Title

Super-ineffability and Normal Welch Games

Journal

Proceedings of the American Mathematical Society, 154(08)

ISSN

0002-9939

Authors

Eshkol, Julian

Foreman, Matthew

Magidor, Menachem

Publication Date

2026-08-01

DOI

10.1090/proc/17657

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 154 Number 8 August 2026 Pages 3523–3540
https://doi org/10 1090/proc/17657
Article electronically published on May 26 2026

SUPER-INEFFABILITY AND NORMAL WELCH GAMES

JULIAN ESHKOL, MATTHEW FOREMAN, AND MENACHEM MAGIDOR

(Communicated by Vera Fischer)

Abstract. This paper generalizes infinite games played with filters on algebras from the *Welch Games* to playing *normal* filters that concentrate on a fixed base set.

Contents

1. Introduction

Work of Galvin, Jech, Prikry and Magidor ([20], [12]) in the 1970s linked games with precipitous ideals. These results motivated Holy, Nielsen, Schlicht and Welch ([15], [29]) to consider games were Player I plays an increasing sequence of κ-algebras and Player II plays an increasing sequence of κ-complete filters on the algebras. They showed that Player II having a winning strategy in the appropriate games was equivalent to various forms of Ramsey Cardinals.

This led to the work of the second and third authors with Zeman ([11]) that characterized exactly when Player II has a winning strategy in terms of precipitous ideals. Versions of the game have different lengths. The assumptions that Player II can win games of particular lengths γ facilitates the definition of a hierarchy of cardinals of provably increasing strength between a weakly compact cardinal and the first measurable cardinal (assuming the non-existence of saturated ideals on the cardinals κ). See [11] for details.[1] However the methods did not allow specifying a set A_0 in advance and building a normal precipitous ideal that concentrates on A_0. This is done by changing the Welch Game slightly to be the *Normal Welch Game*.

Interesting examples of sets A_0 might be the collection of supercompact cardinals less than κ, or the set of measurable cardinals $\alpha < \kappa$ of order 2 or . . . .

Received by the editors July 27, 2025, and, in revised form, December 10, 2025.

2020 *Mathematics Subject Classification.* Primary 03E55.

The first and second authors were partially supported by NSF grant DMS-2505210; the second author was also supported during a visit to the Max Planck Institute of Mathematics, Bonn, Germany. The third author was partially supported by ISF grant 684-17.

[1] The hierarchy is of provable strength, not consistency strength.

The main result in this paper is:

Theorem 1. *Let κ be an inaccessible cardinal with $2^\kappa = \kappa^+$. Let $A_0 \subseteq \kappa$ and suppose that there is no normal κ^+-saturated ideal on κ concentrating on A_0. Let $\gamma \in [\ _1, \kappa]$ be regular. Then the following are equivalent:*

(A) *Player II has a winning strategy in the Normal Welch Game of length γ based on A_0.*

(B) *κ is completely ineffable and A_0 is positive for the Completely Ineffable Ideal and there is a threading ideal $\mathcal{I}$ concentrating on A_0 with a closed dense tree $D \subseteq \mathcal{P}(\kappa)/\mathcal{I}$ that has height γ and is closed under descending sequences of length less than γ. In particular $\mathcal{I}$ is a precipitous ideal extending the Completely Ineffable Ideal.*

Moreover:

(1) *If Player II has a winning strategy in the Welch Game of length   then κ is completely ineffable.*

(2) *If Player II has a winning strategy in the Welch Game $\mathcal{G}^*$ of length   there is a precipitous threading ideal extending the Completely Ineffable Ideal. If the game concentrates on A_0 then the precipitous threading ideal concentrates on A_0.*[2]

Thus one can build models where Player II wins the Normal Welch Game of length of length γ based on sets such as $\{\alpha < \kappa : o(\alpha) = \alpha^{++}\}$ or on the set $\{\lambda < \kappa : \lambda \text{ is } < \kappa - \text{supercompact}\}$—essentially any given definable stationary set that is positive with respect to the Completely Ineffable Ideal. We give the proof of Theorem 1 in Section 5.

Remark 2. In [11] the game $\mathcal{G}^*$ was defined to be the Welch Game of length   where the winning condition for Player II was that there was a κ-complete ultrafilter extending all of Player II's plays at stages $n <$. The argument for Proposition 4.6 of [11] shows that if Player II can win $\mathcal{G}^*$ then they can win the normal version of $\mathcal{G}^*$ (but without a specified A_0). Thus in the proofs of Theorem 1, one can assume Player II is playing normal ideals.

If Player II can win any version of the Welch Games of length any $\gamma \geq \ + 1$, then Player II can win that version of $\mathcal{G}^*$. Hence the corresponding precipitous ideals exist.

If κ is weakly compact then the Kiesler-Tarski Theorem implies that Player II has a winning strategy in the ordinary Welch Game of length . It was a surprise that Theorem 1 implies:

Corollary 3. *Suppose that Player II has a winning strategy in the ordinary Welch Game $\mathcal{G}^*$ played on a regular cardinal κ with $2^\kappa = \kappa^+$ that does not carry a normal κ^+-saturated ideal. Then κ is completely ineffable. In particular if Player II can win the ordinary Welch Game of length γ for any ordinal $\gamma >$ then κ is completely ineffable.*

The converses are also true: if there are dense closed trees, then Player II has a winning strategy in the normal games of the appropriate length that are played on

[2]To emphasize the distinction between the two Games we will use the phrase “ordinary Welch Game when we are not referring to the “normal Welch game.

any set in the dual of the precipitous ideal. (For games $\mathcal{G}^*$ of length , one shows that Player II can win the Galvin-Jech-Magidor game in [12].)

Complete Ineffability and related ideals. There is a large and deep literature about various ideals related to complete ineffability. Some of them correspond to a hierarchy of large cardinal properties called α-*Ramsey* cardinals. We do not summarize the results here but see the papers of Cody and Holy [9], Feng [10], Gitman [13], Gitman and Welch [14], Holy [16], Holy and Lücke [17], and Johnson [22], [23].

The Completely Ineffable Ideal was defined by Baumgartner in [7] and studied extensively in [22] and [23] (as well as other places cited above). The inductive definition is given in Section 2.3 and is used in the proof of Theorem 1.

In [7] Baumgartner defines the notion of an n-ineffable cardinal, and shows that this is equivalent to functions $f : [\kappa]^{n+1}$ 2 having stationary homogeneous sets. He also gives a sequence of ideals (the n-ineffability ideals) for which properness is equivalent to κ being n-ineffable. In Section 4 we show:

Theorem 4. *Let κ be inaccessible. Then*

(A) *If $\mathcal{I}_n$, the n^{th} stage in the construction of the Completely Ineffable Ideal is proper then there is a stationary set of $\delta < \kappa$ that are n-ineffable.*
(B) *If κ is $n+1$-ineffable then there is a stationary set of weakly compact $\lambda < \kappa$ for which the n^{th} stage of the Completely Ineffable Ideal construction is a proper ideal.*

Thus Baumgartner's n-ineffable ideals are not identical with any stage of the construction of the complete ineffable idea. This result contradicts various assertions made without proof in other literature.

2. Background material

The main motivation of this paper is to study the relationship between particular *Challenge and Response* games and precipitous ideals. "Challenge and response games were defined and are studied in a broad literature largely in other contexts ([15], [9], [13], [14], [16], [17], [29] for examples).

The general subject matter of the relationship between *Large Cardinals*, *Extending Filters to Ultrafilters* and *Consistency Strength* has a long and distinguished history. (See [2], [18], [19], [21], [25], and [28].)

Challenge and Response Games. In a Challenge and Response Game an ordinal γ is fixed and Players I and II alternate moves:

I	$\mathcal{C}_0$	$\mathcal{C}_1$	...	$\mathcal{C}_\alpha$	$\mathcal{C}_{\alpha+1}$	...
II	$\mathcal{R}_0$	$\mathcal{R}_1$	...	$\mathcal{R}_\alpha$	$\mathcal{R}_{\alpha+1}$	...

The game proceeds for some length $\ell \leq \gamma$ determined by the play. Player II wins the game if Player II is able to respond successfully according to the rules for the game of length γ. If Player II does not have an acceptable response or the sequence doesn't satisfy the winning condition for sequences of length γ, then they lose. The game is determined by what constitutes the challenges, what constitutes the responses and the definition of a successful sequence of responses. The games in this paper all are concerned with an inaccessible cardinal κ:

The first two games are the main topic of [11].

The ordinary Welch Game $\mathcal{G}_\gamma$: This game was defined by Nielsen and Welch in work that predated [29], In this game:
- The challenges are a sequence of κ-algebras $\mathcal{A}_\alpha$
- The responses are an increasing sequence $\langle U_\alpha : \alpha < \gamma \rangle$ with each U_α a κ-complete ultrafilter on $\mathcal{A}_\alpha$.

The game $\mathcal{G}^*$ a strengthening of the Welch Game of length : This is the same as the game $\mathcal{G}$ except that for Player II to win there must be a κ-complete ultrafilter on $_{n\in} \mathcal{A}_n$ extending $_{n\in} U_n$.

