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Twisted Graded Hecke Algebras for Elementary Abelian Groups

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ABSTRACT OF THE DISSERTATION

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Twisted graded Hecke algebras were introduced by S. Witherspoon in [18] as a common generalization of graded Hecke algebras and twisted symplectic reflection algebras. In this thesis, the structure and representation theory of twisted graded Hecke algebras for $(\mathbb{Z}/\ell\mathbb{Z})^n$ are studied. Such an algebra A_t is finitely generated as a module over its center. Moreover, for a generic central character χ, there exists a unique simple A_t-module on which the center acts by χ.

Contents

Chapter 1

Introduction

In this thesis, we study the twisted graded Hecke algebras associated to the group $(\mathbb{Z}/\ell Z)^{n-1}$. Twisted graded Hecke algebras were introduced by S. Witherspoon in [18] as a common generalization of the graded Hecke algebras of Drinfel'd [6] and Lusztig [11] [12], and the twisted symplectic reflection algebras studied by T. Chmutova [3]. The construction of a twisted graded Hecke algebra for a group G involves the choice of a two-cocycle for G, and the multiplication in G is twisted by this two-cocycle. Part of the motivation for the introduction of twisted graded Hecke algebras by Withersopoon was the appearance of discrete torsion of orbifolds studied by physicists, for example in the paper [5], by Douglas and Fiol.

The main results obtained in this thesis are as follows:

- Let $G = (\mathbb{Z}/\ell\mathbb{Z})^{n-1}$ and A_t the twisted graded Hecke algebra associated to G. We construct an embedding Θ of A_t into the crossed product algebra

  $$\mathbb{C}[X_1^{\pm 1} \dots X_n^{\pm 1}]\#_\alpha G.$$

  This is somewhat analogous to the Dunkl embedding for rational Cherednik algebras, but does not involve differential-difference operators. The embedding Θ

gives a new proof that the algebra A_t has a Poincaré-Birkhoff-Witt (PBW) basis. This embedding in the case $n = 3$ is essentially in the paper [5], although they do not define the algebra A_t.

- We determine the center Z_t of A_t by showing that it is generated by a set of elements $x_1^\ell, \ldots, x_n^\ell$ and w_t, where w_t is an element whose highest degree term is $x_1 \cdots x_n$. In the case $n = 3$ and $\ell = 2$, the element w_t was found in [4], Lemma 7.1. We generalize their formula to any n and ℓ.

- The algebra A_t is finite as a module over its center Z_t, and hence it is a Polynomial Identity (PI) algebra. Moreover, for a generic central character χ, there exists a unique simple A_t-module on which the center Z_t acts via χ. In the case $n = 3$, we give an explicit construction of this simple module for generic parameters t and central character χ. More specifically, we show that the algebra has PI-degree ℓ^2 and we show that, for an idempotent e in A_t and a lifting of χ to a character $\tilde{\chi}$ of $Z_t[x_3]$, the module $A_t e \otimes_{Z_t[x_3]} \mathbb{C}_{\tilde{\chi}}$ is the unique simple module with central character χ for generic t and χ.

- We show that, for $n = 3$, the algebra A_t is Morita equivalent to a deformed Sklyanin algebra studied by C. Walton in [17]. (Our results on the center and representation theory of A_t in this case do not appear to be known for the corresponding deformed Sklyanin algebras.)

The thesis is organized as follows. In chapter 2, we introduce the main object of study, the algebra A_t, which is a quotient of the twisted crossed product algebra $TV\#_\alpha G$ by certain inhomogeneous relations of degree 2. The algebra is shown to be a PBW-deformation of the twisted crossed product algebra $SV\#_\alpha G$. In chapter 3, the center $Z_t := Z(A_t)$ of A_t is determined. In chapter 4, the representation theory of A_t

is considered for $n = 3$ and any ℓ. Finally, it is shown that the algebra A_t is Morita equivalent to a deformed Sklyanin algebra.

Chapter 2

Twisted Graded Hecke Algebras

2.1 Definitions

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. We shall work over the field $\mathbb{C}$ of complex numbers unless otherwise stated; thus, $\otimes$ means $\otimes_{\mathbb{C}}$. Let V be a finite dimensional complex vector space. Denote the tensor algebra of V as TV, the symmetric algebra as SV. Let G be a finite subgroup of $\mathrm{GL}(V)$. Denote the group algebra of G by $\mathbb{C}G$. Recall that G and thus $\mathbb{C}G$, act on the tensor and symmetric algebras, and these actions are induced by the action of G on V. Let $\alpha : G \times G \to \mathbb{C}$ be a function and define the associative algebra $TV\#_\alpha G$ to be $TV \otimes \mathbb{C}G$ as a vector space with multiplication given by

$$(r \otimes g)(s \otimes h) = \alpha(g, h)r(g.s) \otimes gh$$

for all $r, s \in TV$ and $g, h \in G$. We may analogously define $SV\#_\alpha G$. In either case, when the product is associative, this is called a *twisted crossed product algebra*. If $\alpha(g, h) = 1$ for all $g, h \in G$, then we have the ordinary crossed product. Associativity of the product forces the equation

$$\alpha(g,h)\alpha(gh,k) = \alpha(h,k)\alpha(g,hk) \quad \text{for all } g,h,k \in G.$$

In other words, α is a two-cocycle. The defining equation of a coboundary β is

$$\beta(g,h) = \gamma(g)\gamma(h)\gamma(gh)^{-1} \quad \text{for some } \gamma : G \to \mathbb{C}^{\times}$$

Identifying α with its cohomology class, the twisted crossed product algebra is well defined up to isomorphism. We assume that α is a *normalized* cocycle. This means that we always take a representative with the property

$$\alpha(1,g) = \alpha(g,1) = 1 \quad \text{for all } g \in G.$$

We may normalize a representative by multiplication with a coboundary.

The vector space $\mathbb{C}G$ sits inside the twisted crossed product via the isomorphism $\mathbb{C}G \to 1 \otimes \mathbb{C}G \hookrightarrow TV\#_{\alpha}G$. The multiplication in this vector space is not the ordinary group algebra multiplication, but the twisted group algebra multiplication. In this case, $\mathbb{C}G$ is denoted by $\mathbb{C}_{\alpha}G$. The product of g and h in $\mathbb{C}_{\alpha}G$ is $\alpha(g,h)gh$, where the element gh is obtained from the usual product of g and h in the group G.

For each $g \in G$, choose a skew form $\omega_g : V \times V \to \mathbb{C}$. Define

$$A = \frac{TV\#_{\alpha}G}{\langle x \otimes y - y \otimes x - \sum_{g \in G} \omega_g(x,y)g \rangle_{x,y \in V}}$$

The denominator here is the two sided ideal generated by all elements of the form given. Denote the canonical projection of $TV\#_{\alpha}G \twoheadrightarrow A$ by π. It is a filtered algebra homomorphism. A has a filtration given by assigning degree 1 to elements of V and degree 0 to elements of G. In other words

$$F_i(A) = \pi(\bigoplus_{r \le i} V^{\otimes r} \otimes \mathbb{C}G)$$

In the associated graded algebra, $xy - yx = 0$ for all $x, y \in V$, so there is a surjective map

$$SV\#_\alpha G \twoheadrightarrow \mathrm{gr}(A)$$

When this map is an isomorphism, we say that A has the PBW property, or that A is of PBW type (where PBW stands for *Poincaré-Birkhoff-Witt*). The PBW property gives us a basis for the crossed product algebra. If $\{v_1, \dots, v_n\}$ is a basis for V, then $\{v_1^{k_1} \dots v_n^{k_n} \otimes g\}$ is a basis for $TV\#_\alpha G$ where $g \in G$ and $k_1, \dots, k_n \in \mathbb{N}$. When A is of PBW type, we call A a *Twisted Graded Hecke Algebra*, abbreviated TGHA in the remainder of this paper.

The PBW property trivially holds when $\omega_g = 0$ for all $g \in G$. In other words, $SV\#_\alpha G$ is the trivial TGHA for a given V with a given G-action. Denote this algebra by A_0. If some set of skew forms containing at least one nonzero ω_g yields an algebra of PBW type, then we say that the algebra is a *PBW-deformation* of A_0.

Let $n \ge 3, \ell \ge 2, V = \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \cdots \oplus \mathbb{C}x_n \cong \mathbb{C}^n$, and ζ a primitive ℓ-th root of unity. Let $G \cong (\mathbb{Z}/\ell\mathbb{Z})^{n-1}$ be the multiplicative subgroup of $\mathrm{SL}_n(\mathbb{C})$ generated by

$$g_1 = \mathrm{diag}(\zeta, \zeta^{-1}, 1, 1, \dots, 1)$$

$$g_2 = \mathrm{diag}(1, \zeta, \zeta^{-1}, 1, \dots, 1)$$

$$g_3 = \mathrm{diag}(1, 1, \zeta, \zeta^{-1}, \dots, 1)$$

$$\vdots$$

$$g_{n-1} = \mathrm{diag}(1, 1, \dots, 1, \zeta, \zeta^{-1})$$

We write $g_n = g_1^{-1} \cdots g_{n-1}^{-1} = \mathrm{diag}(\zeta^{-1}, 1, \ldots, 1, 1, \zeta)$

Equivalently, G is the subgroup of $\mathrm{SL}_n(\mathbb{C})$ consisting of all diagonal matrices g such that $g^\ell = 1$.

It will be useful to have a notation for "distance modulo n". Let $\|i - j\|_n$ denote the smallest non-negative integer k such that either $i = j + k \mod n$ or $i + k = j \mod n$. For example, $2 \neq \|2 - 0\|_3 = 1$. Define the twisting two-cocycle by

$$\alpha(g_1^{i_1} \cdots g_{n-1}^{i_{n-1}}, g_1^{j_1} \cdots g_{n-1}^{j_{n-1}}) = \zeta^{-\sum_{k=1}^{n-2} i_k j_{k+1}}$$

A few elementary calculations show that the commutation relations for $\mathbb{C}_\alpha G$ are as follows:

$$\begin{aligned} g_{i+1} g_i &= \zeta g_i g_{i+1} \\ g_i g_j &= g_j g_i \qquad \text{for } \|i - j\|_n \neq 1 \end{aligned} \tag{2.1}$$

Definition 1. *Let $t = (t_1, \ldots, t_n) \in \mathbb{C}^n$ be an n-tuple of complex number parameters. Define the algebra A_t to be the quotient of $TV \#_\alpha G$ by the relations*

$$\begin{aligned} x_i x_{i+1} - x_{i+1} x_i &= t_i g_i \\ x_i x_j - x_j x_i &= 0 \qquad \textit{for } \|i - j\|_n \neq 1 \end{aligned} \tag{2.2}$$

where the indices are taken modulo n.

