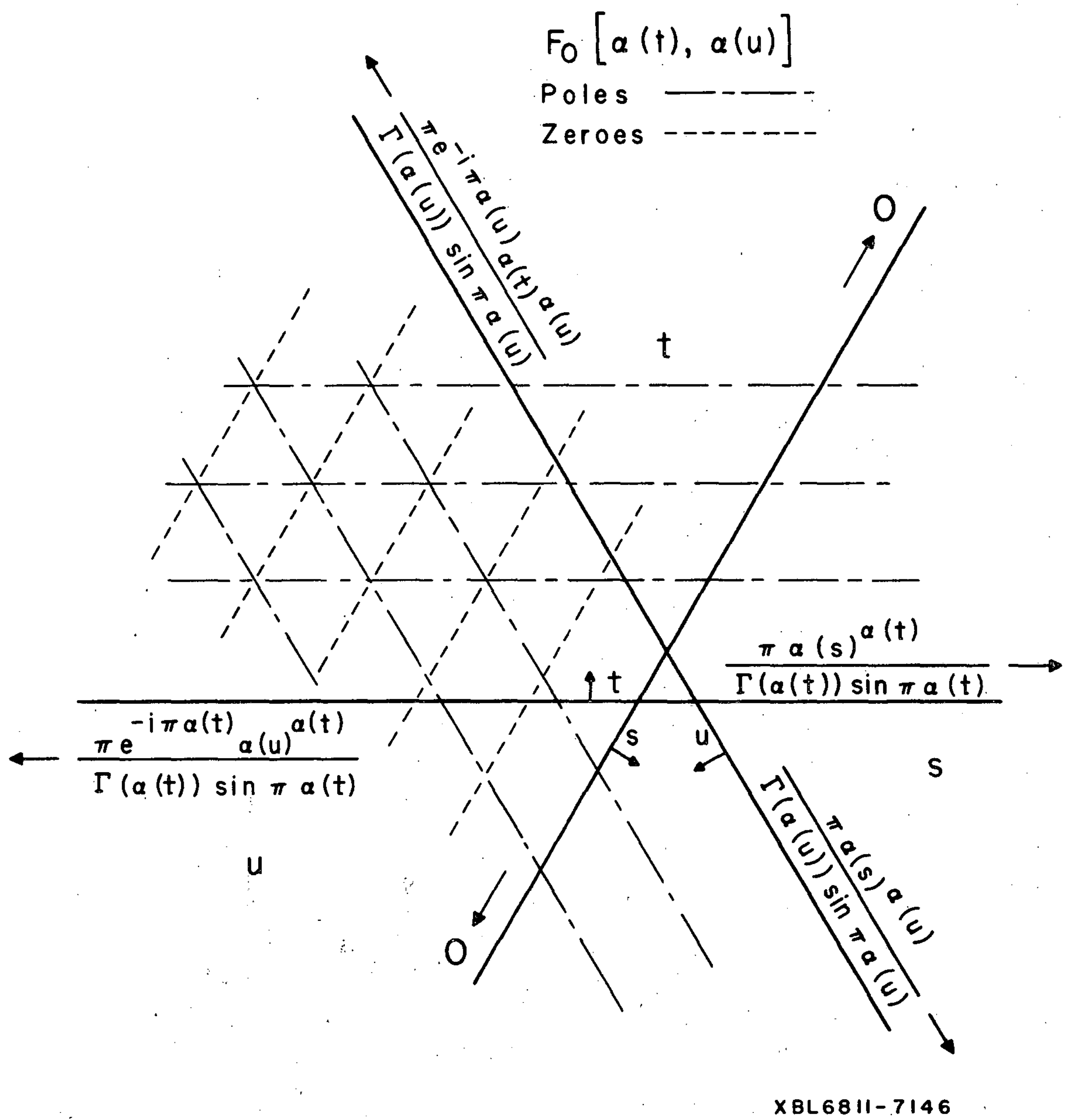
F0 [α(t), α(u)]
Poles
Zeroes
π e^{-iπα(u)} α(t)^{α(u)} / Γ(α(u)) sin π α(u)
O
t
π α(s)^{α(t)} / Γ(α(t)) sin π α(t)
t
s
u
π e^{-iπα(t)} α(u)^{α(t)} / Γ(α(t)) sin π α(t)
s
u
π α(s)^{α(u)} / Γ(α(u)) sin π α(u)
O
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Fig. 3.2.

