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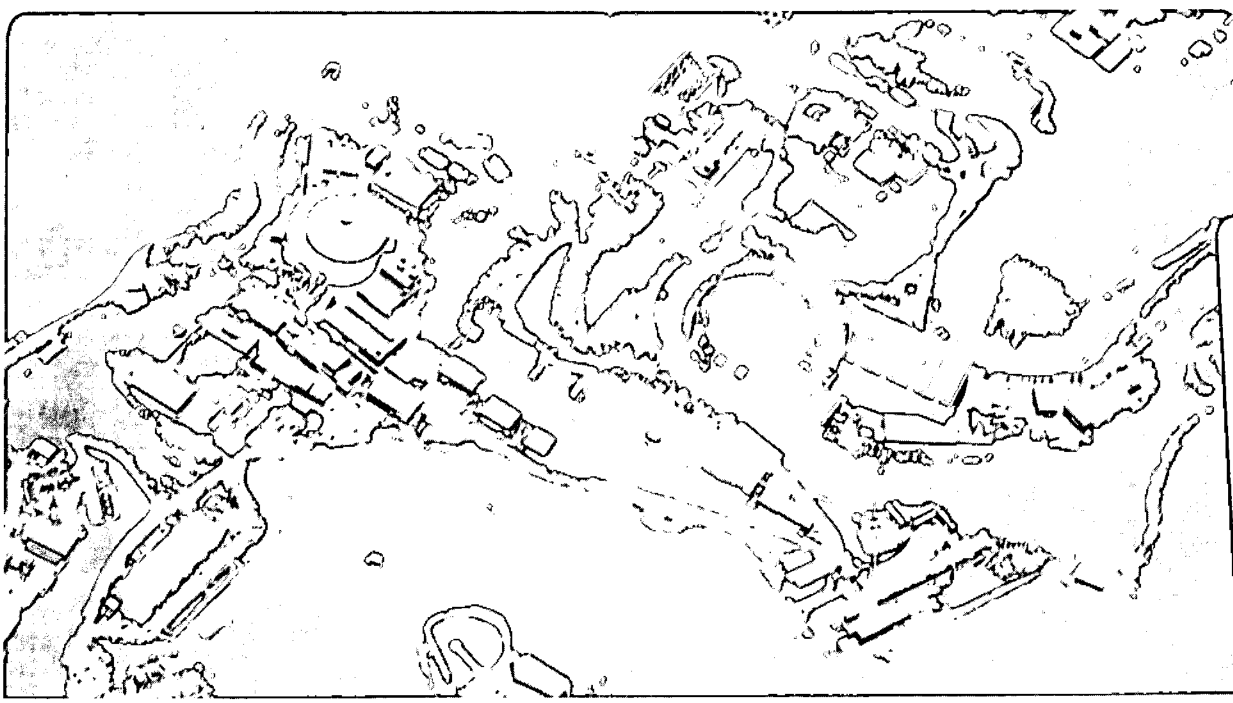

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A NOTE ON ANALYTIC RECOVERY OF TRANSITION PROBABILITIES IN THREE DIMENSIONAL DIFFUSE TOMOGRAPHY[1]

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A Note on Analytic Recovery of Transition Probabilities in Three Dimensional Diffuse Tomography

S. K. Patch

February 27, 1994

Abstract

A recovery algorithm is given for the $2 \times 2 \times 2$ problem in diffuse tomography. This three dimensional algorithm is computationally more complex and yields relatively more information than its two dimensional counterpart.

1 Introduction

The word "tomography" refers to imaging an object by slices. X rays, for example, have high energy and travel straight through the body. Data analysis is linear and yields a scalar valued function. The oxymoron "diffuse tomography" refers to low energy imaging in which the paths of the radiant energy are not necessarily straight and are *unknown*. Data analysis in diffuse tomography is highly nonlinear and yields a vector valued function. Problems in diffuse tomography are highly nonlinear because low energy is used. Clinical applications such as neonatal imaging and annual mammograms are not amenable to high energy techniques which might overexpose the patient to harmful radiation. Experimentalists in the medical arena are presently working with near infrared radiation; mathematicians have done preliminary mathematical analysis of diffuse tomographic methods in [2, 3, 4, 5, 9]. An analytic algorithm for recovering Markov transition probabilities from boundary value data for the smallest nontrivial problem in three dimensions is outlined in this paper. We begin with a brief description of the model.

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Consider an $n \times n \times n$ array of voxels in $\mathbb{R}^3$ enclosing the object to be reconstructed. On each of the $6n^2$ outer faces there are two devices. One device shoots photons across the outside face into the neighboring voxel; the other device detects photons as they leave the system. For each of the $6n^2$ outside faces we collect $6n^2$ pieces of data. Within the array, photons travel in six directions: $n$orth, $s$outh, $e$ast, $w$est, up, and $d$own. They may change direction as long as they travel in one of the six preferred directions. They do not interact and may be absorbed within a voxel. Photons move according to a two step Markov process. The probabilities with which a photon moves to a neighboring voxel depend upon its previous, as well as present, location. In this two step formulation the state space consists of locations. We may redefine the state space so that photons move according to a one step Markov process. In the new state space a single state consists of the photon's location and direction of travel.