The Normal Welch Game of length γ:
- There is a base set $A_0 \subseteq \kappa$
- The challenges are a sequence of κ-algebras $\mathcal{A}_\alpha$ containing A_0 with some normality conditions
- The responses are normal ultrafilters U on $\mathcal{A}_\alpha$ with $A_0 \in U$.

The Ineffable Welch Game of length γ:
- There is a base set A_0.
- The challenges are a sequence of κ-algebras $\mathcal{A}_\alpha$ containing A_0 with some normality conditions
- The responses are sets $B_\alpha \subseteq \kappa$ such that B_α generates an ultrafilter U_α on $\mathcal{A}_\alpha$ modulo the non-stationary ideal and for all $\beta < \alpha$, $U_\beta \subseteq U_\alpha$.

The Holy-Schlicht games (in [15]) are very similar to the *Welch Games*. Holy-Lucke (in [17]) play a similar game to the Normal Welch Game, but the winning conditions are more complicated.

Remark 5. Consider two collections of $<2^\kappa$-directed κ-algebras $\subseteq$ such that for all $\mathcal{A} \in$ there is an $\mathcal{A} \in$ with $\mathcal{A} \subseteq \mathcal{A}$. If Player II has a winning strategy in the game where Player I is required to play κ-algebras in , then Player II has a winning strategy in the game where Player I plays elements of .

2.1. **Standard material.** For the reader's benefit we quickly review some of the standard material and definitions such as if $\mathcal{I}$ is a normal ideal on a cardinal κ then whenever $\langle X_\alpha : \alpha < \kappa \rangle$ be a sequence of elements of $\mathcal{I}^+$:

(1) for all $\alpha < \kappa$, $\Delta_{\alpha<\kappa} X_\alpha \subseteq_{\mathcal{I}} X_\alpha$ and
(2) if $B \subseteq_{\mathcal{I}} X_\alpha$ for all α then $B \subseteq_{\mathcal{I}} \Delta_{\alpha<\kappa} X_\alpha$.

Throughout this paper we will be working in an implicit context:

- κ is a regular uncountable cardinal,
- θ is a regular cardinal at least $(2^\kappa)^+$ and
- Δ is a well-ordering of $H(\theta)$.
- a *κ-algebra* is a κ-complete Boolean subalgebra of $\mathcal{P}(\kappa)$ that has cardinality κ.
- If $\langle *, * \rangle : \kappa \times \kappa \quad \kappa$ is a pairing function then every subset $X \subseteq \kappa$ is associated to a sequence of κ-many subsets $\langle X_\alpha : \alpha < \kappa \rangle$ of κ. If $\mathcal{A}$ is a κ-algebra, then $\mathcal{A}$ can defined to be *normal* if whenever $X \in \mathcal{A}$ for $\alpha < \kappa$, the sets $X_\alpha \in \mathcal{A}$ and $\Delta_\alpha X_\alpha \in \mathcal{A}$. "Normal ultrafilters on a normal κ-algebra can be defined similarly. ([11])

Definition 6 appeared with various variations in several other papers such as [15] [14], [17]:

Definition 6. Let $N \prec \langle H(\theta), \in, \Delta\rangle$ have cardinality κ, have $\kappa \subseteq N$, and $N^{<\kappa} \subseteq N$. Then $\mathcal{A} = \mathcal{P}(\kappa) \cap N$ is the *normal κ-algebra based on N*. If U is an ultrafilter on $\mathcal{A}$ that is normal for sequences $\vec{X}$ of sets with $\vec{X} \in N$, then U is called an *N-normal* ultrafilter on $\mathcal{A}$.

We note that the bulleted definition above of a normal κ-algebra is more general and doesn't mention $H(\theta)$ and works similarly well for the purposes of this paper. This definition is more general than “N-normality but “N-normality is convenient to work with and the various games are generally equivalent with either definition.

2.2. **Previous results using Welch Games to build precipitous ideals.** In [11] it is shown that if Player II can win the game $\mathcal{G}^*$ in a model where $2^\kappa = \kappa^+$ then there is a uniform normal precipitous ideal on κ. Player II having a winning strategy in the game of length $\quad + 1$ (which implies they win $\mathcal{G}^*$) is much stronger than κ being weakly compact (or completely ineffable).

In [11] it is shown that if $2^\kappa = \kappa^+$, there is no κ^+-saturated ideal on κ, $\gamma \in [\ _1, \kappa]$ is regular and Player II has a winning strategy in Welch Game of length γ then there is a normal precipitous ideal $\mathcal{I}$ on κ such that $\mathcal{P}(\kappa)/\mathcal{I}$ has a dense subset D that is isomorphic to a $< \gamma$-closed tree.

Moreover, it is immediate that if there is a dense $< \gamma$-closed tree then player II has a winning strategy in the game of length γ.

Remark 7. The proof of the existence of a precipitous ideal on κ from a winning strategy involves a series of four different auxiliary games: $\mathcal{G}_0, \mathcal{G}_1^-, \mathcal{G}_1$ and $\mathcal{G}_2$. Passing from the game $\mathcal{G}_1^-$ to the game $\mathcal{G}_1$, Player II is required to play normal filters. Player II winning the normal game of length follows from Player II winning the game $\mathcal{G}^*$ using the techniques of Proposition 4.6 of [11]. Moreover games $\mathcal{G}_1$ and $\mathcal{G}_2$ used in [11] require Player II to play normal ultrafilters and win a version of the Normal Welch Game.

Remark 7 will be used later in the paper to show that Player II winning even the ordinary Welch Game implies that κ is completely ineffable.

2.3. **The construction of the Completely Ineffable Ideal.** In [7] Baumgartner gave a construction of an ideal on a regular cardinal κ that is built as an increasing sequence of ideals and on page 101, Baumgartner defined a cardinal κ to be *completely ineffable* if the sequence built in the definition ends with a proper ideal. In [22] page 115, the union of the ideals defined by Baumgartner is given the name the *completely ineffable ideal*. The paper [1] gives many equivalent definitions of complete ineffability that are equivalent to those in [22]. In [17], it is shown that the ideal defined by Baumgartner satisfies a property equivalent to being the *Completely Ineffable Ideal*.

Definition 8. Let $\mathcal{I}$ be a normal, κ-complete ideal on a regular uncountable cardinal κ and $\vec{D} = \langle D_\alpha \subseteq \alpha : \alpha \in C\rangle$ be a sequence of subsets of κ. Then

- $D \subseteq \kappa$ *threads* $\vec{D}$ on $B \subseteq C$ if for all $\alpha \in B, D \cap \alpha = D_\alpha$.
- An *$\mathcal{I}$-thread* is a set D such that

$$\{\alpha \in C : D \cap \alpha = D_\alpha\} \in \mathcal{I}^+.$$

- A set $C \subseteq \kappa$ is $\mathcal{I}$*-ineffable* if every sequence $\langle D_\alpha : \alpha \in C\rangle$ has an $\mathcal{I}$-thread.
- κ is *ineffable* if every sequence $\langle D_\alpha \subseteq \alpha : \alpha \in \kappa\rangle$ has an $\mathcal{I}$-thread with $\mathcal{I}$ being the non-stationary ideal.
- A *threading ideal* on κ is a κ-complete, normal ideal $\mathcal{I}$ such that for all $C \in \mathcal{I}^+$, all $\langle D_\alpha : \alpha \in C\rangle$ with $D_\alpha \subseteq \alpha$ there is a D such that
$$\{\alpha \in C : D \cap \alpha = D_\alpha\} \in \mathcal{I}^+.$$

We now describe a procedure to build a threading ideal. It is equivalent to the procedure defined by Baumgartner in [7].

Definition 9. Define a sequence of normal, κ-complete ideals:

(1) Let $\mathcal{I}_0$ be the non-stationary ideal,
(2) for α a limit ordinal, let $\mathcal{I}_\alpha$ be the normal closure of $\ {}_{\beta<\alpha}\mathcal{I}_\beta$,
(3) given $\mathcal{I}_\alpha$ define $\mathcal{I}_{\alpha+1}$ to be the collection of B that are not $\mathcal{I}_\alpha$-ineffable.

Note that κ is ineffable if and only if the ideal $\mathcal{I}_1$ is a proper ideal.

In [7] Baumgartner stated that Proposition 10 is *easy to see* and gave no proof. We give one here.

Proposition 10. *Let κ be a regular uncountable cardinal. Then for all α, $\mathcal{I}_\alpha$ is a normal ideal on κ.*

$\vdash$ This is shown by induction on $\alpha \geq 1$. Normality at limit stages is immediate from the definition. So it suffices to see that $\mathcal{I}_{\alpha+1}$ is normal if $\mathcal{I}_\alpha$ is.