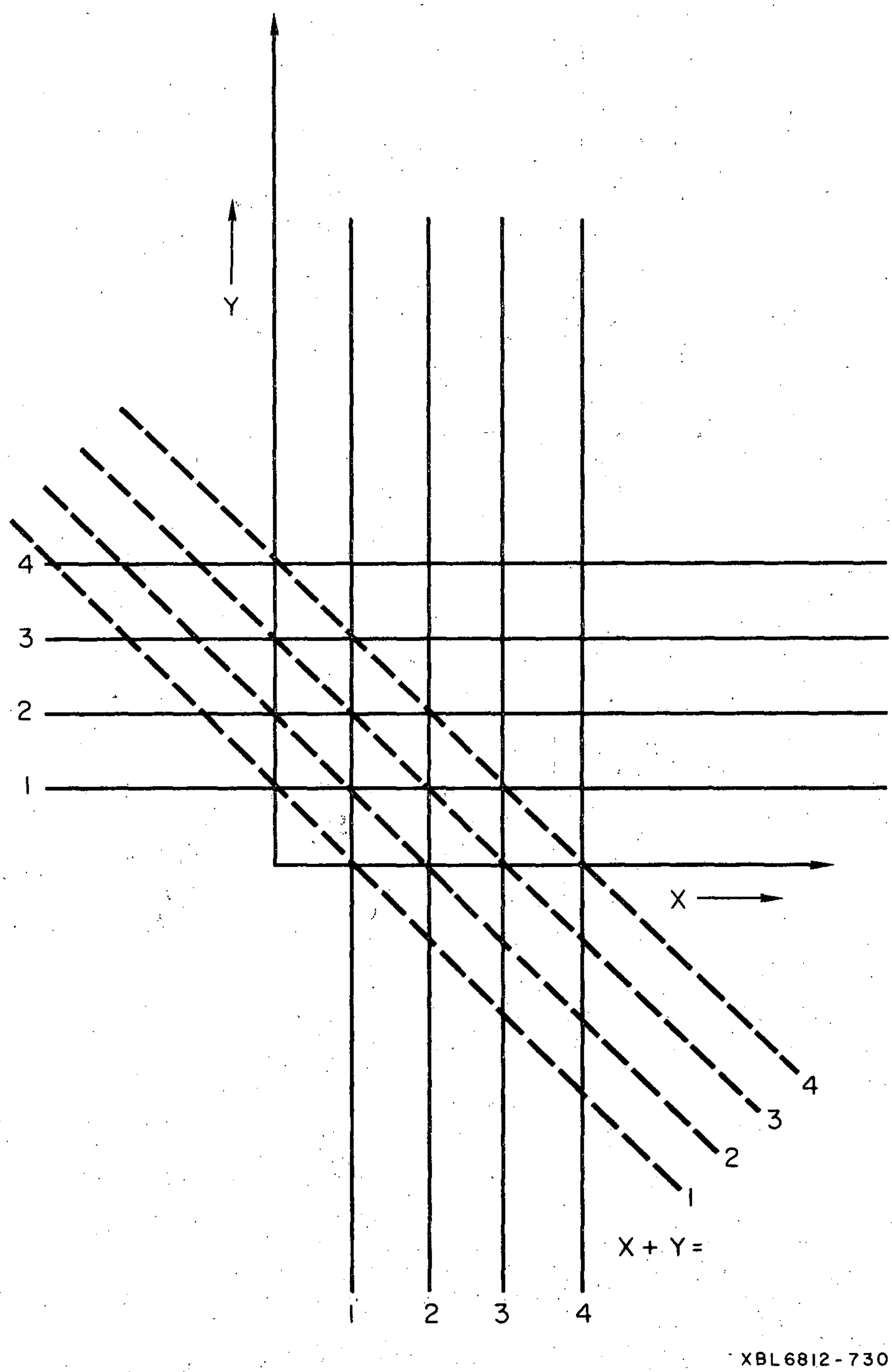

Fig. 3.3.