There are three different types of these Markov states: incoming, outgoing, and hidden. The probabilities with which photons move from one state to another are referred to as transition probabilities. The transition matrix, M, is sparse and may be written as a block matrix with nontrivial subblocks which we refer to as P_{io}, P_{ih}, P_{ho}, and P_{hh}. P_{io}, for example, contains the probabilities with which photons in incoming states move directly to outgoing states. P_{ih} contains the probabilities with which photons in incoming states move to hidden states. P_{ho} and P_{hh} are the transition matrices for photons starting in hidden states travelling to outgoing and hidden states, respectively. P_{io} and P_{hh} are always square matrices. All four of these submatrices of M are sparse block matrices.

The data is written as a $6n^2 \times 6n^2$ data matrix, Q. $Q_{i,j}$ represents the probability that a photon which enters the system at source i exits the system at detector j. Q provides no time-of-flight information. The forward map we wish to invert is a function of the transition probabilities and equals Q. Given Q, we wish to recover the transition probabilities. For a given object the transition probabilities give a discretized "image" of the object. In traditional imaging, we recover a single parameter per voxel. From this information a visual picture of the object is made. In diffuse tomography, however, we want to recover many parameters per voxel. From this information we could make several "pictures" of the object.

2 Forward Problem

The one step Markov transition matrix, M, has a sparse block structure. Ordering the Markov states so that the incoming states precede the hidden states, which precede the outgoing states, gives M the following block structure:

$$M = \begin{bmatrix} 0 & P_{ih} & P_{io} \\ 0 & P_{hh} & P_{ho} \\ 0 & 0 & I \end{bmatrix} \tag{1}$$

Here $M_{i,j}$ equals the probability that a photon in state i moves directly to state j. P_{io}, P_{ih}, P_{hh}, and P_{ho} are one step transition matrices. They are sparse and their nonzero entries are transition

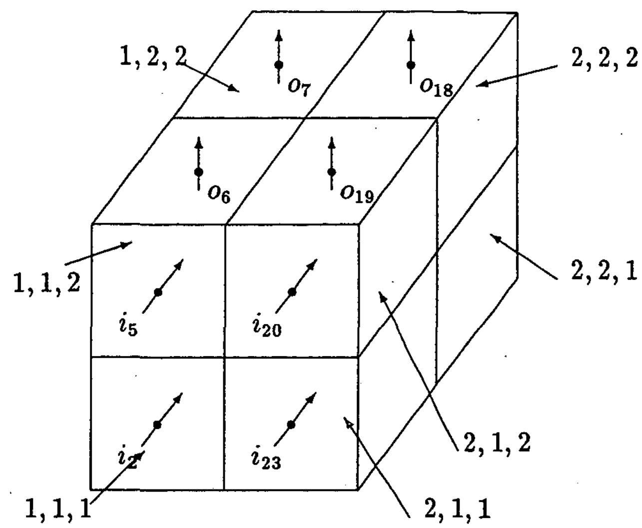

Figure 1: Eight voxels, seven of which are labelled above. Voxel 1,2,1 is hidden from view. Some incoming and outgoing states are labeled as well. A photon which travels north into voxel 112 via incoming state i_5 and then turns upward traveling out of voxel 112 via outgoing state o_6 does so with probability $n112u$.

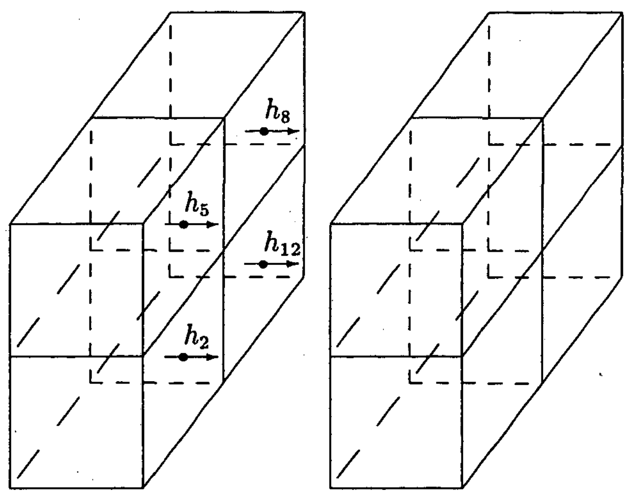

Figure 2: A $2 \times 2 \times 2$ system is split apart so that we can see a few hidden states representing trave from the "leftmost" voxels, 111, 121, 112, and 122, to the "rightmost" voxels, 211, 221, 212, and 222.

probabilities. For example, $P_{io[s,t]}$ = the probability of a photon moving from incoming state s directly to outgoing state t; $P_{ih[s,t]}$ = the probability of a photon moving from incoming state s directly to hidden state t. The one step transition matrices for the $2 \times 2 \times 2$ problem have very special block structures. In this case, P_{ho} and P_{io} have eight 3×3 blocks along their diagonals; P_{ih} and P_{hh} have identical zero structures with 3×3 off diagonal blocks.