In particular, the following equations hold in A_t

$$\begin{aligned} g_i x_i &= \zeta x_i g_i \\ g_i x_{i+1} &= \zeta^{-1} x_{i+1} g_i \\ g_i x_j &= x_j g_i \qquad \text{when } j \neq i, i+1 \mod n \end{aligned} \tag{2.3}$$

For each i, let $\omega_{g_i} : V \times V \to \mathbb{C}$ be the skew form defined by

$$\omega_{g_i}(x_i, x_{i+1}) = t_i = -\omega_{g_i}(x_{i+1}, x_i)$$

and

$$\omega_{g_i}(x_j, x_k) = 0$$

for all other values of j and k. For all other $g \in G$, define ω_g to be identically zero. The algebra A_t may be written in the form that defines a TGHA, once the PBW property is shown:

$$A_t = \frac{TV\#_\alpha G}{\langle x_i \otimes x_j - x_j \otimes x_i - \sum_{g\in G} \omega_g(x_i, x_j)g\rangle_{x_i,x_j\in V}}$$

2.2 Embedding into $\mathbb{C}[X_1^{\pm 1} \dots X_n^{\pm 1}]\#_\alpha G$

In order to show that the PBW property does indeed hold, we will use the following theorem. Let $t_i' = \dfrac{t_i}{\zeta^{-1} - 1}$ for all i.

Theorem 2. *The map*

$$x_i \mapsto X_i + \frac{t_i' g_i}{X_{i+1}},$$

$$g_i \mapsto g_i,$$

where $i = 1, \dots, n$, extends to a unique algebra homomorphism

$$\Theta : A_t \to \mathbb{C}[X_1^{\pm 1} \dots X_n^{\pm 1}]\#_\alpha G.$$

Proof. The five equations below must hold

1. $\Theta(x_i x_{i+1} - x_{i+1} x_i) = \Theta(t_i g_i)$

2. $\Theta(x_i x_j - x_j x_i) = 0$ for $\|i - j\|_n \neq 1$

3. $\Theta(g_i x_i) = \Theta(\zeta x_i g_i)$

4. $\Theta(g_i x_{i+1}) = \Theta(\zeta^{-1} x_{i+1} g_i)$

5. $\Theta(g_i x_j) = \Theta(x_j g_i)$ for $j \neq i, i+1$

To do these calculations, use the following relations in $\mathbb{C}[X_1^{\pm 1} \dots X_n^{\pm 1}]\#_\alpha G$, which hold for all $1 \leq i \leq n$, where the indices are taken modulo n:

$$\begin{aligned} g_i X_i &= \zeta X_i g_i \\ g_i \frac{1}{X_i} &= \zeta^{-1} \frac{1}{X_i} g_i \\ g_i X_{i+1} &= \zeta^{-1} X_{i+1} g_i \\ g_i \frac{1}{X_{i+1}} &= \zeta \frac{1}{X_{i+1}} g_i \end{aligned} \tag{2.4}$$

Also keep in mind that $\frac{t_i' g_i}{X_{i+1}}$ means that g_i appears to the right: $\left(\frac{t_i'}{X_{i+1}}\right) g_i$.

1. Proof that $\Theta(x_i x_{i+1} - x_{i+1} x_i) = \Theta(t_i g_i)$.

$$\begin{aligned} &\Theta(x_i)\Theta(x_{i+1}) - \Theta(x_{i+1})\Theta(x_i) \\ = &\left(X_i + \frac{t_i' g_i}{X_{i+1}}\right)\left(X_{i+1} + \frac{t_{i+1}' g_{i+1}}{X_{i+2}}\right) - \left(X_{i+1} + \frac{t_{i+1}' g_{i+1}}{X_{i+2}}\right)\left(X_i + \frac{t_i' g_i}{X_{i+1}}\right) \end{aligned}$$

$$= X_i X_{i+1} + X_i \left(\frac{t'_{i+1} g_{i+1}}{X_{i+2}}\right) + \left(\frac{t'_i g_i}{X_{i+1}}\right) X_{i+1} + \left(\frac{t'_i g_i}{X_{i+1}}\right)\left(\frac{t'_{i+1} g_{i+1}}{X_{i+2}}\right)$$
$$- X_{i+1} X_i - X_{i+1}\left(\frac{t'_i g_i}{X_{i+1}}\right) - \left(\frac{t'_{i+1} g_{i+1}}{X_{i+2}}\right) X_i - \left(\frac{t'_{i+1} g_{i+1}}{X_{i+2}}\right)\left(\frac{t'_i g_i}{X_{i+1}}\right)$$

$$= t'_i \left(\frac{1}{X_{i+1}}\right) g_i X_{i+1} + t'_i t'_{i+1} \left(\frac{1}{X_{i+1}}\right)\left(\frac{1}{X_{i+2}}\right) g_i g_{i+1}$$
$$- t'_i X_{i+1} \left(\frac{1}{X_{i+1}}\right) g_i - \zeta^{-1} t'_{i+1} t'_i \left(\frac{1}{X_{i+2}}\right)\left(\frac{1}{X_{i+1}}\right) g_{i+1} g_i$$

$$= t'_i \left(\frac{1}{X_{i+1}}\right) g_i X_{i+1} - t'_i X_{i+1} \left(\frac{1}{X_{i+1}}\right) g_i$$
$$= \zeta^{-1} t'_i g_i - t'_i g_i$$
$$= (\zeta^{-1} - 1) t'_i g_i$$
$$= t_i g_i$$
$$= \Theta(t_i g_i)$$

2. Proof that $\Theta(x_i x_j - x_j x_i) = 0$ for $\|i - j\|_n \neq 1$:

$$\Theta(x_i)\Theta(x_j) - \Theta(x_j)\Theta(x_i)$$
$$= \left(X_i + \frac{t'_i g_i}{X_{i+1}}\right)\left(X_j + \frac{t'_j g_j}{X_{j+1}}\right) - \left(X_j + \frac{t'_j g_j}{X_{j+1}}\right)\left(X_i + \frac{t'_i g_i}{X_{i+1}}\right)$$
$$= X_i X_j + \frac{t'_i g_i}{X_{i+1}} X_j + X_i \frac{t'_j g_j}{X_{j+1}} + \frac{t'_i g_i}{X_{i+1}} \frac{t'_j g_j}{X_{j+1}} - X_j X_i - \frac{t'_j g_j}{X_{j+1}} X_i - X_j \frac{t'_i g_i}{X_{i+1}} - \frac{t'_j g_j}{X_{j+1}} \frac{t'_i g_i}{X_{i+1}}$$
$$= 0$$

3. Proof that $\Theta(g_i x_i) = \Theta(\zeta x_i g_i)$:

$$
\begin{aligned}
&\Theta(g_i x_i)\\
&= g_i \left(X_i + \frac{t'_i g_i}{X_{i+1}} \right)\\
&= g_i X_i + g_i \frac{t'_i g_i}{X_{i+1}}\\
&= \zeta X_i g_i + \zeta t'_i \left(\frac{1}{X_{i+1}} \right) g_i g_i\\
&= \zeta \left(X_i + \frac{t'_i g_i}{X_{i+1}} \right) g_i\\
&= \Theta(\zeta x_i g_i)
\end{aligned}
$$

4. Proof that $\Theta(g_i x_{i+1}) = \Theta(\zeta^{-1} x_{i+1} g_i)$:

$$
\begin{aligned}
&\Theta(g_i x_{i+1})\\
&= g_i \left(X_{i+1} + \frac{t'_{i+1} g_{i+1}}{X_{i+2}} \right)\\
&= g_i X_{i+1} + g_i \frac{t'_{i+1} g_{i+1}}{X_{i+2}}\\
&= \zeta^{-1} X_{i+1} g_i + t'_{i+1} g_i \left(\frac{1}{X_{i+2}} \right) g_{i+1}\\
&= \zeta^{-1} X_{i+1} g_i + t'_{i+1} \left(\frac{1}{X_{i+2}} \right) g_i g_{i+1}\\
&= \zeta^{-1} X_{i+1} g_i + \zeta^{-1} t'_{i+1} \left(\frac{1}{X_{i+2}} \right) g_{i+1} g_i\\
&= \zeta^{-1} \left(X_{i+1} + \frac{t'_{i+1} g_{i+1}}{X_{i+2}} \right) g_i\\
&= \Theta(\zeta^{-1} x_{i+1} g_i)
\end{aligned}
$$

5. Proof that $\Theta(g_i x_j) = \Theta(x_j g_i)$ for $j \neq i, i+1$:

$$
\begin{aligned}
&\Theta(g_i x_j) \\
&= g_i\left(X_j + \frac{t'_j g_j}{X_{j+1}}\right) \\
&= g_i X_j + g_i \frac{t'_j g_j}{X_{j+1}} \\
&= X_j g_i + \frac{t'_j g_j}{X_{j+1}} g_i \\
&= \left(X_j + \frac{t'_j g_j}{X_{j+1}}\right) g_i \\
&= \Theta(x_j g_i)
\end{aligned}
$$

This completes the proof that $\Theta : A_t \to \mathbb{C}[X_1^{\pm 1} \dots X_n^{\pm 1}]\#_\alpha G$ is a homomorphism of algebras. □

Theorem 3. *The homomorphism* Θ *is injective.*

Proof. Suppose there is a relation

$$\Theta(\sum_{k_1,\dots,k_n} a_{k_1,\dots,k_n} x_1^{k_1} \cdots x_n^{k_n} \otimes g) = 0.$$

The image of a monomial $x_1^{k_1} \cdots x_n^{k_n} \otimes g$ under Θ has one highest total degree term, which is $X_1^{k_1} \cdots X_n^{k_n} \otimes g$. Therefore, it suffices to consider only terms in the sum with highest total degree. Since the elements $X_1^{k_1} \cdots X_n^{k_n} \otimes g$ for $k_1, \dots, k_n \geq 0$ are linearly independent in $\mathbb{C}[X_1^{\pm 1} \dots X_n^{\pm 1}]\#_\alpha G$, we must have that $a_{k_1,\dots,k_n} = 0$ for all n-tuples $(k_1, \dots, k_n)$ in the sum. This shows that $\ker(\Theta) = 0$. □