Let $\langle A_\beta \mid \beta < \kappa\rangle$ be a sequence of elements of $\mathcal{I}_{\alpha+1} \setminus \mathcal{I}_\alpha$. We show that the diagonal union $\nabla_{\beta<\kappa} A_\beta \in \mathcal{I}_{\alpha+1}$. Let $h : \kappa \times \kappa \quad \kappa$ be a pairing function on κ. Since $\mathrm{NS}_\kappa \subseteq \mathcal{I}_\alpha \subseteq \mathcal{I}_{\alpha+1}$, we may assume without loss of generality that for every $\gamma \in A_\beta$, γ is closed under h. For every $\beta < \kappa$, we can find $\vec{S}^\beta = \langle S^\beta_\gamma \mid \gamma \in A_\beta\rangle$ which witnesses that A_β is not $\mathcal{I}_\alpha$-ineffable. For $\gamma \in \nabla_{\beta<\kappa} A_\beta$, define
$$T_\gamma = \{h(0,\beta) \mid \beta < \gamma,\ \gamma \in A_\beta\} \cup \{h(1, h(\beta,\delta)) : \delta \in S^\beta_\gamma,\ \beta < \gamma, \gamma \in A_\beta\}.$$
That is, T_γ is a subset of γ that codes the set of $\beta < \gamma$ such that $\gamma \in A_\beta$, as well as the relevant S^β_γ's. We claim that the sequence $\langle T_\gamma \mid \gamma \in \nabla_{\beta<\kappa} A_\beta\rangle$ witnesses that $\nabla_{\beta<\kappa} A_\beta \in \mathcal{I}_{\alpha+1}$; that is, that for every $D \subseteq \kappa$, $\{\gamma \in \nabla_{\beta<\kappa} A_\beta \mid D \cap \gamma = T_\gamma\} \in \mathcal{I}_\alpha$.

Suppose not. Let $D \subseteq \kappa$ be such that $B = \{\gamma \in \nabla_{\beta<\kappa} A_\beta \mid D \cap \gamma = T_\gamma\} \in \mathcal{I}^+_\alpha$. Fix $\gamma \in B$ and some $\beta < \gamma$ such that $\gamma \in A_\beta$. Then $h(0,\beta) \in T_\gamma = D \cap \gamma$, so $h(0,\beta) \in D$. This implies that for every $\delta > \gamma$ with $\delta \in B$, one has $\delta \in A_\beta$. Hence, $B \setminus \gamma \subseteq A_\beta$. Let
$$E = \{\rho < \kappa \mid h(1, h(\beta,\rho)) \in D\}.$$
Now for $\delta \in B \setminus \gamma$, $D \cap \delta = T_\delta$. By the definition of T_δ, $E \cap \delta = S^\beta$. But $B \setminus \gamma \notin \mathcal{I}_\alpha$, so $B \setminus \gamma$ and E show that $\vec{S}^\beta$ is not a witness to $A_\beta \in \mathcal{I}_{\alpha+1}$, a contradiction. $\dashv$

Lemma 11 was shown by Baumgartner in [7], pp. 100-101:

Lemma 11. *Let κ be a regular uncountable cardinal. Then there is a threading ideal if and only if there is a stage α such that $\mathcal{I}_\alpha = \mathcal{I}_{\alpha+1}$ and $\mathcal{I}_\alpha$ is a proper ideal. If α is minimal with this property then $\mathcal{I}_\alpha$ is a subset of any threading ideal. (Hence it is called the* Completely Ineffable Ideal.*)*

Versions of the following comment appear in many places, including [21] and [27]:

Remark 12. Let κ be a regular uncountable cardinal and $\mathcal{A}$ be the normal κ-algebra based on $N \prec (H(\theta), \in, \Delta, \langle D_\alpha : \alpha \in C\rangle)$ (as in Definition 6) where $\vec{D} = \langle D_\alpha \subseteq \alpha : \alpha \in C\rangle \in \mathcal{A}$ is a sequence of subsets of κ. Let $X_\beta = \{\alpha : \beta \in D_\alpha\}$. Let U be an N-normal ultrafilter on $\mathcal{A}$ be such that $\{\beta : X_\beta \in U\} \in \mathcal{A}$. Then there is a set $D \in \mathcal{A}$ such that D threads $\vec{D}$ on a set in $U \cap \mathcal{A}$. In particular if κ is a measurable cardinal and U is a normal measure on κ then the dual of U is a threading ideal.

$\vdash$ Let $X_\beta = \{\alpha \in C : \beta \in D_\alpha\}$ then each $X_\beta \in \mathcal{A}$. Let $D = \{\beta : X_\beta \in U\}$. We claim that D threads $\vec{D}$ on $B = C \cap (\Delta_{\beta \in D} X_\beta) \cap \Delta_{\beta \notin D}(\kappa \setminus X_\beta))$.

Suppose that $\alpha \in B$.

- Let $\gamma \in D \cap \alpha$, then $\alpha \in X_\gamma$ by the definition of D. Hence $\gamma \in D_\alpha$.
- Suppose that $\gamma < \alpha$ and $\gamma \notin D \cap \alpha$. Then, since $\gamma \in \Delta_{\beta \notin D}(\kappa \setminus X_\beta)$, we know $\gamma \in \kappa \setminus X_\alpha$. So $\gamma \notin D_\alpha$.

$\dashv$

Remark 13. In [1] it is shown that being completely ineffable is downwards absolute to L. However, by results of Baumgartner [6], if even the first of these ideals ($\mathcal{I}_1$) is proper then κ is ineffable (and hence weakly compact). It is a result of Kunen [26] that if you start with a model of the form $L[\mu]$ where μ is a normal measure on κ, you can iteratively add homogeneous Suslin trees on cardinals up to and including κ. The result is a model where κ is NOT weakly compact and hence is not completely ineffable. However if you shoot a branch through the generic Suslin tree T on κ, then κ is measurable in the resulting model.

It follows that, assuming measurable cardinals are consistent, there are models $M_0 \subseteq M_1 \subseteq M_2$ such that κ is completely ineffable in both M_0 and M_2 but not in M_1.

Hence complete ineffability is not downwards absolute.

2.4. **Complete Ineffability is necessary.** In [11] it is noted that the Keisler-Tarski Theorem showed that in the Welch Game, at any stage α where there is a κ-complete ultrafilter V on a κ-algebra $\mathcal{A}$ and Player I plays a κ-algebra $\mathcal{A} \supseteq \mathcal{A}$ Player II can play a κ-complete ultrafilter U on $\mathcal{A}$ extending V. For this reason if κ is weakly compact, then Player II has a winning strategy in the Welch Game of length .

For normal games this turns out to be false. We show in this section that Player II has a winning semi-tactic if and only if κ is completely ineffable. It follows that Player II can win the game of length if and only if κ is completely ineffable, and that, if κ is completely ineffable, then in a game at a stage $\alpha + 1$ Player II can extend the game by legal plays to length $\alpha + 2, \alpha + 3 \ldots$.

Rather surprisingly, this result can be used to see that if Player II can win the game $\mathcal{G}^*$ then κ is completely ineffable. Thus the hypotheses of theorems in [11] that give precipitous ideals using Welch Games (even the non-normal games) implies that κ is completely ineffable.

The results in this section contain substantial overlap with results in the Nielsen-Welch paper [29], where a close approximation to Theorem 14 is proved in different terminology. This is discussed in Section 2.5, where the similarities and differences are explained. The authors were unaware of [29] when they proved their results in this section.

To state the next theorem, we need to define the notion of a *semi-tactic*. A semi-tactic for a player is a strategy that only depends on their last play and the last play of the opposing player, not on the previous history. There is a special case in the first inning where there is no last play by Player II. For the games played in this paper at stage $\alpha = 0$ we will view the base set A_0 as the previous play by Player II.

More formally, we break the game into *innings*, where an inning consists of the plays $(\mathcal{A}_\alpha, B_\alpha)$ or $(\mathcal{A}_\alpha, U_\alpha)$, depending on whether the game is the Ineffable Welch Game or the Normal Welch Game. Each inning has an even stage where Player I makes their play and an odd stage where Player II then makes their play.

In the Ineffable Welch Game we consider functions *Tac* whose domain consists of pairs:

$$\{(B, \mathcal{A}) : B \in \mathcal{P}(\kappa) \text{ and } \mathcal{A} \text{ is a normal } \kappa\text{-algebra}\}$$

In the game of length $< \gamma$ we will view pairs like these as:

- $B = A_0$ and $\mathcal{A}$ is the play by Player I in inning $\alpha = 0$, or
- a move B by Player II at an odd stage of an inning $\alpha > 0$ and as a move $\mathcal{A}$ by Player I at an even stage of inning $\alpha + 1$.

In the context of Ineffable Welch Games the function *Tac* is a *winning semi-tactic* if whenever:

- If $\alpha = 0$ and $\mathcal{A}_0$ is a legal play for Player I, then $\mathcal{T}(A_0, \mathcal{A}_0)$ is a legal play for Player II, where A_0 is the base set for the game.
- If $\mathcal{A}_{\alpha+1}$ is a legal play at stage $\alpha + 1$ for Player I and B_α is the play at stage α for Player II, then $\mathcal{T}(B_\alpha, \mathcal{A}_{\alpha+1})$ is a legal play for Player II.

We note that semi-tactics are not defined at limit stages $\alpha > 0$. If $\alpha > 0$ then the semi-tactic requires a move by Player II at stage α and the response by Player I at stage $\alpha + 1$ to compute Player II's response at stage $\alpha + 1$.

Remark. It is immediate that if Player II has a winning semi-tactic then they win the game of length . Moreover starting at an inning $\alpha + 1$ Player II having a winning semi-tactic implies that Player II has wins the game of length $\alpha +$.