Notation : The probability that the photon will travel east into pixel $1,1,1$ and continue east into pixel $2,1,1$ is written as $e111e$. The probability that it will turn right and travel out of the system is written as $e111s$ and the probability with which it turns upwards and travel into pixel $1,1,2$ is written as $e111n$.

Submatrices for other systems are not always square. For a $n \times n \times n$ system, there are $6n^2$ incoming and $6n^2$ outgoing states and $6n^3 - 6n^2 = 6n^2(n-1)$ hidden states. Hence for a $n \times n \times n$ problem, P_{io} is $6n^2 \times 6n^2$, P_{ih} is $6n^2 \times 6n^2(n-1)$, P_{hh} is $6n^2(n-1) \times 6n^2(n-1)$, and P_{ho} is $6n^2(n-1) \times 6n^2$. In k dimensions an $n \times n \times \ldots \times n$ system is made up of n^k k-dimensional cubes and has $2k$ large outer faces. Each of these large outer faces contaions $n^{(k-1)}$ faces of individual cubes. Therefore this system has $2kn^{(k-1)}$ incoming and $2kn^{(k-1)}$ outgoing states, and $2kn^k - 2kn^{(k-1)} = 2kn^{(k-1)}(n-1)$ hidden states. P_{io} for this system is $2kn^{(k-1)} \times 2kn^{(k-1)}$, P_{ih} is $2kn^{(k-1)} \times 2kn^{(k-1)}(n-1)$, P_{hh} is $2kn^{(k-1)}(n-1) \times 2kn^{(k-1)}(n-1)$, and P_{ho} is $2kn^{(k-1)}(n-1) \times 2kn^{(k-1)}$.

For $k \in \mathbb{N}$, the i,j entry of the k^{th} power of M is the probability that a photon starting in state i reaches state j after k Markov steps. If i is an incoming state and j is an outgoing state define $Q^k_{i,j}$ as the probability that a photon which entered the system in state i exits the system via state j during the first k transitions. $Q^k_{i,j}$ is the matrix in the upper right corner of M^k and is equal to

$$Q^k = P_{io} + P_{ih} \left(\sum_{n=0}^{k-2} P^n_{hh}\right) P_{ho} \tag{2}$$

Because we have no time-of-flight information the data we collect, Q, may be written as

$$Q = P_{io} + P_{ih} \left(\sum_{n=0}^{\infty} P^n_{hh}\right) P_{ho} = P_{io} + P_{ih} \left(I - P_{hh}\right)^{-1} P_{ho} \tag{3}$$

It is not difficult to show that the sum converges. We say that one solves the forward problem when one calculates Q from P_{io}, P_{hh}, P_{ho}, and P_{ih}. Let f denote the forward map given by 3, so $f(P_{io}, P_{ih}, P_{ho}, P_{hh}) = Q$. For any three dimensional system, there are 36 transition probabilities per voxel since a photon may enter a given voxel via any one of six states and may exit the voxel via six different states. Therefore, $36n^3$ of the entries in P_{io}, P_{hh}, P_{ho}, and P_{ih} are nonzero. The domain of the forward map lies in the unit cube in $\mathbb{R}^{36n^3}$ and satisfies the following conditions

$$
\begin{aligned}
eijke + eijkw + eijkn + eijks + eijku + eijkd &\leq 1 \\
uijke + uijkw + uijkn + uijks + uijku + uijkd &\leq 1 \\
dijke + dijkw + dijkn + dijks + dijku + dijkd &\leq 1 \\
wijke + wijkw + wijkn + wijks + wijku + wijkd &\leq 1 \\
nijke + nijkw + nijkn + nijks + nijku + nijkd &\leq 1 \\
sijke + sijkw + sijkn + sijks + sijku + sijkd &\leq 1
\end{aligned} \tag{4}
$$

for $i, j, k = 1, 2, \ldots, n$. There are similar restrictions upon the range. For an $n \times n \times n$ system f maps these transition probabilities to the $6n^2 \times 6n^2$ matrix Q. Since Q is a transition matrix the image of the forward map lies in $Mat_{\mathbb{R}}(6n^2)$ and for each $Q \in Im(f)$ the following conditions must hold:

$$
0 \leq \sum_{\lambda=1}^{6n^2} Q_{i,\lambda} \leq 1 \qquad i = 1, 2, \ldots, 6n^2 \tag{5}
$$

If $rank(Jac(f(\mathbf{x}))) < 36n^3$ then we cannot hope to invert f at $\mathbf{x}$. If $rank(Jac(f)) = r$ at a generic point,then at most we can recover r pieces of information. At best, we can express the transition probabilities in terms of the data and k independent parameters, where $k = 36n^3 - r$.

In [9] it is shown that any Q generated by the forward map for the $n \times n \times n$ problem contains may rank deficient submatrices and that the forward map is itself rank deficient. For the $2 \times 2 \times 2$ problem the rank of the forward map is generically 240. In the following sections we shall express the unknown transition probabilities in terms of a $36 * 2^3 - 240 = 48$ free parameters and the data.

3 Rewriting the Equations

Given Q we want to recover the one step transition matrices. This is a highly nonlinear problem and requires solving the system of nonlinear equations given in equation 3. Even in the simplest case, the 2×2 problem's counterpart to 3 is an 8×8 array of algebraic equations of degrees 7, 8, and 9.