Corollary 4. *A_t is a PBW-deformation of A_0.*

Proof. The elements $X_1^{k_1} \cdots X_n^{k_n} \otimes g$ for $g \in G$ and each $k_i \in \mathbb{N}$ are linearly independent in $\mathbb{C}[X_1^{\pm 1} \dots X_n^{\pm 1}]\#_\alpha G$. Since Θ is injective, it follows that $\{x_1^{k_1} \cdots x_n^{k_n} \otimes g\}$, where $g \in G$ and each $k_i \in \mathbb{N}$, is a linearly independent set in A_t. This is the PBW basis for A_t. □

Chapter 3

Center of A_t

3.1 Center of A_0

Let $Z_t := Z(A_t)$ denote the center of A_t and $Z_0 := Z(A_0)$ the center of A_0. Let

$$w_0 = x_1 x_2 \cdots x_n \in A_0. \tag{3.1}$$

Lemma 5. *The center of A_0 is*

$$Z_0 = SV^G = \mathbb{C}[x_1^\ell, x_2^\ell, \ldots, x_n^\ell, w_0].$$

Proof. The first step is to show $SV^G \subset Z_0$. Let $p \otimes g \in A_0$ and $q \in SV^G$. Then

$$(p \otimes g) \cdot q = \alpha(g, 1) \cdot p(g.q) \otimes g = pq \otimes g = \alpha(1, g) \cdot qp \otimes g = q \cdot (p \otimes g)$$

This shows that $SV^G \subset Z_0$. To show that $Z_0 \subset SV^G$, take an element $z \in Z_0$. We can write z in the form $\sum_i p_i \otimes g_i$ where $g_i \neq g_j$ for $i \neq j$. Suppose $g_j \neq 1$ for some j. Then there exists $v \in V$ such that $g_j.v \neq v$, and

$$\sum_i p_i v \otimes g_i = vz = zv = \sum_i p_i(g_i.v) \otimes g_i$$

so $g_i.v = v$ for all i, thus $p_j = 0$. It follows that $z \in SV$. We have $g.z = z$ for all $g \in G$ since $(g.z) \otimes g = gz = zg = z \otimes g$.

Finally, we show that $SV^G = \mathbb{C}[x_1^\ell, x_2^\ell, \ldots, x_n^\ell, w_0]$. Note that $w_0 = x_1 x_2 \cdots x_n$ is G-invariant, and x_i^ℓ is G-invariant for all i. If p is a monomial in $\mathbb{C}[x_1, x_2, \ldots, x_n]^G$, we can factor out any of these elements until we are left with $p = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$, where at least one $k_i = 0$, and each $k_i < \ell$. The only such element which is G-invariant is 1. For if $k_i = 0$, then unless $k_{i+1} = 0$, we have that g_i does not fix p. □

3.2 Center of A_t

Lemma 6. *The elements $\{x_i^\ell\}_{i=1}^n$ belong to Z_t.*

Proof. It is clear that $g_j x_i^\ell = x_i^\ell g_j$ for all j and that $x_j x_i^\ell = x_i^\ell x_j$ for all j except possibly $j = i+1, i-1$.

$$\begin{aligned}
&x_{i-1} x_i^\ell \\
&= x_i x_{i-1} x_i^{\ell-1} + [x_{i-1}, x_i] x_i^{\ell-1} \\
&= x_i^2 x_{i-1} x_i^{\ell-2} + x_i [x_{i-1}, x_i] x_i^{\ell-2} + t_{i-1} g_{i-1} x_i^{\ell-1} \\
&= \cdots \\
&= x_i^\ell x_{i-1} + \sum_{k=1}^{\ell} x_i^{k-1} t_{i-1} g_{i-1} x_i^{\ell-k} \\
&= x_i^\ell x_{i-1} + t_{i-1} \sum_{k=1}^{\ell} \zeta^{k-\ell} x_i^{\ell-1} g_{i-1}
\end{aligned}$$

$$= x_i^\ell x_{i-1} + t_{i-1} x_i^{\ell-1} g_{i-1} \left(\sum_{k=1}^{\ell} \zeta^{k-\ell} \right)$$

$$= x_i^\ell x_{i-1}$$

A similar argument holds for x_{i+1}. □

The homogeneous element $w_0 \in Z_0$ of degree n is not in Z_t in general. However, after an adjustment by several lower degree terms involving the parameter t, we can construct an element $w_t \in Z_t$ which is equal to w_0 when $t = 0$.

Example 7. *When $n = 3$ and $\ell = 2$, let*

$$w_t = x_1 x_2 x_3 - \frac{1}{2} (t_1 x_3 g_1 + t_2 x_1 g_2 + t_3 x_2 g_3)$$

It was shown in [4] that w_t is in the center of A_t.

Informally, for any $n \geq 3$ and $\ell \geq 2$, the element w_t which we will define below is just a linear combination of monomials, each of which can be obtained from w_0 by replacing zero or more occurrences of $x_i x_{i+1}$ with $[x_i, x_{i+1}] = t_i g_i$. More precisely, we make the following definitions:

$$I_k = \left\{ \quad (i_1, i_2, \ldots, i_k) \in \mathbb{N}^k \quad \middle| \quad \substack{1 \leq i_1 < i_2 < \cdots < i_k \leq n \\ i_{r+1 (\mathrm{mod}\ k)} \neq i_r + 1 (\mathrm{mod}\ n)} \quad \right\}$$

$$w_{(i_1, i_2, \ldots, i_k)} = \begin{cases} \left(\frac{1}{\zeta - 1}\right)^k x_1 x_2 \cdots \hat{x}_{i_1} \hat{x}_{i_1+1} \cdots \hat{x}_{i_2} \hat{x}_{i_2+1} \cdots \\ \cdots \hat{x}_{i_k} \hat{x}_{i_k+1} \cdots x_n & \text{if } i_k < n \\ \\ \zeta \left(\frac{1}{\zeta - 1}\right)^k x_2 \cdots \hat{x}_{i_1} \hat{x}_{i_1+1} \cdots \hat{x}_{i_2} \hat{x}_{i_2+1} \cdots \\ \cdots \hat{x}_{i_{k-1}} \hat{x}_{i_{k-1}+1} \cdots x_{n-1} & \text{if } i_k = n \end{cases}$$

$$t_{(i_1,i_2,\ldots,i_k)} = t_{i_1}t_{i_2}\cdots t_{i_k}$$

$$g_{(i_1,i_2,\ldots,i_k)} = g_{i_1}g_{i_2}\cdots g_{i_k}$$

$$I_0 = \{0\} \quad t_0 = 1 \quad g_0 = 1$$

Finally, w_t can be defined.

$$w_t = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\mathbf{i}\in I_k} t_{\mathbf{i}} w_{\mathbf{i}} g_{\mathbf{i}} \tag{3.2}$$

Lemma 8. *One has $w_t \in Z_t$*

Proof. The proof proceeds in two parts. Firstly, show that w_t commutes with g_j for each j, and secondly show that w_t commutes with x_j for each j.

Claim 9. *We show that $g_j w_t = w_t g_j$ for all $j \in \{1, \ldots, n\}$.*

Proof. It suffices to check that $g_j(w_{\mathbf{i}}g_{\mathbf{i}}) = (w_{\mathbf{i}}g_{\mathbf{i}})g_j$ for each $\mathbf{i} \in I_k$ for $k \in \{0, 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$.

It is clear that

$$g_j w_0 = w_0 g_j$$

$g_j(w_{\mathbf{i}}) = \zeta(w_{\mathbf{i}})g_j$ if $\mathbf{i}$ contains i_{j+1} and not i_{j-1}. In this case, $g_j(g_{\mathbf{i}}) = \zeta^{-1}(g_{\mathbf{i}})g_j$.

$g_j(w_{\mathbf{i}}) = \zeta^{-1}(w_{\mathbf{i}})g_j$ if $\mathbf{i}$ contains i_{j-1} and not i_{j+1}. In this case, $g_j(g_{\mathbf{i}}) = \zeta(g_{\mathbf{i}})g_j$.

$g_j(w_{\mathbf{i}}) = (w_{\mathbf{i}})g_j$ if $\mathbf{i}$ contains i_{j-1} and i_{j+1}. In this case, $g_j(g_{\mathbf{i}}) = (g_{\mathbf{i}})g_j$.

If $\mathbf{i}$ contains neither i_{j-1}, nor i_{j+1}, then it may contain i_j or not. In either case, $g_j(w_{\mathbf{i}}) = (w_{\mathbf{i}})g_j$ and $g_j(g_{\mathbf{i}}) = (g_{\mathbf{i}})g_j$.

Thus, in all cases $g_j(w_{\mathbf{i}}g_{\mathbf{i}}) = (w_{\mathbf{i}}g_{\mathbf{i}})g_j$.

□

Claim 10. *We show that $x_j w_t = w_t x_j$ for all $j \in \{1, \ldots, n\}$.*

Proof. Writing $x_j w_t$ and $w_t x_j$ in terms of the PBW basis will show that $x_j w_t - w_t x_j$ is zero. In order to do this, it is helpful to partition the set of terms of w_t in such a way that the sum over each set commutes with x_j. Consider the case where $j \in \{3, 4, \ldots, n-3, n-2\}$. To define the partition, first choose two terms of w_t.

$$s = t_{\mathbf{a}} w_{\mathbf{a}} g_{\mathbf{a}} \qquad \text{for } \mathbf{a} = (a_1, \ldots, a_{k_1}) \in I_{k_1}$$

$$q = t_{\mathbf{b}} w_{\mathbf{b}} g_{\mathbf{b}} \qquad \text{for } \mathbf{b} = (b_1, \ldots, b_{k_2}) \in I_{k_2}$$

Note that $w_{\mathbf{a}}$ and $w_{\mathbf{b}}$ are scalar multiples of $\mathbf{x}^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$ and $\mathbf{x}^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}$ respectively, where each μ_i and each ν_i is either zero or one. By using associativity of multiplication, $\mathbf{x}^\mu$ and $\mathbf{x}^\nu$ may be written as

$$\mathbf{x}^\mu = s_1 \left(x_{j-1}^{\mu_{j-1}} x_j^{\mu_j} x_{j+1}^{\mu_{j+1}} \right) s_2$$

$$\mathbf{x}^\nu = q_1 \left(x_{j-1}^{\nu_{j-1}} x_j^{\nu_j} x_{j+1}^{\nu_{j+1}} \right) q_2$$