Theorem 14. *Consider the Ineffable Welch Game. Then:*

(A) *If Player II has a winning strategy in the game of length then the Completely Ineffable Ideal is proper.*

(B) *If the Completely Ineffable Ideal is proper then Player II has winning semi-tactic whose domain consists of all pairs* $(B_\beta, \mathcal{A}_{\beta+1})$ *where there is* some *legal play* $\langle(\mathcal{A}_\alpha, B_\alpha) : \alpha \leq \beta\rangle$*. In particular, Player II has a strategy that cannot lose except in an inning* α *where* α *is a limit ordinal.*

If the Completely Ineffable Ideal is proper then player II can extend any finite game one more step. Hence Player II can win the game of length .

The proof of this theorem follows immediately from Corollary 16 and Proposition 17 below.

Let $\langle \mathcal{I}_\beta : \beta \leq \alpha\rangle$ be the derivation of the Completely Ineffable Ideal. Let $\mathcal{I} = \mathcal{I}_\alpha$. If $\mathcal{I}$ is a proper ideal it is the Completely Ineffable Ideal.

Proposition 15. *Suppose that $\langle(\mathcal{A}_\gamma, B_\gamma) : \gamma < \delta)\rangle$ is a play of the Ineffable Welch Game, U_γ is the ultrafilter on N_γ determined by B_γ, and $U = \quad_{\gamma<\delta} U_\gamma$. If $U \cap \mathcal{I}_\beta \neq \emptyset$ for some $\beta \leq \delta$, then there is a normal κ algebra $\mathcal{A}_\delta$ such that for all legal responses B_δ by Player II there is a κ-algebra $\mathcal{A}$ such that for all legal plays $B_{\delta+1}$ for Player II if U^* is the ultrafilter determined by $B_{\delta+1}$ then U^* contains an element of $\mathcal{I}_{\beta^*}$ for some $\beta^* < \beta$.*

$\vdash$ Let β be the least ordinal such that $U \cap \mathcal{I}_\beta \neq \emptyset$. There are two cases: β is a limit or a successor.

Case 1. β is a successor ordinal $\beta^* + 1$.

Then, since $B \in \mathcal{I}_\beta$ there is a sequence $\vec{D} = \langle D_\alpha : \alpha \in A\rangle$ such that for no $D \subseteq \kappa$ is $\{\alpha \in B : D \cap \alpha = D_\alpha\} \in \mathcal{I}^+_{\beta^*}$. Let $N \prec \langle H(\lambda), \epsilon, \Delta, \langle(\mathcal{A}_\gamma, B_\gamma) : \gamma < \delta)\rangle, \vec{D}\rangle$ have cardinality κ and be closed under $< \kappa$-sequences. Let $\mathcal{A}_\delta = \mathcal{P}(\kappa) \cap N$ and U_δ be the N-normal ultrafilter on $\mathcal{A}_\delta$ determined by a legal response B_δ by Player II. Let $N^* \prec \langle H(\lambda), \epsilon, \Delta, \langle(\mathcal{A}_\gamma, B_\gamma) : \gamma < \delta)\rangle, \vec{D}, \mathcal{A}_\delta, U_\delta\rangle$ and $\mathcal{A} = \mathcal{P}(\kappa) \cap N^*$.

Let $X_\beta = \{\alpha : \beta \in D_\alpha\}$. Since $\{\beta : X_\beta \in U_\delta\} \in \mathcal{A}$, any legal response B by Player II gives an ultrafilter $U^* \supseteq U_\delta$ on $\mathcal{A}$ such that $\{\beta : X_\beta \in U^*\} \in \mathcal{A}$, Hence, by Remark 12, there is a set $D \in \mathcal{A}$ that threads $\vec{D}$ on a set $B^* \in U \cap A$. Since B is threaded on B^*, $B^* \in \mathcal{I}_{\beta^*}$ for some $\beta^* < \beta$.

Case 2. β is a limit ordinal.

Let $B \in U \cap \mathcal{I}_\beta$. There is a sequence of sets $\langle A^*_\rho : \rho < \kappa\rangle$ such that $A^*_\rho \in \mathcal{I}_{\beta\ \rho)}$ and each $\beta(\rho) < \beta$ such that $B \subseteq \nabla_\rho A^*_\rho$. Let $\mathcal{A}_\delta$ be a normal κ-algebra for some N that contains $\{A^*_\rho : \rho < \kappa\} \cup \{\langle A_\rho : \rho < \delta\rangle\} \cup \{U\}$. Let B_δ be a legal response by Player II. Since the filter U_δ generated by B_δ contains B, $B_\delta \subseteq \nabla_{\gamma<\delta} A_\gamma$ and any normal ultrafilter V determined by B has to contain $\nabla_\rho A_\rho$. We claim U_δ contains some A_ρ with $\rho < \beta$. If not, since $\langle A_\rho : \rho \in \kappa\rangle$ in $\mathcal{A}_\delta$ and U_δ is normal $\Delta_{\rho<\kappa}(\kappa \setminus A_\rho) \in U_\delta$. But $\Delta_{\rho<\kappa}(\kappa \setminus A_\rho) \cap \nabla_{\rho<\kappa} A_\rho$ is non-stationary, a contradiction. $\dashv$

Remark. Note that for any α there is a filter V that can be used in the manner that the filter generated by B_δ in the proof of Proposition 15 that is based on an N_α for $\alpha\ > \alpha$.

Corollary 16. *Suppose that Player II can win the Ineffable Welch Game of length (based on some particular set A_0). Then the Completely Ineffable Ideal is proper. If the length of the game is a limit ordinal δ and Player II has a strategy so that the game doesn t end at a successor stage then the dual of every filter played by Player II extends the dual of the Completely Ineffable Ideal.*

By Remark 7 if Player II has a winning strategy in the game $\mathcal{G}^*$ then Player II has a winning strategy in the Normal Welch Game of length . Since Player II winning any game of length $\alpha >$ in the Welch Game implies that Player II can $\mathcal{G}^*$, this Corollary 16 shows that *All* of the hypotheses in the Welch Games that yield precipitous ideals produce precipitous ideals that extend the Complete Ineffability ideal.

In Section 3, Proposition 18 shows that for each limit ordinal γ, Player II has a winning strategy in the Normal Welch Game just in case they do in the Ineffable Welch Game. Thus Corollary 16 holds for the Normal Welch Game as well as the Ineffable Welch game.

A similar technique shows:

If $\mathcal{I}_n$ is not proper then I wins the game of length n.

Proposition 17. *Suppose that the Completely Ineffable Ideal is proper. Let $\langle(\mathcal{A}_\alpha, B_\alpha) : \alpha < \delta+1\rangle$ be a position in the Ineffable Welch Game where Player II has not lost. Suppose that Player I makes a legal play $\mathcal{A}_{\delta+1}$, then Player II has legal response $B_{\delta+1}$. Moreover this response is given by a semi-tactic.*

$\vdash$ Let $\mathcal{I}$ be the Completely Ineffable Ideal and consider $\mathcal{I} \restriction B_\delta$. Writing $\mathcal{A}_{\delta+1} = \langle X_\beta : \beta < \kappa\rangle$, let $D_\alpha = \{\beta < \alpha : \alpha \in X_\beta\}$. Since B_δ is not in $\mathcal{I}$, there is a threading $B_{\delta+1}$ for $\langle D_\alpha : \alpha < \kappa\rangle$ that is positive for $\mathcal{I} \restriction B_\delta$. In particular $B_{\delta+1} \subseteq_{\mathcal{I}} B_\delta$. Since $B_{\delta+1}$ threads $\langle D_\alpha : \alpha < \kappa\rangle$, it generates an ultrafilter $U_{\delta+1}$ on $\mathcal{A}_{\delta+1}$. In particular $B_{\delta+1}$ is a legal play for Player II. $\dashv$

Remark. The definitions of the Normal Welch Game and the Ineffable Welch Game require specifying a base set A_0 to start the game. The base set is not required for the proofs of Theorem 14 or Corollary 16. To see this take $A_0 = \kappa$.

2.5. **The relationship to the Nielsen-Welch paper.** The paper by [29] by Dan Saattrup Nielsen and Philip Welch gives a proof of a theorem (Theorem 3.13) that is almost equivalent to Theorem 14, but is stated in different language and is slightly weaker. We review it here.

They use the language of being a *coherent α-Ramsey* cardinal, coming from the games defined by Holy and Schlicht ([15]). Being *coherent α-Ramsey* can be seen to be equivalent (in the language of [11] and this paper) to Player II having a winning strategy in the Welch Game of length α. Theorem 3.13 in [29] says:

A cardinal is completely ineffable if and only if it is coherent -Ramsey.

(In the Nielsen-Welch paper, there is some inconsistency of use about whether the definitions in Holy-Schlicht refer to the games of length α or $\alpha+1$. The mathematics is incorrect for games of length $+1$ so we interpret the Nielsen-Welch results to mean length α rather than $\alpha+1$. This makes the results appear correct.)

Thus Nielsen-Welch theorem is equivalent to Theorem 14 in the case that $\alpha =$, *except* that the strategy that is given in the Nielsen-Welch paper is not clearly a semi-tactic.

Moreover the Nielsen-Welch theorem is stated only for games involving U_α for ordinals α less than where as Theorem 14 holds for all non-limit α. The authors of this paper have made no attempt to extend the proof in [29] to the general case or to show that the strategies in [29] are semi-tactics.