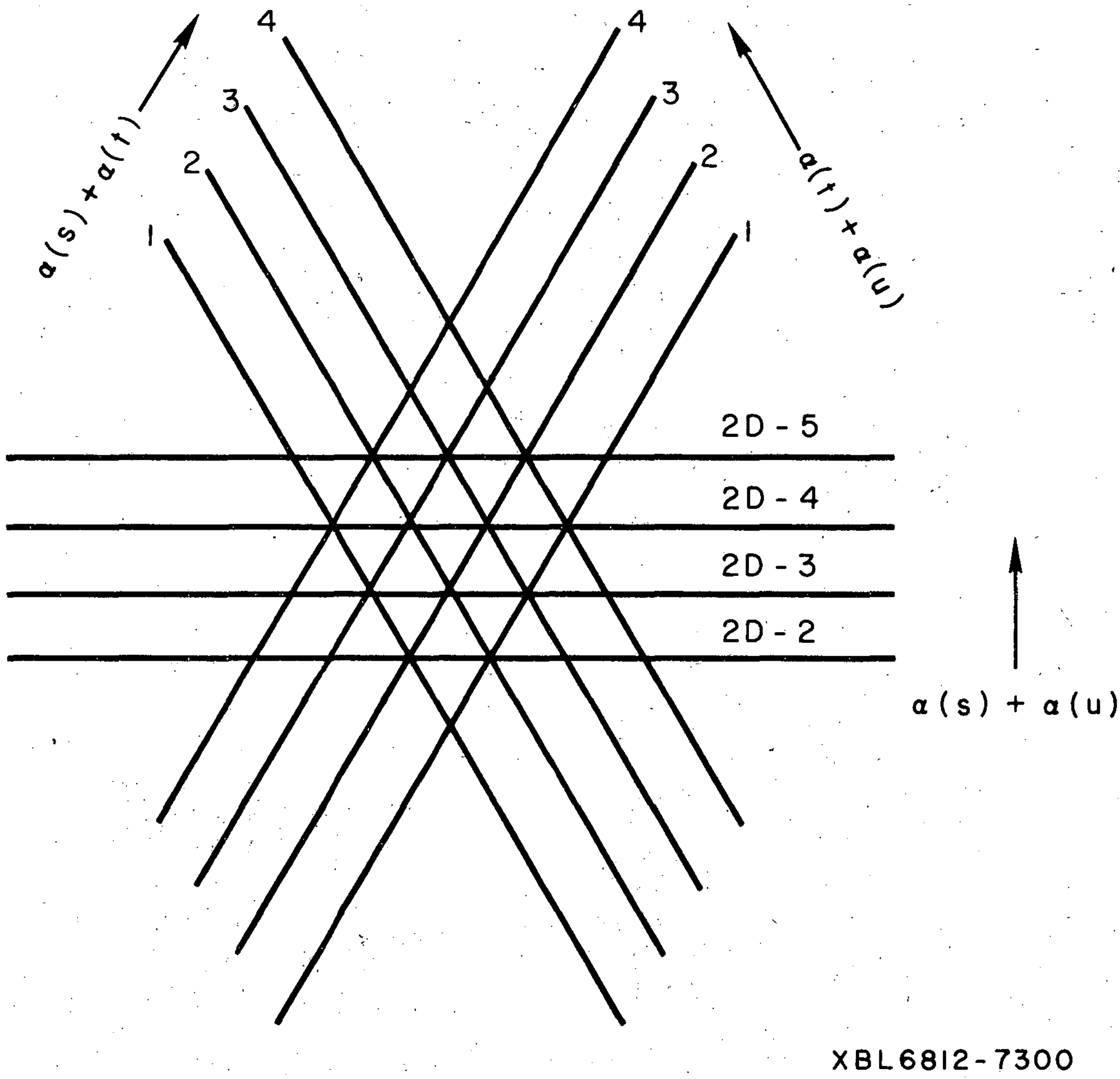

Fig. 3.4.