Place s and q in the same equivalence class if $s_1 = q_1$ and $s_2 = q_2$. It is clear that this is an equivalence relation. The number of equivalence classes depends on n. If s is an element of an equivalence class E, then s is completely determined by $\left(x_{j-1}^{\mu_{j-1}} x_j^{\mu_j} x_{j+1}^{\mu_{j+1}} \right)$, and by whether g_{j-1} or g_j is a factor of s in the case that $\mu_{j-1} = \mu_j = \mu_{j+1} = 0$. For each of the seven possible elements $s \in E$, We rewrite $x_j s$ and $-s x_j$ in terms of the PBW basis, omitting all factors determined by E:

$$1 \begin{cases} x_j(x_{j-1}x_j x_{j+1}) = \underbrace{x_{j-1}x_j^2 x_{j+1}}_{a} - \zeta^{-1} \underbrace{t_{j-1} x_j x_{j+1} g_{j-1}}_{b} \\ -(x_{j-1}x_j x_{j+1}) x_j = - \underbrace{x_{j-1}x_j^2 x_{j+1}}_{a} + \underbrace{t_j x_{j-1} x_j g_j}_{c} \end{cases}$$

$$
2\begin{cases}
x_j\left(\frac{1}{\zeta-1}\right)(t_{j-2}x_jx_{j+1}g_{j-2}) = \left(\frac{1}{\zeta-1}\right)\underbrace{t_{j-2}x_j^2x_{j+1}g_{j-2}}_{d} \\
-\left(\frac{1}{\zeta-1}\right)(t_{j-2}x_jx_{j+1}g_{j-2})x_j = -\left(\frac{1}{\zeta-1}\right)\underbrace{t_{j-2}x_j^2x_{j+1}g_{j-2}}_{d} \\
\qquad\qquad + \left(\frac{1}{\zeta-1}\right)\underbrace{t_{j-2}t_jx_jg_{j-2}g_j}_{e}
\end{cases}
$$

$$
3\begin{cases}
x_j\left(\frac{1}{\zeta-1}\right)(t_{j-1}x_{j+1}g_{j-1}) = \left(\frac{1}{\zeta-1}\right)\underbrace{t_{j-1}x_jx_{j+1}g_{j-1}}_{b} \\
-\left(\frac{1}{\zeta-1}\right)(t_{j-1}x_{j+1}g_{j-1})x_j = -\zeta^{-1}\left(\frac{1}{\zeta-1}\right)\underbrace{t_{j-1}x_jx_{j+1}g_{j-1}}_{b} \\
\qquad\qquad + \left(\frac{1}{\zeta-1}\right)\underbrace{t_{j-1}t_jg_{j-1}g_j}_{f}
\end{cases}
$$

$$
4\begin{cases}
x_j\left(\frac{1}{\zeta-1}\right)(t_jx_{j-1}g_j) = \left(\frac{1}{\zeta-1}\right)\underbrace{t_jx_{j-1}x_jg_j}_{c} - \left(\frac{1}{\zeta-1}\right)\underbrace{t_{j-1}t_jg_{j-1}g_j}_{f} \\
-\left(\frac{1}{\zeta-1}\right)(t_jx_{j-1}g_j)x_j = -\zeta\left(\frac{1}{\zeta-1}\right)\underbrace{t_jx_{j-1}x_jg_j}_{c}
\end{cases}
$$

$$
5\begin{cases}
x_j\left(\frac{1}{\zeta-1}\right)(t_{j+1}x_{j-1}x_jg_{j+1}) = \left(\frac{1}{\zeta-1}\right)\underbrace{t_{j+1}x_{j-1}x_j^2g_{j+1}}_{g} \\
\qquad\qquad -\zeta^{-1}\left(\frac{1}{\zeta-1}\right)\underbrace{t_{j-1}t_{j+1}x_jg_{j-1}g_{j+1}}_{h} \\
-\left(\frac{1}{\zeta-1}\right)(t_{j+1}x_{j-1}x_jg_{j+1})x_j = -\left(\frac{1}{\zeta-1}\right)\underbrace{t_{j+1}x_{j-1}x_j^2g_{j+1}}_{g}
\end{cases}
$$

$$
6\begin{cases}
x_j\left(\frac{1}{\zeta-1}\right)^2(t_{j-2}t_jg_{j-2}g_j) = \left(\frac{1}{\zeta-1}\right)^2\underbrace{t_{j-2}t_jx_jg_{j-2}g_j}_{e} \\
-\left(\frac{1}{\zeta-1}\right)^2(t_{j-2}t_jg_{j-2}g_j)x_j = -\zeta\left(\frac{1}{\zeta-1}\right)^2\underbrace{t_{j-2}t_jx_jg_{j-2}g_j}_{e}
\end{cases}
$$

$$
7\begin{cases}
x_j\left(\frac{1}{\zeta-1}\right)^2(t_{j-1}t_{j+1}g_{j-1}g_{j+1}) = \left(\frac{1}{\zeta-1}\right)^2\underbrace{t_{j-1}t_{j+1}x_jg_{j-1}g_{j+1}}_{h} \\
-\left(\frac{1}{\zeta-1}\right)^2(t_{j-1}t_{j+1}g_{j-1}g_{j+1})x_j = -\zeta^{-1}\left(\frac{1}{\zeta-1}\right)^2\underbrace{t_{j-1}t_{j+1}x_jg_{j-1}g_{j+1}}_{h}
\end{cases}
$$

Note that the terms in (2) and (6) only occur in those equivalence classes having $\mu_{j-2} = 0$. No terms except "d" and "e" terms occur in (2) and (6), and all such terms are included. Similarly, the terms in (5) and (7) only occur if $\mu_{j+2} = 0$. Moreover, no terms except "g" and "h" terms occur in (5) and (7), and all such terms are included. Therefore, if the terms of the RHS sum to zero, then they sum to zero independently of whether x_{j-2} and x_{j+2} occur as factors of s. It is easy to check that the terms associated to each label, "a", "b", etc., sum to zero. Therefore, for $j \in \{3, 4, \ldots, n-3, n-2\}$, we have $w_t x_j = x_j w_t$.

The other cases follow from a certain change of basis. Since the defining relations for A_t depend only on the indices modulo n, There is an automorphism ϕ of A_t sending t_i to $t_{i+1 \pmod n}$, x_i to $x_{i+1 \pmod n}$, and g_i to $g_{i+1 \pmod n}$. Let w_t' be the image of w_t under this automorphism.

Claim 11. *We show that* $w_t = w_t'$

Proof. Let S denote the set of PBW basis elements with nonzero coefficient in w_t. For $k \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, let $c_k = \frac{1}{(\zeta-1)^k}$. Then, an arbitrary term of w_t is of the form

$$t_{\mathbf{i}} w_{\mathbf{i}} g_{\mathbf{i}} = \begin{cases} c_k s & \text{if } i_k < n \\ \zeta c_k s & \text{if } i_k = n \end{cases}$$

for some $s \in S$, $k \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, and $\mathbf{i} = (i_1, \ldots, i_k) \in I_k$. Make the following definition for such indices $\mathbf{i}$:

$$\mathbf{i} + 1 = \begin{cases} (i_1 + 1, i_2 + 1, \ldots, i_k + 1) & \text{if } i_k < n \\ (1, i_1 + 1, i_2 + 1, \ldots, i_{k-1} + 1) & \text{if } i_k = n \end{cases}$$

Then the automorphism can be described as

$$\phi(t_{\mathbf{i}} w_{\mathbf{i}} g_{\mathbf{i}}) = \begin{cases} t_{\mathbf{i}+1} w_{\mathbf{i}+1} g_{\mathbf{i}+1} & \text{if } i_k < n \\ t_{\mathbf{i}+1} w_{\mathbf{i}+1} g_{\mathbf{i}+1} + (\text{ lower degree terms }) & \text{if } i_k = n \end{cases}$$

Note that $\phi(t_{\mathbf{i}}) = t_{\mathbf{i}+1}$ and $\phi(g_{\mathbf{i}}) = g_{\mathbf{i}+1}$ even if $i_k = n$, since $g_i g_j = g_j g_i$ for $\|i-j\|_n \neq 1$.

If i is an entry in $\mathbf{i}$, then we say that x_i and x_{i+1} are "paired away" in $t_{\mathbf{i}} w_{\mathbf{i}} g_{\mathbf{i}}$. We must use caution with this terminology, because if $\mathbf{i}$ contains $i-1$ and $i+1$, then $w_{\mathbf{i}}$ does not have a factor of $x_i x_{i+1}$, however, x_i and x_{i+1} are not paired away.

Let S_i be the subset of S consisting of all monomials whose corresponding term in w_t has x_i and x_{i+1} paired away. Write $S_{(i,\ldots,j)}$ for $S_i \cup \cdots \cup S_j$. Then the set of all monomials with a factor of $x_i x_{i+1}$ is exactly the set $S^c_{i-1,i,i+1}$. For each $i \in \{1, \ldots, n\}$, let $\psi_i : S^c_{i-1,i,i+1} \to S_i$ be the bijective map of sets that removes the factors x_i and x_{i+1}, and inserts the factors t_i and g_i in the PBW order. This map corresponds to pairing away x_i and x_{i+1} for terms of w_t, but it ignores coefficients. Partition S into the following subsets:

$$A_1 = S^c_{n-2} \cap S^c_{n-1,n} \cap S^c_1$$

$$A_2 = S^c_{n-2} \cap S^c_{n-1,n} \cap S_1$$

$$A_3 = S_{n-2} \cap S^c_{n-1,n} \cap S^c_1$$

$$A_4 = S_{n-2} \cap S^c_{n-1,n} \cap S_1$$

$$A_5 = S_{n-1} \cap S^c_1$$

$$A_6 = S_{n-1} \cap S_1$$

$$A_7 = S^c_{n-2} \cap S_n$$

$$A_8 = S_{n-2} \cap S_n$$

It is easy to check that these are pairwise disjoint sets whose union is S. The first four sets are those including x_n as a factor, so after applying ϕ, the commutation relations of A_t apply to give several lower degree terms. Therefore, for a monomial s in any of the first four subsets of S, $\phi(s) \in A_t$ is not even in S. Write φ for the function that adds 1 modulo n to the indices, but rewrites the variables in the PBW order (as if all commutation relations were zero). Notice that ϕ and φ are equal functions on $S_{n-1} \cup S_n = \bigcup_{i=5}^{8} A_i$, because there is no x_1 at the end of the resulting monomial. Notice also that if $s \in S$, then the A_i that contains s is completely determined by which variables are present, so φ applied to this partition remains a partition. Finally, a monomial $s \in S$ has coefficient of the form $\zeta c_k = \frac{\zeta}{(\zeta-1)^k}$ in w_t if and only if $s \in A_7 \cup A_8 = S_n = \phi(A_5 \cup A_6)$. If $s \in \bigcup_{i=1}^{6} A_i$, its coefficient has the form $c_k = \frac{1}{(\zeta-1)^k}$.