3. Equivalence of the games

In this section we prove Proposition 18:

Proposition 18. *Fix a limit ordinal γ and a set $A_0 \subseteq \kappa$. Then the following are equivalent:*

(A) *Player II has a winning strategy in the Ineffable Welch Game of length γ based on A_0.*
(B) *Player II has a winning strategy in the Normal Welch Game of length γ based on A_0.*

$\vdash$ To show that (A) implies (B). Let $\mathcal{A} = \langle X_\beta : \beta < \kappa\rangle$ be a κ-algebra and $D_\alpha = \{\beta < \alpha : \alpha \in X_\beta\}$. Let D thread $\langle D_\alpha : \alpha \in \kappa\rangle$ on a stationary set B and $\underline{D}_\alpha$ be the complement of D_α. Then D^c threads $\langle \underline{D}_\alpha : \alpha < \kappa\rangle$ on B. Hence

$$B \subseteq \Delta_{\beta\in D} X_\beta \cap \Delta_{\beta\notin D}(\kappa \setminus X_\beta) \pmod{NS_\kappa}.$$

It follows that the non-stationary ideal restricted to B induces a normal ultrafilter on the normal κ-algebra generated by $\mathcal{A}$.

Fix a base set A_0. From the definitions of the games if, in the Ineffable Welch Game, Player I plays a sequence of algebras $\langle \mathcal{A}_\alpha : \alpha < \delta\rangle$ and $\langle B_\alpha : \alpha < \delta\rangle$ is the sequence of Player II's responses by a winning strategy $\mathcal{S}$ in the game of length γ then the sequence of ultrafilters $\langle U_\alpha : \alpha < \delta\rangle$ determined by the B_α's is a legal play by Player II in the Normal Welch Game. Hence Player II has a winning strategy in the Normal Welch Game.

To go the other direction, fix a winning strategy $\mathcal{S}$ for Player II in the Normal Welch Game. It requires two moves according to $\mathcal{S}$ to induce one move by a strategy $\mathcal{S}$ in the Ineffable Welch Game of length γ.

Consider the situation where you have two sequences of length $\delta < \gamma$.

$$\overline{NP} = \langle (\mathcal{A}_\alpha, U_\alpha) : \alpha < \delta\rangle \tag{1}$$

$$\overline{IP} = \langle (\mathcal{A}_\alpha, B_\alpha) : \alpha < \delta\rangle \tag{2}$$

such that $\overline{NP}$ is a partial play of the game according to a strategy $\mathcal{S}$ in the Normal Welch Game game based on A_0, and the $\mathcal{A}_\alpha$ are the same in both sequences and arise from a sequence of normal structures $N_\alpha \preceq (H(\theta), \in, \Delta)$. Assume moreover, that each $B_\alpha \subseteq_{NS} A_0$ and induces U_α modulo the non-stationary ideal. We now describe how Player II's strategy $\mathcal{S}$ responds in the Ineffable Welch Game when Player I plays an algebra $\mathcal{A}_\delta$.

Suppose that Player I plays a set $\mathcal{A}_\delta$ that corresponds to some $N_\delta \preceq (H(\theta), \in, \Delta)$. Let $U_\delta = \mathcal{S}\,(\langle \mathcal{A}_\alpha : \alpha \le \delta\rangle)$ Apply the strategy $\mathcal{S}$ in two steps to get plays:

$$(\mathcal{A}_\delta, U_\delta), \quad (\mathcal{A}^*, \mathcal{S}\,(\langle \mathcal{A}_\alpha : \alpha \le \delta\rangle^\frown \mathcal{A}^*))),$$

where $\mathcal{A}^*$ is a normal algebra corresponding to an elementary substructure $M \prec (H(\theta), \in, \Delta, \mathcal{S}\,)$ containing the union $N_\delta \cup \{N_\delta, \mathcal{S}\,, \langle (\mathcal{A}_\alpha, U_\alpha) : \alpha < \delta\rangle, \langle B_\alpha : \alpha < \delta\rangle\}$. Then $U^* = \mathcal{S}\,(\langle \mathcal{A}_\alpha : \alpha \le \delta\rangle^\frown \mathcal{A}^*)))$ is a M-normal ultrafilter containing $U_\delta = \mathcal{S}\,(\langle \mathcal{A}_\alpha : \alpha \le \delta\rangle)$ as a subset. Since N contains an enumeration of U_δ as $\langle X_\alpha : \alpha < \kappa\rangle$, U^* contains some $B \subseteq \Delta_{\alpha<\kappa} X_\alpha$.

Define $\mathcal{S}(\langle U_\alpha : \alpha \le \delta\rangle) = B_\delta$ to be the Δ-least element $B^* \in U^*$ with this property. Then:

- B_δ induces the ultrafilter U_δ.
- B_δ determines a legal move in the Ineffable Welch Game at stage δ in response to $\langle \mathcal{A}_\alpha : \alpha \le \delta\rangle$.

Since S is a winning strategy in the normal game, $A_0 \in U_0$, hence in U_γ, and hence $B_\delta \subseteq_{NS} A_0$.

Applying this inductively shows that (B) implies (A).

Corollary 19. *Let γ be a limit ordinal. Suppose that $\langle (A_\alpha, U_\alpha) : \alpha < \gamma\rangle$ is a play of the Normal Welch Game of length γ where Player II uses a winning strategy $\mathcal{S}$ for the Normal Welch Game. Then there is a sequence $\langle B_{\alpha+2} : \alpha < \gamma\rangle$ of subsets of κ such that*

(1) *$B_{\alpha+2} \in U_{\alpha+2}$ and*
(2) *$B_{\alpha+2}$ induces the ultrafilter $U_{\alpha+1}$.*

4. Comparing complete ineffability and n-ineffability

We will have two notions of threading, one dates to [21], and the other to [6]. We begin with the notion we use to define the Completely Ineffable Ideal.

Remarks similar to the following appear in [17].

Proposition 20. *Suppose that $\mathcal{I}$ is a normal precipitous ideal on a regular cardinal κ. If $\mathcal{I}$ extends the Completely Ineffable Ideal, then for all generic $G \subseteq \mathcal{P}(\kappa)/\mathcal{I}$ if $j : V \quad M \subseteq V[G]$ is the generic embedding with transitive $M \cong Ult(V, G)$ then $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^V$.*

$\vdash$ Let $\dot{S}$ be a term for an element of $\mathcal{P}(\kappa)^M$ and $X \in \mathcal{P}(\kappa)/\mathcal{I}$. It suffices to show that there is a dense set of $Y \in \mathcal{P}(\kappa)/\mathcal{I}$ such that for some $A \in \mathcal{P}(\kappa)^V$, $Y \Vdash \dot{S} = \check{A}$. Then there is a dense set of $X \ \subseteq_{\mathcal{I}} X$ such that for some $f : \kappa \quad \mathcal{P}(\kappa)$

$$X \ \Vdash [f]_G = \dot{S}.$$

Fix such a pair $(X \ , f)$. Then, by the normality of $\mathcal{I}$ we can assume that for all $\alpha \in X \ , f(\alpha) \subseteq \alpha$.

Let $D_\alpha = f(\alpha)$. Since $\mathcal{I}$ extends the Completely Ineffable Ideal $\mathcal{I}$ is a threading ideal. Hence there is a dense open collection of $Y \in P(\kappa)/\mathcal{I}$ below X such that some A threads $\langle D_\alpha : \alpha < \kappa \rangle$ on Y. If Y is in this dense set and $Y \in G$, then $Y \Vdash A = [f]_G$. $\dashv$

All of the precipitous ideals constructed in [11] or in this paper extend the Completely Ineffable Ideal. Hence Proposition 20 applies to them.

Recall the discussion in Remark 13 where the Kunen built a model with an inaccessible κ with a Suslin tree. Forcing with the Suslin tree is κ-c.c. and recovers an elementary embedding $j : V \quad M$ with critical point κ, where M is well-founded. In particular the induced ideal $\mathcal{I}$ on κ is precipitous and forcing with $\mathcal{P}(\kappa)/\mathcal{I}$ adds a new subset of κ. Thus Proposition 20 is not vacuous: it is consistent that there are normal precipitous ideals on inaccessible cardinals that do not concentrate on positives sets for the Completely Ineffable Ideal.

We will use Proposition 21 in the next section. It is similar, but not identical to Lemma 7.1 in [6], and closely related to ideas in [3], [4], [5] and in [8].

Proposition 21. *Let κ be regular and $A \in \mathcal{I}_n^+$, then A is Π^1_{2n}-reflecting. That is, for every Π^1_{2n}-sentence Φ and structure of the form $\mathcal{M} = \langle V_\kappa, \in, B\rangle$ such that $\mathcal{M} \models \Phi$, there is an $\alpha \in A$ such that $\langle V_\alpha, \in, B \cap V_\alpha\rangle \models \Phi$. In particular, Φ reflects to a stationary set.*

$\vdash$ Proceed by induction on n. The case $n = 0$ is trivial since A is stationary. By Remark 23, for $n > 0$ we may assume without loss of generality that every $\alpha \in A$ is inaccessible.