As for two dimensional problems, one may remove the nonlinearities from 3, (or "move" them to the changes of variables) by making several nonlinear changes of variables [6]. First define, (assuming that P_{ho} is invertible),

$$
\begin{aligned}
A &= P_{ho}^{-1} \\
W &= AP_{hh} \\
X &= P_{io}A \\
Y &= P_{io}W - P_{ih}
\end{aligned} \tag{6}
$$

Assuming that A is invertible we can recover, P_{hh}, P_{io}, and P_{ih} in terms of A if we know W, X, and Y. Under these substitutions, the matrix equation 3 may be rewritten:

$$Q(A - W) - (X - Y) = \Theta \tag{7}$$

Recall that Q is the data, so 7 is linear in the unknown matrices, A, W, X, and Y. Furthermore, the new matrices have special block structures. Clearly, A has the same diagonal block structure as P_{ho}. X is also block diagonal. Finally, W and Y have the same off diagonal block structure as P_{hh} and P_{ih}. The variables for each system/column of equations contains three each of the $A_{i,j}s$, $W_{i,j}s$, $X_{i,j}s$, and $Y_{i,j}s$. The $W_{i,j}s$, $X_{i,j}s$, and $Y_{i,j}s$ are functions of $A_{i,j}s$ which correspond to other columns. Although the variables differ from column to column (exactly 288 variables total—no repeats between columns), the columns are only artificially decoupled.

To each column in 7 there corresponds a system of 24 linear equations in the variables which appear in the corresponding columns of $A - W$ and $X - Y$. As far as their zero structures are concerned, the columns of $A - W$ and $X - Y$ come in pairs. The roles of the $A_{i,j}s$ and $W_{i,j}s$ are reversed in the first and last columns of $A - W$ as are the roles of the $X_{i,j}s$ and $Y_{i,j}s$ in the first and last columns of $X - Y$. Hence, one must solve the same matrix equation for the first and last columns of 7. See table 1. We shall consider column three of 7. The third columns of $A - W$ and $X - Y$ are shown below:

$$\begin{bmatrix} A_{1,3} \\ A_{2,3} \\ A_{3,3} \\ -W_{4,3} \\ -W_{5,3} \\ -W_{6,3} \\ \theta \end{bmatrix} \qquad \begin{bmatrix} X_{1,3} \\ X_{2,3} \\ X_{3,3} \\ -Y_{4,3} \\ -Y_{5,3} \\ -Y_{6,3} \\ \theta \end{bmatrix} \tag{8}$$

respectively, where θ is a column vector of eighteen zeros. The 24 equations in column three can be written as a homogeneous matrix equation:

$$
\begin{bmatrix}
Q_{1,1} & Q_{1,2} & Q_{1,3} & Q_{1,4} & Q_{1,5} & Q_{1,6} & -1 & 0 & 0 & 0 & 0 & 0 \\
Q_{2,1} & Q_{2,2} & Q_{2,3} & Q_{2,4} & Q_{2,5} & Q_{2,6} & 0 & -1 & 0 & 0 & 0 & 0 \\
Q_{3,1} & Q_{3,2} & Q_{3,3} & Q_{3,4} & Q_{3,5} & Q_{3,6} & 0 & 0 & -1 & 0 & 0 & 0 \\
Q_{4,1} & Q_{4,2} & Q_{4,3} & Q_{4,4} & Q_{4,5} & Q_{4,6} & 0 & 0 & 0 & -1 & 0 & 0 \\
Q_{5,1} & Q_{5,2} & Q_{5,3} & Q_{5,4} & Q_{5,5} & Q_{5,6} & 0 & 0 & 0 & 0 & -1 & 0 \\
Q_{6,1} & Q_{6,2} & Q_{6,3} & Q_{6,4} & Q_{6,5} & Q_{6,6} & 0 & 0 & 0 & 0 & 0 & -1 \\
Q_{7,1} & Q_{7,2} & Q_{7,3} & Q_{7,4} & Q_{7,5} & Q_{7,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{8,1} & Q_{8,2} & Q_{8,3} & Q_{8,4} & Q_{8,5} & Q_{8,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{9,1} & Q_{9,2} & Q_{9,3} & Q_{9,4} & Q_{9,5} & Q_{9,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{10,1} & Q_{10,2} & Q_{10,3} & Q_{10,4} & Q_{10,5} & Q_{10,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{11,1} & Q_{11,2} & Q_{11,3} & Q_{11,4} & Q_{11,5} & Q_{11,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{12,1} & Q_{12,2} & Q_{12,3} & Q_{12,4} & Q_{12,5} & Q_{12,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{13,1} & Q_{13,2} & Q_{13,3} & Q_{13,4} & Q_{13,5} & Q_{13,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{14,1} & Q_{14,2} & Q_{14,3} & Q_{14,4} & Q_{14,5} & Q_{14,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{15,1} & Q_{15,2} & Q_{15,3} & Q_{15,4} & Q_{15,5} & Q_{15,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{16,1} & Q_{16,2} & Q_{16,3} & Q_{16,4} & Q_{16,5} & Q_{16,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{17,1} & Q_{17,2} & Q_{17,3} & Q_{17,4} & Q_{17,5} & Q_{17,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{18,1} & Q_{18,2} & Q_{18,3} & Q_{18,4} & Q_{18,5} & Q_{18,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{19,1} & Q_{19,2} & Q_{19,3} & Q_{19,4} & Q_{19,5} & Q_{19,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{20,1} & Q_{20,2} & Q_{20,3} & Q_{20,4} & Q_{20,5} & Q_{20,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{21,1} & Q_{21,2} & Q_{21,3} & Q_{21,4} & Q_{21,5} & Q_{21,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{22,1} & Q_{22,2} & Q_{22,3} & Q_{22,4} & Q_{22,5} & Q_{22,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{23,1} & Q_{23,2} & Q_{23,3} & Q_{23,4} & Q_{23,5} & Q_{23,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{24,1} & Q_{24,2} & Q_{24,3} & Q_{24,4} & Q_{24,5} & Q_{24,6} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
A_{1,3} \\ A_{2,3} \\ A_{3,3} \\ W_{4,3} \\ W_{5,3} \\ W_{6,3} \\ X_{1,3} \\ X_{2,3} \\ X_{3,3} \\ Y_{4,3} \\ Y_{5,3} \\ Y_{6,3}
\end{bmatrix}
\tag{9}
$$