Suppose $t_{\mathbf{i}} w_{\mathbf{i}} g_{\mathbf{i}} = c_k s_1$, where $s_1 \in A_1$. then s_1 includes the factors x_1, x_{n-1} and x_n, and thus $\phi(s_1)$ includes the factors x_2, x_n and x_1, in that order. Therefore, when applying ϕ to w_t, we have

$$\begin{aligned}
\phi(c_k s_1) &= \phi(t_{\mathbf{i}} w_{\mathbf{i}} g_{\mathbf{i}}) = c_k t_{\mathbf{i}+1} x_2 \cdots x_n x_1 g_{\mathbf{i}+1} \\
&= c_k t_{\mathbf{i}+1} x_2 \cdots x_1 x_n g_{\mathbf{i}+1} + c_k t_{\mathbf{i}+1} x_2 \cdots \hat{x}_1 \hat{x}_n t_n g_n g_{\mathbf{i}+1} \\
&= c_k t_{\mathbf{i}+1} x_1 x_2 \cdots x_n g_{\mathbf{i}+1} + c_k t_{\mathbf{i}+1} x_2 \cdots \hat{x}_1 \hat{x}_n (t_n g_n) g_{\mathbf{i}+1} - c_k t_{\mathbf{i}+1} (t_1 g_1) \hat{x}_1 \hat{x}_2 \cdots x_n g_{\mathbf{i}+1} \\
&= t_{\mathbf{i}+1} w_{\mathbf{i}+1} g_{\mathbf{i}+1} + c_k \psi_n (t_{\mathbf{i}+1} x_1 x_2 \cdots x_n g_{\mathbf{i}+1}) - c_k \psi_1 (t_{\mathbf{i}+1} x_1 x_2 \cdots x_n g_{\mathbf{i}+1}) \\
&= c_k \varphi(s_1) + c_k \psi_n(\varphi(s_1)) - c_k \psi_1(\varphi(s_1))
\end{aligned}$$

The first term here, $c_k \varphi(s_1)$, is just another term of w_t. The next two terms come from commuting x_1 past x_n and x_2, respectively. If we remove the factor x_2, or x_n, or if we remove both from each monomial in $\varphi(A_1)$, we obtain the sets $\varphi(A_2), \varphi(A_3)$, and $\varphi(A_4)$, respectively. Therefore, taking s_2, s_3, and s_4 in the sets A_2, A_3, and A_4, respectively, we obtain

$$
\begin{aligned}
\phi(s_1) = &\quad \varphi(s_1) + \psi_n(\varphi(s_1)) &-\psi_1(\varphi(s_1))\\
\phi(s_2) = &\quad \varphi(s_2) + \psi_n(\varphi(s_2)) & \\
\phi(s_3) = &\quad \varphi(s_3) &-\psi_1(\varphi(s_3))\\
\phi(s_4) = &\quad \varphi(s_4) &
\end{aligned}
$$

For the first equation, note that $\psi_{n-1} : A_1 \to A_5$ and thus $\psi_n : \phi(A_1) \to \phi(A_5)$ are bijective maps which lower degree by 2. Suppose s_1, and thus $\varphi(s_1)$ each have coefficient c_k in w_t. If $\psi_{n-1}(s_1) = s_5 \in A_5$, then s_5 has coefficient c_{k+1} in w_t and $\phi(s_5) = \varphi(s_5) = \psi_n(\varphi(s_1))$ has coefficient ζc_{k+1} in w_t. Therefore, when applying ϕ to w_t, one portion reads

$$
\begin{aligned}
\phi(c_k s_1 + c_{k+1} s_5) &= c_k\varphi(s_1) + c_{k+1}\varphi(s_5) + c_k\psi_n(\varphi(s_1)) - c_k\psi_1(\varphi(s_1))\\
&= c_k\varphi(s_1) + (c_{k+1} + c_k)\varphi(s_5) - c_k\psi_1(\varphi(s_1))\\
&= c_k\varphi(s_1) + (\zeta c_{k+1})\,\varphi(s_5) - c_k\psi_1(\varphi(s_1))
\end{aligned}
$$

The second equation is similar, except that the useful correspondence is $\psi_n : \phi(A_2) \to \phi(A_6)$. For $\psi_n(\varphi(s_2)) = \phi(s_6) = \varphi(s_6)$, the analogous calculation is:

$$
\begin{aligned}
\phi(c_k s_2 + c_{k+1} s_6) &= c_k\varphi(s_2) + c_{k+1}\varphi(s_6) + c_k\psi_n(\varphi(s_2))\\
&= c_k\varphi(s_2) + (c_{k+1} + c_k)\varphi(s_6)\\
&= c_k\varphi(s_2) + (\zeta c_{k+1})\,\varphi(s_6)
\end{aligned}
$$

So far, every monomial of the form $\varphi(A_1 \cup A_2 \cup A_5 \cup A_6)$ is accounted for, and has the correct coefficient by the equation $c_{k+1} + c_k = \zeta c_{k+1}$. In the same way, use the bijections $\psi_n : A_1 \to A_7$ and $\psi_n : A_3 \to A_8$ to account for other terms, using the same equation in this form: $\zeta c_{k+1} - c_k = c_{k+1}$. The set A_4 stands alone permuting terms of w_t under the automorphism ϕ. Therefore, $\phi(w_t) = w_t'$ is equal to w_t, proving claim (11). □

In turn, the proof of this claim proves claim (10), because it shows that w_t commutes with all x_j for $j \in \{1, \dots, n\}$. □

The proof that $w_t \in Z_t$ is now therefore complete. □

We are now able to state and prove the main theorem of chapter 3:

Theorem 12. $Z_t = \mathbb{C}[x_1^\ell, \dots, x_n^\ell, w_t]$

It remains to show that $Z_t \subset \mathbb{C}[x_1^\ell, \dots, x_n^\ell, w_t]$. In order to do this we use an associated graded argument. Note that the filtration on A_t induces a filtration on Z_t. Denote both filtrations by $F_\bullet$. We say p has filtration degree k if $p \in F_k A_t$, but $p \notin F_{k-1} A_t$. Denote the symbol map for A_t by $\sigma : A_t \twoheadrightarrow \mathrm{gr}(A_t)$, and write σ for its restriction to Z_t also. The map σ takes an element q of filtration degree m to $\sigma_m(q)$, which is defined to be the image of q under the natural projection map $F_m \twoheadrightarrow F_m A_t / F_{m-1} A_t \hookrightarrow \mathrm{gr}(A_t)$. The symbol map is multiplicative, but not additive. Let $q \in A_t$, and let $p \in Z_t$ have filtration degree k.

$$\sigma(p) \cdot \sigma(q) = \sigma(p \cdot q) = \sigma(q \cdot p) = \sigma(q) \cdot \sigma(p) \tag{3.3}$$

Thus $\sigma(p) = \sigma_k(p) \in F_k A_t / F_{k-1} A_t$ is in the center of $\mathrm{gr}(A_t)$, which is just A_0 by the PBW property.

Let M_k be the set of all n-tuples of non-negative integers $\mu = (\mu_1, \dots, \mu_{n+1})$ such that $n\mu_{n+1} + \sum_{i=1}^n \ell\mu_i = k$. Since $Z_0 = \mathbb{C}[x_1^\ell, \dots, x_n^\ell, w_0]$, we can choose a representative ψ of $\sigma_k(p)$ in $F_k A_t$ as follows:

$$\psi = \sum_{\mu \in M_k} \nu_\mu x_1^{\ell\mu_1} \cdots x_n^{\ell\mu_n} w_0^{\mu_{n+1}}$$

where $\{\nu_\mu\}_{\mu \in M_k}$ is a set of complex number coefficients. Let

$$\tilde{\psi} = \sum_{\mu \in M_k} \nu_\mu x_1^{\ell\mu_1} \cdots x_n^{\ell\mu_n} w_t^{\mu_{n+1}}$$

Then $\tilde{\psi} \in Z_t$ and $\sigma(\tilde{\psi}) = \psi$. So $p - \tilde{\psi} \in Z_t$ and it has filtration degree strictly less than k. By induction on the filtration degree, p is a polynomial in $x_1^\ell \cdots x_n^\ell$, and w_t.

Chapter 4

Representation theory of A_t for $n = 3$

4.1 PI Algebras

For the convenience of the reader, we collect here some of the definitions and results on PI algebras which we shall use; more details may be found in [2]. Let $\mathbb{Z}\langle x_1, \ldots, x_n\rangle = \mathbb{Z}\langle \mathbf{x}\rangle$ denote the free algebra generated by $x_1, \ldots, x_n$. A ring R **satisfies** $f \in \mathbb{Z}\langle \mathbf{x}\rangle$ if $f(r_1, \ldots, r_n) = 0 \quad$ for all $r_i \in R$. In this case we call f a polynomial identity of R. The ring R is a **Polynomial Identity ring** or **PI** ring if it satisfies f for some monic f. (The function f is **monic** if at least one of its highest degree monomials has coefficient 1.)

Definition: The **minimal degree** of a PI ring R is the least possible degree of a polynomial identity f. If R is commutative, $f(r_1, r_2) = r_1r_2 - r_2r_1 = 0$ for all $r_1, r_2 \in R$, so commutative rings are examples of PI rings with minimal degree 2.

The following theorem, due to Posner, applies to rings that are prime. A ring

R is **prime** if $a, b \in R$ and $arb = 0$ for all $r \in R$ implies either $a = 0$ or $b = 0$. For example, any integral domain is prime, and any simple ring is prime.

Theorem 13. *(Posner) If R is a prime PI ring with center Z and minimal degree d, let $S = Z \setminus \{0\}, Q = RS^{-1}$, and $F = ZS^{-1}$. Then d is even and $dim_F(Q) = \left(\frac{d}{2}\right)^2$.*

In this case, we define the **PI-degree** of R to be $\sqrt{\dim_F(Q)} = d/2$ and write PI-deg(R) to denote this number. Suppose PI-deg(R) $= n$. If R can be generated as a Z-module by t elements, then $\dim_F(Q) \leq t$. In other words, $n^2 \leq t$.

Proposition 14. *A_t is a prime PI algebra.*

Proof. Since A_t is a finitely generated module over its center, it is a PI algebra. Since the G-action on SV is faithful, the algebra A_0 is prime by ([15], Cor. 12.6). Since $\mathrm{gr}(A_t) = A_0$, it follows that A_t is prime. □

For the sake of generality, The results (15) through (18) below apply to any algebraically closed field k, though in the rest of this paper $\mathbb{C}$ will be used. Assume for these results that R is finitely generated as a k-algebra and that R is finitely generated as a module over its center, Z (so R is a PI algebra).