Assume $\mathcal{M} = \langle V_\kappa, \in, B\rangle \models \Phi$, and assume that for every $\alpha \in A$,

$$\langle V_\alpha, \in, B \cap V_\alpha\rangle \models \neg\Phi.$$

The sentence Φ is of the form $\forall X \exists Y \Psi(X, Y)$, where $\Psi(X, Y)$ is a Π^1_{2n-2} formula. Hence for $\alpha \in A$ there is $T_\alpha \subseteq V_\alpha$ such that

$$\mathcal{M}_\alpha = \langle V_\alpha, \in, B \cap V_\alpha\rangle \models \forall Y \neg\Psi\,(T_\alpha, Y)\,.$$

The set T_α can be coded as S_α, a subset of α. Since $A \in \mathcal{I}_n^+$, there are $S \subseteq \kappa$ and $B \subseteq A$, $B \in \mathcal{I}_{n-1}^+$, such that for all $\alpha \in B$, $S \cap \alpha = S_\alpha$. Then S codes a set $T \subseteq V_\kappa$. We claim that $\mathcal{M} \models \forall Y \neg\Psi(T, Y)$. Note that $\mathcal{M} \models \forall X \exists Y \Psi(X, Y)$. Hence, there is $Y \subseteq V_\kappa$ such that $\mathcal{M} \models \Psi(T, Y)$. By the induction assumption for $n-1$ and B, there is $\alpha \in B$ such that $\mathcal{M}_\alpha \models \Psi(T \cap V_\alpha, Y \cap V_\alpha)$, but $T \cap V_\alpha = T_\alpha$. A contradiction. $\dashv$

4.1. **The relation between *ineffable of order* n and n-*ineffabiltiy*** In this section we refine the notion of ineffability. (Similar ideas were studied without a formal name in [7], [16] and [15].)

Definition 22. Let κ be an regular uncountable cardinal. Say that κ is *ineffable of order* α if the ideal $\mathcal{I}_\alpha$ from Definition 9 is proper.

This definition gives a natural strengthening of the classical notion of *ineffable* cardinals. Namely,

- κ is ineffable if and only if κ is ineffable of order 1, and
- if κ is ineffable of order α, then κ is ineffable of order β for all $\beta < \alpha$.

Remark 23. If κ is ineffable, then κ is weakly compact. A proof of this appears in [24] starting on page 214, and is attributed to Kunen.

In [6], Baumgartner defines another strengthening of ineffability; namely, he defines n-ineffability for $n \in$. Here we show that for $n \in$, the notions of n-ineffability and ineffability of order n have interleaving provable strength.

Definition 24 (Baumgartner)**.** Let κ be a regular uncountable cardinal and $A \subseteq \kappa$. For $n \in$, an (n, A)-sequence is a sequence $\vec{S} = \langle S_{\vec{\beta}} \mid \vec{\beta} \in [A]^n \rangle$ such that $S_{\vec{\beta}} \subseteq (\vec{\beta})_0$.

Definition 25 (Baumgartner)**.** Let κ be a regular uncountable uncountable cardinal. An $A \subseteq \kappa$ is n-*ineffable* if for every (n, A)-sequence $\vec{S} = \langle S_{\vec{\beta}} \mid \vec{\beta} \in [A]^n \rangle$, there is $D \subseteq \kappa$ and a stationary $B \subseteq A$ such that for all $\vec{\beta} \in [B]^n$, $D \cap (\vec{\beta})_0 = S_{\vec{\beta}}$.

If $A = \kappa$ we say that κ is n-ineffable.

Terminology. We will say that D *threads* $\vec{S}$ on $[B]^n$.

Note that κ being 1-ineffable is equivalent to κ being ineffable. Let κ be a regular uncountable cardinal.

Theorem (Baumgartner)**.** *κ is n-ineffable is equivalent to the statement:*

> *For all functions $f : [\kappa]^{n+1}$ 2 there is a stationary homogeneous set $A \subseteq \kappa$.*

In this section we prove that property of n-ineffability intertwines with *ineffability of order* n in strength.

Definition 26. Let $\mathcal{A}$ be a structure such that $\kappa \subseteq \operatorname{dom}(\mathcal{A})$. A subset $I \subseteq \kappa$ is a set of *remarkable n-indiscernibles for* $\mathcal{A}$ if for every $\vec{\alpha}, \vec{\beta} \in [I]^n$, any finite sequence $\vec{\gamma}$ in $(\min\left((\vec{\alpha})_0, (\vec{\beta})_0\right))^{<}$ and every formula Φ in the language of $\mathcal{A}$

$$\mathcal{A} \models \Phi(\vec{\gamma}, \vec{\alpha}) \Leftrightarrow \mathcal{A} \models \Phi(\vec{\gamma}, \vec{\beta}).$$

Lemma 27. *Let $A \subseteq \kappa$ be a stationary set. Then A is n-ineffable if and only if κ is inaccessible and every structure $\mathcal{A}$ with $\kappa \subseteq \operatorname{dom}(\mathcal{A})$ in a language $\mathcal{L}$ of cardinality less than κ and a predicate for A has a stationary set $I \subseteq A$ of remarkable n-indiscernibles.*

$\vdash$ Let $\langle \Phi_\delta(\vec{y}) : \delta < |\mathcal{L}| \rangle$ enumerate the formulas in $\mathcal{L}$ with free variables $\vec{y}$.

Suppose that $A \subseteq \kappa$ is n-ineffable. We show that there is a stationary $\mathcal{I} \subseteq A$ consisting of remarkable indiscernibles. Without loss of generality we can assume that there is a function symbol $\langle \cdot, \cdot \rangle$ in $\mathcal{L}$ interpreted in $\mathcal{A}$ as a bijection $\langle x, \vec{y} \rangle$: $|\mathcal{L}| \times [\kappa]^{<} \quad \kappa$.

Since A is stationary we can assume that A is closed under $\langle x, \vec{y} \rangle$ and its inverse. For $\vec{\beta} \in [A]^n$, let

$$S_{\vec{\beta}} = \{ \langle \delta, \vec{\gamma} \rangle : \vec{\gamma} \in (\vec{\beta})_0 \text{ and } \mathcal{A} \models \Phi_\delta(\vec{\gamma}, \vec{\beta}) \}.$$

By n-ineffability, there is a $D \subseteq \kappa$ and a stationary set $B \subseteq A$ such that for all $\vec{\beta} \in [B]^n$, $D \cap (\vec{\beta})_0 = S_{\vec{\beta}}$. Then B is a set of remarkable n-indiscernibles.

For the other direction fix a sequence $\langle S_{\vec{\beta}} : \vec{\beta} \in [A]^n \rangle$. Suppose that every structure $\mathcal{A}$ and language $\mathcal{L}$ as above has a set $\mathcal{I} \subseteq A$ of remarkable n-indiscernibles that is stationary in κ. Let $\mathcal{A} = \langle H(\kappa), \Delta, \epsilon, A, \mathcal{S} \ldots \rangle$, where $\mathcal{S}(\gamma, \vec{\beta})^{\mathcal{A}}$ if and only if $\gamma \in S_{\vec{\beta}}$.

Let $\mathcal{I} \subseteq A$ be a stationary set of remarkable indiscernibles for $\mathcal{A}$. Let $\vec{\alpha}, \vec{\beta} \in [\mathcal{I}]^n$. Assume that $\alpha_0 \leq \beta_0$. Since $\mathcal{I}$ is a remarkable set of indiscernibles for all $\gamma < \alpha_0$, $\gamma \in S_{\vec{\alpha}}$ if and only if $\gamma \in S_{\vec{\beta}}$. Hence $D = \{ \gamma < \kappa : \text{ for some } \vec{\beta} \in [\mathcal{I}]^n, \gamma \in S_{\vec{\beta}} \}$ threads $\langle S_{\vec{\beta}} : \vec{\beta} \in [I]^n \rangle$. $\dashv$

Proposition 28. *If $A \in \mathcal{I}_n^+$ (as in Definition 9), then A is n-ineffable.*

$\vdash$ We proceed by induction on n.

If $n = 1$, let $\langle S_\alpha \mid \alpha \in A \rangle$ be a $(1, A)$-sequence. Since $A \in \mathcal{I}_1^+$ there is $D \subseteq \kappa$ and a stationary set $B \subseteq A$ such that for all $\alpha \in B$, $D \cap \alpha = S_\alpha$. Then D and B witness that A is 1-ineffable.

For $n > 1$, assume that the proposition is true for $n - 1$. Fix a sequence $\vec{S} = \langle S_{\vec{\beta}} : \beta \in [A]^n \rangle$. For $\alpha \in A$, let $\langle \cdot, \cdot \rangle_\alpha : \alpha \times [\alpha]^{n-1} \quad \alpha$ be a bijection preserving the lexicographical order on $\alpha \times [\alpha]^{n-1}$.

For a fixed $\alpha \in A$ define

- $S^\alpha_{\vec{\beta}} = S_{\vec{\beta} \quad \alpha}$
- $S^*_\alpha = \{ \langle \gamma, \vec{\beta} \rangle : \gamma \in S^\alpha_{\vec{\beta}} \rangle$

Since $A \in \mathcal{I}_n^+$, there are $B \subseteq A$ with $B \in \mathcal{I}_{n-1}^+$ and $D \subseteq \kappa$ such that for all $\alpha \in B, D \cap \alpha = S^*_\alpha$.