We have twelve sets of homogeneous linear equations like 9 corresponding to twelve 24×12 matrices which satisfy the homogeneous equation $Cx = \theta$. Since the trivial solution would not be interesting enough to write about one may safely assume that there must be other solutions. This is indeed the case since the lower left 18×6 submatrix found in equation 9, represents travel into voxels 111 and 112 from the the other six voxels. As shown in [9] this submatrix is of rank four or less. Since the first six equations in 9 are independent, we may solve 9 for at most $6 + 4 = 10$ of the twelve unknowns in terms of the other two.

4 Solving the Equations

Since the $W_{i,j}s$, $X_{i,j}s$, and $Y_{i,j}s$ are already functions of $A_{i,j}s$, it seems natural to solve for them in terms of the $A_{i,j}s$. One can follow this procedure for all 24 columns, reducing the number of unknowns from 288 to 72. The analogous procedure in two dimensions exhausts the supply of independent equations. In three dimensions, however, we have enough information to solve for one third of the $A_{i,j}$s in terms of the remaining $A_{i,j}$s as well.

To solve 9 for the $W_{i,j}s$, $X_{i,j}s$, $Y_{i,j}s$, and $diag(A)$ in terms of the rest of the $A_{i,j}s$, one need only solve:

$$\begin{bmatrix} Q_{1,3} & Q_{1,4} & Q_{1,5} & Q_{1,6} & -1 & 0 & 0 & 0 & 0 & 0 \\ Q_{2,3} & Q_{2,4} & Q_{2,5} & Q_{2,6} & 0 & -1 & 0 & 0 & 0 & 0 \\ Q_{3,3} & Q_{3,4} & Q_{3,5} & Q_{3,6} & 0 & 0 & -1 & 0 & 0 & 0 \\ Q_{4,3} & Q_{4,4} & Q_{4,5} & Q_{4,6} & 0 & 0 & 0 & -1 & 0 & 0 \\ Q_{5,3} & Q_{5,4} & Q_{5,5} & Q_{5,6} & 0 & 0 & 0 & 0 & -1 & 0 \\ Q_{6,3} & Q_{6,4} & Q_{6,5} & Q_{6,6} & 0 & 0 & 0 & 0 & 0 & -1 \\ Q_{13,3} & Q_{13,4} & Q_{13,5} & Q_{13,6} & 0 & 0 & 0 & 0 & 0 & 0 \\ Q_{14,3} & Q_{14,4} & Q_{14,5} & Q_{14,6} & 0 & 0 & 0 & 0 & 0 & 0 \\ Q_{15,3} & Q_{15,4} & Q_{15,5} & Q_{15,6} & 0 & 0 & 0 & 0 & 0 & 0 \\ Q_{16,3} & Q_{16,4} & Q_{16,5} & Q_{16,6} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_{3,3} \\ W_{4,3} \\ W_{5,3} \\ W_{6,3} \\ X_{1,3} \\ X_{2,3} \\ X_{3,3} \\ Y_{4,3} \\ Y_{5,3} \\ Y_{6,3} \end{bmatrix}$$

$$= \begin{bmatrix} -Q_{1,1} & -Q_{1,2} \\ -Q_{2,1} & -Q_{2,2} \\ -Q_{3,1} & -Q_{3,2} \\ -Q_{4,1} & -Q_{4,2} \\ -Q_{5,1} & -Q_{5,2} \\ -Q_{6,1} & -Q_{6,2} \\ -Q_{13,1} & -Q_{13,2} \\ -Q_{14,1} & -Q_{14,2} \\ -Q_{15,1} & -Q_{15,2} \\ -Q_{16,1} & -Q_{16,2} \end{bmatrix} \begin{bmatrix} A_{1,3} \\ A_{2,3} \end{bmatrix} \tag{10}$$