Lemma 15. *(Artin, Tate) Z is a finitely generated k-algebra, hence Noetherian.*

Proof. A proof can be found in ([16], Thm. 4.2.1). □

Proposition 16. *Any simple R-module M is finite dimensional.*

Proof. A proof can be found in ([2], Prop. III.1.1, part 4). □

Proposition 17. *Let R be a prime PI algebra with PI-deg(R) $= n$, and let M be a simple R-module. Then*

1. *$\text{ann}_R(M) \cap Z$ is a maximal ideal $\mathfrak{m}$ of Z, and every maximal ideal of Z arises in this way.*

2. *dim(M) $\leq n$*

3. *If dim(M) $= n$, then M is the unique simple R-module annihilated by $\mathfrak{m}$.*

Proof. The statement (1) is proved in ([2], Prop. III.1.1). Part (2) is proved in ([1], Prop. 3.1a). Lastly, part (3) follows from ([1], Prop. 3.1b), together with ([2], Prop. III.1.6). □

Let U be the set of $\mathfrak{m} \in \text{MaxSpec } Z$ such that there is a simple R-module M with $\dim(M) = n$ and $\mathfrak{m} \subset \text{ann}_R(M)$.

Theorem 18. *U is an open dense subset of MaxSpec(Z).*

The set U in Theorem (18) is known as the *Azumaya locus of R over Z*. The theorem states that for a generic central character χ of Z, there exists a unique simple R-module M with $\dim(M) = n$ and the center of R acts on M via χ. For a proof, consult ([2], III.1.7).

4.2 Generic simple modules for $n = 3$ and $t = 0$

From now on, we assume that $n = 3$. Moreover, we shall consider t_1, t_2, t_3 as variables. In other words, we now consider A_t as an algebra over $\mathbb{C}[t_1, t_2, t_3]$. Let us first examine the simple modules for $A_0 = SV\#_\alpha G$ in the case $n = 3$, so that $G \cong (\mathbb{Z}/\ell\mathbb{Z})^2$. The center of this algebra is $(SV)^G = \mathbb{C}[x_1^\ell, x_2^\ell, x_3^\ell, x_1x_2x_3]$. For a given central character $\chi : (SV)^G \to \mathbb{C}$, if

$$a = \chi(x_1^\ell),\ b = \chi(x_2^\ell),\ c = \chi(x_3^\ell),\ d = \chi(x_1x_2x_3),$$

then the equation $abc = d^\ell$ must hold. Conversely, for all $a, b, c, d \in \mathbb{C}$ satisfying this equation, there is some representation with central character χ. For the remainder of the section, fix a generic central character χ, and let $a, b, c, d \in \mathbb{C}$ be as above. Consider the following G-orbit in V:

$$\mathcal{O}_\chi = \{(x, y, z) \in V | x^\ell = a, y^\ell = b, z^\ell = c, xyz = d\}$$

It is clear that $\mathcal{O}_\chi$ is a non-empty G-stable subset of V on which the action of G is both transitive and free. Thus we may choose an identification of $\mathcal{O}_\chi$ with G. By writing each element of $\mathcal{O}_\chi$ as its corresponding element $g \in G$, the action of G on $\mathcal{O}_\chi$ becomes simply the action of G on itself by left translation. Let $\mathrm{Fun}(\mathcal{O}_\chi)$ denote the vector space of $\mathbb{C}$-valued functions on $\mathcal{O}_\chi$, and denote the characteristic function of $g \in \mathcal{O}_\chi$ by $e_g \in \mathrm{Fun}(\mathcal{O}_\chi)$. The set $\{e_g\}_{g \in \mathcal{O}_\chi}$ forms a basis for the vector space, which therefore has dimension $\ell^2 = |G|$. The G-action on $\mathcal{O}_\chi$ induces an action on $\mathrm{Fun}(\mathcal{O}_\chi)$ by $(g.f)(h) = f(g^{-1}h)$ for all $g \in G$, $f \in \mathrm{Fun}(\mathcal{O}_\chi)$, and $h \in \mathcal{O}_\chi$. In terms of basis elements e_h, the action is given by the formula $g.e_h = e_{gh}$ for all $g \in G$.

We define multiplication on the vector space $\mathrm{Fun}(\mathcal{O}_\chi)$ by pointwise multiplication of functions. Then it is easy to show that the map

$$\mathrm{res}_\chi : SV \to \ \mathrm{Fun}(\mathcal{O}_\chi) : f \mapsto \left(f|_{\mathcal{O}_\chi} : (x, y, z) \mapsto f(x, y, z)\right),$$

is a $\mathbb{C}G$-algebra homomorphism.

Claim 19. *The map res_χ is surjective, and factors through* $(\ker \chi)$*, the ideal in SV generated by* $\ker \chi$*. The dimension of $\frac{SV}{(ker \chi)}$ is therefore greater than or equal to $\ell^2 = |G|$.*

Proof. Let $f \in \ker \chi$. Since $f \in (SV)^G$, it is a polynomial in $x_1^\ell, x_2^\ell, x_3^\ell$, and $x_1x_2x_3$. Therefore, for any $(x, y, z) \in \mathcal{O}_\chi$, $f(x, y, z)$ is a polynomial in a, b, c, and d which is equal to $\chi(f)$. Since $\chi(f) = 0$, the function $\text{res}_\chi(f) : \mathcal{O}_\chi \to \mathbb{C}$ is the zero function. Moreover, any function on a finite subset of a vector space can be extended to a polynomial function on that vector space. Thus, we have a surjective map

$$\text{res}_\chi : \frac{SV}{(\ker\chi)} \twoheadrightarrow \text{Fun}(\mathcal{O}_\chi),$$

which shows that $\dim \frac{SV}{(\ker\chi)} \geq \ell^2$. □

Claim 20. *The domain $\frac{SV}{(ker\chi)}$ is spanned by less than or equal to ℓ^2 elements.*

Proof. Clearly, the set $\{x_1^{p_1}x_2^{p_2}x_3^{p_3} | 0 \leq p_1, p_2, p_3 < \ell\}$ of ℓ^3 monomials spans $\frac{SV}{(\ker\chi)}$. Any element $x_1^{p_1}x_2^{p_2}x_3^{p_3}$ from this spanning set may be written as

$$\frac{1}{\chi(x_1x_2x_3)^{\ell-p_3}}(x_1x_2x_3)^{\ell-p_3}x_1^{p_1}x_2^{p_2}x_3^{p_3} = \frac{\chi(x_3^\ell)}{\chi(x_1x_2x_3)^{\ell-p_3}}x_1^{p_1+\ell-p_3}x_2^{p_2+\ell-p_3}$$

If p_1 or p_2 are greater than or equal to p_3, then factor out x_1^ℓ or x_2^ℓ, respectively, to obtain the additional scalars $\chi(x_1^\ell)$ or $\chi(x_2^\ell)$. □

So in fact, $\frac{SV}{(\ker\chi)}$ is exactly of dimension ℓ^2 and res_χ gives a $\mathbb{C}G$-algebra isomorphism from $\frac{SV}{(\ker\chi)}$ to $\text{Fun}(\mathcal{O}_\chi)$. The simple modules of $SV\#_\alpha G$ for a generic central character χ are precisely the simple modules of $SV/(\ker\chi)\#_\alpha G \cong \text{Fun}(\mathcal{O}_\chi)\#_\alpha G$.

Claim 21. *There is a $\mathbb{C}$-algebra isomorphism $Fun(\mathcal{O}_\chi)\#_\alpha G \cong Mat_{|G|\times|G|}(\mathbb{C})$.*

Proof. We will need the converse part of the statement in Theorem 3.2.1 of [10], originally a statement of Wedderburn.

Theorem 22. *(Wedderburn). If $\mathscr{A}$ is any finite-dimensional simple algebra over $\mathbb{C}$ with unit, then there is a finite-dimensional complex vector space V such that $\mathscr{A} \cong End(V)$.*

The unit in $\mathrm{Fun}(\mathcal{O}_\chi)\#_\alpha G$ is $1\#_\alpha 1$. As vector spaces, one has

$$\mathrm{Fun}(\mathcal{O}_\chi)\#_\alpha G = \mathrm{Fun}(\mathcal{O}_\chi) \otimes_{\mathbb{C}} \mathbb{C}G,$$

so both have dimension $|G|^2 = \ell^4$. By Wedderburn's theorem above, it suffices to show that $\mathrm{Fun}(\mathcal{O}_\chi)\#_\alpha G$ is simple. Suppose I is a nonzero two-sided ideal of $\mathrm{Fun}(\mathcal{O}_\chi)\#_\alpha G$. Let $m = \sum_{k,h} a_{k,h} e_k \otimes h$ be a nonzero element in the ideal. Say $a_{i,g} \neq 0$. By replacing m with mg^{-1}, we can assume $a_{i,1} \neq 0$. Then

$$e_i m = \sum_h a_{i,h} e_i \otimes h$$

is a nonzero element in the ideal. We have,

$$e_i m e_i = \sum_h a_{i,h} e_i e_{hi} \otimes h = a_{i,1} e_i \otimes 1 \neq 0$$

is in the ideal. So $e_i \otimes 1 \in I$. But then $e_{h_1} \otimes h_2 \in I$ for any $h_1, h_2 \in G$, since $h_1 i^{-1}(e_i \otimes 1) h_1^{-1} i h_2 = e_{h_1} \otimes h_2$. This proves the claim.

□

The algebra $\mathrm{Mat}_{|G|\times|G|}(\mathbb{C})$ has a unique simple module, $\mathbb{C}^{|G|}$. So $\frac{SV}{(\ker \chi)}\#_\alpha G$ has a unique simple module of dimension $|G|$. Therefore, by proposition (17), the PI degree of A_t has to be greater than or equal to $|G| = \ell^2$.

4.3 Generic simple modules for $n = 3$

Let $\tilde{Z}_t$ be the subalgebra of A_t generated by Z_t and x_3. Since $x_3^\ell \in Z_t$, $\tilde{Z}_t$ is a finitely generated Z_t-module. Therefore, the Going Up theorem from commutative algebra implies that any homomorphism $\chi : Z_t \to \mathbb{C}$ can be extended to a homomorphism $\tilde{\chi} : \tilde{Z}_t \to \mathbb{C}$.