Define an $((n-1), B)$ sequence by setting $\gamma \in S_{\vec{\beta}}$ if and only if there is an $\alpha \in B$, with $\alpha > \max(\vec{\beta})$ such that $\langle \gamma, \vec{\beta} \rangle \in S^*_\alpha$. This is well defined since D is a threading set.

By induction, there is a stationary $B^* \subseteq B$ and $D^* \subseteq D$ such that D^* threads the $(n-1, B)$-sequence on the stationary set B^*. Then D^* also threads the original (n, A)-sequence on $[B^*]^n$. $\dashv$

Notice that the statement “$A \subseteq \kappa$ is n-ineffable can be expressed as a Π^1_3-sentence in the structure $\langle V_\kappa, \in, A \rangle$: “for every S coding an (n, A)-sequence, there are B and E such that for every club $C \subseteq \kappa$, $E \cap C \neq \emptyset$ and φ, where φ is first-order. So we have the following:

Corollary 29. *If κ is ineffable of order $n > 1$, then there are stationarily many $\delta < \kappa$ which are n-ineffable.*

Note that Corollary 29 is a level by level version of the weaker result, Corollary 4 of [23].

On the other hand:

Proposition 30. *Suppose that κ is $n+1$-ineffable. Then the set of $\delta < \kappa$ which are ineffable of order n is stationary in κ.*

$\vdash$ We consider the fixed structure $\mathcal{M} = \langle H(\kappa^+), \in, \Delta, X\rangle$ where Δ is a well ordering of the structure and $X \subseteq \kappa$. Then $\mathcal{M}$ has definable Skolem functions. By Lemma 27, there is a stationary set $\mathcal{J}$ of remarkable $n+1$-indiscernibles for $\mathcal{M}$. Let $\langle \alpha_i : 0 \leq i \leq n\rangle \in [\mathcal{J}]^{n+1}$. For $i \leq n+1$, let $\mathcal{M}_i$ be the elementary substructure of $\mathcal{M}$ generated by $\alpha_0 \cup \{\alpha_j \mid j < i\}$ and define an embedding $\pi_i : \mathcal{M}_i \quad \mathcal{M}$ by $\pi_i\left(\tau\left(\alpha_0, \ldots, \alpha_{i-1}\right)\right) = \tau\left(\alpha_1, \ldots, \alpha_i\right)$, where τ is one of $\mathcal{M}$'s Skolem functions. Then π_i is a well defined elementary embedding and $j < i < n+1$, $\pi_j = \pi_i \restriction \mathcal{M}_i$. Note that since $\vec{\alpha}$ is a sequence of remarkable indiscernibles, for every $D \subset \alpha_0$, $D \in \mathcal{M}_i$, we have $\pi_i(D) \cap \alpha_0 = D$.

For the rest of this section, let $\mathcal{I}_j$ be the ideal as in Definition 9, constructed relative to α_0 (not κ).

Lemma 31. *Let $i < n+1$ and $A \subseteq \alpha_0$ with $A \in \mathcal{M}_i$. If $\alpha_0 \in \pi_i(A)$ then $A \in \mathcal{I}^+_{n-i}$.*

$\vdash$ Proceeding by reverse induction on i, we show that the lemma holds for every A. If $i = n$, then $\mathcal{I}_0 = NS_\alpha$. If $A \in \mathcal{M}_n$ is not stationary in α_0, then the club C witnessing this is in $\mathcal{M}_n$. But then $\alpha_0 \in \pi_n(C) \cap \pi_n(A) = \emptyset$, a contradiction. Now assume the result holds for $i+1$; we show it for i. Let $A \in \mathcal{M}_i$, $A \subseteq \alpha_0$, be such that $\alpha_0 \in \pi_i(A)$. Suppose that $A \in \mathcal{I}_{n-i}$. Then there is a sequence $\vec{S} = \langle S_\beta \mid \beta \in A\rangle$ such that for $\beta \in A$, $S_\beta \subseteq \beta$ and for every $B \subseteq \alpha_0$ the set $\{\beta < \alpha_0 \mid S_\beta = B \cap \beta\}$ is in $\mathcal{I}_{n-\ i+1)}$. The sequence $\vec{S}$ can be assumed to be in $\mathcal{M}_i$. The domain of $\vec{S}$ is A, so by assumption $\alpha_0 \in \mathrm{dom}(\pi(\vec{S}))$. Let $B = \left(\pi_i(\vec{S})\right)_\alpha$ with $B \subset \alpha_0$. B is definable from the indiscernibles defining $\pi_i(\vec{S})$ (which are $\alpha_1, \ldots \alpha_i$ and in addition α_0). Hence $B \in \mathcal{M}_{i+1}$. Let $C = \{\beta < \alpha_0 \mid \beta \in A,\ B \cap \beta = S_\beta\}$. Then $C \in \mathcal{M}_{i+1}$ as well, so $\pi_{i+1}(C)$ is defined.

$$\pi_{i+1}(B) \cap \alpha_0 = B = \pi_i(\vec{S})_\alpha \ = \pi_{i+1}(\vec{S})_\alpha$$

Thus $\alpha_0 \in \pi_{i+1}(C)$. By the inductive assumption, $C \in \mathcal{I}^+_{n-i-1}$.

Hence we get a contradiction to the assumption that $\vec{S}$ cannot be threaded on a subset of A which in $\mathcal{I}^+_{n-i-1}$. $\dashv$

To finish the proof of Proposition 30, assume that κ is $n+1$-ineffable, and suppose for contradiction that the set of ineffables of order n below κ is non-stationary in κ. Let $X \subseteq \kappa$ be a closed unbounded set with empty intersection with the ineffables of order n. Since κ is $n+1$-ineffable there is a stationary set $I \subseteq \kappa$ of remarkable n-indiscernibles for the structure $\mathcal{M}$. Let $\alpha_0, \ldots \alpha_n \in I \cap X$. .

Let $\pi_n : M_n \quad \mathcal{M}$ be the embedding defined above (given by shifting the ordinals α_i to α_{i+1}). Then $\alpha_0 \in \pi_n(\alpha_0) = \alpha_1$, so by Lemma 31, $\alpha_0 \in (\mathcal{I}_n)^+$. This implies that α_0 is ineffable of order n. But $\alpha \in X$, a contradiction. ⊣

5. The proof of the main theorem

Because of the relationship of playing normal games to the Completely Ineffable Ideal and the structure of the proof in [11] we now explain the full version of the proof of Theorem 1.

The structure of [11] is that it successively defines various games:

- The first is the game G^W_γ which is the Welch Game of length γ. It is called "the game $\mathcal{G}_0$ of length γ. Typically γ is a regular cardinal in the interval $[\ _1, \kappa]$, however it is meaningful in the situation where γ is a successor of a limit ordinal. In this game Players I and II alternate with player I playing an increasing sequence of κ-algebras and Player II playing an increasing sequence of κ-complete ultrafilters on the κ-algebras Player I plays.

  A related special case is called $\mathcal{G}^*$. This game has length and if $\langle \mathcal{A}_i, U_i : i < \ \rangle$ is the play by player II, they win if there is a κ-complete ultrafilter on $\ _n \mathcal{A}_n$ that extends $\ _n U_n$.
- Another game is the game $\mathcal{G}_1$. This is definition 4.4 in [11]. The game $\mathcal{G}_1$ is essentially the same as the Normal Welch Game in this paper. Player II plays ultrafilters U_α on $N_\alpha \cap \mathcal{P}(\kappa)$ where $N \prec H(\theta)$. The ultrafilters are N_α-normal.
- It is then shown that if Player II can win $\mathcal{G}_1$ then they can win another game $\mathcal{G}_2$. (Proposition 4.12 of [11].) The game $\mathcal{G}_2$ is a different version of the Ineffable Welch Game defined in [11]
- Finally it is shown that if there are no κ^+-saturated ideals on κ and Player II wins the game of length γ then there is a precipitous ideal on κ with a dense γ-closed tree of height γ that concentrates on the sets played by player II. The analogous result for the version of $\mathcal{G}^*$ with normal filters implies a precipitous ideal is also shown.

This logical progression shows that once Player II wins the game $\mathcal{G}_2$ there is a precipitous ideal $\mathcal{I}$ on κ where

- $\mathcal{I}$ concentrates on the dual of the first filter played by Player II
- The sets played by Player II in $\mathcal{G}_2$ form a dense set in $\mathcal{P}(\kappa)/\mathcal{I}$ and are a γ-closed tree of height γ

Proposition 18 shows that if Player II has a winning strategy in the Normal Welch Game of length γ, where γ is a limit ordinal, then Player II has a winning strategy in the Ineffable Welch Game of length γ where the associated sequences of filters are the same. But a winning strategy in the Ineffable Welch Game is a winning strategy in the game $\mathcal{G}_2$.

To see that A implies B in Theorem 1, Proposition 18 shows that If Player II has a winning strategy in the Normal Welch Game concentrating on A_0, then Player II can win the game $\mathcal{G}_2$ in a manner that A_0 is in every induced filter. Hence by the techniques in [11] if Player II can win the Normal Welch Game of regular length $\gamma \in [\ _1, \kappa]$, then there is a precipitous ideal $\mathcal{I}$ concentrating on A_0 that has a dense tree $D \subseteq P(\kappa)/\mathcal{I}$ that is closed under descending sequence of length less than γ.