Notation : Denote the determinant of the submatrix of Q taken from rows $[r_1, r_2, \ldots, r_n]$ and columns $[c_1, c_2, \ldots, c_n]$ as $dQ_{[r_1,r_2,\ldots,r_n],[c_1,c_2,\ldots,c_n]}$

column pairs	*nonzero*	*minors*
1,24	$dQ_{[13,14,15,16],[1,22,23,24]}$	$dQ_{[13,14,15,16],[1,2,3,24]}$
2,11	$dQ_{[13,14,15,16],[2,10,11,12]}$	$dQ_{[13,14,15,16],[1,2,3,11]}$
3,4	$dQ_{[13,14,15,16],[3,4,5,6]}$	$dQ_{[13,14,15,16],[1,2,3,4]}$
5,20	$dQ_{[13,14,15,16],[5,19,20,21]}$	$dQ_{[13,14,15,16],[4,5,6,20]}$
6,7	$dQ_{[13,14,15,16],[6,7,8,9]}$	$dQ_{[13,14,15,16],[4,5,6,7]}$
8,17	$dQ_{[1,2,3,4],[8,16,17,18]}$	$dQ_{[1,2,3,4],[7,8,9,17]}$
9,10	$dQ_{[1,2,3,4],[9,10,11,12]}$	$dQ_{[1,2,3,4],[7,8,9,10]}$
12,13	$dQ_{[1,2,3,4],[12,13,14,15]}$	$dQ_{[1,2,3,4],[10,11,12,13]}$
14,23	$dQ_{[1,2,3,4],[14,22,23,24]}$	$dQ_{[1,2,3,4],[13,14,15,23]}$
15,16	$dQ_{[1,2,3,4],[15,16,17,18]}$	$dQ_{[1,2,3,4],[13,14,15,16]}$
18,19	$dQ_{[1,2,3,4],[18,19,20,21]}$	$dQ_{[1,2,3,4],[16,17,18,19]}$
21,22	$dQ_{[1,2,3,4],[21,22,23,24]}$	$dQ_{[1,2,3,4],[19,20,21,22]}$

Table 1: The columns of 7 come in pairs. Each pair is shown in the left hand column. In order to solve a "column of equations" in 7 we require that a minor of Q is nonzero. These minors are displayed to the right of their corresponding column numbers.

The determinant of the lefthand matrix in 10 is $dQ_{[13,14,15,16],[3,4,5,6]}$.

Equation 10 has a unique solution if and only if the determinant of the matrix $dQ_{[13,14,15,16],[3,4,5,6]} \neq 0$. One finds the same sort of requirement for each of the other columns of 7. Although there are only twelve different (and underdetermined!) matrix equations in terms of the unknowns, we must solve 24 different linear systems of equations in order to solve for the $W_{i,j}s$, $X_{i,j}s$, $Y_{i,j}s$ and $diag(A)$ in terms of the rest of the $A_{i,j}s$. Table 1 shows which columns correspond to the same matrix equation and minors of the data we require to be nonzero.

If the data satisfy these requirements then one can solve the 240 independent equations in 288 variables linearly for the nonzero entries in W, X, and Y and $diag(A)$ in terms of the 48 other variables in $A = P_{ho}^{-1}$. (Note that this choice of equations is *not* unique.)

Because the solutions for the transition probabilities in terms of all 72 of the $A_{i,j}$s are much simpler than their solutions in terms of only the off diagonal elements of A, we first solve in terms of all of the entries of A. Sample solutions from each of the four one step transition submatrices shown below:

Since $P_{ho} = A^{-1}$, the solutions for entries of P_{ho} are especially simple:

$$u122w = \frac{dA_{[7,8],[7,8]}}{dA_{[7,8,9],[7,8,9]}} \tag{11}$$

Solutions for variables from a transition submatrix are all of the same form. For example, all of the transition probabilities in P_{ho} are equal to a 2×2 minor of A divided by a 3×3 minor of A.

One of P_{hh}'s nonzero entries may be written as

$$
\begin{aligned}
s112n = \Big(& A_{8,7} dA_{[4,5],[4,5]} \mathrm{dQ}_{[13,14,15],[4,5,8]} + \\
& A_{9,7} dA_{[4,5],[4,5]} \mathrm{dQ}_{[13,14,15],[4,5,9]} + A_{8,7} dA_{[5,6],[4,5]} \mathrm{dQ}_{[13,14,15],[5,6,8]} + \\
& A_{9,7} dA_{[5,6],[4,5]} \mathrm{dQ}_{[13,14,15],[5,6,9]} + A_{7,7} dA_{[4,6],[4,5]} \mathrm{dQ}_{[13,14,15],[4,6,7]} + \\
& A_{7,7} dA_{[5,6],[4,5]} \mathrm{dQ}_{[13,14,15],[5,6,7]} + A_{8,7} dA_{[4,6],[4,5]} \mathrm{dQ}_{[13,14,15],[4,6,8]} + \\
& A_{9,7} dA_{[4,6],[4,5]} \mathrm{dQ}_{[13,14,15],[4,6,9]} + A_{7,7} dA_{[4,5],[4,5]} \mathrm{dQ}_{[13,14,15],[4,5,7]} \Big) / \\
& \left(dA_{[4,5,6],[4,5,6]} \mathrm{dQ}_{[13,14,15],[4,5,6]} \right)
\end{aligned} \tag{12}
$$