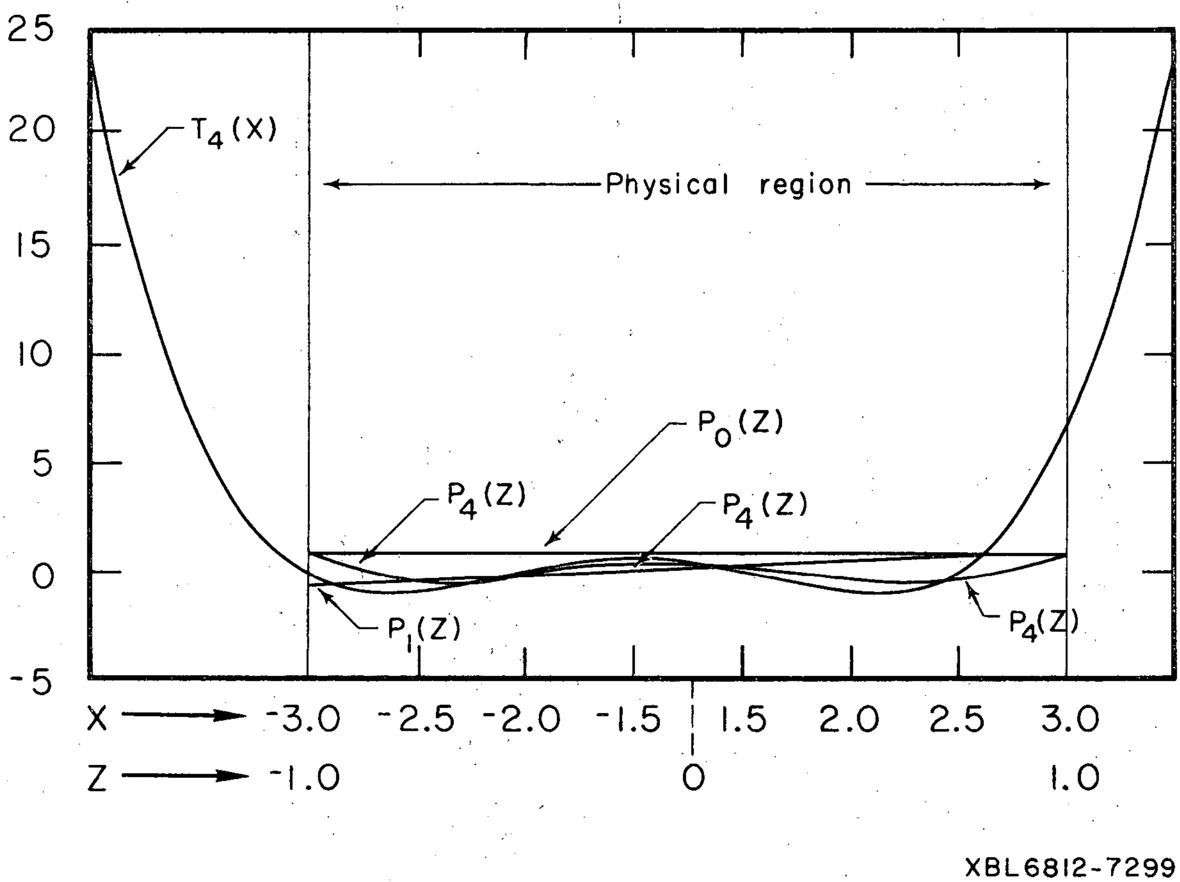
25
20
15
10
5
0
-5
T4(X)
Physical region
P0(Z)
P4(Z)
P4(Z)
P1(Z)
P4(Z)
X
-3.0 -2.5 -2.0 -1.5 1.5 2.0 2.5 3.0
Z
-1.0 0 1.0
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Fig. 3.5.