Let $M(t,\tilde{\chi}) = A_t e \otimes_{\tilde{Z}_t} \mathbb{C}_{\tilde{\chi}}$, where $\mathbb{C}_{\tilde{\chi}}$ is the one-dimensional $\tilde{Z}_t$-module with action given by $\tilde{\chi}$, and e is the idempotent defined by equation (4.1).

$$e = \frac{1}{\ell}\sum_{i=0}^{\ell-1} g_1^i \tag{4.1}$$

The centralizer of $\tilde{Z}_t$ in A_t contains e, so $A_t e$ is a right $\tilde{Z}_t$-module. If $\alpha_t \in A_t$, then write $\alpha_t e$ for the element $\alpha_t e \otimes 1 \in A_t e \otimes_{\tilde{Z}_t} \mathbb{C}_{\tilde{\chi}}$. Since $g_1 e = e$ and $x_3 e = e x_3$, $M(t,\tilde{\chi})$ is spanned by the monomials $x_1^i x_2^j g_2^p e$ for $0 \le i, j, p < \ell$.

Proposition 23. *For generic $(t,\chi), M(t,\tilde{\chi})$ is spanned as a vector space by the ℓ^2 elements $x_2^j g_2^p e$, for $0 \le j, p < \ell$.*

Proof. It must be shown that $x_1^i x_2^j g_2^p e$ can be written as a linear combination of elements of the form $x_2^{j'} g_2^{p'} e$. Say that a term is "o.k." if it is a scalar multiple of $x_2^{j'} g_2^{p'} e$ for some $0 \le j', p' < \ell$. The proof is by induction on i, the exponent of x_1. If $i = 0$, there is nothing to prove. Suppose $i \ge 1$, and assume by induction hypothesis that $x_1^{i'} x_2^j g_2^p e$ can be written as a sum of o.k. terms when $i' < i$.

$$
\begin{aligned}
&x_1^i x_2^j g_2^p e \\
&= \tilde{\chi}(x_3)^{-1}\tilde{\chi}(x_3) x_1^i x_2^j g_2^p e \\
&= \tilde{\chi}(x_3)^{-1} x_1^i x_2^j g_2^p x_3 e \\
&= \zeta^{-p}\tilde{\chi}(x_3)^{-1} x_1^i x_2^j x_3 g_2^p e \\
&= \zeta^{-p}\tilde{\chi}(x_3)^{-1} x_1^{i-1} x_2^{j-1}(x_1x_2x_3) g_2^p e + (\text{ o.k. terms }) \\
&= \zeta^{-p}\tilde{\chi}(x_3)^{-1} x_1^{i-1} x_2^{j-1}(w_t - \frac{t_1}{\zeta-1}x_3g_1 - \frac{t_2}{\zeta-1}x_1g_2 - \frac{t_3\zeta}{\zeta-1}x_2g_3) g_2^p e + (\text{ o.k. terms }) \\
&= \zeta^{-p}\tilde{\chi}(x_3)^{-1} \Bigg(x_1^{i-1}x_2^{j-1} w_t g_2^p e - \frac{t_1}{\zeta-1} x_1^{i-1}x_2^{j-1} x_3 g_1 g_2^p e \\
&- \frac{t_2}{\zeta-1} x_1^{i-1}x_2^{j-1} x_1 g_2 g_2^p e - \frac{t_3\zeta}{\zeta-1} x_1^{i-1}x_2^{j-1} x_2 g_3 g_2^p e \Bigg) + (\text{ o.k. terms }) \\
&= \left(\frac{-t_2}{\zeta-1}\right)\zeta^{-p}\tilde{\chi}(x_3)^{-1} x_1^i x_2^{j-1} g_2^{p+1} e + (\text{ o.k. terms })
\end{aligned}
$$

So we have, for all $0 \le j, p < \ell$,

$$
x_1^i x_2^j g_2^p e + \left(\frac{t_2}{\zeta-1}\right)\zeta^{-p}\tilde{\chi}(x_3)^{-1} x_1^i x_2^{j-1} g_2^{p+1} e = (\text{ o.k. terms })
$$

Or equivalently

$$
\zeta^p \tilde{\chi}(x_3) x_1^i x_2^j g_2^p e + \left(\frac{t_2}{\zeta-1}\right) x_1^i x_2^{j-1} g_2^{p+1} e = (\text{ o.k. terms }) \tag{4.2}
$$

Setting $j + p = \ell - 1$, we get a system of ℓ equations in ℓ unknowns. In matrix form:

$$
A \begin{bmatrix} x_1^i x_2^0 g_2^{\ell-1} e \\ x_1^i x_2^1 g_2^{\ell-2} e \\ \vdots \\ x_1^i x_2^{\ell-1} g_2^0 e \end{bmatrix} = \begin{bmatrix} (\text{ o.k. terms }) \\ (\text{ o.k. terms }) \\ \vdots \\ (\text{ o.k. terms }) \end{bmatrix}
$$

where A can be easily calculated as

$$\begin{bmatrix} \zeta^{-1}\tilde{\chi}(x_3x_2^\ell) & 0 & 0 & \cdots & \frac{t_2}{\zeta-1} \\ \frac{t_2}{\zeta-1} & \zeta^{-2}\tilde{\chi}(x_3) & 0 & \cdots & 0 \\ 0 & \frac{t_2}{\zeta-1} & \zeta^{-3}\tilde{\chi}(x_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{t_2}{\zeta-1} & \tilde{\chi}(x_3) \end{bmatrix}$$

If A is invertible, we may solve this system:

$$\begin{bmatrix} x_1^i x_2^0 g_2^{\ell-1} e \\ x_1^i x_2^1 g_2^{\ell-2} e \\ \vdots \\ x_1^i x_2^{\ell-1} g_2^0 e \end{bmatrix} = A^{-1} \begin{bmatrix} (\text{ o.k. terms }) \\ (\text{ o.k. terms }) \\ \vdots \\ (\text{ o.k. terms }) \end{bmatrix} \tag{4.3}$$

For an arbitrary monomial $x_1^i x_2^j g_2^p e$ (where $j + p$ is not necessarily equal to $\ell - 1$), choose the equation coming from row $j + 1$ of system (4.3), and multiply both sides by $g_2^{p-(\ell-1-j)}$ on the left. This may have a rescaling effect, but still allows us to write an arbitrary monomial with a factor of x_1^i as a linear combination of monomials with lower powers of x_1.

To see that A is invertible, calculate the determinant. Expanding along the first row gives:

$$\begin{aligned}
\det(A) &= \zeta^{-1}\tilde{\chi}(x_3x_2^\ell)\left(\prod_{k=2}^{\ell}\zeta^{-k}\tilde{\chi}(x_3)\right) + (-1)^{\ell-1}\frac{t_2}{\zeta-1}\left(\frac{t_2}{\zeta-1}\right)^{\ell-1} \\
&= \chi(x_2^\ell x_3^\ell)\left(\prod_{k=1}^{\ell}\zeta^{-k}\right) + (-1)^{\ell-1}\frac{t_2}{\zeta-1}\left(\frac{t_2}{\zeta-1}\right)^{\ell-1} \\
&= \zeta^{-\frac{\ell(\ell-1)}{2}}\chi(x_2^\ell x_3^\ell) + (-1)^{\ell-1}\left(\frac{t_2}{\zeta-1}\right)^{\ell} \\
&= (-1)^{\ell-1}\left[\chi(x_2^\ell x_3^\ell) + \left(\frac{t_2}{\zeta-1}\right)^{\ell}\right]
\end{aligned}$$

Since the determinant of A is nonzero for generic (t, χ), the proof is complete. □

Lemma 24. *The module $M(t, \tilde{\chi})$ is nonzero.*

Proof. Let $z_1, \ldots, z_r$ be a maximal set of $\tilde{Z}_t$-linearly independent elements in A_te, and $\alpha_1, \ldots, \alpha_s$ a set of generators of A_te as a $\tilde{Z}_t$-module. For each i such that $\alpha_i \notin \{z_1, \ldots, z_r\}$, there are coefficients $c_{ij}, d_i \in \tilde{Z}_t$, $d_i \neq 0$ with

$$d_i\alpha_i + c_{i1}z_1 + \cdots + c_{ir}z_r = 0 \tag{4.4}$$

For each i such that $\alpha_i \in \{z_1, \ldots, z_r\}$, set $d_i = 1$. Let $d = d_1 \cdots d_s \in \tilde{Z}_t$. Since $\tilde{Z}_t$ is a domain (its associated graded algebra is a domain), $d \neq 0$. Consider the localization.

$$\begin{aligned}
\bigoplus_{i=1}^{r}\tilde{Z}_t\left[d^{-1}\right] &\to A_te\otimes_{\tilde{Z}_t}\tilde{Z}_t\left[d^{-1}\right] \\
(k_1, \ldots, k_r) &\mapsto \sum_{i=1}^{r} z_i\otimes k_i
\end{aligned} \tag{4.5}$$

Claim 25. *The map in* (4.5) *is an isomorphism*

Proof. To show that this map is injective, choose an element $\sum_{i=1}^{r} z_i\otimes k_i$ from the image and suppose $\sum_{i=1}^{r} z_i\otimes k_i = 0$. Then, for all $1 \leq i \leq r$ and for $N \in \mathbb{Z}$ sufficiently large, $d^Nk_i \in \tilde{Z}_t$ and

$$0 = \sum_{i=1}^{r} z_i\otimes k_id^N = \sum_{i=1}^{r} z_i(k_id^N)\otimes 1,$$

so $\sum_{i=1}^{r}(k_i d^N)z_i = 0$. By linear independence, $k_i d^N = 0$ for all i, and so $k_i = 0$ for all i.

To show the map is surjective, choose $i \in \{1, \dots, s\}$ and note that it suffices to show $\alpha_i \otimes 1$ is in the image. Equation (4.4) says $d_i \alpha_i = \sum_{j=1}^{r}(-c_{ij})z_j$. Write λ_j for $\frac{d}{d_i}(-c_{ij}) \in \tilde{Z}_t$, so that $d\alpha_i = \sum_{j=1}^{r} \lambda_j z_j$. Consider the element

$$\left(\frac{\lambda_1}{d}, \frac{\lambda_2}{d}, \dots \frac{\lambda_r}{d}\right) \in \bigoplus_{i=1}^{r} \tilde{Z}_t \left[d^{-1}\right].$$

The image of this element is

$$\sum_{j=1}^{r} z_j \otimes \frac{\lambda_j}{d} = \sum_{j=1}^{r} \lambda_j z_j \otimes \frac{1}{d} = d\alpha_i \otimes \frac{1}{d} = \alpha_i \otimes 1.$$

This completes the proof that (4.5) is an isomorphism. □

The remaining portion of the proof that $M(t, \tilde{\chi}) \neq 0$ will use the fact that, for generic χ, one has $\tilde{\chi}(d) \neq 0$. This follows from:

Claim 26. *If I is a prime ideal in the polynomial ring $\mathbb{C}[X_1, ..., X_n]$ and d is not contained in I, then $V(I)$ is not contained in $V(d)$.*

(Here, $V(I)$ is the zero-locus of I, and $V(d)$ is the zero-locus of d. The claim says that we can find a point on which I vanishes but d does not vanish.)