Using the techniques in Section 5, it is immediate that if Player Il wins the game of length γ then the normal filters that Player II plays are all disjoint from the Completely Ineffable Ideal. Hence it is impossible for Player II to play a filter that intersects the Completely Ineffable Ideal. Since the ideals built in [11] all contain the *impossible ideal* (defined in [11]), the Completely Ineffable Ideal is a subset of $\mathcal{I}$.

To see that B implies A, what's left to prove is that if $\mathcal{I}$ is a normal, κ-complete ideal on κ with a dense γ-closed tree of height γ then Player II wins the normal game of length γ. We show that Player II has a winning strategy $\mathcal{S}$ in the Ineffable Welch Game of length γ.

Suppose that $G \subseteq \mathcal{P}(\kappa)/\mathcal{I}$ is generic and M is the transitive collapse of the generic ultrapower. Then Proposition 20 shows that $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^V$. Suppose that $\langle(\mathcal{A}_\alpha, B_\alpha) : \alpha < \delta\rangle \cup \{\mathcal{A}_\delta\}$ is a partial play of the game. By the distributivity of $\mathcal{P}(\kappa)/\mathcal{I}$ there is a positive set $B \in \mathcal{P}(\kappa)/\mathcal{I}$ such that for all $X \in \mathcal{A}_\delta$ B decides the statement $\check{X} \in \dot{G}$. Then, by the normality of $\mathcal{I}$,

$$B \leq_{\mathcal{I}} \Delta\{X \in \mathcal{A}_\delta : B \vdash \dot{X} \in \hat{G}\}.$$

In particular B induces a normal ultrafilter on $\mathcal{A}_\delta$. Let

$$\mathcal{S}(\langle(\mathcal{A}_\alpha, B_\alpha) : \alpha < \delta\rangle \cup \{\mathcal{A}_\delta\}$$

be the least such element B on the closed tree of height γ.

This builds a $<_{\mathcal{I}}$-decreasing sequence of subsets of $\mathcal{P}(\kappa)$. Since γ is regular and the tree $\mathcal{T}$ is $< \gamma$-closed this strategy is a winning strategy for the game of length γ. $\dashv$

References

[1] F. G. Abramson, L. A. Harrington, E. M. Kleinberg, and W. S. Zwicker, *Flipping properties: a unifying thread in the theory of large cardinals*, Ann. Math. Logic **12** (1977), no. 1, 25–58, DOI 10.1016/0003-4843(77)90005-5. MR460120

[2] Arthur W. Apter and Saharon Shelah, *On the strong equality between supercompactness and strong compactness*, Trans. Amer. Math. Soc. **349** (1997), no. 1, 103–128, DOI 10.1090/S0002-9947-97-01531-6. MR1333385

[3] Joan Bagaria, *Derived topologies on ordinals and stationary reflection*, Trans. Amer. Math. Soc. **371** (2019), no. 3, 1981–2002, DOI 10.1090/tran/7366. MR3894041

[4] Joan Bagaria, Menachem Magidor, and Salvador Mancilla, *The consistency strength of hyperstationarity*, J. Math. Log. **2** (2020), no. 1, 2050004, 35, DOI 10.1142/S021906132050004X. MR4094556

[5] Joan Bagaria, Menachem Magidor, and Hiroshi Sakai, *Reflection and indescribability in the constructible universe*, Israel J. Math. **2 8** (2015), no. 1, 1–11, DOI 10.1007/s11856-015-1191-7. MR3416912

[6] J. E. Baumgartner, *Ineffability properties of cardinals. I*, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vols. I, II, III, Colloq. Math. Soc. János Bolyai, Vol. 10, North-Holland, Amsterdam-London, 1975, pp. 109–130. MR384553

[7] James E. Baumgartner, *Ineffability properties of cardinals. II*, Logic, foundations of mathematics and computability theory (Proc. Fifth Internat. Congr. Logic, Methodology and Philos. of Sci., Univ. Western Ontario, London, Ont., 1975), Univ. Western Ontario Ser. Philos. Sci., Vol. 9, Reidel, Dordrecht-Boston, Mass., 1977, pp. 87–106. MR540770

[8] Brent Cody, *A refinement of the Ramsey hierarchy via indescribability*, J. Symb. Log. **85** (2020), no. 2, 773–808, DOI 10.1017/jsl.2019.94. MR4206111

[9] Brent Cody and Peter Holy, *Ideal operators and higher indescribability*, J. Symb. Log. **88** (2023), no. 2, 835–873, DOI 10.1017/jsl.2022.59. MR4594301

[10] Qi Feng, *A hierarchy of Ramsey cardinals*, Ann. Pure Appl. Logic **49** (1990), no. 3, 257–277, DOI 10.1016/0168-0072(90)90028-Z. MR1077260

[11] Matthew Foreman, Menachem Magidor, and Martin Zeman, *Games with filters I*, J. Math. Log. **24** (2024), no. 3, Paper No. 2450003, 53, DOI 10.1142/S021906132450003X. MR4832745

[12] F. Galvin, T. Jech, and M. Magidor, *An ideal game*, J. Symbolic Logic **43** (1978), no. 2, 284–292, DOI 10.2307/2272827. MR485391

[13] Victoria Gitman, *Ramsey-like cardinals*, J. Symbolic Logic **76** (2011), no. 2, 519–540, DOI 10.2178/jsl/1305810762. MR2830415

[14] Victoria Gitman and P. D. Welch, *Ramsey-like cardinals II*, J. Symbolic Logic **76** (2011), no. 2, 541–560, DOI 10.2178/jsl/1305810763. MR2830435

[15] Peter Holy and Philipp Schlicht, *A hierarchy of Ramsey-like cardinals*, Fund. Math. **242** (2018), no. 1, 49–74, DOI 10.4064/fm396-9-2017. MR3800756

[16] Peter Holy, *Ramsey-like operators*, Fund. Math. **259** (2022), no. 2, 111–149, DOI 10.4064/fm989-5-2022. MR4502088

[17] Peter Holy and Philipp Lücke, *Small models large cardinals and induced ideals*, Ann. Pure Appl. Logic **172** (2021), no. 2, Paper No. 102889, 50, DOI 10.1016/j.apal.2020.102889. MR4156888

[18] T. Jech, M. Magidor, W. Mitchell, and K. Prikry, *Precipitous ideals*, J. Symbolic Logic **45** (1980), no. 1, 1–8, DOI 10.2307/2273349. MR560220

[19] Thomas Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded. MR1940513

[20] Thomas Jech and Karel Prikry, *Ideals over uncountable sets: application of almost disjoint functions and generic ultrapowers*, Mem. Amer. Math. Soc. **18** (1979), no. 214, iii+71, DOI 10.1090/memo/0214. MR519927

[21] Thomas J. Jech, *Some combinatorial problems concerning uncountable cardinals*, Ann. Math. Logic **5** (1972/73), 165–198, DOI 10.1016/0003-4843(73)90014-4. MR325397

[22] C. A. Johnson, *Distributive ideals and partition relations*, J. Symbolic Logic **51** (1986), no. 3, 617–625, DOI 10.2307/2274018. MR853844

[23] C. A. Johnson, *More on distributive ideals*, Fund. Math. **128** (1987), no. 2, 113–130, DOI 10.4064/fm-128-2-113-130. MR918427

[24] A. Kanamori and M. Magidor, *The evolution of large cardinal axioms in set theory*, Higher set theory (Proc. Conf., Math. Forschungsinst., Oberwolfach, 1977), Lecture Notes in Math., vol. 669, Springer, Berlin, 1978, pp. 99–275. MR520190

[25] H. J. Keisler and A. Tarski, *From accessible to inaccessible cardinals. Results holding for all accessible cardinal numbers and the problem of their extension to inaccessible ones*, Fund. Math. **53** (1963/64), 225–308, DOI 10.4064/fm-53-3-225-308. MR166107

[26] Kenneth Kunen, *Saturated ideals*, J. Symbolic Logic **43** (1978), no. 1, 65–76, DOI 10.2307/2271949. MR495118

[27] M. Magidor, *Combinatorial characterization of supercompact cardinals*, Proc. Amer. Math. Soc. **42** (1974), 279–285, DOI 10.2307/2039718. MR327518

[28] K. L. Prikry, *Changing measurable into accessible cardinals*, Dissertationes Math. (Rozprawy Mat.) **68** (1970), 55. MR262075

[29] Dan Saattrup Nielsen and Philip Welch, *Games and Ramsey-like cardinals*, J. Symb. Log. **84** (2019), no. 1, 408–437, DOI 10.1017/jsl.2018.75. MR3922802

Department of Mathematics University of California Irvine Irvine California 92697

Email address: `jeshkol@uci edu`

Department of Mathematics University of California Irvine Irvine California 92697

Email address: `mdforema@ad uci edu`

Einstein Institute of Mathematics Hebrew University of Jerusalem Givat Ram. Jerusalem 9190401 Israel

Email address: `mensara@savion huji acil`