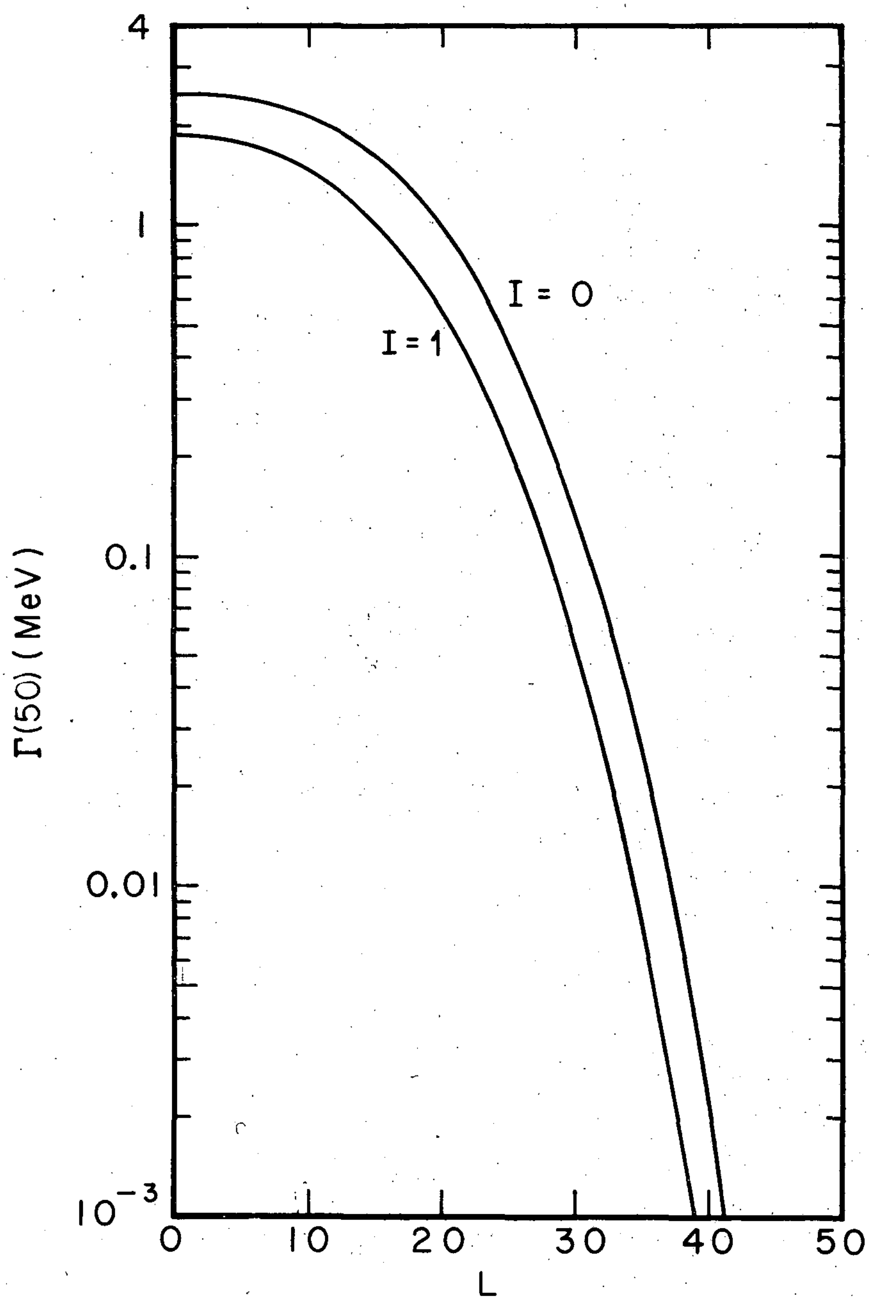

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Fig. 3.6.

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