The solutions for entries of P_{ho} and P_{hh} were quite simple (for MAPLE) to compute. The solutions for transition probabilities in P_{io} and P_{ih} were appeared to be extremely messy at first. By grouping terms in the solutions for entries of P_{io} carefully it is possible to simplify them using matrix expansions of the forms

$$
\begin{aligned}
-dA_{[2,3],[1,3]} A_{3,2} + dA_{[2,3],[2,3]} A_{3,1} + dA_{[2,3],[1,2]} A_{3,3} &= 0 \\
dA_{[1,2],[1,2]} A_{3,3} + dA_{[1,2],[2,3]} A_{3,1} - dA_{[1,2],[1,3]} A_{3,2} &= dA_{[1,2,3],[1,2,3]}
\end{aligned} \tag{13}
$$

The resulting solutions are quite simple:

$$
\begin{aligned}
d112u = -dA_{[4,5],[5,6]} \Big(& \mathrm{dQ}_{[6,13,14,15],[1,2,3,4]} A_{4,4} + \mathrm{dQ}_{[6,13,14,15],[1,2,3,6]} A_{6,4} + \\
& \mathrm{dQ}_{[6,13,14,15],[1,2,3,5]} A_{5,4} \Big) / \mathrm{dQ}_{[13,14,15],[1,2,3]} dA_{[4,5,6],[4,5,6]} - \\
dA_{[4,5],[4,6]} \Big(& \mathrm{dQ}_{[6,13,14,15],[6,19,20,21]} A_{6,5} + \mathrm{dQ}_{[6,13,14,15],[4,19,20,21]} A_{4,5} + \\
& \mathrm{dQ}_{[6,13,14,15],[5,19,20,21]} A_{5,5} \Big) / \mathrm{dQ}_{[13,14,15],[19,20,21]} dA_{[4,5,6],[4,5,6]} + \\
dA_{[4,5],[4,5]} \Big(& \mathrm{dQ}_{[6,13,14,15],[6,7,8,9]} A_{6,6} + \mathrm{dQ}_{[6,13,14,15],[5,7,8,9]} A_{5,6} + \\
& \mathrm{dQ}_{[6,13,14,15],[4,7,8,9]} A_{4,6} \Big) / \mathrm{dQ}_{[13,14,15],[7,8,9]} dA_{[4,5,6],[4,5,6]}
\end{aligned} \tag{14}
$$

The identities giving the Graßmann-Plücker embedding can be used to simplify the solutions for entries of P_{ih} considerably. The method for simplifying these solutions is exactly the same as that used in [6, 5]. One of the simplified solutions for a transition probability in P_{ih} is shown below:

$$
\begin{aligned}
d122d = {} & -\mathrm{dQ}_{[13,14,15],[4,5,6]}\mathrm{dQ}_{[1,2,3],[16,17,18]}\mathrm{dQ}_{[13,14,15],[10,11,12]}dA_{[7,8,9],[7,8,9]} \\
& \Big(\mathrm{dQ}_{[7,13,14,15],[7,8,9,12]}A_{12,10} + \mathrm{dQ}_{[7,13,14,15],[7,8,9,10]}A_{10,10} + \\
& \quad \mathrm{dQ}_{[7,13,14,15],[7,8,9,11]}A_{11,10}\Big) + \\
& \mathrm{dQ}_{[13,14,15],[4,5,6]}\mathrm{dQ}_{[1,2,3],[16,17,18]}\Big(A_{10,10}\Big(dA_{[7,8],[7,8]}\mathrm{dQ}_{[13,14,15],[7,8,10]} + \\
& \quad dA_{[7,9],[7,8]}\mathrm{dQ}_{[13,14,15],[7,9,10]} + dA_{[8,9],[7,8]}\mathrm{dQ}_{[13,14,15],[8,9,10]}\Big) + \\
& A_{12,10}\Big(dA_{[7,9],[7,8]}\mathrm{dQ}_{[13,14,15],[7,9,12]} + \\
& \quad dA_{[8,9],[7,8]}\mathrm{dQ}_{[13,14,15],[8,9,12]} + dA_{[7,8],[7,8]}\mathrm{dQ}_{[13,14,15],[7,8,12]}\Big) + \\
& A_{11,10}\Big(dA_{[7,8],[7,8]}\mathrm{dQ}_{[13,14,15],[7,8,11]} + dA_{[8,9],[7,8]}\mathrm{dQ}_{[13,14,15],[8,9,11]} + \\
& \quad dA_{[7,9],[7,8]}\mathrm{dQ}_{[13,14,15],[7,9,11]}\Big)\Big)\Big(\mathrm{dQ}_{[7,13,14,15],[7,10,11,12]}A_{7,9} + \\
& \quad A_{8,9}\mathrm{dQ}_{[7,13,14,15],[8,10,11,12]} + A_{9,9}\mathrm{dQ}_{[7,13,14,15],[9,10,11,12]}\Big) - \\
& \mathrm{dQ}_{[1,2,3],[16,17,18]}\mathrm{dQ}_{[13,14,15],[10,11,12]}\Big(A_{12,10}\Big(dA_{[7,9],[8,9]}\mathrm{dQ}_{[13,14,15],[7,9,12]} + \\
& \quad dA_{[7,8],[8,9]}\mathrm{dQ}_{[13,14,15],[7,8,12]} + dA_{[8,9],[8,9]}\mathrm{dQ}_{[13,14,15],[8,9,12]}\Big) + \\
& A_{10,10}\Big(dA_{[7,8],[8,9]}\mathrm{dQ}_{[13,14,15],[7,8,10]} + dA_{[8,9],[8,9]}\mathrm{dQ}_{[13,14,15],[8,9,10]} + \\
& \quad dA_{[7,9],[8,9]}\mathrm{dQ}_{[13,14,15],[7,9,10]}\Big) + \\
& A_{11,10}\Big(dA_{[8,9],[8,9]}\mathrm{dQ}_{[13,14,15],[8,9,11]} + dA_{[7,9],[8,9]}\mathrm{dQ}_{[13,14,15],[7,9,11]} + \\
& \quad dA_{[7,8],[8,9]}\mathrm{dQ}_{[13,14,15],[7,8,11]}\Big)\Big)\Big(A_{9,7}\mathrm{dQ}_{[7,13,14,15],[4,5,6,9]} + \\
& \quad A_{7,7}\mathrm{dQ}_{[7,13,14,15],[4,5,6,7]} + \mathrm{dQ}_{[7,13,14,15],[4,5,6,8]}A_{8,7}\Big) + \\
& \mathrm{dQ}_{[13,14,15],[4,5,6]}\mathrm{dQ}_{[13,14,15],[10,11,12]}\Big(A_{10,10}\Big(dA_{[8,9],[7,9]}\mathrm{dQ}_{[13,14,15],[8,9,10]} + \\
& \quad dA_{[7,9],[7,9]}\mathrm{dQ}_{[13,14,15],[7,9,10]} + dA_{[7,8],[7,9]}\mathrm{dQ}_{[13,14,15],[7,8,10]}\Big) + \\
& A_{11,10}\Big(dA_{[7,8],[7,9]}\mathrm{dQ}_{[13,14,15],[7,8,11]} + dA_{[7,9],[7,9]}\mathrm{dQ}_{[13,14,15],[7,9,11]} + \\
& \quad dA_{[8,9],[7,9]}\mathrm{dQ}_{[13,14,15],[8,9,11]}\Big) + \\
& A_{12,10}\Big(dA_{[7,8],[7,9]}\mathrm{dQ}_{[13,14,15],[7,8,12]} + dA_{[8,9],[7,9]}\mathrm{dQ}_{[13,14,15],[8,9,12]} + \\
& \quad dA_{[7,9],[7,9]}\mathrm{dQ}_{[13,14,15],[7,9,12]}\Big)\Big)\Big(A_{8,8}\mathrm{dQ}_{[1,2,3,7],[8,16,17,18]} + \\
& \quad A_{7,8}\mathrm{dQ}_{[1,2,3,7],[7,16,17,18]} + \mathrm{dQ}_{[1,2,3,7],[9,16,17,18]}A_{9,8}\Big) \,/ \\
& (dA_{[7,8,9],[7,8,9]}\mathrm{dQ}_{[13,14,15],[7,8,9]}\mathrm{dQ}_{[13,14,15],[10,11,12]}\mathrm{dQ}_{[13,14,15],[4,5,6]}\mathrm{dQ}_{[1,2,3],[16,17,18]})
\end{aligned}
\tag{15}
$$

These solutions,(15, 16, parameter solution to the 2×2 problem where all of the transitions probabilities can be expressed in terms of the $A_{i,j}$s. In this $2 \times 2 \times 2$ problem, however, we can solve for $diag(A)$ in terms of the remaining $A_{i,j}$s. One of the solutions for a diagonal entry of A is shown below:

$$A_{20,20} = -\frac{\mathrm{dQ}_{[13,14,15,16],[4,5,6,21]} A_{21,20} + \mathrm{dQ}_{[13,14,15,16],[4,5,6,19]} A_{19,20}}{\mathrm{dQ}_{[13,14,15,16],[4,5,6,20]}} \tag{16}$$

Substituting solutions for $diag(A)$ into the solutions for the transition probabilities leaves us with a 48 parameter family of solutions to the $2 \times 2 \times 2$ problem.

5 Conclusion

We have seen that from a strictly mathematical perspective there is a $48 = 288/6$ parameter family of solutions to the $2 \times 2 \times 2$ problem. In two dimensions, there are 64 unknown transition probabilities and a $16 = 64/4$ parameter family of solutions to the 2×2 problem. The extension of the two dimensional recovery algorithm to $n \times n$ systems gives a $8n(n+1)$ parameter family of solutions for the $16n^2$ unknown transition probabilities. (For details, see the upcoming U. C. Berkeley. thesis, "Analytic Recovery of Transition Probabilities in Diffuse Tomography"). The analogous extension of the $2 \times 2 \times 2$ problem will doubtless result in a m parameter family of solutions to the $n \times n \times n$ problem. The author's best guess is that m will be $O(24n^3)$.

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