Proof. By the Hilbert Nullstellensatz, the ideal of all f in $\mathbb{C}[X_1, .., X_n]$ such that $V(f)$ contains $V(I)$ is equal to the radical $\sqrt{I}$ of the ideal I, where $\sqrt{I} = \{f$ such that sufficiently large powers of f are in $I\}$. Thus, if $V(I)$ is contained in $V(d)$, then some power of d has to be in I. But I is a prime ideal, so d is in I, a contradiction. □

Since $\tilde{\chi}(d) \neq 0$, we can extend $\tilde{\chi}$ from $\tilde{Z}_t$ to $\tilde{Z}_t[d^{-1}]$. In this way, $\mathbb{C}_{\tilde{\chi}}$ becomes a left $\tilde{Z}_t[d^{-1}]$-module, and when considered as such, it will be denoted $\mathbb{C}_{\hat{\chi}}$.

$$
\begin{aligned}
M(t,\tilde{\chi}) &= A_t e \otimes_{\tilde{Z}_t} \mathbb{C}_{\tilde{\chi}} \\
&= A_t e \otimes_{\tilde{Z}_t} \tilde{Z}_t[d^{-1}] \otimes_{\tilde{Z}_t[d^{-1}]} \mathbb{C}_{\hat{\chi}} \\
&\cong \bigoplus_{k=1}^{r} \tilde{Z}_t[d^{-1}] \otimes_{\tilde{Z}_t[d^{-1}]} \mathbb{C}_{\hat{\chi}} \\
&= \bigoplus_{k=1}^{r} \mathbb{C}_{\hat{\chi}} \\
&\neq 0
\end{aligned}
$$

□

The following theorem is one of the main results of this thesis.

Theorem 27. *The PI-degree of A_t is ℓ^2. For generic (t,χ), $M(t,\tilde{\chi})$ is the unique simple A_t-module with central character χ.*

Proof. Since we have a nonzero module $M(t,\tilde{\chi})$ with dimension at most ℓ^2 for generic t and χ, it follows from Theorem (18) that the PI-degree of A_t is at most ℓ^2. On the other hand, we have shown in Section 4.2 that the PI-degree of A_t is at least ℓ^2. Therefore, the PI-degree of A_t is equal to ℓ^2, and $M(t,\tilde{\chi})$ must be the unique simple A_t-module with central character χ for generic (t,χ). □

4.4 Morita equivalence with deformed Sklyanin algebras

For a fixed ring R, one can form the category whose objects are all left R-modules, and whose morphisms are all of the R-module homomorphisms. Denote this category by $R-$mod. If there is another ring S, such that $R-$mod and $S-$mod are equivalent as categories, then we say that R and S are *Morita equivalent.* It is a well-known fact that if R is an associative algebra with an idempotent e satisfying $ReR = R$, then R and eRe are Morita equivalent. For a proof, see ([9], Cor 2.3.4)

Definition 28. *For $i = 1, 2, 3$, let $e_i, d_i \in \mathbb{C}$. Let $W = \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}z \cong \mathbb{C}^3$ be a 3-dimensional complex vector space. The deformed Sklyanin algebra S_{def} is the quotient of the tensor algebra TW by the relations*

$$\begin{aligned} ayz + bzy + cx^2 + d_1x + e_1 = 0 \\ azx + bxz + cy^2 + d_2y + e_2 = 0 \\ axy + byx + cz^2 + d_3z + e_3 = 0 \end{aligned} \tag{4.6}$$

where $(a, b, c) \in \mathbb{C}^3$ is generic.

Here, "generic" means that not all of $a, b,$ and c are cube roots of unity, and (a, b, c) does not lie on any coordinate axis, as in ([17], Def. IV.2).

Theorem 29. *Let $n = 3$ and A_t the twisted graded Hecke algebra with parameters $t_1, t_2, t_3 \in \mathbb{C}$. Then A_t is Morita equivalent to S_{def}, where the parameters for S_{def} are*

$$\begin{array}{lll} a = 1 & b = \zeta & c = 0 \\ d_1 = 0 & d_2 = 0 & d_3 = 0 \\ e_1 = -\zeta t_1 & e_2 = -\zeta t_2 & e_3 = -\zeta t_3 \end{array} \tag{4.7}$$

More specifically, for the idempotent element e defined in equation (4.1), we have $A_t e A_t = A_t$ and $eA_te \cong S_{def}$.

Proof. Let us first check that $A_t e A_t = A_t$. It suffices to show that 1 is a linear combination of elements of the form $g_2^j e g_2^k$ for $0 \leq j, k \leq \ell - 1$. This will also show that $\mathbb{C}_\alpha G e \mathbb{C}_\alpha G = \mathbb{C}_\alpha G$. Since $g_2 g_1 = \zeta g_1 g_2$, we have

$$g_2^j e g_2^k = g_2^j \left(\frac{1}{\ell} \sum_{i=0}^{\ell-1} g_1^i \right) g_2^k = \frac{1}{\ell} \sum_{i=0}^{\ell-1} \zeta^{ij} g_1^i g_2^{j+k} \tag{4.8}$$

Define $e_{j,k}$ to be equal to $g_2^j e g_2^k$. Then equation (4.8) implies $e_{j,-j} = \frac{1}{\ell}\sum_i \zeta^{ij} g_1^i$. Consider the following matrix equation:

$$\frac{1}{\ell}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta & \zeta^2 & \cdots & \zeta^{\ell-1} \\ 1 & \zeta^2 & \zeta^4 & \cdots & \zeta^{2(\ell-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^{\ell-1} & \zeta^{(\ell-1)2} & \cdots & \zeta^{(\ell-1)(\ell-1)} \end{bmatrix} \begin{bmatrix} 1 \\ g_1 \\ g_1^2 \\ \vdots \\ g_1^{\ell-1} \end{bmatrix} = \begin{bmatrix} e_{0,0} \\ e_{1,-1} \\ e_{2,-2} \\ \vdots \\ e_{\ell-1,1-\ell} \end{bmatrix} \tag{4.9}$$

By the Vandermonde formula, The determinant of this matrix is

$$\prod_{0\le i<j\le \ell-1} (\zeta^j - \zeta^i) \neq 0$$

Therefore, by solving the corresponding system of $\ell - 1$ simultaneous equations and using the first equation, 1 can be written as a linear combination of the elements $e_{j,-j} \in \mathbb{C}_\alpha G e \mathbb{C}_\alpha G$. This shows that $\mathbb{C}_\alpha G e \mathbb{C}_\alpha G = \mathbb{C}_\alpha G$ and that $A_t e A_t = A_t$.

As the next step, we show that $e\mathbb{C}_\alpha G e$ is one-dimensional. Any element of $e\mathbb{C}_\alpha G e$ can be written as a linear combination of the elements $e g_1^j g_2^k e$ for $0 \le j, k \le \ell - 1$. But $g_1 e = e = e g_1$. So we need only consider elements of the form $e g_2^k e$. However, we have that

$$e g_2^k e = \frac{e}{\ell}\left(\sum_{i=0}^{\ell-1} \zeta^{ik} g_1^i\right) g_2^k = \frac{e}{\ell}\left(\sum_{i=0}^{\ell-1} \zeta^{ik}\right) g_2^k = \begin{cases} e & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

Therefore $e\mathbb{C}_\alpha G e = \mathbb{C} e \cong \mathbb{C}$

Let

$$y_1 = x_1 g_2$$
$$y_2 = x_2 g_3 \tag{4.10}$$
$$y_3 = x_3 g_1$$

Let S be the subalgebra of A_t generated by y_1, y_2, y_3. Observe that g_1, g_2, and g_3 each commute with each of y_1, y_2, and y_3. For we have

$$g_1 y_1 = g_1 x_1 g_2 = \zeta x_1 g_1 g_2 = x_1 g_2 g_1$$
$$g_1 y_2 = g_1 x_2 g_3 = \zeta^{-1} x_2 g_1 g_3 = x_2 g_3 g_1 \tag{4.11}$$
$$g_1 y_3 = g_1 x_3 g_1 = x_3 g_1 g_1$$

The calculations for g_2 and g_3 are essentially the same.

Definition 30. *Let q be any complex number. For elements a_1 and a_2 of an algebra A, define the q-commutator of a_1 with a_2 to be $[a_1, a_2]_q = a_1 a_2 - q a_2 a_1$.*

Let us compute the ζ-commutation relations for the y_i, keeping in mind the ordinary commutation relations (2.1), (2.2), and (2.3) for A_t.

$$\begin{aligned}
&[y_1, y_2]_\zeta \\
&= x_1 g_2 x_2 g_3 - \zeta x_2 g_3 x_1 g_2 \\
&= \zeta x_1 x_2 g_2 g_3 - x_2 x_1 g_3 g_2 \\
&= x_1 x_2 g_3 g_2 - x_2 x_1 g_3 g_2 \\
&= (x_1 x_2 - x_2 x_1) g_3 g_2 \\
&= [x_1, x_2] \alpha(g_3, g_2) g_1^{-1} \\
&= \zeta t_1 g_1 g_1^{-1} \\
&= \zeta t_1
\end{aligned} \tag{4.12}$$

We can compute the other two $\zeta-$commutators similarly. The three resulting equations are:

$$\begin{aligned} [y_1, y_2]_\zeta &= \zeta t_1 \\ [y_2, y_3]_\zeta &= \zeta t_2 \\ [y_3, y_1]_\zeta &= \zeta t_3 \end{aligned} \tag{4.13}$$

These are exactly equations (4.6) with parameters set as in (4.7), if x, y, and z are set to y_3, y_1 and y_2, respectively. Therefore S is the deformed Sklyanin algebra S_{def}.

As shown in (4.11), g_1 and g_2 commute with the generators of S. As algebras, therefore, $A_t \cong S \otimes \mathbb{C}_\alpha G$. So $eA_te = S \otimes e\mathbb{C}_\alpha Ge \cong S$, since $e\mathbb{C}_\alpha Ge \cong \mathbb{C}$. This shows that eA_te is isomorphic to S_{def}.

□

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